INFINITE ABELIAN GROUPS

Volume II

László Fuchs

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Volume II

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INFINITE ABELIAN GROUPS

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VOLUME II

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PREFACE

The theory of abelian groups is a branch of algebra which deals with commutative groups. Curiously enough, it is rather independent of general group theory: its basic ideas and methods bear only a slight resemblance to the noncommutative case, and there are reasons to believe that no other condition on groups is more decisive for the group structure than commutativity.

The present book is devoted to the theory of abelian groups. The study of abelian groups may be recommended for two principal reasons: in the first place, because of the beauty of the results which include some of the best examples of what is called algebraic structure theory; in the second place, it is one of the principal motives of new research in module theory (e.g., for every particular theorem on abelian groups one can ask over what rings the same result holds) and there are other areas of mathematics in which extensive use of abelian group might be very fruitful (structure of homology groups, etc.).

It was the author's original intention to write a second edition of his book "Abelian Groups" (Budapest, 1958). However, it soon became evident that in the last decade the theory of abelian groups has moved too rapidly for a mere revised edition, and consequently, a completely new book has been written which reflects the new aspects of the theory. Some topics (lattice of subgroups, direct decompositions into subsets, etc.) which were treated in "Abelian Groups" will not be touched upon here.

The twin aims of this book are to introduce graduate students to the theory of abelian groups and to provide a young algebraist with a reasonably comprehensive summary of the material on which research in abelian groups can be based. The treatment is by no means intended to be exhaustive or even to yield a complete record of the present status of the theory—this would have been a Sisyphean task, since the subject has become so extensive and is growing almost from day to day. But the author has tried to be fairly complete in what he considers as the main body of up-to-date abelian group theory, and the reader should get a considerable amount of knowledge of the central ideas, the basic results, and the fundamental methods. To assist the reader in this, numerous exercises accompany the text; some of them are straightforward, others serve as additional theory or contain various complements. The exercises are not used in the text except for other exercises, but the reader is advised to attempt some exercises to get a better understanding of the theory. No mathematical knowledge is presupposed beyond the rudiments of abstract algebra, set theory, and topology; however, a certain maturity in mathematical reasoning is required.

The selection of material is unavoidably somewhat subjective. The main emphasis is on structural problems, and proper place is given to homological questions and to some topological considerations. A serious attempt has been made to unify methods, to simplify presentation, and to make the treatment as self-contained as possible. The author has tried to avoid making the discussion too abstract or too technical. With this view in mind, some significant results could not be treated here and maximum generality has not been achieved in those places where this would entail a loss of clarity or a lot of technicalities.

Volume I presents what is fundamental in abelian groups together with the homological aspects of the theory, while Volume II is devoted to the structure theory and to applications. Each volume has a Bibliography listing those works on abelian groups which are referred to in the text. The author has tried to give credit wherever it belongs. In some instances, however, especially in the exercises, it was nearly impossible to credit ideas to their original discoverers. At the end of each chapter, some comments are made on the topics of the chapter, and some further results and generalizations (also to modules) are mentioned which a reader may wish to pursue. Also, research problems are listed which the author thought interesting.

The system of cross-references is self-explanatory. The end of a proof is marked with the symbol □. Problems which, for some reason or other, seemed to be difficult are often marked by an asterisk, as are some sections which a beginning reader may find it wise to skip.

The author is indebted to a number of group theorists for comments and criticisms; sincere thanks are due to all of them. Special thanks go to B. Charles for his numerous helpful comments. The author would like to express his gratitude to the Mathematics Departments of University of Miami, Coral Gables, Florida, and Tulane University, New Orleans, Louisiana, for their assistance in the preparation of the manuscript, and to Academic Press, Inc., for the publication of this book in their prestigious series.

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SEPARABLE p-GROUPS

Our discussion of the major classes of abelian groups begins with the theory of torsion groups. Recall that torsion groups decompose in a unique way into direct sums of p-groups belonging to different primes p, hence the structure theory of torsion groups at once reduces to p-groups. A further reduction to reduced p-groups is immediate. This chapter and the next one are devoted to the theory of reduced p-groups.

In the present chapter we confine ourselves primarily to p-groups without elements of infinite height—for the sake of brevity, we shall call them *separable* p-groups—and leave the general case to the next chapter. Separable p-groups are fundamental in the general theory of p-groups (e.g., the Ulm factors of p-groups are necessarily separable). Every separable p-group is a pure subgroup between its basic subgroup B and the "largest" separable p-group $\bar{B}$ with the same basic subgroup B. These (so-called torsion-complete) p-groups $\bar{B}$ have numerous remarkable properties which we shall develop in **68–71**; in particular, they admit satisfactory complete systems of invariants. Essentially, for no major class of separable p-groups other than direct sums of cyclic groups and torsion-complete groups is a satisfactory structure theory known.

65. LEMMAS ON p-GROUPS

Our study of p-groups starts with p-groups A without elements of infinite height. The absence of elements of infinite height means, in other words, that the first Ulm subgroup of A vanishes, $A^1 = 0$. This section is devoted to some preliminary results mainly on such p-groups.

First, we introduce a terminology. An arbitrary group A is said to be *separable* if every finite subset $\{a_1, \cdots, a_n\}$ of A can be embedded in a direct summand S of A such that S is a direct sum of groups of rank 1. Clearly, S may then be chosen as a finite direct sum of rank 1 groups; for p-groups S, this amounts to being finitely cogenerated [see (25.1)]. Because of their structure, divisible groups are separable; and it is easy to see that a group is separable exactly if its reduced part is separable.

Proposition 65.1. *A reduced p-group is separable if and only if it contains no element* $\neq 0$ *of infinite height.*

If A is a separable reduced p-group, then every element of A is contained in a finite direct summand of A, hence A has no elements of infinite height. Conversely, if A has no elements of infinite height and $\{a_1, \cdots, a_n\}$ is a finite subset of A, then $\langle a_1, \cdots, a_n\rangle$ is a finite subgroup with elements of bounded heights, thus by (27.8) it can be embedded in a bounded direct summand S of A. That S is a direct sum of rank 1 groups is shown by (17.2).□

In view of (65.1), p-groups without elements of infinite height will be called, for the sake of brevity, *separable p-groups*. [For separable torsion-free groups, see **87**.]

Evidently, a p-group is Hausdorff in its p-adic topology exactly if it is separable. Though we shall have occasion to consider other topologies as well, *p-groups will always be regarded as equipped with their p-adic topologies*, unless specifically stated to the contrary.

In accordance with this, for a subset X of a p-group A, X^- will denote the topological closure of X in the p-adic topology. Manifestly, X^- is a subgroup of A whenever X is one; as a matter of fact, X^-/X is then exactly the first Ulm subgroup of A/X.

Notice that, in a separable p-group A, the subgroups $A[p^n]$ for $n = 0, 1, \cdots$ are all closed. If G is a closed subgroup of A, then $G[p^n] = G \cap A[p^n]$—as the intersection of two closed subgroups—is likewise closed. For pure subgroups, we have the converse statement:

Lemma 65.2. *A pure subgroup G of a separable p-group A is closed if and only if, for some* $n \geqq 1$, $G[p^n]$ *is closed in* $A[p^n]$.

If $G[p^n]$ is closed in $A[p^n]$ for some n, then $G[p]$ is closed in $A[p]$. Assuming G is not closed, i.e., $(A/G)^1 \neq 0$, there exists a coset $a + G$ of order p and of infinite height in A/G. By (28.1), a may be chosen of order p. For every k, there are $x \in A$ and $g \in G$ such that $p^k x = a + g$, whence $p^{k+1}x = pg \in G$, and so some $h \in G$ satisfies $p^{k+1}h = pg$. Therefore, $p^k(x - h) = a + (g - p^k h)$ with $g - p^k h \in G[p]$ implies a is contained in the closure of $G[p]$, i.e., $a \in G[p] \leqq G$, a contradiction.□

Let A be a reduced p-group and $a \in A$ of order p^n. With a we associate the increasing sequence

(1) $$H(a) = (h^*(a), h^*(pa), \cdots, h^*(p^n a) = \infty)$$

of ordinals and the symbol ∞; here h^* denotes the generalized height at the prime p as defined in **37**; i.e., $h^*(a) = \sigma$ whenever $a \in p^\sigma A \setminus p^{\sigma+1}A$, while we set $h^*(0) = \infty$. We shall call $H(a)$ the *indicator* or the *Ulm sequence* of a. It is often convenient to continue this sequence with symbols ∞ *ad infinitum*:

$$H(a) = (h^*(a), h^*(pa), \cdots, h^*(p^n a), h^*(p^{n+1}a), \cdots). \tag{2}$$

It will always be clear from the context which of the forms (1) and (2) is to be used.

Evidently, for every element a, $h^*(p^i a) < h^*(p^{i+1}a)$ holds provided $p^i a \neq 0$. If

$$h^*(p^i a) + 1 < h^*(p^{i+1}a),$$

then the indicator of a is said to have a *gap* between $h^*(p^i a)$ and $h^*(p^{i+1}a)$. If $o(a) = p^n$, then there is a gap between $h^*(p^{n-1}a)$ and $h^*(p^n a) = \infty$.

The following lemma describes precisely what indicators look like. Recall that the σth Ulm–Kaplansky invariant $f_\sigma(A)$ of a p-group A was defined as the rank of $p^\sigma A[p]/p^{\sigma+1}A[p]$ [since there is no danger of ambiguity, we write simply $p^\sigma A[p]$ for $(p^\sigma A)[p]$].

Lemma 65.3 (Kaplansky [3]). *Let A be a reduced p-group and $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ a strictly increasing sequence of ordinals. Then $(\sigma_0, \sigma_1, \cdots, \sigma_{n-1}, \sigma_n = \infty)$ is the indicator of some $a \in A$ if and only if the following condition holds:*

(*) *If there is a gap between σ_i and σ_{i+1}, then the σ_ith Ulm–Kaplansky invariant of A is different from* 0.

Assume first that in (1) a gap occurs between $h^*(p^i a)$ and $h^*(p^{i+1}a)$ for some $a \in A$. This means there is a $b \in A$ such that $p^{i+1}a = pb$, with $h^*(b) > h^*(p^i a) = \sigma_i$. Now $c = p^i a - b$ is of order p and of height $\min(h^*(p^i a), h^*(b)) = \sigma_i$, hence the group

$$p^{\sigma_i}A[p]/p^{\sigma_i+1}A[p] \neq 0,$$

and its rank $f_{\sigma_i}(A) \neq 0$.

Conversely, let $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}, \sigma_n = \infty$ satisfy condition (*). Then we can choose an $a_{n-1} \in A[p]$ of height $\sigma_{n-1} \neq \infty$. If there is no gap between σ_{n-2} and σ_{n-1}, then we pick an $a_{n-2} \in A$ of height σ_{n-2} such that $pa_{n-2} = a_{n-1}$. If there is a gap between them, then (*) ensures the existence of a $b \in A[p]$ of height σ_{n-2}. There is a $c \in A$ of height $> \sigma_{n-2}$ such that $pc = a_{n-1}$. Now $a_{n-2} = b + c$ will be of height σ_{n-2} and will satisfy $pa_{n-2} = a_{n-1}$. Thus proceeding, we obtain successively elements $a_{n-1}, a_{n-2}, \cdots, a_0$ of A such that $pa_i = a_{i+1}$ and $h^*(a_i) = \sigma_i$ $(i = 0, \cdots, n-1)$. Obviously, $a = a_0$ has the indicator $(\sigma_0, \sigma_1, \cdots, \sigma_{n-1}, \infty)$.□

Lemma 65.4 (Baer [5]). *Let A be a p-group and $(r_0, r_1, \cdots, r_{n-1}, r_n = \infty)$ the indicator of $a \neq 0$ in A, where r_i $(i < n)$ are integers. Set $r_0 = k_1$, and let*

$$r_{n_1} = n_1 + k_2, \cdots, r_{n_{t-1}} = n_{t-1} + k_t, r_{n_t} = r_n = \infty,$$

be the r_i before which gaps occur. Then

$$0 < n_1 < \cdots < n_t, \qquad \text{and} \qquad 0 \leqq k_1 < \cdots < k_t,$$

and there exist elements $c_1, \cdots, c_t \in A$ such that:

(i) *they are independent and $o(c_i) = p^{n_i + k_i}$ $(i = 1, \cdots, t)$;*
(ii) *$C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_t \rangle$ is a direct summand of A;*
(iii) *$a = p^{k_1}c_1 + \cdots + p^{k_t}c_t$ holds.*

Proof by induction on the exponent $e(a)$ of a. If this is 1, then the indicator of a is (k_1, ∞) and everything follows from (27.2). Thus let $e(a) = m + 1$ and assume (65.4) true for elements of exponents $\leqq m$. The inequalities for the n_i and k_i are obvious. Consider $p^{n_{t-1}}a$, which is of height $n_{t-1} + k_t$, and let $c_t \in A$ be chosen such that

$$p^{n_{t-1}+k_t}c_t = p^{n_{t-1}}a.$$

Then $\langle c_t \rangle$ is pure in A and is of order $p^{n_t + k_t}$. Now $a' = a - p^{k_t}c_t$ is of exponent $\leqq n_{t-1} \leqq m$ and—as is easily verified—has the indicator $(r_0, r_1, \cdots, r_{n_{t-1}-1}, \infty)$. By induction hypothesis, for a' there are elements $c_1, \cdots, c_{t-1} \in A$ of the desired kind. $\langle c_t \rangle$ intersects trivially $\langle c_1 \rangle \oplus \cdots \oplus \langle c_{t-1} \rangle = C'$, since $p^{n_t+k_t-1}c_t$ is of order p and of height $n_t + k_t - 1$, while C' does not contain such an element. $C = C' \oplus \langle c_t \rangle$ must be pure in A, since the height of an element in the socle of C is exactly the minimum of the heights of its C'- and $\langle c_t \rangle$-coordinates; thus the elements in the socle of C have the same height in C as in A, whence (27.5) completes the proof.□

The indicators can be partially ordered componentwise. That is,

$$H(a) \leqq H(b) \qquad \text{means} \qquad h^*(p^i a) \leqq h^*(p^i b) \qquad \text{for} \quad i = 0, 1, 2, \cdots.$$

Lemma 65.5. *Let A be a p-group such that $A^1 = 0$ and let $a, b \in A$. There is an endomorphism of A mapping a upon b if and only if $H(a) \leqq H(b)$.*

Since endomorphisms never decrease heights, the necessity is obvious. Therefore, assume $H(a) \leqq H(b)$. By the preceding lemma, we can embed a and b in direct summands $C = \langle c_1 \rangle \oplus \cdots \oplus \langle c_t \rangle$ and $D = \langle d_1 \rangle \oplus \cdots \oplus \langle d_s \rangle$ of A, respectively. It suffices to exhibit an $\eta: C \to D$ carrying a into b. Write a in the form (65.4)(iii) and, analogously,

$$b = p^{l_1}d_1 + \cdots + p^{l_s}d_s,$$

where $e(d_j) = m_j + l_j$, $0 < m_1 < \cdots < m_s$, $0 \leqq l_1 < \cdots < l_s$. Notice that

$$\infty = h^*(p^{n_t}a) \leqq h^*(p^{n_t}b)$$

implies $p^{n_t}b = 0$, thus $n_t \geqq m_s$, and therefore we may let η map c_t upon

$$p^{l_j - k_t} d_j + \cdots + p^{l_s - k_t} d_s,$$

where j is the least index with $n_t \geqq m_j$ and $k_t \leqq l_j$ [then the image of c_t will be of exponent $\leqq e(c_t) = n_t + k_t$]. Next we consider

$$a' = a - p^{n_t}c_t \qquad \text{and} \qquad b' = b - p^{l_j}d_j - \cdots - p^{l_s}d_s,$$

rather than a and b. Then $H(a') \leqq H(b')$ and, by induction, $\langle c_1 \rangle \oplus \cdots \oplus \langle c_{t-1} \rangle$ has a homomorphism into $\langle d_1 \rangle \oplus \cdots \oplus \langle d_{j-1} \rangle$ carrying a' into b'. This extends to a homomorphism $\eta \colon C \to D$ [with η acting on c_t as indicated] such that $\eta a = b$.$\square$

Following Kaplansky [3], we call a reduced p-group *fully transitive* if, for any two a,b of its elements with $H(a) \leqq H(b)$, there exists an endomorphism of the group sending a upon b. (65.5) asserts that separable p-groups are fully transitive.

The class of fully transitive p-groups includes, besides the separable groups, other important classes of groups, like the totally projective p-groups.

EXERCISES

1. Show that an arbitrary group is separable if and only if its reduced part is separable.
2. A pure subgroup is dense in A exactly if its socle is dense in $A[p]$. [For density, cf. **66**, too.]
3. (Charles [4]) Let A be a p-group and $a \in A$ with $\langle a \rangle \cap A^1 = 0$. Then a can be embedded in a minimal pure subgroup of A and any two minimal pure subgroups containing a are isomorphic over $\langle a \rangle$.
4. Show that the indicators of elements in a separable p-group form a distributive lattice under the partial order defined above.
5. Describe all possible indicators in a finite group of type $(p^{k_1}, \cdots, p^{k_m})$ with $k_1 < \cdots < k_m$.
6. (a) Prove (65.5) for automorphisms, replacing inequality by equality.
   (b) Improve on (65.5) by replacing the condition on the separability of A by $\langle a \rangle \cap A^1 = 0$.
7.* (Kaplansky [3]) By making use of the structure theory of countable p-groups, show that countable p-groups are fully transitive.

66. SUBSOCLES

In the preceding chapters, we have had several occasions to consider the socles of p-groups; as a matter of fact, some of our proofs relied heavily upon properties of the socle. The socle and its subgroups play a rather significant role in the theory.

A subgroup S of the socle $A[p]$ of a p-group A will be called a *subsocle* of A. A subsocle S is said *to support* a subgroup C of A if $C[p] = S$. Evidently, we say that a subsocle S is *closed* if $S^- = S$, and *dense* if $S^- = A[p]$, closures being taken in the p-adic topology. Finally, S is said to be *discrete* if $S \cap p^nA = 0$ for some n, i.e., the elements of S are of bounded heights.

We start with two lemmas on the socles of summands.

Lemma 66.1 (Enochs [1]). *Let $A = B \oplus C$ be a p-group and G a pure subgroup of A such that $G[p] = B[p]$. Then $A = G \oplus C$.*

Clearly, $G \cap C = 0$ and $G + C$ contains the socle of A. Write $a \in A[p]$ in the form $a = b + c$ ($b \in B[p] = G[p]$, $c \in C[p]$); then the height of a in $G + C$ is $\geqq \min(h(b), h(c))$ = height of a in A. Thus $G + C$ is pure in a and so $G + C = A$ follows.□

Note that (66.1) implies that *a pure subgroup supported by the socle of a summand is an isomorphic summand.*

Lemma 66.2 (Crawley [4]). *Let*

$$A = U \oplus V \oplus W = B \oplus C$$

be a p-group such that

$$U[p] \leqq B[p] \leqq U[p] \oplus V[p].$$

Then there exists a subgroup B^* of B such that $B^*[p] = V[p] \cap B[p]$, B^* is isomorphic to a subgroup of V, and

$$A = U \oplus B^* \oplus C.$$

The projection of A onto B maps U isomorphically onto a subgroup B' of B such that $B'[p] = U[p]$. Clearly, B' is pure in B, and so in A, thus (66.1) implies $A = B' \oplus V \oplus W$. Hence $B = B' \oplus B^*$ with $B^* = (V \oplus W) \cap B$. By hypothesis on $B[p]$, $B^*[p] = V[p] \cap B[p]$. Now $B^* \cap W = 0$ implies that B^* is isomorphic to a subgroup of V. Finally, application of (66.1) yields $A = B' \oplus B^* \oplus C = U \oplus B^* \oplus C$.□

For subsocles, one of the basic questions is to find out which of them support pure subgroups. If the subsocle is discrete, then is is clear that it supports a pure subgroup, but this is not true for arbitrary subsocles [cf. Ex. 9]. However, for dense subsocles we can prove the following result.

Theorem 66.3 (Hill and Megibben [3]). *Let S be a dense subsocle of the p-group A. There is a subgroup C of A maximal with respect to the property $C[p] = S$; such a C is pure and dense in A.*

The existence of a subgroup C of the stated kind follows at once from Zorn's lemma. We use induction to prove $C \cap p^n A \leqq p^n C$. For $n = 1$, let $pa = c \in C$ with $a \in A$. If $a \notin C$, then by maximality there is a $b \in \langle C, a \rangle$ of order p not in S; write $b = -c' + ka$ for some $c' \in C$ and integer k $(1 \leqq k \leqq p - 1)$ which may be assumed 1, without loss of generality. Then $pc' = p(a - b) = pa = c$. Assume now $C \cap p^n A \leqq p^n C$ true for a certain $n \geqq 1$, and let $a \in A$ satisfy $p^{n+1} a \in C$. By what has been shown, $p^{n+1} a = pc$ for some $c \in C$. The density of S in $A[p]$ and $p^n a - c \in A[p]$ imply the existence of a $d \in S$ such that $p^n a - c - d \in p^n A$. By induction hypothesis, some $c_1 \in C$ satisfies $p^n c_1 = c + d$. Hence $p^{n+1} c_1 = pc = p^{n+1} a$, and the purity of C follows.

From (28.1) we know that elements of order p in A/C can be represented by elements of order p in A. By the density of S, the elements of order p in A/C are thus of infinite height in A/C. Hence A/C is divisible [see **20**(C)] and C is dense in A.□

Pure subgroups with the same socle need not be isomorphic. Moreover, we have:

Theorem 66.4 (Hill and Megibben [3]). *Let A be a reduced p-group of the power of the continuum $\mathfrak{c}$ with a countable basic subgroup. If S is a proper dense subsocle of A such that $|S| = \mathfrak{c}$, then S supports $2^{\mathfrak{c}}$ pairwise nonisomorphic pure subgroups of A.*

Hypotheses imply that A is unbounded. In view of (66.3) there exists a pure subgroup C which is maximal with respect to the property of being supported by S. C has again a countable basic subgroup [for this extends to a basic subgroup of A], whence it follows at once that $pC \cap S$ is of the power $\mathfrak{c}$. We can, therefore, pick an independent set $L = \{c_i\}_{i \in I}$ of elements c_i $(\in C)$ which are of order p^2 such that $|I| = \mathfrak{c}$. By assumption, there is a $b \in A[p] \setminus S$. We define $2^{\mathfrak{c}}$ sets $L' = \{c_i'\}_{i \in I}$ such that, for each $i \in I$, we put either $c_i' = c_i$ or $c_i' = c_i + b$. Since the socle of $\langle L' \rangle$ is contained in S, there is a subgroup C' of A containing L' and being maximal with respect to $C'[p] = S$. By (66.3), C' is pure. In this way, to every L' we can find a pure subgroup C' containing L' and supported by S.

If L' and L'' are different sets of the stated kind, then the corresponding pure subgroups C' and C'' are distinct, since for no i may both c_i and $c_i + b$ belong to a subgroup supported by S. Therefore, A contains [at least] $2^{\mathfrak{c}}$ different pure subgroups C' supported by S.

By the countability of the basic subgroup B' of C', $\mathrm{Hom}(B', A) =$

$\prod_n \prod A[p^n]$ is of the power $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$. The continuum is an upper bound for $\mathrm{Hom}(C', A)$, as is clear from the exact sequence

$$0 = \mathrm{Hom}(C'/B', A) \to \mathrm{Hom}(C', A) \to \mathrm{Hom}(B', A).$$

This shows that among our groups C' there are at most $\mathfrak{c}$ isomorphic ones, and hence the set of nonisomorphic C' is of the power $2^{\mathfrak{c}}$.□

It is straightforward to check that the preceding proof (due to Cutler and Winthrop [1]) carries over to any reduced p-group A whenever $|A| = 2^{\mathfrak{n}}$, where $\mathfrak{n}$ is the cardinality of the basic subgroup B of A. Needless to say, $\mathfrak{c}$ in (66.4) must then be replaced by $|A|$, and in the final part of the argument the generalized continuum hypothesis is needed in order to infer that $2^{|B|} < 2^{|C|}$.

An immediate corollary to (66.4) is a result by Leptin [1]: *there exist* $2^{\mathfrak{c}}$ *pairwise nonisomorphic p-groups without elements of infinite height whose cardinality is* $\mathfrak{c}$; moreover, all of these may be assumed to have the basic subgroup $\bigoplus_{n=1}^{\infty} Z(p^n)$.

EXERCISES

1. (Hill and Megibben [1]) A neat subgroup supported by a dense subsocle is pure.
2. If every closed subsocle supports a pure subgroup, then every subsocle does the same.
3. Let A be a direct sum of cyclic and quasicyclic p-groups. Then every subsocle of A supports a pure subgroup.
4. Give the details of proof for the assertion stated after the proof of (66.4).
5.* (Hill and Megibben [1]) Prove that there exist $2^{\mathfrak{c}}$ nonisomorphic separable p-groups A with the same basic subgroup B such that $|A| = \mathfrak{c}$ and $\bar{B}/A \cong Z(p^{\infty})$. [For $\bar{B}$, cf. **68**.]
6.* Assume $\mathfrak{m}$ is a cardinal number such that $\mathfrak{n} < \mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$, for some cardinal $\mathfrak{n}$. Then there exist $2^{\mathfrak{m}}$ nonisomorphic separable p-groups of cardinality $\mathfrak{m}$. [*Hint*: (66.4) and (68.2).]
7. (a) Two pure subgroups of a p-group have isomorphic basic subgroups if they have the same socle.

   (b) In a direct sum of cyclic p-groups, two pure subgroups with the same socle are isomorphic.
8.* (Hill [2]) Give an example of a separable p-group that contains nonisomorphic pure subgroups with the same socle. [*Hint*: let

$$B' = \bigoplus_n \langle a_{2n-1} \rangle \qquad \text{and} \qquad B'' = \bigoplus_n \langle a_{2n} \rangle,$$

with $o(a_k) = p^{2k}$, and set $C = \bigoplus_n \langle a_{2n-1} + pa_{2n} \rangle$; then in the torsion-complete group $\bar{B}' \oplus \bar{B}''$ [see **68**] the subgroups $G = B' \oplus \bar{B}''$ and $H =$

$\bar{C} + B''$ have the same socle; under an isomorphism $G \to H$, G must have an element carried outside of H.]

9. (Megibben [1]) Show that a subsocle of a separable p-group need not support a pure subgroup. [*Hint*: $\bar{B}''[p]$ in H as in Ex. 8.]
10. (Irwin and Swanek [1]) (a) If G is a pure subgroup of the p-group A such that $A[p]/G[p]$ supports a pure subgroup of $A/G[p]$, then G is a summand of A. [*Hint*: if $K/G[p]$ is pure supported by $A[p]/G[p]$, then $K = G[p] \oplus H$ for some H and $A = G \oplus H$.]
(b) Let $0 \to G \to C \to A \to 0$ be a pure-projective resolution of the p-group A, where A is not a direct sum of cyclic groups. Then $C[p]/G[p]$ does not support a pure subgroup of $C/G[p]$.
11. (Dieudonné [2]) Let $C = \prod_{k=1}^{\infty} \langle c_k \rangle$, where $o(c_k) = p$, and $C_n = \prod_{k>n}^{\infty} \langle c_k \rangle$. For every $x \in c_n + C_n$, $x \neq 0$, take a generator a_{ni} and put $p^n a_{ni} = x$. Define A as the group generated by C and all these a_{ni} $(n = 1, 2, \cdots)$. Show that:
(a) a_{ni} is of order p^{n+1} in A;
(b) C is a subgroup of A such that A/C is a direct sum of cyclic groups;
(c) Every nonzero x in $c_n + C_n$ is of height n, and A does not contain elements of infinite height;
(d) If S is any subgroup of elements of bounded height, then $S \cap C$ is finite;
(e) A is not a direct sum of cyclic groups. [*Hint*: use (17.1) to contradict (d).]

67. FULLY INVARIANT AND LARGE SUBGROUPS

Recall that a subgroup G of a group A is called *fully invariant* [*characteristic*] if every endomorphism [automorphism] of A carries G into itself. Clearly, fully invariant subgroups are characteristic, but the converse is not true as is shown by examples of suitable 2-groups [and torsion-free groups].

Example (Kaplansky [3]). Let $A = \langle a_1 \rangle \oplus \langle a_2 \rangle \oplus \langle a_3 \rangle$ with $o(a_i) = 2^i$, and consider the subgroup G generated by all $g \in A$ such that

$$o(g) = 4, \qquad h(g) = 0, \qquad \text{and} \qquad h(2g) = 2.$$

It is easy to list these g: $a_1 + 2a_2 \pm 2a_3$ and $a_1 \pm 2a_3$. Every automorphism of A carries a generator of G into a generator; thus G is characteristic. But it is not fully invariant, since $a_1 \notin G$, while the projection $A \to \langle a_1 \rangle$ of the given decomposition maps G onto $\langle a_1 \rangle$.

No full description of fully invariant subgroups of p-groups is known so far, but in certain special cases such a characterization is possible [see Baer [3], Shiffman [1], and Kaplansky [3]]. These special cases include the most important classes of p-groups such as separable p-groups and totally projective p-groups.

In order to describe the fully invariant subgroups in certain p-groups, let

$$\mathbf{u} = (\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots)$$

be an increasing sequence of ordinals and symbols ∞. With $\mathbf{u}$ there is associated a subgroup

(1) $$A(\mathbf{u}) = A(\sigma_0, \cdots, \sigma_n, \cdots) = \{a \in A \mid H(a) \geqq \mathbf{u}\}$$

of A; this is evidently a fully invariant subgroup of A. Note that

(A) $A(\mathbf{u}) = \bigcap_n p^{-n}(p^{\sigma_n}A)$;
(B) if $A = \oplus A_i$, then $A(\mathbf{u}) = \oplus A_i(\mathbf{u})$;
(C) under any homomorphism $A \to C$, $A(\mathbf{u})$ is mapped into $C(\mathbf{u})$;
(D) $A(\mathbf{u} \cap \mathbf{v}) = A(\mathbf{u}) + A(\mathbf{v})$, where $\mathbf{u} \cap \mathbf{v}$ means pointwise minimum.

We say that $\mathbf{u}$ satisfies the *gap condition* if a gap occurs between σ_n and σ_{n+1} only if the group A has an element of order p and of height σ_n; in other words, $f_{\sigma_n}(A) \neq 0$.

Theorem 67.1 (Kaplansky [3]). *Let A be a fully transitive p-group. A subgroup G of A is fully invariant if and only if it is of the form* (1) *with the σ_n satisfying the gap condition. Every fully invariant G can be written uniquely in this form.*

Every subgroup G of the form (1) is evidently fully invariant.

Assume, conversely, G is fully invariant and define σ_n as the minimum of the heights $h^*(p^n g)$ with g running over G. The sequence $\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots$ is obviously increasing. To verify the gap condition, let $\sigma_i + 1 < \sigma_{i+1}$ for some i. Some $g \in G$ exists with $h^*(p^i g) = \sigma_i$, and by definition, $h^*(p^{i+1}g) \geqq \sigma_{i+1}$. By (65.3), A contains an element of order p and of height σ_i.

The inclusion $G \leqq A(\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots)$ is obvious. For every n, we establish the existence of a $g \in G$ such that $h^*(p^i g) = \sigma_i$ for $i = 0, 1, \cdots, n-1$. If there is no gap in the sequence $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$, and if $g \in G$ is such that $h^*(p^{n-1}g) = \sigma_{n-1}$, then we must have $h^*(p^i g) = \sigma_i$ for $i = 0, 1, \cdots, n-1$. If the sequence $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ does contain gaps, and if the first gap occurs between σ_{j-1} and σ_j, then there is a $g_j \in G$ such that $h^*(p^i g_j) = \sigma_i$ for $i = 0, 1, \cdots, j-1$. If the second gap lies between σ_{k-1} and σ_k ($j < k$), then some $g' \in G$ exists with $h^*(p^i g') = \sigma_i$ for $i = j, \cdots, k-1$. By (65.3) there is a $g_k \in A$ such that $h^*(p^i g_k) \geqq \max(h^*(p^i g'), \sigma_i + 1)$ for $i = 0, 1, \cdots, j-1$, and $h^*(p^i g_k) = h^*(p^i g')$ for $i \geqq j$; because of full transitivity and (65.5), $g_k \in G$. Thus proceeding, for every gap in $\sigma_0, \sigma_1, \cdots, \sigma_{n-1}$ we construct elements $g_j, g_k, \cdots, g_l \in G$ such that $g = g_j + g_k + \cdots + g_l$ will satisfy: $h^*(p^i g) = \sigma_i$ for $i = 0, 1, \cdots, n-1$. Thus $H(g) \leqq H(a)$ for every $a \in A$ $(\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots)$ of order $\leqq o(g)$. Full transitivity shows $a \in G$, i.e., G is of the form (1).

If $(\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots)$ and $(\sigma'_0, \sigma'_1, \cdots, \sigma'_n, \cdots)$ are different sequences both satisfying the gap condition, then let n be the first index with $\sigma_n \neq \sigma'_n$, say,

$\sigma_n < \sigma'_n$. There exists an $a \in A$ such that $h^*(p^i a) = \sigma_i$ for $i = 0, 1, \cdots, n$. This a belongs to $A(\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots)$ but not to $A(\sigma'_0, \sigma'_1, \cdots, \sigma'_n, \cdots)$, hence the uniqueness statement is obvious.□

There is a type of fully invariant subgroup in p-groups which is of particular interest.

Following Pierce [1], we call a fully invariant subgroup G of a p-group A *large* if

$$G + B = A, \qquad \text{for every basic subgroup } B \text{ of } A.$$

Clearly, the following hold:

(a) 0 is a large subgroup if and only if A is bounded.

(b) Every fully invariant subgroup of a bounded group is large.

(c) For every n, $p^n A$ is a large subgroup of A.

(d) If G is large in A, then so is $p^n G$ for every n. [To see this, all that we have to show is $p^n G + B = A$, which follows from $A = p^n A + B = p^n(G + B) + B$.]

We also have the following less trivial result:

(e) *A^1 is contained in every large subgroup of A.*

If $a \in A^1$ and G is large in A, then write $a = b + g$ with $b \in B, g \in G$. Embed b in a finite direct summand B' of B and write $A = B' \oplus A'$. If $\pi: A \to A'$, is the projection in the last decomposition, then $\pi b = 0$ implies $\pi a = \pi g \in G$, by full invariance. But $(1 - \pi)a = 0$ as an element of infinite height in B'. Thus $a = \pi g \in G$, in fact.

Our main purpose is to single out the large subgroups from among the fully invariant subgroups. The following condition, called the *Pierce condition*, for a subgroup G is fundamental:

($*$) *For every nonnegative integer k, there is an integer $n \geqq 0$ such that $a \in A$, $e(a) \leqq k$, and $h(a) \geqq n$ imply $a \in G$; i.e.,*

$$p^n A[p^k] \leqq G.$$

By making use of this condition, we prove:

Theorem 67.2 (Pierce [1]). *For a fully invariant subgroup G of a reduced p-group A, the following conditions are eauivalent:*

(i) *G is a large subgroup of A;*

(ii) *$G = A(r_0, r_1, \cdots, r_n, \cdots)$, where no symbol ∞ occurs if A is not bounded; here r_n are nonnegative integers or ∞;*

(iii) *The Pierce condition holds for G.*

Assume (i) for G. In view of (b) and (67.1), for the verification of (ii), it suffices to investigate the case when A is unbounded. In this case, A contains a basic subgroup $B \neq A$, and since $p^nG + B = A$ for every n by (d), we must have $p^nG \neq 0$ for every n. Hence, if A is separable, $r_n = \min_{g \in G} h(p^n g) < \omega$ for every n. If A is not separable, then in a similar fashion we obtain $p^nG > A^1$, whence $r_n < \omega$ for every n. Consequently, $G \leqq A\,(r_0, r_1, \cdots, r_n, \cdots)$. To establish equality [even if A fails to be fully transitive], we pick, if possible, an $a \in A\ (r_0, r_1, \cdots, r_n, \cdots)$ not in G. By adding to a an element of A^1 if necessary, we may assume that $H(a) = (s_0, s_1, \cdots, s_{n-1}, s_n = \infty)$ with $s_0, \cdots, s_{n-1} < \omega$. Since $r_0, r_1, \cdots, r_{n-1}$ are finite ordinals, as in the proof of (67.1) one can construct a $g \in G$ such that $H(g) = (r_0, r_1, \cdots, r_{n-1}, \infty)$. In view of $s_i \leqq r_i$ $(i = 0, \cdots, n-1)$, the obvious generalization of (65.5) implies, by the full invariance of G, that $a \in G$. Hence (ii) follows from (i).

Next let G satisfy (ii). To verify (iii), we may again restrict ourselves to unbounded groups A. Given k, define $n = r_{k-1} - k + 1$. Then none of $r_{k-2} - k + 2$, $r_{k-3} - k + 3, \cdots, r_0$ exceeds n. Therefore, if $a \in A$ satisfies $e(a) \leqq k$ and $h(a) \geqq n$, then $h^*(p^i a) \geqq n + i \geqq r_i$ for $i = 0, 1, \cdots, k-1$, and $h^*(p^i a) = \infty$ for $i \geqq k$. Thus $H(a) \geqq (r_0, r_1, \cdots, r_n, \cdots)$ and $a \in G$.

Finally, assume (iii); we have to prove $G + B = A$ for B basic in A. Let $a \in A$ with $e(a) = k$, and choose n corresponding to this k in the Pierce condition. We know from the divisibility of A/B that $a = b + p^n c$ for some $b \in B$ and $c \in A$. Here b may be assumed to satisfy $e(b) \leqq k$, for $0 = p^k a = p^k b + p^{n+k} c$ implies $p^k b = p^{n+k} b'$ for some $b' \in B$, and so $a = (b - p^n b') + p^n(b' + c)$ shows that b can be replaced by $b - p^n b'$ of exponent $\leqq k$. But then $e(p^n c) \leqq k$, too, and therefore, by (iii), $p^n c \in G$. This proves $G + B = A$. $\square$

Notice that (ii) is equivalent to

$$G = \bigcap_n p^{-n}(p^{r_n} A). \tag{2}$$

In order to clarify the meaning of the Pierce condition, let us prove:

Proposition 67.3 (Pierce [1]). *In a reduced p-group A, a subgroup satisfies the Pierce condition exactly if it contains a large subgroup of A.*

Again, only the case of unbounded A is of interest. The implication in one direction is evident, so we prove only that if G satisfies the Pierce condition, then it contains a large subgroup of A. By hypothesis, there exists a sequence $n_1 \leqq n_2 \leqq \cdots \leqq n_k \leqq \cdots$ of natural integers such that $e(a) \leqq k$, $h(a) \geqq n_k$ implies $a \in G$. Since A is unbounded, there exists an increasing sequence of nonnegative integers $r_0, r_1, \cdots, r_k, \cdots$ satisfying the gap condition of (65.3) and, in addition, $r_i \geqq n_{i+1} + i$ for $i = 0, 1, \cdots$ To prove $A(r_0, r_1, \cdots, r_n, \cdots) \leqq G$, we use induction on the exponent k of $a \in A(r_0, r_1, \cdots, r_n, \cdots)$. If $k = 0$, then $a = 0 \in G$. Let $k \geqq 1$. Since $h(p^{k-1}a) \geqq r_{k-1}$, some $b \in A$ satisfies $p^{r_{k-1}}b = p^{k-1}a$. Now $c = p^{r_{k-1}-k+1}b \in G$, because $e(c) = k$ and $h(c) \geqq r_{k-1} - k + 1 \geqq n_k$. From $p^{k-1}(a - c) = 0$ and

$$h(p^i(a - c)) \geqq \min(r_i, r_{k-1} - k + 1 + i) = r_i \qquad (i = 0, \cdots, k - 2),$$

we see that $a - c$ is of exponent $<k$ and contained in $A(r_0, r_1, \cdots, r_n, \cdots)$, so $a - c \in G$ by induction hypothesis. Therefore $a \in G$, establishing the stated inclusion. □

The following interesting fact on large subgroups is worth mentioning.

Proposition 67.4. *For a large subgroup G of a p-group A, A/G is a direct sum of cyclic groups.*

Since $G + B = A$ for any basic subgroup B of A, we have $A/G \cong B/(G \cap B)$. Write $B = \bigoplus_{i \in I} \langle b_i \rangle$ and notice that the projection of G into the summand $\langle b_i \rangle$ of A is $G \cap \langle b_i \rangle$, whence $G \cap B = \bigoplus_{i \in I} (G \cap \langle b_i \rangle)$ follows [cf. (9.3)]. Therefore, $B/(G \cap B)$ is the direct sum of the groups $\langle b_i \rangle/(G \cap \langle b_i \rangle)$ for $i \in I$. □

EXERCISES

1. (a) Fully invariant [characteristic] subgroups of torsion groups are direct sums of fully invariant [characteristic] p-subgroups.
   (b) Determine the fully invariant [characteristic] subgroups of a divisible torsion group.
2. (a) In a p-group A, the only pure, fully invariant subgroups are 0, A and the maximal divisible subgroup of A.
   (b) For a pure subgroup G and a large subgroup $A(\mathbf{u})$ of A, we have
   $$G(\mathbf{u}) = G \cap A(\mathbf{u}).$$
3. Let A be a separable p-group and $a \in A$. Give the minimal fully invariant subgroup containing a in the form (1).
4. (Kaplansky [3]) For a fully invariant subgroup G of a fully transitive p-group A, the following holds:
   (a) there is a countable subset X such that G is the minimal fully invariant subgroup containing X;
   (b) if there is such a finite subset X, then there exists also one consisting of a single element.
5. (a) Give an example where $A(\mathbf{u}) = A(\mathbf{v})$, but $\mathbf{u} \neq \mathbf{v}$.
   (b) Show that if $A(\mathbf{u}) = A(\mathbf{v})$ and if $\mathbf{u}$ satisfies the gap condition, then $\mathbf{u} \leqq \mathbf{v}$.
   (c) Write the following fully invariant subgroups of A in form (1): 0, A^σ, $p^n A^\sigma$, $A[p^k]$, $p^\sigma A[p^k]$, $p^{-m}(p^\sigma A[p^k])$, where σ is an ordinal.

6.* (Megibben [5]) Prove that the following group is not fully transitive: $A = G \oplus H$, where $G^1 \cong H^1 \cong Z(p)$, G/G^1 is torsion-complete, and H/H^1 is a direct sum of cyclic groups. [*Hint*: show that G^1 is fully invariant in A, using the fact that every homomorphism $G/G^1 \to H/H^1$ is small; cf. **69**, Ex. 6.]

7. (a) Determine the number of fully invariant subgroups and the length of a maximal chain of fully invariant subgroups in a bounded p-group.
(b) The same for an unbounded separable p-group.

8. Let A be a fully transitive p-group.
(a) Given a family G_i $(i \in I)$ of fully invariant subgroups of A in the form (1), find the forms (1) for their union and intersection. [*Hint*: pointwise and adjust to have an increasing sequence.]
(b) Show that the fully invariant subgroups form a distributive, complete sublattice of the lattice of all subgroups of A.

9. In an unbounded separable p-group, a fully invariant subgroup is large if and only if it is unbounded.

10. The large subgroups form a sublattice [but not always a complete sublattice] of the lattice of all fully invariant subgroups.

11. Show that a large subgroup $G = A(r_0, r_1, \cdots, r_n, \cdots)$ of A can be written in the form

$$G = \sum_n p^{r_n - n}A[p^{n+1}] = \sum_n p^{r_n - n}(A[p^{r_n+1}]).$$

12. (Pierce [1]) Let A be a p-group and B a basic subgroup of A such that $B \neq A$. If G is fully invariant in A and $G + B = A$ for this B, then G is large in A.

13. (Pierce [1]) If $\alpha: A \to C$ is a homomorphism and G a large subgroup of C, then $\alpha^{-1}G$ contains a large subgroup of A.

14. (Pierce [1]) If G is a large subgroup of the separable p-group A and B is basic in A, then $G \cap B$ is a basic in G. [*Hint*: for purity show $B \cap pG = p(B \cap G)$ and use (d).]

15. (Pierce [1]) A large subgroup of a large subgroup of A is large in A.

16. A homomorphism $\phi: A \to C$ is small (see **46**) if and only if Ker ϕ contains a large subgroup of A.

68. TORSION-COMPLETE GROUPS

Apart from direct sums of cyclic p-groups, the most significant class of separable p-groups is the class of so-called torsion-complete groups, first studied by Kulikov [2] [under the name of *closed p-groups*]. These groups can be described easily in terms of cardinal invariants, and they are fundamental in the study of p-groups, in view of the fact that every separable p-group is a pure and dense subgroup of a torsion-complete p-group.

In this section, we shall adhere to the following notation: B_n will denote a direct sum of cyclic groups of order p^n, say, $B_n = \bigoplus_{\mathfrak{m}_n} Z(p^n)$, and B will denote the direct sum $\bigoplus_{n=1}^{\infty} B_n$.

By a *torsion-complete p-group* is meant the torsion part $T(\hat{B})$ of the p-adic completion $\hat{B}$ of a direct sum B of cyclic p-groups. $T(\hat{B})$ is uniquely determined by B, so we may denote it by $\bar{B}$; this notation will be standard:

$$\bar{B} = T(\hat{B}).$$

A complete p-adic group is the p-adic completion of any of its basic subgroups, so a torsion-complete p-group has the form $\bar{B}$ for each of its basic subgroups B.

Since $\hat{B}$ is a subgroup of $\prod_n B_n$, so is $\bar{B}$. Hence the elements g of $\bar{B}$ can be written uniquely in the form

$$(1) \qquad g = (b_1, b_2, \cdots, b_n, \cdots) \qquad \text{with} \quad b_n \in B_n.$$

Naturally, b_n can be identified with the infinite vector $(0, \cdots, 0, b_n, 0, \cdots)$. The order p^m of g is the l.u.b. of the orders of $b_1, \cdots, b_n, \cdots$; thus the sequence of coordinates of $g \in \bar{B}$ is bounded. In view of the structure of B_n, for any bounded sequence $\{b_n\}$ [say, $p^m b_n = 0$ for all n], we have $h(b_n) \geqq n - m$, thus for g in (1), $h(g - b_1 - \cdots - b_{n-1}) \geqq n - m$ and so $g \in \hat{B}$. Consequently, (1) *represents an element g of $\bar{B}$ if and only if the sequence $b_1, b_2, \cdots, b_n, \cdots$ is bounded.* It is convenient to regard $\bar{B}$ as the set of all bounded sequences (1).

All bounded p-groups are examples for torsion-complete p-groups. The simplest unbounded example is when $B = \bigoplus_{n=1}^{\infty} \langle a_n \rangle$ with $\langle a_n \rangle$ cyclic of order p^n. The elements of $\bar{B}$ are now $b = (k_1 a_1, \cdots, k_n a_n, \cdots)$, where $k_n \in Z/(p^n)$ and there is an $m \geqq 0$ such that $p^m k_n = 0$ for all n.

From the definition we can derive easily:

(a) *B is a basic subgroup of $\bar{B}$.* In fact, in view of (39.6), B is a basic subgroup of $\hat{B}$, and hence one of $T(\hat{B})$, this being pure in $\hat{B}$.

(b) *Two torsion-complete p-groups $\bar{B}$ and $\bar{B}'$ are isomorphic if and only if their basic subgroups B and B' are isomorphic.* The "only if" part is a trivial consequence of (a) and the uniqueness (up to isomorphism) of basic subgroups.

(c) *$\bar{B} = B$ if and only if B is bounded.* If B is bounded, then $\hat{B} = B$, and so $\bar{B} = B$, too, while if B is unbounded, then $\hat{B}$ contains elements of finite order, not in B.

(d) $\overline{B \oplus B'} = \bar{B} \oplus \bar{B}'$. This follows easily from the definition.

An essential consequence of (b) is that the sequence $\mathfrak{m}_1, \cdots, \mathfrak{m}_n, \cdots$ of cardinal invariants of B is at the same time *a complete (and independent)* *system of invariants for* $\bar{B}$. This immediately solves the structure problem for torsion-complete p-groups.

We wish to clarify the relation between arbitrary p-groups A and torsion-complete p-groups. Let $B = \bigoplus_n B_n$ [with the adopted notation] denote a basic subgroup of A. Following (32.4), we write $B_n^* = B_{n+1} \oplus B_{n+2} \oplus \cdots$, and set $A_n = B_n^* + p^n A$. Then $A_{n+1} \leqq A_n$ for every n, and we have a sequence of direct decompositions

$$A = B_1 \oplus \cdots \oplus B_n \oplus A_n \qquad (n = 1, 2, \cdots), \tag{2}$$

such that each is obtained from its predecessor by splitting the last summand. Therefore, to every $a \in A$ there is a sequence $b_1, \cdots, b_n, \cdots$ $(b_n \in B_n)$ such that, for every n, $a = b_1 + \cdots + b_n + a_n$ holds for some $a_n \in A_n$. This gives rise to a correspondence

$$\eta: a \mapsto (b_1, \cdots, b_n, \cdots) \qquad (b_n \in B_n), \tag{3}$$

which is obviously a homomorphism of A into $\prod_{n=1}^{\infty} B_n$. Manifestly, the order of a is an upper bound for the orders of $b_1, \cdots, b_n, \cdots$. This means η can be viewed as a homomorphism of A into $\bar{B}$. This η essentially fixes the elements of B, since $\eta b_n = (0, \cdots, 0, b_n, 0, \cdots)$. From the separability of $\bar{B}$ we conclude $A^1 \leqq \operatorname{Ker} \eta$. On the other hand, if $\eta a = 0$, then $a = a_n \in A_n$ for every n, thus $h(a) = h(a_n) \geqq n + 1 - e(a)$ for every n, so $h(a) = \infty$ and $a \in A^1$. Consequently, $\operatorname{Ker} \eta = A^1$.

Theorem 68.1. *Let A be a p-group and B a basic subgroup of A. Then η in* (3) *is a homomorphism of A onto a pure subgroup of $\bar{B}$ containing B; the kernel of η is A^1.*

Only the purity of ηA in $\bar{B}$ must be verified. By **34**(F), ηB is a basic subgroup of ηA, so $\eta A / \eta B$ is divisible. Hence ηA is pure in $\bar{B}$. □

Corollary 68.2 (Kulikov [2]). *A separable p-group A with basic subgroup B is isomorphic to a pure [and dense] subgroup of $\bar{B}$ containing B.* □

In view of this, separable p-groups may be, and several times will be, identified with pure and dense subgroups of torsion-complete p-groups.

We interrupt, for a moment, the discussion of our main topic, in order to prove the following useful generalization of (17.3).

Proposition 68.3. *Let A be a separable p-group and B a basic subgroup of A. If A/B is countable, then A is a direct sum of cyclic groups.*

The group A may be thought of as a pure subgroup of $\bar{B}$. There is a countable set $a_1, \cdots, a_m, \cdots$ in A which, together with B, generates A. As in (1), we may write $a_m = (b_{m1}, \cdots, b_{mn}, \cdots)$ with $b_{mn} \in B_n$. Each B_n is a direct sum of cyclic groups of order p^n, so there is a direct decomposition

$B'_n = B'_n \oplus B''_n$ such that $b_{mn} \in B'_n$ for every m, and B'_n is countable. Putting $B' = \bigoplus_n B'_n$, we obtain $B = B' \oplus B''$ and $A = A' \oplus B''$, where $A' = \langle B', a_1, \cdots, a_m, \cdots \rangle$. Here A' is countable, so by (17.3), it is a direct sum of cyclic groups. □

The next theorem deals with various algebraic characterizations of torsion-complete groups.

Theorem 68.4. *For a reduced p-group A, the following conditions are equivalent:*

(i) *A is torsion-complete;*

(ii) *A is the torsion part of an algebraically compact group;*

(iii) *A is pure-injective for the class of p-groups, i.e., A has the injective property relative to the class of pure-exact sequences of p-groups;*

(iv) *A is a direct summand in every p-group in which it is contained as a pure subgroup.*

The proof is cyclic. (i) implies (ii), since $\hat{B}$ is algebraically compact.

Next let A be the torsion part of an algebraically compact group C. Then C is pure-injective, thus for every pure-exact row of p-groups

$$\begin{tikzcd}
0 \arrow[r] & G \arrow[r] \arrow[d, "\eta"'] & H \arrow[r] \arrow[dl, dashed, "\chi"] & K \arrow[r] & 0 \\
 & C & & &
\end{tikzcd}$$

and for every $\eta: G \to C$, there exists a homomorphism $\chi: H \to C$ making the diagram commute. Since G,H are p-groups, η,χ can equally well be considered as maps into A, so (iii) follows.

If (iii) holds for A and A is pure in a p-group G, then we conclude that the identity map $A \to A$ factors as $A \to G \to A$, i.e., A is a direct summand in G.

Finally, assume (iv) for A. If we had $A^1 \neq 0$, then A would be pure, but not a direct summand, in the torsion part G of the pure-injective hull of A [by (41.8), G/A is divisible and every divisible subgroup $\neq 0$ of G intersects A^1 nontrivially]. Hence $A^1 = 0$, and so by (68.2), A is a pure subgroup of $\bar{B}$, where B is basic in A. By hypothesis, $\bar{B} \cong A \oplus \bar{B}/A$. The second summand must vanish, because $\bar{B}$ is reduced and $\bar{B}/A$ is divisible. Hence (i) follows. □

Our discussion of torsion-complete groups relies heavily on the theory of algebraically compact groups. It should be pointed out that it is easy to give an independent treatment.

One can start by defining $\bar{B}$ as the torsion part of $\prod B_n$ [notice that its independence of the decomposition $B = \bigoplus B_n$ must also be checked]. In order to prove the equivalence of (i), (iii), and (iv) in (68.4), we can argue as follows.

To verify (iii) for $A = \bar{B}$, let G be a pure subgroup of a p-group H and $\eta: G \to \bar{B}$. If

ε_n denotes the coordinate-projection $\bar{B} \to B_n$, then the kernel of $\varepsilon_n \eta: G \to B_n$ contains $p^n G$, so it induces a homomorphism $G/p^n G \to B_n$. By (27.10), $G/p^n G$ is a summand of $H/p^n G$, so the last homomorphism can be extended to a homomorphism $H/p^n G \to B_n$. If this is preceded by the canonical map $H \to H/p^n G$, then we obtain an extension $\chi_n: H \to B_n$ of $\varepsilon_n \eta$, for every n. The map

$$\chi: h \mapsto (\chi_1 h, \cdots, \chi_n h, \cdots)$$

is clearly an extension of η to a homomorphism of H into $\bar{B}$.

The proof of the implication (iii) $\Rightarrow$ (iv) is as above.

Finally, to show that (iv) implies (i), note that if $0 \neq a \in A^1[p]$, then A will be a pure subgroup in the p-group $C = \langle A, x_1, \cdots, x_n, \cdots \rangle$, where $px_1 = a$, $px_n = x_{n-1}$ for every $n \geq 2$. This is impossible, since by hypothesis $C = A \oplus G$ for some G, and the coordinates of $a, x_1, \cdots, x_n, \cdots$ in A must generate a quasicyclic subgroup of A. Hence A is separable, so it is pure in a torsion-complete p-group $\bar{B}$. By hypothesis, it is a summand of $\bar{B}$. The complement must vanish, since $\bar{B}$ is reduced and $\bar{B}/A$ is divisible, so $A = \bar{B}$, indeed.

Obviously, (iv) is equivalent to the condition:

$$\operatorname{Pext}(X, A) = 0 \qquad \text{for every } p\text{-group} \quad X.$$

In particular, we have $\operatorname{Pext}(Z(p^\infty), A) = 0$. For every p-group X, there is a pure-exact sequence $0 \to C \to X \to \bigoplus Z(p^\infty) \to 0$, with C a direct sum of cyclic groups, so from (53.7) we obtain the exact sequence

$$\operatorname{Pext}(\bigoplus Z(p^\infty), A) = \prod \operatorname{Pext}(Z(p^\infty), A) \to \operatorname{Pext}(X, A) \to \operatorname{Pext}(C, A) = 0.$$

Consequently, we have proved:

Corollary 68.5. *A reduced p-group A is torsion-complete if and only if*

$$\operatorname{Pext}(Z(p^\infty), A) = 0. \square$$

On using (38.3) and (68.4), it is straightforward to show:

Corollary 68.6. *The torsion part A of a direct product $\prod A_i$ of p-groups A_i is torsion-complete if and only if all the A_i are torsion-complete.* $\square$

A partial analog of (39.2) is the following result.

Corollary 68.7 (Irwin and O'Neill [1]). *A subgroup G of a torsion-complete p-group A is again torsion-complete whenever A/G is reduced.*

From the exactness of $0 \to G \to A \to A/G \to 0$, we infer the exactness of

$$\operatorname{Hom}(Z(p^\infty), A/G) \to \operatorname{Ext}(Z(p^\infty), G) \to \operatorname{Ext}(Z(p^\infty), A).$$

The first group vanishes if A/G is reduced, hence $\operatorname{Ext}(Z(p^\infty), G)$ can be regarded as a subgroup of $\operatorname{Ext}(Z(p^\infty), A)$. If the latter group has vanishing first Ulm subgroup, then the same holds for the former group, i.e., $\operatorname{Pext}(Z(p^\infty), A) = 0$ implies $\operatorname{Pext}(Z(p^\infty), G) = 0$. $\square$

Remark. The reader might find it interesting to learn that the torsion-complete p-groups can also be characterized as p-groups A satisfying

$$\mathrm{Pext}(\bar{B}, A) = 0,$$

for all torsion-complete p-groups $\bar{B}$ [see Griffith [9]]. It suffices to verify that if A satisfies this condition, then necessarily $\mathrm{Pext}(Z(p^\infty), A) = 0$.

By way of contradiction, suppose that $\mathrm{Pext}(Z(p^\infty), A) \neq 0$. Select a cardinal number $\mathfrak{n}$ such that $|A| < \mathfrak{n}$ and $\mathfrak{n}^{\aleph_0} = 2^{\mathfrak{n}}$ ($= \mathfrak{m}$); such an $\mathfrak{n}$ does exist, for instance, $\mathfrak{n} = |A| + 2^{|A|} + 2^{2^{|A|}} + \cdots$. Let $\bar{B}$ be the torsion-complete p-group with the basic subgroup $B = \oplus_{\mathfrak{n}} \oplus_{i=1}^{\infty} Z(p^i)$. From the pure-exact sequence $0 \to B \to \bar{B} \to \bar{B}/B \to 0$, we obtain the exact sequence

$$\mathrm{Hom}(B, A) \to \mathrm{Pext}(\bar{B}/B, A) \to \mathrm{Pext}(\bar{B}, A) = 0.$$

Here

$$|\mathrm{Hom}(B, A)| = \prod_{\mathfrak{n}} \prod_{i=1}^{\infty} |A[p^i]| \leq |A|^{\mathfrak{n}} = 2^{\mathfrak{n}},$$

while $\bar{B}/B = \oplus_{\mathfrak{m}} Z(p^\infty)$ implies

$$|\mathrm{Pext}(\bar{B}/B, A)| = \prod_{\mathfrak{m}} |\mathrm{Pext}(Z(p^\infty), A)| \geq 2^{\mathfrak{m}}.$$

By the last exact sequence, $2^{\mathfrak{n}} \geq 2^{\mathfrak{m}}$, in contradiction to $\mathfrak{m} = 2^{\mathfrak{n}} < 2^{\mathfrak{m}}$.

Finally, we prove the following result:

Proposition 68.8. *If G is a pure subgroup of the torsion-complete p-group A, then A/G is the direct sum of a divisible group and a torsion-complete group.*

Because of (53.7), the pure-exact sequence $0 \to G \to A \to A/G \to 0$ implies that $\mathrm{Pext}(Z(p^\infty), A) \to \mathrm{Pext}(Z(p^\infty), A/G)$ is an epimorphism. By (68.5), the first group is 0, whence the assertion is clear. □

This implies, in particular, that the closure G^- of a pure subgroup G of a torsion-complete p-group A satisfies: G^-/G is divisible, and hence G^- is likewise pure [cf. quasi-complete groups in **74**]. Evidently, A/G^- is now torsion-complete, hence by (68.7), G^- is again torsion-complete. In view of its purity, we have:

Corollary 68.9. *In a torsion-complete p-group, the closure of a pure subgroup is a summand.* □

EXERCISES

1. For a direct sum B of cyclic p-groups, show that $\bar{B}/\bar{B}[p]$ is isomorphic to the torsion-complete group with $B/B[p]$ as basic subgroup.
2. (Fuchs [3]) (a) Let A be a separable p-group and B an upper basic subgroup of A. If $B \neq A$, then $|A/B| \geq \aleph_1$ and there is a decomposition

$A = A' \oplus A''$ such that A' is a direct sum of cyclic groups and $|A''| = |A/B|$. [*Hint*: argue as in (68.3).]
(b) Every separable p-group A can be written in the form $A = A' \oplus A''$, where A' is a direct sum of cyclic groups and in A'' every basic subgroup is both upper and lower.

3. Using the notation of the text, show that $\prod B_n/\bar{B}$ is a torsion-free algebraically compact group which is not divisible unless 0.
4. An element $g = (b_1, \cdots, b_n, \cdots) \in \bar{B}$ with $e(g) = n$ generates a direct summand of $\bar{B}$ exactly if $e(b_n) = n$.
5. (Leptin [1]) Let A be a separable p-group and $a = (b_1, \cdots, b_n, \cdots) \in A$, where $b_n \in B_n$, $\bigoplus B_n = B$ is basic in A. Then the maximal n with $h(b_n) = h(a)$ does not depend on the choice of B; it is an invariant of a in A.
6. (Leptin [1]) (a) Let $M \cup M' = N \cup N'$ be partitions of the set of positive integers [into disjoint sets]. For a subset X of integers, write $B_X = \bigoplus_{n \in X} Z(p^n)$. Prove that
$$A_M = B_M \oplus \bar{B}_{M'} \qquad \text{and} \qquad A_N = B_N \oplus \bar{B}_{N'}$$
are isomorphic if and only if M and N differ only in a finite set. [*Hint*: in the contrary case write $A = B_L \oplus C \cong \bar{B}_L \oplus C'$ for an infinite L, and show that the projection $A \to B_L$ is monic on $\bar{B}_L$.]
(b) Use (a) to produce continuously many nonisomorphic groups with the basic subgroup $\bigoplus_{n=1}^{\infty} Z(p^n)$ which are direct sums of cyclic groups and a torsion-complete group.
7. Prove that for an unbounded B,
$$|\bar{B}| = |B|^{\aleph_0}.$$
[*Hint*: for the nontrivial part of the set-theoretic argument see Example 2 in 75.]
8. (a) Large subgroups of torsion-complete p-groups are torsion-complete. [*Hint*: (67.4) and (68.7).]
(b) If C is a torsion-complete subgroup of a p-group A such that A/C is bounded, then A is torsion-complete.
9. Call a torsion group A *torsion-complete* if all of its p-components are torsion-complete.
(a) Prove (68.4) for reduced torsion groups A after replacing "p-groups" in (iii) and (iv) by "torsion groups."
(b) Drop the condition of reducedness and prove the same as in (a) with (i) replaced by: (i′) A is the direct sum of a divisible and torsion-complete group.
10. Prove that a torsion group A is the direct sum of a divisible and a torsion-complete group if and only if $\operatorname{Pext}(Q/Z, A) = 0$.

11. (a) Kernels of endomorphisms of torsion-complete groups are again torsion-complete.
(b) Give an example where the endomorphic image of a torsion-complete p-group is not torsion-complete. [*Hint*: (36.1).]
12. The torsion part of an inverse limit of torsion-complete p-groups is likewise torsion-complete. [*Hint*: (68.4), (12.3), and (39.4).]
13. In the notation of the text,

$$\hat{B} = \text{Ext}(Q/Z, \bar{B})$$

holds. [*Hint*: **54** (H).]
14. Let T be a separable p-group with basic subgroup B such that $\bar{B}/T$ is of finite rank r. Show that

$$\text{Pext}(Z(p^\infty), T) = \bigoplus_r J_p.$$

[*Hint*: (56.5).]
15. If T is as in Ex. 14 and if S is a separable p-group which contains T as a pure dense subgroup such that $r(S/T) = r$, then $S \cong \bar{B}$. [*Hint*: (68.5) and $\text{Pext}(Z(p^\infty), S)$ is 0 or infinite.]

69. FURTHER CHARACTERIZATIONS OF TORSION-COMPLETE p-GROUPS

In the preceding section, the torsion-complete p-groups were characterized in various ways in terms of algebraic properties involving other groups, too. Here our main objective will be to obtain intrinsic algebraic characterizations for them.

Our starting point is the following result:

Theorem 69.1 (Leptin [1]). *Two pure and dense subgroups A, A' of a torsion-complete p-group $\bar{B}$ are isomorphic if and only if $\bar{B}$ has an automorphism carrying one upon the other. Moreover, any isomorphism between A and A' can be uniquely extended to an automorphism of $\bar{B}$.*

It suffices to prove that if α is an isomorphism between A and A', then $\bar{B}$ has a unique automorphism $\bar{\alpha}$ inducing α. If α is viewed as a homomorphism of A into $\bar{B}$, then the pure-exactness of $0 \to A \to \bar{B} \to \bar{B}/A \to 0$ implies, because of (68.4)(iii), the existence of a homomorphism $\bar{\alpha}\colon \bar{B} \to \bar{B}$ such that $\bar{\alpha}|A = \alpha$. Similarly, there is a homomorphism $\bar{\beta}\colon \bar{B} \to \bar{B}$ such that $\bar{\beta}|A' = \alpha^{-1}$. Now both $\bar{\beta}\bar{\alpha}$ and $\bar{\alpha}\bar{\beta}$ are endomorphisms of $\bar{B}$, the former is the identity on A, the latter is the identity on A', thus both are identities on the basic subgroups of A and A', respectively. From our hypothesis we infer that these are at the same time basic subgroups of $\bar{B}$, whence (34.1) implies $\bar{\beta}\bar{\alpha} = 1_{\bar{B}} = \bar{\alpha}\bar{\beta}$, i.e., $\bar{\alpha}$

is an automorphism of $\bar{B}$. The uniqueness of $\bar{\alpha}$ is likewise a consequence of (34.1), since any two extensions of α induce the same map on a basic subgroup of $\bar{B}$.□

This result has the significant consequence that the question of isomorphy between separable p-groups is equivalent to the existence problem for certain automorphisms of torsion-complete p-groups. Unfortunately, the latter problem seems to be as difficult to solve as the former one.

Another consequence of (69.1) is that *an isomorphism between two basic subgroups of* $\bar{B}$ *can be extended, in a unique way, to an automorphism of* $\bar{B}$. In particular, this also shows that a torsion-complete group is of the form $\bar{B}$ for *any* basic subgroup B.

The italicized remark leads to a new characterization of torsion-complete groups.

Theorem 69.2 (Leptin [1], Enochs [2]). *A reduced p-group A is torsion-complete if and only if every isomorphism between basic subgroups extends to an automorphism of A.*

In order to prove the "if" part, suppose A has the indicated property. If A is bounded, then it has only one basic subgroup [cf. (35.4)], and there is nothing to prove. Assume A is unbounded, and let B be a basic subgroup of A. Now $\bar{B} \neq B$, and let $x_1, \cdots, x_n \in \bar{B} \backslash B$. In view of the divisibility of $\bar{B}/B$, there exists a pure subgroup B' of $\bar{B}$, containing B and $x_1, \cdots, x_n$, such that B'/B is a finite direct sum of groups $Z(p^\infty)$. By (68.3), B' is again a direct sum of cyclic groups, so it is again basic in $\bar{B}$. Owing to (69.1), $\bar{B}$ has an automorphism α carrying B' onto B. Now αB is again basic in $\bar{B}$ and is contained in B, so it is basic in A, too. By hypothesis, there is an automorphism β of A such that $\beta | B = \alpha | B$.

Next we define a map $\phi \colon \bar{B} \to A$ as follows: Let ϕ map every $b \in B$ onto itself. If $x \in \bar{B}$, then let x be equal to some member x_i in a finite set $x_1, \cdots, x_n$ as in the preceding paragraph, and define $\phi x = \phi x_i = \beta^{-1}\alpha x_i$. This definition is unambiguous: if B'' is a basic subgroup of $\bar{B}$ larger than B', and if α' is an automorphism of $\bar{B}$ mapping B'' onto B and β' is an automorphism of A with $\beta' | B = \alpha' | B$, then $\alpha'\alpha^{-1}B = \alpha'B' \leqq \alpha'B'' = B$ and $\beta'\beta^{-1}$ agrees with $\alpha'\alpha^{-1}$ on $\alpha B = \beta B$, and hence on B. We conclude $\beta'\beta^{-1}(\alpha x_i) = \alpha' x_i$, i.e., $\beta^{-1}\alpha x_i = \beta'^{-1}\alpha' x_i$. Thus ϕ is a well-defined homomorphism $\bar{B} \to A$. If $x \in \operatorname{Ker} \phi$, then $\beta^{-1}\alpha x = 0$, $\alpha x = 0$, and $x = 0$, that is, ϕ is monic.

Therefore, $\phi\bar{B}$ is a subgroup of A such that $\phi B = B$. This means $\phi\bar{B}$ is pure in A, and (68.4)(iv) implies that $\phi\bar{B}$ is a direct summand of A. But A is reduced and $A/\phi\bar{B}$ is divisible, so $\phi\bar{B} = A$, proving that ϕ is an isomorphism and A is torsion-complete.□

If we confine our attention to separable p-groups, the last theorem can be improved.

Theorem 69.3 (Leptin [3]). *Suppose A is a separable p-group and B is a basic subgroup of A such that every automorphism of B is extendible to an automorphism of A. Then either $A = B$ or $A = \bar{B}$.*

We may think of A as being embedded in $\bar{B}$ as a pure subgroup containing B. If $A \neq B$, then B is unbounded. Write $B = \bigoplus_n B_n'$ with $B_n' \neq 0$ a direct sum of a set of $Z(p^{i_n})$ and $i_1 < \cdots < i_n < \cdots$. In every B_n' we select a direct summand $\langle b_n \rangle$, and define

$$x_n = (0, \cdots, 0, p^{i_n - 1}b_n, p^{i_{n+1}-1}b_{n+1}, \cdots) \in \bar{B}\backslash B.$$

x_n is of order p and of height $i_n - 1$. If $y \in \bar{B}\backslash B$ is of order p, then its height must be $i_n - 1$ for some n, thus y is of the form

$$y = (0, \cdots, 0, p^{i_n - 1}c_n, p^{i_{n+1}-1}c_{n+1}, \cdots),$$

with $c_m \in B_m'$ and infinitely many nonvanishing coordinates; let its nonzero coordinates be

$$p^{i_n - 1}c_n, \quad p^{i_{n_2}-1}c_{n_2}, \quad p^{i_{n_3}-1}c_{n_3}, \cdots.$$

Notice that $\langle c_{n_j} \rangle$ is a direct summand of B_{n_j}', and hence of $B_{n_j}' \oplus \cdots \oplus B_{n_{j+1}-1}'$. In the latter group,

$$c_{n_j}' = b_{n_j} + p^{i_{n_j+1} - i_{n_j}}b_{n_j+1} + \cdots + p^{i_{n_{j+1}} - i_{n_j}}b_{n_{j+1}-1}$$

also generates a direct summand of the same order, thus B has an automorphism α mapping c_{n_j}' upon c_{n_j} for every j. In view of (69.2), α can be extended to an automorphism $\bar{\alpha}$ of $\bar{B}$. From the vector representation it follows immediately that $\bar{\alpha}y = x_n$.

If A has the property stated in the theorem, then there exists an automorphism ϕ of A such that $\phi | B = \alpha$. Also, ϕ can be extended to an automorphism of $\bar{B}$, and the uniqueness of this extension shows $\bar{\alpha} | A = \phi$, i.e., A is left fixed under $\bar{\alpha}$. The inverse of the automorphism $\bar{\alpha}$ also leaves A fixed, hence we infer that to any two elements y and y' of $\bar{B}\backslash B$ which are of order p and of the same height there exists an automorphism β of $\bar{B}$ such that $\beta A = A$ and $\beta y = y'$.

Now let $a \in A\backslash B$ be of order p. It is easy to see that the coset $a + B$ contains, for every n, an element a_n of order p and of height $i_n - 1$. Given $y \in \bar{B}\backslash B$ of order p, its height is $i_n - 1$ for some n, hence, by the last paragraph, $\bar{B}$ has an automorphism β such that $\beta A = A$ and $\beta a_n = y$. Therefore, $y \in A$, and so A contains the socle of $\bar{B}$. The purity of A in $\bar{B}$ concludes the proof that $A = \bar{B}$. □

EXERCISES

1. Show that every height-preserving automorphism of the socle of $\bar{B}$ can be extended to an automorphism of $\bar{B}$.
2. (Enochs [3]) If A is a reduced p-group such that every automorphism of A^1 extends to one of A, then $A^1 = 0$.
3. (Leptin [3]) A reduced p-group A is torsion-complete if isomorphisms of some fixed basic subgroup B of A with basic subgroups of B are induced by automorphisms of A.
4. (Mader [6]) (a) Let B be a direct sum of cyclic p-groups and $\hat{B}$ its p-adic completion. Two pure fully invariant subgroups of $\hat{B}$ containing B are isomorphic if and only if they coincide.
   (b) Let A be an arbitrary group such that $p^{\omega}A = 0$ and p-basic subgroups B of A are p-groups [where p is a fixed prime]. A has the property that every isomorphism between its p-basic subgroups is induced by an automorphism of A exactly if A is isomorphic to a p-pure fully invariant subgroup between $\bar{B}$ and $\hat{B}$.
5. Let $\bar{B}$ be a torsion-complete p-group with the basic subgroup $B = \bigoplus B_n$ (canonical form). Let $\{c_i\}_{i \in I}$ be a basis of B and write $\eta c_i = (b_{i1}, \cdots, b_{in}, \cdots)$ with $b_{in} \in B_n$ for a homomorphism $\eta\colon B \to \bar{B}$. For $b = (b_1, \cdots, b_n, \cdots) \in \bar{B}$, define formally $\eta b = \sum_{n=1}^{\infty} \eta b_n$ where the b_ns are replaced by their linear combinations in terms of the c_i. Show that the infinite sum makes sense and defines an extension of η to $\bar{B}$. [*Hint*: for every k, almost all the kth coordinates of ηb_n vanish.]
6. (Megibben [5]) Let A be an unbounded torsion-complete p-group and C an arbitrary separable p-group. There exists a homomorphism of A into C which is not small if and only if C contains an unbounded torsion-complete p-subgroup. [*Hint*: exhibit an unbounded torsion-complete group in the image of a nonsmall homomorphism.]
7. (Richman [1]) A p-group T is called *thin* if every homomorphism of a torsion-complete p-group into T is small [cf. the definition of slenderness in **94**]. Show that:
   (a) the class of thin groups is closed under subgroups, direct sums and extensions;
   (b) countable reduced p-groups are thin;
   (c) every homomorphism $\bar{B} \to B$ is small.

70. TOPOLOGICAL COMPLETENESS OF TORSION-COMPLETE GROUPS

Separable p-groups A can be equipped with various topologies which can be derived from the standard p-adic topology and which are particularly interesting from the point of view of torsion-complete p-groups.

Our starting point is the p-adic topology of separable p-groups A, which will be denoted by τ_A [or simply τ]. For every k, the subgroup $A[p^k]$ has a topology, inherited from τ_A, where

$$p^nA \cap A[p^k] = p^nA[p^k] \qquad (n = 0, 1, 2, \cdots)$$

form a subbase of neighborhoods of 0. These induce a topology $\tau_A^{(k)}$ on A, which is thus again a linear topology with $p^nA[p^k]$ $(n = 0, 1, 2, \cdots)$ as a subbase. Finally, we introduce the topology

$$\tau_A^* = \bigcap_k \tau_A^{(k)},$$

i.e., the τ_A^*-open sets are exactly those which are $\tau_A^{(k)}$-open for every k. Notice that

$$\tau_A \leqq \tau_A^* \leqq \cdots \leqq \tau_A^{(k)} \leqq \cdots \leqq \tau_A^{(1)}. \tag{1}$$

It is easy to conclude that τ_A^* is again linear. The topological group (A, τ_A^*) is the inductive [i.e., direct] limit of the topological groups $A[p^k]$ furnished with the topologies inherited from τ_A; the topology τ_A^* may, therefore, be referred to as the *inductive p-adic topology* of A. (70.1) will more explicitly describe this topology.

We shall find the following observations useful.

(a) If A is bounded, then all the topologies in (1) are discrete and hence equal. However, if A is unbounded, then the topologies τ_A^*, $\tau_A^{(k)}$ $(k = 1, 2, \cdots)$ are all different. [From the following discussion it will turn out that τ_A and τ_A^*, too, are distinct.]

(b) Large subgroups of A are closed in each of the topologies in (1). This follows at once from (2) in **67** where the subgroups $p^{-n}(p^{r_n}A)$ [containing the open subgroups $p^{r_n}A$] are open.

(c) If C is a pure subgroup of A, then its p-adic topology τ_C is known to coincide with the topology inherited from τ_A. Obviously, the same holds for $\tau_C^{(k)}$ and $\tau_A^{(k)}$, for every k, whence we conclude that for a pure subgroup C of A, the topology τ_C^* is the same as the one inherited from τ_A^*.

Our first result describes the open subgroups in τ_A^*.

Proposition 70.1 (B. Charles). *A subgroup G of A is open in the inductive p-adic topology of A if and only if it contains a large subgroup of A, i.e., it satisfies the Pierce condition.*

If G satisfies the Pierce condition, then, given k, choose n correspondingly. Thus $p^nA[p^k] \leqq G$, and G is open in $\tau_A^{(k)}$. This holds for every k, therefore G is open in τ_A^*. Conversely, if G is open in τ_A^*, then it is open in every $\tau_A^{(k)}$, thus to any k there is an n such that $p^nA[p^k] \leqq G$. This amounts to the Pierce condition. □

Consequently, *the inductive p-adic topology is a* **D**-*topology* [see 7] *where the dual ideal* **D** *is the set of all subgroups of A satisfying the Pierce condition.* For an unbounded A, there are continuously many large subgroups, and it is not difficult to show that τ_A^* need not satisfy the second axiom of countability. However, the following is true:

Lemma 70.2 (Cutler and Stringall [1]). *To every Cauchy net* $\{a_i\}_{i\in I}$ *in the inductive p-adic topology of A, there exists an m such that* $\{p^m a_i\}_{i\in I}$ *converges to* 0.

We may assume the index set I is partially ordered according to the large subgroups G_i of A such that $i \leqq j$ in I is equivalent to $G_i \geqq G_j$. In view of our discussion in **13**, we shall restrict ourselves to neat Cauchy nets, i.e., $a_i - a_j \in G_i$ holds for every $j \geqq i$.

Our first observation is that all the a_j with $j \geqq i$ have the same order mod G_i; in fact, $a_i \equiv a_j \bmod G_i$.

Deny our assertion, i.e., suppose the existence of a steadily increasing sequence $n_1 < n_2 < \cdots$ of integers and a sequence $i_1 < i_2 < \cdots$ of indices in I such that the exponent of $a_j \bmod G_{i_k}$ is n_k, whenever $j \geqq i_k$ $(k = 1, 2, \cdots)$. Without loss of generality, the n_k may be assumed to satisfy the inequalities $n_{k+1} > n_k + k$. Clearly, $p^{n_k-1}a_j + G_{i_k}$ is the same coset for every $j \geqq i_k$, and since A/G_{i_k} does not contain elements of infinite height [see (67.4)], there is a finite upper bound h_k for the heights $h(p^{n_k-1}a_j)$ with $j \geqq i_k$. Now define a large subgroup of A, $G = A(r_0, r_1, \cdots, r_k, \cdots)$ such that $r_k > h_{k+1}$ for every k; this G is labeled by an index in I, say $G = G_{i_0}$. There is an integer n such that $p^n a_j \in G_{i_0}$ for all $j \geqq i_0$, and we choose a k such that $n_k \geqq n$. Pick a $j \in I$ satisfying both $j \geqq i_0$ and $j \geqq i_{k+1}$. By the definition of G_{i_0}, $p^{n_k}a_j \in G_{i_0}$ implies that

$$p^{n_{k+1}-1}a_j = p^{n_{k+1}-n_k-1}(p^{n_k}a_j)$$

must be of height $\geqq r_{n_{k+1}-n_k-1} \geqq r_k > h_{k+1}$, in contradiction to the definition of h_{k+1}. This establishes our assertion that $\{p^m a_i\}_{i\in I}$ is a 0-net for some m. □

It is a consequence of this lemma that we can limit our considerations of Cauchy nets $\{a_i\}_{i\in I}$ in τ_A^* to bounded ones, i.e., those which satisfy $p^m a_i = 0$ for some m and for all i. In fact, if $\{a_i\}_{i\in I}$ is an arbitrary neat Cauchy net of A in τ_A^*, then by (70.2), $p^m a_i \in G_i$ for some m and for all i. Along with G_i, also $p^m G_i$ is a large subgroup; put $p^m G_i = G_{i(m)}$, $i(m) > i$. Now $a_i - a_{i(m)} \in G_i$ implies $p^m a_i - p^m a_{i(m)} \in G_{i(m)}$, and so $p^m a_i \in G_{i(m)} = p^m G_i$, i.e., $p^m a_i = p^m g_i$ for some $g_i \in G_i$. The net $\{a_i - g_i\}_{i\in I}$ is again a neat Cauchy net in τ_A^*, is bounded by p^m, and differs from the given $\{a_i\}$ in a 0-net $\{g_i\}_{i\in I}$. Therefore, *we may and shall confine ourselves from now on to bounded Cauchy nets in* τ_A^*.

Moreover, a further reduction is possible. To a bounded Cauchy net

$\{a_i\}_{i\in I}$ in τ_A^*, there will correspond a bounded Cauchy sequence in τ_A in the following way. We select the p^nA from among the G_i, say $p^nA = G_{i_n}$ $(n = 0, 1, \cdots)$, and define a correspondence

$$\psi : \{a_i\}_{i\in I} \mapsto \{a_{i_n}\}_{n=0,1,\cdots}.$$

In other words, we just keep the a_i corresponding to the subgroups p^nA. Given any bounded Cauchy sequence $\{a'_n\}$ in τ_A, say $p^m a'_n = 0$ for all n, there exists a Cauchy net $\{a_i\}$ mapped by ψ upon it: for $i \in I$, choose r_i to be the smallest integer satisfying $p^{r_i}A[p^m] \leqq G_i$, and then define $a_i = a'_{r_i}$. The Cauchy net character of $\{a_i\}$ is readily checked. On the other hand, if $\{a_{i_n}\}$ is a 0-sequence, i.e., $a_{i_n} \in p^nA$ for every n, then to any given $i \in I$ and integer n there is an index $j \in I$ with $j > i$ and $j > i_n$, and the Cauchy character of the given net shows that $a_j \in p^nA$ and $a_i - a_j \in G_i$. Hence a_i is of height $\geqq n$ mod G_i, and since A/G_i is separable, $a_i \in G_i$, that is to say, $\{a_i\}_{i\in I}$ is a 0-net. Hence ψ maps only 0-nets upon 0-sequences. We thus arrive at the following result:

Proposition 70.3. *For a separable p-group A, there exists a natural isomorphism [induced by ψ] between the quotient group of all bounded Cauchy nets* mod *bounded* 0*-nets in τ_A^* and the quotient group of all bounded Cauchy sequences* mod *bounded* 0*-sequences in τ_A.* □

In view of the foregoing considerations, it is clear that *completeness in τ_A^* can be investigated by means of bounded Cauchy sequences in τ_A.*

A bounded Cauchy sequence in τ_A is immediately seen to be a Cauchy sequence in some $\tau_A^{(m)}$; also, the converse follows easily. Likewise, a Cauchy sequence in $\tau_A^{(m)}$ is Cauchy in $\tau_A^{(m+1)}$; but the converse of this is no longer true. However, we have:

Lemma 70.4. *Let m be an integer $\geqq 1$. A separable p-group A is complete in $\tau_A^{(m+1)}$ exactly if it is complete in $\tau_A^{(m)}$.*

Assume A complete in $\tau_A^{(m+1)}$, and let $\{a_n\}$ be a Cauchy sequence in $\tau_A^{(m)}$. Then it is Cauchy in $\tau_A^{(m+1)}$, too, and thus it tends to some a in $\tau_A^{(m+1)}$. For some large n, $p^m a_n = p^m a_{n+k}$ for every $k \geqq 1$. Given an integer s, for large k we have $a - a_{n+k} \in p^sA[p^{m+1}]$. Hence $p^m a - p^m a_n$ is of infinite height in A, and so $p^m(a - a_{n+k}) = 0$ for every $k \geqq 0$. This shows that a is the limit of $\{a_n\}$ in $\tau_A^{(m)}$, too.

Conversely, suppose A complete in $\tau_A^{(m)}$, and let $\{a_n\}$ be a neat Cauchy sequence in $\tau_A^{(m+1)}$. Now $\{pa_n\}$ is a neat Cauchy sequence in $\tau_A^{(m)}$, and hence it has a limit a in $\tau_A^{(m)}$. Since pA is closed in $\tau_A^{(m)}$, $a \in pA$, i.e., $a = pb$ for some $b \in A$. Hence there exist $c_n \in A$ such that $pa_n - pb = p^{n+1}c_n$ with $p^{m+n+1}c_n = 0$. The sequence $\{a_n - b - p^n c_n\}$ is Cauchy in $\tau_A^{(m+1)}$, moreover in $\tau_A^{(m)}$, since the members are of order $\leqq p \leqq p^m$. Consequently, it has a limit $b' \in A$, i.e., $a_n - b - b' - p^n c_n = p^n c_n'$ for some $c_n' \in A$ with $p^{n+m} c_n' = 0$. Hence

$$a_n - b - b' \in p^n A[p^{m+1}]$$

and $b + b'$ is the limit of $\{a_n\}$ in $\tau_A^{(m+1)}$.□

An argument similar to the second part of the preceding proof applies to establish the nontrivial part of

Lemma 70.5 (Enochs [2]). *For a separable p-group A, $A[p^m]$ and $A[p^{m+1}]$ are simultaneously complete in the topologies induced by τ_A [by τ_A^*], for every integer $m \geqq 1$.*□

A comparison of the completeness in the various topologies considered so far yields the following result.

Theorem 70.6. *For a separable p-group A, the following conditions are equivalent:*

(i) *every bounded Cauchy sequence in the p-adic topology τ_A has a limit in A;*
(ii) *A is complete in the inductive p-adic topology τ_A^*;*
(iii) *A is complete in $\tau_A^{(m)}$ for some [and hence for every] m;*
(iv) *$A[p^m]$ is complete in the topology τ_A for some [and hence for every] m.*

Conditions (i) and (ii) are equivalent in view of (70.3). (i) holds exactly if (iii) holds for every m, which is the same as (iii) for some m, as is shown by (70.4). A similar remark applies to (iv) because of (70.5).□

We hasten to show that the groups covered by (70.6) are old acquaintances of ours:

Theorem 70.7 (Kulikov [2]). *Let A be a separable p-group. Every bounded Cauchy sequence in the p-adic topology converges in A if and only if A is torsion-complete.*

Let $B = \bigoplus_n B_n$ with $B_n = \bigoplus Z(p^n)$ be a basic subgroup of A. Then A can be regarded as a pure subgroup of $\bar{B}$ containing B, i.e., every $a \in A$ can be identified with a vector $a = (b_1, b_2, \cdots, b_n, \cdots) \in \bar{B}$ with $b_n \in B_n$, where $p^m b_n = 0$ for every n if $o(a) = p^m$.

Now assume $A = \bar{B}$, and let

$$a_k = (b_{k1}, \cdots, b_{kn}, \cdots) \qquad \text{with} \quad k = 1, 2, \cdots$$

be a bounded Cauchy sequence in $\bar{B}$, where $b_{kn} \in B_n$. Assuming it neat, we have

$$a_{k+l} - a_k = (b_{k+l,1} - b_{k1}, \cdots, b_{k+l,n} - b_{kn}, \cdots) \in p^k \bar{B} \qquad \text{for all} \quad k, l \geqq 1,$$

which implies $b_{k+l,1} = b_{k1}, \cdots, b_{k+l,k} = b_{kk}$. This shows that the first k coordinates of $a_k, a_{k+1}, \cdots$ are the same and, in addition, $b_{k+l,n} - b_{kn} \in p^k B_n$ for each n. Define a as the "diagonal" element

$$a = (b_{11}, b_{22}, \cdots, b_{nn}, \cdots)$$

which belongs to $\bar{B}$ in view of the boundedness of the orders of a_k. Since $a - a_k = (0, \cdots 0, b_{k+1,k+1} - b_{k,k+1}, b_{k+2,k+2} - b_{k,k+2}, \cdots)$ obviously belongs to $p^k\bar{B}$, a is the limit of the given Cauchy sequence.

Next suppose every bounded Cauchy sequence converges in A, and let $c = (b_1, b_2, \cdots, b_n, \cdots) \in \bar{B}$. If $p^m c = 0$, then $h(b_n) \geqq n - m$ for every $n \geqq m$, thus $a_k = b_1 + \cdots + b_{m+k-1}$ $(k = 1, 2, \cdots)$ is a [neat] bounded Cauchy sequence in A. If $a \in A$ is its limit, then both a and c are its limit in $\bar{B}$, hence $c = a \in A$ and $A = \bar{B}$.□

In view of the foregoing results, the embedding theorem (68.2) may be given a topological interpretation.

First, let B be a direct sum of cyclic p-groups, equipped with the inductive topology τ_B^*. We can form the completion of B in τ_B^*. Because of (70.3), this completion consists of the limits of all bounded Cauchy sequences in τ_B mod 0-sequences. Consequently, (70.7) shows that a completion of B is the torsion-complete group $\bar{B}$. Moreover, it follows that the identity map of B extends to an isomorphism between the completion and $\bar{B}$ [this is a topological isomorphism if $\bar{B}$ is furnished with the topology $\tau_{\bar{B}}^*$]. Hence we are justified to call $\bar{B}$ the *torsion-completion* of B.

Naturally, we can start with an arbitrary separable p-group A and form its completion $\bar{A}$ in τ_A^*. Since any basic subgroup B of A, with its τ_B^*-topology, is a dense τ_A^*-subspace of A, it is clear that the inclusion map $B \to A$ extends to an isomorphism $\bar{B} \to \bar{A}$. Therefore:

Proposition 70.8. *If A is a separable p-group and B is a basic subgroup of A, then the τ_A^*-completion $\bar{A}$ of A is isomorphic to $\bar{B}$.*□

EXERCISES

1. The topologies $\tau_A^{(k)}$ $(k = 1, 2, \cdots)$ and τ_A^* are functorial in the category of separable p-groups [i.e., every group homomorphism in the category is continuous].
2. A subgroup C of A is closed in τ_A^* if and only if $C[p^k]$ is closed in $A[p^k]$ in τ_A, for every k.
3. (a) An unbounded separable p-group A contains proper subgroups C, open in τ_A^*, such that A/C is divisible.

   (b) Basic subgroups of A are dense in τ_A^*.

4. (B. Charles) Let A and C be separable p-groups, furnished with the inductive p-adic topology and the discrete topology, respectively. Show that a homomorphism $\phi: A \to C$ is continuous exactly if it is a small homomorphism.
5. Suppose A is a separable p-group and τ'_A is a topology on A which is coarser than τ_A. If every bounded τ'_A-Cauchy sequence has a τ'_A-limit in A, then A is torsion-complete. [*Hint:* Remark in **39**.]
6. A closed subgroup of a torsion-complete p-group is again torsion-complete (in its own topology!). [*Hint:* Ex. 5.]
7. Prove that, for an unbounded p-group A, τ_A^* does not satisfy the axioms of countability. [*Hint:* idea of proof of (70.2).]
8. Show that in a torsion-complete p-group, an infinite series $a_1 + \cdots + a_n + \cdots$ with $p^m a_n = 0$ for some m and for all n converges exactly if the sequence $\{a_n\}$ tends to 0.
9. If $0 \to G \to A$ is a pure-exact sequence of separable p-groups, then the induced sequence $0 \to \bar{G} \to \bar{A}$ of completions in τ^* is splitting-exact.
10. An unbounded separable p-group can be complete in infinitely many different linear topologies.

71. DIRECT DECOMPOSITIONS OF TORSION-COMPLETE GROUPS

Our study begins with the following technical result:

Lemma 71.1 (Hill [10]). *Let A be a torsion-complete p-group and ϕ a homomorphism of $A[p]$ into the direct sum $C = \bigoplus_i C_i$ of separable p-groups C_i. If ϕ does not decrease heights, then there exist an integer m and a finite number $C_{i_1}, \cdots, C_{i_k}$ of the C_i such that*

$$\phi(p^m A[p]) \leqq C_{i_1} \oplus \cdots \oplus C_{i_k}.$$

If the conclusion is false, then we can find a strictly increasing sequence $m_1 < m_2 < \cdots$ of integers and a sequence $a_k \in p^{m_k} A[p]$ such that the projections $\pi_i: C \to C_i$ satisfy

$$p^{m_k} C \cap \langle \pi_i \phi a_1, \cdots, \pi_i \phi a_{k-1} \quad \text{for all} \quad i \in I \rangle = 0,$$

and

$$\phi a_k \notin C_{i_1} \oplus \cdots \oplus C_{i_r},$$

where $\{i_1, \cdots, i_r\}$ is the minimal subset of I with $\langle \phi a_1, \cdots, \phi a_{k-1} \rangle \leqq C_{i_1} \oplus \cdots \oplus C_{i_r}$. Visibly, the sequence $g_k = a_1 + \cdots + a_k$ $(k = 1, 2, \cdots)$ converges to a limit $g \in A$. However, the sequence $\phi g_k = \phi a_1 + \cdots + \phi a_k$ $(k = 1, 2, \cdots)$ cannot have a limit in C. For, if $c \in C$ was one, then $h(\phi a_k) \geqq m_k$

would imply $c - (\phi a_1 + \cdots + \phi a_k) \in p^{m_k+1}C$ for every k, whence it is easy to see that $\pi_i c \neq 0$ for infinitely many $i \in I$, which is absurd.□

The special case $A \leqq C$ gives the more interesting

Proposition 71.2 (Enochs [1]). *If a torsion-complete p-group A is contained in the direct sum $C = \bigoplus_i C_i$ of separable p-groups C_i, then there exist an integer m and a finite number $C_{i_1}, \cdots, C_{i_k}$ of the C_i such that*

$$p^m A[p] \leqq C_{i_1} \oplus \cdots \oplus C_{i_k}.\square$$

A satisfactory description of direct decompositions of torsion-complete groups is given in the next theorem.

Theorem 71.3 (Kulikov [2]). *If a torsion-complete p-group A is the direct sum of infinitely many subgroups A_i, then the A_i are torsion-complete and for a sufficiently large integer m,*

$$p^m A_i = 0 \qquad \textit{for almost all} \quad i.$$

If $B = \bigoplus_i C_i$ is a direct decomposition of a basic subgroup of $\bar{B}$ such that, for some m, $p^m C_i = 0$ holds for almost all i, then $\bar{B} = \bigoplus_i C_i^-$, where $C_i^- \cong \bar{C}_i$.

If $A = \bigoplus A_i$ is torsion-complete, then by (68.7) so are the A_i and a simple appeal to (71.2) proves the first assertion.

Let $B = \bigoplus_i C_i$ be a direct decomposition of a basic subgroup of $\bar{B}$ such that $C_{i_1}, \cdots, C_{i_k}$ are unbounded and all other C_i satisfy $p^m C_i = 0$. Write $B = C_{i_1} \oplus \cdots \oplus C_{i_k} \oplus C_0$, where C_0 is the direct sum of the rest, $p^m C_0 = 0$. Since an obvious induction applies, it suffices to prove that $B = C_1 \oplus C_2$ implies $\bar{B} = C_1^- \oplus C_2^-$ with $C_j^- \cong \bar{C}_j$ $(j = 1, 2)$. There is a direct decomposition $B = \bigoplus_n B_n$ with $B_n = \bigoplus Z(p^n)$ such that $B_n = (B_n \cap C_1) \oplus (B_n \cap C_2)$, for every n, and so every $a \in \bar{B}$ can be written in the form

$$a = (b_{11} + b_{12}, \cdots, b_{n1} + b_{n2}, \cdots) = (b_{11}, \cdots, b_{n1}, \cdots) + (b_{12}, \cdots, b_{n2}, \cdots),$$

with $b_{nj} \in B_n \cap C_j$. In the last sum, the first vector is in C_1^-, and the second in C_2^-. It is clear that $a = (b_1, \cdots, b_n, \cdots) \in \bar{B}$ belongs to C_j^- exactly if every $b_n \in B_n \cap C_j$, which proves not only $C_1^- \cap C_2^- = 0$, but also the isomorphisms $C_j^- \cong \bar{C}_j$.□

Corollary 71.4 (Kulikov [2]). *Any two direct decompositions of a torsion-complete group have isomorphic refinements.*

Assume A is a torsion-complete p-group, and let $A = \bigoplus_i A_i = \bigoplus_j C_j$ be two direct decompositions of A. Let B_i and B'_j denote basic subgroups of A_i and C_j, respectively. Then $B = \bigoplus_i B_i$ and $B' = \bigoplus_j B'_j$ are basic subgroups of A, and thus $B \cong B'$. From (18.2) we conclude the existence of groups B_{ij} and B'_{ji} such that

$$B_i = \bigoplus_j B_{ij}, \qquad B'_j = \bigoplus_i B'_{ji}, \qquad \text{and} \qquad B_{ij} \cong B'_{ji}.$$

In view of (71.3), there is an integer m satisfying $p^m A_i = p^m C_j = 0$ for almost all i and j. Since $p^m A_i = 0$ or $p^m C_j = 0$ implies $p^m B_{ij} = 0$, we see that there exist but a finite number of B_{ij} with $p^m B_{ij} \neq 0$. Define $A_{ij} = B_{ij}^-$ and $C_{ji} = B_{ji}'^-$. Then $A_{ij} \cong \bar{B}_{ij} \cong \bar{B}'_{ji} \cong C_{ji}$ and $A_i = \bigoplus_j A_{ij}$, $C_j = \bigoplus_i C_{ji}$, as is seen from (71.3).□

EXERCISES

1. Give an example to show that (71.1) need not hold for arbitrary reduced p-groups C_ι.
2. A torsion-complete p-group is contained in a direct sum of cyclic groups if and only if it is bounded.
3. If A is torsion-complete and $A \leqq \bigoplus C_i$, where the C_i are separable p-groups, then $A \cap C_i$ is a torsion-complete subgroup of C_i, for each i.
4. (Irwin and O'Neill [1]) If a direct sum of separable p-groups contains an unbounded torsion-complete p-group, then one of the summands contains such a subgroup.
5. (a) Show that (71.2) holds with $[p]$ replaced by $[p^k]$ for any $k \geqq 1$.
   (b) Give an example to show that (71.2) fails to hold if $[p]$ is omitted in the conclusion.
6. Under the hypotheses of (71.2), A has a direct decomposition $A = G \oplus H$ such that G is bounded and $H[p] \leqq C_{i_1} \oplus \cdots \oplus C_{i_k}$ for a finite number of indices $i_1, \cdots, i_k$.
7. Extend (71.1)–(71.3) to arbitrary torsion-complete groups.
8. Every subgroup C of a torsion-complete p-group can be embedded in a direct summand of power $\leqq |C|^{\aleph_0}$. [*Hint*: embed C in a pure subgroup and take closure.]
9. A torsion-complete p-group $\bar{B}$ fails to have a proper pure subgroup isomorphic to $\bar{B}$ exactly if its Ulm–Kaplansky invariants are finite.

72. THE EXCHANGE PROPERTY

Our results on torsion-complete groups indicate that these groups behave nicely in direct decompositions. Crawley and Jónsson [1] have noticed that these groups enjoy a very strong property, called the exchange property. This section is devoted to the discussion of this significant phenomenon. We shall refrain from entering into a detailed study of the exchange property in general; we concentrate on torsion-complete groups.

A group A is said to have the *exchange property* if it satisfies the following condition: whenever A appears as a direct summand of a group M which is a direct sum of subgroups C_i:

$$M = A \oplus N = \bigoplus_{i \in I} C_i, \tag{1}$$

there are always subgroups E_i of C_i such that

$$M = A \oplus \bigoplus_{i \in I} E_i. \tag{2}$$

And we say A has the *finite exchange property* if it has the said property for finite index sets I.

Our discussion begins with a few elementary observations.

(a) E_i is a summand of C_i, $C_i = E_i \oplus E_i'$, for all i.

(b) If E_i' is defined as in (a), then $A \cong \bigoplus_{i \in I} E_i'$.

(c) *The group $A = G \oplus H$ has the exchange property if and only if both G and H have it.*

To prove this, let A have the exchange property, and suppose $M = G \oplus N = \bigoplus C_i$. Then $H \oplus M = A \oplus N = H \oplus \bigoplus C_i$ implies $H \oplus M = A \oplus K \oplus \bigoplus E_i$ for some $K \leqq H$ and $E_i \leqq C_i$. We must have $K = 0$ because of $K \leqq A$; thus $H \oplus M = H \oplus G \oplus \bigoplus_i E_i$. Since $G \oplus \bigoplus E_i \leqq M$, the last equality implies $M = G \oplus \bigoplus E_i$.

Conversely, suppose G and H have the exchange property. Then (1) implies $M = G \oplus \bigoplus E_i$ for suitable $E_i \leqq C_i$. Hence $M/G = \bar{H} \oplus \bar{N} = \bigoplus \bar{E}_i$ where bars indicate cosets mod G. We infer $M/G = \bar{H} \oplus \bigoplus \bar{E}_i'$ for some $\bar{E}_i' \leqq \bar{E}_i$. Since G is a summand of M, there are subgroups $E_i' \leqq E_i$ such that $G \oplus E_i'$ correspond to $\bar{E}_i'$. From (9.4) we get $M = G \oplus H \oplus \bigoplus E_i' = A \oplus \bigoplus E_i'$, as desired.

(d) *If A has the exchange property for all groups M in (1) with the C_i restricted to groups isomorphic to subgroups of A, then A has the exchange property.*

Suppose (1) holds with arbitrary C_i, where A is assumed to have this weaker exchange property. Let $\pi\colon M \to A$ be the projection with kernel N and define $K_i = \operatorname{Ker}(\pi \mid C_i)$. Then $K = \bigoplus K_i \leqq N$, and passing mod K, we have $M/K = \bar{A} \oplus N/K = \bigoplus (C_i/K_i)$, where $\bar{A} = (A + K)/K \cong A$ and C_i/K_i has been identified with $(C_i + K)/K$ under the canonical map. Now the C_i/K_i are isomorphic to subgroups of A, hence there exist groups $E_i/K_i \leqq C_i/K_i$ such that $M/K = \bar{A} \oplus \bigoplus (E_i/K_i)$. It is straightforward to check that $M = A \oplus \bigoplus E_i$ which completes the proof.

It is now easy to conclude:

(e) A torsion group has the exchange property if and only if each of its p-components has the exchange property.

The following two theorems state that some classes of groups have the exchange property.

Theorem 72.1 (Crawley and Jónsson [1]). *If the reduced part of A is a torsion group with bounded p-components, then A has the exchange property.*

In view of (c) and (e), it suffices to show that divisible groups and groups which are direct sums of copies of the same $Z(p^n)$ have the exchange property.

Let A be divisible and assume (1). Select a subgroup E of M which is maximal with respect to the properties: $E = \bigoplus_i E_i$ with $E_i \leqq C_i$, and $E \cap A = 0$. We want to prove (2) for these E_i. For the natural homomorphism $\phi\colon M \to M/E$, $\phi(A) \cong A$ is a subgroup of $M/E = \bigoplus_i (C_i/E_i)$, where C_i/E_i has been identified with $(C_i + E)/E$. The maximal choice of E ensures that $\phi(A) \cap (C_i/E_i)$ is an essential subgroup of C_i/E_i. Thus

$$\bigoplus_i [\phi(A) \cap (C_i/E_i)],$$

and *a fortiori* $\phi(A)$, is essential in M/E. Since $\phi(A)$ has no proper essential extension, $\phi(A) = M/E$, whence $M = A \oplus E$, in fact.

Next let $A = \bigoplus Z(p^n)$ with fixed p^n. Assume (1) with the C_i satisfying $p^n C_i = 0$. The argument in the preceding paragraph can be repeated, except for the last sentence. Instead, one should observe that $p^n(M/E) = 0$ implies, in view of the structure of A, that M/E can not be a proper essential extension of $\phi(A) \cong A$. Hence again $\phi(A) = M/E$ and $M = A \oplus E$.$\square$

Turning our attention to other classes of groups, first of all it is clear that a group has the exchange property exactly if its reduced part has it. Therefore, only reduced groups need to be considered. So far, no reasonable characterization has been obtained for reduced groups with the exchange property.

The exchange property can be established for algebraically compact groups, too [it suffices to investigate complete groups]. The exchange properties for complete and torsion-complete groups can be discussed analogously; for the sake of convenience, we confine ourselves to torsion-complete groups and delegate the parallel discussion for complete groups to the exercises.

We can now prove the principal result of this section.

Theorem 72.2 (Crawley and Jónsson [1]). *Torsion-complete groups have the exchange property.*

Let A be a torsion-complete p-group, and assume (1) with the C_i isomorphic to subgroups of A. Thus the C_i are separable p-groups; therefore, (71.2) implies the existence of an integer m such that

$$p^m A[p] \leqq C_1 \oplus \cdots \oplus C_k = C'$$

[finite direct sum]. From (27.7) we infer that A has a direct decomposition $A = A_1 \oplus A_2$ with $p^m A_2 = 0$ such that $A_1[p] \leqq C'$, and in view of (c) and (72.1), we can leave A_2 out of consideration. Clearly, A_1 is torsion-complete, and therefore so is $\pi A_1 \cong A_1$, where π denotes the obvious projection of M onto C'. Because of the purity of A_1 in M, it is readily verified that πA_1 is pure in C', whence $C' = \pi A_1 \oplus N'$ for some $N' \leqq C'$. In view of the purity of A_1 and πA_1 in M and the equality of their socles, (66.1) implies that in the direct sum $\oplus C_i$, the partial sum $C_1 \oplus \cdots \oplus C_k$ can be replaced by $A_1 \oplus N'$. Consequently, we need only verify the finite exchange property for torsion-complete p-groups.

Starting anew with a torsion-complete p-group A and with a group $M = A \oplus N = C_1 \oplus \cdots \oplus C_k$, where $C_1, \cdots, C_k$ are separable p-groups, we first form the torsion-completion of M:

$$\bar{M} = A \oplus \bar{N} = \bar{C}_1 \oplus \cdots \oplus \bar{C}_k.$$

Let $B = \oplus_n B_n$ [with B_n a direct sum of cyclic groups of orders p^n] be a basic subgroup of A. Each B_n has the exchange property, as was shown in (72.1), so we can produce successively for every n, a decomposition

$$\bar{M} = B_1 \oplus \cdots \oplus B_n \oplus \bar{C}_1^{(n)} \oplus \cdots \oplus \bar{C}_k^{(n)},$$

where each $\bar{C}_i^{(n)}$ is a summand of $\bar{C}_i^{(n-1)}$. Clearly, a complement of $\bar{C}_i^{(n)}$ in $\bar{C}_i^{(n-1)}$ must be a direct sum of cyclic groups of orders p^n. Hence we conclude that $\bar{M}$ has a basic subgroup of the form $B \oplus D_1 \oplus \cdots \oplus D_k$ where, for every i, D_i is a summand of a basic subgroup of $\bar{C}_i$. Owing to (71.2), we have $\bar{M} = A \oplus E_1 \oplus \cdots \oplus E_k$, where $E_i = D_i^- \cong \bar{D}_i$. Therefore,

$$M = A \oplus [(E_1 \oplus \cdots \oplus E_k) \cap M],$$

where the second summand is the intersection of $E_1 \oplus \cdots \oplus E_k$ with $C_1 \oplus \cdots \oplus C_k$ in $\bar{C}_1 \oplus \cdots \oplus \bar{C}_k$, so it is equal to $(E_1 \cap C_1) \oplus \cdots \oplus (E_k \cap C_k)$. This completes the proof. □

The following example will show that unbounded direct sums of cyclic p-groups, in general, fail to have the exchange property.

Example (Crawley and Jónsson [1]). Let $B = \oplus_{n=1}^{\infty} \langle b_n \rangle$ and $C = \oplus_{n=1}^{\infty} \langle c_n \rangle$, where $o(b_n) = o(c_n) = p^n$ for every n. The direct sum $M = B \oplus C$ can also be decomposed as follows:

$$M = A \oplus B = C \oplus D, \qquad \text{where} \quad A = \bigoplus_{n=1}^{\infty} \langle c_n + pb_{n+1} \rangle \quad \text{and} \quad D = \bigoplus_{n=1}^{\infty} \langle b_n + pc_{n+1} \rangle.$$

Assume we can find $C' \leq C$ and $D' \leq D$ such that $M = A \oplus C' \oplus D'$. Then $A \oplus C' \leq A + C \leq A + pB$ shows that every b_n can be written in the form $b_n = a + pb + d$, with $a \in A$, $b \in B$, $a + pb \in A \oplus C'$, and $d \in D'$. Here $p^n b_n = 0$ implies $p^n d = 0$. From $M = A \oplus B$, we see that $p^k(b_n - a) = p^{k+1}b + p^k d$ has height k for $k = 0, 1, \cdots, n-1$, thus $h(p^k d) = k$. Hence it is immediate that $\langle d \rangle$ is a pure subgroup, and so a summand of D'. Thus for every n, D' has a direct summand of order p^n. The comparison of basic subgroups leads us to the conclusion that $C' = 0$. But $A + D$ contains neither b_1 nor c_1; this shows that the exchange property fails for A.

EXERCISES

1. (a) Assertion (c) fails to hold for infinite sums.
(b) Let $A = \bigoplus A_j$ be such that for every subgroup X of A,
$$X = \bigoplus (A_j \cap X)$$
holds. Assuming the exchange property for every A_j, prove it for A.
2. Show that no unbounded p-group that is a direct sum of cyclic groups has the exchange property.
3. An infinite direct sum of unbounded torsion-complete p-groups does not have the exchange property [for fixed p].
4. (Crawley and Jónsson [1]) Show that if a group A has the exchange property for an index set I of cardinality 2, then it has the finite exchange property.
5. (Crawley and Jónsson [1]) If a group A has the exchange property for all index sets I such that $|I| \leqq |A|$, then A has the exchange property.
6. (Crawley and Jónsson [1]) For an indecomposable group, the finite exchange property implies the exchange property.
7. The group Q_p has the exchange property. [*Hint*: if $Q_p \oplus N = C \oplus D$, then using projections, either $Q_p \to C \xrightarrow{\sigma} Q_p$ or $Q_p \to D \to Q_p$ is an automorphism of Q_p; if the first is, C is replaceable by $Q_p \oplus \operatorname{Ker} \sigma$.]
8.* (Warfield [2]) An indecomposable group has the exchange property exactly if its endomorphism ring is a local ring. [*Hint*: for sufficiency, argue as in Ex. 7; if $\xi - \eta = 1_A$, where the endomorphisms ξ, η are not automorphisms, then
$$M = A \oplus A = \operatorname{Im}(\xi \oplus \eta)\nabla_A \oplus \operatorname{Im} \Delta_A$$
and neither the first nor the second summand generates with the third the group M.]
9. (Warfield [3]) A complete group has the exchange property. [*Hint*: using (39.9), reduce to finite exchange property and to p-adic modules; argue as in the proof of (72.2), only the existence of a basic subgroup of the stated kind needs some extra consideration.]
10. (a) By making use of Ex. 9, show that every pure, fully invariant subgroup of a complete group shares the finite exchange property. [*Hint*: as in the proof of (72.2) pass to completions, apply Ex. 9 and (9.3).]
(b) Derive (72.2) from part (a).
11. Let A be a separable p-group with the exchange property and let A be a summand of $\bigoplus C_i$, where the C_i are separable p-groups. Then there is an integer m such that $p^m A[p]$ is contained in a finite direct sum of the C_i. [*Hint*: Example and (c).]

73. DIRECT SUMS OF TORSION-COMPLETE GROUPS

As we have indicated, so far the direct sums of cyclic p-groups and the torsion-complete p-groups are essentially the only classes of separable p-groups with satisfactory structure theory. It is reasonable to expect that there is a common generalization of these classes for which some of the results can be carried over.

Kolettis [2] came up with the idea of investigating p-groups of the form $B \oplus C$, where B is a direct sum of cyclic groups and C is torsion-complete. More natural generalizations are the direct sums of torsion-complete groups; they have recently received a great deal of attention, so that their theory has been nicely developed. This class of groups constitutes the topic of this section. Needless to say, there is no loss of generality in limiting our considerations to p-groups.

Our discussion starts with some preliminaries which are of independent interest. A p-group A will be called *pure-complete* if every subsocle of A supports a pure subgroup of A. In view of (66.3), it suffices to know that all closed subsocles of A support pure subgroups.

It is easy to see that summands G of pure-complete groups A are again pure-complete. In fact, if $\pi: A \to G$ is a projection onto G and C is a pure subgroup of A supported by a subsocle S of G, then πC will be a pure subgroup of G supported by S.

Lemma 73.1 (Hill and Megibben [3]). *If $A = \bigoplus_{n=1}^{\infty} A_n$ where every A_n is torsion-complete, then A is pure-complete.*

If A is pure-complete and C is a direct sum of cyclic p-groups, then $A \oplus C$ is likewise pure-complete.

First of all, we show that a torsion-complete p-group A is necessarily pure-complete. We prove somewhat more: if S is a subsocle of A and G is a pure subgroup of A with $G[p] \leqq S$, then A contains a pure subgroup H containing G and supported by S. Let L' be a p-basis for G and let L be a p-independent set in A maximal with respect to the properties: $L' \subseteq L$ and $\langle L \rangle[p] \leqq S$. Then $B' = \langle L \rangle$ is a summand of a basic subgroup B of A, say $B = B' \oplus B''$, and we have $A = \bar{B}' \oplus \bar{B}''$. Here $S \leqq \bar{B}'[p]$, since $\langle L \rangle[p]$ is dense in S. A simple appeal to (66.3) establishes the existence of an H with the desired properties.

Now suppose $A = \bigoplus_{n=1}^{\infty} A_n$ with torsion-complete p-groups A_n, and let S be a subsocle of A. Then $S_n = S \cap (A_1 \oplus \cdots \oplus A_n)$ is a subsocle of the torsion-complete group $A_1 \oplus \cdots \oplus A_n$; hence it supports a pure subgroup G_n of $A_1 \oplus \cdots \oplus A_n$. By the preceding paragraph, these G_n can be selected so as to satisfy $G_1 \leqq \cdots \leqq G_n \leqq \cdots$. Evidently, $G = \bigcup_n G_n$ is pure and supported by S.

A direct sum C of cyclic p-groups may be viewed as a union of an ascending chain $C_1 \leqq \cdots \leqq C_n \leqq \cdots$ of bounded pure subgroups. If A is pure-complete and S is a subsocle of $A \oplus C$, then write $S_n = (A \oplus C_n) \cap S$, and suppose that a monotone sequence $G_1 \leqq \cdots \leqq G_n$ of pure subgroups $G_j \leqq A \oplus C_j$ has been selected such that $G_j[p] = S_j$ $(j = 1, \cdots, n)$. Then $(G_n + S_{n+1})/G_n$ is a discrete subsocle of $(A \oplus C_{n+1})/G_n$; thus it supports a pure subgroup G_{n+1}/G_n. It is immediately seen that $G_{n+1}[p] = S_{n+1}$ and G_{n+1} is pure in $A \oplus C$. The union $G = \bigcup_n G_n$ is as desired, so $A \oplus C$ is again pure-complete.□

If C is a summand of a group A, then there is a projection $\pi\colon A \to C$. This π, restricted to the socle, gives rise to a projection $A[p] \to C[p]$ which does not decrease heights. Hence, a necessary condition for a subsocle S of a p-group A to support a summand of A is the existence of a projection of $A[p]$ onto S which does not decrease heights. Interestingly enough, the converse can easily be established for pure-complete groups:

Lemma 73.2 (Hill [10]). *Let π be a projection of the socle of a pure-complete p-group A that does not decrease heights. Then* Im π *supports a direct summand of A.*

Let G and H be pure subgroups of A supported by Im π and Im$(1 - \pi)$, respectively. Then $A[p]$ is the socle of $G \oplus H$, and in order to verify the equality $G \oplus H = A$, it is enough to show that no $a \in A[p]$ has a greater height in A than in $G \oplus H$. But this is in fact true, since the height of a in $G \oplus H$ is $\min\{h(\pi a), h((1 - \pi)a)\}$ [the heights being the same in G, H as in A], where $h(\pi a)$, $h((1 - \pi)a)$ are both greater than or equal to the height of a in A, due to the nature of π.□

Turning our attention to the main topic of this section, our first task is to find criteria under which a separable p-group belongs to the class of direct sums of torsion-complete groups. No satisfactory criterion is known in the general case, but for countable p-groups, we have a rather simple result [where completeness is to be understood in the inductive p-adic topology]:

Proposition 73.3. *A separable p-group A is the direct sum of countably many torsion-complete groups if and only if it satisfies*:

(i) *$A[p]$ is the union of an ascending chain of complete subsocles S_n* $(n = 1, 2, \cdots)$;

(ii) *every complete subsocle of A supports a pure subgroup.*

If $A = \bigoplus_{n=1}^{\infty} A_n$, where each A_n is torsion-complete, then the subsocles $S_n = A_1[p] \oplus \cdots \oplus A_n[p]$ satisfy (i), while (ii) follows from (73.1).

Conversely, if A satisfies (i) and (ii), then let G_n be a pure subgroup of A supported by S_n. By (70.6), G_n is torsion-complete. It contains a summand G'_{n-1} supported by S_{n-1}, $G_n = G'_{n-1} \oplus A_n$, where obviously, A_n is likewise torsion-complete [we may set $G_1 = A_1$]. These A_n ($n = 1, 2, \cdots$) generate their direct sum in A, which contains the socle of A. As $A_1 \oplus \cdots \oplus A_n$ is a summand of A, $\bigoplus_{n=1}^{\infty} A_n$ must be pure in and hence equal to A.□

We find the following immediate consequence worthwhile recording.

Corollary 73.4 (Irwin, Richman, and Walker [1]). *Every summand G of a countable direct sum A of torsion-complete groups is again a direct sum of countably many torsion-complete groups.*

If S_n is as in (73.3)(i), then $T_n = G \cap S_n$ is closed in S_n and hence complete. Obviously, $G[p] = \bigcup T_n$. Furthermore, G as a summand of a pure-complete group is likewise pure-complete. A simple reference to (73.3) completes the proof.□

The socles play a crucial role in direct sums of torsion-complete groups. Convincing evidence is furnished by the next theorem.

Theorem 73.5 (Hill [5]). *Let $A = \oplus_{i \in I} A_i$ and $G = \bigoplus_{j \in J} G_j$ be pure subgroups of some p-group H such that $A[p] = G[p]$, where each of A_i, G_j is torsion-complete. Then $A \cong G$.*

A simple consequence of (71.1) is that, for every i, there exist a decomposition $A_i = B_i \oplus C_i$ and a finite subset $J(i)$ of J such that B_i is bounded and $C_i[p] \leqq \bigoplus_{j \in J(i)} G_j$. By (73.1), the last group has a pure subgroup C_i', which is by (70.6) a summand, supported by $C_i[p]$. Furthermore, G has a summand B_i' whose socle is $B_i[p]$. From (66.1) we derive the isomorphisms $C_i' \cong C_i$ and $B_i' \cong B_i$. The B_i', C_i' ($i \in I$) generate their direct sum E in G, and the claimed isomorphism will follow if we can show that $E = G$.

Knowing that $E[p] = G[p]$ and $E \leqq G$, we need only verify that no element x of the socle has a smaller height in E than in G. If the height of x in A is k, then the coordinates of x in B_i', C_i' all have heights $\geqq k$, so their sum has height $\geqq k$ in E.□

The last theorem has an important consequence concerning the isomorphism of direct sums of torsion-complete groups.

Let A be a direct sum of torsion-complete p-groups. Since it is separable, by virtue of (7.2) we see that A is a [Hausdorff] metric space in the metric $\delta(a, b) = \|a - b\| = \exp(-h(a - b))$, where $a, b \in A$. The socle $A[p]$ is a metric vector space over the prime field of characteristic p, and every isomorphism of A with a group G induces an isometry of $A[p]$ with $G[p]$.

Theorem 73.6 (Hill [5]). *Let A and G be direct sums of torsion-complete p-groups. A and G are isomorphic if and only if there is an isometry between their socles $A[p]$ and $G[p]$.*

Our claim will be a simple consequence of (73.5), if we can show that A and G can be embedded as pure subgroups with the same socle in a p-group. Let ϕ be an isometry from $A[p]$ to $G[p]$; in other words, it is a height-preserving isomorphism from $A[p]$ to $G[p]$. If B is a basic subgroup of A, then $\phi(B[p])$ supports a basic subgroup C of G, and there is an isomorphism $\phi^*: B \to C$ extending ϕ. Moreover, ϕ^* can be extended, in view of (68.4), to an isomorphism $\overline{\phi}: \bar{B} \to \bar{C}$, and so $\overline{\phi}$ will embed A in $\bar{C}$ as a pure subgroup with the same socle as G.□

Our discussion continues with summands of direct sums of torsion-complete groups. From (73.4) we know that they are themselves direct sums of torsion-complete groups whenever the direct sums have a countable number of components. The countability hypothesis can be removed as is shown by the following theorem.

Theorem 73.7 (Irwin, Richman, and Walker [1], Hill [10]). *A direct summand of a direct sum of torsion-complete groups is again a direct sum of torsion-complete groups.*

Suppose that $A = \bigoplus_{\sigma \in I} A_\sigma$ is a p-group where the A_σ are torsion-complete; anticipating a transfinite induction, we have chosen our index set I as the set of all ordinals less than the first ordinal ω_τ of some cardinality $\aleph_\tau$. Letting $A = G \oplus H$, say with projection $\pi: A \to G$, we intend to show that G is a direct sum of torsion-complete groups.

To start with, we decompose every $A_\sigma = B_\sigma \oplus C_\sigma$ in such a way that B_σ is bounded and $\pi C_\sigma[p]$ is contained in the direct sum of a finite number of A_ρ; (71.1) guarantees that this can be done. Now $B = \bigoplus_\sigma B_\sigma$ is a direct sum of cyclic groups, and we have $A = B \oplus \bigoplus_\sigma C_\sigma = G \oplus H$.

The next step is to establish the existence of a chain

$$\varnothing = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\sigma \subseteq \cdots \qquad (\sigma < \omega_\tau)$$

of subsets of I such that: (i) $\sigma \in I_{\sigma+1}$; (ii) $I_\sigma = \bigcup_{\rho<\sigma} I_\rho$ if σ is a limit ordinal; (iii) $|I_\sigma| \leqq \aleph_0 |\sigma|$; and in addition,

$$\text{(iv)} \qquad \pi\Big(\bigoplus_{\rho \in I_\sigma} C_\rho[p]\Big) \leqq \bigoplus_{\rho \in I_\sigma} A_\rho = A_\sigma^*.$$

All that we need to do is indicate how to obtain $I_{\sigma+1}$ from I_σ, whenever $I_\sigma \neq I$. Take the first ordinal σ' not in I_σ [this is $\geqq \sigma$] and choose a finite subset $I(\sigma')$ of I, satisfying $\pi C_{\sigma'}[p] \leqq \bigoplus_{\rho \in I(\sigma')} A_\rho$. Repeat this process with each $\rho \in I(\sigma') \backslash I_\sigma$, then again, ω times, and define $I_{\sigma+1}$ to be the union of I_σ with $I(\sigma')$ and all occurring finite subsets. That $I_{\sigma+1}$ will satisfy conditions (i), (iii), and (iv) is evident.

We return now to the group B, and represent it as the union of an ascending chain of direct summands

$$0 = B'_0 \leqq B'_1 \leqq \cdots \leqq B'_\sigma \leqq \cdots \qquad (\sigma < \omega_\tau),$$

subject to the conditions:

$$B'_\sigma \leqq \bigoplus_{\rho \in I_\sigma} B_\rho \qquad \text{and} \qquad \pi B'_\sigma[p] \leqq A^*_\sigma.$$

This can be done, since each cyclic summand of B and its image under π must belong to A^*_σ for some σ. It is routine to check that there are pure subgroups B^*_σ of A^*_σ such that $B^*_\sigma[p] = B'_\sigma[p]$. These B^*_σ are again direct sums of cyclic groups such that, for each σ,

$$C^*_\sigma = B^*_\sigma \oplus \bigoplus_{\rho \in I_\sigma} C_\rho$$

is a summand of A^*_σ satisfying $\pi C^*_\sigma[p] \leqq A^*_\sigma$. It is clear that $\bigcup_\sigma C^*_\sigma$ is equal to A, since it is a pure subgroup supported by $A[p]$. Hence $\bigcup_\sigma \pi C^*_\sigma[p] = G[p]$.

The initial step in the induction is based on the fact that by (73.1) A^*_1 is pure-complete. Thus from (73.2) we derive the existence of a direct summand K_1 of A^*_1 which is supported by $\pi C^*_1[p]$, $A^*_1 = K_1 \oplus L_1$. By virtue of (73.4), both K_1 and L_1 are countable direct sums of torsion-complete p-groups.

We are now ready for a transfinite induction. Suppose that for every ordinal $\tau' < \tau$ we have established that: (1) every projection of the socle of a direct sum of $\aleph_{\tau'}$ torsion-complete groups that does not decrease heights supports a summand of the group; and (2) every summand of a direct sum of $\aleph_{\tau'}$ torsion-complete groups can be written as a direct sum of at most $\aleph_{\tau'}$ torsion-complete groups. This means that for every $\sigma < \omega_\tau$ we can write $A^*_\sigma = K_\sigma \oplus L_\sigma$, where $K_\sigma[p] = \pi C^*_\sigma[p]$ and L_σ is a direct sum of less than $\aleph_\tau$ torsion-complete groups. Obviously,

$$A^*_{\sigma+1} = K_\sigma \oplus \left(L_\sigma \oplus \bigoplus_{\rho \in I_{\sigma+1} \setminus I_\sigma} A_\rho\right).$$

Let ϕ denote the projection of $A^*_{\sigma+1}$ onto the second summand. Then $K_{\sigma+1}[p] = K_\sigma[p] \oplus H$, where $H = \phi\pi C^*_{\sigma+1}[p]$ supports, by induction hypothesis, a summand G_σ of A which is a direct sum of less than $\aleph_\tau$ torsion-complete groups. It is readily verified that these G_σ $(\sigma < \omega_\tau)$ generate their direct sum in A and that $\bigoplus_{\sigma < \omega_\tau} G_\sigma = G'$ is a pure subgroup of A. Now we notice that $G'[p]$ is contained in $\pi A = G$ and contains all $K_\sigma[p]$, whence the equality $G'[p] = G[p]$ becomes evident. Finally, (66.1) implies $G \cong G' = \bigoplus_\sigma G_\sigma$, completing the proof.□

We conclude this section with a refinement theorem.

Theorem 73.8 (Crawley and Jónsson [1], Enochs [1], Hill [10]). *If A is a direct sum of torsion-complete groups, then any two direct decompositions of A have isomorphic refinements.*

By (73.7), it suffices to consider direct decompositions $A = \bigoplus_{i \in I} A_i = \bigoplus_{j \in J} A'_j$, where all A_i, A'_j are torsion-complete. We may assume that A is itself a p-group.

First, suppose that I and J are countable, and for the sake of simplicity, let $I = J = \{1, 2, \cdots, n, \cdots\}$. The subsocles

$$S_n = (A_1 \oplus \cdots \oplus A_n)[p] \cap (A'_1 \oplus \cdots \oplus A'_n)[p] \qquad (n = 1, 2, \cdots)$$

form an ascending chain whose union is $A[p]$. Every S_n is complete; thus $S_{n+1} = S_n \oplus S_n^*$ for some complete subsocle S_n^*. If C_n and C'_n are [torsion-complete] summands of $A_1 \oplus \cdots \oplus A_n$ and $A'_1 \oplus \cdots \oplus A'_n$, respectively, supported by S_n^*, then $A = \bigoplus_{n=1}^{\infty} C_n = \bigoplus_{n=1}^{\infty} C'_n$ is easily verified. By (66.1), $C_n \cong C'_n$, so it follows from (71.4) that any two refinements of the last two decompositions of A have isomorphic refinements. Consequently, it suffices to find isomorphic refinements for the decompositions $A = \bigoplus_{n=1}^{\infty} A_n = \bigoplus_{n=1}^{\infty} C_n$, where, for every n, $C_1 \oplus \cdots \oplus C_n$ is a summand of $A_1 \oplus \cdots \oplus A_n$. We construct successively groups $C_{ji} \cong A_{ij}$ $(i \leq j)$ such that

$$C_j = C_{j1} \oplus \cdots \oplus C_{jj} \qquad \text{and} \qquad A_i = A_{ii} \oplus A_{i,\, i+1} \oplus \cdots$$

for all i, j. Setting $C_1 = C_{11} = A_{11}$ and $A_1 = A_{11} \oplus A'_{12}$, assume that we have selected $C_{ji} \cong A_{ij}$ for $i \leq j \leq n$, such that $C_j = C_{j1} \oplus \cdots \oplus C_{jj}$, $A_i = A_{ii} \oplus \cdots \oplus A_{in} \oplus A'_{in}$ for all $j, i \leq n$, and $C_1 \oplus \cdots \oplus C_n \oplus A'_{1n} \oplus \cdots \oplus A'_{nn} = A_1 \oplus \cdots \oplus A_n$. Adding A_{n+1} to both sides and applying the exchange property for $C_1 \oplus \cdots \oplus C_{n+1}$, we obtain decompositions $A'_{in} = A_{i,\, n+1} \oplus A'_{i,\, n+1}$ $(i \leq n)$ and $A_{n+1} = A_{n+1,\, n+1} \oplus A'_{n+1,\, n+1}$ such that

$$C_{n+1} \cong A_{1,\, n+1} \oplus \cdots \oplus A_{n+1, n+1}$$

and

$$C_1 \oplus \cdots \oplus C_n \oplus C_{n+1} \oplus A'_{1,\, n+1} \oplus \cdots \oplus A'_{n+1,\, n+1} = A_1 \oplus \cdots \oplus A_n \oplus A_{n+1}.$$

Since $\bigoplus_j A_{ij}$ is pure in A_i and cannot have a smaller socle, they are equal. This settles the case of countable index sets.

Turning to the general case, according to (71.1) we may decompose $A_i = B_i \oplus C_i$, where B_i is bounded and $C_i[p] = \bigoplus_{j \in J(i)} A_j$ for some finite subset $J(i)$ of J. In this way, a new decomposition $A = B \oplus \bigoplus_{i \in I} C_i$ arises where $B = \bigoplus_{i \in I} B_i$ is a direct sum of cyclic groups; moreover, by changing B, it may be assumed that, for every $i \in I$, C_i is an unbounded torsion-complete group. In a similar fashion, there is a decomposition $A'_j = B'_j \oplus C'_j$ such that B'_j is bounded and $C'_j[p] \leq \bigoplus_{i \in I(j)} C_i$ for a suitable finite subset $I(j)$ of I. We can write $A = B' \oplus \bigoplus_{j \in J} C'_j$, where B' is a direct sum of cyclic groups and C'_j are unbounded torsion-complete groups. We are going to show that the decompositions

$$A = B \oplus \bigoplus_{i \in I} C_i = B' \oplus \bigoplus_{j \in J} C'_j \tag{1}$$

have isomorphic refinements. It is easy to check that this will complete the proof.

We start with the construction of two transfinite sequences of sets

$$\varnothing = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_\sigma \subseteq \cdots \qquad \text{and} \qquad \varnothing = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_\sigma \subseteq \cdots$$

such that (I) $\bigcup_\sigma I_\sigma = I$ and $\bigcup_\sigma J_\sigma = J$; (II) $|I_{\sigma+1} \backslash I_\sigma|$ and $|J_{\sigma+1} \backslash J_\sigma| \leqq \aleph_0$; (III) $I_\sigma = \bigcup_{\rho<\sigma} I_\rho$ and $J_\sigma = \bigcup_{\rho<\sigma} J_\rho$ if σ is a limit ordinal, and

$$\text{(IV)} \qquad \bigoplus_{j \in J_\sigma} C'_j[p] \leqq \bigoplus_{i \in I_\sigma} C_i[p] \leqq B' \oplus \bigoplus_{j \in J_\sigma} C'_j[p] \qquad \text{for every } \sigma.$$

We need only construct $I_{\sigma+1}$ and $J_{\sigma+1}$ out of I_σ and J_σ. Note that $I_\sigma = I$ exactly if $J_\sigma = J$. If there is an index $i \in I \backslash I_\sigma$, then define $K_0 = \{i\}$, $L_0 = J(i)$, and $K_n = \bigcup_{j \in L_{n-1}} I(j)$, $L_n = \bigcup_{i \in K_n} J(i)$, and set

$$I_{\sigma+1} = I_\sigma \cup \bigcup_{n=1}^{\infty} K_n, \qquad J_{\sigma+1} = J_\sigma \cup \bigcup_{n=1}^{\infty} L_n.$$

It is evident that both (II) and (IV) [the latter with σ replaced by $\sigma + 1$] are satisfied. Writing $I_\sigma^* = I \backslash I_\sigma$, $J_\sigma^* = J \backslash J_\sigma$, we have

$$A = B \oplus \bigoplus_{i \in I_\sigma} C_i \oplus \bigoplus_{i \in I_\sigma^*} C_i = B' \oplus \bigoplus_{j \in J_\sigma} C'_j \oplus \bigoplus_{j \in J_\sigma^*} C'_j.$$

From (IV) and (66.2), we derive the existence of a subgroup E_σ of A, isomorphic to a subgroup of B' and supported by $\bigoplus_{i \in I_\sigma} C_i \cap B'[p]$, such that

$$A = B \oplus \bigoplus_{j \in J_\sigma} C'_j \oplus E_\sigma \oplus \bigoplus_{i \in I_\sigma^*} C_i. \tag{2}$$

Comparing this decomposition with its analog for $\sigma + 1$, and noticing that $E_\sigma[p] \leqq E_{\sigma+1}[p]$, we conclude again from (66.2) that some subgroup F_σ of $E_{\sigma+1}$ satisfies

$$A = B \oplus \bigoplus_{j \in J_{\sigma+1}} C'_j \oplus E_\sigma \oplus F_\sigma \oplus \bigoplus_{i \in I_{\sigma+1}^*} C_i.$$

Hence we obtain

$$\bigoplus_{j \in J_{\sigma+1} \backslash J_\sigma} C'_j \oplus F_\sigma \cong \bigoplus_{i \in I_{\sigma+1} \backslash I_\sigma} C_i. \tag{3}$$

Setting $F_0 = 0$, a transfinite induction shows that $E_\sigma[p] = \bigoplus_{\rho<\sigma} F_\rho[p]$ and $\bigoplus_{\rho<\sigma} F_\rho$ is pure in A for every σ. If $I_\tau = I$, $J_\tau = J$ for an ordinal τ, then $F = \bigoplus_{\rho<\tau} F_\rho$ is pure in A and $F[p] = E_\tau[p]$; thus (66.1) shows

$$A = B \oplus \bigoplus_{j \in J} C'_j \oplus F.$$

Hence,

$$B \oplus \bigoplus_{\rho<\tau} F_\rho \cong B'. \tag{4}$$

Since the two decompositions in (3) involve but countably many torsion-complete groups, they have isomorphic refinements. In view of (4), it follows at once that the decompositions in (1) possess isomorphic refinements.□

Exercises

1. (Hill and Megibben [3]) If A is a pure-complete p-group and C is a torsion-complete p-group with finite Ulm–Kaplansky invariants, then $A \oplus C$ is again pure-complete.
2.* (Hill and Megibben [3]) The direct sum of two pure-complete p-groups need not be pure-complete. [*Hint*: using (66.4), construct two non-isomorphic pure subgroups G and H of A with the same socle such that G is quasi-complete in the sense of **74**; from (74.3) conclude that H too is quasi-complete; by (74.2) G and H are pure-complete, but in the direct sum $A \oplus A$ the subsocle $S = \{(x, x) \mid x \in G[p] = H[p]\}$ does not support a pure subgroup.]
3. (Cutler [1]) If pA is a direct sum of torsion-complete p-groups, then so is A.
4. (Hill [10]) There exist a countable direct sum A of torsion-complete p-groups and a separable p-group C such that $A[p]$ and $C[p]$ are isometric vector spaces, but A and C fail to be isomorphic. [*Hint*: Ex. 8 in **66**.]
5. By a *contraction* of a metric space is meant a mapping of the space into itself that does not increase distances. Show that if π is a contraction of the vector space $A[p]$ of a pure-complete p-group A which is at the same time a projection, then $\operatorname{Im} \pi$ supports a summand of A.
6. (Hill [10]) A subsocle S of a direct sum A of cyclic p-groups supports a summand of A exactly if S is the image of a projection of $A[p]$ which does not decrease heights.
7. (Kolettis [2]) If A is a direct sum of torsion-complete p-groups almost all of which are bounded, then every summand of A is again of the same sort.
8. (Enochs [1]) In any two decompositions of a p-group into direct sums of torsion-complete groups, the numbers of unbounded components are the same if at least one of them is infinite.

9. Let G be a summand of a direct sum of torsion-complete p-groups A_i $(i \in I)$, where exactly $\mathfrak{m}$ of the A_i are unbounded, $\mathfrak{m} \geqq \aleph_0$. Then any direct decomposition of G has at most $\mathfrak{m}$ unbounded summands.

74. QUASI-COMPLETE GROUPS

In this section, we discuss a noteworthy property of torsion-complete groups. Head [1] was the first to call attention to it, and it turned out that this property is shared by a somewhat wider class of groups.

A reduced torsion group A will be called *quasi-complete* if the closure G^- [naturally, in the Z-adic topology of A] of every pure subgroup G of A is again pure in A.

Let us begin the discussion with a few simple observations, clarifying the concept.

(a) *A reduced torsion group A is quasi-complete exactly if, for every pure subgroup G of A, "G is closed" is equivalent to "A/G is reduced."* Note that $G^-/G = (A/G)^1$ implies that G^- is pure in A exactly if $(A/G)^1$ is pure in A/G. This is the case if and only if $(A/G)^1$ is divisible.

(b) *Quasi-complete groups are separable.* This is clear from $0^- = A^1$.

(c) *A separable torsion group is quasi-complete if G^- is pure for all unbounded pure subgroups G.* In fact, bounded pure subgroups are direct summands, and hence necessarily closed in separable torsion groups.

(d) *A torsion group is quasi-complete if and only if each of its p-components is quasi-complete.*

(e) *Torsion-complete groups are quasi-complete.* This is obvious from (68.9).

(f) *In a quasi-complete group, the closure of a pure subgroup is again quasi-complete.*

The following characterization of quasi-completeness is of particular interest.

Theorem 74.1 (Irwin, Richman, and Walker [1], Koyama [1]). *A reduced p-group A is quasi-complete if and only if, for every pure subgroup C of A and for every subsocle S with $C[p] \leqq S$, there exists a pure subgroup G of A which contains C and is supported by S.*

Let A be quasi-complete and G a subgroup of A which is maximal among the pure subgroups of A containing C and being supported by subgroups of S. To verify $G[p] = S$, by way of contradiction, assume the existence of an $x \in S \backslash G$. If the coset $x + G$ is of finite height n in A/G, then write $x + G =$

$p^n y + G$ and notice that $\langle y + G\rangle$ is a summand of A/G. Hence $G \oplus \langle y\rangle$ is a pure subgroup supported by the subsocle $G[p] \oplus \langle x\rangle \leqq S$. If $x + G$ is of infinite height, then quasi-completeness implies that it is contained in a subgroup $H/G \cong Z(p^\infty)$. Now H is pure and supported by $G[p] \oplus \langle x\rangle \leqq S$. This contradiction establishes $G[p] = S$.

Conversely, if A has the stated property, then first choosing $S = A^1[p]$, we infer A^1 is divisible and hence 0. Next, for any pure C, choose G to be pure, containing C, and having the socle $C^-[p]$. Then G/C is pure in A/C and has $(A/C)^1[p]$ for its socle. By **20**(C), G/C is divisible and $G = C^-$. Thus A is quasi-complete.□

Corollary 74.2 (Hill and Megibben [2]). *Quasi-complete groups are pure-complete.*□

A different sort of characterization of quasi-completeness is given in the next result. Closures will be understood in $\bar{B}$, where B is a basic subgroup of A, and A will be regarded as a subgroup of $\bar{B}$.

Proposition 74.3 (Hill and Megibben [2]). *A separable p-group A is quasi-complete if and only if*

$$A[p] + S^- = \bar{B}[p]$$

holds for every nondiscrete subsocle S of A.

Let A be quasi-complete and S a nondiscrete subsocle of A. Following the pattern of proof of (33.1), one can select a pure subgroup C of A which is a direct sum of cyclic groups and $C[p]$ is dense in S. Since S is not discrete, C must be unbounded, so by (35.1), it contains a basic subgroup C' such that $C/C' \cong Z(p^\infty)$. Let $c \in C\backslash C'$ be of order p^2 and b any element of order p in $\bar{B}$. One can easily construct a direct sum D of cyclic groups, which is pure in A and satisfies $D[p] = C'[p]$ and $b + c \in D^-$. By quasi-completeness, the closure $D^- \cap A$ of D in A is pure in A, whence $pc \in D^- \cap A$ implies that $pc = pa$, for some $a \in D^- \cap A$. We obtain

$$b = (b + c - a) - (c - a) \in D^-[p] + A[p] \leqq S^- + A[p],$$

whence $\bar{B}[p] \leqq S^- + A[p]$.

Conversely, suppose the stated condition is satisfied, and let G be an unbounded pure subgroup. We have to verify that the closure $G^- \cap A$ of G in A is pure in A. If $pa \in G^- \cap A$, for some $a \in A$, then let $g \in G$, $h \in G^-$ satisfy $ph = pa + g$; such elements exist, since $\bar{B}$ is torsion-complete, and thus G^-/G is divisible. Hence $p(h - a) = g = pg'$, for some $g' \in G$, and $h - (a + g') \in \bar{B}[p]$ is, by hypothesis, of the form $a' + h'$, with $a' \in A[p]$ and $h' \in G^-[p]$. This shows $h - h' = a + a' + g' \in G^- \cap A$ is a solution of $px \equiv pa \bmod G$. Hence $(G^- \cap A)/G$ must be neat in A/G, and thus divisible, and so $G^- \cap A$ is pure in A.□

An easy consequence of the preceding theorem is as follows:

Corollary 74.4 (Hill and Megibben [2]). *Let B be a basic subgroup of the quasi-complete p-group A, $B \leqq A \leqq \bar{B}$. If A contains a basic subgroup of an unbounded summand H of $\bar{B}$, then $A + H = \bar{B}$.*

If A contains a basic subgroup of H, then the quotient $(A + H)/A \cong H/(A \cap H)$ is divisible. Therefore, $A + H$ is pure in $\bar{B}$. (74.3) implies $\bar{B}[p] \leqq A + H$, whence $A + H = \bar{B}$, in fact.□

The following result displays another remarkable feature of quasi-complete groups.

Proposition 74.5 (Hill and Megibben [2]). *For the quasi-completeness of a separable p-group A, it is necessary and sufficient that, for every unbounded pure subgroup G of A, A/G be the direct sum of a divisible and a torsion-complete p-group.*

The sufficiency immediately follows from (a). For the proof of necessity, assume A is quasi-complete, and let H be the closure of an unbounded pure subgroup G of A, the closure being taken in $\bar{B}$. Then H is a summand of $\bar{B}$ [cf. (68.9)], and since A contains a basic subgroup [of G and hence] of H, from (74.4) we deduce $A + H = \bar{B}$. In view of quasi-completeness, $(A \cap H)/G = (A/G)^1$ is divisible. Therefore, A/G is isomorphic to a direct sum of a divisible group and $A/(A \cap H) \cong (A + H)/H = \bar{B}/H$, a summand of $\bar{B}$.□

Corollary 74.6. *If A is a quasi-complete but not a torsion-complete p-group, then in any direct decomposition of A, one of the summands is bounded.*

Let A be quasi-complete and $A = G \oplus H$ with unbounded G and H. Because of (74.5), both G and H must be torsion-complete.□

Another simple consequence of (74.4) which is needed in the sequel is proved next.

Proposition 74.7 (Hill and Megibben [2]). *Let A be a quasi-complete p-group and $B = B' \oplus B''$ a direct decomposition of a basic subgroup B of A into unbounded summands. Then A is a subdirect sum of $\bar{B}'$ and $\bar{B}''$ with pure kernels.*

First of all, it is clear that $(A \cap \bar{B}') \oplus (A \cap \bar{B}'') \leqq A \leqq \bar{B}' \oplus \bar{B}''$. A reference to (74.4) shows that $A + \bar{B}' = \bar{B} = A + \bar{B}''$. Hence A is a subdirect sum of $\bar{B}'$ and $\bar{B}''$, where the kernels $A \cap \bar{B}'$ and $A \cap \bar{B}''$ are pure, being the closures of B' and B'' in A.□

The two preceding results make it possible to show that the quasi-complete p-groups which are not torsion-complete have final ranks at most continuum; thus there are only a few of them.

Theorem 74.8. *A quasi-complete p-group of final rank $>2^{\aleph_0}$ is necessarily torsion-complete.*

Let A be a quasi-complete p-group of final rank $>2^{\aleph_0}$, and B a basic subgroup of A. Let B' be an unbounded countable summand of B, $B = B' \oplus B''$. By (74.7), A is a subdirect sum of $\bar{B}'$ and $\bar{B}''$ with kernels $A \cap \bar{B}'$, $A \cap \bar{B}''$ pure in A. From our discussion in **8** we know that A consists of all pairs (b', b'') with $b' \in \bar{B}'$, $b'' \in \bar{B}''$ such that b', b'' are mapped by the canonical mappings $\bar{B}' \to \bar{B}'/(A \cap \bar{B}')$, $\bar{B}'' \to \bar{B}''/(A \cap \bar{B}'')$ upon elements corresponding to each other under some isomorphism $\bar{B}'/(A \cap \bar{B}') \cong \bar{B}''/(A \cap \bar{B}'')$. These quotients are divisible of cardinality $\leqq 2^{\aleph_0}$, since $|\bar{B}'| = 2^{\aleph_0}$. Therefore, choosing representatives of $\bar{B}''$ mod $A \cap \bar{B}''$ and expressing them as infinite vectors as in **68**, we can split $B'' = C_1 \oplus C_2$ in such a way that all coordinates of these infinite vectors come from C_1 and still $|C_1| \leqq 2^{\aleph_0}$, and furthermore $\bar{C}_2 \leqq A \cap \bar{B}''$, so $\bar{C}_2$ is a summand of A. Its complement has a basic subgroup $\cong B' \oplus C_1$ whose cardinality is $\leqq 2^{\aleph_0}$, so the complement itself is, because of (34.4), at most of cardinality $2^{\aleph_0}$. If A has final rank $>2^{\aleph_0}$, then $\bar{C}_2$ must have a final rank $>2^{\aleph_0}$, so it cannot be bounded. (74.6) implies that A is torsion-complete.□

The following example, due to Hill and Megibben [2], show that there do exist quasi-complete p-groups which are not torsion-complete.

Example. Let $\bar{B}$ be a torsion-complete group, with a basic subgroup $B \cong \bigoplus_{n=1}^{\infty} Z(p^n)$. Clearly, $|\bar{B}| = 2^{\aleph_0}$ and $\bar{B}$ has $2^{\aleph_0}$ countable subsets. Every unbounded summand of $\bar{B}$ is determined by a basic subgroup, which is countable, thus $\bar{B}$ has [at most, and so exactly] $2^{\aleph_0}$ unbounded summands. We can index these summands H_σ by ordinals $\sigma < \omega_1$, where ω_1 is the first ordinal of cardinality $2^{\aleph_0}$. Let $C = \bigoplus_{n=1}^{\infty} \langle c_n \rangle$ be a countable subsocle of $\bar{B}$ such that $B[p] \cap C = 0$. We wish to select elements $h_{\rho n}$ and subgroups T_σ of $\bar{B}[p]$ in such a way that for every $\sigma < \omega_1$ we have:

(i) $h_{\rho n} \in H_\rho$ for every $\rho < \sigma$ and $n = 1, 2, \cdots$;
(ii) $T_\sigma = B[p] \oplus \bigoplus_{\rho<\sigma} \bigoplus_{n=1}^{\infty} \langle h_{\rho n} - c_n \rangle$;
(iii) $T_\sigma \cap C = 0$.

Starting with $T_0 = B[p]$, we use transfinite induction. Assume that $h_{\rho n}$ have been selected for all $\rho < \sigma$ and $n = 1, 2, \cdots$, such that (ii) and (iii) hold. Note that

$$|T_\sigma + C| \leqq |\sigma|\aleph_0 + \aleph_0 < 2^{\aleph_0} \qquad \text{and} \qquad |H_\sigma[p]| = 2^{\aleph_0},$$

thus $H_\sigma[p]$ contains elements $h_{\sigma n}$ $(n = 1, 2, \cdots)$ independent mod $T_\sigma + C$. If $T_{\sigma+1}$ is defined with these $h_{\sigma n}$, as given in (ii), then (iii) will also be satisfied. Obviously,

$$T \cap C = 0 \qquad \text{for} \quad T = B[p] \oplus \bigoplus_{\rho < \omega_1} \bigoplus_{n=1}^{\infty} \langle h_{\rho n} - c_n \rangle.$$

Let T^* be a subsocle of $\bar{B}[p]$ such that $T \leqq T^*$ and T^* is maximal with respect to $T^* \cap C = 0$; thus $T^* + C = \bar{B}[p]$. Then T^* is dense in $\bar{B}[p]$ and, by (66.3), it supports a pure subgroup A of $\bar{B}[p]$. Clearly, A is not torsion-complete, since its socle is not closed in $\bar{B}[p]$, and hence not complete. In showing that A is quasi-complete, (74.3) will be made use of. If S is a non-discrete subsocle of A, then S^- supports an unbounded pure subgroup H of $\bar{B}$. By (65.2), H is closed in $\bar{B}$, thus by (68.9), H is one of the H_σ. By (i) and (ii) we obtain

$$A[p] + S^- = A[p] + H_\sigma[p] \geqq T^* + C = \bar{B}[p],$$

and (74.3) shows that A is quasi-complete, in fact.

By making use of the theory of quasi-complete groups developed here, the following characterization of torsion-complete p-groups can be established.

Proposition 74.9 (Koyama [1]). *A reduced p-group A is torsion-complete if and only if, for every pure subgroup G of A, G^- is a summand of A.*

The necessity has been proved in (68.9).

Conversely, if A has the stated property, then (a) implies A is quasi-complete. It suffices to look at unbounded groups A. If $B = B' \oplus B''$ is as in (74.7), then because the closure $A \cap \bar{B}'$ of B' in A is a summand, we have $A = (A \cap \bar{B}') \oplus H$ for some unbounded H [with basic subgroup $\cong B''$]. A simple appeal to (74.6) shows that A is torsion-complete.□

EXERCISES

1. In any reduced torsion-free group, the closure of a pure subgroup is always pure.
2. (Head [1]) A quasi-complete p-group is a direct sum of cyclic groups if and only if it is bounded.
3. The direct sum of a quasi-complete group and a bounded group is again quasi-complete.
4. Every closed subsocle of a quasi-complete p-group supports a quasi-complete pure subgroup.
5.* (Hill and Megibben [2]) Let $\bar{B}$ be an unbounded torsion-complete p-group of cardinality $2^{\aleph_0}$ and C any countable subsocle of $\bar{B}$. There exists a pure, quasi-complete subgroup A of $\bar{B}$ such that $A \cap C = 0$ and $\bar{B}/A$ is divisible.
6. Assuming the continuum hypothesis, show that every quasi-complete p-group which is not torsion-complete is of final rank $2^{\aleph_0}$.

75. DIRECT DECOMPOSITIONS OF p-GROUPS

Separable torsion groups have numerous direct decompositions; as a matter of fact, every element can be embedded in a finite summand of the group. We have, however, failed to investigate what can be asserted about direct summands of larger cardinalities or about the existence of direct decompositions into a large number of summands. No systematic survey of the various possibilities exists up to now, and therefore in the discussion of direct decompositions one has to be content with mentioning a few independent results.

(A) We start with a curious phenomenon, connected with our Problem 5.

Theorem 75.1 (Hill [18], Crawley and Megibben [1]). *There exists a p-group A of cardinality $\aleph_1$, not a direct sum of cyclic groups, such that every countable subgroup of A is contained in a direct summand of A that is a direct sum of cyclic groups.*

For every ordinal σ, less than the first uncountable ordinal ω_1, choose an unbounded separable, countable p-group C_σ, and form the groups

$$G = \prod_\sigma C_\sigma \qquad \text{and} \qquad H = \bigoplus_\sigma C_\sigma .$$

For every $\sigma < \omega_1$, put $G^{(\sigma)} = \prod_{\rho < \sigma} C_\rho$ and $\bar{G}^{(\sigma)} = \prod_{\sigma \leqq \rho} C_\rho$, so that $G = G^{(\sigma)} \oplus \bar{G}^{(\sigma)}$ for every $\sigma < \omega_1$. For a subgroup X of G, let $X^{(\sigma)} = X \cap G^{(\sigma)}$ and $\bar{X}^{(\sigma)} = X \cap \bar{G}^{(\sigma)}$. We may regard the elements g of G as infinite vectors whose σth coordinates $g(\sigma)$ belong to C_σ.

For every limit ordinal $\lambda < \omega_1$, we choose a sequence $x_{\lambda 1}, \cdots, x_{\lambda n}, \cdots \in G$ such that

(i) $x_{\lambda n}(\sigma) = 0$ for all n and all $\sigma \geqq \lambda$;
(ii) for every n and each $\sigma < \lambda$, $x_{\lambda n}(\rho) = 0$, for almost all $\rho < \sigma$;
(iii) $X_\lambda = \langle x_{\lambda 1}, \cdots, x_{n\lambda}, \cdots \rangle$ satisfies $(X_\lambda + H^{(\lambda)})/H^{(\lambda)} \cong Z(p^\infty)$.

It is easy to find such $x_{\lambda n}$ [select ordinals $\sigma_1, \cdots, \sigma_n, \cdots$ tending to λ and choose $x_{\lambda n}$ with support $\{\sigma_n, \sigma_{n+1}, \cdots\}$ such that $p x_{\lambda n}(\sigma_m) = x_{\lambda, n-1}(\sigma_m)$ for $m \geqq n$].

Let A be the subgroup of G which is generated by H and the X_λ with limit ordinals $\lambda < \omega_1$. Then A is a separable p-group of cardinality $\aleph_1$ which satisfies $A = \bigcup_{\sigma < \omega_1} A^{(\sigma)}$. Here each $A^{(\sigma)}$ is countable [and hence a direct sum of cyclic groups], every countable subset of A is contained in some $A^{(\sigma)}$ $(\sigma < \omega_1)$, and since $A = A^{(\sigma)} \oplus \bar{A}^{(\sigma)}$ for every $\sigma < \omega_1$, all that remains to be shown is that A itself fails to be a direct sum of cyclic groups.

By way of contradiction, suppose $A = \bigoplus_{i \in I} K_i$ with K_i countable. Starting with an arbitrary K_{i_0}, there is a limit ordinal $\lambda_1 < \omega_1$ such that $K_{i_0} \leqq \bigcup_{\sigma < \lambda_1} A^{(\sigma)}$. There is a countable subset $I_1 \subset I$ and a limit ordinal

$\lambda_2 < \omega_1$ such that $\bigcup_{\sigma<\lambda_1} A^{(\sigma)} \leqq \bigoplus_{i \in I_1} K_i \leqq \bigcup_{\sigma<\lambda_2} A^{(\sigma)}$. Repeating this process, we find a countable subset J of I and a limit ordinal $\lambda < \omega_1$ such that $\bigoplus_{i \in J} K_i = \bigcup_{\sigma<\lambda} A^{(\sigma)}$. The latter group is thus a summand of A and hence of $A^{(\lambda)}$. Since $A^{(\lambda)} = X_\lambda + \bigcup_{\sigma<\lambda} A^{(\sigma)}$ and the quotient $A^{(\lambda)}/\bigcup_{\sigma<\lambda} A^{(\sigma)} \neq 0$ is an epic image of $(X_\lambda + H^{(\lambda)})/H^{(\lambda)} \cong Z(p^\infty)$, the group $A^{(\lambda)}$ must have a summand $Z(p^\infty)$. This contradiction shows that A cannot be a direct sum of countable groups, in fact.□

(B) Call a group A *quasi-indecomposable* if for every direct decomposition of A the following holds: to every cardinal $\mathfrak{p} < |A|$ there is a summand S in the given decomposition such that $|S| > \mathfrak{p}$. It is obvious from the definition that:

(a) A countable p-group is quasi-indecomposable exactly if it is unbounded.

(b) Let $|A| = \aleph_\sigma$, where σ is not a limit ordinal. A is quasi-indecomposable exactly if in every direct decomposition of A, one of the summands has the same cardinality as A.

Theorem 75.2 (Kulikov [1], [2]). *For every infinite cardinal* $\mathfrak{m}$, *there exist quasi-indecomposable separable p-groups of cardinality* $\mathfrak{m}$.

Ignoring the case $\mathfrak{m} = \aleph_0$ settled by (a), we distinguish three mutually exclusive cases:

Case I. There is an infinite cardinal $\mathfrak{n}$ such that $\mathfrak{n} < \mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$.

Define $B_n = \bigoplus_{\mathfrak{n}} Z(p^n)$, and let $\bar{B}$ be the torsion-complete group with the basic subgroup $B = \bigoplus_n B_n$. Since $|\bar{B}| = \mathfrak{n}^{\aleph_0}$, there is a subgroup A of $\bar{B}$ such that $B \leqq A$ and $A/B \cong \bigoplus_{\mathfrak{m}} Z(p^\infty)$. Evidently, $|A| = \mathfrak{m}$. Because of $|B| = \mathfrak{n}$, every direct decomposition of A has at most $\mathfrak{n}$ nonzero summands. Clearly, each of these cannot be of power $\leqq \mathfrak{p}$ if $\mathfrak{p} < \mathfrak{m}$. Hence A is quasi-indecomposable.

Case II. For every $\mathfrak{n} < \mathfrak{m}$, $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$ holds, and $\mathfrak{m}$ is not a limit cardinal.

Let $\mathfrak{m} = \aleph_{\sigma+1}$ and $\mathfrak{n} = \aleph_\sigma$, where $\mathfrak{n}^{\aleph_0} = \mathfrak{n}$. A result of Hausdorff [*Jahresber. Deut. Math. Ver.* **13** (1904), 569–571] states that for every pair of ordinals ρ, σ the equality $\aleph_{\sigma+1}^{\aleph_\rho} = \aleph_\sigma^{\aleph_\rho} \cdot \aleph_{\sigma+1}$ holds. For $\rho = 0$, this yields $\mathfrak{m}^{\aleph_0} = \mathfrak{m}$. Now, let A be the torsion-complete group with the basic subgroup $B = \bigoplus_n B_n$, where $B_n = \bigoplus_{\mathfrak{m}} Z(p^n)$. Manifestly, $|A| = \mathfrak{m}$ and fin $r(A) = \mathfrak{m}$. If $A = \bigoplus_{i \in I} A_i$ is any direct decomposition of A and if m is an integer such that almost all $p^m A_i$ are 0 [see (71.3)], then one of these components $p^m A_i$ must be of cardinality $\mathfrak{m}$. This establishes the quasi-indecomposability of A.

Case III. For every $\mathfrak{n} < \mathfrak{m}$, $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$ holds, and $\mathfrak{m}$ is a limit cardinal.

Write $\mathfrak{m}$ as the sum of ordinals $\mathfrak{m}_\sigma$ $(\sigma < \tau)$ such that $[\mathfrak{m}_\sigma^{\aleph_0} = \mathfrak{m}_\sigma$ and] the $\mathfrak{m}_\sigma$ tend to $\mathfrak{m}$. For each σ, take the torsion-complete group $A(\sigma)$ with basic subgroup $B(\sigma) = \bigoplus_{n=1}^{\infty} \bigoplus_{\mathfrak{m}_\sigma} Z(p^n)$, and define $A = \bigoplus_\sigma A(\sigma)$. Since the cardinality of A is $\mathfrak{m}$, only the quasi-indecomposability of A needs to be verified. Let $A = \bigoplus_{i \in I} G_i$ be a direct decomposition of A. Given σ, (71.2) implies the existence of an integer m such that $p^m A(\sigma)[p]$ is contained in the direct sum of a finite number of the G_i, and it is evident that one of these must have cardinality $\geqq \mathfrak{m}_\sigma$. Since the $\mathfrak{m}_\sigma$ tend to $\mathfrak{m}$, it follows that A is quasi-indecomposable.

Cases I–III cover all possibilities, hence the theorem has been proved. □

(C) Let $\mathfrak{m}$ be an infinite cardinal. A group A is said to be $\mathfrak{m}$-*indecomposable* if it cannot be written as a direct sum of $\mathfrak{m}$ nonzero groups. It is an absolutely trivial observation that $\mathfrak{m}$-indecomposability implies $\mathfrak{n}$-indecomposability for every $\mathfrak{n} > \mathfrak{m}$, and that groups of cardinality $< \mathfrak{m}$ are $\mathfrak{m}$-indecomposable.

Nontrivial examples of $\mathfrak{m}$-indecomposable p-groups are as follows.

Example 1. Let the cardinals $\mathfrak{m}$, $\mathfrak{n}$ satisfy $\mathfrak{n} < \mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$. Define A as the torsion-complete group with the basic subgroup $B = \bigoplus_{n=1}^{\infty} B_n$, where $B_n = \bigoplus_{\mathfrak{n}} Z(p^n)$. Now $\mathfrak{n}^{\aleph_0} \leqq \mathfrak{m}^{\aleph_0} \leqq (\mathfrak{n}^{\aleph_0})^{\aleph_0}$ implies $|A| = \mathfrak{n}^{\aleph_0} = \mathfrak{m}^{\aleph_0}$. Suppose that $A = \bigoplus_{i \in I} S_i$. Here $|I| \leqq |B| = \mathfrak{n}\aleph_0$, so that A is $\mathfrak{m}$-indecomposable whenever $\mathfrak{n}$ is infinite. If $\mathfrak{n}$ is finite, then all bounded summands of A are finite, and from (71.3) it follows at once that I must then be finite.

Example 2. Let $\mathfrak{m}$ be a cardinal for which $\mathfrak{n} < \mathfrak{m}$ always implies $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$, but there exist countably many cardinals $\mathfrak{n}_1 \leqq \cdots \leqq \mathfrak{n}_n \leqq \cdots$, all less than $\mathfrak{m}$, such that $\sum_{n=1}^{\infty} \mathfrak{n}_n = \mathfrak{m}$. By a set-theoretical result [Hausdorff, *Mengenlehre*, 3rd ed. (1935), p. 36], the inequality $\sum \mathfrak{n}_n = \mathfrak{m} < \prod \mathfrak{n}_n$ holds. Obviously,

$$\mathfrak{m}^{\aleph_0} \leqq \left(\prod \mathfrak{n}_n\right)^{\aleph_0} = \prod \mathfrak{n}_n^{\aleph_0} \leqq \prod \mathfrak{m}^{\aleph_0} = \mathfrak{m}^{\aleph_0}.$$

No generality is lost in assuming $\mathfrak{n}_n^{\aleph_0} = \mathfrak{n}_n$; thus $\prod \mathfrak{n}_n = \mathfrak{m}^{\aleph_0}$. Define A as the torsion-complete group with the basic subgroup $B = \bigoplus_{n=1}^{\infty} B_n$, where $B_n = \bigoplus_{\mathfrak{n}_n} Z(p^n)$. Evidently, $|A| = \mathfrak{m}^{\aleph_0}$. If $A = \bigoplus_{i \in I} G_i$ and if m is an integer such that almost all $p^m G_i$ vanish, then $|I| \leqq \mathfrak{n}_m$. This establishes the $\mathfrak{m}$-indecomposability for A.

We now prove that for all other cardinals $\mathfrak{m}$, $\mathfrak{m}$-indecomposable p-groups must be of cardinality $< \mathfrak{m}$.

Theorem 75.3 (Szele [16], Fuchs [5]). *$\mathfrak{m}$-indecomposable [reduced] p-groups of cardinality $\geqq \mathfrak{m}$ exist if and only if $\mathfrak{m}$ satisfies one of the following conditions*:

(i) *there is a cardinal $\mathfrak{n}$ with $\mathfrak{n} < \mathfrak{m} \leqq \mathfrak{n}^{\aleph_0}$;*

(ii) *$\mathfrak{n} < \mathfrak{m}$ implies $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$ and there is a countable set of cardinals $\mathfrak{n}_1 \leqq \cdots \leqq \mathfrak{n}_n \leqq \cdots$ such that $\mathfrak{n}_n < \mathfrak{m}$ and $\sum_{n=1}^{\infty} \mathfrak{n}_n = \mathfrak{m}$.*

To complete the proof, we consider cardinals $\mathfrak{m}$ such that, for every cardinal $\mathfrak{n} < \mathfrak{m}$, $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$ holds, and for any countable set $\mathfrak{n}_1, \cdots, \mathfrak{n}_n, \cdots$ of cardinals less than $\mathfrak{m}$, $\sum \mathfrak{n}_n < \mathfrak{m}$ is satisfied. It remains to show that for such an $\mathfrak{m}$ there do not exist $\mathfrak{m}$-indecomposable p-groups of cardinality $\geqq \mathfrak{m}$.

Assume that $\mathfrak{m}$ is such a cardinal and A is an $\mathfrak{m}$-indecomposable p-group of cardinality $\geqq \mathfrak{m}$. Let $B = \bigoplus_{n=1}^{\infty} B_n$ with $B_n = \bigoplus_{\mathfrak{n}_n} Z(p^n)$ be a basic subgroup of A. Since B_n is a summand of A, we must have $\mathfrak{n}_n < \mathfrak{m}$ for every n, whence by hypothesis on $\mathfrak{m}$, $\sum \mathfrak{n}_n < \mathfrak{m}$ follows. On the other hand, from (34.4) we conclude that $\mathfrak{m} \leqq |A| \leqq |B|^{\aleph_0} = (\sum \mathfrak{n}_n)^{\aleph_0}$, whence, again by our hypothesis on $\mathfrak{m}$, $\mathfrak{m} \leqq \sum \mathfrak{n}_n$ results. The contradiction shows the claimed nonexistence.□

Notice that in the course of the proof we have found that for an $\mathfrak{m}$-indecomposable p-group A necessarily $\mathfrak{n}_n < \mathfrak{m}$, thus $|B| \leqq \mathfrak{m}$. This gives the upper bound $|A| \leqq |B|^{\aleph_0} \leqq \mathfrak{m}^{\aleph_0}$ for the cardinality of $\mathfrak{m}$-indecomposable p-groups A. This and another consequence of the proof are displayed as:

Corollary 75.4. *Every $\mathfrak{m}$-indecomposable p-group is of cardinality $\leqq \mathfrak{m}^{\aleph_0}$. If there exists an $\mathfrak{m}$-indecomposable p-group of cardinality $\geqq \mathfrak{m}$, then there is also one that is separable and of power $\mathfrak{m}^{\aleph_0}$.*□

EXERCISES

1. (Hill [18]) For every uncountable cardinal $\mathfrak{m}$, there exists a group A that is not a direct sum of cyclic groups, but every countable subset of A is contained in a countable summand of A which is a direct sum of cyclic groups. [*Hint:* direct sums of groups in (75.1).]
2. (Crawley and Megibben [1]) Modify the proof of (75.1) by replacing the C_σ by unbounded reduced countable groups and choosing X_λ with $Z(p^\infty)$ replaced by a suitable divisible group. Then conclude that the arising group A is not a direct sum of countable groups but every countable subset of A is contained in a countable summand of A.
3. A torsion-complete p-group A with $\operatorname{fin} r(A) = |A|$ is quasi-indecomposable.
4. Every unbounded quasi-complete p-group with a countable basic subgroup is quasi-indecomposable. [*Hint:* (74.6).]
5. A group A is $\mathfrak{m}$-indecomposable if and only if its divisible part has rank $< \mathfrak{m}$ and its reduced part is $\mathfrak{m}$-indecomposable.
6.* (Khabbaz [1]) A reduced p-group A is $\mathfrak{m}$-indecomposable if and only if its initial Ulm factor A_0 is $\mathfrak{m}$-indecomposable.
7. Let $\mathfrak{n}$ be a cardinal such that $\mathfrak{n} < \mathfrak{m} < \mathfrak{n}^{\aleph_0}$, for some cardinal $\mathfrak{n} \geqq \aleph_0$. There exists an $\mathfrak{m}$-indecomposable p-group of cardinality $\mathfrak{p}$, for every cardinal $\mathfrak{p}$ satisfying $\mathfrak{m} < \mathfrak{p} \leqq \mathfrak{m}^{\aleph_0}$. [*Hint:* modify Example 1.]

8. (a) There exist no countably infinite reduced $\aleph_0$-indecomposable p-groups. [*Hint:* Zippin's theorem (76.2).]
(b)* (Khabbaz [1]) There exist no $\mathfrak{m}$-indecomposable reduced p-groups of power $\mathfrak{m} > \aleph_0$ if and only if $\mathfrak{m}$ is a cardinal such that $\mathfrak{n} < \mathfrak{m}$ implies $\mathfrak{n}^{\aleph_0} < \mathfrak{m}$.
9. (Fuchs [5]) A torsion group T is $\mathfrak{m}$-indecomposable exactly if so are its p-components T_p and the basic subgroups B_p of T_p satisfy $|B_p| = \mathfrak{m}$ for at most a finite number of primes p while $\sum_q |B_q| < \mathfrak{m}$, the summation being extended for the remaining primes q.
10.* Using the existence theorem (76.1), show that the set of nonisomorphic $\mathfrak{m}$-indecomposable p-groups of power $\geqq \mathfrak{m}$ is, if not empty, of cardinality $2^{\mathfrak{p}}$, where $\mathfrak{p} = \mathfrak{m}^{\aleph_0}$.

NOTES

H. Prüfer's papers, [1]–[3], were undoubtedly the first systematic treatments on infinite abelian p-groups; in [2] he published the most important theorem (17.3), together with (17.2), restricted to the countable case. It was noticed by R. Baer, a decade later, that countability is not essential in (17.2): bounded p-groups of arbitrary cardinalities are direct sums of cyclic groups. Already Prüfer [1] recognized that p-groups without elements of infinite height need not be direct sums of cyclic groups if their cardinality is the continuum. Szele [11] pointed out that the same holds for cardinality $\aleph_1$, clarifying the set-theoretic background of (17.3).

The structure theory of separable p-groups has been revitalized by Kulikov [1, 2]. It was he who proved the general criterion (17.1) for direct sums of p-groups, introduced the notion of basic subgroup, and discovered the torsion-complete p-groups [he called them closed p-groups]. In addition, he proved that all separable p-groups are pure subgroups between their basic subgroups B and the corresponding $\bar{B}$—this result is still one of the most important informations available for arbitrary separable p-groups. (68.4) has several contributors: the equivalence of (i) with (iv) was proved by Kulikov [2] for separable p-groups and extended by Papp [1] to arbitrary reduced p-groups, while the injective property (iii) has been noticed in a special case by Leptin [1]. The completeness of $\bar{B}$ with respect to bounded Cauchy sequences was already pointed out by Kulikov [2]; but the important observation that this completeness is to be viewed in the topology defined by the large subgroups was made by B. Charles in 1967 only.

The first step to combine the theory of direct sums of cyclic p-groups with torsion-complete groups was made by Kolettis [2], who considered direct sums of cyclic p-groups and a torsion-complete group. The more general, and more natural, class of direct sums of torsion-complete p-groups first appeared almost simultaneously in a paper by Enochs [1] and in a problem by Fuchs [23]. In the 1960s several results for this class were established in rapid succession. Hill [5] and [10] made important contributions, but so far no satisfactory characterization is known for this class; (73.3) is a very modest attempt to fill this gap.

The search for satisfactory invariants of separable p-groups has not been very successful as yet. Leptin's theorem (69.1) is most interesting, but it fails to produce more invariants. Beaumont and Pierce [8] introduced new invariants, but many more are needed. The question of quasi-isomorphic invariants has also been considered, basically without any real benefit for the general structure theory.

A more tractable class is that of quasi-complete p-groups, discovered by Head [1]. This is, however, an extremely isolated class, in contrast to the pure-complete p-groups, which constitute a fairly large class. Except for some examples and a few sporadic results, not much is known about this latter class.

Surprising examples of separable p-groups A have been given by Corner [7]. He proved that to every positive integer m there exists an A such that the direct sum of n copies of A is isomorphic to the direct sum of k copies of A exactly if $n \equiv k \bmod m$.

The exchange property has been investigated for general algebraic systems by Crawley and Jónsson [1]; they proved most of the theorems in **72**. The related question of isomorphic refinement for certain p-groups was discussed by Crawley [4]. It was not so easy to exhibit a p-group which fails to have the isomorphic refinement property (cf. Corner and Crawley [1]). Both the exchange property and the isomorphic refinement for groups and modules were discussed by Warfield in a series of interesting papers [2, 3], [*Pacific J. Math.* **31** (1969), 263–276] etc. *Inter alia* he proved: (i) injective modules have the exchange property; (ii) an indecomposable module has the exchange property exactly if its endomorphism ring is a local ring; (iii) a module has the exchange property if and only if its endomorphism ring as a module over itself has this property; (iv) direct sums of modules with the exchange property have the isomorphic refinement property.

Problem 51. Characterize the separable p-groups by invariants.

Problem 52. Determine the cardinality of the set of nonisomorphic separable p-groups of cardinality $\leqq \mathfrak{m}$, for every infinite cardinal $\mathfrak{m}$.

Problem 53. Construct large systems of p-groups such that all homomorphisms between different members are small.

Problem 54. Is a separable p-group A torsion complete if it contains a torsion-complete subgroup G such that A/G is torsion-complete?

Problem 55. (Pierce [1]) Do there exist essentially indecomposable p-groups of arbitrarily large final ranks? [A p-group is called *essentially indecomposable* if in any direct decomposition of the group, one of the summands is bounded.]

Problem 56. For which cardinals $\mathfrak{m}$ does a p-group exist which is not a direct sum of cyclic groups, but every subgroup of cardinality $< \mathfrak{m}$ decomposes into the direct sum of cyclic groups?

Nunke [6] established the existence of such p-groups for $\mathfrak{m} = \aleph_n$ where $n = 1, 2, \cdots$.

Problem 57. Which groups have the exchange property?

By a result of Warfield, A has it exactly if $\mathsf{R} = \mathsf{E}(A)$ has it as an R-module.

Problem 58. A group A is said to have the *substitution property* if $M = A \oplus H = B \oplus K$ with $A \cong B$, implies that $M = C \oplus H = C \oplus K$, for some subgroup C of M. Which groups have this property?

Crawley [3] proved it for p-groups with finite Ulm–Kaplansky invariants. For a general discussion of this property cf. Fuchs [*Monatsh. Math.* **75** (1971), 198–204].

XII

p-GROUPS WITH ELEMENTS OF INFINITE HEIGHT

In the preceding chapter we have investigated p-groups without elements $\neq 0$ of infinite height, while the present chapter penetrates deeply into the realm of p-groups in general.

In contrast to the theory of separable p-groups, the emphasis here is on Ulm factors and Ulm–Kaplansky invariants, and the discussions are centered around two fundamental questions:

(1) Which well-ordered sequences of p-groups are Ulm sequences?
(2) Which is a possibly largest class of p-groups whose members are distinguishable *via* the Ulm–Kaplansky invariants?

The results here are as favorable as one could wish: full answers can be given to both questions.

The first problem can be solved by utilizing the properties of Ulm factors established in (37.6). But the proof is long and tedious, so, to avoid repetition of lengthy arguments, we will choose to derive the main result (76.1) from the more general theorem (105.3) on arbitrary groups. Special attention will be given to the case of countable p-groups.

There is very interesting and deep material on the second problem. This can be viewed from different perspectives, thus permitting us to choose a more transparent discussion. The starting point is, naturally, Ulm's classical result (77.3) on countable p-groups, bringing the structure theory of countable p-groups to a most satisfactory conclusion. It continues with the theory of direct sums of countable p-groups [in **78**] and culminates in showing that an answer to the second question is furnished by the class of totally projective p-groups.

The heart of the matter is naturally to explore the properties of the groups in this class. This remarkable class will be characterized in seven different ways, each revealing an interesting aspect of these groups [see Theorems 81.9, 82.3, 82.5, and 83.5 for more details]. The totally projective p-groups, however, do not exhaust the class of all p-groups, and not much is known of p-groups beyond this class.

76. EXISTENCE THEOREMS ON p-GROUPS

Let A be a reduced p-group. It defines a well-ordered sequence of subgroups:

$$A = A^0 > A^1 > \cdots > A^\sigma > \cdots > A^\tau = 0 \qquad \text{for some ordinal} \quad \tau,$$

where—we recall—$A^1 = \bigcap_{n=1}^{\infty} p^n A$ is the subgroup of all elements of A which are of infinite height in A, $A^{\sigma+1} = (A^\sigma)^1$, and $A^\rho = \bigcap_{\sigma<\rho} A^\sigma$ if ρ is a limit ordinal. A^σ was called the *σth Ulm subgroup* of A, and the quotient $A_\sigma = A^\sigma/A^{\sigma+1}$ the *σth Ulm factor* of A. The well-ordered sequence

$$(1) \qquad A_0, A_1, \cdots, A_\sigma, \cdots \qquad (\sigma < \tau)$$

was said to be the *Ulm sequence* of A; τ is the *Ulm type* of A.

Note that from **37** it follows that the Ulm factors of A must be separable p-groups which are unbounded with the possible exception of the last one, namely $A_{\tau-1}$, if this exists at all. More properties of the Ulm sequence were stated in (37.6).

It is natural to ask for necessary and sufficient conditions on (1) that it be the Ulm sequence of a p-group A. Necessary conditions have been listed in (37.6) for arbitrary groups A, and in (105.3) we are going to prove that if they are satisfied, then there is always a group A with the given sequence as Ulm sequence. For p-groups A, these conditions can slightly be simplified, and we can state the following existence theorem [B_σ denotes a basic subgroup of A_σ]:

Theorem 76.1 (Kulikov [3], Fuchs [2]). *Suppose we are given a well-ordered sequence* (1) *of p-groups A_σ ($\sigma < \tau$) and a cardinal number $\mathfrak{m}$. There exists a reduced p-group A of cardinality $\mathfrak{m}$, of Ulm type τ, and with* (1) *as Ulm sequence if and only if the following conditions are satisfied:*

(a) *for every $\sigma < \tau$, A_σ is separable;*
(b) $\sum_{0 \leqq \sigma < \tau} |A_\sigma| \leqq \mathfrak{m} \leqq \prod_{0 \leqq n < \min(\omega,\tau)} |A_n|$;
(c) $r(B_{\sigma+1}) \leqq \text{fin}\, r(A_\sigma)$ *for every* $\sigma + 1 < \tau$;
(d) $\sum_{\rho<\sigma<\tau} |A_\sigma| \leqq |A_\rho|^{\aleph_0}$ *for every* $0 \leqq \rho < \tau$.

We derive this theorem as a corollary to (105.3). All that we have to do is show that for p-groups, (d) is equivalent to $\sum_{\rho<\sigma<\tau} |B_\sigma| \leqq |B_\rho|^{\aleph_0}$. Notice that (34.4) implies that $|A_\rho|^{\aleph_0} = |B_\rho|^{\aleph_0}$, and we have

$$\sum_{\rho<\sigma<\tau} |B_\sigma| \leqq \sum_{\rho<\sigma<\tau} |A_\sigma| \leqq \sum_{\rho<\sigma<\tau} |B_\sigma|^{\aleph_0} \leqq \Big(\sum_{\rho<\sigma<\tau} |B_\sigma|\Big)^{\aleph_0},$$

the last inequality being a consequence of the inequality

$$\sum_{i \in I} \mathfrak{m}_i^{\aleph_0} \leqq \Big(\sum_{i \in I} \mathfrak{m}_i\Big)^{\aleph_0},$$

which holds for arbitrary cardinals $\mathfrak{m}_i$ and index set I. The stated equivalence of inequalities is evident.□

One might modify the posed question and ask for conditions on the Ulm–Kaplansky invariants of A rather than on its Ulm factors. Since the Ulm–Kaplansky invariants are less informative than the Ulm factors, the conditions for the Ulm–Kaplansky invariants can easily be derived from the stronger result (76.1). It should, however, be pointed out that the Ulm type τ has to be replaced by the *length* of A which was defined in **37** as the smallest ordinal ρ satisfying $p^\rho A = 0$.

An important special case is when we limit our considerations to countable p-groups A. Then all A_σ are countable separable p-groups, so they are, in view of Prüfer's theorem (17.3), direct sums of cyclic groups. The Ulm type τ of A must, of course, be a countable ordinal, and it is easy to check [see Ex. 2] that (c) is now equivalent to the condition that A_σ is unbounded for every $\sigma + 1 < \tau$. We are thus led to the following:

Corollary 76.2 (Zippin [1]). *There exists a countable reduced p-group A of Ulm type τ and with the Ulm sequence A_σ $(0 \leqq \sigma < \tau)$ exactly if:*

(i) *τ is a countable ordinal;*
(ii) *every A_σ is a countable direct sum of cyclic p-groups such that, for $\sigma + 1 < \tau$, A_σ is unbounded.*

Because of the significance of this theorem, we give an independent proof. The necessity of conditions (i) and (ii) is immediate [the only nontrivial argument follows from the easy (37.2)], so we need only establish the sufficiency of the conditions.

Suppose we are given a sequence A_σ $(0 \leqq \sigma < \tau)$ of p-groups satisfying (i) and (ii). We use transfinite induction on τ. If $\tau = 1$, then the sequence has one term only, A_0, and $A = A_0$ is as desired. So assume that $\tau \geqq 2$ and the assertion holds for ordinals less than τ. We distinguish several cases.

Case I. $\tau - 2$ exists and $A_{\tau-1} = \langle a \rangle$ is a cyclic group, say of order p^k. Write $A_{\tau-2} = \bigoplus_{n=1}^{\infty} \langle b_n \rangle$, where $o(b_n) = k_n$, and define $A'_{\tau-2} = \bigoplus_{n=1}^{\infty} \langle c_n \rangle$, where $o(c_n) = k + k_n$. By induction hypothesis, there exists a countable reduced p-group C whose Ulm sequence is $A_0, A_1, \cdots, A'_{\tau-2}$. Define A as the quotient of C modulo the subgroup generated by all $p^{k_n} c_n - p^{k_m} c_m$ $(n, m = 1, 2, \cdots)$. Let $p^{k_n} c_n = c \in A$; then the quotient $A/\langle c \rangle$ is clearly a countable p-group with the Ulm sequence $A_0, A_1, \cdots, A_{\tau-2}$. Since the set $\{k_n\}$ is unbounded and every $c_n \in A^{\tau-2}$, we must have $c \in A^{\tau-1}$. It is easy to show [see, e.g., Example in **35**] that $\langle c \rangle$ will be of order p^k in A and $A^{\tau-1} = \langle c \rangle$. Thus A is of type τ and is as desired.

Case II. $\tau - 2$ exists and $A_{\tau-1} = \bigoplus_{i \in I} \langle a_i \rangle$, where I is countable. We decompose every A_σ $(\sigma < \tau - 1)$ into a direct sum, $A_\sigma = \bigoplus_{i \in I} A_{\sigma i}$ such that no

$A_{\sigma i}$ is bounded. By Case I, for every $i \in I$, there exists a countable reduced p-group G_i whose Ulm sequence is $A_{0i}, \cdots, A_{\tau-2,i}, \langle a_i \rangle$. From (37.5) it follows that $A = \bigoplus_{i\in I} G_i$ has the given sequence for its Ulm sequence.

Case III. $\tau - 1$ is a limit ordinal and $A_{\tau-1} = \langle a \rangle$, $o(a) = p^k$. We can select a sequence $\tau_1, \cdots, \tau_i, \cdots (i < \omega)$ of ordinals tending to τ. We decompose every A_σ $(\sigma < \tau - 1)$ into a direct sum, $A_\sigma = \bigoplus_{i=1}^{\infty} A_{\sigma i}$ such that $A_{\tau_i i} = \langle b_i \rangle$ is cyclic, $A_{\sigma i} = 0$ for $\sigma > \tau_i$, and $A_{\sigma i}$ is unbounded for $\sigma < \tau_i$. Without loss of generality, the b_i can be assumed to satisfy the following condition: for every ordinal $\rho < \tau - 1$ and for every integer m, there is an i such that $\tau_i > \rho$ and $o(b_i) = p^{k_i} > p^m$. In view of the induction hypothesis, there exists, for every i, a countable reduced p-group G_i of type $\tau_i + 1$ with the Ulm sequence $A_{0i}, \cdots, A_{\sigma i}, \cdots, \bar{A}_{\tau_i i} = \langle c_i \rangle$, where $o(c_i) = p^{k+k_i}$. Define A as the quotient of $\bigoplus_{i=1}^{\infty} G_i$ mod the subgroup generated by all $p^{k_i}c_i - p^{k_j}c_j$ $(i, j = 1, 2, \cdots)$. In the same way as in Case I, it follows that A will have the prescribed Ulm sequence.

Case IV. $\tau - 1$ is a limit ordinal and $A_{\tau-1} = \bigoplus_{i\in I} \langle a_i \rangle$. This case can be reduced to the preceding one, by imitating the method of Case II.

Case V. τ is a limit ordinal. We write each A_σ $(\sigma < \tau)$ as an infinite direct sum $A_\sigma = A_{\sigma\sigma} \oplus \cdots \oplus A_{\sigma\rho} \oplus \cdots$ $(\sigma \leqq \rho < \tau)$ such that every $A_{\sigma\rho}$ is unbounded. By induction, there exists, for every $\sigma < \tau$, a countable reduced p-group G_σ whose Ulm sequence is $A_{0\sigma}, A_{1\sigma}, \cdots, A_{\sigma\sigma}$. Then $A = \bigoplus_{0 \leqq \sigma < \tau} G_\sigma$ will have the given Ulm sequence. □

This is a good opportunity to determine the cardinality of the set of nonisomorphic p-groups of power $\leqq \mathfrak{n}$.

Theorem 76.3 (Kulikov [3]). *For every prime p, the set of nonisomorphic p-groups of cardinality not exceeding an infinite cardinal* $\mathfrak{n}$ *has the cardinality* $2^{\mathfrak{n}}$.

We noticed in **14** that the set in question can not have a cardinality larger than $2^{\mathfrak{n}}$. Conversely, let τ be the first ordinal of cardinality $\mathfrak{n}$, and set

$$G_1 = \bigoplus_{n=1}^{\infty} C_{2n-1}, \qquad G_2 = \bigoplus_{n=1}^{\infty} C_{2n}, \qquad \text{where} \quad C_i = \bigoplus_{\mathfrak{n}} Z(p^i).$$

From (76.1) we derive the existence of a reduced p-group A of cardinality $\mathfrak{n}$ and of type τ whose Ulm sequence is A_σ $(0 \leqq \sigma < \tau)$ where each A_σ is isomorphic either to G_1 or to G_2. There are $2^{\mathfrak{n}}$ different ways of choosing the Ulm sequence for A in the indicated manner and different choices give rise obviously to nonisomorphic A. Therefore, the cardinality in question is $\geqq 2^{\mathfrak{n}}$. □

One final remark is in order. The last proof can slightly be modified by choosing A_0 constantly isomorphic to $G_1 \oplus G_2$, and admitting a choice between G_1 and G_2 for the other A_σ. In this way we obtain $2^{\mathfrak{n}}$ nonisomorphic p-groups A of cardinality $\mathfrak{n}$ with the additional property that their basic subgroups are isomorphic to $G_1 \oplus G_2 = \bigoplus_{n=1}^{\infty} \bigoplus_{\mathfrak{n}} Z(p^n)$ [cf. **34**(F)]. All these A have final rank equal to $\mathfrak{n}$. [This strengthened form of (76.3) was needed in the proof of (47.6).]

Exercises

1. Derive the inequality
$$\sum_{0 \leqq \sigma < \tau} |A_\sigma| \leqq \prod_{0 \leqq n < \min(\omega, \tau)} |A_n|$$
from the inequality (d).
2. Show that (c) implies the unboundedness of all A_σ with $\sigma + 1 < \tau$.
3. Give an example of a reduced p-group A of type ω and of cardinality $2^{\aleph_0}$ whose Ulm factors are countable.
4. The set of nonisomorphic countable reduced p-groups of a fixed countable type $\geqq 1$ has the cardinality $2^{\aleph_0}$.
5. (a) If every element of the reduced p-group A can be embedded in a countable summand of A, then the Ulm type of A is at most ω_1 [=the first ordinal of cardinality $\aleph_1$].
(b) Prove the existence of a p-group A with the property indicated in (a) whose Ulm type is exactly ω_1. [*Hint:* $A = \bigoplus_{0 \leqq \sigma < \omega_1} G_\sigma$, where G_σ is countable of type σ.]
(c) Establish the analogous propositions for any infinite cardinal $\aleph_\rho$ rather than $\aleph_0$.
6. Show that every ordinal is the Ulm type [length] of some reduced p-group.
7. There exists a [reduced] p-group A of cardinality $2^{\mathfrak{n}}$ such that every [reduced] p-group of cardinality $\leqq \mathfrak{n}$ is isomorphic to a summand of A.
8. Let G be a separable p-group of final rank $\mathfrak{n}$. Give an estimate for the cardinality of the set of nonisomorphic reduced p-groups A such that $A_0 \cong G$. [*Hint:* (34.5) and (76.3).]
9.* Given a well-ordered sequence of cardinal numbers $\mathfrak{m}_\sigma$ $(0 \leqq \sigma < \tau)$, find necessary and sufficient conditions for the existence of a p-group A of length τ such that, for every σ, the Ulm–Kaplansky invariant of A equals $\mathfrak{m}_\sigma$.

77. ULM'S THEOREM

One of the fundamental questions in the theory of p-groups is to find out to what extent the groups are determined by their Ulm sequences. One of the most celebrated results in the theory of abelian groups is Ulm's theorem, which states that countable reduced p-groups are uniquely determined, up to isomorphism, by their Ulm sequences. This section is devoted to the proof of this theorem and to some of its consequences.

The methods of establishing the isomorphism between countable reduced p-groups with the same Ulm sequence rely heavily on the technique of successive extensions of isomorphisms between certain subgroups. It is for this reason that we first study extensions of isomorphisms between subgroups.

Let A be a p-group and G a subgroup of A. An element $a \in A \backslash G$ is called *proper with respect to* G if for the [generalized] height h^* [at p] we have

$$h^*(a) \geqq h^*(b) \qquad \text{for all} \quad b \in a + G.$$

In other words, a has maximal height in its coset mod G. Thus an element $a \in A$ of height σ is proper with respect to G if and only if

$$a \notin p^{\sigma+1}A + G.$$

It is easy to see that this is equivalent to the equality of $h^*(a)$ with the height of $a + G$ in A/G. Note that if a is proper with respect to G, then

$$h^*(a + g) = \min(h^*(a), h^*(g)) \qquad \text{for all} \quad g \in G.$$

Evidently, if G is a finite subgroup of A then every coset mod G contains an element proper with respect to G. The same holds if $G = p^{\rho}A$ [cf. (37.1)].

Let A be a reduced p-group and G a subgroup of A. For every ordinal σ, we can define

$$G(\sigma) = (p^{\sigma+1}A + G) \cap p^{\sigma}A[p].$$

This is obviously a subgroup of $p^{\sigma}A[p]$ containing $p^{\sigma+1}A[p]$. From our remark made above it is evident that *an element a of order p and of height σ belongs to $G(\sigma)$ if and only if it is not proper with respect to G.*

The cardinal number

$$f_{\sigma}(A, G) = r(p^{\sigma}A[p]/G(\sigma))$$

is uniquely determined by A and G. It is called the *σth Ulm–Kaplansky invariant of A relative to G* (Hill [24]). We see that $f_{\sigma}(A, G) \leqq f_{\sigma}(A)$ for every subgroup G of A and for every ordinal σ. Evidently, $f_{\sigma}(A, 0) = f_{\sigma}(A)$.

It is straightforward to see that if an isomorphism ψ between the p-groups A and C carries the subgroup G of A onto a subgroup H of C, then necessarily $G(\sigma)$ is mapped onto $H(\sigma)$. Consequently, ψ induces, for every σ, an isomorphism

$$\psi(\sigma)\colon p^{\sigma}A[p]/G(\sigma) \to p^{\sigma}C[p]/H(\sigma).$$

In particular, we have

$$f_{\sigma}(A, G) = f_{\sigma}(C, H) \qquad \text{for every} \quad \sigma.$$

Let again A and C be p-groups, G and H subgroups of A and C, respectively. An isomorphism $\phi: G \to H$ is called *height-preserving* if

$$h^*(\phi g) = h^*(g) \qquad \text{for every} \quad g \in G.$$

It should be emphasized that the heights are always computed in C and A, respectively.

Manifestly, the restriction of any isomorphism between A and C is height-preserving.

The crucial step in the proof of Ulm's theorem is singled out as our next lemma, due to Kaplansky and Mackey [1]. It is formulated in a stronger form to facilitate later discussions.

Lemma 77.1. *Let A and C be reduced p-groups, and ϕ a height-preserving isomorphism between a subgroup G of A and a subgroup H of C. Suppose that*

$$(1) \qquad f_\sigma(A, G) \leqq f_\sigma(C, H) \qquad \textit{for every} \quad \sigma,$$

and

$$\alpha_\sigma : p^\sigma A[p]/G(\sigma) \to p^\sigma C[p]/H(\sigma)$$

are arbitrary monomorphisms for all σ.

If $a \in A$ is proper with respect to G and $pa \in G$, then ϕ can be extended to a height-preserving isomorphism

$$\phi^* : \langle G, a \rangle \to \langle H, c \rangle$$

for a suitable $c \in C$ such that, for every σ, α_σ carries $\langle G, a\rangle(\sigma)/G(\sigma)$ onto $\langle H, c\rangle(\sigma)/H(\sigma)$.

Setting $h^*(a) = \sigma$, we may assume that a is chosen in its coset mod G, in addition to being proper, to satisfy also $h^*(pa) > \sigma + 1$ whenever this is possible. We distinguish two cases according as this is impossible or possible.

Case I. $h^*(pa) = \sigma + 1$. Pick a $c \in C$ of height σ such that $pc = \phi(pa)$. If $c \in H$, then $\phi g = c$ for some $g \in G$, $pg = \phi^{-1}(pc) = pa$, hence $p(a - g) = 0$, where $h^*(a - g) = \min(h^*(a), h^*(g)) = \sigma$, since a was proper with respect to G. Hence $a' = a - g$ is likewise proper with respect to G, but $h^*(pa') > \sigma + 1$, in contradiction to the choice of a. Thus $c \notin H$. If c is not proper with respect to H, i.e., if $h^*(c + h) > \sigma$ for some $h \in H$, where necessarily $h^*(h) = \sigma$, then $\sigma < h^*(c + h) < h^*(pc + ph) = h^*(pa + p\phi^{-1}h)$ shows that $a' = a + \phi^{-1}h$ satisfies $h^*(a') = \min(h^*(a), h^*(h)) = \sigma$ and $h^*(pa') > \sigma + 1$, contrary to the choice of a. Hence $c \notin H$ and c is proper with respect to H.

Case II. $h^*(pa) > \sigma + 1$. Write $pa = pb$ with $b \in p^{\sigma+1}A$. Then $a - b \in A[p]$ has height σ and is proper with respect to G, for otherwise $h^*(a - b + g) > \sigma$ with $g \in G$ would imply $h^*(a + g) > \sigma$, a contradiction. By an earlier remark,

$a - b \notin G(\sigma)$, thus there is a $u \in p^\sigma C[p]$ such that $\alpha_\sigma: a - b + G(\sigma) \mapsto u + H(\sigma)$. This u is evidently proper with respect to H. Since ϕ is height-preserving, we can find a $d \in p^{\sigma+1}C$ such that $pd = \phi(pa) \in H$. Now $c = d + u$ is proper with respect to H and satisfies $h^*(c) = \sigma$, $pc = \phi(pa) \in H$.

In both cases, ϕ will be extended to an isomorphism $\phi^*: \langle G, a\rangle \to \langle H, c\rangle$ by letting the selected c correspond to a. To see that ϕ^* is still height-preserving, observe that for every $g \in G$,

$$h^*(a + g) = \min(h^*(a), h^*(g)) = \min(h^*(c), h^*(\phi g)) = h^*(c + \phi g).$$

To check the last condition note that if $\rho < \sigma$, then $a \in p^{\rho+1}A$ implies $\langle G, a\rangle(\rho) = G(\rho)$, and if $\rho > \sigma$, then the same equality is a consequence of a being proper with respect to G. Similar equality holds for H and c. If $\rho = \sigma$, then in Case I, no $g \in G$ and $a' \in p^{\sigma+1}A$ can satisfy $p(a - g + a') = 0$, hence again $\langle G, a\rangle(\sigma) = G(\sigma)$, and similarly $\langle H, c\rangle(\sigma) = H(\sigma)$. In Case II, $\langle G, a\rangle(\sigma) = \langle a - b\rangle \oplus G(\sigma)$ and $\langle H, c\rangle(\sigma) = \langle u\rangle \oplus H(\sigma)$, and the choice of u guarantees that α_σ induces an isomorphism as desired. □

Naturally, if the monomorphisms α_σ are not preassigned, then in Case II any $u \in p^\sigma C[p] \setminus H(\sigma)$ can be chosen.

Note that the last part of the preceding proof indicates that the Ulm–Kaplansky invariants of A relative to G and $\langle G, a\rangle$ are the same except possibly the σth which can be related as follows: $f_\sigma(A, \langle G, a\rangle) + 1 = f_\sigma(A, G)$. Hence the following is obvious:

Corollary 77.2. *Under the hypotheses of Lemma* 77.1, *equality in* (1) *implies*

$$f_\sigma(A, \langle G, a\rangle) = f_\sigma(C, \langle H, c\rangle)$$

for all σ. □

Now we are ready to verify the fundamental

Theorem 77.3 (Ulm [1]). *The countable reduced p-groups, A and C, are isomorphic if and only if they have the same Ulm type* τ *and, for each ordinal* $\sigma < \tau$, *the Ulm factors* A_σ *and* C_σ *are isomorphic.*

This is the case if and only if their corresponding Ulm–Kaplansky invariants are equal.

Since the groups A, C are countable p-groups, A_σ and C_σ are direct sums of cyclic p-groups. Note that the numbers of cyclic summands of order p^n in decompositions of A_σ and C_σ are equal to the $(\omega\sigma + n - 1)$th Ulm–Kaplansky invariants of the groups A and C. Therefore, it will be enough to show that A and C are necessarily isomorphic whenever their corresponding Ulm–Kaplansky invariants are the same.

The elements of A and C can be arranged in sequences of type ω:

$$A = \{a_1, \cdots, a_n, \cdots\}, \qquad C = \{c_1, \cdots, c_n, \cdots\}.$$

Suppose that $0 = G_0 < G_1 < \cdots < G_n$ and $0 = H_0 < H_1 < \cdots < H_n$ are finite subgroups in A and C, respectively, such that, for $i = 0, 1, \cdots, n$, there exists a height-preserving isomorphism $\phi_i \colon G_i \to H_i$ satisfying $\phi_n | G_i = \phi_i$. If n is even, we select the first a_k in the sequence which does not belong to G_n, and wish to extend ϕ_n to a height-preserving isomorphism ϕ_{n+1} of $G_{n+1} = \langle G_n, a_k \rangle$ with a subgroup H_{n+1} of C, containing H_n. This is possible by virtue of (77.1), since the adjunction of a_k to a finite group G_n can be realized *via* successive adjunctions of elements a satisfying the hypotheses (77.1). If n is odd, we select the first c_l not in H_n, and extend ϕ_n^{-1} to an isomorphism of $H_{n+1} = \langle H_n, c_l \rangle$ with a subgroup G_{n+1} of A. After each step, (1) will be satisfied as guaranteed by (77.2). It is clear that the ϕ_n ($n = 0, 1, \cdots$) will eventually define an isomorphism between A and C. □

We wish to record a corollary to the proof.

Corollary 77.4 (Zippin [1]). *If A and C are countable p-groups with the same Ulm–Kaplansky invariants, then every isomorphism between $p^\rho A$ and $p^\rho C$ (for any ordinal ρ) extends to an isomorphism between A and C.*

Though this will follow at once from the more general (83.4), we give an independent proof.

All that we have to check is that if in (77.1), a happens to belong to $p^\rho A$, then $c = \psi a \in p^\rho C$ can be chosen where $\psi \colon p^\rho A \to p^\rho C$ is the given isomorphism. Note that ψ is height-preserving: if a is of height σ in $p^\rho A$, then it is of height $\rho + \sigma$ in A. Thus, we need only check that ψa is proper with respect to H. If this were false, i.e., $h^*(\psi a + h) > h^*(\psi a)$ for some $h \in H$, then $\psi a + h \in p^\rho C$ would imply $h^*(a + \psi^{-1} h) = h^*(\psi a + h) > h^*(a)$, a contradiction. □

The theorems (17.3), (76.2), and (77.3) by Prüfer, Zippin, and Ulm, respectively, yield a complete classification of countable reduced p-groups. As a matter of fact, these three theorems constitute a satisfactory structure theory for countable reduced p-groups.

A few words may be inserted here about the invariants.

Let A be a p-group of Ulm type τ. Its Ulm–Kaplansky invariants $f_\sigma(A)$ can be arranged in a matrix form

$$\text{(2)} \qquad \mathbb{M}(A) = \begin{bmatrix} f_0(A) & f_1(A) & \cdots & f_n(A) & \cdots \\ f_\omega(A) & f_{\omega+1}(A) & \cdots & f_{\omega+n}(A) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{\omega\rho}(A) & f_{\omega\rho+1}(A) & \cdots & f_{\omega\rho+n}(A) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where the rows can be indexed by ordinals $\sigma < \tau$ and the columns by nonnegative integers; we may say briefly this is a $\tau \times \omega$-matrix. If A is countable, then τ is countable and every cardinal $f_\sigma(A)$ is either a nonnegative integer or is equal to $\aleph_0$. The unboundedness of the σth Ulm factor A_σ $(\sigma + 1 < \tau)$ is equivalent to the fact that in the σth row, infinitely many cardinals are different from 0. In this way, with every countable reduced p-group A of type τ there is associated a matrix of type $\tau \times \omega$ satisfying the following conditions:

(i) τ is a countable ordinal;
(ii) the entries are nonnegative integers or $\aleph_0$;
(iii) every row, except possibly for the last one, if this exists, contains infinitely many entries $\neq 0$.

Now Ulm's theorem (77.3) can be interpreted as the assertion that the matrices (2) form complete systems of invariants for countable reduced p-groups [that is to say, two such groups are isomorphic if and only if the corresponding matrices are equal]. Since it is easy to decide whether or not two matrices with cardinal numbers as entries are equal, our matrices yield satisfactory systems of invariants. Zippin's theorem (76.2) states that every $\tau \times \omega$-matrix satisfying (i)–(iii) corresponds to a countable reduced p-group; therefore, the matrices under consideration yield independent systems of invariants.

The discovery of this one-to-one correspondence between countable reduced p-groups and matrices satisfying (i)–(iii) is undoubtedly one of the most significant achievements in abelian group theory.

We can, of course, change our point of view and consider the Ulm–Kaplansky invariants $f_\sigma(A)$ of A arranged in a well-ordered sequence

$$f_0(A), \cdots, f_n(A), \cdots, f_\omega(A), \cdots, f_\sigma(A), \cdots \qquad (\sigma < \tau),$$

where τ designates this time the length of A. Conditions (i) and (ii) continue to hold, while (iii) has to be replaced now by

(iii′) if $\sigma + \omega \leq \tau$, then infinitely many of $f_\sigma(A), \cdots, f_{\sigma+n}(A), \cdots$ $(n < \omega)$ are different from 0.

As an application of the theory we prove two results.

Proposition 77.5. *Every countably infinite, reduced p-group decomposes into the direct sum of infinitely many nontrivial groups.*

Let A be as stated, and let A_σ $(0 \leqq \sigma < \tau)$ be its Ulm sequence. In view of (17.2), we may confine ourselves to the case when A is unbounded. Invoking Prüfer's theorem (17.3), we decompose each A_σ into an infinite direct sum of unbounded groups $G_{\sigma i}$, $A_\sigma = \bigoplus_{i=1}^{\infty} G_{\sigma i}$; if $A_{\tau-1}$ exists and is bounded, we set $G_{\sigma 1} = A_{\tau-1}$ and $G_{\sigma i} = 0$ for $i \geqq 2$. Zippin's theorem (76.2) ensures the existence of a countable reduced p-group G_i, for each i, whose Ulm sequence is $G_{\sigma i}$ $(0 \leqq \sigma < \tau)$. Then $\bigoplus_{i=1}^{\infty} G_i$ is a countable reduced p-group with the same Ulm sequence as A; hence (77.3) implies $A \cong \bigoplus_{i=1}^{\infty} G_i$. $\square$

Proposition 77.6 (Baer [1]). *A countable reduced p-group A has the property that any two direct decompositions of A have isomorphic refinements exactly if the Ulm type of A is equal to* 1.

If the Ulm type of A is 1, then (18.2) implies that A does have the mentioned property.

If the Ulm type of A is greater than 1, then we decompose its Ulm factors $A_\sigma = G_\sigma \oplus H_\sigma$ $(0 \leqq \sigma < \tau)$ in such a way that both G_σ and H_σ are unbounded [whenever A_σ is unbounded], and G_σ and H_σ have no cyclic summands of the same order in common. Let G and H be countable reduced p-groups with the Ulm sequences G_σ $(0 \leqq \sigma < \tau)$ and H_σ $(0 \leqq \sigma < \tau)$, respectively. Furthermore, let G' and H' be countable reduced p-groups whose Ulm sequences are G_0, H_σ $(1 \leqq \sigma < \tau)$ and H_0, G_σ $(1 \leqq \sigma < \tau)$, respectively. By Ulm's theorem (77.3), $A \cong G \oplus H \cong G' \oplus H'$. These direct decompositions cannot have isomorphic refinements, since no [cyclic] summands $\neq 0$ of G and H' [G' and H] can be isomorphic, and G, G' are not isomorphic.□

We now exhibit examples to convince the reader that Ulm's theorem fails to hold in general for p-groups of cardinality $\geqq \aleph_1$.

Note that the two equivalent formulations of Ulm's theorem [the Ulm factors or the Ulm–Kaplansky invariants determine the countable reduced p-groups within isomorphism] are no longer equivalent if we pass to p-groups of larger cardinalities. The reason for this is obviously the failure of Prüfer's theorem in the uncountable case. [We recollect that a torsion-complete p-group $\bar{B}$ and its basic subgroup B have the same Ulm–Kaplansky invariants.] Consequently, the following examples show the stronger version that not even the Ulm factors determine the uncountable p-groups, in general.

Example 1. Let A_n $(n < \omega)$ be a sequence of unbounded countable p-groups. By (76.1), there exist p-groups A and A', with the given sequence as Ulm sequence whose cardinalities are $\aleph_0$ and $\aleph_1$, respectively.

Example 2. Let C be a direct sum of cyclic p-groups such that $|C| = \aleph_1$. If A and A' are defined as in Example 1, then $A \oplus C$ and $A' \oplus C$ are both of cardinality $\aleph_1$ and have the same Ulm sequence. They fail to be isomorphic, since their first Ulm subgroups are of cardinality $\aleph_0$ and $\aleph_1$, respectively.

Example 3 (Kulikov [2]). Let A_0 be the torsion-complete group with the basic subgroup $B = \bigoplus_{n=1}^{\infty} Z(p^n)$, and let A_1 be the direct sum of $2^{\aleph_0}$ cyclic groups of order p. Two p-groups, A and C, will be defined with A_0, A_1 as Ulm sequence. We have to refer to the specific construction to be given in (105.1): in the definition of A, we never put $pv_{j1} = 0$ in (5), while for C, we put $pv_{j1} = 0$ for continuously many indices j. Both A and C are of power $2^{\aleph_0}$ and have the same Ulm sequence, but because of $|A[p]/A^1| = \aleph_0$ and $|C[p]/C^1| = 2^{\aleph_0}$, they are not isomorphic.

Exercises

1. If G, H are subgroups of A and if $G \leqq H$, then $f_\sigma(A, G) \geqq f_\sigma(A, H)$ for every σ.
2. Show that for every ρ, $f_\sigma(A, p^\rho A) = 0$ or $f_\sigma(A)$ according as $\rho \leqq \sigma$ or $\rho > \sigma$.
3. Let A and C be countable p-groups. State conditions, in terms of their Ulm–Kaplansky invariants, for C to be isomorphic to a direct summand of A.

4. If A and C are countable p-groups, each isomorphic to a direct summand of the other, then $A \cong C$.
5. (Kaplansky [3]) If A is a countable p-group and if C satisfies $A \oplus A \cong C \oplus C$, then $A \cong C$.
6. A group A is said to have the *cancellation property for countable p-groups*, if for countable p-groups G, H, the isomorphism $A \oplus G \cong A \oplus H$ implies $G \cong H$. Show that a countable reduced p-group A has this property if and only if all of its Ulm–Kaplansky invariants are finite.
7. Find necessary and sufficient conditions for a countable reduced p-group A, in terms of its Ulm–Kaplansky invariants, to satisfy $A \oplus A \cong A$.
8. (a) A countable reduced p-group A has a direct summand which is an unbounded direct sum of cyclic groups if and only if A is unbounded.
(b) Give a full survey on the direct summands of Prüfer's example in **35**.
9. (a) A countable reduced p-group A of Ulm type τ has a direct summand of Ulm type σ exactly if $\sigma \leqq \tau$.
(b) A countable reduced p-group A of length τ has a summand of length σ if and only if either σ is a limit ordinal $\leqq \tau$ or $f_{\sigma-1}(A) \neq 0$.
10. If A is a countable p-group of Ulm type τ, then we can write
$$A = \bigoplus_{\sigma \leqq \tau} G_\sigma,$$
where the Ulm type of G_σ is equal to σ. If τ is a limit ordinal, summation over all $\sigma < \tau$ suffices.
11. (a) Given a countable ordinal τ, there exists a countable p-group $G(\tau)$ of Ulm type τ such that every countable reduced p-group of Ulm type $\leqq \tau$ is isomorphic to a summand of $G(\tau)$.
(b) There exists a p-group of power $\aleph_1$ and of Ulm type ω_1 such that every countable reduced p-group is isomorphic to a summand of this group. [*Hint*: $\bigoplus G(\tau)$.]
12. (Irwin and Walker [2]) Let A be a p-group and σ an ordinal. Show that for every ordinal $\rho < \sigma$, $f_\rho(A) = f_\rho(H)$, where H is any $p^\sigma A$-high subgroup of A.
13. (Hill and Megibben [4]) Generalize (77.4) to the case when only $A/p^\rho A$ and $C/p^\rho C$ are assumed to be countable.

78. DIRECT SUMS OF COUNTABLE p-GROUPS

It has been observed by Kolettis [1] that much of the theory of countable p-groups can be carried over to their direct sums. In this section, we wish to discuss this wider class of p-groups.

Evidently, direct sums of cyclic p-groups belong to the class of direct sums of countable p-groups. By Prüfer's theorem (17.2), a separable direct sum of countable p-groups must be a direct sum of cyclic groups. In view of (37.5), we infer:

Lemma 78.1. *The Ulm factors of a direct sum of countable p-groups are direct sums of cyclic groups.* □

The converse is not true: there exist reduced p-groups all of whose Ulm factors are direct sums of cyclic groups, and which fail to be direct sums of countable groups [cf. Ex. 2].

It is routine to check that Ulm subgroups A^σ of direct sums A of countable p-groups are again such groups, and so are the subgroups $p^\sigma A$.

Obviously, a basic subgroup of a countable reduced p-group is always of the same cardinality as the group itself. Thus we have $|B| = |A|$ for a basic subgroup B of a direct sum A of countable reduced p-groups. Noting that $|B| = \sum_{k<\omega} f_k(A)$ if $|B|$ is infinite, we obtain

$$|A| = \sum_{k<\omega} f_k(A)$$

for any direct sum A of countable reduced p-groups provided that A is infinite.

Since the Ulm types of countable p-groups are countable, and since the Ulm type of a direct sum is equal to the supremum of the Ulm types of the components, we obtain:

Lemma 78.2. *An ordinal τ is the Ulm type of some direct sum of countable p-groups exactly if* $\tau \leqq \omega_1$. □

In a later result, (82.4), we shall give a criterion for a reduced p-group to be a direct sum of countable p-groups. Our next purpose is to establish a structure theorem by showing that the Ulm–Kaplansky invariants form a complete system of invariants for direct sums of countable p-groups.

The following proof of (78.4) is based on the observation made by Richman and Walker [3] that no group theory is actually needed in extending Ulm's theorem from countable p-groups to their direct sums. In fact, the following simple set-theoretical lemma will furnish us with all that is needed for the proof of (78.4).

Lemma 78.3 (Richman and Walker [3]). *Let $\mathfrak{m}$ be an infinite cardinal, X a set, and f_i, g_i $(i \in I)$ functions from X to the cardinals satisfying*:

(a) $\sum_{i\in I} f_i(x) = \sum_{i\in I} g_i(x)$ *for every* $x \in X$;
(b) $\sum_{x\in X}[f_i(x) + g_i(x)] \leqq \mathfrak{m}$ *for every* $i \in I$.

Then there exists a partition $I = \bigcup_{j\in J} I_j$ *such that*:

(1) $|I_j| \leqq \mathfrak{m}$ *for every* $j \in J$;
(2) $\sum_{i\in I_j} f_i(x) = \sum_{i\in I_j} g_i(x)$ *for every* $j \in J$ *and every* $x \in X$.

It will be convenient to assume that I is the set of ordinals less than the first ordinal ω_τ of cardinality $|I|$. Since there is nothing to verify if $\sum_{i \in I} f_i(x) \leqq \mathfrak{m}$, we shall assume that $\sum_{i \in I} f_i(x) > \mathfrak{m}$.

By (a), there is a subset L_1 of I such that $f_0(x) \leqq \sum_{i \in L_1} g_i(x)$ for every $x \in X$, and because of (b), L_1 can also be chosen so as to satisfy $0 \in L_1$ and $|L_1| \leqq \mathfrak{m}$. Similarly, there is a subset L_2 of I such that $L_1 \subseteq L_2$, $|L_2| \leqq \mathfrak{m}$, and $\sum_{i \in L_2} f_i(x) \geqq \sum_{i \in L_1} g_i(x)$ for all $x \in X$, and there exists another subset $L_3 \subseteq I$ such that $L_2 \subseteq L_3$, $|L_3| \leqq \mathfrak{m}$, and $\sum_{i \in L_2} f_i(x) \leqq \sum_{i \in L_3} g_i(x)$ for all $x \in X$, etc. The union $I_0 = \bigcup_{n=1}^{\infty} L_n$ will then satisfy (1) and (2) for $j = 0$.

Since $\sum_{i \in I_0} f_i(x) \leqq \mathfrak{m}$, but $\sum_{i \in I} f_i(x) > \mathfrak{m}$, (a) continues to hold with I replaced by $I \backslash I_0$. Starting with the first index in $I \backslash I_0$, we repeat transfinitely the process of the preceding paragraph to obtain I_1, etc. After ω_τ steps we arrive at a desired partition of I. $\square$

We are now able to derive:

Theorem 78.4 (Kolettis [1]). *Two direct sums of countable reduced p-groups are isomorphic if and only if their corresponding Ulm-Kaplansky invariants are equal.*

Let A and C be such groups, say of lengths τ, satisfying $f_\sigma(A) = f_\sigma(C)$ for all $\sigma < \tau$. Ignoring the finite case, $|A| = \sum_{k < \omega} f_k(A) = \sum_{k < \omega} f_k(C) = |C|$, whence it is clear that if one of A and C is countable, then so is the other, and the isomorphy $A \cong C$ follows from (77.3).

If they are not countable, we can then write

$$A = \bigoplus_{i \in I} A_i \qquad \text{and} \qquad C = \bigoplus_{i \in I} C_i,$$

where A_i, C_i are countable reduced p-groups and $|A| = |I| = |C|$. We apply (78.3) to the case when $\mathfrak{m} = \aleph_0$ and X is the set of ordinals $\sigma < \tau$, while $f_i(\sigma)$ and $g_i(\sigma)$ are the σth Ulm–Kaplansky invariants of A_i and C_i, respectively. Conditions (a) and (b) are clearly fulfilled, and with the partition $I = \bigcup_{j \in J} I_j$ satisfying (1) and (2), we form the groups $G_j = \bigoplus_{i \in I_j} A_i$ and $H_j = \bigoplus_{i \in I_j} C_i$. Visibly, G_j and H_j are countable p-groups with the same Ulm–Kaplansky invariants, hence $G_j \cong H_j$ for every j. Since $A = \bigoplus_{j \in J} G_j$ and $C = \bigoplus_{j \in J} H_j$, the claimed isomorphism is evident. $\square$

Direct summands of direct sums of countable p-groups are likewise such groups—this is an immediate consequence of (9.10). But their subgroups need not again be direct sums of countable p-groups, as is shown by the following example.

Example (Nunke [6]). Let A be as in the example of **35**, and let $\bar{B}$ be the torsion-completion of $B = \bigoplus_{n < \omega} Z(p^n)$. We claim that

$$G = \operatorname{Tor}(A, \bar{B}) \leqq \bigoplus_{\mathfrak{c}} A \qquad (\text{where} \quad \mathfrak{c} = 2^{\aleph_0}),$$

but G is not a direct sum of countable p-groups.

There is a monomorphism $\alpha: \bar{B} \to \oplus_{\mathfrak{c}} Z(p^\infty)$ which induces by (63.1) a monomorphism

$$\alpha_*: \mathrm{Tor}(A, \bar{B}) \to \mathrm{Tor}(A, \bigoplus_{\mathfrak{c}} Z(p^\infty)) \cong \bigoplus_{\mathfrak{c}} \mathrm{Tor}(A, Z(p^\infty)) \cong \bigoplus_{\mathfrak{c}} A.$$

(64.2) implies $p^\omega \mathrm{Tor}(A, \bar{B}) = \mathrm{Tor}(p^\omega A, p^\omega \bar{B}) = 0$; thus if G was a direct sum of countable p-groups, it would be a direct sum of cyclic groups. From the pure-exact sequence $0 \to B \to \bar{B} \to \oplus_{\mathfrak{c}} Z(p^\infty) \to 0$, we obtain the exact sequence $0 \to C = \mathrm{Tor}(A, B) \to G \to \oplus_{\mathfrak{c}} A \to 0$. Here $|p^\omega(G/C)| = \mathfrak{c}$, while if G was a direct sum of cyclic groups, then a countable subgroup C would be embeddable in a countable summand D of G, and $p^\omega(G/C) = p^\omega(D/C)$ must be countable, a contradiction.

We turn our attention to the existence problem. The basic question is: which well-ordered sequences

$$\mathfrak{m}_0, \mathfrak{m}_1, \cdots, \mathfrak{m}_\sigma, \cdots \qquad (\sigma < \tau)$$

of cardinals can be realized as Ulm–Kaplansky invariants of a direct sum of countable p-groups? By (78.2), an obvious necessary condition is that $\tau \leqq \omega_1$. To help answer the question fully, we consider functions g from the ordinals $< \tau$ to the cardinals where τ is a given ordinal [for the time being, $\tau \leqq \omega_1$ can be supposed]. g is said to be *τ-admissible* if the following conditions are fulfilled:

(i) $\tau = \sup\{\sigma + 1 \mid g(\sigma) \neq 0\}$,
(ii) $\sum_{\rho \geqq \sigma + \omega} g(\rho) \leqq \sum_{n < \omega} g(\sigma + n)$ for all σ with $\sigma + \omega < \tau$.

A function g satisfying condition (i) is called *of length* τ.
The following lemma will be needed [proof by E. A. Walker].

Lemma 78.5. *If τ is a limit ordinal, then a τ-admissible function is the sum of admissible functions of lengths $< \tau$.*

Let g be a τ-admissible function, τ a limit ordinal. Consider the cardinals $\mathfrak{m}_\rho = \sum_{\rho \leqq \sigma < \tau} g(\sigma)$, for all $\rho < \tau$; there is a smallest among them. Let λ be the index of the first smallest $\mathfrak{m}_\rho$, and set $J = \{\nu \mid \lambda \leqq \nu < \tau\}$.

For every limit ordinal $\rho < \tau$, we consider the cardinal

$$\mathfrak{n}_\rho = \min_{k < \omega} \sum_{k \leqq n < \omega} g(\rho + n).$$

Evidently, $\mathfrak{n}_\rho \geqq |J|$. It is an elementary exercise in cardinal arithmetic to show that the restriction f_ρ of g to the interval $[\rho, \rho + \omega)$ can be decomposed in the form

$$f_\rho = \sum_{\nu \in J} f_{\rho\nu},$$

subject to the condition

$$\sum_{k \leqq n < \omega} f_{\rho\nu}(\rho + n) \geqq \mathfrak{n}_\rho \qquad \text{for all} \quad k < \omega.$$

[This is equivalent to the fact that a direct sum of cyclic p-groups of final rank n can be decomposed into the direct sum of n or less summands, each having final rank n.] Putting $f_{\rho\nu}(\sigma) = 0$ whenever $\sigma < \rho$ or $\sigma \geqq \rho + \omega$, define

$$h_\nu = \sum_\rho f_{\rho\nu} \qquad \text{for every} \quad \nu \in J.$$

Then obviously $g = \sum_{\nu \in J} h_\nu$ and $\sum_{n<\omega} h_\nu(\sigma + n) \geqq \sum_{\rho \geqq \sigma+\omega} g(\rho)$ for every σ and every ν. Finally, let

$$g_\nu(\sigma) = \begin{cases} h_\nu(\sigma) & \text{if} \quad \sigma < \nu, \\ \sum\limits_{\mu \leqq \nu} h_\mu(\sigma) & \text{if} \quad \sigma = \nu, \\ 0 & \text{if} \quad \sigma > \nu. \end{cases}$$

Then g_ν has length $\nu + 1 < \tau$, and $\sum_\nu g_\nu = g$. In view of the inequalities

$$\sum_{n<\omega} g_\nu(\sigma + n) = \sum_{n<\omega} h_\nu(\sigma + n) \geqq \sum_{\rho \geqq \sigma+n} g(\rho) \geqq \sum_{\rho \geqq \sigma+n} g_\nu(\rho)$$

for every σ with $\sigma + \omega < \nu + 1$, we infer that g_ν is $(\nu + 1)$-admissible. □

The existence theorem on direct sums of countable p-groups can now be established without difficulty.

Theorem 78.6 (Kolettis [1]). *Let g be a function from the ordinals $<\tau$ to the cardinals. There exists a direct sum A of countable reduced p-groups whose length is τ and which satisfies*

$$f_\sigma(A) = g(\sigma) \qquad \textit{for all} \quad \sigma < \tau$$

if and only if $\tau \leqq \omega_1$ and g is τ-admissible.

To establish necessity, we need to show that $f_\sigma(A)$ is τ-admissible. (i) is clear. To check (ii), note that for every σ and for every direct sum A of countable reduced p-groups, $|p^\sigma A| = \sum_{n<\omega} f_{\sigma+n}(A)$ holds whenever $p^\sigma A$ is infinite. On the other hand, for every σ and for every p-group A, $\sum_{\sigma \leqq \rho} f_\rho(A) \leqq |p^\sigma A|$, whence (ii) follows at once.

Conversely, suppose g is a τ-admissible function where $\tau \leqq \omega_1$. If τ is a limit ordinal, then from (78.5) we obtain τ_i-admissible functions g_i with $\tau_i < \tau$ such that $\sum_{i \in I} g_i = g$. If A_i is a direct sum of countable reduced p-groups whose Ulm–Kaplansky invariants are precisely $g_i(\sigma)$ with $\sigma < \tau_i$, then manifestly $A = \bigoplus_{i \in I} A_i$ will satisfy $f_\sigma(A) = g(\sigma)$ for every $\sigma < \tau$. If $\tau = \rho + n$, where ρ is a limit ordinal and n a positive integer, then let C, of length ρ, be a direct sum of countable p-groups such that $f_\sigma(C) = g(\sigma)$ for $\sigma < \rho$. To construct a group A such that

$$A/p^\rho A \cong C \qquad \text{and} \qquad p^\rho A \cong \bigoplus_{k=0}^{n-1} \bigoplus_{g(\rho+k)} Z(p^{k+1}),$$

all that we have to do is separate off a direct sum of as many as $g(\rho) + \cdots + g(\rho + n - 1)$ countable summands C_i of length ρ from C [by (ii), this is possible], and replace these C_i by groups A_i with $A_i/p^\rho A_i \cong C_i$, $p^\rho A_i \cong Z(p^{k+1})$ [in view of (76.2) such A_i do exist] such that

$$\bigoplus_i p^\rho A_i \cong \bigoplus_{k=0}^{n-1} \bigoplus_{g(\rho+k)} Z(p^{k+1}).$$

The direct sum A of all these A_i and the untouched components of C will have the required properties. □

EXERCISES

1. Every Ulm factor of a direct sum of countable p-groups of length ω_1 is at least of cardinality $\aleph_1$.
2. The Ulm factors of the group A' in Example 1 of **77** are direct sums of cyclic groups, but A' is not a direct sum of countable p-groups. [*Hint*: its basic subgroup is countable.]
3. Let $A = \bigoplus_{i \in I} A_i$ with countable reduced p-groups A_i and uncountable I. Show that, for any ordinal σ, the number of A_i of lengths $> \sigma$ is an invariant of A provided this number is uncountable.
4. For every cardinal $\mathfrak{n} > \aleph_0$, there exists a direct sum A of countable p-groups, of cardinality $\mathfrak{n}$, such that every direct sum of countable p-groups of cardinality $\mathfrak{n}$ is isomorphic to a summand of A. A is unique up to isomorphism.
5. The cardinality of the set of nonisomorphic direct sums of countable p-groups of an infinite cardinality $\aleph_\sigma$ is $(|\sigma| + \aleph_0)^{\aleph_1}$.
6. If A and C are direct sums of countable p-groups, each isomorphic to a summand of the other, then $A \cong C$.
7. If A is a direct sum of countable p-groups and $A \oplus A \cong C \oplus C$, then $A \cong C$.
8. (Kolettis [1], Charles [6]) Let A be a p-group such that A^1 is countable and $A/A^1 = A_0$ is a direct sum of cyclic groups. Show that A is a direct sum of countable p-groups. [*Hint*: there is a countable pure subgroup C' of A containing A^1; embed C' in C such that C/A^1 is a summand of A_0 and show that C is a summand of A.]
9. (Hill [8]) Prove (78.4) in the following way: Let $A = \bigoplus_{i \in I} A_i$ and $C = \bigoplus_{j \in J} C_j$ (A_i and C_j are countable reduced p-groups), and let ϕ be a height-preserving isomorphism between the socles $A[p]$ and $C[p]$. Let $\sim$ be the smallest equivalence relation on J for which $j \sim j'$ whenever both C_j and $C_{j'}$ intersect nontrivially the same $\phi A_i[p]$. Show that the equivalence classes are countable, group the C_j accordingly, apply ϕ^{-1} to get a grouping of the A_i, and verify that the direct sums in corresponding groups are isomorphic.

79. NICE SUBGROUPS

Prior to undertaking the general theory of those p-groups which can be characterized by their Ulm–Kaplansky invariants, we enter into the discussion of some important types of subgroups. This section and the next one are devoted to their more or less systematic study.

A close analysis of the Kaplansky–Mackey proof of Ulm's theorem [see (77.3)] led P. Hill to the discovery of a significant type of subgroup which embodies the properties of finite subgroups relevant to the proof. The abundance of such subgroups in a p-group, in the precise sense to be described in **82**, will guarantee that the group is uniquely determined by its Ulm–Kaplansky invariants.

A subgroup N of a p-group A is said to be *nice* if every nonzero coset of A mod N contains an element which is proper with respect to N. In other words, to every $a + N$ ($a \in A \backslash N$) there is an $x \in N$ such that

$$h_A^*(a + x) = h_{A/N}^*(a + N),$$

where the indices [which we can suppress from now on, there being no danger of ambiguity] show in which group heights are computed.

The following simple remark is useful in checking whether or not a subgroup is nice.

Lemma 79.1. *If N is a subgroup of the p-group A such that the cosets* mod *N whose heights are limit ordinals contain elements proper with respect to N, then N is a nice subgroup of A.*

Every element in a coset of height 0 has 0 height. Applying transfinite induction, suppose that cosets of heights $\leqq \sigma$ contain elements proper with respect to N, and let $h^*(a + N) = \sigma + 1$. Then $pb + N = a + N$ for some $b \in A$ with $h^*(b + N) = \sigma$, where $h^*(b) = \sigma$ may be assumed. Thus $h^*(pb) \geqq \sigma + 1$, and since strict inequality is impossible, pb is proper with respect to N. The case of limit ordinals is taken care of by hypothesis. □

Nice subgroups are characterized in the following lemma.

Lemma 79.2 (Hill [24]). *A subgroup N of a p-group A is nice exactly if*

$$(1) \qquad p^\sigma(A/N) = (p^\sigma A + N)/N \qquad \textit{for every ordinal} \quad \sigma.$$

Observe that $p^\sigma(A/N)$ is the set of all elements in A/N whose heights are $\geqq \sigma$, while $(p^\sigma A + N)/N$ is the image of $p^\sigma A$ under the canonical map $A \to A/N$. Hence the inclusion $\geqq$ is trivial for every subgroup N of A. Now, N is nice in A if and only if every coset in $p^\sigma(A/N)$ can be represented by an element of A whose height is likewise $\geqq \sigma$. Thus the reverse inclusion holds exactly if N is nice. □

In other words, (79.2) says that N is nice in A if and only if for the exact sequence $0 \to N \xrightarrow{\alpha} A \xrightarrow{\beta} C \to 0$ [where α is the inclusion map] and for every ordinal σ, the map $p^\sigma A \to p^\sigma C$ induced by β is surjective.

To explore this new concept, we mention the following properties; one encounters no difficulty in proving them.

(a) Direct summands are nice subgroups.

(b) Finite extensions of nice subgroups are nice.

(c) Subgroups closed in the p-adic topology of A are nice. [Note that a subgroup N is closed in the p-adic topology exactly if all cosets ($\neq N$) mod N are of finite height.]

(d) Let N_i be a subgroup of A_i for every $i \in I$. Then $\bigoplus_{i \in I} N_i$ is nice in $\bigoplus_{i \in I} A_i$ if and only if each N_i is nice in A_i.

(e) For every ordinal σ, $p^\sigma A$ is nice in A [cf. (37.1)].

(f) A nice subgroup N of A need not be nice in a subgroup of A containing N. [Note that in a divisible group, all subgroups are nice.]

(g) The property of being a nice subgroup is not transitive [see Ex. 9], but the following is true: a subgroup N of $p^\sigma A$ is nice in $p^\sigma A$ if and only if it is nice in A. This follows at once from the obvious relation $h_A^*(a) = \sigma + h_{p^\sigma A}^*(a)$ for all $a \in p^\sigma A$. Moreover, we have: if N is nice in A, then $N \cap p^\sigma A$ is nice in $p^\sigma A$.

The following lemma furnishes us with a frequently needed information about nice subgroups.

Lemma 79.3. *Let N and M be subgroups of a p-group A such that $N \leqq M \leqq A$. Then the following hold*:

(i) *if M is nice in A, then M/N is nice in A/N;*

(ii) *if N is nice in A and M/N is nice in A/N, then M is nice in A.*

To verify (i), let $h^*(a + M/N) = \sigma$. We induct on σ and assume, in view of (79.1), that σ is a limit ordinal and all cosets mod M/N of heights $< \sigma$ contain cosets mod N proper with respect to M/N. Then to every ordinal $\rho < \sigma$, there exists an $x_\rho \in M$ such that $h^*(a + x_\rho + N) > \rho$. Therefore, $h^*(a + M) \geqq \sigma$, and since the opposite inequality is trivially true, $h^*(a + M) = \sigma$. As M was supposed to be nice in A, $h^*(a + x) = \sigma$ for some $x \in M$. It follows that $h^*(a + x + N) = \sigma$, since $>$ is impossible.

Under the hypotheses of (ii), $h^*(a + M) = \sigma$ implies that there is a $y \in M$ satisfying $h^*(a + y + N) = \sigma$, and an $x \in N$ satisfying $h^*(a + y + x) = \sigma$. $\square$

From (79.3) we infer:

(h) If N is a nice subgroup of A, then under the natural correspondence between subgroups of A containing N and subgroups of A/N, nice subgroups correspond to nice subgroups.

EXERCISES

1. If M and N are subgroups of A and $a \in A$ is proper with respect to both M and N, then a is proper with respect to $M \cap N$.
2. (a) For a nice subgroup of a reduced p-group A, A/N is necessarily reduced.
   (b) A basic subgroup of a p-group A is nice exactly if it is the reduced part of A.
3. (a) In a separable p-group, a subgroup is nice if and only if it is closed in the p-adic topology.
   (b) In a torsion-complete p-group, a pure subgroup is nice exactly if it is a summand.
4. A pure subgroup need not be nice.
5. In a p-group A, all subgroups are nice if and only if A is the direct sum of a bounded group and a divisible group.
6. The union of an ascending chain of nice subgroups [direct summands] need not be nice. [*Hint*: Ex. 2(b).]
7. (a) Let A be as in the example of 35. Show that $A[p]$ is not a nice subgroup. [*Hint*: examine the coset $a_1 + A[p]$.]
   (b) Fully invariant subgroups need not be nice.
8. Give an example where $N < M < A$, M/N is nice in A/N, but M fails to be nice in A.
9. Let B an unbounded direct sum of cyclic p-groups and $\bar{B}$ its torsion-completion. Show that:
   (a) $\bar{B}[p]$ is nice in $\bar{B}$;
   (b) $B[p]$ is nice in $\bar{B}[p]$, but not in $\bar{B}$.
   Conclude that the property of being nice is not transitive.
10. (Hill [24]) Let σ be an arbitrary ordinal. For a subgroup N of A to be nice, it is necessary and sufficient that $N \cap p^\sigma A$ is nice in $p^\sigma A$ and $(N + p^\sigma A)/p^\sigma A$ is nice in $A/p^\sigma A$.
11. The extensions of a p-group A by a p-group C in which A is a nice subgroup form a subgroup of $\mathrm{Ext}(C, A)$. [*Hint*: the Baer sum of extensions.]

80. ISOTYPE AND BALANCED SUBGROUPS

Isotype subgroups have been introduced by Kulikov [3]. They refine the concept of purity in precisely the same spirit as generalized heights improve on the concept of height.

A subgroup C of a p-group A is said to be *isotype* if

$$p^\sigma C = C \cap p^\sigma A \qquad \text{for every ordinal} \quad \sigma.$$

It is evident that isotype subgroups are pure, and separable subgroups are isotype exactly if they are pure. In particular, basic subgroups are always isotype.

Observe that C is isotype in A exactly if the exactness of $0 \to C \to A \to A/C \to 0$ implies that the induced sequence

$$0 \to p^\sigma C \to p^\sigma A \to p^\sigma(A/C)$$

is exact, for every ordinal σ.

We begin with a few remarks about isotype subgroups.

(A) A subgroup C of A is isotype if and only if the generalized heights of its elements are the same in C as in A.

(B) C is isotype in A if and only if $p^\sigma C$ is pure in $p^\sigma A$ for every σ. In particular, direct summands are isotype.

(C) C is isotype in A if and only if

$$p^\sigma C[p] = C \cap p^\sigma A[p] \qquad \text{for every} \quad \sigma,$$

that is, the generalized heights of the elements of the socle of C are the same in C as in A. The argument of **26**(h) can be adapted to prove this.

(D) For every ordinal $\sigma, f_\sigma(C) \leqq f_\sigma(A)$ if C is isotype in A.

(E) An isotype subgroup C of A is isotype in every subgroup of A containing C.

(F) If $C \leqq B \leqq A$ such that C is isotype in B and B is isotype in A, then C is isotype in A.

(G) If C is pure in A and $(A/C)^1 = 0$, then C is isotype in A. [Observe that $A^1 \leqq C$.]

(H) If C is isotype in A, then for each ordinal σ, $C/p^\sigma C$ is isotype in $A/p^\sigma C$. Also, if $C/p^\sigma C$ is viewed as a subgroup of $A/p^\sigma A$ in the natural way, then $C/p^\sigma C$ is isotype in $A/p^\sigma A$. In fact, these follow at once from (37.1).

(I) The union of an ascending chain of isotype subgroups is likewise isotype. Hence isotypeness is an inductive property.

A simple way of constructing nontrivial isotype subgroups is exhibited by the next result [generalizing (27.7)].

Proposition 80.1 (Irwin and Walker [2]). *For every ordinal ρ, $p^\rho A$-high subgroups are isotype.*

Let C be $p^\rho A$-high in A. We use transfinite induction to establish the inclusion $C \cap p^\sigma A \leqq p^\sigma C$. This holds trivially for $\sigma = 0$, and it is true for a limit ordinal σ whenever it holds for all ordinals less than σ. So, it remains only to ascertain that if it is true for σ, then it also holds for $\sigma + 1$. It suffices to look at the case when $\sigma < \rho$. Let $c \in C \cap p^{\sigma+1}A$; thus $pa = c$ for some $a \in p^\sigma A$. A glance at (9.8) reveals that $a \in C \oplus p^\rho A$, i.e., $a = c' + a'$ for some $c' \in C$ and $a' \in p^\rho A$. Since $pc' + pa' = pa \in C$, we have $pa' = 0$ and $pc' = c$. Now $\sigma < \rho$ implies $c' = a - a' \in C \cap p^\sigma A = p^\sigma C$, whence $c \in p^{\sigma+1}C$, in fact. □

Next we proceed to combine the concepts of nice and isotype subgroups.

Call a subgroup B of a p-group A *balanced* if it is both nice and isotype. The meaning of *balanced-exact* sequences is obvious. In view of remarks made earlier, it is fairly evident that the definition of balanced subgroups can be rephrased in terms of exact sequences: B is a balanced subgroup of A if and only if the exact sequence

(1) $$0 \to B \to A \xrightarrow{\alpha} C \to 0$$

implies that, for every ordinal σ, the induced sequence

(2) $$0 \to p^\sigma B \to p^\sigma A \to p^\sigma C \to 0$$

is again exact. We thus have the following commutative diagram where all the columns and the first two rows are exact:

(3) $$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & p^\sigma B \arrow[r] \arrow[d] & p^\sigma A \arrow[r] \arrow[d] & p^\sigma C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & B/p^\sigma B \arrow[r] \arrow[d] & A/p^\sigma A \arrow[r] \arrow[d] & C/p^\sigma C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

By virtue of the 3×3-lemma, the third row is exact:

(4) $$0 \to B/p^\sigma B \to A/p^\sigma A \to C/p^\sigma C \to 0.$$

Actually, the exactness of (4) is equivalent to the exactness of (2).

From what has been said, it follows readily that if B is balanced in A, then $p^\sigma B$ is balanced in $p^\sigma A$ and $B/p^\sigma B$ is balanced in $A/p^\sigma A$, for every σ.

To grasp the idea involved better, let us point out, before entering into the discussion of balanced subgroups, that if one tries to verify the exactness of (2) by transfinite induction, then he notices at once that:

1. for the exactness at $p^\sigma B$, no assumption is needed on B;
2. for the exactness at $p^\sigma A$, the hypothesis that B was isotype is needed only when passing from $\sigma - 1$ to σ;
3. for the exactness at $p^\sigma C$, the assumption that B is nice in A is necessary only in passing from smaller ordinals to a limit ordinal σ.

In short, isotypeness is needed to force exactness at nonlimit ordinals, while niceness is needed at limit ordinals.

Example 1. A nontrivial example for a balanced subgroup is the following. Let $\bar{B}$ be an unbounded torsion-complete p-group and $0 \to C \to A \to \bar{B} \to 0$ a pure-projective resolution for $\bar{B}$. Then C is closed in the p-adic topology of A, hence it is nice in A. Furthermore, by (G) it is isotype, too. Thus it is balanced, but it fails to be a summand of A.

Example 2. Let C be the group of Example 3 in **77**, and X the direct sum of continuously many copies of the group A of Example in **35**. For every $c \in C[p]$, we take a homomorphism ϕ_c mapping a copy A_c of A into C such that some $a_c \in A_c$ of height $h^*(c)$ is sent upon c. In this way, an epimorphism ϕ: $X \to C$ is obtained whose kernel is balanced [to see this, follow (30.1)]. Ker ϕ is not a summand, since C is not a direct sum of countable p-groups.

From the discussions in **81** and **82** it will follow that every p-group C, which is not totally projective, can be embedded in a nonsplitting exact sequence $0 \to B \to A \to C \to 0$ where A is totally projective and B is balanced in A. This will justify a claim that examples for balanced subgroups are abundant.

Next, we list a few useful properties of balanced subgroups.

(a) Direct summands are always balanced subgroups.

(b) *If B is balanced in A and if $B < G < A$, then B is balanced in G.*

By (E), it suffices to verify that B is nice in G. Assume $g \in G$ satisfies $h^*_{G/B}(g + B) = \sigma$ for some limit ordinal σ, and that cosets of G mod B of heights $<\sigma$ contain elements of G proper with respect to B. Then for every $\rho < \sigma$, there is a $b_\rho \in B$ such that $g + b_\rho \in p^\rho G$. If $b \in B$ satisfies $g + b \in p^\sigma A$, then $b - b_\rho \in B \cap p^\rho A = p^\rho B$ and so $g + b = (g + b_\rho) + (b - b_\rho) \in p^\rho G$, whence $h^*_G(g + b) = \sigma$.

(c) *If B is balanced in A and $C < B$, then B/C is balanced in A/C.*

Because of (79.3), we need only show that B/C is isotype in A/C. If $b + C \in p^\sigma(A/C)$ for $b \in B$, then some $a \in p^\sigma A$ and $c \in C$ satisfy $b + c = a \in B \cap p^\sigma A = p^\sigma B$. Hence $b + C \in p^\sigma B + C \leqq p^\sigma(B/C)$ and B/C is isotype in A/C.

(d) *If $C < B < A$, where C is balanced in A and B/C is balanced in A/C, then B is balanced in A.*

To check isotypeness, let $b \in B \cap p^\sigma A$. Then $b + C \in B/C \cap p^\sigma(A/C) =$

$p^\sigma(B/C)$ implies that $b - b_1 \in C$ holds for some $b_1 \in p^\sigma B$. Therefore, $b - b_1 \in C \cap p^\sigma A = p^\sigma C$ and $b \in p^\sigma B$.

(e) *If $C < B < A$, where C is balanced in B and B in A, then C is balanced in A.*

In view of (F), we only prove that C is nice in A. Let $a \in A$ satisfy $h^*(a + C) = \sigma$. If $a \in B$, then (c) implies $a + C$ has height σ in B/C; thus $h^*(a + c) = \sigma$ for some $c \in C$. If $a \notin B$, then $h^*(a + B) \geqq \sigma$ and $h^*(a + b) = h^*(a + B)$ for some $b \in B$. Clearly, $h^*(-a - b + a + C) \geqq \sigma$, whence $h^*(-b + c) \geqq \sigma$ and $h^*(a + c) = h^*(a + b - b + c) \geqq \sigma$ for some $c \in C$. Consequently, $h^*(a + c) = \sigma$, and the assertion follows.

The most frequently used characterizations of balanced subgroups are collected in the following result.

Proposition 80.2. *For an exact sequence* (1), *the following conditions are equivalent:*

(i) *B is a balanced subgroup of A;*
(ii) *for every ordinal σ,* (2) *is exact;*
(iii) *for every ordinal σ,* (4) *is exact;*
(iv) *$\alpha(p^\sigma A[p]) = p^\sigma C[p]$ holds for every σ.*

The equivalence of (i)–(iii) has already been indicated above. So suppose (ii) and pick a $c \in p^\sigma C[p]$. There is an $a \in p^\sigma A$ for which $\alpha a = c$. Here $pa \in p^{\sigma+1}A \cap B = p^{\sigma+1}B$, thus $pb = pa$ for a suitable $b \in p^\sigma B$. Evidently, $a - b \in p^\sigma A[p]$ is mapped by α upon c. Hence $p^\sigma C[p] \leqq \alpha(p^\sigma A[p])$; the inclusion in the opposite direction is trivial.

Supposing (iv), the first stage is the verification of the inclusion $p^\sigma C[p^k] \leqq \alpha(p^\sigma A)$ for $k = 1, 2, \cdots$. In case $k = 1$, this follows from (iv), so assume it is true for every σ and for $k - 1$. If $c \in p^\sigma C$ is of order p^k ($k > 1$), then there is an $x \in A$ such that $\alpha x = c$, and by induction hypothesis, some $y \in p^{\sigma+1}A$ satisfies $\alpha y = pc \in p^{\sigma+1}C[p^{k-1}]$. Choose an $a_0 \in p^\sigma A$ with $pa_0 = y$, and notice that $\alpha(x - a_0) = c - \alpha a_0 \in p^\sigma C[p]$. Consequently, $c - \alpha a_0 = \alpha a_1$ for some $a_1 \in p^\sigma A$, and $c = \alpha a$ with $a = a_0 + a_1 \in p^\sigma A$. Hence (2) is exact at $p^\sigma C$.

To conclude the proof, it remains to check $B \cap p^\sigma A = p^\sigma B$. This being true for $\sigma = 0$, we suppose it holds for σ and want to verify it for $\sigma + 1$. To this end, pick a $b \in B \cap p^{\sigma+1}A$ and an $a_0 \in p^\sigma A$ with $pa_0 = b$. Since $\alpha a_0 \in p^\sigma C\,[p]$, we can find an $a_1 \in p^\sigma A[p]$ such that $\alpha a_1 = \alpha a_0$. Thus $a = a_0 - a_1 \in B \cap p^\sigma A = p^\sigma B$, whence $b = pa \in p^{\sigma+1}B$. The induction step for limit ordinals is evident. □

The next lemma is one of the crucial steps in the proof of the important theorem (81.9).

Lemma 80.3. *Given a commutative diagram*

$$\begin{tikzcd} & 0 \arrow[r] & N \arrow[r] \arrow[d, "\psi"] & G \arrow[d, "\phi"] & \\ 0 \arrow[r] & B \arrow[r] & A \arrow[r, "\alpha"] & C \arrow[r] & 0 \end{tikzcd} \tag{5}$$

where B is balanced in A and the two rows are exact, suppose that ψ does not decrease heights in G. If $g \in G$ is proper with respect to N and $pg \in N$, then ψ can be extended to a map

$$\psi^*: \langle N, g\rangle \to A,$$

such that $\alpha\psi^* g = \phi g$ and ψ^* does not decrease heights.

If $h^*(g) = \sigma$, then some $a \in p^\sigma A$ satisfies $\alpha a = \phi g$. Visibly, $pa - \psi(pg) \in B \cap p^{\sigma+1}A = p^{\sigma+1}B$, whence the existence of a $b \in p^\sigma B$ with $pb = pa - \psi(pg)$ is immediate. To define ψ^*, we set $\psi^* g = a - b \in A$. It is readily seen that this gives rise to a homomorphism of $\langle N, g\rangle$ into A satisfying $\alpha\psi^* g = \alpha a = \phi g$. Obviously, $h^*(a - b) \geqq \sigma = h^*(g)$, and to check that ψ^* does not decrease heights, it is enough to show that $h^*(g + x) \leqq h^*(a - b + \psi x)$ for all $x \in N$. Since g was proper with respect to N, $h^*(g + x) = \min(h^*(g), h^*(x))$. On the other hand, $h^*(a - b + \psi x) \geqq \min(h^*(a - b), h^*(\psi x)) \geqq \min(\sigma, h^*(x))$ which completes the proof. □

If N is a nice subgroup of finite [or countable] index in G and if (5) is as stated in (80.3), then a repeated application of (80.3) yields an extension $\bar{\psi}: G \to A$ of ψ satisfying $\alpha\bar{\psi} = \phi$.

EXERCISES

1. (Kulikov [3]) A subgroup C of A is isotype in A if and only if, for some ordinal ρ, $p^\rho C$ is isotype in $p^\rho A$ and $C/p^\rho C$ is isotype in $A/p^\rho C$.
2. A subgroup C of A is isotype in A exactly if, for every ordinal σ, $p^{\sigma+1}C = p^\sigma C \cap p^{\sigma+1}A$.
3. (Irwin and Walker [2]) Every subgroup of a p-group A whose elements are of heights $\leqq \sigma$ can be embedded in an isotype subgroup of A whose elements are again of heights $\leqq \sigma$.
4. A p-group A of length τ has an isotype subgroup of length $\sigma \leqq \tau$ if and only if either σ is a limit ordinal or $f_{\sigma-1}(A) \neq 0$.
5. Let A be a p-group of length $\leqq \omega_1$. Show that every countable subgroup of A can be embedded in a countable isotype subgroup. [*Hint*: as in (26.2).]
6. Let A be a group generated by $a_0, a_1, \cdots, a_n, \cdots$ and $b_0, b_1, \cdots, b_n, \cdots$ subject to the relations

$$pa_0 = b_0, \quad pb_0 = 0, \qquad p^n a_n = a_0, \quad p^n b_n = b_0 \qquad \text{for every } n.$$

Show that $B = \langle b_0, b_1, \cdots, b_n, \cdots \rangle$ is pure in A, but it is not isotype.

7. (Irwin and Walker [2]) Let A be a p-group such that $|p^{\omega_1}A| = \aleph_0$. Prove that A has a pure subgroup containing $p^{\omega_1}A$ which is not isotype.
8. If B is a balanced subgroup of A, then (1) implies that the induced sequence $0 \to D_B \to D_A \to D_C \to 0$ of maximal divisible subgroups is again exact.
9. For a basic subgroup of a p-group to be balanced, it is necessary and sufficient that it be a summand.
10. If B is a balanced subgroup of A and if $\langle a + B \rangle$ is a cyclic summand of A/B, then $\langle B, a \rangle$ is a balanced subgroup of A.
11. (a) A balanced subgroup of countable index in A is a summand of A. [*Hint*: apply the remark after (80.3) to the case $N = 0$, $C = G$, $\phi = 1_C$.] (b) In a countable p-group, a subgroup is balanced exactly if it is a summand.
12. Let N be a nice subgroup in a p-group G and ψ a homomorphism of N into a p-group A which does not decrease heights in G. If $N < M \leqq G$ such that M/N is countable, then ψ extends to a homomorphism of M into A which does not decrease heights either. [*Hint*: successive extension, for $|M : N| = p$ extend as in (80.3).]
13. If B is a balanced subgroup of G, then $\mathrm{Tor}(B, X)$ is a balanced subgroup of $\mathrm{Tor}(G, X)$ for every group X. [*Hint*: (63.2) and (64.2).]

81. p-GROUPS WITH NICE COMPOSITION SERIES

Several attempts have made been to generalize Ulm's theorem to various p-groups which are not necessarily direct sums of countable p-groups. Evidently, the problem—which appears very naturally—consists in finding a broadest class of p-groups in which the individual members are distinguishable *via* their Ulm–Kaplansky invariants. In recent years, a great amount of work has been done in this direction, and as a result, a rich theory is available to us; and above all: the mentioned problem can fully be answered.

The theory can be approached from different sides, each emphasizing some specific aspect of the class in question. Our starting point is close to the one adapted by P. Hill. We then proceed to obtain various equivalent characterizations of the class and leave the proof of the generalized Ulm theorem to the end of our discussion.

If one examines the details of the proof of Ulm's theorem (77.3), he can then easily notice that the most essential feature—though not quite sufficient for the proof—was that every height-preserving isomorphism of a finite subgroup could be extended a step further, eventually reaching the whole group. This extensibility property is enjoyed by all nice subgroups, as we learned in (77.1). Bearing this in mind, we wish to focus our attention on groups which can be reached, possibly transfinitely, in a similar fashion through nice subgroups.

Let A be a p-group, and

(1) $$0 = N_0 < N_1 < \cdots < N_\lambda < \cdots < N_\mu = A$$

a well-ordered strictly ascending chain of subgroups of A such that:

(a) $N_0 = 0$ and $N_\mu = A$;
(b) each N_λ is a nice subgroup of A;
(c) $|N_{\lambda+1} : N_\lambda| = p$ for every $\lambda < \mu$;
(d) $N_\lambda = \bigcup_{\kappa<\lambda} N_\kappa$ if λ is a limit ordinal.

Such a chain will be called a *nice composition series* for A.

We hasten to show that the countable p-groups and their direct sums belong to the class of groups with nice composition series:

Lemma 81.1. *Direct sums of countable p-groups have nice composition series.*

This is fairly obvious upon a little reflection, although to write out the proof in all of its details is somewhat cumbersome.

To start off, let A be a countable p-group. It can oviously be obtained as a union of an ascending chain, of type ω, of its finite subgroups. Intercalating subgroups between adjacent terms, if necessary, one obtains an ascending chain $0 = N_0 < N_1 < \cdots < N_k < N_{k+1} < \cdots < A$, where $\bigcup_{k<\omega} N_k = N_\omega = A$ and N_k is of index p in N_{k+1} for every k. Since finite subgroups are nice, this will, in fact, be a nice composition series for A.

If A is a direct sum of countable p-groups, then we well-order the index set and write $A = \bigoplus_{\sigma<\tau} A_\sigma$ with countable A_σ, for some ordinal τ. If $0 = N_{\sigma 0} < N_{\sigma 1} < \cdots < N_{\sigma\omega} = A_\sigma$ is a nice composition series for A_σ, then

$$0 = N_{00} < N_{01} < \cdots < A_0 < A_0 \oplus N_{11} < \cdots < A_0 \oplus A_1 < \cdots$$
$$< \bigoplus_{\rho<\sigma} A_\rho \oplus N_{\sigma k} < \cdots < \bigoplus_{\sigma<\tau} A_\sigma = A$$

will be one for A. □

We will find it convenient to list the following properties which are either immediately verified or follow from the proof of (81.1).

(A) If B is a balanced subgroup of A, and both B and A/B possess nice composition series, then so does A. The same holds if $B = p^\sigma A$ for some ordinal σ.

(B) For A to have a nice composition series, it is enough to ensure the existence of a well-ordered ascending chain (1) where (a), (b), and (d) are satisfied, while (c) is replaced by the condition that $N_{\lambda+1}/N_\lambda$ is countable. This is a simple consequence of **79**(b).

(C) Direct sums of p-groups with nice composition series have again nice composition series.

A careful analysis can convince the reader that the definition of nice composition series has been phrased in a somewhat stronger form than it will be needed in the proofs of the following theorems. As a matter of fact, condition (b) can be replaced by a condition requiring only that the cosets $a + N_\lambda$ with $a \in N_{\lambda+1} \backslash N_\lambda$ contain elements proper with respect to N_λ for each λ [moreover, it suffices to suppose this for just one out of the $p - 1$ cosets].

The next theorem and its corollaries point out remarkable features ot p-groups with nice composition series.

Theorem 81.2. *Let A and C be reduced p-groups, and let ϕ be a height-preserving isomorphism between a nice subgroup G of A and a subgroup H of C. Suppose that:*

(α) *A/G has a nice composition series;*

(β) *the relative Ulm–Kaplansky invariants satisfy $f_\sigma(A, G) \leqq f_\sigma(C, H)$ for every σ.*

Then ϕ extends to a height-preserving isomorphism ϕ^* of A into C.

For each σ, we choose an arbitrary monomorphism α_σ: $p^\sigma A[p]/G(\sigma) \to p^\sigma C[p]/H(\sigma)$. We also select a nice composition series from G to A:

$$G = N_0 < N_1 < \cdots < N_\lambda < \cdots < N_\mu = A.$$

Consider the set of all pairs $(N_\lambda, \phi_\lambda)$ such that:

1. ϕ_λ is a height-preserving isomorphism of N_λ with a subgroup M_λ of C;
2. the restriction of ϕ_λ to G is equal to ϕ;
3. for each σ, α_σ induces an isomorphism $N_\lambda(\sigma)/G(\sigma) \to M_\lambda(\sigma)/H(\sigma)$.

This set is partially ordered in the obvious way. By Zorn's lemma, there is a maximal pair (N_ν, ϕ_ν) in this set. From condition 3 we infer that $f_\sigma(A, N_\nu) \leqq f_\sigma(C, M_\nu)$ for every σ. Because of condition 1, we are now in the situation of (77.1), so ϕ_ν can be extended to a height-preserving isomorphism $\phi_{\nu+1}$ of $N_{\nu+1}$ with a subgroup $M_{\nu+1}$ of C, such that condition 3 is still satisfied. This would contradict the choice of (N_ν, ϕ_ν), unless $N_\nu = A$. □

Corollary 81.3. *Let A be a reduced p-group with a nice composition series. A is isomorphic to an isotype subgroup of a p-group C if and only if*

$$f_\sigma(A) \leqq f_\sigma(C) \qquad \textit{for all} \quad \sigma.$$

That the condition is necessary is clear from **80**(D). To see its sufficiency, simply apply (81.2) with $G = 0 = H$ to obtain an embedding of A in C with preservation of generalized heights. □

Corollary 81.4. *Let A and C be p-groups, and η a homomorphism of a nice subgroup G of A into C which does not decrease heights. If A/G has a nice composition series, then η can be extended to a homomorphism $\eta^*: A \to C$ which does not decrease heights either.*

Note that η induces a height-preserving automorphism $\phi: G \oplus C \to G \oplus C$ *via* $(g, c) \mapsto (g, c + \eta g)$. Since $G \oplus C$ is nice in $A \oplus C$ and $(A \oplus C)/(G \oplus C)$ has a nice composition series, (81.2) is applicable, and we arrive at a height-preserving isomorphism ϕ^* of $A \oplus C$ into [moreover, onto] itself. Now the composite of the natural injection $A \to A \oplus C$, ϕ^*, and the natural projection $A \oplus C \to C$ will produce an η^* as required. □

There is a stronger version of the condition of existence of nice composition series. We say that the p-group A has a *nice system* [Hill [17] calls this condition *the third axiom of countability*] if A has a system $\mathbf{N}$ of nice subgroups such that:

(a′) $0 \in \mathbf{N}$;

(b′) if $\{N_i\}_{i \in I}$ is any subset in $\mathbf{N}$, then $\sum_{i \in I} N_i \in \mathbf{N}$;

(c′) given any $N \in \mathbf{N}$ and a countable subset X of A, there exists an $M \in \mathbf{N}$ satisfying

$$\langle N, X\rangle \leqq M \qquad \text{and} \qquad |M/N| \leqq \aleph_0.$$

It is an easy exercise in transfinite arithmetic to verify the existence of nice composition series in groups having a nice system. Surprisingly, all groups with nice composition series have nice systems—this is not so easy to prove, but will follow from (81.9).

If A is a countable p-group, then $\{0, A\}$ is a nice system in A. If A is a direct sum of countable p-groups A_i $(i \in I)$, then the subgroups $N_J = \bigoplus_{i \in J} A_i$, for all subsets J of I, form a nice system in A.

The following lemma will be needed.

Lemma 81.5 (Hill [24]). *The class of groups with nice systems is closed under direct sums and direct summands.*

Let $\mathbf{N}_i$ be a system of nice subgroups of A_i $(i \in I)$ satisfying conditions (a′)–(c′), and let $\mathbf{N}$ be the set of all subgroups N of $A = \bigoplus_{i \in I} A_i$ which are of the form $N = \bigoplus_{i \in I} N_i$ with $N_i \in \mathbf{N}_i$. By **79**(d), each N is nice in A. Trivially, (a′) and (b′) are fulfilled for this $\mathbf{N}$. Let X be a countable subset in A, and let $N \in \mathbf{N}$. There is a countable subset J of I such that $X \subseteq \bigoplus_{i \in J} A_i$, and for each $i \in J$, there is a subgroup $M_i \in \mathbf{N}_i$ such that $N_i \leqq M_i$, M_i/N_i is countable, and M_i contains the projection of X in A_i. Putting $M_i = N_i$ for $i \in I \backslash J$, $M = \bigoplus_{i \in I} M_i$ is as required by (c′).

Next, let $A = B \oplus C$ and let $\mathbf{N}$ be a collection of nice subgroups in A satisfying (a′)–(c′). Define $\mathbf{N}'$ to consist of all $N' \leqq B$ such that, for some

$N \in \mathbf{N}$, $N = N' \oplus (N \cap C)$ holds. Again by **79**(d), N' is nice in B. That $\mathbf{N}'$ satisfies conditions (a′) and (b′) is fairly obvious. Suppose X is a countable subset of B and $N' \in \mathbf{N}'$; let $N \in \mathbf{N}$ satisfy $N = N' \oplus (N \cap C)$. Choose $N_1 \in \mathbf{N}$ so as to have $\langle N, X\rangle) \leqq N_1$ and $|N_1/N| \leqq \aleph_0$. There are subgroups B_1, C_1 such that $N' \leqq B_1 \leqq B$ and $N \cap C \leqq C_1 \leqq C$ with $|B_1/N'| \leqq \aleph_0$ and $|C_1/(N \cap C)| \leqq \aleph_0$ satisfying $N_1 \leqq B_1 \oplus C_1$. One can find an N_2 in $\mathbf{N}$ with $\langle N, B_1\rangle \leqq N_2$ and $|N_2/N| \leqq \aleph_0$. Proceeding in a similar fashion, we infer that there are sequences $N_n \in \mathbf{N}$, $B_n \leqq B$, and $C_n \leqq C$ ($n = 1, 2, \cdots$) with the following properties:

$$N_n \leqq B_n \oplus C_n, \qquad \langle N, B_n\rangle \leqq N_{n+1}, \qquad \text{and} \qquad |N_n/N| \leqq \aleph_0.$$

Clearly, $M = \bigcup_{n<\omega} N_n \in \mathbf{N}$ satisfies $M = (M \cap B) \oplus (M \cap C)$, whence $M \cap B \in \mathbf{N}'$ follows. To complete the proof, notice that $\langle N', X\rangle \leqq M \cap B$ and $|(M \cap B)/N'| \leqq |M/N| \leqq \aleph_0$.□

So far we have no examples at hand for p-groups with nice systems other than direct sums of countable p-groups; as a matter of fact, (81.5) does not produce more examples. In order to ascertain that many more p-groups have nice systems, and the lengths of such groups are not bounded by any ordinal, we shall build groups by an inductive process. As we shall see, they will play a pertinent role in the theory.

Starting with the group $H_0 = 0$, we are going to construct, for every ordinal σ, a p-group H_σ such that:

(i) H_σ is of length σ;
(ii) $p^\sigma H_{\sigma+1}$ is cyclic of order p and $H_{\sigma+1}/p^\sigma H_{\sigma+1} \cong H_\sigma$;
(iii) $H_\sigma = \bigoplus_{\rho<\sigma} H_\rho$ if σ is a limit ordinal;
(iv) every Ulm–Kaplansky invariant of H_σ is $\leqq |\sigma|$.

Since $H_0 = 0$, from (ii) it follows that for any integer $n > 0$, H_n is a cyclic group of order p^n. Furthermore, (iii) shows that $H_\omega = \bigoplus_{n<\omega} H_n$. In general, (iii) tells us how to obtain H_σ for limit ordinals σ provided all H_ρ with $\rho < \sigma$ have already been defined.

In order to define H_σ for isolated ordinals σ, two cases will be distinguished, according as σ is not or is a successor of a limit ordinal. Suppose first that $H_{\sigma+1}$ is known; then the construction of $H_{\sigma+2}$ is easy. Let C be a cyclic group of order p^2 and γ an epimorphism of C onto a cyclic group C' of order p. The induced map $\gamma_*\colon \mathrm{Ext}(H_\sigma, C) \to \mathrm{Ext}(H_\sigma, C')$ is epic, so there exists a p-group $H_{\sigma+2}$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & H_{\sigma+2} & \longrightarrow & H_\sigma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\alpha} & & \| & & \\ 0 & \longrightarrow & C' = p^\sigma H_{\sigma+1} & \longrightarrow & H_{\sigma+1} & \longrightarrow & H_\sigma & \longrightarrow & 0 \end{array} \tag{2}$$

commutative and its first row exact. Here $p^\sigma H_\sigma = 0$ implies $p^\sigma H_{\sigma+2} \leqq C$. But $p^\sigma H_{\sigma+2} \leqq pC$ would imply $p^\sigma H_{\sigma+1} = 0$, a contradiction. Hence $H_{\sigma+2}$ satisfies both (i) and (ii).

It remains to define $H_{\sigma+1}$ for a limit ordinal σ. Note that $H_\sigma \cong \bigoplus_{\rho<\sigma} (H_{\rho+1}/p^\rho H_{\rho+1})$, where all $p^\rho H_{\rho+1}$ are cyclic of order p. Therefore, using the codiagonal map ∇: $\bigoplus_\rho p^\rho H_{\rho+1} \to C'$, one can define $H_{\sigma+1}$ as a pushout to obtain a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{\rho<\sigma} p^\rho H_{\rho+1} \arrow[r] \arrow[d, "\nabla"] & \bigoplus\limits_{\rho<\sigma} H_{\rho+1} \arrow[r] \arrow[d] & H_\sigma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C' \arrow[r] & H_{\sigma+1} \arrow[r] & H_\sigma \arrow[r] & 0
\end{tikzcd} \tag{3}
$$

with exact rows. Since ∇ is epic, $C' \leqq p^\sigma H_{\sigma+1}$, while the converse inclusion follows from $p^\sigma H_\sigma = 0$. Consequently, $H_{\sigma+1}$ satisfies (i) and (ii).

Now (iv) follows readily from the definition, *via* a trivial transfinite induction.

Remark. For the orientation of the reader, let us point out at once that the group $H_{\sigma+2}$ as defined above depends on the choice of γ. But different choices of γ lead to isomorphic groups $H_{\sigma+2}$, since another choice of γ simply means that γ is preceded by an automorphism of C which [being a multiplication by an integer prime to p] clearly extends to an isomorphism between the two $H_{\sigma+2}$. Similar remarks apply to the construction of $H_{\sigma+1}$ in (3). Consequently, *the groups H_σ as defined above, are unique up to isomorphism.*

It is easy to recognize $H_{\omega+1}$ as an old acquaintance of ours. In fact, $H_{\omega+1}$ is isomorphic to the group A of the example in **35**, due to H. Prüfer [this being a simplest example for a nonseparable reduced p-group]. Because of this resemblance, the group H_σ as defined above will be referred to in the sequel as the *(generalized) Prüfer group of length* σ. Nunke [7] was the first to call attention to their importance.

We can now verify that these groups do have nice systems.

Lemma 81.6. *The generalized Prüfer groups have nice systems.*

Proof by transfinite induction. The limit case is obvious from (81.5), because H_σ is then the direct sum of H_ρ with $\rho < \sigma$. If H_σ has a system $\mathbf{N}$ of nice subgroups satisfying (a′)–(c′), then the preimages of the subgroups in $\mathbf{N}$ under the epimorphism $H_{\sigma+1} \to H_\sigma$ whose kernel $p^\sigma H_{\sigma+1}$ is nice in $H_{\sigma+1}$ render a system in $H_{\sigma+1}$ of the desired type. □

The next two results point out important features of the Prüfer groups H_σ.

Lemma 81.7 (Nunke [7]). *If A is a p-group and if $a \in p^\sigma A[p^n]$, then there is a homomorphism*

$$\phi\colon H_{\sigma+n} \to A \qquad \textit{such that} \quad \phi h = a,$$

where h is a generator of $p^\sigma H_{\sigma+n}$.

The correspondence $h \mapsto a$ gives rise to a homomorphism $\eta\colon p^\sigma H_{\sigma+n} \to \langle a \rangle$ which does not decrease heights. Since $p^\sigma H_{\sigma+n}$ is nice in $H_{\sigma+n}$ such that the corresponding quotient group has a nice composition series, by (81.4) η extends to a homomorphism ϕ as desired. $\square$

Lemma 81.8. *Suppose A is a reduced p-group of length τ. There exist a direct sum H of generalized Prüfer groups of lengths $\leqq \tau$ and an epimorphism $\phi\colon H \to A$ whose kernel is a balanced subgroup of H.*

For every nonzero $a \in p^\sigma A[p^n]$ $(\sigma + n < \tau)$ we select a generalized Prüfer group $H_a \cong H_{\sigma+n}$ and a homomorphism $\phi_a\colon H_a \to A$ as stated in (81.7). Setting $H = \bigoplus_{a \in A} H_a$, we obtain an epimorphism $\phi = \nabla(\bigoplus \phi_a)$ of H onto A. Evidently, $\phi(p^\sigma H[p]) = p^\sigma A[p]$ for every σ, thus (80.2) implies that Ker ϕ is balanced in H. $\square$

Now the principal result of this section can easily be proved.

Theorem 81.9. *The following conditions on a reduced p-group A are equivalent:*

(α) *A has a nice system;*

(β) *A has a nice composition series;*

(γ) *A has the projective property relative to all balanced-exact sequences $0 \to B \to G \to C \to 0$ of p-groups;*

(δ) *A is a summand of a direct sum of generalized Prüfer groups.*

We have noticed that the implication $(\alpha) \Rightarrow (\beta)$ is trivial.

Supposing (β), we want to prove (γ). Starting with a diagram

$$\begin{tikzcd} & & & A \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 0 \arrow[r] & B \arrow[r] & G \arrow[r, "\alpha"] & C \arrow[r] & 0 \end{tikzcd}$$

where B is balanced in G, we select a nice composition series (1) for A. The map ψ can be defined successively, *via* transfinite recursion, for the N_λ: at limit ordinals we take unions, while passing from N_λ to $N_{\lambda+1}$, we simply refer to (80.3).

Next assume (γ) and prove (δ). From (81.8) we know that to every p-group A, there are a direct sum H of generalized Prüfer groups and a balanced-exact sequence $0 \to B \to H \xrightarrow{\phi} A \to 0$. Applying ($\gamma$), it follows that $\phi\psi = 1_A$ for a suitable homomorphism $\psi\colon A \to H$, i.e., A is isomorphic to a summand of H.

Finally, we derive (α) from (δ). In view of (81.5) and (81.6), there is nothing to prove. $\square$

A straightforward transfinite induction can convince us that the Ulm factors of the generalized Prüfer groups are direct sums of cyclic groups.

Since this property is inherited by direct sums and direct summands, we obtain at once:

Proposition 81.10. *All the Ulm factors of a p-group A satisfying one* [*and hence all*] *of the conditions of* (81.9) *are direct sums of cyclic groups.* □

For the structure theorem on groups covered by (81.9), we refer to (83.3).

It is natural to ask: *which reduced p-groups have the injective property relative to the class of balanced-exact sequences of p-groups?* Unexpectedly, this question does not lead to any new class: the answer is given by the class of torsion-complete p-groups [Griffith [9]].

To prove this, notice that all pure-exact sequences $0 \to A \to G \to C \to 0$ of p-groups with $C^1 = 0$ are necessarily balanced-exact. Therefore, if A is injective relative to all balanced-exact sequences of p-groups, then $\text{Pext}(C, A) = 0$ for all separable p-groups C. The choice $C = \bar{B}$ implies at once that A must be torsion-complete, as was shown in **68** [see Remark].

EXERCISES

1.* (a) Using (82.4) show that an unbounded torsion-complete p-group can not have any nice composition series.
(b) The torsion part of a direct product of infinitely many unbounded reduced p-groups with nice composition series never has such a series.
2. Let A be a p-group and n an integer. Prove that $p^n A$ has a nice composition series exactly if A has one.
3. Let the p-group A have a nice system. If B is a finite subgroup and C is of finite index in A, then both A/B and C have nice systems.
4. (a) For every infinite ordinal σ, the generalized Prüfer group H_σ of length σ has cardinality $|\sigma|$.
(b) Show that $f_\rho(H_\sigma) \neq 0$ for all $\rho < \sigma$.
5. Prove that the Prüfer groups H_σ are determined, up to isomorphism, by (ii) and (iii).
6. Let σ be an isolated ordinal. Any summand of a direct sum of generalized Prüfer groups of lengths $\leqq \omega_\sigma$ is a direct sum of groups of cardinalities $< \aleph_\sigma$. [*Hint*: (9.10).]
7. (a) For every ordinal σ, there exists an extension G_σ of Z by H_σ such that
$$p^\sigma G_\sigma \cong Z \qquad \text{and} \qquad G_\sigma / p^{\sigma+n} G_\sigma \cong H_{\sigma+n} \qquad \text{for every integer} \quad n \geqq 0.$$
(b) Show that $G_{\sigma+n} \cong G_\sigma$, for every ordinal σ and integer n.
8. (Nunke [5]) If G_σ is defined as in Ex. 7, then for any group A,
$$\text{Hom}(G_\sigma, A) \cong p^\sigma A.$$
9. For every p-group A and every ordinal $\sigma \geqq \omega$, there is an epimorphism $\partial\colon \text{Tor}(H_\sigma, A) \to A$. [*Hint*: apply (63.1) to $0 \to Z \to G_\sigma \to H_\sigma \to 0$, noting that $G_\sigma \otimes A \to H_\sigma \otimes A$ must be an isomorphism.]

10. Verify (81.4) by making use of (80.3) rather than (81.2).
11. (a) Show that a p-group has a nice composition series [a nice system] exactly if its reduced part has one.
    (b) Divisible p-groups have the projective property relative to all balanced-exact sequences of p-groups.
    (c) Extend (81.9) to nonreduced p-groups.
12. (R. B. Warfield) A short exact esquence of p-groups relative to which the groups of (81.9) have the projective property is balanced-exact.
13. Show that if A is a reduced p-group which contains a subgroup C with a nice system such that $A/C = Z(p^\infty)$, then A, too, has a nice system.

82. TOTALLY PROJECTIVE p-GROUPS

A most important type of p-group has been discovered by Nunke [5] *via* homological considerations. It was observed a few years later by P. Hill that these p-groups are distinguishable through their Ulm–Kaplansky invariants, and they form a largest such class.

Let σ denote an ordinal. A p-group A is called *p^σ-projective* if

$$p^\sigma \operatorname{Ext}(A, C) = 0 \qquad \text{for all groups} \quad C.$$

A reduced p-group A is said to be *totally projective* if

$$p^\sigma \operatorname{Ext}(A/p^\sigma A, C) = 0 \qquad \text{for all ordinals} \quad \sigma \quad \text{and groups} \quad C.$$

In other words, A is totally projective if and only if $A/p^\sigma A$ is p^σ-projective for every ordinal σ.

For technical purposes, we record here a few remarks which are immediate consequences of the definitions.

(A) The class of p^σ-projective [totally projective] p-groups is closed under taking arbitrary direct sums and summands.

(B) A totally projective p-group of length $\leqq \sigma$ is p^σ-projective.

(C) A p-group A is p^n-projective, n a nonnegative integer, exactly if $p^n A = 0$ [cf. **52**(D)].

(D) Direct sums of cyclic p-groups are totally projective and p^ω-projective [cf. (53.3)].

(E) If A is totally projective, then so is $A/p^\rho A$ for every ordinal ρ.

Our main objective is to investigate totally projective p-groups. To this end, we require more information about p^σ-projective and totally projective p-groups. We may profit a great deal from the following two lemmas.

Lemma 82.1 (Nunke [5], Irwin, Walker, and Walker [1]). *If S is a subsocle of the p-group A such that A/S is p^σ-projective, then A is $p^{\sigma+1}$-projective.*

In view of (C), this is true for finite ordinals σ, so assume $\sigma \geqq \omega$. By (51.3) we obtain the exact sequence

$$\mathrm{Hom}(S, C) \xrightarrow{\phi} \mathrm{Ext}(A/S, C) \xrightarrow{\psi} \mathrm{Ext}(A, C) \to \mathrm{Ext}(S, C) \to 0$$

for any C. Here $p\ \mathrm{Ext}(S, C) = 0$ implies $p\ \mathrm{Ext}(A, C) \leqq \mathrm{Im}\ \psi$. We need only verify that $p^{\sigma+1}\ \mathrm{Im}\ \psi = 0$, since then $p^{\sigma+1}\ \mathrm{Ext}(A, C) = 0$ [note that $1 + \sigma = \sigma$ since $\sigma \geqq \omega$].

More generally, we will show that if T is a subsocle of a p-group E satisfying $p^\sigma E = 0$, then $p^{\sigma+1}(E/T) = 0$. [We can apply this to the situation $T = \mathrm{Im}\ \phi$ and $E = \mathrm{Ext}(A/S, C)$.] Clearly, $p^\sigma(E/T) \leqq E[p]/T$, whence $p^{\sigma+1}(E/T) = 0$, indeed. □

The same argument applies to establish the following fact on total projectivity.

Lemma 82.2 (Nunke [5]). *If A is a p-group such that $p^{\sigma+1}A = 0$ and $A/p^\sigma A$ is totally projective, then A is totally projective.* □

Now the following remark can be made:

(F) For the total projectivity of a p-group A, it suffices to know that $p^\sigma\ \mathrm{Ext}(A/p^\sigma A, C) = 0$ for all limit ordinals σ and for all groups C. In fact, from (82.2) we can infer that $p^\sigma\ \mathrm{Ext}(A/p^\sigma A, C) = 0$ for all C implies $p^{\sigma+1}\ \mathrm{Ext}(A/p^{\sigma+1}A, C) = 0$ for all C.

We can now learn precisely what the totally projective p-groups are.

Theorem 82.3 (Hill [24]). *A p-group is totally projective exactly if it satisfies one of the equivalent conditions of* (81.9).

Our first task is to show that a direct summand of a direct sum of generalized Prüfer groups is totally projective. Because of (A), it suffices to establish the total projectivity of the generalized Prüfer groups H_σ. We induct on σ. Passage from σ to $\sigma + 1$ is trivial by virtue of (82.2). For limit ordinals H_σ, when H_σ is the direct sum of all H_ρ with $\rho < \sigma$, the argument is equally trivial from (A).

Conversely, let A be totally projective. We want to verify that A has one or another of the equivalent properties listed in (81.9). Again, we apply induction, this time on the length σ of A. If σ is an integer n, then $p^n A = 0$, A is a bounded group, so it is a direct sum of cyclic p-groups and hence belongs to the class of groups covered by (81.9).

Suppose the assertion has been verified for totally projective p-groups of lengths $\leqq \sigma$ and A is a totally projective p-group of length $\sigma + 1$. Now $A/p^\sigma A$ has, by induction hypothesis, a nice composition series, and so does the elementary p-group $p^\sigma A$. Consequently, A satisfies (β) and hence all conditions of (81.9).

Finally, assume σ is a limit ordinal and the assertion holds for all ordinals $<\sigma$. Let A be a totally projective p-group of length σ. We need only show that a balanced-exact sequence $E: 0 \to B \to G \to A \to 0$ is necessarily splitting. From **80** we recall that, for every $\rho < \sigma$, the sequence $0 \to B/p^\rho B \to G/p^\rho G \to A/p^\rho A \to 0$ is exact such that (3) commutes. Starting from the top and bottom rows and the natural maps ϕ and ψ, we obtain a diagram

$$\begin{tikzcd}
E: & 0 \arrow[r] & B \arrow[r, "\alpha"] \arrow[d, "\phi"'] & G \arrow[r, "\beta"] \arrow[d, dashed, "\lambda"] & A \arrow[r] \arrow[d, equal] & 0 \\
E_\rho \psi: & 0 \arrow[r] & B/p^\rho B \arrow[r, "\gamma"] \arrow[d, equal] & H \arrow[r, "\delta"] \arrow[d, "\mu"] & A \arrow[r] \arrow[d, "\psi"] & 0 \\
E_\rho: & 0 \arrow[r] & B/p^\rho B \arrow[r, "\varepsilon"] & G/p^\rho G \arrow[r, "\xi"] & A/p^\rho A \arrow[r] & 0
\end{tikzcd}$$

where the middle row is defined as $E_\rho \psi$; hence the two lower squares commute. Since H is a pullback and the natural map $v: G \to G/p^\rho G$ followed by ξ equals $\psi\beta$, there exists a unique $\lambda: G \to H$ such that $v = \mu\lambda$ and $\delta\lambda = \beta$. For the commutativity of the diagram, it remains only to prove that $\gamma\phi = \lambda\alpha$. Now $\mu\gamma\phi = \varepsilon\phi = v\alpha = \mu\lambda\alpha$ and $\delta\gamma\phi = 0 = \beta\alpha = \delta\lambda\alpha$ show that $\gamma\phi$ and $\lambda\alpha$ are maps $B \to H$ which become equal if followed by μ or δ. Since H was a pullback, $\gamma\phi = \lambda\alpha$, indeed.

It follows from a remark in **80** that $B/p^\rho B$ is balanced in $G/p^\rho G$. Therefore, E_ρ must split owing to the induction hypothesis. Consequently, the middle row also splits, and in view of the exactness of the sequence

$$\mathrm{Ext}(A, p^\rho B) \xrightarrow{\kappa_*} \mathrm{Ext}(A, B) \xrightarrow{\phi_*} \mathrm{Ext}(A, B/p^\rho B),$$

we conclude that $E \in \mathrm{Im}\, \kappa_*$, where $\kappa: p^\rho B \to B$ is the inclusion map. By the obvious analog of (56.1), $\mathrm{Im}\, \kappa_* \leqq p^\rho\, \mathrm{Ext}(A, B)$, therefore $E \in \bigcap_{\rho<\sigma} p^\rho\, \mathrm{Ext}(A, B) = p^\sigma\, \mathrm{Ext}(A, B)$. The last group is 0, whence E is splitting. □

Capitalizing on (82.3), we can get important information about how totally projective p-groups are related to direct sums of countable p-groups.

Theorem 82.4 (Nunke [7]). *A reduced p-group is a direct sum of countable p-groups if and only if it is a totally projective p-group of length* $\leqq \omega_1$.

From (81.1), (81.9), and (82.3) it is clear that a direct sum of countable p-groups is totally projective; its length must be $\leqq \omega_1$. Conversely, (82.3) and (81.9) imply that a totally projective p-group A of length $\leqq \omega_1$ is a summand of a direct sum of generalized Prüfer groups of lengths $<\omega_1$. These generalized Prüfer groups are countable, hence by (9.10), A is a direct sum of countable p-groups. □

In the light of (82.4), we can, in fact, regard the totally projective p-groups as generalizations of direct sums of countable p-groups. While the lengths of direct sums of countable p-groups are limited to ordinals $\leqq \omega_1$, totally projective p-groups can have arbitrarily large ordinals as lengths. The interrelation between the two classes will become even more relevant in the next section where the structure theorem on totally projective p-groups will be proved.

The class of totally projective p-groups has the following remarkable characterization. [Here a class of groups is always understood to contain along with A all groups isomorphic to A.]

Theorem 82.5 (Parker and Walker [1]). *The class of totally projective p-groups is the smallest class $\mathscr{C}$ of groups with the following properties:*

(1) *$\mathscr{C}$ contains the cyclic group of order p;*

(2) *$\mathscr{C}$ is closed under the formations of direct sums and direct summands;*

(3) *for any group A and ordinal σ, $A \in \mathscr{C}$ exactly if both $p^\sigma A \in \mathscr{C}$ and $A/p^\sigma A \in \mathscr{C}$.*

It follows by a fairly trivial induction that all Prüfer groups H_σ belong to $\mathscr{C}$, and (2) guarantees that all totally projective p-groups are members of $\mathscr{C}$. Since the class of totally projective p-groups has the listed properties [see (E), **79**(g) and **81**(A)], the assertion follows. □

EXERCISES

1. (Irwin, Walker, and Walker [1]) If A is p^σ-projective, then $\mathrm{Ext}(A, C) \cong \mathrm{Ext}(A, C/p^\sigma C)$ for every group C. [*Hint*: analog of (56.1).]
2. (Nunke [5]) (a) A p^σ-projective group A has length $\leqq \sigma$. [*Hint*: to show that $p^\sigma A \neq 0$ is absurd, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(A/p^\sigma A, C/p^\sigma C) \arrow[r] \arrow[d] & \mathrm{Hom}(A, C/p^\sigma C) \arrow[r, "\alpha"] \arrow[d, "\gamma"] & \mathrm{Hom}(p^\sigma A, C/p^\sigma C) \arrow[d] \\
\mathrm{Ext}(A/p^\sigma A, p^\sigma C) \arrow[r] & \mathrm{Ext}(A, p^\sigma C) \arrow[r, "\beta"] \arrow[d] & \mathrm{Ext}(p^\sigma A, p^\sigma C) \rightarrow 0 \\
 & p^\sigma \mathrm{Ext}(A, C) = 0 &
\end{tikzcd}$$

with exact rows and columns; since γ is onto and $\alpha = 0$, conclude that $\beta = 0$; but C can be chosen so a to have $\mathrm{Ext}(p^\sigma A, p^\sigma C) \neq 0$.]
(b) A totally projective p-group is p^σ-projective if and only if its length is $\leqq \sigma$.
(c) H_σ is p^ρ-projective exactly if $\sigma \leqq \rho$.

3. (Nunke [7]) (a) A p-group A is totally projective if and only if both $p^\sigma A$ and $A/p^\sigma A$ are totally projective (σ is any fixed ordinal).
(b) Let σ be a countable ordinal. A is a direct sum of countable p-groups exactly if both $p^\sigma A$ and $A/p^\sigma A$ are direct sums of countable p-groups.
4. If A is totally projective and $p^n A \leqq C \leqq A$, for some n, then C is totally projective. [*Hint*: $p^\omega C = p^\omega A$ and Ex. 3(a).]
5. The class of totally projective p-groups remains unchanged if in the definition, C is restricted to p-groups. [*Hint*: check the proofs.]
6. If $0 \to B \to G \to A \to 0$ is a balanced-exact sequence and if $\mathbf{u} = (\sigma_0, \cdots, \sigma_m, \cdots)$ is any increasing sequence of ordinals and ∞, then $0 \to B(\mathbf{u}) \to G(\mathbf{u}) \to A(\mathbf{u}) \to 0$ is again balanced-exact. [*Hint*: for $a \in A(u)$ choose x in a totally projective p-group T with $H(x) = H(a)$, extend $x \mapsto a$ to a map $T \to A$ which factors through $G \to A$.]
7. Given an arbitrary reduced p-group A, there is a totally projective p-group G and an epimorphism $\phi: G \to A$ with Ker ϕ balanced in G.
8. (Nunke [5]) Show that if A is any p^σ-projective p-group then so is Tor(A, C), for every group C. [*Hint*: by making use of the isomorphism

$$\operatorname{Ext}(A, \operatorname{Ext}(C, G)) \cong \operatorname{Ext}(\operatorname{Tor}(A, C), G),$$

a trivial proof can be given.]
9. (Nunke [5], Irwin, Walker, and Walker [1]) An exact sequence E: $0 \to G \to H \to J \to 0$ is called *p^σ-pure exact* [and G a *p^σ-pure* subgroup of H] if E represents an element of p^σ Ext(J, G). Verify the following properties [we assume $C \leqq B \leqq A$]:

(i) If C is p^σ-pure in A, then it is p^σ-pure in B.
(ii) If C is p^σ-pure in B and B is p^σ-pure in A, then C is p^σ-pure in A.
(iii) If B is p^σ-pure in A, then B/C is p^σ-pure in A/C.
(iv) If C is p^σ-pure in A and B/C in A/C, then B is p^σ-pure in A.

[A word of warning: p-pure in the sense of **26** is p^ω-pure in the new sense.]
10. (Griffith [10]) If $0 \to B \to A \xrightarrow{\alpha} C \to 0$ is a p^σ-pure exact sequence, then

$$\alpha(p^\rho A[p]) = p^\rho C[p] \qquad \text{for every} \quad \rho < \sigma.$$

[*Hint*: if $c \in p^\rho C[p]$, choose a homomorphism $\phi: H_{\rho+1} \to C$ with $\phi(p^\rho H_{\rho+1}) = \langle c \rangle$; because of $\rho + 1 \leqq \sigma$, $H_{\rho+1}$ is p^σ-projective, so some $\psi: H_{\rho+1} \to A$ satisfies $\alpha\psi = \phi$.]
11. (Irwin, Walker, and Walker [1]) If C is p^σ-pure in A, then

$$p^\rho C = C \cap p^\rho A \qquad \text{for every} \quad \rho \leqq \sigma.$$

[*Hint*: from Ex. 10 argue like in (80.2); or give a direct proof as follows: induct on σ: if C is $p^{\sigma+1}$-pure in A, then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & C & \to & A & \to & G & \to & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle p} & & \\ 0 & \to & C & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & G & \to & 0 \end{array}$$

where C is p^σ-pure in H and the last vertical map is multiplication by p; if $c \in C \cap p^{\sigma+1}A$, then $pa = c$, for some $a \in p^\sigma A$, and since $A \leqq H \oplus G$, $a = (h, g)$, it follows easily that $ph = c$, $pg = 0$, $\beta h = 0$, $h \in C$; since necessarily $h \in p^\sigma H$, $h \in p^\sigma C$, and $c \in p^{\sigma+1}C$.]

12.* (Nunke [5]) For any p-group A and ordinal $\sigma \geqq \omega$, Ker ∂ is p^σ-pure in $\mathrm{Tor}(H_\sigma, A)$, where the notations are those of Ex. 9 in **81**. [*Hint*: from the diagram

$$\begin{array}{ccccc} p^\sigma \operatorname{Ext}(A, C) \to \operatorname{Hom}(Z, \operatorname{Ext}(A, C)) & \xrightarrow{\delta} & \operatorname{Ext}(H_\sigma, & \operatorname{Ext}(A, C)) \\ \wr\| & & & \wr\| \\ \operatorname{Ext}(A, C) & \xrightarrow{\partial^*} & \operatorname{Ext}(\operatorname{Tor}(H_\sigma, A), C) & \end{array}$$

conclude that Ker $\partial^* = p^\sigma \operatorname{Ext}(A, C)$ for every C; the choice $C = \operatorname{Ker} \partial$ shows that Ker ∂ must be p^σ-pure in $\mathrm{Tor}(H_\sigma, A)$.]

13. (Nunke [5]) (a) There are enough p^σ-projectives: for every p-group A and ordinal σ, there is a p^σ-pure exact sequence $0 \to K \to T \to A \to 0$ with T p^σ-projective. [*Hint*: ∂: $\mathrm{Tor}(H_\sigma, A) \to A$, Exs. 8 and 12.]
(b) If A is p^σ-projective, then it is a summand of $\mathrm{Tor}(H_\sigma, A)$.

14. (Nunke [5]) Let Γ be the smallest class of p-groups which satisfies the following conditions:
(i) $0 \in \Gamma$ and Γ is closed under taking subgroups;
(ii) if B is an elementary p-subgroup of A and $A/B \in \Gamma$, then $A \in \Gamma$;
(iii) Γ is closed under arbitrary direct sums.
Show that a p-group A belongs to Γ if and only if it is a subgroup of a p^σ-projective group, for some ordinal σ. [*Hint*: for the "if" part we use induction, noting Ex. 13.]

83. SIMPLY PRESENTED p-GROUPS

In this section, we give an account of another characterization of totally projective p-groups, based on a very special sort of presentation. This is one of the characterizations which is most closely related to the structure theorem, since it will admit a relatively easy method to reach the main result on totally projective p-groups.

A p-group A will be said to be *simply presented* [a *T-group* in the terminology of Crawley and Hales [1]], if it can be generated by a set of elements, $X = \{x_i\}_{i \in I}$, subject only to defining relations of the form:

$$p^m x_i = 0 \qquad \text{or} \qquad p^n x_i = x_j \qquad (i \neq j),$$

where m and n are positive integers.

Evidently, a relation like $p^m x_i = 0$ can be replaced by a set of relations $px_i = y_1, py_1 = y_2, \cdots, py_{m-1} = 0$, after adjoining new generators $y_1, y_2, \cdots, y_{m-1}$ to X. Moreover, we may and shall assume that for every generator x_i of A having order p^m, there are generators $y_1, y_2, \cdots, y_{m-1}$ with the relations as given above. The same remark applies to the relations $p^n x_i = x_j$. Thus we see that no generality is lost by assuming that all relations are of the form $px_i = 0$ or $px_i = x_j$ $(i \neq j)$.

It can clearly happen that distinct elements of X become equal in the group A or a generator collapses to 0 in A. We can eliminate these cases by omitting duplications and vanishing generators, and at the same time suitably modifying the defining relations. Since it is not absolutely clear that during this procedure nothing can go wrong with the defining relations, more care must be exercised. More precisely, we intend to verify that every simply presented p-group A can also be presented by a set X of generators and a set Σ of relations such that:

(i) for every $x \in X$, $x \neq 0$ in A;
(ii) if x, y are distinct elements in X, then $x \neq y$ in A;
(iii) all relations are of the form $px = 0$ or $px = y$ where $x, y \in X$.

Such a presentation will be referred to as *faithful.*

Let A be a simply presented p-group: $A = \langle X'; \Sigma' \rangle$, where X' is a set of generators and Σ' is a set of defining relations of the form $px' = 0$ or $px' = y'$. Let X be a subset of X' such that (i) and (ii) hold for X and every $x' \in X'$ is equal in A to some $x \in X$. A relation $px = 0$ [or $px = y$] will be included in Σ exactly if $x \in X$ and $px = 0$ in A [$x, y \in X$ and $px = y$ in A]. Then the group $B = \langle X; \Sigma \rangle$ obviously has an epimorphism ϕ onto A. There is another map, namely, $\psi: A \to B$, where every $x' \in X'$ is sent into $x \in X$ such that $x' = x$ in A. Since $\psi\phi = 1_B$ and ϕ is epic, ϕ is an isomorphism, establishing the existence of faithful presentation.

Henceforth, all presentations will be assumed to be faithful, unless stated otherwise.

A faithful presentation of a simply presented group $A = \langle X; \Sigma \rangle$ gives rise to a natural partial order in X: for $x, y \in X$ define

$$y < x \qquad \text{if} \quad p^n x = y \quad \text{for some} \quad n > 0,$$

that is, if $px = x_1$, $px_1 = x_2$, $\cdots$, $px_{n-1} = y$ are in Σ for suitable x_1, $\cdots$, $x_{n-1} \in X$. Manifestly, X satisfies the minimum condition.

The simplest examples for simply presented p-groups [with faithful presentations] are as follows.

Example 1. Cyclic groups of order p^n are simply presented:

$$Z(p^n) = \langle x_1, x_2, \cdots, x_n; px_1 = 0, px_2 = x_1, \cdots, px_n = x_{n-1}\rangle.$$

Example 2. Quasicyclic groups are simply presented:

$$Z(p^\infty) = \langle x_1, \cdots, x_n, \cdots; px_1 = 0, px_2 = x_1, \cdots, px_n = x_{n-1}, \cdots\rangle.$$

Example 3. The Prüfer group $H_{\omega+1}$ is simply presented:

$$H_{\omega+1} = \langle a_0, a_1, a_2, a_{21}, \cdots, a_n, a_{n1}, \cdots, a_{n,n-1}, \cdots; pa_0 = 0, pa_1 = a_0, \\ pa_2 = a_{21}, pa_{21} = a_0, \cdots, pa_n = a_{n1}, \cdots, pa_{n,n-1} = a_0, \cdots\rangle.$$

Our study of simply presented p-groups starts with a list of elementary results [see Crawley and Hales [1]].

(a) *The direct sum of simply presented groups is again simply presented.*

This is clear, since the set-union of the generating systems together with the set-union of the defining relations yields a presentation for the direct sum.

From the definition of generalized Prüfer groups H_σ it is evident that if H_σ is simply presented, then so is $H_{\sigma+1}$. An obvious transfinite induction, using (a) at limit ordinals, leads us to the result:

Proposition 83.1 *The generalized Prüfer groups are simply presented.* □

(b) *Each nonzero element a of a simply presented p-group A can be written uniquely in the form*

$$a = s_1x_1 + \cdots + s_kx_k \qquad (k \geqq 1), \tag{1}$$

where x_1, $\cdots$, x_k are distinct elements of X and $0 < s_i < p$ for $i = 1, \cdots, k$.

From (i)–(iii), the existence of such a representation is evident. Suppose that $a = s_1x_1 + \cdots + s_kx_k = t_1x_1 + \cdots + t_kx_k$ holds for distinct $x_1, \cdots, x_k \in X$ and for some integers s_i, $t_i = 0, 1, \cdots, p-1$. Let x_1 be maximal among $x_1, \cdots, x_k$ in the natural partial order of X. There is a homomorphism ϕ of $\langle x_1, \cdots, x_k\rangle$, and hence one of A, into $Z(p^\infty)$ which sends x_1 upon an element of order p and $x_2, \cdots, x_k$ onto 0. Now $\phi a = s_1(\phi x_1) = t_1(\phi x_1)$ implies $s_1 = t_1$, and a simple induction on k completes the proof.

(c) *If* (1) *is the unique representation of a nonzero element a in a simply presented group A, then $a \in p^\sigma A$ if and only if $x_i \in p^\sigma A$ for $i = 1, \cdots, k$.*

Proof of necessity by induction on σ. The assertion being trivially true for $\sigma = 0$, suppose that $a = pb$ where $b = r_1y_1 + \cdots + r_ly_l \in p^\sigma A$ for some σ. If $y_1, \cdots, y_l \in X$ are distinct and $0 < r_j < p$ for $j = 1, \cdots, l$, then by induction hypothesis $y_j \in p^\sigma A$ for $j = 1, \cdots, l$. Hence $a = r_1(py_1) + \cdots + r_l(py_l)$, which can be written as $t_1 z_1 + \cdots + t_m z_m$, where the zs are distinct elements of X, the ts are positive integers $<p$, and every z is of the form $p^n y_j$ ($n \geqq 1$). Hence $z_1, \cdots, z_m \in p^{\sigma+1}A$. From (b) we obtain that these zs are equal to the xs, whence $x_1, \cdots, x_k \in p^{\sigma+1}A$ follows.

Given $y \in X$, define

$$X_y = \{x \in X \mid y \leqq x\}.$$

(d) *If A is a simply presented p-group and M is the set of minimal elements in X, then*

$$A = \bigoplus_{y \in M} \langle X_y \rangle. \tag{2}$$

For distinct y and $z \in M$, the sets X_y and X_z are disjoint, and by the minimum condition in X, every $x \in X$ belongs to some X_y. Since all defining relations involve elements of the same X_y only, it is clear that A is the direct sum of the groups $\langle X_y \rangle$.

(e) *Let A be a simply presented reduced p-group whose length is a limit ordinal. Then A is the direct sum of simply presented p-groups of smaller lengths.*

In the representation (2) of A, the last nonzero Ulm subgroup of $\langle X_y \rangle$ is, by (c), generated by y. Thus the lengths of $\langle X_y \rangle$ are nonlimit ordinals; hence they are less than the length of A.

(f) *If A is a simply presented p-group and Y is a subset of X, then $N = \langle Y \rangle$ is a nice subgroup of A.*

Let $a \in A \backslash N$ and write $a = s_1 x_1 + \cdots + s_k x_k + t_1 y_1 + \cdots + t_l y_l$, where x_i and y_j are distinct elements of $X \backslash Y$ and Y, respectively, and s_i, t_j are positive integers $<p$. We claim that $b = s_1 x_1 + \cdots + s_k x_k \in a + N$ is proper with respect to N. In fact, for any $c = r_1 y_1' + \cdots + r_m y_m' \in N$ [written in the form (1)], the height of $a + c$ is, owing to (c), equal to $\min_{i,j}\{h^*(x_i), h^*(y_j')\} \leqq \min_i h^*(x_i) = h^*(b)$.

The following assertion is now immediate:

Lemma 83.2 (Crawley and Hales [1], Hill [17]). *Every simply presented p-group has a nice system.* $\square$

(g) *If A is an infinite reduced simply presented p-group, then*

$$|A| = \sum_\sigma f_\sigma(A).$$

Proof by induction on the length τ of A. The case where τ is a limit ordinal is easy because of (e). Let τ be an isolated ordinal. The case where τ

is finite is trivial, since then A is a direct sum of cyclic groups. If $\tau = \rho + n$, where ρ is a limit ordinal >0 and n an integer >0, then $A/p^\rho A$ is infinite and clearly $|A/p^\rho A| \geqq |p^\rho A| \geqq \sum_{\rho \leqq \sigma < \tau} f_\sigma(A)$. Owing to $f_\sigma(A/p^\rho A) = f_\sigma(A)$ for $\sigma < \rho$, the desired equality follows.

We now have sufficient apparatus to establish the main structure theorem; it was proved, in the following form, by Crawley and Hales [1], and in an equivalent form [namely, for p-groups with nice systems] by Hill [24].

Theorem 83.3 (Crawley and Hales [1], Hill [24]). *Two simply presented reduced p-groups are isomorphic if and only if their corresponding Ulm–Kaplansky invariants are equal.*

Some advantage is to be gained by proving this result in the following strengthened form, due to Hill [24]; the idea of using Zorn's lemma in the proof, rather than transfinite induction, is due to E. A. Walker.

Theorem 83.4. *Let A and C be p-groups, G and H nice subgroups of A and C, respectively, such that $A/G = \langle X, \Sigma_X \rangle$ and $C/H = \langle Y, \Sigma_Y \rangle$ are simply presented. Suppose that:*

(a′) *there is a height-preserving isomorphism ϕ from G onto H;*
(b′) *the relative Ulm–Kaplansky invariants are equal:*

$$f_\sigma(A, G) = f_\sigma(C, H) \qquad \textit{for every} \quad \sigma.$$

Then ϕ extends to an isomorphism $\phi^*\colon A \to C$.

For each σ, we select an arbitrary but fixed isomorphism

$$\alpha_\sigma : p^\sigma A[p]/G(\sigma) \to p^\sigma C[p]/H(\sigma),$$

and consider pairs $(G_\lambda, \phi_\lambda)$ subject to the following postulates:

(α) $G_\lambda = \langle G, X_\lambda \rangle$ for some subset X_λ of X;
(β) ϕ_λ is a height-preserving isomorphism of G_λ with a subgroup $H_\lambda = \langle H, Y_\lambda \rangle$, where Y_λ is a subset of Y;
(γ) the restriction of ϕ_λ to G equals ϕ;
(δ) α_σ induces an isomorphism of $G_\lambda(\sigma)/G(\sigma)$ with $H_\lambda(\sigma)/H(\sigma)$, for every σ.

If the set of these pairs is partially ordered in the obvious way, Zorn's lemma yields a maximal pair (G^*, ϕ^*), where ϕ^* is a height-preserving isomorphism of $G^* = \langle G, X^* \rangle$ with $H^* = \langle H, Y^* \rangle$, say. Note that by ($\delta$), $f_\sigma(A, G^*) = f_\sigma(C, H^*)$. If some $x \in X$ is missing from G^*, then $px \in G^*$ may be assumed, and by a trivial application of (77.1), we extend ϕ^* to a height-preserving isomorphism ϕ_0^* of $G_0^* = \langle G^*, x \rangle$ with a subgroup H_0^* of C, containing H^*, such that (δ) is preserved. There are a finite number of generators, $y_1, \cdots, y_m$ in Y such that $H_0^* \leq H_1^* = \langle H^*, y_1, \cdots, y_m \rangle$. Now we invert the procedure: H_1^* is a finite extension of H^*, so it is nice in C, thus (77.1) is successively applicable to H_1^* and ϕ_0^{*-1} and the generators $y_1, \cdots, y_m$, and we obtain a finite extension G_1^* of G_0^* together with a height-preserving isomorphism $\phi_1^*: G_1^* \to H_1^*$ such that $\phi_1^* | G_0^* = \phi_0^*$ and $\alpha_\sigma(G_1^*(\sigma)/G(\sigma)) = H_1^*(\sigma)/H(\sigma)$. We come back to A and select $x_1, \cdots, x_n \in X$ such that $G_1^* \leqq \langle G_0^*, x_1, \cdots, x_n \rangle = G_2^*$. Then we repeat this process, alternately in C and A, to get an ascending chain of subgroups $G^* < G_0^* \leqq G_1^* \leqq \cdots$ of A along with height-preserving isomorphisms ϕ_n^* of G_n^* with subgroups H_n^* of C satisfying the requisite conditions, and $\phi_n^* | G_{n-1}^* = \phi_{n-1}^*$. If we set $G^{**} = \bigcup_{n<\omega} G_n^*$ and $H^{**} = \bigcup_{n<\omega} H_n^*$, then the pair (G^{**}, ϕ^{**}), where ϕ^{**} is the map $G^{**} \to H^{**}$ induced by the ϕ_n^*, also belongs to the set of pairs considered. Consequently, $G^* = A$, and for reasons of symmetry, $H^* = C$. □

We have not gone far enough yet to announce that simply presented reduced p-groups are exactly the totally projective p-groups. In view of (83.1) and (83.2), just one further step remains to be taken: to show that direct summands of simply presented p-groups are likewise simply presented. Instead of proving this directly, we digress to total projectivity and show, by transfinite induction on the length σ of A, that a totally projective p-group A is simply presented.

If σ is a limit ordinal, then A is the direct sum of totally projective p-groups of lengths $<\sigma$ and, by virtue of (a), there is nothing to prove. So let A be a totally projective p-group of length $\sigma + 1$. By hypothesis, $A/p^\sigma A$ is simply presented; let M be the set of minimal elements in a generating system X of $A/p^\sigma A$, and put $M_\rho = \{x \in M | h^*(x) \geqq \rho\}$ for $\rho < \sigma$. Then $|M_\rho| \geqq r(p^\sigma A)$, where $p^\sigma A$ is an elementary p-group. Choose a basis $\{u_i\}_{i \in I}$ for $p^\sigma A$ and define a group C as generated by $\{X, u_i\ (i \in I)\}$ and subject to the relations $pu_i = 0$ for all i and the relations between the elements of X in $A/p^\sigma A$ with the exception that $px = 0$ $(x \in M)$ is replaced by $px = u_i$ for some i. We may assume that for every $i \in I$ and for every $\rho < \sigma$ there is a relation $px = u_i$ with some $x \in M_\rho$. Then C will be simply presented such that $p^\sigma C \cong p^\sigma A$ and $C/p^\sigma C \cong A/p^\sigma A$. Invoking (83.4), the isomorphism $C \cong A$ follows. This completes the proof of

Theorem 83.5 (Crawley and Hales [1]). *A reduced p-group is simply presented if and only if it is totally projective.* □

It would perhaps make good sense to interrupt the discussion at this point and to summarize briefly the various characterizations of totally projective p-groups. Our results show that for a reduced p-group A, the following are equivalent:

1. A has a nice composition series;
2. A has a nice system;
3. A is simply presented;
4. A is a direct summand of a direct sum of generalized Prüfer groups;

5. A has the projective property relative to balanced-exact sequences of p-groups;
6. A is totally projective;
7. A belongs to the smallest class of groups that contains $Z(p)$, is closed under taking direct sums and direct summands, and which contains a group G exactly if it contains both $p^\sigma G$ and $G/p^\sigma G$, for any ordinal σ.

Having obtained a great deal of insight into the structure of totally projective p-groups, we wish to find out what sequences of cardinals are eligible to be the Ulm–Kaplansky invariants of a totally projective p-group.

Theorem 83.6 (Crawley and Hales [1], Hill [24]). *Let g be a function from the ordinals $\sigma < \tau$ [where τ is a given ordinal] to the cardinals. There exists a totally projective p-group A of length τ such that*

$$f_\sigma(A) = g(\sigma) \qquad \textit{for all} \quad \sigma < \tau,$$

exactly if g is a τ-admissible function.

First, let A be a totally projective p-group of length τ. Then (i) in the definition of τ-admissible functions in **78** is obvious for $f_\sigma(A)$. To verify (ii), induction can be used on the length τ. The case of finite or limit ordinals being obvious, set $\tau = \rho + n$ with ρ a limit ordinal >0 and n an integer >0. Then for all $\sigma < \rho$, $p^\sigma A/p^\rho A$ is infinite, and $|p^\sigma A/p^\rho A| \geqq |p^\rho A| \geqq \sum_{\rho \leqq \sigma < \tau} f_\sigma(A)$. This and (g) imply that $f_\sigma(A)$ is, in fact, τ-admissible.

Let g be a τ-admissible function and write $\tau = \sigma + n$, where σ is a limit ordinal and n is an integer. If $\sigma = 0$, then g is of finite length, and $A \cong \bigoplus_{i=0}^{n-1} \bigoplus_{g(i)} Z(p^{i+1})$ has $g(0), \cdots, g(n-1)$ for its sequence of Ulm–Kaplansky invariants. If both $\sigma > 0$ and $n > 0$, then use induction to conclude the existence of a totally projective p-group G of length σ whose ρth Ulm–Kaplansky invariant is $g(\rho)$, for every $\rho < \sigma$. Following the pattern of proof of (83.5), a totally projective p-group A of length $\sigma + n$ can be constructed such that $A/p^\sigma A \cong G$ and $p^\sigma A \cong \bigoplus_{i=0}^{n-1} \bigoplus_{g(\sigma+i)} Z(p^{i+1})$, provided the inequality $\sum_{\pi < \rho < \sigma} g(\rho) \geqq \sum_{i=0}^{n-1} g(\sigma + i)$ is satisfied for every $\pi < \sigma$. But this is a simple consequence of (ii) in **78**.

If, finally, $\tau = \sigma$ is a limit ordinal, then we argue as follows. Owing to (78.5), the τ-admissible function g can be written as a sum $\sum g_i$ of τ_i-admissible functions g_i of lengths $\tau_i < \tau$. Now if A_i is a totally projective p-group with $g_i(\sigma)$ as the σth Ulm–Kaplansky invariant, then $A = \bigoplus_{i \in I} A_i$ will belong to the given function g. $\square$

To conclude the theory of totally projective p-groups, we prove the following result which may be interpreted as saying that the class of totally projective p-groups is the largest class to which Ulm's theorem can be extended:

Theorem 83.7. *Any class of reduced p-groups, which contains all totally projective p-groups, is closed under the formation of direct sums and in which* *the Ulm–Kaplansky invariants distinguish between nonisomorphic groups coincides with the class of totally projective p-groups.*

Let A be a member of the given class and let τ be the length of A. Set $\mathfrak{m} = |A| \cdot \aleph_0$, and consider the direct sum

$$H = \bigoplus_{\mathfrak{m}} \bigoplus_{\sigma \leqq \tau} H_\sigma$$

of generalized Prüfer groups. It is clear that $f_\sigma(H) = \mathfrak{m}$ for all $\sigma < \tau$; hence $A \oplus H$ and H have the same Ulm–Kaplansky invariants. By hypothesis, $A \oplus H$ is still in the given class and $A \oplus H \cong H$. All this amounts to saying that A is a summand of a direct sum of generalized Prüfer groups. $\square$

Theorems (83.3), (83.6), and (83.7) accomplish our ultimate aim in the theory of totally projective p-groups. One should, however, be aware of the fact that the theory of totally projective p-groups has by no means been fully exploited, and if we drop total projectivity, we are able to say virtually nothing about the group structure.

EXERCISES

1. A separable p-group is simply presented exactly if it is a direct sum of cyclic p-groups. [*Hint*: structure theorem.]
2. If $A = \langle X; \Sigma \rangle$ is a simply presented p-group and Y is a subset of X, then $A/\langle Y \rangle$ is simply presented.
3. For any ordinal σ, a p-group A is simply presented if and only if both $p^\sigma A$ and $A/p^\sigma A$ are simply presented.
4. (Crawley and Hales [2]) If A_i $(i \in I)$ is a family of simply presented p-groups such that $p^{\sigma_i} A_i = \langle a_i \rangle$ is cyclic of order p^n for every $i \in I$, then the group $\bigoplus_{i \in I} A_i/G$ is again simply presented, where G denotes the subgroup of $\bigoplus A_i$ generated by all $a_i - a_j$ $(i, j \in I)$.
5. (E. A. Walker) For a reduced p-group A, define a group G as generated by g_a for all $a \in A$, subject to the relations $g_0 = 0$, $p^n g_a = g_b$ if and only if $p^n a = b$ in A. Then G is simply presented and the kernel of the epimorphism $G \to A$ induced by $g_a \mapsto a$ is balanced in G. [*Hint*: check (80.2)(iv).]
6. (Hill [15]) Totally projective p-groups are fully transitive. [*Hint*: (81.4).]
7. (L. Fuchs and E. A. Walker) For every totally projective p-group A and for every increasing sequence of ordinals and ∞, $\mathbf{u} = (\sigma_0, \cdots, \sigma_n, \cdots)$, both $A(\mathbf{u})$ and $A/A(\mathbf{u})$ are totally projective. [*Hint*: reduce to Prüfer groups, and induct on the length, noting that $(A/p^\sigma A)(\mathbf{u}) = (A(\mathbf{u}) + p^\sigma A)/p^\sigma A$ for a group A of length $\sigma + 1$ and Ex. 3 in **81**.]
8. (a) (Nunke [7]) Let A be a p-group with a finite number of Ulm factors, all of which are direct sums of cyclic groups. Prove that A is a direct sum of countable p-groups.

(b) (Hill and Megibben [4]) A reduced p-group of finite Ulm type whose Ulm factors, except possibly the last, are direct sums of cyclic groups is uniquely determined by its Ulm factors.

9. (Hill and Megibben [4]) Let A be a p-group and σ a countable ordinal such that $p^\sigma A = \bigoplus_{i\in I} C_i$ and $A/p^\sigma A$ is a direct sum of countable p-groups. Show that $A = E \oplus \bigoplus_{i\in I} E_i$, where $p^\sigma E = 0$ and for each $i \in I$, $p^\sigma E_i = C_i$ and $|E_i| \leqq \max(|C_i|, \aleph_0)$. [*Hint*: reduce to the case when $p^\sigma A$ is a direct sum of cyclic groups; then A is a direct sum of countable p-groups and use (76.2) to construct a required decomposition.]
10. Prove (83.6) by using the same steps as in the proof of (76.2).
11. If A and C are totally projective p-groups, each isomorphic to an isotype subgroup of the other, then $A \cong C$. [*Hint*: **80**(D) and (83.3).]
12. If A is a totally projective p-group and $A \oplus A \cong C \oplus C$, then $A \cong C$.

84. SUMMABLE p-GROUPS

We conclude our discussion of p-groups with an interesting class, discovered by Honda [3]. This class includes the direct sums of countable p-groups.

Let A be a reduced p-group, say of length τ. Define S_σ by

$$p^\sigma A[p] = p^{\sigma+1}A[p] \oplus S_\sigma \qquad \text{for} \quad \sigma < \tau. \tag{1}$$

Thus the nonzero elements of S_σ are precisely of height σ. The direct sum $\bigoplus_{\sigma<\tau} S_\sigma$ obviously has the property that if $a = a_{\sigma_1} + \cdots + a_{\sigma_k}$, where $0 \neq a_{\sigma_i} \in S_{\sigma_i}$ with different σ_i, then $h^*(a) = \min_i \sigma_i$.

The direct sum $\bigoplus_{\sigma<\tau} S_\sigma$ fails, in general, to exhaust $A[p]$; a simple counterexample is an unbounded torsion-complete p-group A, where $\bigoplus_{n<\omega} S_n$ will support just a basic subgroup of A. Since even a direct sum of cyclic groups can contain a basic subgroup different from the group, it follows easily that $\bigoplus_{\sigma<\tau} S_\sigma$ can be equal to $A[p]$ for some choice of S_σ in (1), but unequal to $A[p]$ for a different choice. In view of this, we define A *summable* [see Hill and Megibben [5]] if, for suitable choices of S_σ in (1), one has

$$A[p] = \bigoplus_{\sigma<\tau} S_\sigma. \tag{2}$$

From this definition it is clear that:

(A) A direct sum of cyclic p-groups is summable.

(B) A direct sum of summable groups is again summable.

A subgroup G of A is said to be *height-finite* if the heights of its elements [all heights are computed in A] assume but a finite number of different values.

The following is a useful criterion for summability.

Theorem 84.1 (Honda [3]). *A reduced p-group A of countable length τ is summable if and only if there is an ascending chain*

$$0 \leqq G_1 \leqq G_2 \leqq \cdots \leqq G_n \leqq \cdots$$

of height-finite subgroups of $A[p]$ such that

$$\bigcup_{n<\omega} G_n = A[p].$$

If A is summable and τ is countable, then we can list all S_σ ($\sigma < \tau$) from (2) in a sequence $S_{\sigma_1}, S_{\sigma_2}, \cdots, S_{\sigma_n}, \cdots$ of type ω. Evidently $G_n = S_{\sigma_1} \oplus \cdots \oplus S_{\sigma_n}$ ($n = 1, 2, \cdots$) are height-finite with $A[p]$ as their union.

Conversely, suppose there is an ascending chain of height-finite subgroups G_n of A whose union is $A[p]$. Inasmuch as $p^\sigma A \cap G_1$ is a summand of G_1, starting with the maximal height σ_1 of elements $\neq 0$ in G_1, we can find a decomposition $G_1 = K_{\sigma_1} \oplus \cdots \oplus K_{\sigma_k}$, where $\sigma_1 > \cdots > \sigma_k$ and every nonzero element of K_{σ_j} has height σ_j. By induction, we can construct in this way, for every n, decompositions

$$G_n = \bigoplus_{\sigma<\tau} K_\sigma^{(n)}$$

such that $K_\sigma^{(n)} = 0$ for almost all σ, $K_\sigma^{(n)} \leqq p^\sigma A$, $K_\sigma^{(n)} \cap p^{\sigma+1}A = 0$, and $K_\sigma^{(n-1)} \leqq K_\sigma^{(n)}$. Setting $S_\sigma = \bigcup_n K_\sigma^{(n)}$, one gets $A[p] = \bigoplus_{\sigma<\tau} S_\sigma$ as is easily verified. $\square$

Since the socle of a countable reduced p-group is the union of an ascending chain of finite subsocles, from (84.1) and (B) we are at once led to

Corollary 84.2. *Countable reduced p-groups and their direct sums are summable.* $\square$

Using this corollary, summable p-groups of any length $\leqq \omega_1$ can be constructed. A surprising and not at all trivial result is that there are not any summable p-groups of length $> \omega_1$.

Theorem 84.3 (Honda [3]). *A summable p-group A satisfies $p^{\omega_1}A = 0$.*

Suppose that A is summable, but $p^{\omega_1}A \neq 0$. Then we can write

$$A[p] = \bigoplus_{\sigma<\omega_1} S_\sigma \oplus p^{\omega_1}A[p],$$

where the nonzero elements in S_σ are precisely of height σ. Let $0 \neq a \in A$ have height ω_1, and let $b_1, \cdots, b_n, \cdots \in A$ be such that $pb_n = a$ for every n and $\sigma_1 < \cdots < \sigma_n < \cdots$, where $\sigma_n = h(b_n)$. In addition, these b_n may be supposed to be chosen in such a way that in the above decomposition of $A[p]$, $b_n - b_{n-1}$ has nonzero coordinates in the S_σ with $\sigma < \sigma_n$ only. Since these σ_n are countable, there exist a countable ordinal σ_0 such that $\sigma_0 > \sigma_n$ for all n

and a $b_0 \in A$ of height σ_0, satisfying $pb_0 = a$. Clearly, $b = b_0 - b_1 \in A[p]$ is of height σ_1, and selecting another b_0 if necessary, $b \in \oplus_{\sigma<\omega_1} S_\sigma$ may be assumed. Considering that $b = (b_0 - b_n) + (b_n - b_1)$ and $h^*(b_0 - b_n) = \sigma_n$, and since $b_n - b_1$ has 0 coordinate in S_{σ_n}, it is clear that b has a nonzero coordinate in S_{σ_n} for every n. This contradiction establishes that $p^{\omega_1}A = 0$ for every summable p-group A.$\square$

Subgroups of summable groups are not necessarily summable [see Ex. 7]. For certain subgroups, however, summability is inherited.

Proposition 84.4 (Hill and Megibben [5]). *Isotype subgroups of countable length of summable groups are again summable.*

Let C be an isotype subgroup of countable length ρ in the summable group A. Then there is a $p^\rho A$-high subgroup E of A such that $C \leqq E$. The proof will be broken down into two steps.

First, we show E summable. Writing $A[p]$ in the form (2), there is a $p^\rho A$-high subgroup E' of A such that $E'[p] = \bigoplus_{\sigma<\rho} S_\sigma$. Since by (80.1), E' is isotype in A, the summability of E' is evident. The canonical homomorphism $A \to A/p^\rho A$ maps $p^\rho A$-high subgroups isomorphically and by (37.1) in a height-preserving manner onto subgroups of $A/p^\rho A$. It follows easily that under this mapping, the socles of $p^\rho A$-high subgroups have the same image. The summability of E is now an easy consequence of that of E'.

The proof is now reduced to the case when C is isotype in a summable group E of countable length ρ. By (84.1), $E[p]$ is the union of an ascending chain of height-finite subgroups G_n. Now $C[p]$ is the union of the ascending chain of subgroups $G_n \cap C$ $(n = 0, 1, \cdots)$ which are height-finite, C being isotype in A.$\square$

We refer to Hill [16] for an example of a summable group which fails to be a direct sum of countable p-groups.

EXERCISES

1. A separable p-group is summable if and only if it is a direct sum of cyclic groups.
2. A subsocle S of a reduced p-group A is called *summable* if $S = \bigoplus_{\sigma<\tau} T_\sigma$, where all nonzero elements of T_σ are of height σ. Show that all countable subsocles of A are summable.
3. Every subsocle of a summable p-group of countable length is summable. [*Hint*: proof of (84.4).]
4. (Hill and Megibben [5]) If A is a reduced p-group and $A[p] = S \oplus p^{\omega_1}A[p]$, then S is never summable unless $p^{\omega_1}A = 0$. [*Hint*: proof of (84.3).]
5. (Hill and Megibben [5]) A summable p-group A satisfies

$$p^{\omega_1} \operatorname{Ext}(Z(p^\infty), A) = 0.$$

[*Hint*: this holds by (56.3) if the length of A is $<\omega_1$; if A has length ω_1 and $E: 0 \to A \to G \to Z(p^\infty) \to 0$ is a nonsplitting extension in $p^{\omega_1}\text{Ext}(Z(p^\infty), A)$, then $G[p] = A[p] \oplus \langle g \rangle$ for some $g \in G$; if $h^*(g) = \omega_1$, G is summable, which is impossible; if $h^*(g) < \omega_1$, consider $E\eta: 0 \to A \to G' \to Z(p^\infty) \to 0$, where η is an endomorphism of $Z(p^\infty)$ with kernel $Z(p)$, and show that $E\eta$ is nonsplitting, but G' is summable.]

6. (Hill and Megibben [5]) A summable p-group is a direct summand of every group in which it is p^{ω_1}-high. [*Hint*: Ex. 5.]
7. (Griffith [10]) Let A be a p^{ω_1}-high subgroup of the Prüfer group H_{ω_1+1}. Show that:
   (a) A is isomorphic to an isotype subgroup of H_{ω_1}.
   (b) A is not summable. [*Hint*: Ex. 4.]

NOTES

Prüfer's famous group $H_{\omega+1}$ was the first known example of a reduced p-group with elements $\neq 0$ of infinite height (see Prüfer [2]). A decade later, Ulm [1] proved, by using ingenious matrix-theoretical methods, that the Ulm factors [these are now direct sums of cyclic groups] of countable reduced p-groups determine the groups up to isomorphism. Zippin [1] gave a purely group-theoretical proof for this result, and at the same time he established the existence theorem (76.2). Surprisingly, no fruit has been born out of this remarkable theory for a long time, and even applications have been very scarce.

Attempts at generalizations started in the early 1950s: the existence theorem (76.1) has been established for arbitrary p-groups with prescribed Ulm factors, independently and almost simultaneously by Kulikov [3] and Fuchs [2]. A more difficult task was to extend Ulm's theorem to larger classes of p-groups.

Actually, the proof of Ulm's theorem, given by Kaplansky and Mackey [1], has opened the door to various generalizations. The first major step was taken by Kolettis [1] who carried over the Ulm–Zippin theory to direct sums of countable p-groups. At this stage it became evident that there must be a broader class of p-groups in which the Ulm–Kaplansky invariants distinguish between the groups. It was E. A. Walker who conjectured that the totally projective p-groups, introduced by Nunke [5], form such a class. Parker and Walker [1] succeeded in extending Ulm's theorem to totally projective p-groups of lengths $\leq \omega_1\omega$ only. Major credit must be given to Nunke [7] who explored the properties of totally projective p-groups.

Shortly afterwards, P. Hill announced [*Notices Amer. Math. Soc.* **14** (1967), 940] Ulm's theorem for totally projective p-groups. His paper [24] [not yet published, but closely followed by Griffith [10]] is based on nice systems and on a delicate analysis of extensions of isomorphisms. In a different direction, Crawley and Hales [1, 2] discovered the class of simply presented p-groups [they called them T-groups] and showed that they can be adequately classified by their Ulm–Kaplansky invariants. The observation that the existence of a single nice composition series suffices to ensure total projectivity seems to be new.

In our presentation, we have tried to explore briefly the various aspects of the theory of totally projective p-groups, and have avoided a more comprehensive study of powerful techniques, in order to move towards our goal more rapidly. However, we must refer the reader to the original memoirs for additional materials.

Recently Warfield [5] has succeeded in extending the main results on totally projective p-groups to mixed modules over discrete valuation rings which are simply presented in the sense of **83**. If one notices the simple fact that the simply presented torsion-free groups are precisely the completely decomposable groups [which are also describable by means of cardinal invariants], then one can wonder what a more general theory is which covers all these special cases. He introduced new invariants for the modules M, namely the ranks of the groups $p^\sigma M/(p^{\sigma+1}M + T_\sigma)$ [where T_σ stands for the torsion part of $p^\sigma M$] for limit ordinals σ, and showed that a slightly larger class of p-groups can be classified *via* their Ulm–Kaplansky invariants and these new invariants, taken for limit ordinals σ not cofinal with ω. In addition to totally projective p-groups, this class includes all $p^\sigma A$-high subgroups of totally projective p-groups A for limit ordinals σ.

There are various other important aspects of the theory of p-groups which are completely omitted from our discussions. Of these, the generalized Kulikov's criterion by Megibben [9] is most interesting, and so is his generalization of the theory of basic subgroups (cf. Megibben [12]). The latter theory has been extended recently by P. Crawley, covering p-groups of arbitrary limit length λ where λ is cofinal with ω. For some topological aspects, we refer to P. F. Dubois [1], Mines [1], and Waller [1]. The final rank of p-groups has been generalized by Cutler and Dubois [1]. Properties of torsion groups which are expressible in certain infinitary languages have been investigated by J. Barwise and P. Eklof [*Ann. Math. Logic* **2** (1970–71), 25–68].

Problem 59. Characterize the p-groups which can be embedded in direct sums of countable reduced p-groups [or, more generally, in totally projective p-groups].

Problem 60. Let G be a fully invariant subgroup of the p-group A. Find criteria under which a homomorphism $G \to A$, not decreasing heights, is induced by an endomorphism of A.

Problem 61. Investigate p-groups for which any two (finite) direct decompositions have isomorphic refinements.

Problem 62. Which subgroup of $\mathrm{Ext}(C, A)$ corresponds to the extensions in which A is nice?

Problem 63. Imitating the definition of pure-complete p-groups, call a p-group A *isotype-complete* if every subsocle of A which supports an isotype subgroup in a suitable p-group containing A as an isotype subgroup also supports an isotype subgroup of A. Investigate isotype-complete p-groups.

Problem 64. Find a complete system of invariants for p-groups of finite type whose Ulm factors are torsion-complete.

Richman [3] discussed this problem in case the first Ulm subgroup was an elementary p-group.

Problem 65. Which are the compact abelian groups whose duals are totally projective p-groups?

XIII

TORSION-FREE GROUPS

Whereas there are fairly large classes of torsion groups whose structures can be described in terms of satisfactory invariants, there are only a very few and rather restricted classes of torsion-free groups for which satisfactory structure theory is known. These include the torsion-free groups of rank 1 and their direct sums, but no other major classes; even for groups of finite rank no useful complete systems of invariants are known. Naturally, one can establish certain schemes for constructing torsion-free groups, which provide a certain amount of information about their structures, but the schemes so far known fail to give an acceptable solution to the basic problem of deciding when two groups given by different schemes are isomorphic.

The most fundamental difference in the behavior of torsion and torsion-free groups lies perhaps in their direct decompositions. While indecomposable torsion groups ought to be of rank 1, there are as many indecomposable torsion-free groups of rank $\mathfrak{m}$ as groups of the same rank [at least, up to the first inaccessible cardinal]. Moreover, even in the finite-rank case, a torsion-free group can have direct decompositions, of surprisingly different nature, into indecomposable groups.

We shall present several results on the direct products of rank 1 groups and on their subgroups, including some interesting facts, discovered only recently. The attractive theory of slender groups will also be discussed.

85. TORSION-FREE GROUPS OF RANK 1

In torsion-free groups the concept corresponding to the height is of utmost importance in distinguishing between elements. We start our discussion with this concept.

In this section, $p_1, p_2, \cdots, p_n, \cdots$ will denote the sequence of all rational primes, say, in increasing order of magnitude.

Given a prime p, the largest integer k such that $p^k \mid a$ holds in the torsion-free group A ($a \in A$) is called the *p-height* $h_p(a)$ of a; if no such maximal integer k exists, then we set $h_p(a) = \infty$ [see **1**]. The sequence of p-heights

$$\chi(a) = (h_{p_1}(a), \cdots, h_{p_n}(a), \cdots)$$

is said to be the *characteristic* or the *height-sequence* of a; since this depends on A, we sometimes write $\chi_A(a)$ to emphasize the role of A. Thus a characteristic is an ordered sequence of nonnegative integers and symbols ∞, and two characteristics $(k_1, \cdots, k_n, \cdots)$ and $(l_1, \cdots, l_n, \cdots)$ are equal if and only if $k_n = l_n$ for every n.

The following remarks are obvious consequences of the definition.

(a) $\chi(-a) = \chi(a)$ for all $a \in A$.

(b) If $\chi(a) = (k_1, \cdots, k_n, \cdots)$, then a is divisible by the integer $m = p_1^{l_1} \cdots p_r^{l_r}$ exactly if $l_i \leqq k_i$ for $i = 1, \cdots, r$.

(c) If $\chi(a) = (k_1, \cdots, k_n, \cdots)$, then

$$\chi(p_n a) = (k_1, \cdots, k_{n-1}, k_n + 1, k_{n+1}, \cdots)$$

[we set $\infty + 1 = \infty$].

(d) Every sequence $(k_1, \cdots, k_n, \cdots)$ of nonnegative integers and symbols ∞ is a characteristic, namely, in the subgroup R of Q, generated by all $p_n^{-l_n}$ with $l_n \leqq k_n$, for all n, the element 1 will have this characteristic. With this R, we can write $Ra = \langle a \rangle_*$ if $\chi(a) = (k_1, \cdots, k_n, \cdots)$ in a group A.

If we mean by $(k_1, \cdots, k_n, \cdots) \geqq (l_1, \cdots, l_n, \cdots)$ that $k_n \geqq l_n$ for all n, then the set of all characteristics becomes a [complete, distributive] lattice under the pointwise operations:

$$(k_1, \cdots, k_n, \cdots) \cap (l_1, \cdots, l_n, \cdots) = (\min(k_1, l_1), \cdots, \min(k_n, l_n), \cdots),$$

where for $\cup$, "min" is replaced by "max." In this lattice, $(0, \cdots, 0, \cdots)$ is the minimum and $(\infty, \cdots, \infty, \cdots)$ the maximum element. For torsion-free A, we clearly have:

(e) $\chi_C(c) \leqq \chi_A(c)$ for all c in a subgroup C of A; and C is pure in A if and only if equality prevails for all $c \in C$.

(f) $\chi(b + c) \geqq \chi(b) \cap \chi(c)$ for all $b, c \in A$.

(g) If $A = B \oplus C$ and $b \in B$, $c \in C$, then $\chi(b + c) = \chi(b) \cap \chi(c)$.

(h) For any homomorphism $\alpha: A \to B$ and for any $a \in A$, $\chi_A(a) \leqq \chi_B(\alpha a)$.

(j) If C is a pure subgroup of A and a^* is a coset of A mod C, then

$$\chi_{A/C}(a^*) = \bigcup_{a \in a^*} \chi_A(a).$$

A concept which is derived from the characteristic and which is basic for torsion-free groups is the type. Two characteristics, $(k_1, \cdots, k_n, \cdots)$ and

$(l_1, \cdots, l_n, \cdots)$, will be considered as *equivalent* if $k_n \neq l_n$ holds only for a finite number of n such that in case $k_n \neq l_n$ both k_n and l_n are finite. In other words, $\sum_n |k_n - l_n|$ is finite [if we set $\infty - \infty = 0$]. An equivalence class of characteristics is called a *type*. If $\chi(a)$ belongs to the type $\mathbf{t}$, then we say that *a is of type* $\mathbf{t}$ and write $\mathbf{t}(a) = \mathbf{t}$, or, more explicitly, $\mathbf{t}_A(a)$ if it is necessary to indicate that the type of a is computed in A.

We shall represent a type $\mathbf{t}$ by a characteristic in this class. In other words, we write $\mathbf{t} = (k_1, \cdots, k_n, \cdots)$ and keep in mind that here the characteristic can be replaced by an equivalent one. Since the equivalence of characteristics is compatible with the lattice operations introduced above, the set of types is again a (distributive) lattice. Thus, by definition, $\mathbf{t} \geqq \mathbf{s}$ holds between the types $\mathbf{t}$ and $\mathbf{s}$ if there are characteristics $(k_1, \cdots, k_n, \cdots)$ and $(l_1, \cdots, l_n, \cdots)$ in $\mathbf{t}$ and $\mathbf{s}$, respectively, such that $(k_1, \cdots, k_n, \cdots) \geqq (l_1, \cdots, l_n, \cdots)$.

(A) If a and b are dependent in A [i.e., $ma = rb$ for integers $m, r \neq 0$], then $\mathbf{t}(a) = \mathbf{t}(b)$.

(B) $\mathbf{t}(b + c) \geqq \mathbf{t}(b) \cap \mathbf{t}(c)$ for all $b, c \in A$.

(C) If $A = B \oplus C$ and $b \in B$, $c \in C$, then $\mathbf{t}(b + c) = \mathbf{t}(b) \cap \mathbf{t}(c)$.

(D) For $\alpha: A \to B$ and $a \in A$, $\mathbf{t}(a) \leqq \mathbf{t}(\alpha a)$.

With every type $\mathbf{t}$, we can associate two fully invariant subgroups of the torsion-free group A which are useful tools in the theory. The set $A(\mathbf{t})$ of elements a in A whose types are $\geqq \mathbf{t}$ form, in view of (B), a subgroup of A, which must be pure because of (A). Furthermore, the elements of types $> \mathbf{t}$ generate a subgroup in A which will be denoted by $A^*(\mathbf{t})$. The inclusion

$$A^*(\mathbf{t}) \leqq A(\mathbf{t})$$

is obvious.

A torsion-free group A in which all the elements $\neq 0$ are of the same type $\mathbf{t}$ is called *homogeneous*. In this case, $A(\mathbf{t}) = A$ and $A^*(\mathbf{t}) = 0$. Evidently, every rank 1 group is homogeneous.

The study of torsion-free groups is based, to a certain extent, on our knowledge of rank 1 groups. The rest of this section is devoted to torsion-free groups of rank 1.

From **24** we know that every torsion-free group R of rank 1 is isomorphic to a subgroup of Q. In other words, a torsion-free group of rank 1 is essentially a rational group.

Since R is homogeneous, we can assign the type $\mathbf{t}$ to R, where $\mathbf{t}$ is the type common to all nonzero elements of R. Isomorphic groups are obviously associated with the same type. We next show that if R and S are rank 1 groups of the same type $\mathbf{t}$, then they are necessarily isomorphic:

Theorem 85.1 (Baer [6]). *Two torsion-free groups of rank* 1 *are isomorphic if and only if they are of the same type. Every type is realized by a rational group.*

Let $a \in R$ and $b \in S$ be arbitrary nonzero elements, and write $\chi(a) = (k_1, \cdots, k_n, \cdots)$, $\chi(b) = (l_1, \cdots, l_n, \cdots)$. If these characteristics belong to the same type **t**, then there are but a finite number of indices, say $n_1, \cdots, n_t$, where $k_{n_i} \neq l_{n_i}$. These k_{n_i} and l_{n_i} are nonnegative integers, so we can divide a by $p_{n_1}^{k_{n_1}} \cdots p_{n_t}^{k_{n_t}}$ and b by $p_{n_1}^{l_{n_1}} \cdots p_{n_t}^{l_{n_t}}$ to obtain elements $c \in R$ and $d \in S$ in whose characteristics 0 stands at the places $n_1, \cdots, n_t$. Evidently, $\chi(c) = \chi(d)$. Consequently, the equation $mx = nc$ is solvable in R if and only if the equation $my = nd$ admits a solution in S. Such equations have at most one solution in torsion-free groups, therefore, letting $x \leftrightarrow y$, a one-to-one correspondence arises between R and S which is readily seen to be addition preserving.

The second assertion follows at once from (d).□

Thus we have established a one-to-one correspondence between torsion-free groups of rank 1 and types. Since types are described in terms of sequences of nonzero integers and ∞, and since the equality of two types can easily be decided, (85.1) is a satisfactory structure theorem for rank 1 torsion-free groups.

Example 1. We clearly have $\mathbf{t}(Z) = (0, \cdots, 0, \cdots)$, $\mathbf{t}(Q) = (\infty, \cdots, \infty, \cdots)$. Also, $\mathbf{t}(Q_p) = (\infty, \cdots, \infty, 0, \infty, \cdots)$, $\mathbf{t}(Q^{(p)}) = (0, \cdots, 0, \infty, 0, \cdots)$, where 0 and ∞, respectively, stand at the nth place if $p = p_n$.

Example 2. The type of rationals with square-free denominators is $(1, 1, \cdots, 1, \cdots)$.

Corollary 85.2. *The cardinality of the set of all nonisomorphic torsion-free groups of rank* 1 *is the power of the continuum.*

The set of all characteristics is of the power $2^{\aleph_0}$, so the set of types cannot have a larger cardinality. On the other hand, distinct characteristics consisting of 0s and ∞s only belong to different types, thus the cardinality in question is at least $2^{\aleph_0}$. From (85.1) we know that every type is realized by a rank 1 group; this completes the proof.□

If $\chi = (k_1, \cdots, k_n, \cdots)$ and $\chi_1 = (l_1, \cdots, l_n, \cdots)$ are two characteristics, then their *product* is defined as

$$\chi\chi_1 = (k_1 + l_1, \cdots, k_n + l_n, \cdots),$$

where, naturally, the sum of ∞ and anything is ∞. A characteristic χ is *idempotent* [i.e., $\chi^2 = \chi$] exactly if, for every n, either $k_n = 0$ or $k_n = \infty$. The multiplication of characteristics is compatible with the equivalence relation introduced above, so we may speak of the *product* $\mathbf{tt}_1$ of types $\mathbf{t}, \mathbf{t}_1$, and of an *idempotent type* $\mathbf{t}^2 = \mathbf{t}$.

We also define the *quotient* $\chi_1:\chi_2$ of two characteristics $\chi_1 \geqq \chi_2$ as the largest characteristic χ such that $\chi\chi_2 \leqq \chi_1$. It is easy to see that this χ exists and satisfies: $\chi'\chi_2 \leqq \chi_1$ is equivalent to $\chi' \leqq \chi$. The quotient of types is defined analogously.

We conclude this section with two elementary results.

Proposition 85.3. *If A and C are torsion-free groups of rank* 1, *then $A \otimes C$ is of rank* 1 *and*

$$\mathbf{t}(A \otimes C) = \mathbf{t}(A) \cdot \mathbf{t}(C).$$

From (60.6) we infer that $A \otimes C$ is a subgroup of $Q \otimes Q \cong Q$, thus it is of rank 1. Write $\chi(a_0) = (k_1, \cdots, k_n, \cdots)$ and $\chi(c_0) = (l_1, \cdots, l_n, \cdots)$ for some nonzero $a_0 \in A$, $c_0 \in C$. Then $a_0 \otimes c_0 \in A \otimes C$ will obviously be divisible by p_n^m for $m \leqq k_n + l_n$, thus $\chi(a_0 \otimes c_0) \geqq \chi(a_0)\chi(c_0)$. Since, for rationals r, s with $ra_0 \in A$, $sc_0 \in C$, the correspondence $ra_0 \otimes sc_0 \mapsto rs$ is a bilinear mapping of $A \times C$ into the rational group R, where $\chi_R(1) = \chi(a_0)\chi(c_0)$, from the definition of tensor products we see that $\chi(a_0 \otimes c_0) > \chi_R(1)$ is impossible.□

Proposition 85.4. *If A and C are torsion-free groups of rank* 1, *then* $\mathrm{Hom}(A, C)$ *is* 0 *if* $\mathbf{t}(A)$ *is not* $\leqq \mathbf{t}(C)$, *and is a torsion-free group of rank* 1 *and of type* $\mathbf{t}(C)\colon \mathbf{t}(A)$ *if* $\mathbf{t}(A) \leqq \mathbf{t}(C)$.

From (D) we infer that $\mathrm{Hom}(A, C) \neq 0$ only if $\mathbf{t}(A) \leqq \mathbf{t}(C)$. In this case we choose nonzero $a \in A$ and $c \in C$ with $\chi(a) \leqq \chi(c)$, and notice that there is a unique $\eta\colon A \to C$ such that $\eta a = c$. It is easy to check that this η will have the characteristic $\chi(c)\colon\chi(a)$ and there are no homomorphisms of A into C other than rational multiples of η.□

Exercises

1. Let $a \in A$ with torsion-free A. The set $S(a)$ of all integers $n > 0$ with $n \mid a$ satisfies the following conditions:
   (i) $1 \in S(a)$;
   (ii) $n \in S(a)$ and $m \mid n$ implies $m \in S(a)$;
   (iii) $m, n \in S(a)$ implies $[m, n] \in S(a)$.
   Show that, for $a, b \in A$, $S(a) = S(b)$ if and only if $\chi(a) = \chi(b)$.
2. For every type $\mathbf{t} \neq (\infty, \cdots, \infty, \cdots)$, there exists a countably infinite set of characteristics of type $\mathbf{t}$.
3. If $\mathbf{t} = (k_1, \cdots, k_n, \cdots)$, then $A(\mathbf{t})$ is the pure subgroup generated by $\bigcap_n p_n^{k_n} A$.
4. If C is pure in A, then $C(\mathbf{t})$ is pure in $A(\mathbf{t})$.
5. Give examples such that:
   (a) the elements of types $> \mathbf{t}$ do not form a subgroup;
   (b) $A^*(\mathbf{t})$ is not pure in A;
   (c) A contains elements of type $\mathbf{t}$ and $A^*(\mathbf{t}) = A(\mathbf{t})$.

6. If A contains elements of type $\mathbf{t}$, then $A(\mathbf{t})$ is generated by elements of type $\mathbf{t}$.
7. Groups A, B of rank 1 are not isomorphic if and only if there exists an infinite set Ω of prime powers p^k such that exactly one of $\bigcap p^kA$ and $\bigcap p^kB$ [with $p^k \in \Omega$] is different from 0.
8. (a) Let A, B be of rank 1. B is isomorphic to a subgroup of A if and only if $\mathbf{t}(B) \leqq \mathbf{t}(A)$.
   (b) A group of rank 1 contains either a finite or a continuum number of pairwise nonisomorphic subgroups.
9. Characterize the p-adic and the Z-adic completions of rank 1 groups.
10. (Baer [6]) If A is of finite rank, then the set of types of elements in A satisfy both the maximum and the minimum conditions. [*Hint*: ranks of $A(\mathbf{t})$.]
11. (Baer [6]) There exists a torsion-free group A of rank 2 such that the set of types of elements in A is infinite. [*Hint*: in $Qa \oplus Qb$ choose A such that $\chi(a + p_n b) = (0, \cdots, 0, \infty, 0, \cdots)$ with ∞ at the nth place, for all primes p_n.]
12. The cardinality of the set of nonisomorphic torsion-free groups of finite rank is the continuum.
13. Describe the quotient groups of torsion-free groups of rank 1.

86. COMPLETELY DECOMPOSABLE GROUPS

There are only a very few classes of torsion-free groups which are easy to handle. One of these is the class of completely decomposable groups.

A torsion-free group A is said to be *completely decomposable* if it is a direct sum of rank 1 groups. Free groups and divisible groups are trivial examples for completely decomposable groups.

Proposition 86.1 (Baer [6]). *Any two direct decompositions of a completely decomposable group into direct sums of rank* 1 *groups are isomorphic.*

Suppose $A = \bigoplus_{i \in I} A_i$, where the A_i are rational groups. It is readily seen directly or from (9.3) that for any type $\mathbf{t}$, $A(\mathbf{t})$ and $A^*(\mathbf{t})$ are the direct sums of those A_i which are of types $\geqq \mathbf{t}$ and $> \mathbf{t}$, respectively. Thus

$$A_{\mathbf{t}} = A(\mathbf{t})/A^*(\mathbf{t})$$

is isomorphic to the direct sum of the A_i which are exactly of type $\mathbf{t}$. We see that the rank of $A_{\mathbf{t}}$ is precisely the number of the components A_i of type $\mathbf{t}$. Since $A_{\mathbf{t}}$ is defined independently of the direct decomposition of A, the assertion is immediate from the uniqueness of the rank. □

In particular, *the ranks $r(A_{\mathbf{t}})$ taken for all types* $\mathbf{t}$ *form a complete and independent system of invariants for completely decomposable groups A.* This settles the structure problem for completely decomposable groups.

Let C be a pure subgroup of the torsion-free group A. We say $a \in A \backslash C$ is *proper with respect to C* if

$$\chi(a) \geqq \chi(a + c) \qquad \text{for all} \quad c \in C.$$

This amounts to $\chi_A(a) = \chi_{A/C}(a + C)$. The pure subgroup C of A is said to be *balanced* [or *regular*] if every coset of A mod C contains an element proper with respect to C.

The reader can easily check for himself that this definition of balanced subgroup is in accordance with the definition given in **80** for p-groups. [Note that isotypeness reduces to purity in torsion-free groups.]

Direct summands are always balanced, but the converse is not true [see Ex. 2]. However, additional conditions may force a balanced subgroup to be a summand, as we shall see in (86.3).

Theorem 86.2. *Completely decomposable groups have the projective property relative to all balanced-exact sequences of torsion-free groups.*

The projective property is inherited by direct sums, thus it suffices to establish the assertion for rank one groups. Let R be of rank 1 in the diagram

$$\begin{tikzcd}
& & & R \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\
0 \arrow[r] & C \arrow[r] & A \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

where all groups are torsion-free and C is balanced in A. Pick any nonzero $x \in R$ and let $a \in A$ be an element in the coset $\phi(x)$ which is proper with respect to C. In view of $\chi(x) \leqq \chi(\phi(x)) = \chi(a)$, the correspondence $x \mapsto a$ extends to a homomorphism $\psi: R \to A$ rendering the triangle commutative.□

The converse of (86.2) holds true [see Ex. 17]. The following corollary is an immediate consequence of (86.2).

Corollary 86.3 (Lyapin [5]). *If C is a balanced subgroup of the torsion-free group A such that A/C is completely decomposable, then C is a summand of A.*□

The next lemma is technical.

Lemma 86.4 (Baer [6]). *Let C be a pure subgroup of A. If every element of the coset $a + C$ has the same type in A as the coset $a + C$ has in A/C, then $a + C$ contains an element proper with respect to C.*

From $\chi(a) \leqq \chi(a + C)$ and $\mathbf{t}(a) = \mathbf{t}(a + C)$, it is clear that there exist a finite number of elements $a_1, \cdots, a_k \in a + C$ such that $\chi(a_1) \cup \cdots \cup \chi(a_k) = \chi(a + C)$. Therefore, it will be enough to show that if $a \equiv b \bmod C$ with equivalent $\chi(a), \chi(b)$, then there exists a $g \in a + C$ satisfying $\chi(g) \geqq \chi(a) \cup \chi(b)$. Hypothesis implies the existence of relatively prime integers m, n such that $\chi(ma) = \chi(nb)$. If the integers u, v satisfy $mu + nv = 1$, then $g = mua + nvb \equiv a \bmod C$ obviously satisfies $\chi(g) \geqq \chi(ma) \geqq \chi(a) \cup \chi(b)$.□

It is now easy to verify the following useful result.

Proposition 86.5 (Baer [6]). *Let C be a pure subgroup of the torsion-free group A such that:*

(a) *A/C is completely decomposable and homogeneous of type* $\mathbf{t}$;
(b) *all the elements in $A \backslash C$ are of type* $\mathbf{t}$.

Then C is a summand of A.

From (86.4) we know that C is a balanced subgroup of A. The rest follows at once from (86.3).□

Our next result generalizes the familiar theorem that subgroups of free groups are again free.

Theorem 86.6 (Baer [6], Kolettis [3]). *If A is completely decomposable and homogeneous of type* $\mathbf{t}$, *then every subgroup C of A which is homogeneous of type* $\mathbf{t}$ [*in particular, every pure subgroup of A*] *is completely decomposable.*

The proof is the same as that of (14.5). We write $A = \bigoplus_\sigma G_\sigma$, where each G_σ is a rational group of type $\mathbf{t}$ and the index σ ranges over all ordinals less than some ordinal τ. Define

$$A_\sigma = \bigoplus_{\rho < \sigma} G_\rho \qquad \text{and} \qquad C_\sigma = A_\sigma \cap C.$$

Clearly, $C_\sigma = C_{\sigma+1} \cap A_\sigma$, thus

$$C_{\sigma+1}/C_\sigma \cong (C_{\sigma+1} + A_\sigma)/A_\sigma \leqq A_{\sigma+1}/A_\sigma \cong G_\sigma.$$

We claim that C_σ is a summand of $C_{\sigma+1}$. In fact, if $C_{\sigma+1} \neq C_\sigma$, then all the elements of $C_{\sigma+1}$ are of type $\mathbf{t}$ and the type of $C_{\sigma+1}/C_\sigma$ is $\mathbf{t}$ [this group is isomorphic to a subgroup of G_σ and is an epimorphic image of $C_{\sigma+1}$]. Hence (86.5) implies $C_{\sigma+1} = C_\sigma \oplus B_\sigma$ for some $B_\sigma \leqq C$. It follows that the B_σ ($\sigma < \tau$) generate their direct sum which must be C.□

We turn our attention to the fundamental result:

Theorem 86.7 (Baer [6], Kulikov [4], Kaplansky [4]). *Direct summands of completely decomposable torsion-free groups are completely decomposable.*

We start with the finite rank case. Let $A = H_1 \oplus \cdots \oplus H_n = B \oplus C$, where each H_i is completely decomposable, of finite rank, and homogeneous of type $\mathbf{t}_i$. We use induction on n. If $n = 1$, then A is homogeneous and the complete decomposability of B follows from (86.6). Suppose the types $\mathbf{t}_1, \cdots, \mathbf{t}_n$ ($n > 1$) are distinct and $\mathbf{t}_n$ is a maximal among them. Then, by (9.3), $H_n = A(\mathbf{t}_n) = B(\mathbf{t}_n) \oplus C(\mathbf{t}_n)$, whence $B = B(\mathbf{t}_n) \oplus B'$ and $C = C(\mathbf{t}_n) \oplus C'$ for some $B' \leqq B$ and $C' \leqq C$. Therefore, $A = B' \oplus C' \oplus H_n$ and $H_1 \oplus \cdots \oplus H_{n-1} \cong B' \oplus C'$. Induction hypothesis implies that $B(\mathbf{t}_n)$ and B' are completely decomposable, and so is therefore B.

To settle the general case, we refer to (87.5) [in whose proof only the preceding paragraph will be made use of]. Every summand of a completely decomposable group is by (9.10) a direct sum of countable groups. By (87.5), these summands are separable and thus, on account of (87.1), completely decomposable.□

For later purposes we prove here the following lemma.

Lemma 86.8. *Let A be a completely decomposable homogeneous group of finite rank. Then every pure subgroup of A is a summand of A.*

The group A may be written in the form $A = Ra_1 \oplus \cdots \oplus Ra_k$, where R is a subgroup of Q. A straightforward generalization of (15.3) shows that if $n_1, \cdots, n_k$ are relatively prime integers, then there exist $b_1, \cdots, b_k \in A$ such that

$$A = Rb_1 \oplus \cdots \oplus Rb_k \qquad \text{with} \quad b_1 = n_1 a_1 + \cdots + n_k a_k .$$

Now if C is a pure subgroup of rank 1 in A, then it can be written in the form $C = Rc$ with $c = n_1 a_1 + \cdots + n_k a_k$ where $n_1, \cdots, n_k$ are relatively prime integers, and the preceding sentence implies that C is a summand of A. If C is a pure subgroup of any rank and if C' is a rank 1 pure subgroup of C, then by what has been proved, C' is a summand of A, that is, $A = C' \oplus A'$ and hence $C = C' \oplus C''$ for some $A' < A$ and $C'' = C \cap A'$. Here A' is by (86.6) again completely decomposable, and since C'' is pure in A', a simple induction concludes the proof.□

Exercises

1. (Haimo [1]) Let B, C be pure subgroups of A such that $C \leqq B$. Verify the following:
   (a) If C is balanced in A, then it is balanced in B.
   (b) If B is balanced in A, then B/C is balanced in A/C.
   (c) If C is balanced in A and B/C balanced in A/C, then B is balanced in A.

2. Let F be a free group and G a subgroup such that F/G is homogeneous of type $(0, \cdots, 0, \cdots)$, but not free. Show that G is balanced in F, but it is not a summand.
3. (a) The set of types of elements in a completely decomposable group is closed under intersection.
   (b) Determine the least upper bound for the numbers of different types in completely decomposable groups of rank n.
4. (de Groot [1]) If two groups are completely decomposable and each is isomorphic to a pure subgroup of the other, then they are isomorphic. [*Hint*: if $\phi: A \to B$ is a pure embedding, then $\phi A(\mathbf{t}) = \phi A \cap B(\mathbf{t})$, the same for $A^*(\mathbf{t})$, and so ϕ induces an embedding of $A(\mathbf{t})/A^*(\mathbf{t})$ in $B(\mathbf{t})/B^*(\mathbf{t})$.]
5. (Baer [6]) If C is pure in A such that $C^*(\mathbf{t}) = C \cap A^*(\mathbf{t})$ and $A(\mathbf{t})/A^*(\mathbf{t})$ is completely decomposable, then $C(\mathbf{t})$ is the direct sum of $C^*(\mathbf{t})$ and a completely decomposable group.
6*. A pure subgroup of a completely decomposable group of finite rank need not be completely decomposable. [*Hint*: some pure subgroup of rank 2 in $\langle p_1^{-\infty}a_1, p_2^{-\infty}a_1\rangle \oplus \langle p_1^{-\infty}a_2, p_3^{-\infty}a_2\rangle \oplus \langle p_2^{-\infty}a_3, p_3^{-\infty}a_3\rangle$, where p_1, p_2, p_3 are different primes (for notation see **88**).]
7. If A is completely decomposable of finite rank and B an essential subgroup of A, then A/B is finite if and only if the elements of B are of the same type in B as in A.
8. (Fuchs, Kertész, and Szele [2]) A torsion-free group A has the property that all pure subgroups of A are summands of A if and only if the reduced part of A is a homogeneous completely decomposable group of finite rank. [*Hint*: reduce to the finite rank case and use (86.8).]
9. Any two direct decompositions of a completely decomposable group have isomorphic refinements.
10. (Griffith [3]) Every torsion-free group A is a pure-essential extension of some completely decomposable subgroup C such that $|A| \leqq |C|^{\aleph_0}$. [*Hint*: select a maximal pure completely decomposable subgroup.]
11. (Procházka [10]) If A is completely decomposable such that the different types of the rank one components are inversely well ordered, then pure subgroups of A are again completely decomposable.
12*. (Procházka [14]) A reduced torsion-free group A has the property that $A \cong C \oplus A/C$ for all pure subgroups C of A if and only if A is a completely decomposable group of finite rank such that the types of the rank 1 summands are totally ordered.
13. (Fuchs [19]) Verify (86.7) for countable groups as follows. Write $A(\mathbf{t}) = A^*(\mathbf{t}) \oplus A_{\mathbf{t}}$, and similarly for B, C.
   (a) For any finite set $\mathbf{t}_1, \cdots, \mathbf{t}_n$ of types, show that $A = B_{\mathbf{t}_1} \oplus C_{\mathbf{t}_1} \oplus \cdots \oplus B_{\mathbf{t}_n} \oplus C_{\mathbf{t}_n} \oplus (\bigoplus_{\mathbf{t} \neq \mathbf{t}_i} A_{\mathbf{t}})$. [*Hint*: induct starting with a smallest $\mathbf{t}_i$.]

(b) Setting $B' = B \cap [C_{\mathbf{t}_1} \oplus \cdots \oplus C_{\mathbf{t}_n} \oplus (\bigoplus_{\mathbf{t} \neq \mathbf{t}_i} A_{\mathbf{t}})]$ and similarly for C', show that $A = A_{\mathbf{t}_1} \oplus \cdots \oplus A_{\mathbf{t}_n} \oplus B' \oplus C'$.
(c) Prove that $B = X_{\mathbf{t}_1} \oplus \cdots \oplus X_{\mathbf{t}_n} \oplus B'$ with $X_{\mathbf{t}_i} \cong B_{\mathbf{t}_i}$ and $X_{\mathbf{t}_i} \leqq A_{\mathbf{t}_1} \oplus \cdots \oplus A_{\mathbf{t}_n}$.
(d) Show that passing from n to $n+1$, the $X_{\mathbf{t}_i}$s need not be changed.
(e) B will be the direct sum of the $X_{\mathbf{t}_i}$.

14. (Baer [6]) Define classes Γ_σ of torsion-free groups as follows: Let Γ_1 be the class of all countable torsion-free groups. If $\sigma > 1$ is any ordinal and if Γ_ρ have been defined for all $\rho < \sigma$, then we let $A \in \Gamma_\sigma$ if A contains a pure subgroup C of finite rank such that A/C is a direct sum of groups each of which belongs to a class Γ_ρ with $\rho < \sigma$.
Prove that a homogeneous group A of type $\mathbf{t}$ is completely decomposable exactly if, for all pure subgroups C of finite rank, A/C is homogeneous of type $\mathbf{t}$ and A belongs to some class Γ_σ. [*Hint*: for sufficiency, induct on σ.]
15. Show that the direct product of infinitely many cyclic groups does not belong to any class Γ_σ. [*Hint*: (19.2).]
16. For every torsion-free group A, there exist a completely decomposable group G and an epimorphism $\phi: G \to A$ such that Ker ϕ is balanced in G. [*Hint*: select monomorphisms $\phi_i: R_i \to A$ such that $\{\mathrm{Im}\, \phi_i\}$ be the set of all pure subgroups of rank 1 in A, and set $G = \bigoplus_i R_i$.]
17. A torsion-free group has the projective property relative to all balanced-exact sequences of torsion-free groups if and only if it is completely decomposable. [*Hint*: (86.2), Ex. 16 and (86.7).]

87. SEPARABLE GROUPS

Next we turn our attention to a class of torsion-free groups which is more general than completely decomposable groups. The groups in this class may be regarded as "locally" completely decomposable in the following sense.

A torsion-free group A is called *separable* [see Baer [6]] if every finite subset of elements of A is contained in a completely decomposable direct summand of A. Clearly, this summand may then be assumed to be of finite rank.

Let us start with the following elementary facts.

(a) A is separable if and only if its reduced part is separable.
(b) Direct sums of separable groups are again separable.
(c) If A is separable, then for every type $\mathbf{t}$, $A^*(\mathbf{t})$ is a pure subgroup of A.
(d) *Every full invariant subgroup C of a separable group A is likewise separable.*

In fact, if $c_1, \cdots, c_k \in C$, then by definition there exists a decomposition $A = A_1 \oplus \cdots \oplus A_n \oplus A'$ such that $A_1, \cdots, A_n$ are of rank 1 and $c_1, \cdots, c_k \in A_1 \oplus \cdots \oplus A_n$. Hence (9.3) yields

$$C = (C \cap A_1) \oplus \cdots \oplus (C \cap A_n) \oplus (C \cap A'),$$

where each of $C \cap A_1, \cdots, C \cap A_n$ is either 0 or of rank 1, and $c_1, \cdots, c_k$ are in their direct sum.

(e) *If C is a fully invariant pure subgroup of a separable group A, then A/C is separable.*

Clearly, A/C is torsion-free. Given $a_1 + C, \cdots, a_k + C$, there is a decomposition $A = A_1 \oplus \cdots \oplus A_n \oplus A'$ such that $A_1, \cdots, A_n$ are of rank 1 and $a_1, \cdots, a_k$ belong to $A_1 \oplus \cdots \oplus A_n$. Let $A_1 \cap C = \cdots = A_m \cap C = 0$ and $A_{m+1}, \cdots, A_n \leqq C$. Then

$$A/C = (A_1 + C)/C \oplus \cdots \oplus (A_m + C)/C \oplus (A' + C)/C,$$

where the direct sum of the first m summands contains the given cosets $a_1 + C, \cdots, a_k + C$.

(f) *If A is separable and $\mathbf{t}$ is any type, then $A(\mathbf{t})/A^*(\mathbf{t})$ is separable.*

This follows at once from (d), (c), and (e).

Important examples for separable torsion-free groups [that are not completely decomposable] are certain direct products of rank 1 groups. For instance, the proof of (19.2) shows that *direct products of infinite cyclic groups are always separable.* The next result indicates, however, that only the uncountable separable groups yield something new.

Theorem 87.1 (Baer [6]). *A countable separable torsion-free group is completely decomposable.*

Let A be such a group and $a_1, \cdots, a_n, \cdots$ a generating system of A. There is a completely decomposable direct summand A_1 of A which is of finite rank and contains a_1. Assume that we have constructed an ascending chain $A_1 \leqq \cdots \leqq A_{n-1}$ of completely decomposable direct summands of finite rank in A such that $a_1, \cdots, a_i$ are contained in A_i ($i = 1, \cdots, n-1$). There is then a completely decomposable direct summand A_n of finite rank of A which contains both [a maximal independent system of] A_{n-1} and a_n. Evidently, the union of the ascending chain $A_1 \leqq \cdots \leqq A_n \leqq \cdots$ is A. Write $B_1 = A_1$ and $A_n = A_{n-1} \oplus B_n$ for $n \geqq 1$, to obtain $A = \bigoplus_{n=1}^{\infty} B_n$; here B_n is completely decomposable as a summand of A_n [see (86.7), proven part].□

Homogeneous separable groups are particularly interesting in view of the next two results.

Proposition 87.2 (Baer [6]). *A homogeneous group A is separable exactly if every finite rank pure subgroup of A is a summand of A.*

Assuming A separable, a pure subgroup C of finite rank in A can be embedded in a finite rank completely decomposable summand B of A. By homogeneity, (86.8) implies C is a summand of B and hence of A. Conversely, if pure subgroups C of finite rank are summands of A, then every pure subgroup of C is a summand of C, and it follows at once that C is completely decomposable.□

Corollary 87.3. *Pure subgroups of homogeneous separable groups are separable.*

Let C be a pure subgroup in a homogeneous separable group A. If $c_1, \cdots, c_k \in C$, then $\langle c_1, \cdots, c_k \rangle_*$ is by (87.2) a summand of A and hence of C. (87.2) also implies C separable.□

So far no satisfactory description of the structure of separable torsion-free groups is known. In the homogeneous case, some information is given by the next result.

Proposition 87.4. *A homogeneous torsion-free group of type* $\mathbf{t}$ *is separable if and only if it is isomorphic to a pure subgroup of a group* $(\prod_i R_i)(\mathbf{t})$ *where* R_i *are of rank* 1 *and of type* $\mathbf{t}$.

First we establish the separability of $G = (\prod R_i)(\mathbf{t})$. This G is clearly homogeneous of type $\mathbf{t}$. We can write $\prod R_i = \prod(Ra_i)$ with R a rational group of type $\mathbf{t}$, thus a $g \in G$ has the form $g = (\cdots, r_i a_i, \cdots)$ with $r_i \in R$. If g is such that $\langle g \rangle_* = Rg$, then the denominators of r_i are divisible only by primes p at which $\mathbf{t}$ is ∞, so we can write $r_i = s_i n_i$, where s_i is a rational number whose numerator and denominator are divisible only by primes at which $\mathbf{t}$ is ∞ and n_i is an integer divisible by primes only at which $\mathbf{t}$ is finite. If there is an index j with $|n_j| = 1$, then $\prod R_i = Rg \oplus H_j$, where H_j is the product of the Ra_i with $i \neq j$. If there is no such j, then—as in the proof of (19.2)—we can find a finite number of elements $h_1, \cdots, h_k$ in G such that $\prod R_i = Rh_1 \oplus \cdots \oplus Rh_k \oplus H'$, where $g \in Rh_1 \oplus \cdots \oplus Rh_k$ and H' is the product of almost all Ra_i. By induction on n, follows the existence of a similar decomposition of $\prod R_i$ with $g_1, \cdots, g_n \in Rh_1 \oplus \cdots \oplus Rh_k$, where $g_1, \cdots, g_n$ are preassigned elements of G. Since here the types of Rh_i are $\mathbf{t}$ [summands of type $< \mathbf{t}$, if any, can be omitted; actually, there are no such summands because of (96.2)], we obtain $G = (\prod R_i)(\mathbf{t}) = Rh_1 \oplus \cdots \oplus Rh_k \oplus H'(\mathbf{t})$, establishing the separability of G. Now the sufficiency part of our assertion is a simple consequence of (87.3).

To prove the converse, let A be separable and homogeneous of type $\mathbf{t}$. For every $a \neq 0$ in A, $\langle a \rangle_*$ is a summand of A as is shown by (87.2), so we have an epimorphism $\pi_a \colon A \to R$ [R a rational group of type $\mathbf{t}$] with $\pi_a a \neq 0$. Consequently, A is a subdirect sum of rational groups R_i of types $\mathbf{t}$; moreover, it is obviously contained in $(\prod R_i)(\mathbf{t})$. Since $\pi_a a$ has the same characteristic as a, the purity of A in $\prod R_i$ follows.□

Our main result on separable groups is the following theorem.

Theorem 87.5 (Fuchs [26]). *Direct summands of separable groups are separable.*

Let $A = B \oplus C$ be a direct decomposition of the separable group A. Given $b_1, \cdots, b_k \in B$, there exists a completely decomposable direct summand G of finite rank of A such that $b_1, \cdots, b_k \in G$. Write $A = G \oplus H$ for some $H \leqq A$.

Set $G = G_1 \oplus \cdots \oplus G_n$, where each G_i is completely decomposable and homogeneous of type $\mathbf{t}_i$. The most essential part of the proof is to verify the existence of a direct decomposition $A = G' \oplus H'$ such that $G \leqq G' = G'_1 \oplus \cdots \oplus G'_n$, each G'_i is completely decomposable of finite rank and homogeneous of type $\mathbf{t}_i$, and $H' = (H' \cap B) \oplus (H' \cap C)$. We induct on n.

If $n = 1$, then G is homogeneous, say of type $\mathbf{t}$. The groups $A(\mathbf{t}) = G \oplus H(\mathbf{t})$ and $A(\mathbf{t})/A^*(\mathbf{t}) = \bar{G} \oplus H(\mathbf{t})/H^*(\mathbf{t})$ are separable [see (d) and (f)], where the overbar indicates the obvious image of G in $A(\mathbf{t})/A^*(\mathbf{t})$. By (87.3), the summands in

$$A(\mathbf{t})/A^*(\mathbf{t}) = B(\mathbf{t})/B^*(\mathbf{t}) \oplus C(\mathbf{t})/C^*(\mathbf{t})$$

are separable; therefore, there exist decompositions

$$B(\mathbf{t})/B^*(\mathbf{t}) = \bar{B}_1 \oplus \cdots \oplus \bar{B}_r \oplus \bar{X} \qquad \text{and} \qquad C(\mathbf{t})/C^*(\mathbf{t}) = \bar{C}_1 \oplus \cdots \oplus \bar{C}_s \oplus \bar{Y}$$

such that $\bar{B}_i$, $\bar{C}_j$ are of type $\mathbf{t}$, of rank 1, and $\bar{G}$ is contained in

$$\bar{G}' = \bar{B}_1 \oplus \cdots \oplus \bar{B}_r \oplus \bar{C}_1 \oplus \cdots \oplus \bar{C}_s.$$

In view of (86.5), there exist rank 1 subgroups B_i and C_j of B and C, respectively, which map upon $\bar{B}_i$, $\bar{C}_j$. Clearly, $B_1, \cdots, B_r$ are contained in a completely decomposable summand J of finite rank of A, where, without loss of generality, $J(\mathbf{t}) = J$ may be assumed. $\bar{B}_1 \oplus \cdots \oplus \bar{B}_r$ is a summand of $A(\mathbf{t})/A^*(\mathbf{t})$ and hence of $J/J^*(\mathbf{t})$, so $B_1 \oplus \cdots \oplus B_r \oplus J^*(\mathbf{t})$ is a summand of J. We infer $B_1 \oplus \cdots \oplus B_r$ is a summand of A, and hence of B, i.e.,

$$B = B_1 \oplus \cdots \oplus B_r \oplus B_0 \qquad \text{for some} \quad B_0 \leqq B.$$

In a similar fashion, we infer

$$C = C_1 \oplus \cdots \oplus C_s \oplus C_0 \qquad \text{for some} \quad C_0 \leqq C.$$

There are rational groups $\bar{K}_1, \cdots, \bar{K}_l$ of type $\mathbf{t}$ such that $\bar{G}' = \bar{G} \oplus \bar{K}_1 \oplus \cdots \oplus \bar{K}_l$, and if $K_1, \cdots, K_l$ are chosen to be of rank 1 and to map upon $\bar{K}_1, \cdots, \bar{K}_l$, then $G' = G \oplus K_1 \oplus \cdots \oplus K_l$ will be a completely decomposable subgroup of A mapping upon $\bar{G}'$. We intend to show:

$$A = G' \oplus B_0 \oplus C_0.$$

First, $G' + A^*(\mathbf{t}) = (B_1 \oplus \cdots \oplus B_r \oplus C_1 \oplus \cdots \oplus C_s) + A^*(\mathbf{t})$ and $A^*(\mathbf{t}) \leqq B_0 \oplus C_0$ imply that G', B_0, and C_0 together generate $B \oplus C = A$. Furthermore, if $g \in G' \cap (B_0 \oplus C_0)$, then $g = a + b + c$ [with $a \in A^*(\mathbf{t})$, $b \in B_1 \oplus \cdots \oplus B_r$, $c \in C_1 \oplus \cdots \oplus C_s$] implies $b + c = g - a \in B_0 \oplus C_0$. Hence $g - a = 0$ and $g \in G' \cap A^*(\mathbf{t}) = 0$. Thus we have embedded G in a completely decomposable homogeneous summand G' of A with a complement $H' = B_0 \oplus C_0$, where $B_0 \leqq B$, $C_0 \leqq C$.

If $n > 1$, then let $\mathbf{t}_n$ be a maximal one among the types $\mathbf{t}_1, \cdots, \mathbf{t}_n$ which may be assumed to be distinct. By induction, there exists a direct decomposition $A = K' \oplus L$ such that K' contains $K = G_1 \oplus \cdots \oplus G_{n-1}$ and is completely decomposable of finite rank with summands of types $\mathbf{t}_1, \cdots, \mathbf{t}_{n-1}$ only, and $L = (L \cap B) \oplus (L \cap C)$. From $A(\mathbf{t}_n) = K'(\mathbf{t}_n) \oplus L(\mathbf{t}_n) = L(\mathbf{t}_n)$ we infer $G_n \leqq L$, and thus G_n is a summand of L. It is readily checked from the definition that a direct summand L of a separable group A with a finite rank complement is again separable. By the preceding paragraph, there exist a completely decomposable homogeneous G_n' of finite rank and a subgroup H' of L such that $L = G_n' \oplus H'$, where $G_n \leqq G_n'$ and

$$H' = (H' \cap L \cap B) \oplus (H' \cap L \cap C) = (H' \cap B) \oplus (H' \cap C).$$

Choose $G' = K' \oplus G_n'$, and then $A = G' \oplus H'$ is a desired decomposition of A.

Now the proof of (87.5) can easily be completed. Notice that

$$A = G' \oplus (H' \cap B) \oplus (H' \cap C)$$

implies

$$B = (H' \cap B) \oplus B' \qquad \text{and} \qquad C = (H' \cap C) \oplus C'$$

with $B' = [G' \oplus (H' \cap C)] \cap B$ and $C' = [G' \oplus (H' \cap B)] \cap C$. Consequently,

$$A = (H' \cap B) \oplus (H' \cap C) \oplus B' \oplus C',$$

where $b_1, \cdots, b_k \in B'$. The isomorphism $G' \cong B' \oplus C'$ together with the proven part of (86.7) implies that B' is completely decomposable, and hence B is separable. $\square$

An obvious generalization of separability is $\mathfrak{m}$-separability for infinite ordinals $\mathfrak{m}$. A group A is *$\mathfrak{m}$-separable* if every subset of cardinality $< \mathfrak{m}$ in A can be embedded in a completely decomposable summand of A. Virtually nothing is known about these groups.

Griffith [5] calls a group A *coseparable* if every subgroup B with A/B finitely generated contains a summand C of A with finitely generated complement. He studied W-groups [see **99**] by using this concept.

EXERCISES

1. (Baer [6]) A torsion-free group A is separable if and only if:
   (i) every finite subset of A belongs to a direct summand which is a direct sum of homogeneous groups;
   (ii) $A(\mathbf{t})/A^*(\mathbf{t})$ is separable for every type $\mathbf{t}$. [*Hint*: summands in (i) are separable.]
2. Every pure subgroup of finite rank is a summand of A if and only if the reduced part of A is a homogeneous separable group.
3*. (R. Nunke) Using the notation and results in **94** show that $A = pP + S$, for any prime p, is not separable. [*Hint*: for any $\phi\colon A \to Z$, $\phi(p, \cdots, p, \cdots) \in pZ$, since Z is slender.]
4. (Charles [8]) A homogeneous group A of the type of Z is separable exactly if, for every prime p, $\bigcap \phi^{-1}(pZ) = pA$, where ϕ runs over all $\phi\colon A \to Z$. [*Hint*: for sufficiency show that every pure subgroup of rank 1 is a summand.]
5*. Show that the group $(\prod R_i)(\mathbf{t})$ in (87.4) is not a summand of $\prod R_i$ unless they are equal; assume that the cardinality is nonmeasurable. [*Hint*: **94**, Ex. 8.]
6. (E. Sąsiada) There exist nonisomorphic separable groups which are isomorphic to pure subgroups of each other. [*Hint*: $\prod Z$ and $\prod Z \oplus (\bigoplus Z)$.]
7. For every group A, the group $\mathrm{Hom}(A, Z)$ is separable. [*Hint*: (87.4).]
8. The tensor product of two separable torsion-free groups is again separable.
9. (Baer [6]) Call an element $a \in A$ *primitive* of type $\mathbf{t}$ if $a \in A(\mathbf{t})\backslash A^*(\mathbf{t})$ and a is proper with respect to $A^*(\mathbf{t})$. A set $\{a_1, \cdots, a_k\}$ is primitive if its members are primitive of different types. If A is separable, then:
   (a) $\{a_1, \cdots, a_k\}$ is a primitive set exactly if the a_i are of different types and the pure subgroup they generate is a direct summand of A such that $\langle a_1, \cdots, a_k\rangle_* = \langle a_1\rangle_* \oplus \cdots \oplus \langle a_k\rangle_*$;
   (b) given two primitive sets, $\{a_1, \cdots, a_k\}$ and $\{b_1, \cdots, b_k\}$, there is an automorphism ϕ of A satisfying $\phi a_i = b_i$ for every i if and only if $\chi(a_i) = \chi(b_i)$ for every i.

88. INDECOMPOSABLE GROUPS

A group is called *indecomposable* if it has only trivial direct summands. Among the torsion groups, only the cocyclic groups are indecomposable, while no mixed group is indecomposable [cf. (27.4)]. Our present aim is to get information about torsion-free indecomposable groups.

The first question which comes to mind is to find out what cardinals are ranks of indecomposable groups. Our main result [see next section] will show that there exist indecomposable groups of every rank less than the first strongly inaccessible cardinal.

The simplest way of constructing indecomposable groups up to rank $2^{\aleph_0}$ is *via* the p-adic integers.

Theorem 88.1 (Baer [6]). *p-pure subgroups of the p-adic integers are indecomposable.*

Let $A \neq 0$ be a p-pure subgroup of J_p and let $0 \neq \pi \in A$. If $\pi = s_k p^k + s_{k+1}p^{k+1} + \cdots$ $(s_k \neq 0)$ is the canonical form of π, then by p-purity, $s_k + s_{k+1}p + \cdots \in A$. Therefore, A contains a p-adic unit, and thus $A + pJ_p = J_p$. By p-purity, $pA = A \cap pJ_p$, whence

$$A/pA = A/(A \cap pJ_p) \cong (A + pJ_p)/pJ_p = J_p/pJ_p \cong Z(p).$$

From $A = B \oplus C$ with $B \neq 0$, $C \neq 0$, we would obtain $A/pA \cong B/pB \oplus C/pC \cong Z(p) \oplus Z(p)$, since B, C are again p-pure in J_p. This contradiction shows A is indecomposable.□

Since J_p is of rank $2^{\aleph_0}$, (88.1) yields the existence of indecomposable groups of any rank up to the continuum.

Our next purpose is to obtain more explicit examples for indecomposable groups. In our constructions, we shall often use the practice—as a matter of convenience—of writing expressions like $p^{-\infty}a$ as an abbreviation for what would more properly be written as an infinite set: $p^{-1}a, \cdots, p^{-n}a, \cdots$. Also, we will find it convenient to construct groups A by starting from independent elements a_n of a Q-vectorspace V and then describing the generators of A in V, or by starting from a direct sum and then adjoining some elements of its divisible hull. Unexplained letters, like a_n, b_n, e_n, will denote independent elements in some Q-vectorspace; in any case, our notation will be self-explanatory.

Example 1. Every torsion-free group of rank 1 is indecomposable.

Example 2. Consider a (finite or infinite) set $\{q, p_1, \cdots, p_n, \cdots\}$ of primes and for each n, let

$$E_n = \langle p_n^{-\infty} e_n \rangle \qquad \text{and} \qquad G_n = \langle p_n^{-\infty} e_n, q^{-1} e_n \rangle.$$

Then E_n is of index q in G_n. Define $A[\leqq \bigoplus_n G_n]$ as follows:

$$A = \langle \bigoplus_n E_n; q^{-1}(e_1 + e_2), \cdots, q^{-1}(e_1 + e_n), \cdots \rangle.$$

To show A indecomposable, suppose $A = B \oplus C$. Notice that no E_n has a nontrivial homomorphism into G_m if $n \neq m$, since the elements of E_n are divisible by every power of p_n, while G_m does not contain any such element except for 0. Therefore, the subgroups E_n are fully invariant in A, and from (9.3) we obtain $E_n = (E_n \cap B) \oplus (E_n \cap C)$. Here one of the summands vanishes, because E_n—as a group of rank 1—is indecomposable; that is to say, either $E_n \leqq B$ or $E_n \leqq C$. Assume, e.g., $E_1 \leqq B$ and $E_n \leqq C$ for some $n > 1$. Write $q^{-1}(e_1 + e_n) = b + c$ $(b \in B,\ c \in C)$; hence $qb = e_1$ and $qc = e_n$, which is impossible, since no e_n is divisible by q in A. Therefore, all E_n are contained in B or in C, whence $C = 0$ or $B = 0$, $\bigoplus E_n$ being an essential subgroup of A.

Example 3. The last example can be modified by defining A as the group

$$A = \langle \bigoplus_n E_n, q^{-\infty}(e_1 + e_2), \cdots, q^{-\infty}(e_1 + e_n), \cdots \rangle.$$

The same proof applies to show that this A is indecomposable.

Example 4. The same holds for the group

$$A = \langle \bigoplus_n E_n, q^{-1}(e_1 - e_2), \cdots, q^{-1}(e_n - e_{n+1}), \cdots \rangle,$$

as is readily seen.

In our constructions of indecomposable groups, the following concept is fundamental. A set $\{G_i\}_{i \in I}$ of torsion-free groups $\neq 0$ is said to be a *rigid system* [see Fuchs [18]] if

$$\mathrm{Hom}(G_i, G_j) \cong \begin{cases} \text{a subgroup of } Q & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

That is, groups in a rigid system have no endomorphisms other than multiplications by rational numbers and no nontrivial homomorphisms into other groups of the system. In particular, groups in a rigid system have no idempotent endomorphisms $\neq 0$ and 1, thus they are necessarily indecomposable. A group G is *rigid* if the singleton $\{G\}$ is a rigid system.

The simplest example for a rigid system is a set of rank 1 groups with incomparable types. In fact, this is immediate from (85.4). An upper bound for the cardinality of a rigid system of rank 1 groups is the continuum [cf. (85.2)]; moreover, we have:

Lemma 88.2. *There exists a rigid system $\{A_i\}_{i \in I}$ of rank 1 groups such that $|I| = 2^{\aleph_0}$.*

Let $p_1, q_1, p_2, q_2, \cdots, p_n, q_n, \cdots$ be different primes, and consider all sets $S_i = \{r_1, r_2, \cdots, r_n, \cdots\}$, where, for each n, $r_n = p_n$ or $r_n = q_n$. Clearly, i runs over an index set I of the power of the continuum, and no S_i contains another S_j. For each S_i, define A_i to be a rational group of type $\mathbf{t}_i = (k_1, \cdots, k_n, \cdots)$, where $k_n = \infty$ exactly if the nth prime belongs to S_i and $k_n = 0$ otherwise. Then the $\mathbf{t}_i$ are pairwise incomparable, hence $\{A_i\}$ is a rigid system. □

We now prove a result which is the key lemma in several cases to establish indecomposability.

Lemma 88.3 (Fuchs [18]). *Let E_0, E_i $(i \in I)$ be a rigid system of groups, and let p_i $(i \in I)$ be [not necessarily distinct] primes such that there are elements $u_i \in E_0$ and $v_i \in E_i$ $(i \in I)$ not divisible by p_i. Then the group*

$$A = \langle E_0 \oplus \bigoplus_i E_i;\ p_i^{-1}(u_i + v_i) \text{ for all } i \in I \rangle$$

is indecomposable. The same holds if the p_i^{-1} are replaced by $p_i^{-\infty}$ or by n_i^{-1} with integers $n_i > 1$ whose prime divisors do not divide u_i and v_i.

Hypothesis ensures the full invariance of E_0, E_i in A. Therefore, if $A = B \oplus C$, then $E_i = (E_i \cap B) \oplus (E_i \cap C)$ for $i \in I \cup 0$. Hence the indecomposability of E_i implies that each of E_i $(i \in I \cup 0)$ is contained entirely either in B or in C. If $E_0 \leqq B$ and $E_j \leqq C$ for some $j \in I$, then $p_j^{-1}(u_j + v_j) = b + c$ for some $b \in B$, $c \in C$, and the divisibility relations $p_j \mid u_j, v_j$ follow. These fail to hold in the direct sum of the Es, and it is straightforward to check that they are contradictory in A, too. Consequently, all of E_i $(i \in I \cup 0)$ belong to the same summand of A; therefore, A is indecomposable.□

With the aid of (88.2) and (88.3), it is easy to construct indecomposable groups of any rank $\leqq 2^{\aleph_0}$. It suffices to choose a rigid system as in (88.2) so that some prime p does not occur among the p_n, q_n; then the groups A_i as defined there can serve as E_0 and E_i of (88.3), while all the primes of (88.3) are equal to p.

Evidently, (88.3) is of no help in constructing homogeneous indecomposable groups of rank $\geqq 2$. It should, however, be pointed out that (88.1) does yield such groups. More explicit examples are as follows.

Example 5. Let G be a completely decomposable homogeneous group of finite rank $r \geqq 2$ and of type $(0, \cdots, 0, \infty, 0, \cdots)$ with ∞ at the place of the prime p; thus we can write

$$G = Q^{(p)}a_1 \oplus \cdots \oplus Q^{(p)}a_r.$$

Choose $r - 1$ algebraically independent [over Q] p-adic units $\pi_2, \cdots, \pi_r$, and let $\pi_1 = 1$. For $n = 1, 2, \cdots$, we set

$$x_n = p^{-n}(a_1 + \pi_{2n}a_2 + \cdots + \pi_{rn}a_r) \in G, \tag{1}$$

where $\pi_{in} = s_{i0} + s_{i1}p + \cdots + s_{i,n-1}p^{n-1}$ is the $(n-1)$st partial sum of the standard form $\pi_i = s_{i0} + s_{i1}p + \cdots + s_{in}p^n + \cdots$ $(0 \leqq s_{in} < p)$. Define a subgroup A of G as

$$A = \langle a_1, \cdots, a_r, x_1, x_2, \cdots, x_n, \cdots\rangle.$$

A is of rank r, and we will show that it is rigid. First, observe that $px_{n+1} = x_n + s_{2n}a_2 + \cdots + s_{rn}a_r$ for every n, and thus every element of A not in $A_0 = \langle a_1, \cdots, a_r\rangle$ has the form $kx_n + k_2a_2 + \cdots + k_ra_r$ for some integers $k, k_2, \cdots, k_r$ with $p \nmid k$ and for some $n \geqq 1$. Furthermore, if

$$p^m(kx_n + k_2a_2 + \cdots + k_ra_r) = l_2a_2 + \cdots + l_ra_r \qquad (l_i \in Z),$$

then substituting (1), the coefficient of a_1 must vanish. Hence $k = 0$ and $\langle a_2, \cdots, a_r\rangle$ is [p-pure and so] pure in A. To show A homogeneous of type $(0, \cdots, 0, \cdots)$, assume $p^{-n}(k_1a_1 + \cdots + k_ra_r) \in A$ for all n. Then

$$p^{-n}(k_1a_1 + \cdots + k_ra_r - k_1p^nx_n) \in A$$

and substitution from (1) imply, in view of the purity of $\langle a_2, \cdots, a_r\rangle$ in A, that $p^n | k_i - k_1\pi_{in}$ for all n. Hence the sequence $\{k_i - k_1\pi_{in}\}_n$ is a 0-sequence, i.e., $k_i = k_1\pi_i$ for $i = 2, \cdots, r$. Thus π_i is rational, in contradiction to the algebraic independence of the π_i.

To verify that A is rigid, let $\eta \neq 0$ be an endomorphism of A. Without loss of generality, η may be assumed to map A_0 into itself [otherwise replace η by some $m\eta \neq 0$]. η is determined by the images

$$\eta: a_i \mapsto \sum_{j=1}^{r} t_{ij} a_j \qquad (t_{ij} \in \mathbf{Z}).$$

In fact,

$$\eta x_n = p^{-n} \sum_i \pi_{in} \eta a_i = p^{-n} \sum_j \left(\sum_i \pi_{in} t_{ij}\right) a_j = k_n x_n + l_{2n} a_2 + \cdots + l_{rn} a_r$$

for some integers $k_n, l_{2n}, \cdots, l_{rn}$. We obtain

$$\sum_{i=1}^{r} \pi_{in} t_{i1} = k_n \qquad \text{and} \qquad p^{-n} \sum_{j=2}^{r} \left[\sum_{i=1}^{r} \pi_{in} t_{ij} - k_n \pi_{jn}\right] a_j \in \langle a_2, \cdots, a_r\rangle,$$

whence the coefficients in brackets are divisible by p^n. Letting $n \to \infty$, for $j = 2, \cdots, r$, the equations

$$\sum_{i=1}^{r} \pi_i t_{ij} - \kappa \pi_j = 0 \qquad \text{with} \quad \sum_{i=1}^{r} \pi_i t_{i1} = \kappa$$

arise. By the algebraic independence of the π_i, we obtain $t_{jj} = t_{11}$, while $t_{ij} = 0$ for $i \neq j$. This shows that η acts like multiplication by the integer t_{11}, and $\mathbf{Z}$ is the endomorphism ring of our group A.

Using more algebraically independent p-adic units $\pi_2, \cdots, \pi_r, \pi'_2, \cdots, \pi'_s$, we can construct A as above and another group A' of rank s with the aid of $\pi'_1 = 1, \pi'_2, \cdots, \pi'_s$. A word-by-word repetition of the proof in the preceding paragraph will convince us that every $\eta: A \to A'$ must be the zero map. Since there are continuously many algebraically independent p-adic units, it is clear that we can construct a rigid system of homogeneous groups of the type $(0, 0, \cdots, 0, \cdots)$ and of finite ranks $\geqq 2$, where the cardinality of the system is $2^{\aleph_0}$. We state this result as

Theorem 88.4. *There exists a rigid system $\{A_i\}_{i \in I}$ of groups A_i of any finite rank $r \geqq 2$ which are homogeneous of type* $(0, 0, \cdots, 0, \cdots)$ *such that* $|I| = 2^{\aleph_0}$.□

A property that is considerably stronger than indecomposability is the following. Call a group A *purely indecomposable* if every pure subgroup of A

is indecomposable. (88.1) shows that pure subgroups of the p-adic integers are purely indecomposable. A fairly satisfactory information about purely indecomposable groups is given in the next theorem.

Theorem 88.5 (Griffith [1]). *A reduced torsion-free purely indecomposable group is isomorphic to a subgroup of $J = \prod_p J_p$; in particular, its cardinality is at most continuum.*

Let A be a reduced torsion-free purely indecomposable group and $a \neq 0$ in A. Since A is Hausdorff in its Z-adic topology, A can be regarded as a pure subgroup in its Z-adic completion $\hat{A}$ [see **39**]. By (40.1) one can write $\hat{A} = \prod_p \hat{A}_p$ with modules $\hat{A}_p$ over the p-adic integers, and accordingly, $a = (\cdots, a_p, \cdots)$ with $a_p \in \hat{A}_p$. Every $a_p \neq 0$ is contained in a pure Q_p^*-submodule $C_p \cong J_p$ of $\hat{A}_p$, which is, by algebraic compactness, a summand of $\hat{A}_p$. Thus a can be embedded in a summand $C = \prod_p C_p$ of $\hat{A}$ [where $C_q = 0$ if $a_q = 0$], $\hat{A} = C \oplus D$. By torsion-freeness, $G = (C \cap A) \oplus (D \cap A)$ is pure in $\hat{A}$ and hence in A. By hypothesis, G is indecomposable, thus from $a \in C \cap A$ we infer $D \cap A = 0$. Therefore, the projection $\hat{A} \to C$ maps A isomorphically upon a subgroup of $\prod_p C_p$; this group is isomorphic to a summand of J.□

EXERCISES

1. (a) (Bognár [1]) Show that if $p_1, \cdots, p_n$ are distinct odd primes, then the group
$$\langle p_1^{-\infty} a_1, \cdots, p_n^{-\infty} a_n, \tfrac{1}{2}(a_1 + a_2), \tfrac{1}{2}(a_1 + a_3), \cdots, \tfrac{1}{2}(a_{n-1} + a_n)\rangle$$
is indecomposable.
(b) The same fails to work if 2 is replaced by an odd prime.
2. (de Groot [2]) If $p, p_1, \cdots, p_n$ are different primes, then the group
$$\langle p_1^{-\infty} a_1, \cdots, p_n^{-\infty} a_n, p^{-1}(a_1 + \cdots + a_n)\rangle$$
is indecomposable.
3. Let p, q, r be different primes $\geqq 5$. Verify that the group
$$A = \langle p^{-\infty} a, p^{-\infty} q^{-\infty} b, r^{-\infty} c, 2^{-1}(b + c), 3^{-1}(a + c)\rangle$$
is indecomposable. [*Hint*: in a nontrivial decomposition, $\langle b, c\rangle_*$ must be one summand; compare the coordinates of a and $3^{-1}(a + c)$.]
4. There are $2^{2^{\aleph_0}}$ nonisomorphic indecomposable groups of rank $\leqq 2^{\aleph_0}$. How many of them are of finite [or countable] rank?
5. (a) The group A in Example 5 stays indecomposable if it is tensored by any rank 1 group R such that $pR \neq R$.
(b) Establish the existence of homogeneous indecomposable groups of any finite rank $\geqq 2$ for arbitrary types $\mathbf{t} \neq (\infty, \cdots, \infty, \cdots)$.

6. (a) Every subgroup of rank $\leqq r-1$ of the group A in Example 5 is free.
(b) For every pure subgroup C of rank $r-1$ of this A, $A/C \cong Q^{(p)}$.
7. (a) (Fuchs and Loonstra [2]) There exists a torsion-free group G of rank 2 with the following properties:
(i) every subgroup of rank 1 is cyclic;
(ii) every torsion-free quotient group of rank 1 is divisible;
(iii) the endomorphism ring of G is isomorphic to $\mathbf{Z}$.
Show that G satisfies $G/nG \cong Z(n)$ for every n. [*Hint*: modify Example 5.]
(b)* Verify the existence, for every $r \geqq 2$, of a group G of rank $2r$ such that:
(i′) subgroups of rank $\leqq r$ are free;
(ii′) torsion-free quotient groups of rank $\leqq r$ are divisible;
(iii′) as (iii) above.
8. (A. L. S. Corner) Given an integer $n \geqq 2$, there exists a torsion-free group A of rank n with the following properties:
(i) all subgroups of rank $n-1$ are free;
(ii) all quotient groups of rank 1 are divisible;
(iii) the endomorphism ring of A is isomorphic to $\mathbf{Z}$.
9. (Procházka [2]) Find a torsion-free group A of rank 2 such that for any cyclic subgroup C of A, A/C does not split as a mixed group.
10. If the reduced group $A = B \oplus C$ contains a subgroup $G \cong J_p$, then either $G \leqq B$ or $G \cap B = 0$. [*Hint*: $G \cap B$ is pure in G, thus $G/(G \cap B)$ is divisible.]
11. Let A be a torsion-free group such that $|A/pA| \leqq p$ for every prime and A contains no subgroup which is p-divisible for infinitely many primes. Prove that A is purely indecomposable. [*Hint*: use (35.2) to show that $|G/pG| \leqq p$ for all pure G.]
12. Give an example for a purely indecomposable group that is not isomorphic to any pure subgroup of $J = \prod_p J_p$.
13. (de Groot [2]) (a) A torsion-free group is purely indecomposable if any two independent elements have incomparable types.
(b) Let V be a $\mathbf{Q}$-vectorspace with basis $\{a_1, \cdots, a_n, \cdots\}$. Arrange the linear combinations $b = k_1 a_{n_1} + \cdots + k_r a_{n_r}$ with $n_1 < \cdots < n_r$, k_i integers $\neq 0$ and $k_1 > 0$, $(k_1, \cdots, k_r) = 1$, in a sequence. Assign to the jth b in the sequence the character $(0, \cdots, 0, \infty, 0, \cdots)$ with ∞ only at the jth place to obtain a subgroup A of V. Prove that A is purely indecomposable.
14. (a) With the aid of (88.2) extend Ex. 13(b) to obtain groups A up to the power of the continuum such that any two isomorphic pure subgroups of A are equal.
(b) Show that a group of cardinality $> 2^{\aleph_0}$ cannot have the property in (a).

15. (Griffith [1]) Show that there exist $2^{2^{\aleph_0}}$ nonisomorphic purely indecomposable groups. [*Hint*: Ex. 14(a).]
16*. (Griffith [1]) Let A be a reduced torsion-free group all of whose p-basic subgroups are either Z or 0. Then A is purely indecomposable exactly if the endomorphisms $\neq 0$ of pure subgroups of A are monic. [*Hint*: $\hat{A}$ is a summand of J.]
17. (D. W. Dubois [1]) A torsion-free group A is said to be *cohesive* if A/G is divisible for all pure subgroups $G \neq 0$ of A. Show that:
(a) pure subgroups of the p-adic integers J_p are cohesive;
(b) reduced cohesive groups are purely indecomposable;
(c) A is cohesive if and only if for each prime p, either A is p-divisible or A is isomorphic to a p-pure subgroup of J_p. [*Hint*: if $pA \neq A$, there exists a nontrivial homomorphism $A \to J_p$ which must be monic, p-basic subgroup of A must be cyclic.]

89.* LARGE INDECOMPOSABLE GROUPS

We have come to the problem of constructing indecomposable groups of large cardinalities. The construction is based on a transfinite induction whose idea goes back to Fuchs [18]; the set-theoretic background has been furnished by Corner [8]. The procedure works only for cardinals less than the first strongly inaccessible cardinal; a cardinal number $\mathfrak{m}^* > \aleph_0$ is said to be *strongly inaccessible* if:

(a) $\sum_{i \in I} \mathfrak{m}_i < \mathfrak{m}^*$ whenever $\mathfrak{m}_i < \mathfrak{m}^*$ for every $i \in I$ and the index set I is of cardinality $< \mathfrak{m}^*$;

(b) $2^{\mathfrak{n}} < \mathfrak{m}^*$ for every cardinal $\mathfrak{n} < \mathfrak{m}^*$.

The strongly inaccessible cardinals are extraordinarily large, and their existence is believed to be independent of the standard axioms of set theory. Sometimes additional axioms are stipulated in theorems involving inaccessible cardinals.

In constructing large indecomposable groups we shall need a particular result from set theory. Let τ be an ordinal number > 1. A *regressive function* for τ is a function f_τ on the set $W^*(\tau)$ of nonzero ordinals $< \tau$ with values in the set $W(\tau)$ of all ordinals $< \tau$ such that

$$f_\tau(\sigma) < \sigma \qquad \text{for all} \quad \sigma \in W^*(\tau).$$

For regressive functions we prove:

Lemma 89.1 (Corner [8]). *Given an ordinal τ such that $\aleph_\tau$ is less than the first inaccessible cardinal, there exists a regressive function f_τ for τ satisfying*

$$(1) \qquad |f_\tau^{-1}(\sigma)| \leqq 2^{\aleph_\sigma} \qquad \textit{for all} \quad \sigma \in W(\tau).$$

The proof runs by transfinite induction. If $\tau < \omega_1$, then $|\tau| \leqq \aleph_0$, and the function: $f_\tau(\sigma) = 0$ for all $\sigma \in W^*(\tau)$ obviously satisfies inequality (1). We may henceforth suppose that $\tau \geqq \omega_1$ and that for every ordinal $\rho < \tau$ there is a regressive function f_ρ satisfying (1).

Case 1. τ is an isolated ordinal, say, $\tau = \rho + 1$, for some ordinal ρ. We then set $f_\tau(\sigma) = f_\rho(\sigma)$ for $0 < \sigma < \rho$ and $f_\tau(\rho) = 0$. This is a regressive function for τ of the desired kind.

For limit ordinals τ we distinguish two subcases, according as τ is singular [i.e., there is an increasing sequence σ_α $(\alpha < \rho)$ of ordinals $< \tau$, for some $\rho < \tau$, such that $\lim_{\alpha<\rho} \sigma_\alpha = \tau$] or regular [i.e., it is not singular].

Case 2. τ is a regular limit ordinal. By hypothesis, $\aleph_\tau$ is not strongly inaccessible, and since (a) holds for $\mathfrak{m}^* = \aleph_\tau$, there must exist a $\rho < \tau$ such that $\aleph_\tau \leqq 2^{\aleph_\rho}$. As $\rho + 1 < \tau$, by induction hypothesis a regressive function $f_{\rho+1}$ satisfying (1) exists, and we can define:

$$f_\tau(\sigma) = f_{\rho+1}(\sigma) \quad \text{for} \quad 0 < \sigma \leqq \rho, \qquad \text{and} \quad f_\tau(\sigma) = \rho \quad \text{for} \quad \rho < \sigma < \tau,$$

in order to obtain a regressive function f_τ for τ. This f_τ obviously satisfies (1).

Case 3. τ is a singular limit ordinal. Choosing σ_α as above, we suppose $0 < \sigma_0 < \cdots < \sigma_\alpha < \sigma_{\alpha+1} < \cdots$. In this case, $W(\tau)$ is the set-union of the pairwise disjoint sets $W(\sigma_0)$, $W(\sigma_1)\backslash W(\sigma_0), \cdots, W(\sigma_{\alpha+1})\backslash W(\sigma_\alpha), \cdots$. Each of these is well-ordered and, for instance, $W(\sigma_{\alpha+1})\backslash W(\sigma_\alpha)$ is order-isomorphic to $W(\rho_\alpha)$ for some ordinal $\rho_\alpha \leqq \sigma_{\alpha+1}$. Consequently, f_{ρ_α} gives rise to a regressive function g_{ρ_α} on $W(\sigma_{\alpha+1})\backslash W(\sigma_\alpha)$ in the obvious fashion. Using these g_{ρ_α} together with f_ρ, we can define:

$$f_\tau(\sigma) = \begin{cases} g_{\rho_\alpha}(\sigma) & \text{whenever} \quad \sigma_\alpha < \sigma < \sigma_{\alpha+1}, \\ \sigma_{f_\rho(\alpha)} & \text{whenever} \quad \sigma = \sigma_\alpha. \end{cases}$$

It is now straightforward to check that if f_{ρ_α} and f_ρ all satisfy (1), then so does f_τ. $\square$

We are now ready to prove the main result on indecomposable groups of large cardinalities.

Theorem 89.2. *For every infinite cardinal* $\mathfrak{m}$ *less than the first strongly inaccessible cardinal, there exists a rigid system consisting of* $2^{\mathfrak{m}}$ *groups, each of which has cardinality* $\mathfrak{m}$.

The case $\mathfrak{m} = \aleph_0$ is settled by (88.2); thus $\mathfrak{m} \geqq \aleph_1$ may be assumed. Let $P(\mathfrak{m})$ denote a system $\{B_i\}_{i \in I}$ of groups with the following properties:

(i) $|B_i| = \mathfrak{m}$ for every $i \in I$;

(ii) $|I| = 2^{\mathfrak{m}}$;

(iii) for every $i \in I$, there are chosen an element $b_i \in B_i\backslash 2B_i$ and a pure subgroup $\bar{B}_i < B_i$ such that $B_i(\mathbf{t}_0)$, $\langle b_i \rangle$, and $\bar{B}_i$ generate their direct sum in B_i [where $\mathbf{t}_0 = (\infty, 0, \cdots, 0, \cdots)$], and $|\bar{B}_i/2\bar{B}_i| = \mathfrak{m}$;

(iv) for $B_i^* = \langle B_i, 2^{-\infty}b_i, 2^{-\infty}\bar{B}_i\rangle$, we have

$$\operatorname{Hom}(B_i, B_j^*) \cong \begin{cases} 0 & \text{if } i \neq j, \\ \text{a subgroup of } Q & \text{if } i = j. \end{cases}$$

We apply transfinite induction on the index τ of $\mathfrak{m} = \aleph_\tau$ to establish the existence of such a system $P(\mathfrak{m})$ for every cardinal $\mathfrak{m}$ ($\geqq \aleph_1$), less than the first inaccessible cardinal. This will suffice, since $P(\mathfrak{m})$ is clearly a rigid system.

1°. The first step is to establish a system for $\mathfrak{m} = \aleph_1$. Arguing as in (88.2), it is easy to construct a rigid system of rank 1 groups A_k such that $2A_k \neq A_k \neq 3A_k$ with k running over an index set K of cardinality $\aleph_1$. We form subsets K_i ($i \in I$) of K subject to the following conditions:

(α) $|K_i| = \aleph_1$;
(β) $|I| = 2^{\aleph_1}$;
(γ) some fixed, but otherwise arbitrary index k_0 belongs to all K_i;
(δ) $K_i \subseteq K_j$ implies $i = j$.

Using standard set-theoretic arguments, this can be achieved easily. For every $k \in K$, pick an $a_k \in A_k \backslash 3A_k$ and, for every $i \in I$, set

$$B_i = \langle \bigoplus_{k \in K_i} A_k, 3^{-\infty}(a_{k_0} + a_k) \text{ for all } k \neq k_0 \text{ in } K_i \rangle.$$

Then $|B_i/2B_i| = \aleph_1$, and we can therefore choose b_i and $\bar{B}_i$ as required in (iii). To show that the system $\{B_i\}_{i \in I}$ satisfies (iv), too, let $\eta: B_i \to B_j^*$. If $i \neq j$, there is an index k_1 in K_i but not in K_j. Since B_j^* contains no elements $\neq 0$ of type $\geqq \mathbf{t}(A_{k_1})$, we have necessarily $\eta A_{k_1} = 0$. But then $\eta(a_{k_0} + a_{k_1}) = \eta a_{k_0}$ is of type $\geqq \mathbf{t}(A_{k_0})$ and infinitely divisible by 3, so it must vanish: $\eta a_{k_0} = 0$. It follows in like manner that $\eta a_k = 0$ for all $k \in K_i$, that is, η vanishes on a maximal independent set of B_i, and thus $\eta = 0$. On the other hand, if $i = j$, then every A_k in B_i being fully invariant, we obtain $\eta a_k = r_k a_k$ for some rational number r_k, for every $k \in K_i$. Manifestly, $\eta(a_{k_0} + a_k) = r_{k_0}(a_{k_0} + a_k) + (r_k - r_{k_0})a_k$ is infinitely divisible by 3, and thus $(r_k - r_{k_0})a_k$ must vanish in B_i^*. Consequently, $r_k = r_{k_0}$ is a constant, and η is nothing else than multiplication by this constant.

2°. The second step is to show how to obtain a system $P(\mathfrak{n})$ from $P(\mathfrak{m})$, where $\mathfrak{n}$ is a cardinal such that $\mathfrak{m} < \mathfrak{n} \leqq 2^{\mathfrak{m}}$. Let $\{B_i\}_{i \in I}$ be a system $P(\mathfrak{m})$. Select some $i_0 \in I$ and form subsets I_j of I, indexed by a set J, such that the analogs of (α)–(δ) hold with $\aleph_1$ replaced by $\mathfrak{n}$. By means of these I_j ($j \in J$), we define

$$C_j = \langle \bigoplus_{i \in I_j} B_i, 2^{-\infty}(b_{i_0} + b_i) \text{ for all } i \neq i_0 \text{ in } I_j \rangle.$$

In accordance with (iii), we pick an element c_j in $\bar{B}_{i_0} \backslash 2\bar{B}_{i_0}$ and choose for $\bar{C}_j$ the direct sum of the $\bar{B}_i$ ($i \in I_j$) with $\bar{B}_{i_0}$ omitted. Then (i)–(iii) will clearly be fulfilled.

To prove (iv), let $\eta: C_j \to C_l^* = \langle C_l, 2^{-\infty}c_l, 2^{-\infty}\bar{C}_l\rangle$. If $j \neq l$, and if $m \in I_j \backslash I_l$, then $\eta B_m \leqq C_l^* \leqq \oplus_{i \in I_l} B_i^*$; hence (iv) for $\{B_i\}$ implies $\eta B_m = 0$. Therefore, $\eta(b_{i_0} + b_m) = \eta b_{i_0}$ is of type $\geqq \mathbf{t}_0$. Clearly, η maps B_{i_0} into $B_{i_0}^*$; hence $\eta b_{i_0} = rb_{i_0}$ for some rational r. But the type of rb_{i_0} in C_l^* is $\geqq \mathbf{t}_0$ only if $r = 0$; therefore $\eta B_{i_0} = 0$. Similarly, $\eta B_i = 0$ follows for all $i \in I_j$, i.e., $\eta = 0$. If $j = l$, then η must act as a multiplication by a rational number r_i on B_i, $\eta b_i = r_i b_i$ for every $i \in I_j$. From $\eta(b_{i_0} + b_i) = r_{i_0}(b_{i_0} + b_i) + (r_i - r_{i_0})b_i$, we infer that the type of $(r_i - r_{i_0})b_i$ is $\geqq \mathbf{t}_0$, whence $r_i = r_{i_0}$ follows, showing that η acts as a multiplication by r_{i_0}.

3°. The final and most difficult step is to construct a system $P(\mathfrak{m})$ for a limit cardinal $\mathfrak{m}$. Let $\mathfrak{m} = \aleph_\tau$, and suppose that for every $\sigma < \tau$ we have constructed a system $P(\aleph_\sigma)$ of groups $\{B_i^\sigma\}$ satisfying (i)–(iv). Moreover, we may assume that $B_j^{\sigma+1}$ in $P(\aleph_{\sigma+1})$ is constructed from groups B_i^σ as described in 2° [in particular, $B_i^\sigma < B_j^{\sigma+1}$ for properly chosen indices i, j] and, for a limit ordinal $\sigma < \tau$, every group in $P(\aleph_\sigma)$ is the union of an ascending chain $A_k < B_i^{(1)} < B_j^{(2)} < \cdots < B_l^\rho < \cdots$ with $\rho < \sigma$, where A_k is of rank 1 and B_l^ρ is a member of $P(\aleph_\rho)$.

The groups B_m^τ of $P(\aleph_\tau)$ will be constructed as unions of transfinite sequences

$$(2) \qquad A_k < B_i^{(1)} < \cdots < B_j^\sigma < \cdots \qquad \text{for} \quad \sigma < \tau$$

with inclusions valid between members of systems $P(\aleph_\sigma)$ for $\sigma < \tau$. Observe that different sequences may lead to the same group B_m^τ [this occurs trivially if, e.g., two sequences are equal from some term on]. Also, the same A_k may appear in many ways in B_m^τ; if A_k is viewed as a subgroup of the union B_m^τ of (2), then we can more explicitly write it as $A_{kij\ldots l\ldots}$.

First, we show how to select a $b_m^\tau \in B_m^\tau$ and a subgroup $\bar{B}_m^\tau < B_m^\tau$ as required by (iii). Let $f_\tau = f$ be a regressive function for τ satisfying (1). For the sake of convenience, the generalized continuum hypothesis will now be assumed, i.e., $2^{\aleph_\rho} = \aleph_{\rho+1}$ [but our remark below shows that it is easy to get rid of it].

Since f satisfies $|f^{-1}(\sigma)| \leqq \aleph_{\sigma+1}$ for every $\sigma < \tau$, it is possible to define a mapping h_σ from the limit ordinals $\rho \leqq \tau$ with $f(\rho) = \sigma$ into an index set L of cardinality $\aleph_{\sigma+1}$ such that $h_\sigma(\rho) \neq h_\sigma(\rho')$ whenever $\rho \neq \rho'$. Let a basis $\{a_l^{\sigma+1}\}$ of $\bar{B}_{i_0}^{\sigma+1}/2\bar{B}_{i_0}^{\sigma+1}$ be indexed by L [let c_j of 2° be in this basis, but no index assigned to it]. We stipulate that for limit ordinals $\rho < \tau$, the b_j^ρ have been chosen such that b_j^ρ is an element $a_l^{\sigma+1}$ in some $\bar{B}_{i_0}^{\sigma+1}$ whose index l corresponds to ρ under h_σ. Now b_m^τ will be picked according to the same rule. Finally, we define $\bar{B}_m^\tau$ as a union of a chain $\bar{B}_i^{(1)} < \cdots < \bar{B}_j^\sigma < \cdots$ $(\sigma < \tau)$ corresponding to (2) with the proviso that $i \neq i_0$, $j \neq j_0$, $\cdots$ in order to guarantee that $\bar{B}_m^\tau$ be independent of $B_m^\tau(\mathbf{t}_0)$.

From our definition it is now clear that in condition (iii), we have introduced the b_i in order to use them in the construction of the next system [see 2°]. Evidently, the $\bar{B}_i^\sigma$ for the distinguished index $i = i_0$ are needed to select b_j^ρ for limit ordinals ρ, while the other $\bar{B}_i^\sigma$ serve to construct the $\bar{B}_i^\rho$ for larger ρ.

Next, we verify (i)–(iv) for the system $\{B_m^\tau\}$ thus constructed. As in (2), $|B_j^\sigma| = \aleph_\sigma$ holds for every σ, it is clear that $|B_m^\tau| = \sum_{\sigma<\tau} \aleph_\sigma = \aleph_\tau$, i.e., (i) is satisfied. It is straightforward from the definitions that (iii) also holds. To check (iv), let $\eta: B_m^\tau \to B_n^{\tau *}$. Clearly, η maps every $A_{kij\ldots l\ldots} \leqq B_m^\tau$ onto 0 or onto a subgroup of $B_n^{\tau *}$, isomorphic to A_k. If η is nontrivial on $A_{kij\ldots l\ldots}$, then $\eta A_{ki\ldots l\ldots} = A_{k'i'\ldots l'\ldots}$ implies $k = k'$, $i = i'$, $\cdots$, $l = l'$, $\cdots$, because the first indices with $l \neq l'$ would lead to the contradiction that $\eta: B_l^\sigma \to B_{l'}^{\sigma *}$ is not trivial. Hence $\eta \neq 0$ implies that $m = n$ and that η maps every $A_{ki\ldots l\ldots}$ into itself. It is now easy to conclude that η is then a multiplication by a rational number.

To establish (ii), we argue as follows. First, we divide the set of the A_k ($k \neq k_0$) into $2^{\aleph_1}$ pairwise disjoint subsets, each of the cardinality $2^{\aleph_1}$, and consider only B_l^σ whose subgroups A_k ($k \neq k_0$) belong to the same subset. This process is repeated for the $B_i^{(1)}$ considered, i.e., we divide every subset of the groups $B_i^{(1)}$ constructed from the same subset of the A_k [except for $B_{i_0}^{(1)}$] into $2^{\aleph_2}$ pairwise disjoint subsets, each of cardinality $2^{\aleph_2}$, and construct only B_l^σ ($\sigma \geqq 2$) containing $B_i^{(1)}$ ($i \neq i_0$) from the same subset, etc. It is clear that if B_m^τ and B_n^τ are such that at some stage $\sigma < \tau$, $B_l^{\sigma+1}$ [$< B_m^\tau$] is constructed from groups B_j^σ which do not occur in B_n^τ, then B_m^τ and B_n^τ can not be isomorphic. In this way, in constructing the mentioned groups of $P(\aleph_\tau)$, we can choose A_k from $2^{\aleph_1}$ disjoint subsets, then the $B_i^{(1)}$ from $2^{\aleph_2}$ disjoint subsets, etc., which shows that at least $\prod_\sigma 2^{\aleph_\sigma} = 2^{\Sigma \aleph_\sigma} = 2^{\aleph_\tau}$ nonisomorphic groups B_m^τ will arise. Since we can not have more than $2^{\aleph_\tau}$ nonisomorphic groups of cardinality $\aleph_\tau$, (ii) is proved.□

Remark. In the final part of the proof, the generalized continuum hypothesis was assumed. We can avoid referring to it simply by passing from a system $P(\mathfrak{m})$ immediately to $P(2^{\mathfrak{m}})$, which does not effect our argument in 3°.

By making use of large rigid systems, one can construct—to every infinite cardinal $\mathfrak{m}$ less than the first strongly inaccessible cardinal—a torsion-free group of cardinality $\mathfrak{m}$ which has $2^{\mathfrak{m}}$ pairwise nonisomorphic, indecomposable summands of cardinality $\mathfrak{m}$. [See Fuchs [28].]

Exercises

1. Show that in (89.2) one may, moreover, assume that the endomorphism rings of all the groups are isomorphic to $\mathbf{Z}$.
2. Use (88.4) to establish a system $P(\aleph_1)$ of groups B_i with (i)–(iv) such that all B_i are homogeneous of type $(0, 0, \cdots, 0, \cdots)$. [*Hint*: in the definition of B_i in 1°, apply the method of Example 5 in **88** to replace $3^{-\infty}(a_{k_0} + a_k)$.]
3*. Prove that (89.2) holds even if all the groups are required to be homogeneous of type $(0, 0, \cdots, 0, \cdots)$.

4*. (A. L. S. Corner) The same as Ex. 3 with any type $\neq(\infty, \infty, \cdots, \infty, \cdots)$.

5*. Let A_i $(i \in I)$ be a rigid system of slender groups [see **94**] such that $2A_i \neq A_i$. With $a_i \in A_i \backslash 2A_i$, define $A = \langle \Pi A_i, \frac{1}{2}(a_i + a_j)$ for all $i, j \in I\rangle$, and show that A is indecomposable. [*Hint*: the direct sum and the new generators all belong to the same summand; apply **94**, Ex. 8.]

6. Let g_i $(i \in I)$ be elements in an indecomposable group G, independent mod $2G$. Let H be a torsion-free group with $\mathrm{Hom}(G, H) = 0$ and h_i $(i \in I)$ elements independent mod $2H$. If all nonzero summands of the group $\langle H, \frac{1}{2}h_i\ (i \in I)\rangle$ intersect the subgroup $\langle h_i\ (i \in I)\rangle$ nontrivially, then

$$A = \langle G \oplus H, \tfrac{1}{2}(g_i + h_i) \text{ for all } i \in I\rangle$$

is indecomposable. [*Hint*: if $A = X \oplus Y$, G is contained in X; if in A/G the coset of $\frac{1}{2}\sum n_i h_i$ belongs to Y, look at $\frac{1}{2}\sum n_i(g_i + h_i)$.]

7. (Sąsiada [3]) (a) In Ex. 6, let the group G and the index set I be of cardinality $\mathfrak{m}$, and for H take the direct product of $\mathfrak{m}$ copies of a suitable rational group R. On using **94**, Ex. 8, show that A will be indecomposable of cardinality $2^{\mathfrak{m}}$.

   (b)* Apply the method of (a) transfinitely in order to obtain indecomposable groups up to cardinality $\aleph_{\omega_1}$.

90. DIRECT DECOMPOSITIONS OF FINITE RANK GROUPS

From the point of view of direct decompositions, the existence of large indecomposable torsion-free groups is a major difference between torsion and torsion-free groups. What is probably even more surprising is the occurrence of various unexpected phenomena in direct decompositions of torsion-free groups already in the finite rank case. Some of these will be surveyed in this section.

It is evident that a torsion-free group A of finite rank decomposes into the direct sum of a finite number of indecomposable groups. The question we pose is this: is there any sort of isomorphy or uniqueness in the decompositions of A into indecomposable summands? In view of the following results, our answer will be a definite "no" to this question.

Our first result shows that even the number of summands can be different in the decompositions. [In the sequel, "summand" means "nonzero summand."]

Theorem 90.1. *For every integer $n \geqq 2$, there exists a group of finite rank which has direct decompositions into 2 and into n indecomposable summands.*

Let $p, q, p_1, \cdots, p_{n-1}$ be different primes and $n \geqq 3$. Define

$$A = \langle p_1^{-\infty} a_1\rangle \oplus \cdots \oplus \langle p_{n-1}^{-\infty} a_{n-1}\rangle$$
$$\oplus \langle p_1^{-\infty} b_1, \cdots, p_{n-1}^{-\infty} b_{n-1}, p^{-1}q^{-1}(b_1 + b_2), \cdots, p^{-1}q^{-1}(b_1 + b_{n-1})\rangle,$$

with independent elements $a_1, \cdots, a_{n-1}, b_1, \cdots, b_{n-1}$. The first $n-1$ summands are of rank 1, the indecomposability of the last one follows from (88.3). If s, t are integers such that $ps - qt = 1$, then choose $c_i = pa_i + tb_i$ and $d_i = qa_i + sb_i$ for $i = 1, \cdots, n-1$, and let

$$C = \langle p_1^{-\infty}c_1, \cdots, p_{n-1}^{-\infty}c_{n-1}, p^{-1}(c_1 + c_2), \cdots, p^{-1}(c_1 + c_{n-1})\rangle,$$

$$D = \langle p_1^{-\infty}d_1, \cdots, p_{n-1}^{-\infty}d_{n-1}, q^{-1}(d_1 + d_2), \cdots, q^{-1}(d_1 + d_{n-1})\rangle.$$

Then C and D are contained in A; clearly, they are indecomposable. Since $a_i = sc_i - td_i$, $b_i = pd_i - qc_i$, and $b_1 + b_i = p(d_1 + d_i) - q(c_1 + c_i)$, it follows at once that all the generators of A are in $C + D$, whence $A = C \oplus D$.□

Restriction to direct decompositions with a fixed number of indecomposable summands does not help claim isomorphy—as is shown by the following striking result.

Theorem 90.2 (Corner [1]). *Given the integers n, k such that $n \geqq k \geqq 1$, there exists a torsion-free group A of rank n with the following property: to every partition of n into k integers $r_i \geqq 1$, $n = r_1 + \cdots + r_k$, there is a decomposition*

$$A = A_1 \oplus \cdots \oplus A_k,$$

where A_i is indecomposable of rank r_i $(i = 1, \cdots, k)$.

Let $p, p_1, \cdots, p_{n-k}, q_1, \cdots, q_{n-k}$ be different primes. With independent elements $u_1, \cdots, u_k, x_1, \cdots, x_{n-k}$, define

$$A = \langle p^{-\infty}u_1, \cdots, p^{-\infty}u_k, p_1^{-\infty}x_1, \cdots, p_{n-k}^{-\infty}x_{n-k},$$
$$q_1^{-1}(u_1 + x_1), \cdots, q_{n-k}^{-1}(u_1 + x_{n-k})\rangle.$$

To show that this A has the indicated property, let $n = r_1 + \cdots + r_k$ be a partition of n with $r_i \geqq 1$. Notice that if $s_1, \cdots, s_k$ are integers such that $s_1 + \cdots + s_k = 1$, then the system

$$\begin{array}{rcl} s_1v_1 + s_2v_2 + \cdots + s_kv_k &=& u_1 \\ -v_1 \quad + v_2 \qquad\qquad &=& u_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots & & \\ -v_1 \qquad\qquad\quad + v_k &=& u_k \end{array}$$

with the unknowns $v_1, \cdots, v_k$ has determinant 1. Hence this system has a solution $v_1, \cdots, v_k \in A$ such that $\langle v_1, \cdots, v_k\rangle = \langle u_1, \cdots, u_k\rangle$. With these v_i, put

$$A_i = \langle p^{-\infty}v_i, p_j^{-\infty}x_j, q_j^{-1}(v_i + x_j) \text{ for } j = t_i + 1, \cdots, t_{i+1}\rangle,$$

where $t_1 = 0$ and $t_i = (r_1 - 1) + \cdots + (r_{i-1} - 1)$. In view of

$$u_1 + x_j = (v_i + x_j) + s_1 v_1 + \cdots + s_{i-1} v_{i-1} + (s_i - 1)v_i + \cdots + s_k v_k,$$

we find that if $s_1, \cdots, s_k$ are chosen so as to satisfy

$$s_i \equiv \begin{cases} 1 \bmod q_j & \text{for} \quad j = t_i + 1, \cdots, t_{i+1}, \\ 0 \bmod q_j & \text{otherwise,} \end{cases}$$

then all the A_i $(i = 1, \cdots, k)$ will be contained in A, and every generator of A will belong to $A_1 + \cdots + A_k$. This gives at once $A = A_1 \oplus \cdots \oplus A_k$, where, by construction, $r(A_i) = r_i$. The indecomposability of the A_i follows from (88.3). [Numbers s_i can be chosen, for instance, as $s_i = m_{t_i+1} \hat{q}_{t_i+1} + \cdots + m_{t_{i+1}} \hat{q}_{t_{i+1}}$, where $\hat{q}_j = q_1 \cdots q_{j-1} q_{j+1} \cdots q_{n-k}$ and $\sum_j m_j \hat{q}_j = 1$.]□

In our endeavor to find some sort of uniqueness, we can make an additional restriction and compare merely decompositions into indecomposable summands with the same distribution of ranks. This is still not enough to ensure isomorphy:

Theorem 90.3 (Fuchs and Loonstra [1]). *Given any integer $m \geqq 2$, there exist two torsion-free indecomposable groups, A and C, of rank 2, such that*

$$A \oplus \cdots \oplus A \cong C \oplus \cdots \oplus C \qquad (n \text{ summands})$$

if and only if $n \equiv 0 \bmod m$.

We slightly modify the technique of the foregoing proofs in order to take care of the claimed nonisomorphy as well. Starting with two disjoint, infinite sets P_1 and P_2 of primes such that $p \notin P_1 \cup P_2$ for a prime p to be suitably chosen later on, we form the following rank 1 groups:

$$X_i = \langle p_1^{-1} x_i \mid p_1 \in P_1 \rangle, \qquad Y_i = \langle p_2^{-1} y_i \mid p_2 \in P_2 \rangle \qquad \text{for} \quad i = 1, \cdots, n.$$

Then $p \nmid x_i$ in X_i and $p \nmid y_i$ in Y_i, thus the groups

$$A_i = \langle X_i \oplus Y_i,\ p^{-1}(x_i + y_i) \rangle \qquad (i = 1, \cdots, n)$$

are indecomposable; they are isomorphic. Choose groups $U_i \cong X_i$ and $V_i \cong Y_i$ $(i = 1, \cdots, n)$, and let $u_i \leftrightarrow x_i$, $v_i \leftrightarrow y_i$ $(u_i \in U_i, v_i \in V_i)$ under some fixed isomorphisms. For any choice of $k = 1, \cdots, p - 1$, we can form the isomorphic groups

$$C_i = \langle U_i \oplus V_i,\ p^{-1}(u_i + kv_i) \rangle \qquad (i = 1, \cdots, n),$$

which are likewise indecomposable.

Suppose the existence of an isomorphism

$$\varphi: A = A_1 \oplus \cdots \oplus A_m \to C = C_1 \oplus \cdots \oplus C_m.$$

Our choice of X_i and Y_i guarantees that $X = X_1 \oplus \cdots \oplus X_m$ and $Y = Y_1 \oplus \cdots \oplus Y_m$ are mapped by φ onto $U = U_1 \oplus \cdots \oplus U_m$ and $V = V_1 \oplus \cdots \oplus V_m$, respectively. Thus

$$(1) \qquad \varphi\colon x_i \mapsto \sum_{j=1}^{m} r_{ij} u_j \qquad \text{and} \qquad y_i \mapsto \sum_{j=1}^{m} s_{ij} v_j,$$

where r_{ij}, s_{ij} must be integers such that the matrices $[r_{ij}]$ and $[s_{ij}]$ are invertible, i.e., the determinants

$$(2) \qquad \det[r_{ij}] = \pm 1, \qquad \det[s_{ij}] = \pm 1.$$

An isomorphism must map an element divisible by p upon an element with the same property, therefore

$$(3) \qquad \varphi\colon x_i + y_i \mapsto \sum_{j=1}^{m} r_{ij}(u_j + kv_j) + \sum_{j=1}^{m} (s_{ij} - kr_{ij})v_j,$$

together with the independence of the v_j, implies

$$(4) \qquad s_{ij} \equiv kr_{ij} \bmod p \qquad \text{for all} \quad i, j.$$

On the other hand, (4) and (2) collectively imply that φ, as given in (1), is, in fact, an isomorphism between A and C [namely, $p | x_i + y_i$ in A for all i implies $p | u_j + kv_j$ in C for all j, and conversely].

Our goal is to choose the matrices $[r_{ij}]$, $[s_{ij}]$, and the integer k such that $A_1 \oplus \cdots \oplus A_n \cong C_1 \oplus \cdots \oplus C_n$ hold for $n = m$ but not for $m \nmid n$. From (2) and (4) we deduce that $k^m = \pm 1 \bmod p$. In order to exclude isomorphy for $n < m$, k will be chosen so as to satisfy

$$(5) \quad k^m \equiv -1 \bmod p, \qquad \text{but} \qquad k^n \not\equiv \pm 1 \bmod p \qquad \text{for} \quad n = 1, \cdots, m-1.$$

[We dropped the case $k^m \equiv 1 \bmod p$, because this can occur, in the presence of the other condition, only for odd m.] To find a solution for (5), let p be subject to the condition $p \equiv 1 \bmod 2m$; by Dirichlet's theorem on primes in arithmetic progressions, such a prime p does exist. Let t be a primitive root mod p; then $k = t^{(p-1)/2m}$ satisfies (5). Note that (5) implies that $k^n \equiv \pm 1 \bmod p$ if and only if m divides n.

With this k, we construct $[r_{ij}]$ and $[s_{ij}]$ as follows. Because of $k^m \equiv -1 \bmod p$, the determinants in (2) must have different signs. Let l satisfy $kl \equiv 1 \bmod p$, and define $[r_{ij}]$ as the lower triangular $m \times m$ matrix

$$[r_{ij}] = \begin{bmatrix} 1 & & & & & & & \\ l & 1 & & & & & & \\ 0 & l & 1 & & & & & \\ l & 1 & l & 1 & & & & \\ 0 & l & 1 & l & 1 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & 1 & l & 1 & \\ \cdots & \cdots & \cdots & \cdots & l & 1 & l & -1 \end{bmatrix}$$

where in the main diagonal we have 1s except for the last entry, and below the diagonal we have alternately l and 0 in the first column, l and 1 in the other columns. Then $\det[r_{ij}] = -1$. In accordance with (4), we multiply the above matrix by k and replace each entry by a congruent one to get a matrix whose determinant is 1:

$$[s_{ij}] = \begin{bmatrix} k & & & & & & & k' \\ 1 & k & & & & & & \\ 0 & 1 & k & & & & & \\ 1 & k & 1 & k & & & & \\ 0 & 1 & k & 1 & k & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & k & 1 & k & \\ \cdots & \cdots & \cdots & \cdots & 1 & k & 1 & -k \end{bmatrix}$$

which is obtained from $[r_{ij}]$ by substituting k for 1 and 1 for l throughout and by putting $k' = (-1)^{m+1}(k^m + 1)$ in the upper right corner. Consequently, r_{ij}, s_{ij}, and k can be chosen to satisfy (2), (4), and (5), and therefore the groups have all the stated properties. $\square$

Our final attempt in the search for uniqueness is the cancellation property. A group B is said to have the *cancellation property* if, for any groups H and K, $B \oplus H \cong B \oplus K$ implies $H \cong K$. An equivalent formulation is: for any groups H and K, $B \oplus H = C \oplus K$ with $B \cong C$ implies $H \cong K$. The next result shows that not even the rank one groups necessarily have the cancellation property.

Theorem 90.4 (Fuchs and Loonstra [1]). *Given an integer m, there exists a torsion-free group A of rank 3 which has decompositions*

$$A = B_i \oplus C_i \qquad (i = 1, \cdots, m)$$

such that $B_1 \cong \cdots \cong B_m$ are of rank 1 and $C_1, \cdots, C_m$ are pairwise non-isomorphic, indecomposable groups of rank 2.

Like in the proof of (90.3), we start off with two disjoint, infinite sets P_1 and P_2 of primes, and a prime p not in P_1 and P_2, and let

$$B_1 = \langle p_1^{-1}b \mid p_1 \in P_1\rangle, \qquad X = \langle p_1^{-1}x \mid p_1 \in P_1\rangle, \qquad Y = \langle p_2^{-1}y \mid p_2 \in P_2\rangle.$$

Then $C_1 = \langle X \oplus Y, p^{-1}(x + y)\rangle$ will be indecomposable of rank 2. Define

$$A = B_1 \oplus C_1.$$

If q_i, r_i, s_i, t_i are integers such that $q_i t_i - r_i s_i = 1$, and if we put

$$b_i = q_i b + s_i x, \qquad x_i = r_i b + t_i x \qquad (i = 2, \cdots, m),$$

then $B_i \oplus X_i = B_1 \oplus X$ holds for the pure subgroups $B_i = \langle b_i \rangle_*$, $X_i = \langle x_i \rangle_*$ isomorphic to B_1. We intend to choose B_i and X_i so that, for some integers k_i $(1 < k_i < p)$, we have

$$(6) \qquad A = B_i \oplus C_i \qquad \text{with} \quad C_i = \langle X_i \oplus Y,\ p^{-1}(k_i x_i + y)\rangle$$

with indecomposable C_i. Then $k_i x_i + y = k_i r_i b + (k_i t_i - 1)x + (x + y)$ must be divisible by p, thus

$$(7) \qquad k_i r_i \equiv 0 \quad \text{and} \quad k_i t_i \equiv 1 \qquad \text{mod } p.$$

Given k_i with $1 < k_i < p$, we can choose $r_i = p$, $q_i = k_i$, and s_i, t_i so as to satisfy $q_i t_i - r_i s_i = 1$; then (7) holds and thus (6) will be fulfilled.

In order to select nonisomorphic C_i, notice that an isomorphism $\varphi: C_i \to C_j$ must map X_i upon X_j and Y upon itself such that $x_i \mapsto \pm x_j$ and $y \mapsto \pm y$ [the only automorphisms of X_i and Y are multiplications by ± 1]. Thus $k_i x_i + y$ maps upon $\pm(k_i x_j \pm y)$. The preservation of divisibility forces $k_j \equiv \pm k_i$ mod p. Consequently, if we choose $[k_1 = 1]$, $k_2 = 2, \cdots, k_m = m$, and $p > 2m - 1$, then $k_j \not\equiv \pm k_i$ mod p, and therefore no two of $C_1, \cdots, C_m$ are isomorphic.□

Notice that (90.4) also shows that *for any rank $r \geqq 3$ and to any integer m there is a torsion-free group of rank r which has at most m pairwise nonisomorphic decompositions into indecomposable summands.* It is natural to wonder whether there exist finite rank groups with infinitely many decompositions.

Exercises

1. Given $m > n \geqq 2$, there is a finite rank group which decomposes into the direct sum of m and also into the direct sum of n indecomposable summands.
2. Let $r_1, \cdots, r_{k-1}$ be positive integers and $r = r_1 + \cdots + r_{k-1}$. There exists a group A of rank $2r$ such that $A = A_1 \oplus \cdots \oplus A_{k-1} \oplus A_k = B \oplus C$, where A_i is indecomposable of rank r_i $(i = 1, \cdots, k-1)$, while the groups A_k, B, C are indecomposable of rank r. [*Hint*: in the proof of (90.1), replace the rank 1 groups by groups in a suitable rigid system.]
3. (Corner [1]) Show that in any direct decomposition of the group A, as defined in the proof of (90.2), into indecomposable groups, there are always exactly k nonzero components.
4. Given integers n, k, and r such that $kr \leqq n$, construct a group A of rank n with the following property: to a partition $n = r_1 + \cdots + r_k$ with $r_i \geqq 1$ there is a decomposition of A into indecomposable summands of rank r_i exactly if each $r_i \geqq r$. [*Hint*: in the proof of (90.2), replace $\langle p^{-\infty} u \rangle$ by a suitable rigid group of rank r.]

5. Show that there is no group of rank 4 which can be written as the direct sum of two rank 1 groups and a rank 2 group and also as a direct sum of a rank 1 group and an indecomposable group of rank 3. [*Hint*: look at the types of the rank 1 subgroups.]
6. (Fuchs and Loonstra [1]) To every $m \geqq 2$, there exist pairwise non-isomorphic indecomposable groups $B, C_1, \cdots, C_m$ of rank 2 such that $B \oplus \cdots \oplus B \cong C_1 \oplus \cdots \oplus C_m$ (m summands), but the direct sum of $n(<m)$ copies of B is not isomorphic to the direct sum of any n groups of $C_1, \cdots, C_m$. [*Hint*: argue as in (90.3).]
7. Show that (90.3) prevails if the A_i and C_i are required to have any rank $r > 2$. [*Hint*: replace X_i, Y_i by a rigid system of groups with the only automorphisms ± 1.]
8. For every odd integer $m > 0$ and every positive integer t there exist t groups of rank 2 such that any two of them behave like A and C in (90.3). [*Hint*: use a product of primes rather than p.]
9. For every integer $n \geqq 1$, there is an indecomposable group A of rank n such that $A \oplus B \cong A \oplus C$ for suitable nonisomorphic indecomposable groups B and C of finite ranks. [*Hint*: see (90.4).]
10. (Sąsiada [7]) Let $p = 5$, and $q = 3$, $r = 2$ or $q = 2$, $r = 3$. Set

$$X_i = \langle p^{-\infty} x_i, q^{-\infty} y_i, r^{-1}(x_i + y_i)\rangle$$

for $i = 1, \cdots, k$. Show that every indecomposable summand of $A = X_1 \oplus \cdots \oplus X_k$ is isomorphic to X_1. [*Hint*: $A = U \oplus V$, $U \neq 0$, some $n_1 x_1 + \cdots + n_k x_k = u_0 \in U$ (or y_i rather than x_i) with relatively prime coefficients; if $\sum r^{-1} n_i(x_i + y_i) = u + v$, then $r^{-1}u - u_0$ is a linear combination of the y_i and $\langle u_0, u\rangle_*$ is a summand of U.]
11. (Fuchs and Loonstra [2]) Let A be a torsion-free group of rank 1. We say it has the *lifting property* if the automorphisms of A/nA are induced by automorphisms of A, for every $n > 0$.
Prove that for a torsion-free noncyclic group A of rank 1 a necessary and sufficient condition to have the lifting property is that the following condition is satisfied for all n: for every group G and subgroup G_0 with $G/G_0 \cong Z(n)$, all subdirect sums of A and G with kernels nA and G_0 are isomorphic. [*Hint*: for necessity use Ex. 7(a) in **88**, and show that any isomorphism between two subdirect sums must carry nA and G_0 into themselves.]
12*. (Fuchs and Loonstra [2]) A torsion-free noncyclic group A of rank 1 has the cancellation property if it has the lifting property. [*Hint*: if $A \oplus H = C \oplus K$ with $A \cong C$, reduce the problem to the case when none of the groups contains another; if B is the projection of C in H and $D = (A \oplus B) \cap K$, then $B \cong D \cong A$, and show that H and K are isomorphic to subdirect sums of A and $H/(H \cap K) \cong K/(H \cap K)$ with kernels of finite index; cf. Ex. 11.]

13. (a) (A. L. S. Corner) Let A be a torsion-free group of finite rank n, and B, C isomorphic pure subgroups of rank $n-1$. Verify the isomorphism $A/B \cong A/C$. [*Hint*: reduce to the case $n=2$, $B \neq C$, and calculating the torsion group $A/(B+C)$ in two ways, conclude that $\mathbf{t}(A/B):\mathbf{t}(C) = \mathbf{t}(A/C):\mathbf{t}(B)$.]
(b) Cancellation by a torsion-free group of finite rank is permitted if the complements are of rank 1.

91. DIRECT DECOMPOSITIONS OF COUNTABLE GROUPS

Our next program is to study the direct decompositions of torsion-free groups of countable rank. In contrast to the finite rank case, countable groups need not be direct sums of indecomposable groups. Some rather paradoxical examples of such groups will appear in (91.5) and (91.6). But first we concentrate on the case when the groups are better behaved and admit direct decompositions into indecomposable groups. In view of our experience with groups of finite rank, it is no wonder that nearly everything conceivable can occur in the countable case.

The first question we take up is again concerned with the number of summands.

Theorem 91.1 (Corner [1]). *There exists a group A which has two direct decompositions*:

$$A = B \oplus C = \bigoplus_{n=-\infty}^{\infty} E_n,$$

where B, C are indecomposable of rank $\aleph_0$ and the E_n are indecomposable of rank 2.

Let $\{p_n\}_n$, $\{q_n\}_n$, and $\{r_n\}_n$ be three, pairwise disjoint, infinite sets of primes, where $n = 0, \pm 1 \pm 2, \cdots$. With independent b_n and c_n, define

$$B = \langle p_n^{-\infty} b_n, q_n^{-1}(b_n + b_{n+1}) \text{ for all } n \rangle,$$

$$C = \langle p_n^{-\infty} c_n, r_n^{-1}(c_n + c_{n+1}) \text{ for all } n \rangle.$$

Both B and C are of rank $\aleph_0$, and by (88.3), they are indecomposable. Choose integers k_n [to be specified later], and with

$$u_n = (1+k_n)b_n - k_n c_n, \qquad v_n = k_n b_n + (1-k_n)c_n$$

define

$$E_n = \langle p_n^{-\infty} u_n, p_{n+1}^{-\infty} v_{n+1}, q_n^{-1} r_n^{-1}(u_n + v_{n+1}) \rangle \qquad \text{for} \quad n = 0, \pm 1, \cdots.$$

Then the E_n are indecomposable of rank 2. Clearly, the direct sum of $\langle b_n \rangle_*$ and $\langle c_n \rangle_*$ is equal to the direct sum of $\langle u_n \rangle_*$ and $\langle v_n \rangle_*$. From

$$u_n + v_{n+1} = (b_n + b_{n+1}) + k_n(b_n - c_n) + (k_{n+1} - 1)(b_{n+1} - c_{n+1})$$

it is apparent that for $q_n | u_n + v_{n+1}$ it is necessary to choose k_n such that $k_n \equiv 0$ and $k_{n+1} \equiv 1 \bmod q_n$, and similarly, from

$$u_n + v_{n+1} = (c_n + c_{n+1}) + (1 + k_n)(b_n - c_n) + k_{n+1}(b_{n+1} - c_{n+1})$$

it follows that for $r_n | u_n + v_{n+1}$ the k_n must satisfy $k_{n+1} \equiv 0$ and $k_n \equiv -1 \bmod r_n$. Thus if

$$k_n \equiv \begin{cases} 0 \bmod q_n r_{n-1}, \\ 1 \bmod q_{n-1}, \\ -1 \bmod r_n, \end{cases}$$

then the E_n will be subgroups of $A = B \oplus C$. But if the k_n are chosen in this way, then $q_n | u_n + v_{n+1}$ will imply $q_n | b_n + b_{n+1}$ and $r_n | u_n + v_{n+1}$ will imply $r_n | c_n + c_{n+1}$ in $\bigoplus E_n$, proving that $A = \bigoplus E_n$.□

A consequence of (91.1) is that, in contrast to (86.7), *summands of direct sums of finite rank groups need not be again direct sums of finite rank groups.*

The method of (90.1) carries over, with minor modifications, to yield an analogous conclusion for the case of countable rank.

Theorem 91.2. *There exists a group G which can be written as $G = A \oplus B$ and also as $G = C \oplus D$, where B, C, and D are indecomposable of rank $\aleph_0$ and A is completely decomposable of rank $\aleph_0$.*

Let p, q, and p_n ($n = 1, 2, \cdots$) be different primes. For independent a_n, b_n, define

$$A = \bigoplus_{n=1}^{\infty} \langle p_n^{-\infty} a_n \rangle \qquad \text{and} \qquad B = \langle p_n^{-\infty} b_n,\ p^{-1} q^{-1}(b_n - b_{n+1}) \text{ for all } n \rangle.$$

Then A is completely decomposable, and reasoning as in (88.3), we derive that B is indecomposable. If s, t are integers such that $ps - qt = 1$, then with $c_n = pa_n + tb_n$ and $d_n = qa_n + sb_n$ we set

$$C = \langle p_n^{-\infty} c_n,\ p^{-1}(c_n - c_{n+1}) \text{ for all } n \rangle,$$
$$D = \langle p_n^{-\infty} d_n,\ q^{-1}(d_n - d_{n+1}) \text{ for all } n \rangle,$$

which groups are indecomposable. The equality $A \oplus B = C \oplus D$ follows as in the proof of (90.1).□

It is not hard to modify our last construction in order to exhibit an example where A is completely decomposable of any finite rank $r \geqq 1$. Set

$$A = \langle p_1^{-\infty} a_1 \rangle \oplus \cdots \oplus \langle p_r^{-\infty} a_r \rangle,$$
$$B = \langle p_1^{-\infty} b_1, \cdots, p_r^{-\infty} b_r,\ X,\ Y,\ p^{-1}(b_n + x), q^{-1}(b_n + y) \text{ for } n = 1, \cdots, r \rangle,$$

where p, q, $p_1, \cdots, p_r$ are different primes, X and Y are indecomposable groups of rank $\aleph_0$ such that $\langle p_1^{-\infty}b_1\rangle, \cdots, \langle p_r^{-\infty}b_r\rangle$, X, Y form a rigid system, and $p \nmid x \in X$, $q \nmid y \in Y$; the existence of such X and Y is guaranteed by the results in **88**. Then the full invariance of X, Y in B ensures the indecomposability of B. If s, t, c_n, d_n have the same meaning as above, then

$$C = \langle p_1^{-\infty}c_1, \cdots, p_r^{-\infty}c_r, X, p^{-1}(qc_n - x) \text{ for } n = 1, \cdots, r\rangle,$$

$$D = \langle p_1^{-\infty}d_1, \cdots, p_r^{-\infty}d_r, Y, q^{-1}(pd_n + y) \text{ for } n = 1, \cdots, r\rangle$$

are readily checked to be indecomposable and to satisfy $A \oplus B = C \oplus D$.

Also, (90.2) has an analog in the countable case.

Theorem 91.3 (Corner [1]). *There exists a group A of rank $\aleph_0$ such that for every sequence of positive integers $r_1, \cdots, r_n, \cdots$, infinitely many of which are >1, there exist indecomposable subgroups A_n of rank r_n in A such that*

$$A = \bigoplus_{n=1}^{\infty} A_n.$$

Let $r_1', \cdots, r_n', \cdots$ denote those r_n which exceed 1. Let further p, p_n, q_n $(n = 1, 2, \cdots)$ be distinct primes and u_n, x_n $(n = 1, 2, \cdots)$ independent elements. Put

$$A = \bigoplus_{n=1}^{\infty} B_n \qquad \text{with} \quad B_n = \langle p^{-\infty}u_n, p_n^{-\infty}x_n, q_n^{-1}(u_n + x_n)\rangle.$$

Notice that by the proof of (90.2), a direct sum of m of these B_n decomposes into the direct sum of an indecomposable group of rank $m - 1$ [of the same form as A_i in the proof of (90.2)] and $m - 1$ groups of rank 1 [of the form $\langle p^{-\infty}v\rangle$]. Decomposing the groups $C_n = B_{t_n+1} \oplus \cdots \oplus B_{t_{n+1}}$ with $t_1 = 0$, $t_n = (r_1' - 1) + \cdots + (r_{2n-2}' - 1)$ in this way, we get a decomposition of A into indecomposable summands infinitely many of which are of rank 1 and the rest are of ranks $r_1' + r_2' - 1, \cdots, r_{2n-1}' + r_{2n}' - 1, \cdots$; all these are of the same form as A_i in the proof of (90.2). Applying (90.2), the direct sum of the group of rank $r_{2n-1}' + r_{2n}' - 1$ and a rank 1 group can be redecomposed into a direct sum of two indecomposable groups of ranks r_{2n-1}' and r_{2n}', respectively. It is clear that these direct sums can be taken in such a way that as many rank 1 summands of A are left over as the number of 1s among the r_n. □

The group of (91.3) admits continuously many nonisomorphic direct decompositions into indecomposable summands; in fact, there exist continuously many distinct subsets of the positive integers ≥ 2, and owing to (91.3) each subset gives rise to a direct decomposition into indecomposable summands whose ranks are exactly the elements of the given subset. The following example is a countable group with continuously many nonisomorphic, indecomposable direct summands.

Theorem 91.4 (Fuchs [28]). *There exists a group A of rank $\aleph_0$ such that*

$$A = B_j \oplus C_j \qquad \text{with} \quad B_j \cong C_j$$

holds for continuously many, pairwise nonisomorphic indecomposable groups B_j.

Let $P_1, \cdots, P_i, \cdots, Q_1, \cdots, Q_i, \cdots$ be pairwise disjoint, infinite sets of primes and p, q, r odd primes not in their union. Choosing independent elements a_i, b_i, c_i, d_i $(i = 1, 2, \cdots)$, we define

$$B = \langle P_i^{-1}a_i,\ Q_i^{-1}b_i,\ p^{-1}(a_i + a_{i+1}), q^{-1}(b_i + b_{i+1}), r^{-1}(a_i + b_i) \text{ for all } i\rangle,$$

$$C = \langle P_i^{-1}c_i,\ Q_i^{-1}d_i,\ p^{-1}(c_i + c_{i+1}), q^{-1}(d_i + d_{i+1}), r^{-1}(c_i + d_i) \text{ for all } i\rangle,$$

where $P_i^{-1}a$ stands for $\{p_i^{-1}a \text{ for all } p_i \in P_i\}$. Then $B \cong C$ are indecomposable, and we let $A = B \oplus C$. For each i, choose an integer k_i [to be specified later], and define

$$u_i = a_i, \qquad v_i = k_i b_i + (k_i^2 - 1)d_i,$$

$$x_i = k_i a_i + c_i, \qquad y_i = b_i + k_i d_i$$

for all i. Our present aim is to choose the k_i such that $A = U \oplus X$ hold, where

$$U = \langle P_i^{-1}u_i,\ Q_i^{-1}v_i,\ p^{-1}(u_i + u_{i+1}), q^{-1}(v_i + v_{i+1}), r^{-1}(u_i + k_i v_i) \text{ for all } i\rangle,$$

$$X = \langle P_i^{-1}x_i,\ Q_i^{-1}y_i,\ p^{-1}(x_i + x_{i+1}), q^{-1}(y_i + y_{i+1}), r^{-1}(x_i + k_i y_i) \text{ for all } i\rangle.$$

Investigating divisibility by p, q, and r, our usual technique shows that for $A = U \oplus X$ it is necessary and sufficient that the k_i are subject to the conditions:

$$(1) \qquad k_i \equiv k_{i+1} \bmod pq \qquad \text{and} \qquad k_i^2 \equiv 1 \bmod r \qquad \text{for all} \quad i.$$

Pick an integer l such that $l \equiv 1 \bmod pq$ and $l \equiv -1 \bmod r$. For each choice of $k_i = 1$ or l, the sequence of the k_i will satisfy (1) and we get a decomposition $A = U \oplus X$ with indecomposable $U \cong X$.

We fix $k_1 = 1$, and show that if the sequence $k_2, \cdots, k_i, \cdots$ is different from $k_2', \cdots, k_i', \cdots$, then the corresponding groups U and U' are not isomorphic. Manifestly, an alleged isomorphism $\varphi\colon U \to U'$ must act as follows on the generators: $u_i \mapsto \pm u_i'$, $v_i \mapsto \pm v_i'$. From

$$p \mid u_i + u_{i+1} \mapsto \pm(u_i' \pm u_{i+1}'), \qquad q \mid v_i + v_{i+1} \mapsto \pm(v_i' \pm v_{i+1}'),$$

$$r \mid u_1 + v_1 \mapsto \pm(u_1' \pm v_1')$$

we infer that the signs of u_i' and v_i' must be throughout the same, say $+1$. Therefore, $u_i \mapsto u_i'$ and $v_i \mapsto v_i'$ imply $r \mid u_i + k_i v_i \mapsto u_i' + k_i' v_i'$ for every i. This is impossible if one of k_i, k_i' is 1 and the other is l, because of $l \not\equiv 1 \bmod r$. To complete the proof, it remains only to observe that there are continuously many ways of choosing the sequence $k_2, \cdots, k_i, \cdots$. $\square$

So far we have failed to exhibit any countable torsion-free group which is not a direct sum of indecomposable groups. The next result provides us with such an example; moreover, it asserts a most remarkable fact: the existence of groups with no indecomposable summands at all.

Theorem 91.5 (Corner [3]). *There exists a countable group which has no indecomposable direct summand* $\neq 0$.

The proof is based on the existence theorem (110.1). We therefore first give a ring R and then define A as a group having R for its endomorphism ring.

Let Λ be the semigroup whose elements λ_r are indexed by the nonnegative rational numbers r and whose multiplication rule is

$$\lambda_r \lambda_s = \lambda_{\max(r,s)}.$$

Let R be the semigroup ring of Λ over the ring Z of integers. Then R is a countable [commutative] ring with the identity λ_0 whose additive group is freely generated by the λ_r. Invoking (110.1), we infer the existence of a countable torsion-free group A whose endomorphism ring is isomorphic to R.

Let $\varepsilon = n_1\lambda_{r_1} + \cdots + n_k\lambda_{r_k}$ with $0 \neq n_j \in \mathrm{Z}$ and $r_1 < \cdots < r_k$ be an idempotent of R. We have

$$\varepsilon^2 = \sum_{j=1}^{k} n_j(2n_1 + \cdots + 2n_{j-1} + n_j)\lambda_{r_j},$$

and equating coefficients, we obtain $2n_1 + \cdots + 2n_{j-1} + n_j = 1$ for $j = 1, \cdots, k$. Hence $n_j = (-1)^{j-1}$ for $j = 1, \cdots, k$, and thus $\varepsilon = \lambda_{r_1} - \lambda_{r_2} + - \cdots + (-1)^{k-1}\lambda_{r_k}$. Pick rationals s and t such that $r_1 < s < t < r_2$ [if $k = 1$, the last inequality is meaningless], and let $\xi = \lambda_s - \lambda_t$. This is an idempotent of R such that $\varepsilon\xi = \xi\varepsilon = \xi \neq \varepsilon$. Thus to every idempotent $\varepsilon \neq 0$ of R there exists a nonzero idempotent $\xi \in \mathrm{R}$ such that $\xi = \varepsilon\xi = \xi\varepsilon \neq \varepsilon$.

To show that A has no indecomposable summand $\neq 0$, let $B \neq 0$ be a summand of A and $\varepsilon\colon A \to B$ a corresponding projection. Then $C = \xi A$ is a nonzero summand of A which is, because of $\varepsilon\xi = \xi$, contained in B. This yields $B = C \oplus (\varepsilon - \xi)B$ with nonzero summands.□

Another interesting phenomenon is shown in the following theorem.

Theorem 91.6 (Corner [4]). *Given a positive integer m, there exists a countable torsion-free group A such that the direct sum of n_1 copies of A is isomorphic to the direct sum of n_2 copies of A exactly if $n_1 \equiv n_2 \bmod m$.*

We remark at the outset that then $A \cong A \oplus \cdots \oplus A$ with $m + 1$ summands, but not for a fewer number (≥ 2) of summands. Conversely, if A has this property then it satisfies the conditions of (91.6); in fact, to prove the equivalence, one has merely to add a certain number of copies of A to both sides of an isomorphism and replace $m + 1$ summands A by a single A.

The construction of our group A will again be based on (110.1). Thus, first a ring R will be defined whose additive group is a countable free group.

Let Λ be a semigroup with 1 which is generated by symbols ρ_i, σ_i ($i = 0, 1, \cdots, m$) subject to the relations

$$\rho_j \sigma_i = 1 \quad \text{or} \quad 0 \qquad \text{according as} \qquad i = j \quad \text{or} \quad i \neq j.$$

Let S be the semigroup ring of Λ over Z such that the 0 of the semigroup Λ is identified with the 0 of the ring S. The additive group of S is free; as a matter of fact, the different nonvanishing products of the ρ_i and σ_i form a basis. Every such product is seen to be of the form

$$\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\rho_{j_l}\cdots\rho_{j_2}\rho_{j_1}, \tag{2}$$

where $k, l \geqq 0$ and the indices i, j are from $0, 1, \cdots, m$. The principal ideal I generated by $\tau = 1 - \sigma_0\rho_0 - \cdots - \sigma_m\rho_m$ in S is additively generated by elements of the form

$$\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\tau\rho_{j_l}\cdots\rho_{j_2}\rho_{j_1}, \tag{3}$$

since $\rho_i\tau = 0 = \tau\sigma_i$ for all i. It is a trivial observation that if the basis elements (2) with $k, l \geqq 1$ and $i_k = j_l = 0$ are replaced by the elements of the form (3), then the new system is still an additive basis for S. Consequently, the additive group of the ring $\mathrm{R} = \mathrm{S}/\mathrm{I}$ is free, and R is obviously countable.

Our next step is to define a group-homomorphism ψ: $\mathrm{S} \to Z(m)$ which vanishes on I and satisfies

$$\psi(1) = 1_m \qquad \text{and} \qquad \psi(\xi\eta) = \psi(\eta\xi) \qquad \text{for all} \quad \xi, \eta \in \mathrm{S}.$$

Let ψ act on the generators (2) as follows:

$$\psi(\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}\rho_{j_l}\cdots\rho_{j_2}\rho_{j_1}) = \begin{cases} 1_m & \text{if} \quad k = l \quad \text{and} \quad i_t = j_t \quad \text{for all} \quad t, \\ 0 & \text{otherwise.} \end{cases}$$

This ψ trivially vanishes on the elements (3) except when $k = l$ and $i_t = j_t$ ($t = 1, \cdots, k$) in which case its value is $1_m - (m+1)1_m$ which is likewise 0 in $Z(m)$. It suffices to verify $\psi(\xi\eta) = \psi(\eta\xi)$ when ξ is of the form (2) and η is ρ_i or σ_i; this can be left to the reader as an exercise.

In view of (110.1), there is a torsion-free countable group A whose endomorphism ring is isomorphic to the ring R as defined above; moreover, we identify R with the endomorphism ring of A. Then $\sigma_i\rho_i = \varepsilon_i$ ($i = 0, 1, \cdots, m$) are pairwise orthogonal idempotents with sum 1, thus $A = \varepsilon_0 A \oplus \varepsilon_1 A \oplus \cdots \oplus \varepsilon_m A$. But $\sigma_i\rho_i = \varepsilon_i$ and $\rho_i\sigma_i = 1$ imply $\varepsilon_i A \cong A$ [see **106**(g)], hence A is isomorphic to the direct sum of $m + 1$ copies of itself. If A is isomorphic to the direct sum of $n + 1$ copies of itself ($0 < n \leqq m$), then in view of **106**(g) this means the existence of two endomorphisms α, β of A such that $\alpha\beta = 1$ and $\beta\alpha = \varepsilon_0 + \cdots + \varepsilon_n$. But $1_m = \psi(\alpha\beta) = \psi(\beta\alpha) = \psi(\varepsilon_0 + \cdots + \varepsilon_n) = (n+1)1_m$ holds for no $0 < n < m$, completing the proof.□

EXERCISES

1. Show that (91.1) continues to hold if the E_n are required to be of any finite or countably infinite rank $r > 2$. [*Hint*: replace the groups $\langle p_n^{-\infty} b_n \rangle$ by a suitable rigid system.]
2. Prove that (91.2) holds if the complete decomposability of A is replaced by the condition that A is the direct sum of indecomposable groups of prescribed ranks $r_1, r_2, \cdots, r_n, \cdots$.
3. (Corner [1]) (a) Let A be a direct sum of infinitely many indecomposable groups of finite rank. If almost all these are of rank 1, then in every decomposition of A into indecomposable groups, almost all components must be of rank 1. [*Hint*: A is completely decomposable modulo a subgroup of finite rank.]
(b) Conclude that in (91.3) the condition that infinitely many r_n be > 1 is relevant.
(c) Notice that (a) fails to hold if the components are not all of finite rank.
4. Extend (91.4) to $m \geqq 2$ isomorphic components, in the spirit of (90.3).
5. Verify the existence of a countable group A which has continuously many nonisomorphic decompositions $A = B_j \oplus C_j$ with indecomposable B_j and C_j such that all B_j are isomorphic. [*Hint*: the method of (90.4).]
6. Show that every summand of the group A in the proof of (91.5) is fully invariant.
7. (Corner [4]) There exist countable torsion-free groups A, B, C such that $A \cong B \oplus C$, $B \cong A \oplus C$, but A and B are not isomorphic. [*Hint*: take $m = 2$ in (91.6).]
8. (Corner [4]) For every integer $m \geqq 2$ there exist countable torsion-free groups A, B such that the direct sum of n copies of A is isomorphic to the direct sum of n copies of B exactly if $m \mid n$.
9. (Corner [4]) There exists a countable torsion-free group A such that A is isomorphic to the direct sum of any finite number of copies of A, but not to the direct sum of infinitely many copies of A. [*Hint*: an infinite direct sum of copies of A has continuously many endomorphisms.]
10*. (Fuchs [28]) Let $\mathfrak{m}$ be any infinite cardinal less than the first strongly inaccessible cardinal. Prove the analog of (91.4) for a group of cardinality $\mathfrak{m}$ with $2^{\mathfrak{m}}$ nonisomorphic B_j. [*Hint*: (89.2).]

11. For every infinite cardinal $\mathfrak{m}$ there exists a group of rank $\mathfrak{m}$ which has no indecomposable direct summand. [*Hint*: direct sums of groups in (91.5); prove this for a finite sum, a countable sum, and apply (9.10).]

92.* QUASI-DIRECT DECOMPOSITIONS

The results of the preceding sections show that in the study of direct decompositions of torsion-free groups there is not much hope for getting any sort of uniqueness in the traditional sense. Jónsson [2] suggested a basically new idea and showed that it leads to a uniqueness theorem in a somewhat weaker sense.

This idea lies in replacing the isomorphism by a weaker notion, called quasi-isomorphism. Let the torsion-free groups A and C be of finite rank such that A is contained in the divisible hull D of C [for convenience, we suppose that everything takes place in a given divisible group]. Then A is said to be *quasi-contained* in C [notation: $A \prec C$] if $nA \leqq C$ for some integer $n > 0$, and *quasi-equal* to C, $A \approx C$, if $A \prec C$ and $C \prec A$. It is easy to see that, in this case, quasi-equality simply means that $A \cap C$ is of finite index both in A and in C.

Furthermore, A and C are called *quasi-isomorphic*, $A \sim C$, if A and C are isomorphic to quasi-equal subgroups of some divisible group D.

In the second part of this section we shall extend these notions to torsion-free groups of arbitrary rank.

Since every subgroup of finite index in a rank 1 group A is isomorphic to A, we see that, *for groups of rank* 1, *quasi-isomorphism implies isomorphism.* More generally we have:

Proposition 92.1 (Procházka [12]). *Let A be a torsion-free group of finite rank such that $|A/pA| \leqq p$ for each prime p. If a torsion-free group C is quasi-isomorphic to A, then $C \cong A$.*

It suffices to prove the isomorphism $C \cong A$ for a subgroup C of prime index p in A. Then $pA \neq A$ and $pA \leqq C$, whence hypothesis implies $C = pA$. Since $pA \cong A$, the assertion is evident. □

A group A is a *quasi-direct sum* of subgroups $C_1, \cdots, C_k$ of its divisible hull D, if $A \approx C_1 \oplus \cdots \oplus C_k$ holds. In this case, we also refer to the last quasi-equality as a *quasi-direct decomposition* of A, and call the groups C_i *quasi-direct summands* of A. It is straightforward to check:

(a) *$A \oplus B \approx A \oplus C$ implies $B \sim C$;*
(b) *if $A \approx B \oplus C$ and if $B \prec X \prec A$, then $X \approx B \oplus (X \cap C)$.*

A group having only trivial quasi-direct decompositions is called *strongly indecomposable.*

Example 1. The group A of Example 2 in **88** is quasi-equal to the direct sum $\oplus_n E_n$, but this fails for A in Example 5 which group is strongly indecomposable.

Example 2. Rigid groups are strongly indecomposable.

Let A still be torsion-free of finite rank and D a divisible hull of A. An endomorphism ϕ of D is said to be a *quasi-endomorphism* of A if it satisfies

$$\phi A \prec A.$$

The quasi-endomorphisms of A form a subring $\tilde{\mathrm{E}}(A)$ of the ring of all endomorphisms of D. The units of $\tilde{\mathrm{E}}(A)$ are the *quasi-automorphisms* of A. Note that:

(c) *The ring* $\tilde{\mathrm{E}}(A)$ *is a* Q-*algebra with identity and every left ideal of* $\tilde{\mathrm{E}}(A)$ *is a* Q-*vectorspace.* Since division by an integer $n \neq 0$ is clearly a quasi-endomorphism, the assertions are evident.

(d) $\tilde{\mathrm{E}}(A)$ *satisfies the minimum condition on left* [*and right*] *ideals.* This follows from (c) and the finite dimensionality of D.

(e) *For every idempotent* $\varepsilon \in \tilde{\mathrm{E}}(A)$, $A \approx \varepsilon A \oplus (1 - \varepsilon)A$ *is a quasi-direct decomposition of* A; *and if* $A \approx B \oplus C$, *then for some idempotent* $\varepsilon \in \tilde{\mathrm{E}}(A)$, $B \approx \varepsilon A$ *and* $C \approx (1 - \varepsilon)A$. The proof is straightforward and is left to the reader.

The next lemma holds for all finite decompositions, but we confine ourselves to two summands only.

Lemma 92.2 (J. D. Reid [1]). *The correspondence*

$$A \approx \varepsilon_1 A \oplus \varepsilon_2 A \mapsto \tilde{\mathrm{E}}(A) = \tilde{\mathrm{E}}(A)\varepsilon_1 \oplus \tilde{\mathrm{E}}(A)\varepsilon_2$$

between the quasi-direct decompositions of the torsion-free group A *of finite rank and the decompositions of* $\tilde{\mathrm{E}}(A)$ *into direct sums of left ideals is one-to-one. Here* ε_1, $\varepsilon_2 = 1 - \varepsilon_1$ *are idempotent quasi-endomorphisms of* A.

Proof as in **106**(e). □

It is well known that an Artinian ring with identity does not decompose into a direct sum of left ideals $\neq 0$ if and only if it is a local ring [i.e., its non-units form an ideal]. Thus, from (d) and the last lemma we obtain at once the following important information.

Proposition 92.3 (J. D. Reid [2]). *The quasi-endomorphism ring of a torsion-free group* A *of finite rank is local exactly if* A *is strongly indecomposable.* □

The next lemma is crucial in our study of quasi-direct decompositions. Its resemblance to the exchange property in **72** is apparent.

Lemma 92.4. *A strongly indecomposable group A of finite rank has the following property: if B, $C_1, \cdots, C_m$ are torsion-free of finite rank and if*

$$G \approx A \oplus B \approx C_1 \oplus \cdots \oplus C_m, \tag{1}$$

then there exist an index i and a subgroup C_i' of C_i such that

$$G \approx A \oplus C_1 \oplus \cdots \oplus C_i' \oplus \cdots \oplus C_m.$$

Let ε, π, and θ_i be the idempotent quasi-endomorphisms of G associated with the quasi-direct decompositions in (1). Then $\varepsilon = \varepsilon\theta_1 + \cdots + \varepsilon\theta_m$ restricted to A is a quasi-automorphism of A, and hence (92.3) implies that one of $\varepsilon\theta_i$, say $\varepsilon\theta_1$, is a quasi-automorphism of A [for convenience, $\theta_i | A$ is denoted as θ_i]. Consequently, $\theta_1 A$ is quasi-isomorphic to A, and since ε maps $\theta_1 A$ onto a group quasi-equal to A, we have $G \approx \theta_1 A \oplus B$. Hence $C_1 \approx \theta_1 A \oplus C_1'$ with $C_1' = C_1 \cap B$, and since θ_1 maps A isomorphically onto $\theta_1 A$, we obtain $G \approx A \oplus C_1' \oplus C_2 \oplus \cdots \oplus C_m$.$\square$

Evidently, every torsion-free group of finite rank is quasi-equal to a finite direct sum of strongly indecomposable groups. In contrast to (90.3), this is unique as is shown by the following basic result on quasi-direct decompositions.

Theorem 92.5 (Jónsson [2]). *Let A be a torsion-free group of finite rank and*

$$A \approx A_1 \oplus \cdots \oplus A_m \approx C_1 \oplus \cdots \oplus C_n,$$

where each of A_i and C_j is strongly indecomposable. Then $m = n$ and, after suitably rearranging the components, $A_i \sim C_i$ for every i.

From (92.4) and from the strong indecomposability of the occurring groups, it is immediate that A_1 can replace one of the C_i, say C_1, i.e., $A \approx A_1 \oplus C_2 \oplus \cdots \oplus C_n$. Now (a) gives $A_1 \sim C_1$. We also conclude that $A_2 \oplus \cdots \oplus A_m \sim C_2 \oplus \cdots \oplus C_n$, or $A_2 \oplus \cdots \oplus A_m \approx C_2' \oplus \cdots \oplus C_m'$ with $C_i' \sim C_i$ $(i = 2, \cdots, n)$. A trivial induction completes the proof.$\square$

Example 3. The groups occurring in the proofs in **90** are all quasi-isomorphic to completely decomposable groups.

Turning to the infinite rank case, we define a torsion-free group A of arbitrary rank to be *quasi contained* in C, $A \prec C$, if A is contained in the divisible hull D of C such that, for every summand E of finite rank of D, $A \cap E \prec C \cap E$ holds in the old sense. It is immediate how the definitions of *quasi-equality*, *quasi-isomorphism*, etc., read in the general case. [This definition of quasi-isomorphism for torsion-free groups of infinite rank is different from definitions used by various authors, e.g., by J. D. Reid [4].]

Example 4. The group A of (91.1) is quasi-isomorphic to a completely decomposable group. However, this fails to hold for the group

$$B = \langle p_n^{-\infty} b_n, q_n^{-\infty}(b_n + b_{n+1}) \text{ for all integers } n \rangle.$$

If we wish to extend the preceding results to the general case, a certain restriction of the direct sums is inevitable. $A \approx B \oplus C$ will be called an *admissible* quasi-direct decomposition of A, in notation: $A \approx B \mathbin{\hat{\oplus}} C$, if for the projections π, ρ of the divisible hull of A onto the divisible hulls of B and C, respectively, $n\pi A \leqq B$, $nB \leqq A$ for some $n \neq 0$, and $m\rho A \leqq C, mC \leqq A$ for some $m \neq 0$. An infinite quasi-direct sum, $A \approx \bigoplus_{i \in I} A_i$, is admissible if, for every projection π_i, the analogous property holds; in this case we write $A = \hat{\bigoplus}_{i \in I} A_i$. The reader can easily convince himself that all quasi-direct decompositions of a group of finite rank are necessarily admissible.

The following are easy consequences of the definitions:

(a′) If $X \approx A \mathbin{\hat{\oplus}} B$ and $X \approx A \mathbin{\hat{\oplus}} C$, then $B \sim C$.

(b′) If $A \approx B \mathbin{\hat{\oplus}} C$ and if $B \prec X \prec A$, where X is of finite rank, then $X \approx B \mathbin{\hat{\oplus}} (X \cap C)$.

(c′) If $A_i \approx C_i$ for all $i \in I$, then $A = \bigoplus_{i \in I} A_i$ satisfies $A \approx \hat{\bigoplus}_{i \in I} C_i$.

In the remainder of this section we develop a theory for quasi-decompositions which is analogous to those in **86** and **87**. To achieve reasonable generality, we shall admit groups of any finite rank where in **86** and **87** only rank one groups were considered.

With this in mind, we call the torsion-free group A *completely quasi-decomposable* if $A \approx \hat{\bigoplus}_{i \in I} A_i$, where each A_i is of finite rank; thus exclusively, admissible quasi-direct decompositions will be considered. Without changing the content of this definition, we may assume, in addition, that the A_i are strongly indecomposable. If we do so, we obtain the exact analog of (86.1):

Theorem 92.6 (Viljoen [2]). *Let*

$$A \approx \hat{\bigoplus_{i \in I}} A_i \qquad \text{and} \qquad A \approx \hat{\bigoplus_{j \in J}} C_j$$

be two admissible quasi-direct decompositions of a torsion-free group A, where the groups A_i and C_j are strongly indecomposable of finite rank. Then there exists a one-to-one correspondence f between I and J such that

$$A_i \sim C_{f(i)} \qquad \text{for every} \quad i \in I.$$

If I is finite, then so is J, and our assertion reduces to (92.5). Next suppose I is countably infinite, say, $I = \{1, 2, \cdots, n, \cdots\}$. Without loss of generality, $J = I$ may be assumed. Clearly, A_1 is contained in the divisible hull of

$C_1 \oplus \cdots \oplus C_m$ for some m. Hence (b′) shows that A_1 is a quasi-direct summand of $C_1 \oplus \cdots \oplus C_m$. By (92.4), one of these C_i, say C_1, can be replaced by A_1 to obtain $A \approx A_1 \oplus \bigoplus_{n \geqq 2} C_n$. Hence (a′) implies $A_1 \sim C_1$ and, replacing the C_n, if necessary, by isomorphic copies, we have $A' = \hat{\bigoplus}_{n \geqq 2} A_n$ and $A' \approx \hat{\bigoplus}_{n \geqq 2} C_n$. Next we argue with C_2 as we did with A_1 to infer $C_2 \sim A_2$, say. Thus proceeding, using the first remaining A_n and C_n alternately, we arrive at a desired correspondence.

If the index sets I, J are uncountable, then the method of proof of (9.10) will yield partitions $I = \bigcup_{k \in K} I_k$ and $J = \bigcup_{k \in K} J_k$ into pairwise disjoint and countable subsets I_k of I and J_k of J, respectively, such that $\bigoplus_{i \in I_k} A_i$ and $\bigoplus_{j \in J_k} C_j$ are quasi-isomorphic for every $k \in K$. A simple appeal to the countable case completes the proof.□

A torsion-free group A is said to be *quasi-separable* if every finite subset $\{a_1, \cdots, a_k\}$ of A is contained in a suitable quasi-direct summand of finite rank of A [or, equivalently, in a suitable completely quasi-decomposable quasi-summand of A]; here again, all occurring quasi-direct sums are admissible. It is an easy exercise to prove the analog of Baer's theorem (87.1): *a countable quasi-separable group is completely quasi-decomposable.* It is relatively easy to establish the result analogous to (87.5):

Theorem 92.7 (Viljoen [2]). *Every summand in an admissible quasi-direct decomposition of a quasi-separable group is itself quasi-separable.*

Let $A \approx B \hat{\oplus} C$ be quasi-separable, and $b_1, \cdots, b_k \in B$. Then $b_1, \cdots, b_k \in X$ for a finite rank quasi-direct summand X of A, $A \approx X \hat{\oplus} Y$. Write $X \approx X_1 \oplus \cdots \oplus X_m$ with strongly indecomposable X_j. An easy verification shows that (92.4) continues to hold if the finiteness of the rank of B and C_i is dropped. Consequently, each of X_j has the property stated in (92.4), and it follows readily [cf. **72**(c)] that the same holds for X. Therefore, there exist subgroups $B' \leqq B$ and $C' \leqq C$ such that $A \approx X \hat{\oplus} B' \hat{\oplus} C'$. Since B is an admissible quasi-direct summand, it follows easily that $B \approx B' \hat{\oplus} [B \cap (X \hat{\oplus} C')]$, where the second quasi-direct summand is of finite rank and contains $b_1, \cdots, b_k$, establishing the quasi-separability for B.□

The last result enables us to prove the following analog of (86.7).

Corollary 92.8. *Quasi-direct summands of completely quasi-decomposable groups are again completely quasi-decomposable.*

An obvious modification of (9.10) leads us to conclude that a quasi-direct summand is quasi-isomorphic to a direct sum of countable groups. By (92.7) and the remark preceding it, our claim is evident in the countable case.□

It was pointed out by Walker [7] that if the category of torsion-free groups is replaced by its quotient category modulo finite groups, then in the new category (92.5) is equivalent to the uniqueness, up to isomorphism, of direct decompositions of objects into indecomposable objects.

EXERCISES

1. Under the hypotheses of (92.1), nA ($n > 0$) are the only subgroups of finite index in A.
2. Let A be an indecomposable torsion-free group of finite rank such that $|A/pA| \leqq p$ for every prime p. Prove that A is strongly indecomposable.
3. Let $A = A_1 \oplus \cdots \oplus A_m = C_1 \oplus \cdots \oplus C_n$, where A_i and C_j are indecomposable of finite rank satisfying the condition in Ex. 2. Then the two decompositions of A are isomorphic.
4. Let A be a torsion-free group of finite rank and F, F' free subgroups of A of rank n. Then A/F and A/F' are isomorphic to subgroups of $\bigoplus_{i=1}^n Q/Z$ such that $A/F \oplus G \cong A/F' \oplus G'$ for suitable finite groups G and G'.
5. Let A be a torsion-free group of finite rank. If ϕ is an isomorphism of A into itself, then ϕA is of finite index in A. [*Hint*: if F is a free subgroup of A of the same rank as A, then $F \cap \phi F$ is of finite index in F; apply Ex. 4 to $\phi A/(F \cap \phi F) \leqq A/(F \cap \phi F)$, noting that $A/F \cong \phi A/\phi F$.]
6. (Viljoen [2]) Let $A = \langle a_n\ (n \geqq 1);\ p^{-n}(a_1 + a_{n+1})$ for all $n\rangle$, where p is a prime. Show that $A \approx \bigoplus \langle a_n \rangle$, but the last decomposition is not admissible.
7. Prove (a′)–(c′).
8. Extend (92.2) to admissible finite decompositions $A \approx \varepsilon_1 A \mathbin{\hat{\oplus}} \cdots \mathbin{\hat{\oplus}} \varepsilon_n A$ of groups A of infinite rank.
9. (J. D. Reid [2]) Let $\tilde{\mathrm{E}}(A)$ have the minimum condition on left ideals.
   (a) Every quasi-direct decomposition of A has only finitely many summands.
   (b) A is strongly indecomposable if and only if $\tilde{\mathrm{E}}(A)$ is local.
   (c) (92.4) holds for A.
10. (a) Show that no subgroup C of A with bounded quotient A/C can be expressed as the direct sum of finite rank groups if A is the group of (91.2).
    (b) If quasi-isomorphism of A with C is defined by $C \cong A' \leqq A$ with A/A' bounded, then a countable quasi-separable group need not be completely quasi-decomposable.
11. (Procházka [12]) If A and C are quasi-isomorphic torsion-free groups of finite rank, then $\mathrm{Ext}(A, G) \cong \mathrm{Ext}(C, G)$ for every group G.

93.* COUNTABLE TORSION-FREE GROUPS

For torsion-free groups of rank >1, so far no satisfactory structure theory has been obtained. Though several schemes are known for constructing torsion-free groups of at most countable rank, they fail to answer the isomorphy problem in a satisfactory way. Namely, different constructions may lead to isomorphic groups, and the invariants are equivalence classes of matrices or other quantities where the question of whether or not two of these belong to the same equivalence class is as difficult to decide as the isomorphy of the groups. In view of the theoretical importance of the method, we present a theory as developed by Kurosh [2], Malcev [1], and Derry [1] in the finite-rank case.

In this theory, the fundamental idea is to localize the structure problem of torsion-free groups A to $\mathbf{Q}_p^*$-modules $J_p \otimes A$, which are easy to handle up to countable rank.

We introduce the following notations. For torsion-free A, we write

$$A_p = Q_p \otimes A, \qquad A_p^* = J_p \otimes A, \qquad E = Q \otimes A, \qquad \text{and} \qquad E_p^* = K_p \otimes A,$$

where K_p denotes the additive group of the p-adic number field. Under the natural identification, E is a divisible hull of A, and we have $A \leqq A_p \leqq A_p^* \leqq E_p^*$; we may write, for instance, the elements of A_p as finite sums $\sum q_i a_i$ with $q_i \in \mathbf{Q}_p$ and $a_i \in A$.

Lemma 93.1. *If the A_p are regarded as subgroups of* E, *then* $A = \bigcap_p A_p$.

Let $x \in \bigcap_p A_p$ and, for some p, write $x = \sum q_i a_i$, with $q_i \in \mathbf{Q}_p$, $a_i \in A$. There is an integer s, prime to p, such that $sx \in A$. Let s have the prime factors $p_1, \cdots, p_m$, and select an integer s_j, prime to p_j, such that $s_j x \in A$, for $j = 1, \cdots, m$. As the integers $s, s_1, \cdots, s_m$ are relatively prime, it is an easy matter to establish the inclusion $x \in A$.□

Lemma 93.2 (Derry [1]). *For every prime p,* $E \cap A_p^* = A_p$.

Write $x \in A_p^*$ in the form $x = \sum_{i=1}^n \pi_i a_i$ with $\pi_i \in \mathbf{Q}_p^*$ and $a_i \in A$. If $a_1, \cdots, a_k$ are independent in A and $a_{k+1}, \cdots, a_n$ depend on them, then we may assume, without loss of generality, that $\pi_{k+1}, \cdots, \pi_n$ are rational integers. If $x \in E$ too, then x depends on a maximal independent system $\{a_1, \cdots, a_k, a'_{k+1}, \cdots\}$ of A, say

$$x = \sum_{i=1}^{k} m_i a_i + \sum_{i=k+1}^{r} m_i a'_i \qquad \text{for rational numbers} \quad m_i.$$

Since a maximal independent system of A becomes, after identifying $1 \otimes a$ with a, a maximal independent system in the $\mathbf{K}_p$-module E_p^* [where $\mathbf{K}_p$ stands for the field of p-adic numbers], there is essentially only one dependence relation for x. Thus $\pi_1, \cdots, \pi_n$ must be rationals and $x \in A_p$. This proves $E \cap A_p^* \leqq A_p$.□

Our last preparatory result is a structure theorem.

Lemma 93.3 (Prüfer [3]). *A reduced, countably generated torsion-free* Q_p^**-module is free.*

A torsion-free Q_p^*-module of rank 1 is either cyclic or injective, since all Q_p^*-modules $\neq K_p$ between J_p and K_p are isomorphic to J_p. From the algebraic compactness of J_p we easily deduce that pure torsion-free Q_p^*-modules of rank 1 are summands. Consequently, finitely generated torsion-free Q_p^*-modules are free and the same holds for reduced Q_p^*-modules of finite rank. The assertion follows then in the same way as in the proof of (19.1).□

From now on we suppose that the rank r of A is at most countable. $\{a_i\}$, with $i = 1, \cdots, r$ or $i = 1, 2, \cdots$, will denote a maximal independent system in E [preferably in A], according as r is finite or not.

The Q_p^*-module A_p^* is again of rank r, thus by (93.3) its reduced part is free. Consequently, there exist $v_n, w_m \in A_p^*$ such that

$$A_p^* = \bigoplus_n \mathsf{K}_p v_n \oplus \bigoplus_m \mathsf{Q}_p^* w_m, \tag{1}$$

where n and m range over index sets, say, of cardinality k_p and l_p, respectively, where $k_p + l_p = r$. [To simplify notation, the dependence of v_n and w_m on p is not shown.] Thus we can write

$$a_i = \sum_n \alpha_{in} v_n + \sum_m \beta_{im} w_m \qquad (\alpha_{in}, \beta_{im} \in \mathsf{K}_p), \tag{2}$$

where for a fixed i almost all of α_{in}, β_{im} vanish. [If $a_i \in A$, then $\beta_{im} \in \mathsf{Q}_p^*$.] This gives rise to an $r \times r$ row-finite matrix

$$\mathbb{M}_p = \left[\begin{array}{ccccccccc} \alpha_{11} & \cdots & \alpha_{1n} & \cdots & \beta_{11} & \cdots & \beta_{1m} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{i1} & \cdots & \alpha_{in} & \cdots & \beta_{i1} & \cdots & \beta_{im} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right] \qquad (\alpha_{in}, \beta_{im} \in \mathsf{K}_p), \tag{3}$$

$$\underbrace{\qquad\qquad\qquad}_{k_p} \quad \underbrace{\qquad\qquad\qquad}_{l_p}$$

which is invertible over K_p, since the system $\{v_n, w_m\}$ is likewise maximal independent over K_p in A_p^*. We are thus led to a correspondence

$$A \mapsto \{\mathbb{M}_2, \mathbb{M}_3, \cdots, \mathbb{M}_p, \cdots\}, \tag{4}$$

where, for each prime p, $\mathbb{M}_p$ is a row-finite, invertible matrix with p-adic numbers as entries.

The correspondence (4) not only depends on the choice of $\{a_i\}$, but also on the selection of $\{v_n, w_m\}$ for each p. We are now going to investigate how

(3) changes under the transition to another maximal independent set $\{a_i'\}$ and to another system $\{v_n', w_m'\}$. To simplify notation, we write $\mathbf{a}$, $\mathbf{a}'$ for the r-dimensional column vectors with coordinates a_i and a_i', respectively, and $[{}^{\mathbf{v}}_{\mathbf{w}}]$, $[{}^{\mathbf{v}'}_{\mathbf{w}'}]$ for the column vectors whose coordinates are v_n, w_m and v_n', w_m', respectively.

There is an invertible, row-finite $r \times r$ matrix $\mathbb{B}$ with rational entries carrying $\mathbf{a}$ into $\mathbf{a}'$, i.e.,

$$\mathbf{a}' = \mathbb{B}\mathbf{a}.$$

Similarly, for some invertible, row-finite $r \times r$ matrix $\mathbb{C}_p$ with p-adic entries, we have

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \mathbb{C}_p \begin{bmatrix} \mathbf{v}' \\ \mathbf{w}' \end{bmatrix}.$$

Here $\mathbb{C}_p$ must have an inverse of the same type, thus it is of the form

$$\mathbb{C}_p = \begin{bmatrix} \mathbb{V} & 0 \\ \mathbb{W}_1 & \mathbb{W}_2 \end{bmatrix} \tag{5}$$

where $\mathbb{V}$ is invertible of type $k_p \times k_p$, $\mathbb{W}_2$ is an $l_p \times l_p$-matrix over Q_p^*, admitting an inverse with entries again in Q_p^*, and $\mathbb{W}_1$ is an $l_p \times k_p$-matrix over K_p; all of $\mathbb{V}$, $\mathbb{W}_1$, $\mathbb{W}_2$ are row-finite. The matrices $\mathbb{C}_p$ of the type (5) form a group $\Gamma_p(k_p, l_p)$ under matrix multiplication.

Evidently, $\mathbf{a} = \mathbb{M}_p[{}^{\mathbf{v}}_{\mathbf{w}}]$ transforms into $\mathbf{a}' = \mathbb{B}\mathbf{a} = \mathbb{B}\mathbb{M}_p\mathbb{C}_p[{}^{\mathbf{v}'}_{\mathbf{w}'}]$, i.e., correspondence (4) becomes $A \mapsto \{\mathbb{M}_2', \cdots, \mathbb{M}_p', \cdots\}$, where $\mathbb{M}_p' = \mathbb{B}\mathbb{M}_p\mathbb{C}_p$ for each prime p. In view of this, two matrix-sequences $\{\mathbb{M}_2, \cdots, \mathbb{M}_p, \cdots\}$ and $\{\mathbb{M}_2', \cdots, \mathbb{M}_p', \cdots\}$ will be called *equivalent* if there exists an invertible, row-finite $r \times r$ matrix $\mathbb{B}$ with rational entries and there are matrices $\mathbb{C}_p \in \Gamma_p(k_p, l_p)$, one for each prime p, such that

$$\mathbb{M}_p' = \mathbb{B}\mathbb{M}_p\mathbb{C}_p \qquad \text{for every prime} \quad p.$$

This equivalence enables us to establish the following result.

Theorem 93.4. *The ranks k_p and l_p [satisfying $k_p + l_p = r(A)$], for all primes p, and the equivalence class of the matrix-sequence in (4) form a complete system of invariants for countable torsion-free groups A.*

The invariance of k_p, l_p and the equivalence class of the matrix-sequence in (4) is clear from the foregoing considerations. Thus we need only verify the isomorphy of two groups, A and A', under the assumption that they have the same ranks k_p and l_p, for every p, and the corresponding matrix-sequences $\{\mathbb{M}_2, \cdots, \mathbb{M}_p, \cdots\}$ and $\{\mathbb{M}_2', \cdots, \mathbb{M}_p', \cdots\}$ are equivalent; moreover, it suffices to consider the case when these sequences are equal. We form A_p', $A_p'^*$, E', $E_p'^*$ for A'. In view of the definition, E and E' have maximal independent systems $\{a_i\}$ and $\{a_i'\}$, and there are direct decompositions like (1) for A_p and A_p' such that for the corresponding vectors we have $\mathbf{a} = \mathbb{M}_p[{}^{\mathbf{v}}_{\mathbf{w}}]$ and $\mathbf{a}' = \mathbb{M}_p[{}^{\mathbf{v}'}_{\mathbf{w}'}]$.

The correspondence $a_i \mapsto a_i'$ (for all i) extends uniquely to an isomorphism $\varphi: E \to E'$. This, in turn, induces a unique $\mathbf{Q}_p^*$-isomorphism $\varphi_p^*: E_p^* \to E_p'^*$. Since the transition matrices from $\mathbf{a}$ to $[{}^{\mathbf{v}}_{\mathbf{w}}]$ and from $\mathbf{a}'$ to $[{}^{\mathbf{v}'}_{\mathbf{w}'}]$ are identical, φ_p maps v_n onto v_n' and w_m onto w_m', whence $\varphi_p | A_p^*$ is an isomorphism between A_p^* and $A_p'^*$. Therefore, φ_p and hence φ maps $A_p = E \cap A_p^*$ isomorphically upon $A_p' = E \cap A_p'^*$ [cf. (93.2)]. We find that φ maps $A = \bigcap_p A_p$ isomorphically upon $A' = \bigcap_p A_p'$ [cf. (93.1)].$\square$

Turning matters around, we can ask what matrix-sequences correspond to countable torsion-free groups. The answer is given in the next theorem.

Theorem 93.5 (Kurosh [2]). *For every prime p, let k_p and l_p be finite or countably infinite cardinals such that $k_p + l_p = r$ is independent of p, and let $\{\mathbb{M}_2, \cdots, \mathbb{M}_p, \cdots\}$ be a sequence of matrices with $\mathbb{M}_p$ of type* (3). *Then this sequence corresponds to a countable torsion-free group A of rank r if and only if it is equivalent to a sequence $\{\mathbb{M}_2', \cdots, \mathbb{M}_p', \cdots\}$, where, for each p, the elements β_{im}' in the last l_p columns of $\mathbb{M}_p'$ are p-adic integers.*

The necessity follows by selecting $\{a_i\}$ in A. For sufficiency, suppose we are given k_p, l_p and the sequence $\{\mathbb{M}_2, \cdots, \mathbb{M}_p, \cdots\}$, where $\mathbb{M}_p$ are of the form (3) with $\beta_{im} \in \mathbf{Q}_p^*$. Starting with a divisible group $E = \bigoplus Qa_i$ of rank r, form $E_p^* = K_p \otimes E = \bigoplus K_p a_i$, for every prime p; E can be viewed as a subgroup of E_p^*. The matrix $\mathbb{M}_p$ being row-finite and invertible, there are k_p- and l_p-dimensional column vectors $\mathbf{v}$ and $\mathbf{w}$, respectively, such that $\mathbf{a} = \mathbb{M}_p[{}^{\mathbf{v}}_{\mathbf{w}}]$ for the column vector $\mathbf{a}$ of the a_i. Define A_p^* by (1) with the aid of the coordinates of $\mathbf{v}$ and $\mathbf{w}$, and let $A_p = E \cap A_p^*$. We contend that $A = \bigcap_p A_p$ is a group with the desired properties.

The choice of $\mathbb{M}_p$ implies $a_i \in A_p^*$, whence $a_i \in A$, A is of rank r, and E is a divisible hull of A. Hence $J_p \otimes A \leqq A_p^*$ is evident. Also, $Q_p \otimes A \leqq E$ and $Q_p \otimes A \leqq J_p \otimes A$ imply $Q_p \otimes A \leqq E \cap A_p^* = A_p$. To show that the last inclusion is actually an equality, write $b \in A_p$ in the form $b = p^k n^{-1} a$ with $(n, p) = 1$ and $a \in A$. If $k \geqq 0$, then from $n^{-1} \in \mathbf{Q}_p$ we deduce $b \in Q_p \otimes A$, while if $k < 0$, then from $p^k a \in A_q$ for every $q \neq p$ and from $p^k a = nb \in A_p$ we obtain $p^k a \in A$; thus $b \in Q_p \otimes A$ and $Q_p \otimes A = A_p$. To complete the proof of (93.5), we must still verify that $A_p^* \leqq J_p \otimes A$. For $c \in A_p^*$ write $c = \sum_{i=1}^m \rho_i a_i$ with $\rho_i \in \mathbf{K}_p$, where $\rho_i = p^{-l} s_i + \sigma_i$ with rational integers s_i and p-adic integers σ_i. Therefore, $c - p^{-l} \sum s_i a_i = \sum \sigma_i a_i \in J_p \otimes A \leqq A_p^*$, whence $p^{-l} \sum s_i a_i \in A_p^* \cap E = A_p \leqq J_p \otimes A$, and the inclusion $c \in J_p \otimes A$ follows.$\square$

Example. Let A be completely decomposable of finite rank r, $A = A_1 \oplus \cdots \oplus A_r$ with $r(A_i) = 1$. Then A_p^* is the direct sum of k_p copies of K_p and $l_p = r - k_p$ copies of J_p, where k_p is the number of A_i with $pA_i = A_i$. If $\{a_1, \cdots, a_r\}$ with $a_i \in A_i$ is chosen as a maximal independent system, then in (1) we may choose v_n as some a_i and w_m as some $p^{-e} a_i$ [with $e = h_p(a_i)$], so that $\mathbb{M}_p$ will be a monomial matrix whose nonzero entries are 1 or powers of p. The types of the A_i can be read off from these powers.

In conclusion, let us emphasize again that—as far as the structure theory is concerned—the foregoing theory is of minor practical value: it fails to furnish us with a useful way of deciding the isomorphy of two countable torsion-free groups. As a matter of fact, the results are not satisfactory even in the rank 2 case.

EXERCISES

1. Let A be of finite rank and B a p-basic subgroup of A. Show that $l_p = r(B_p)$ and $k_p = r(A/B_p)$.
2. Determine the matrix-sequences (4) corresponding to the groups in **88**.
3. Let C be a subgroup of the countable torsion-free group A. Then $k_p(C) \leqq k_p(A)$ for every prime p. The same inequality holds for l_p whenever C is pure in A.
4. A group A of finite rank decomposes into a nontrivial direct sum if and only if in its equivalence class of matrix-sequences there is a sequence $(\mathbb{M}_2, \cdots, \mathbb{M}_p, \cdots)$ such that the $\mathbb{M}_p$ can be brought after the same row and column arrangements to the form
$$\mathbb{M}_p = \begin{bmatrix} \mathbb{N}_p & 0 \\ 0 & \mathbb{L}_p \end{bmatrix}$$
where $\mathbb{N}_p$ and $\mathbb{L}_p$ are square matrices whose orders, m and $r - m$, are independent of p; here $0 < m < r$.
5. (Kurosh [2]) (a) Let A be of finite rank r and $k_p = r - 1$, $l_p = 1$, for some p. If A decomposes into a direct sum, then in a suitable basis, $\mathbb{M}_p$ will have the form as given in Ex. 4, and in any other equivalent matrix, the column of the β contains at most $r - m$, rationally independent p-adic numbers.
(b) Conclude in this way the existence of indecomposable torsion-free groups of any finite rank.
6. The cardinals k_p and l_p are quasi-isomorphic invariants.
7. (Rotman [4]) Let A be a torsion-free group of finite rank n and $0 = A_0 < A_1 < \cdots < A_n = A$ a sequence of pure subgroups of A such that all factors in the sequence are of rank 1. Then the number of p-divisible factors, for every prime p, is an invariant of A.
8. (Szekeres [1]) The subfield of K_p obtained by adjoining to Q the β_{im} in the last l_p columns of $\mathbb{M}_p$ is an invariant of A.

94. SLENDER GROUPS

A remarkable class of torsion-free groups was discovered by J. Łoś: this is the class of slender groups.

Let P denote the direct product of a countable set of infinite cyclic groups,
$$P = \prod_{n=1}^{\infty} \langle e_n \rangle \qquad \text{where} \quad o(e_n) = \infty,$$

and set $S = \oplus_{n=1}^{\infty} \langle e_n \rangle$. The elements x of P can be written either as infinite vectors $x = (k_1 e_1, \cdots, k_n e_n, \cdots)$ with $k_n \in \mathbf{Z}$ or as formal infinite sums $x = \sum_{n=1}^{\infty} k_n e_n$ with $k_n \in \mathbf{Z}$.

A torsion-free group G is called *slender* if, for every homomorphism $\eta: P \to G$, $\eta e_n = 0$ for almost all n. From the definition it is evident that the following holds.

(a) Subgroups of slender groups are slender.

(b) The group Q of rationals is not slender.

(c) The group P is not slender.

(d) The group J_p of p-adic integers is not slender. In fact, the free group S has a homomorphism ϕ into J_p such that $\phi e_n \neq 0$ for all n. The purity of S in P, together with the pure-injectivity of J_p, implies that ϕ extends to an $\eta: P \to J_p$.

(e) *A slender group can not contain any subgroup isomorphic to Q, P, or J_p.* In particular, a slender group does not contain any algebraically compact group $\neq 0$ [cf. (40.4)].

The following special case of the main theorem (94.4) is a simple consequence of our last remark.

(f) *If η is a homomorphism of P into a slender group G such that $\eta S = 0$, then $\eta = 0$. Hence an epic image of P in a slender group G is finitely generated.*

Im η is now an epimorphic image of the algebraically compact group P/S [see (42.2)], thus it is, by (54.1), a cotorsion group. A torsion-free cotorsion group is, because of (54.5), algebraically compact, hence Im $\eta \leqq G$ implies Im $\eta = 0$. The final assertion is a simple corollary to the first, on using the definition of slenderness.

A useful consequence of (f) is recorded as

Lemma 94.1. *Every homomorphism η of P into a slender group is completely determined by $\eta \mid S$.* □

Let us point out that, as a consequence of (94.1), for every $\eta: P \to G$ with slender G, we can write

$$\eta\left(\sum_{n=1}^{\infty} k_n e_n\right) = \sum_{n=1}^{\infty} k_n(\eta e_n),$$

where in the last sum almost all summands vanish.

By making use of the following result, one can construct a number of slender groups.

Proposition 94.2 (Sąsiada [4]). *A countable group [more precisely, a group of power less than the continuum] is slender if and only if it is reduced.*

It suffices to verify the "if" part.

Let G be a reduced group of power $<2^{\aleph_0}$ and $\eta: P \to G$ a homomorphism such that $\eta e_n \neq 0$, for infinitely many n. Omitting the $\langle e_n \rangle$ with $\eta e_n = 0$, we may assume $\eta e_n \neq 0$, for all n. Reducedness implies $\bigcap_m mG = 0$, thus there exists a sequence of integers $1 = k_1 < \cdots < k_n < \cdots$ such that $\eta(k_n! e_n) \notin k_{n+1}G$, for $n = 1, 2, \cdots$. The set of elements $(g_1, \cdots, g_n, \cdots) \in P$ with $g_n = 0$ or $k_n! e_n$ is of the power of the continuum; therefore, there are two different elements of P in this set which have the same image under η. Their difference $a = (h_1, \cdots, h_n, \cdots) \neq 0$ is of the form $h_n = 0$ or $\pm k_n! e_n$, and clearly $\eta a = 0$. This is impossible, since if m is the first index with $h_m \neq 0$, then $\eta h_m \notin k_{m+1}G$ and

$$\eta h_m = \eta a - \eta(0, \cdots, 0, h_{m+1}, \cdots) = -\eta(0, \cdots, 0, h_{m+1}, \cdots) \in k_{m+1}G. \square$$

The following result enables us to construct slender groups of arbitrarily large cardinalities.

Theorem 94.3 (Fuchs [16]). *Direct sums of slender groups are slender.*

Let G_i $(i \in I)$ be slender groups and $G = \bigoplus_i G_i$; π_i will denote the ith coordinate projection $G \to G_i$. Given $\eta: P \to G$, (f) implies that the $\pi_i \eta P$ are finitely generated free subgroups of G, and $\operatorname{Im} \eta \leqq \bigoplus_i \pi_i \eta P$. Here $\operatorname{Im} \eta$ is finitely generated, for otherwise P would have an epimorphism onto a countable free group, contrary to (94.2). Hence almost all $\pi_i \eta = 0$, and η may be viewed as a map into the direct sum of a finite number of G_i, say $G_1 \oplus \cdots \oplus G_m$. Since almost all $\pi_1 \eta e_n, \cdots, \pi_m \eta e_n$ $(n = 1, 2, \cdots)$ vanish, almost all $\eta e_n = 0$. This completes the proof of slenderness of G. $\square$

An immediate corollary to this theorem is that subgroups of direct sums of countable reduced groups are slender.

For an index set I of cardinality $\aleph_\sigma$, we denote by P_σ and S_σ the direct product and the direct sum, respectively, of torsion-free groups A_i, with $i \in I$.

Recall that a cardinal $\mathfrak{m}$ is said to be *measurable* if a set X of cardinality $\mathfrak{m}$ admits a countably additive measure μ such that μ assumes only values 0 and 1, and

$$\mu(X) = 1, \qquad \mu(x) = 0 \qquad \text{for all} \quad x \in X.$$

From the definition of measurability it is clear that if a cardinal is not measurable, then neither are all smaller cardinals. Thus if there exists a measurable cardinal at all, then there is a smallest one and all larger cardinals are measurable. It is not known whether or not the existence of measurable cardinals can be derived from the usual axioms of set theory. Assuming the existence of strongly inaccessible cardinals, it can be shown that many of them are nonmeasurable. [Cf. K. Kuratowski and A. Mostowski, "Set Theory," North-Holland Publ., Amsterdam, 1968.]

The main result on slender groups is our next theorem.

Theorem 94.4 (J. Łoś). *Let G be a slender group and η: $P_\sigma \to G$. Then the following holds:*

(i) *For almost all i, $\eta A_i = 0$.*
(ii) *If $\aleph_\sigma$ is nonmeasurable and $\eta S_\sigma = 0$, then $\eta = 0$.*

To prove (i), suppose we had $\eta A_i \neq 0$ for infinitely many indices i, say for $i = i_1, \cdots, i_n, \cdots$. Then choosing $e_n \in A_{i_n}$ with $\eta e_n \neq 0$, the restriction of η to $P' = \prod_{n=1}^{\infty} \langle e_n \rangle$ would be a forbidden homomorphism into G.

The proof of (ii) is likewise indirect, but much more sophisticated. Suppose the existence of an $a \in P_\sigma$ such that $\eta a \neq 0$ and $\eta S_\sigma = 0$. We introduce a G-valued measure ν on the subsets of I in the following way. For a subset J of I, let $\nu(J) = \eta a_J$, where a_J has the same jth coordinates as a for $j \in J$ and 0 coordinates elsewhere. For pairwise disjoint subsets $J_1, \cdots, J_k$ of I,

$$\nu(J_1 \cup \cdots \cup J_k) = \eta(a_{J_1 \cup \cdots \cup J_k}) = \eta(a_{J_1} + \cdots + a_{J_k}) = \nu(J_1) + \cdots + \nu(J_k),$$

so that ν is additive. In order to show that it is countably additive [i.e., σ-additive], let $J_1, \cdots, J_k, \cdots$ be a countable set of pairwise disjoint subsets of J, and J_0 the complement of their union in J. Then $P_0 = \prod_{k=0}^{\infty} \langle a_{J_k} \rangle$ is a subgroup of P_σ. We conclude that almost all of $\eta a_{J_k} = \nu(J_k)$ vanish, and we must have $\eta a = \sum_{k=0}^{\infty} \eta a_{J_k}$, in view of (94.1). Consequently, ν is countably additive.

Consider all subsets J of I such that $\nu(J') = 0$ for every subset J' of J. These J form a countably additive ideal **I** in the Boolean algebra **B** of all subsets of I, and ν induces a countably additive G-valued measure $\bar{\nu}$ on **B**/**I**. Let $\bar{J}_k$ be pairwise disjoint elements in **B**/**I**. Choose subsets $\bar{J}_k'$ of $\bar{J}_k$ with $\bar{\nu}(\bar{J}_k') \neq 0$, and representatives J_k' of $\bar{J}_k'$ in I such that J_k' are still pairwise disjoint. The preceding paragraph shows that $\nu(J_k') \neq 0$ can hold only for a finite set of indices k; in other words, **B**/**I** is a finite Boolean algebra. Thus it has but a finite number of atoms; at these $\bar{\nu}$ is not zero. We can derive from $\bar{\nu}$ a $\{0, 1\}$-valued measure $\bar{\mu}$ on **B**/**I** if we select an atom in **B**/**I** and define $\bar{\mu}(\bar{J})$ to be 1 or 0 according as $\bar{J}$ does or does not contain the selected atom. In the obvious manner, $\bar{\mu}$ gives rise to a measure μ on **B**, that is, the index set I is measurable. This is a contradiction to our hypothesis, hence (ii) is proved. □

Remark. Let us point out that (ii) is the best possible result in the sense that *for measurable $\aleph_\sigma$, $\eta S_\sigma = 0$ no longer implies $\eta = 0$.* To show this, we construct for a measurable index set I an epimorphism η of $P' = Z^I$ onto Z which vanishes on $S' = \oplus Z$. Let μ be a countably additive $\{0, 1\}$-valued measure on I, and for every $a = (\cdots, n_i e_i, \cdots) \in P'$ define $X_n(a) = \{i \in I \mid n_i = n\}$. Then the $X_n(a)$ are pairwise disjoint subsets of I whose union is I, hence exactly one of them, say, $X_m(a)$ is of measure 1. We set $\eta a = m$. From the properties of μ it is readily checked that η preserves addition and $\eta S' = 0$.

The last theorem has a number of corollaries. Let us mention here some of them.

Corollary 94.5. *If G is a slender group and A_i ($i \in I$) are torsion-free groups such that the index set I is not measurable, then there is a natural isomorphism*

$$\mathrm{Hom}\left(\prod_{i \in I} A_i, G\right) \cong \bigoplus_{i \in I} \mathrm{Hom}(A_i, G).$$

From (94.4) we know that every $\eta: \Pi A_i \to G$ is essentially a homomorphism $A_1 \oplus \cdots \oplus A_n \to G$ for some finite subset $\{1, \cdots, n\}$ of I [depending on η].□

Corollary 94.6 (Zeeman [1]). *If I is not measurable, then there are natural isomorphisms*

$$\mathrm{Hom}\left(\bigoplus_{i \in I} Z, Z\right) \cong \prod_{i \in I} Z \quad \textit{and} \quad \mathrm{Hom}\left(\prod_{i \in I} Z, Z\right) \cong \bigoplus_{i \in I} Z.$$

This follows from (43.1) and (94.5), respectively.□

The last result expresses a remarkable duality between direct sums and products of infinite cyclic groups.

Corollary 94.7. *If G_i ($i \in I$) are torsion-free groups and I is nonmeasurable, then every slender summand of $\prod_{i \in I} G_i$ is isomorphic to a summand of the direct sum of a finite number of the G_i.*

If H is a slender summand of $G = \prod_{i \in I} G_i = H \oplus K$, then in view of (94.4) the projection $\pi: G \to H$ sends almost all components G_i and their product onto 0. This means that this product is contained in K, and so, if we pass modulo this product, we obtain $H \oplus K' \cong G_{i_1} \oplus \cdots \oplus G_{i_n}$ for some $K' \leqq K$ and a finite subset $\{i_1, \cdots, i_n\}$ of I.□

EXERCISES

1. (G. A. Reid [1]) A torsion-free group G is slender exactly if, for all $\eta: P \to G$, $\mathrm{Im}\,\eta$ is finitely generated. [*Hint*: finitely generated groups are slender.]
2. Show that G is slender if both its pure subgroup H and the quotient G/H are slender.
3. Imitating the construction of the classes Γ_σ in **86**, Ex. 14, but using only reduced groups, show that all the groups in the classes are slender.
4. (Nunke [2]) Prove that J_p is not slender by showing that

$$(k_1 e_1, \cdots, k_n e_n, \cdots) \mapsto \sum_{n=1}^{\infty} p^n k_n$$

is a homomorphism of P into J_p.

5. (T. Yen) Let P' denote the subgroup of P for which P'/S is the maximal divisible subgroup of P/S. A group G is slender exactly if, for every $\eta: P' \to G$, $\eta e_n = 0$ for almost all n. [*Hint*: $(a_1, \cdots, a_n, \cdots) \mapsto (a_1, \cdots, n!a_n, \cdots)$ is a monomorphism of P into P'.]
6. Prove that (94.5) fails to hold [with the natural isomorphism!] if G is not slender.
7. Prove (f) by selecting some $x_1 = (k_1 e_1, \cdots, k_n e_n, \cdots) \in P$ with $\eta x_1 \neq 0$, and showing that η induces a forbidden homomorphism of $\prod \langle x_n \rangle$ into G, where $x_n = x_1 - k_1 e_1 - \cdots - k_{n-1} e_{n-1}$.
8. (J. Łoś) Let G_i $(i \in I)$ be slender groups with nonmeasurable I. Then $\prod G_i$ has no proper summand containing $\bigoplus G_i$. [*Hint*: argue as in (94.7).]
9*. Let $\aleph_\sigma$ be nonmeasurable and assume $2^{\aleph_\sigma} = \aleph_{\sigma+1}$. Establish the existence of an $\aleph_{\sigma+1}$-pure subgroup which is not $\aleph_{\sigma+2}$-pure. [*Hint*: let $\bigoplus_{\mathbf{K}} Z < Z^{\mathfrak{m}}$, where $\mathbf{K}$ is the ideal of all subsets of cardinality $< \mathfrak{m} = \aleph_{\sigma+1}$; Ex. 8.]
10. (Corner [2]) Let $\mathfrak{m}, \mathfrak{n}$ be infinite cardinals satisfying $\mathfrak{n} \leqq \mathfrak{m} \leqq 2^{\mathfrak{n}}$. Show that $Z^{\mathfrak{m}}$ has a subgroup of rank $\mathfrak{n}$ which is not contained in any proper summand.
11. (a) Let $P' = Z^{\mathfrak{m}}$ with nonmeasurable cardinal $\mathfrak{m} \geqq \aleph_0$. Show that every summand of P' is again a product of infinite cyclic groups.
    (b) In any direct decomposition of P', there are but a finite number of nonzero summands, and one of these is isomorphic to P'. [*Hint*: (94.6).]
12. (G. A. Reid [1]) Consider the class of groups which are of the form $\mathrm{Hom}(A, Z)$ for some nonmeasurable A. Show that this class is closed under direct sums and direct products.
13. (G. A. Reid [1]) Let $A = Z^{\aleph_1}/\bigoplus_{\mathbf{K}} Z$, where $\mathbf{K}$ is the ideal of all countable subsets. Show that A is $\aleph_1$-free and satisfies $\mathrm{Hom}(A, Z) = 0$.
14. (a) (Łoś [1]) The direct product of countable torsion-free groups decomposes into the direct sum of countable groups if and only if almost all components are divisible. [*Hint*: the reduced part must be slender.]
    (b) The direct product of infinitely many torsion-free groups $\neq 0$ can not be imbedded in a direct sum of countable reduced torsion-free groups.
15. (Sąsiada [7]) Let p, q, and r be as in **90**, Ex. 10. Show that if $X_i = \langle p^{-\infty} x_i, q^{-\infty} y_i, r^{-1}(x_i + y_i) \rangle$ $(i = 1, 2, \cdots)$, then every countable summand of $\prod_{i=1}^{\infty} X_i$ is isomorphic to a finite sum $\bigoplus_{i=1}^{n} X_i$. [*Hint*: (94.7) and Ex. 10 in **90**.]
16. (Sąsiada [7]) Let $A \oplus B = C \oplus D$, where the groups A, B, C, D are indecomposable of ranks 1, 3, 2, and 2, respectively. Assume that every indecomposable summand of $C \oplus \cdots \oplus C$ [any finite number of summands] is isomorphic to C; and the same for D. [Such a group is X_i in Ex. 10 of **90**.] With $C_i \cong C$ and $D_i \cong D$, set $G = \prod_{i=1}^{\infty} C_i \oplus \bigoplus_{i=1}^{\infty} D_i$ and $H = G \oplus A$. Prove that each of G and H is isomorphic to a summand

of the other, but they fail to be isomorphic. [*Hint*: write $\prod C_i \oplus \bigoplus D_i = \prod C_i' \oplus \bigoplus D_i' \oplus A$; from the slenderness of $\bigoplus D_i' \oplus A$, conclude that for some n,

$$\bigoplus_{i=1}^{n} C_i \oplus \bigoplus_{i=1}^{\infty} D_i = K \oplus \bigoplus_{i=1}^{\infty} D_i' \oplus A,$$

where K is a countable summand of $\prod C_i$; apply Ex. 15 to K to obtain that $r(K)$ is even and $K \oplus L \oplus A = \bigoplus_{i=1}^{n} C_i \oplus \bigoplus_{j=1}^{m} D_j$; here $r(L)$ is even, a contradiction.]

95. CHARACTERIZATION OF SLENDER GROUPS BY SUBGROUPS

This section is devoted to an interesting characterization of slender groups. It has been discovered by Nunke [2] that the slender groups can be singled out from among torsion-free groups as those not containing either Q, P, or J_p, for any prime p. The proof of this is based on the description of the epimorphic images of the direct product $P = \prod_{n=1}^{\infty} \langle e_n \rangle$ of infinite cyclic groups $\langle e_n \rangle$.

Notice that if x_n $(n = 1, 2, \cdots)$ are elements of P, then an infinite sum

$$x = \sum_{n=1}^{\infty} s_n x_n \qquad (s_n \in \mathbf{Z}) \tag{1}$$

makes sense in P if and only if, for each positive integer m, almost all of $s_n x_n$ have vanishing mth coordinates. If the mth coordinates of almost all x_n are 0, for every m, then (1) makes sense for every choice of the coefficients $s_n \in \mathbf{Z}$, in which case we may denote the subgroup X consisting of all these sums by $\prod_{n=1}^{\infty} \langle x_n \rangle$, and say that X is a *product* in P.

Lemma 95.1. *Let X be a product in P. There are elements $a_n \in P$ and integers k_n $(n = 1, 2, \cdots)$ such that*

$$P = \prod_{n=1}^{\infty} \langle a_n \rangle \qquad \textit{and} \qquad X = \prod_{n=1}^{\infty} \langle k_n a_n \rangle,$$

where $k_n | k_{n+i}$ if $k_n \neq 0$ for all n, $i \geqq 1$.

If X is a product of a finite number of cyclic groups, then by (19.2) and (15.4), the assertion is evident. Suppose X is an infinite product. If all the elements of X have 0 for their first coordinates, then set $a_1 = e_1$ and $k_1 = 0$. Otherwise let $x = (l_1 e_1, \cdots, l_n e_n, \cdots) \in X$ be such that the g.c.d. $l > 0$ of its coordinates is the smallest among the elements of X. Then $l = (l_1, \cdots, l_m)$ for some m, and because of (15.3) we can find $e_1', \cdots, e_m' \in P$ such that $\langle e_1' \rangle \oplus \cdots \oplus \langle e_m' \rangle = \langle e_1 \rangle \oplus \cdots \oplus \langle e_m \rangle$ with $le_1' = l_1 e_1 + \cdots + l_m e_m$. Now the first coordinate of $x = (l_1' e_1', \cdots, l_n' e_n', \cdots)$ [where $e_n' = e_n$, for $n > m$] is l, and as in the proof of (15.4) it follows that all the e_n'-coordinates of the elements of X are divisible by l. Setting $a_1 = (e_1', l_2' l^{-1} e_2', \cdots, l_n' l^{-1} e_n', \cdots)$ and $k_1 = l$, we obtain $P = \langle a_1 \rangle \oplus P'$ and $X = \langle k_1 a_1 \rangle \oplus X'$ with X' a product in $P' = \prod_{n=2}^{\infty} \langle e_n' \rangle$. Selecting an $a_2 \in P'$ and $k_2 \in \mathbf{Z}$ in the same way for P' and X', and then continuing this process *ad infinitum*, we find $a_n \in P$ and $k_n \in \mathbf{Z}$ of the stated kind.□

From this result, it is now easy to conclude what epimorphic images of P are like

Proposition 95.2 (Nunke [2]). *Every epimorphic image of P is the direct sum of a cotorsion group and a direct product of* [*at most countably many*] *infinite cyclic groups.*

Let K be a subgroup of P. For each n, select an

$$x_n = (0, \cdots, 0, s_n e_n, s_{n+1} e_{n+1}, \cdots) \in K$$

such that s_n divides t_n whenever $a = (0, \cdots, 0, t_n e_n, t_{n+1} e_{n+1}, \cdots) \in K$, with the proviso that $x_n = 0$ if $t_n = 0$ for all such $a \in K$. Then we can form the product $X = \prod_{n=1}^{\infty} \langle x_n \rangle$ in P which contains K as a subgroup. We choose $a_n \in P$ and $k_n \in \mathbf{Z}$ as stated in (95.1), and let P_1, P_2 be the product of the $\langle a_n \rangle$ with $k_n \neq 0$ and $k_n = 0$, respectively. Then $P = P_1 \oplus P_2$ with $X \leqq P_1$, and it suffices to prove that P_1/K is cotorsion. From (95.1) we see that $P_1/X \cong \prod_n Z(k_n)$, hence it is algebraically compact. Since $\bigoplus_n \langle x_n \rangle \leqq K$, the quotient X/K as an epimorphic image of an algebraically compact group is cotorsion [cf. (42.2) and (54.1)]. Consequently, **54**(D) implies P_1/K is cotorsion.□

Hence we derive the characterization of slender groups which has already been announced in the opening paragraph of this section.

Theorem 95.3 (Nunke [2]). *A torsion-free group is slender if and only if it contains no copy of Q, P, or J_p for any prime p.*

The necessity being obvious, assume no subgroup of G is isomorphic to Q, P, or J_p. If $\eta: P \to G$, then (95.2) and (54.5) imply that $\operatorname{Im} \eta$ is the direct sum of an algebraically compact torsion-free group and a product of infinite cyclic groups. The absence of Q and J_p implies that the first summand vanishes [see (40.4)], and the absence of P shows that $\operatorname{Im} \eta$ is a free group of finite rank. By (94.2), this is slender; therefore, $\eta e_n = 0$ for almost all n.□

The following consequence of (95.2) is also worth while mentioning.

Corollary 95.4. *A torsion-free group A containing a slender subgroup G such that A/G is reduced torsion is itself slender.*

Let $\eta: P \to A$ be a homomorphism and $\varphi: A \to A/G$ the natural map. (95.2) implies that Im $\varphi\eta$ is cotorsion. Reduced cotorsion torsion groups are bounded [see (54.4)], hence there exists a positive integer n such that $n\eta: P \to G$. In view of the slenderness of G the assertion is immediate.□

EXERCISES

1. (Nunke [4]) Every subgroup of infinite rank in P is isomorphic to a subgroup containing S.
2. (Nunke [4]) Let Z be given the discrete topology and P the corresponding product topology.
   (a) Every homomorphism $P \to Z$ is continuous.
   (b) The endomorphisms of P are continuous.
   (c) The topology of P is independent of the way P is represented as a product of infinite cyclic groups.
3. (G. A. Reid [1]) Show that in (95.3) none of the groups Q, P, J_p can be omitted. [*Hint*: $J_p \leqq P$ is impossible, since P is $\aleph_1$-free. To negate a monomorphism $\lambda: P \to J_p$, (i) show that for every n, $\lambda e_k \in p^n J_p$ for almost all k; (ii) select $k_1 < k_2 < \cdots$ and $n_1 < n_2 < \cdots$ such that $\lambda(m_i e_{n_i}) \equiv p^{k_i} \bmod p^{k_{i+1}} J_p$ for suitable $m_i \in \mathrm{Z}$; (iii) find integers l_i satisfying $\lambda(\sum l_i m_i e_{n_i}) = 0$.]
4. (Specker [1]) Call $a = \sum m_k e_k \in P$ *monotone* if $0 < m_1 \leqq \cdots \leqq m_k \leqq \cdots$. A subgroup T of P is *monotone* if (i) $\sum m_k e_k \in T$ implies $\sum n_k e_k \in T$ for $n_k = \max(1, |m_1|, \cdots, |m_k|)$; and (ii) if $\sum n_k e_k \in T$ and $0 \leqq m_k \leqq n_k$ for all k, then $\sum m_k e_k \in T$.
   (a) Show that for a real number $r \geqq 0$, the set of all $a = \sum n_k e_k$ with $|n_k| \leqq sk^r$, for some constant $s > 0$ depending on a, is a monotone subgroup T_r of P.
   (b) Every monotone subgroup contains an element $a = \sum m_k e_k$ with $m_k \geqq 0$ and $m_k | m_{k+1}$ for all k.
5. (G. A. Reid [1]) Every monotone subgroup T of P, different from P, is slender. [*Hint*: choose $a = \sum m_k e_k \notin T$, and find some $b = \sum \pm m'_k e_k \in P$ with m'_k a subsequence of the m_k which maps outside of T.]
6. (Fuchs [27]) (a) A monotone subgroup T of P, different from the subgroup T_0 [of bounded elements], satisfies: if G is a slender group of cardinality $< 2^{\aleph_0}$, then every homomorphism $\eta: T \to G$ maps almost all e_k upon 0. [*Hint*: argue as in (94.2).]
   (b)* Every such η is determined by $\eta | S$.
7. (Specker [1]) Every monotone subgroup T, different from T_0, contains a subgroup of power $\aleph_1$ which is not free. [*Hint*: the pure subgroup generated by the e_k and some $\sum m_k e_k \in T$ with $m_k | m_{k+1}$.]
8. (a) (Fuchs [27]) If T_1, T_2 are monotone subgroups of P and if $\chi: T_1 \to T_2$ is a homomorphism such that $S \leqq \mathrm{Im}\, \chi$, then $T_1 \leqq T_2$.

(b) (Specker [1]) Two monotone subgroups of P are isomorphic if and only if they are equal.

9*. (Specker [1]) The set of nonisomorphic monotone subgroups of P is of the power $2^{\mathfrak{c}}$ where $\mathfrak{c} = 2^{\aleph_0}$. [*Hint*: find continuously many elements in P such that none of them is contained in the monotone subgroup generated by the rest.]

10. (Chase [4]) Let C be a countable pure subgroup of P such that $S \leqq C$, and let X be any countable torsion-free group. There exists a pure subgroup A of P such that $C \leqq A$ and $A/C \cong X$. [*Hint*: P/C is algebraically compact torsion-free, it has the same structure as P/S [cf. **42**, Ex. 7], so X can be embedded as a pure subgroup.]

11. (Chase [4]) Let X_σ $(\sigma < \omega_1)$ be countable torsion-free groups. There exists an ascending chain of free pure subgroups A_σ $(\sigma < \omega_1)$ of P such that:
    (i) $A_0 = S$;
    (ii) $A_\sigma = \bigcup_{\rho < \sigma} A_\rho$ whenever σ is a limit ordinal;
    (iii) $A_{\sigma+1}/A_\sigma \cong X_\sigma$ for every $\sigma < \omega_1$.

12. (Griffith [10]) (a) Let A be a group of cardinality $\aleph_1$ which is the direct sum of countable groups C_i. If A is the union of a well-ordered ascending chain A_σ $(\sigma < \omega_1)$ of countable groups such that (ii) in Ex. 11 holds, then for every countable ordinal ρ there exists a limit ordinal σ such that A_σ is a summand of A. [*Hint*: show that some A_σ is the direct sum of some subset of $\{C_i\}$.]
    (b) Using the idea of Ex. 11, construct a torsion-free group which is not free, but which is the union of an ascending chain of countable free, pure subgroups.

96. VECTOR GROUPS

By a *vector group* we shall mean a direct product of rank one torsion-free groups, i.e., a group

$$V = \prod_{i \in I} R_i,$$

where the R_i are rational groups. This section is devoted to some questions concerning vector groups.

Our discussion begins with the following lemma.

Lemma 96.1. *If the vector group $V = \prod_{i \in I} R_i$ has a nontrivial homomorphism η into the vector group $W = \prod_{j \in J} S_j$, then*

$$\mathbf{t}(R_i) \leqq \mathbf{t}(S_j) \qquad \textit{for some} \quad i \in I \quad \textit{and} \quad j \in J.$$

If $\eta \neq 0$, then for some $j \in J$, the composite map $V \to W \to S_j$ is not zero, so we may assume, without loss of generality, that $W = S$ is a rational group. If $S \cong Q$, there is nothing to prove. Supposing S is slender, we pick a $v \in V$ such that $\eta v \neq 0$ in S. Write $v = (\cdots, v_i, \cdots)$ with $v_i \in R_i$ and collect into one summand the R_i for which $\chi(v_i)$ is the same, say $T_\chi = \prod_{\chi(v_i)=\chi} R_i$, to obtain $V = \prod_\chi T_\chi$ with χ running over all characters $\chi \geqq \chi(v)$. [The character $(\infty, \cdots, \infty, \cdots)$ is assigned to the 0 coordinates.] The last product contains at most continuously many summands T_χ, thus from (94.4) we conclude the existence of a finite set $\chi_1, \cdots, \chi_k$ such that $\eta T_{\chi_1}, \cdots, \eta T_{\chi_k}$ are not 0, but $\eta(\prod' T_\chi) = 0$, where the prime indicates that $\chi \neq \chi_1, \cdots, \chi_k$. For our $v = (\cdots, v_\chi, \cdots)$ with $v_\chi \in T_\chi$, we thus have $\eta v = \eta v_{\chi_1} + \cdots + \eta v_{\chi_k}$. This is not 0, so at least one of the terms, say, $\eta v_{\chi_1} \neq 0$. We infer from $\chi(\eta v_{\chi_1}) \geqq \chi(v_{\chi_1}) = \chi_1$ that the type of S is at least equal to the type corresponding to χ_1, which is the type of every R_i in T_{χ_1}.□

The following result is not so obvious as it seems at first glance.

Proposition 96.2 (Mishina [4]). *Every rank one summand of a vector group $V = \prod_{i \in I} R_i$ is isomorphic to some R_i.*

We collect the components R_i which are of the same type $\mathbf{t}$ and denote their product by $V_\mathbf{t} = \prod_{\mathbf{t}(R_i)=\mathbf{t}} R_i$. Then $V = \prod_\mathbf{t} V_\mathbf{t}$ which decomposition contains at most continuously many nonzero $V_\mathbf{t}$. Write $V = A \oplus C$ with A of rank 1. Then for the obvious projection ε: $V \to A$, there are at most finitely many $\varepsilon V_{\mathbf{t}_1} \neq 0, \cdots, \varepsilon V_{\mathbf{t}_k} \neq 0$, and ε sends the product of the other $V_\mathbf{t}$ onto 0. Thus this product is contained in C, and we obtain $V_{\mathbf{t}_1} \oplus \cdots \oplus V_{\mathbf{t}_k} \cong A \oplus C'$ for some $C' \leqq C$. From (96.1) we know that $\mathbf{t}(A) \geqq \mathbf{t}_j$ for $j = 1, \cdots, k$. But the maximal type of elements in $V_{\mathbf{t}_j}$ is $\mathbf{t}_j$; therefore, the type of A cannot be larger than all of $\mathbf{t}_1, \cdots, \mathbf{t}_k$.□

We next establish the uniqueness of the representations of vector groups as products of rational groups.

Theorem 96.3 (Sąsiada [5]). *If*

$$V = \prod_\mathbf{t} V_\mathbf{t} = \prod_\mathbf{t} W_\mathbf{t},$$

where $V_\mathbf{t}$ and $W_\mathbf{t}$ are direct products of groups of rank 1 and of type $\mathbf{t}$, and $\mathbf{t}$ runs over different types, then

$$V_\mathbf{t} \cong W_\mathbf{t} \qquad \text{for every type} \quad \mathbf{t}.$$

Let $V_\mathbf{t} = \prod_{i \in I} R_i$ with R_i of rank 1 and of fixed type $\mathbf{t}$. If also $V_\mathbf{t} = \prod_{j \in J} S_j$ with $r(S_j) = 1$, then the S_j are all of type $\mathbf{t}$. Moreover, $|I| = |J|$ holds under either one of the following assumptions:

1. *the generalized continuum hypothesis;*
2. *$V_\mathbf{t}$ is not divisible and I is not measurable.*

Let $\pi_t: V \to V_t$ and $\rho_t: V \to W_t$ be the coordinate projections. By virtue of (96.1), every homomorphism of V_t into W_s with s not $\geqq t$ is trivial, thus $V_t \leqq \prod_{s \geqq t} W_s$. Write $v_t \in V_t$ in the form $v_t = w_t + w'_t$ with $w_t \in W_t$, $w'_t \in \prod_{s > t} W_s$. Again by (96.1), the latter group has no nontrivial homomorphism either into V_t or into W_t, whence $\pi_t w'_t = 0 = \rho_t w'_t$. This shows $v_t = \pi_t v_t = \pi_t w_t$ and $w_t = \rho_t w_t = \rho_t v_t$; consequently, the composite map

$$V_t \xrightarrow{\rho_t} W_t \xrightarrow{\pi_t} V_t$$

is the identity map of V_t. Changing the roles, the isomorphism $V_t \cong W_t$ is immediate.

Because of (96.2), each S_j is of type **t**. For finite I, $|I| = r(V_t) = |J|$. Suppose I is infinite. Then in case 1, $|I| = |J|$ is a consequence of $2^{|I|} = |V_t| = 2^{|J|}$. In case 2, a rational group R of type **t** is slender, whence by (94.5) we have

$$\operatorname{Hom}(V_t, R) \cong \bigoplus_{i \in I} \operatorname{Hom}(R_i, R) \cong \bigoplus_{j \in J} \operatorname{Hom}(S_j, R).$$

Here the two last groups are of ranks $|I|$ and $|J|$, respectively. [Notice that $2^{|I|} = 2^{|J|}$ implies that, together with I, J too is nonmeasurable.]□

We turn to the problem of finding conditions under which vector groups are separable. We start with two preparatory lemmas; the complete answer will be given in (96.6).

Lemma 96.4. *Let $V = \prod_{i \in I} R_i$ be a vector group of infinite rank where all the R_i are rational groups of the same type* **t**. *The following conditions are equivalent:*

(a) *V is separable;*
(b) *V is homogeneous;*
(c) *the type* **t** *is idempotent.*

If V is separable and $0 \neq v \in V$, then v can be embedded in a completely decomposable summand of V. Here the rational summands must be, because of (96.2), again of type **t**; hence (a) implies (b).

Now assume (b). If $\mathbf{t} = (k_1, \cdots, k_n, \cdots)$ satisfied $0 < k_n < \infty$ for an infinity $n_1 < n_2 < \cdots$ of indices, then V would contain an element $v = (\cdots, v_i, \cdots)$ $(v_i \in R_i)$ where $p_{n_j} \nmid v_{i_j}$ for different indices $i_1, \cdots, i_j, \cdots \in I$, and the contradiction $\mathbf{t}(v) < \mathbf{t}$ would arise.

Finally, if **t** is idempotent, then every $v \in V$ is infinitely divisible by primes at which **t** is ∞. Therefore, $V(\mathbf{t}) = V$, and (a) follows at once from (87.4).□

Lemma 96.5. *Let $V = \prod_{i \in I} R_i$, where the R_i are rational groups of different types $\mathbf{t}_i$. If V is separable, then:*

(i) *the set $\{\mathbf{t}_i\}_{i \in I}$ satisfies the minimum condition;*
(ii) *there is no infinite subset of incomparable types in $\{\mathbf{t}_i\}$.*

Conversely, if (i) *and* (ii) *hold and if all the* $\mathbf{t}_i$ *are idempotent, then* V *is separable.*

Let $\mathbf{t}_1 \geqq \cdots \geqq \mathbf{t}_n \geqq \cdots$ and $W = \prod_{n=1}^{\infty} R_n$ with $\mathbf{t}(R_n) = \mathbf{t}_n$. Pick some $w \in W$ with nonzero coordinate in each R_n. If V is separable, then so is W [cf. (87.5)]; thus w belongs to a completely decomposable summand of finite rank of W. Here the types of the rational components must be, in view of (96.2), among the $\mathbf{t}_n$, hence $\mathbf{t}(w) = \mathbf{t}_m$ for some m. This, together with $\mathbf{t}(w) \leqq \mathbf{t}_n$, for all n, proves (i).

If $\mathbf{t}_1, \cdots, \mathbf{t}_n, \cdots$ are incomparable and $w \in W = \prod_{n=1}^{\infty} R_n$ has nonzero coordinates in each R_n, where $\mathbf{t}(R_n) = \mathbf{t}_n$, then let $w \in W'$ where W' is a direct sum of rational groups of types $\mathbf{t}_1, \cdots, \mathbf{t}_m$, say, and $W = W' \oplus W''$. The full invariance of the R_n in W implies that W' is fully invariant in W and $R_n \leqq W''$ for $n > m$. This is in contradiction to the fact that nth projection $W \to R_n$ sends $w \in W'$ on its nonzero nth coordinate.

Suppose the $\mathbf{t}_i$ are idempotent and (i), (ii) are satisfied. For the separability of V, it clearly suffices to show that to every $0 \neq v \in V$, there exists a completely decomposable summand of finite rank of V which contains v and has a complement which is the product of almost all R_i. Write $v = (\cdots, v_i, \cdots)$ with $v_i \in R_i$. Since the product of isomorphic rank 1 groups of idempotent type is by virtue of (96.4) separable, we may for the sake of simplicity suppose that the types of the coordinates $v_i \neq 0$ are different. Notice that among the types $\mathbf{t}(v_i)$ with $v_i \neq 0$ there are minimal ones and they are finite in number, say $\mathbf{t}_1, \cdots, \mathbf{t}_n$. Then we can decompose $V = V_1 \oplus \cdots \oplus V_n$ such that each V_k is the product of some subset of the R_i and $v = v' + \cdots + v^{(n)}$, where each $v^{(k)} \in V_k$ has exactly one minimal among the types of its coordinates. Evidently, we can reduce the problem to the $v^{(k)}$; in other words, we may assume without loss of generality that $v \in V$ has exactly one minimal type among the $\mathbf{t}(v_i)$, say $\mathbf{t}(v_0)$. Write $v_i = n_i a_i$ with $n_i \in \mathbf{Z}$ such that for every prime p, $h_p(a_i) = 0$ or ∞. If $|n_0| = 1$, then $\langle v_0 \rangle_*$ can be replaced by $\langle v \rangle_*$ in $\prod R_i$. If $|n_0| > 1$, then an argument similar to the one in the proof of (19.2) yields the separability of V.□

Now we are in a position to single out the separable vector groups.

Theorem 96.6 (Mishina [5], Król [1]). *The vector group* $V = \prod_{i \in I} R_i$ [*where* R_i *are rational groups of types* $\mathbf{t}_i$] *is separable if and only if the following conditions are satisfied*: (i) *and* (ii) *in* (96.5), *and*

(iii) *the set of* R_i *with nonidempotent* $\mathbf{t}_i$ *is finite.*

Suppose V separable and deny (iii). First we show that $\mathbf{t}_1 < \cdots < \mathbf{t}_n < \cdots$ cannot hold for nonidempotent types $\mathbf{t}_n = \mathbf{t}(R_n)$. Otherwise, there exist a sequence $0 < m_1 < \cdots < m_k < \cdots$ of integers and elements $v_n \in R_n$ such that $p_{m_n} \nmid v_n$ and v_n is divisible, but not infinitely divisible, by p_{m_k} for $k > n$. Then $\mathbf{t}(v) < \mathbf{t}_1$, for $v = (v_1, \cdots, v_n, \cdots) \in \prod_{n=1}^{\infty} R_n$, and obviously, v cannot be contained in a direct sum of rational groups of types $\mathbf{t}_1, \cdots, \mathbf{t}_n$ for any n. Consequently, $\prod_{n=1}^{\infty} R_n$ is not separable, and so the nonidempotent types $\mathbf{t}_i$ must satisfy the maximum condition. A partially ordered set satisfying both the maximum and the minimum conditions and containing no infinitely many incomparable elements is necessarily finite. Hence (iii) holds.

Turning to sufficiency, we assume (i)–(iii) and write $V = R_1 \oplus \cdots \oplus R_n \oplus \prod_{i \in I'} R_i$, where I' contains only R_i with idempotent types. The separability of the last product is a consequence of (96.5), and thus V, too, is separable. □

EXERCISES

1. Derive (96.1) for a nonmeasurable index set I from (94.5).
2. Show that (96.2) for a nonmeasurable index set I follows at once from (94.7).
3. (Mishina [1], Łoś [1]) A vector group is completely decomposable if and only if almost all components are isomorphic to Q.
4. (Mishina [4]) Let $V = \prod_{i \in I} R_i$ be a vector group with nonmeasurable index set I. Prove that every slender summand of V is isomorphic to the direct sum of a finite number of the R_i. [*Hint*: (94.7) and (86.7).]
5. For a rational group $R \not\cong Q$ and for a nonmeasurable index set I, $R^I \oplus \bigoplus_I R$ can never be a vector group, unless I is finite.
6. Let R be a nondivisible rational group of idempotent type, and $\mathfrak{m}$ a nonmeasurable cardinal number. There exist groups A of rank $2^{\mathfrak{m}}$ such that
$$\mathrm{Hom}(A, R) \cong A.$$
7. No reduced vector group has cotorsion subgroups $\neq 0$. [*Hint*: torsion-free quotients of rank 1 of J_p are divisible.]
8. Let $\mathbf{t}$ be an idempotent type, and $V = \prod R_i$, where R_i are rational groups of type $\mathbf{t}$. Prove the separability of V by putting a ring structure R with 1 on the R_i and then arguing as in (19.2) for R-modules. [Note that R will be a principal ideal ring; cf. (121.1).]
9. (Beaumont [5]) Let R_i ($i \in I$) be rational groups of the same type $\mathbf{t}$, and let $\mathbf{t}^0$ be the maximal idempotent type such that $\mathbf{t}^0 \leqq \mathbf{t}$ [i.e., $\mathbf{t}^0 = \mathbf{t} : \mathbf{t}$]. Show that $\prod R_i$ contains elements of any type $\mathbf{s}$ satisfying $\mathbf{t}^0 \leqq \mathbf{s} \leqq \mathbf{t}$, whenever I is infinite.
10. (Balcerzyk, Białynicki-Birula, and Łoś [1]) Let R_i ($i \in I$) be a family of reduced rank 1 groups with nonmeasurable index set I. Suppose that for $i, j \in I$, either $R_i \cong R_j$ or $\mathbf{t}(R_j) : \mathbf{t}(R_i)$ is not $\leqq \mathbf{t}(R_j)$. Prove that:
(a) $\mathrm{Hom}(\prod R_i, \prod R_i) \cong \bigoplus_i R_i^0$, where $\mathbf{t}(R_i^0) = \mathbf{t}(R_i) : \mathbf{t}(R_i)$;
(b) $\mathrm{Hom}(\bigoplus R_i^0, \prod R_i) \cong \prod R_i$;
(c) every summand of $\prod_i R_i$ is isomorphic to the product of a subset of the R_i.

97. FINITE-VALUED FUNCTIONS INTO A GROUP

Direct products of isomorphic rational groups are particularly interesting types of vector groups. We devote this section to the discussion of certain subgroups of such vector groups. Actually, our results can be stated more generally for direct products of copies of an arbitrary group A. The case $A = Z$ is most important, and the general case can easily be reduced to this one.

Let A be an arbitrary group and I any infinite index set. The set of all functions $f\colon I \to A$ such that f assumes but a finite number of distinct values in A is clearly an abelian group $\bar{A}$, namely, a subgroup of the cartesian power A^I of A. Manifestly, every $f \in \bar{A}$ can be written uniquely in *the canonical form*

$$(1) \qquad f = a_1 h_{X_1} + \cdots + a_k h_{X_k} \qquad (k \geqq 0),$$

where $a_1, \cdots, a_k$ are elements $\neq 0$ of A, the subsets $X_1, \cdots, X_k$ of I are pairwise disjoint, while the h_X are characteristic functions of subsets X of I:

$$h_X(i) = \begin{cases} 1 & \text{if} \quad i \in X, \\ 0 & \text{if} \quad i \notin X. \end{cases}$$

A subgroup S of $\bar{A}$ is said to be a *Specker group* (*over* A) if $f \in S$ implies $Ah_{X_1}, \cdots, Ah_{X_k} \leqq S$, i.e., $ah_{X_1}, \cdots, ah_{X_k} \in S$ for all $a \in A$, where f is as in (1). It is clear that every Specker group is the union of its subgroups of the form Ah_X for certain $X \subseteq I$.

Our main purpose is to describe the structure of Specker groups. We say that a subgroup F of $\bar{A}$ has a *characteristic A-basis* if

$$F = \bigoplus_{j \in J} Ah_j$$

for suitable characteristic functions h_j. The main result states that Specker groups over A have characteristic A-bases.

First, the case $A = Z$ will be settled. We assemble some useful properties of Specker groups over Z in the following lemma. Observe that Z carries a ring structure, hence so does Z^I.

Lemma 97.1. *For a subgroup S of the group $\bar{Z} = Z^I$, the following conditions are equivalent*:

(i) *S is a Specker group*;
(ii) *$f \in S$ implies $h_X \in S$, where X is the support of f*;
(iii) *S is pure in $\bar{Z}$ and is a subring of $\bar{Z}$.*

From $h_X = h_{X_1} + \cdots + h_{X_k}$ it is clear that (i) implies (ii). To show the converse, we induct on k. For $k = 1$, there is nothing to prove. If $k > 1$ and if (ii) holds, then

$$f - n_k h_X = (n_1 - n_k)h_{X_1} + \cdots + (n_{k-1} - n_k)h_{X_{k-1}},$$

and induction hypothesis implies $h_{X_1}, \cdots, h_{X_{k-1}} \in S$. Hence $n_k h_{X_k} \in S$ and $h_{X_k} \in S$, so (i) follows.

Assuming (i), the purity of S is evident. Thus for (iii) it remains to verify that h_X, $h_Y \in S$ implies $h_X h_Y \in S$. This is a simple consequence of $h_X h_Y = h_{X \cap Y}$ and the canonical form $h_W + 2h_{X \cap Y}$ of $h_X + h_Y$, where $W = (X \backslash Y) \cup (Y \backslash X)$. Conversely, suppose (iii) and let $f \in S$ be as in (1). If $k = 1$ $h_{X_1} \in S$ follows from purity. If $k > 1$,

$$f^2 - n_k f = (n_1^2 - n_k n_1)h_{X_1} + \cdots + (n_{k-1}^2 - n_k n_{k-1})h_{X_{k-1}}$$

has a canonical form of length $\leqq k - 1$, so $h_1 = h_{X_1 \cup \cdots \cup X_{k-1}} \in S$ by induction. Similarly, $h_2 = h_{X_2 \cup \cdots \cup X_k} \in S$, thus $h_X = h_1 + h_2 - h_1 h_2 \in S$ and (ii) follows.$\square$

In view of this lemma we see that the Specker subgroups of $\bar{\mathrm{Z}}$ are rings generated by idempotents h_X. Before stating the main theorem (97.3), we first prove an auxiliary result in general, on commutative rings generated by idempotents. Let R be a commutative and associative ring with 1. Assume that R as a ring is generated by a set E of idempotents and the additive group R^+ of R is torsion-free. If E^* is the set of all finite products of elements of E, then the elements of E^* are again idempotent [the empty product is 1], and it is clear that every $\alpha \in \mathrm{R}$ is of the form $\alpha = \sum_{i=1}^m n_i \varepsilon_i$, with $n_i \in \mathrm{Z}$, $\varepsilon_i \in E^*$. Moreover, here $\varepsilon_1, \cdots, \varepsilon_m$ can be chosen so as to form a set of orthogonal idempotents. In fact, given any set $\eta_1, \cdots, \eta_k$ of idempotents in R, the elements $\varepsilon_{i_1 \cdots i_k} = \eta_1^{(i_1)} \cdots \eta_k^{(i_k)}$ for all $i_k = 0$ and 1 are mutually orthogonal idempotents, where $\eta_j^{(0)} = 1 - \eta_j$ and $\eta_j^{(1)} = \eta_j$. Since $\eta_j = \sum \varepsilon_{i_1 \cdots i_{j-1} 1\, i_{j+1} \cdots i_k}$ for every j, α is a linear combination of these εs. Furthermore, if $n_i = n_j$, then replacing $n_i \varepsilon_i + n_j \varepsilon_j$ by $n_i \varepsilon$, where $\varepsilon = \varepsilon_i + \varepsilon_j$ is an idempotent, orthogonal to all ε_l with $l \neq i, j$, we can achieve that $\alpha = \sum_{i=1}^m n_i \varepsilon_i$ with pairwise orthogonal idempotents $\varepsilon_i \in E^*$ and different $n_i \in \mathrm{Z}$. This form of α is obviously unique. In view of torsion-freeness, $\alpha \in n\mathrm{R}$ exactly if $n | n_i$ for every i.

Lemma 97.2 (G. M. Bergman). *Let* R *be a commutative ring with identity* 1 *whose additive group* R^+ *is torsion-free. If* R *is generated as a ring by a set E of idempotents, then* R^+ *is a free abelian group on a basis whose elements are finite products of elements of E.*

We pick a well-ordering for E, say $\varepsilon_0, \cdots, \varepsilon_\sigma, \cdots (\sigma < \tau)$. The set E^* of all finite products $\varepsilon_{\sigma_1} \cdots \varepsilon_{\sigma_k}$ $(\tau > \sigma_1 > \cdots > \sigma_k)$ of elements of E will be ordered lexicographically, reading from the left; the empty product 1 is the minimal element in this ordering. Our claim is that *the elements of E^* which are not equal to linear combinations of elements of E^* smaller in the ordering form a free basis for* R^+.

If $\tau = 0$, then E is empty and so $\mathsf{R} \cong \mathsf{Z}$ or 0. As a basis of induction, we assume that $\tau > 0$ and the indicated rule yields a basis for every ring where the generating set of idempotents is well-ordered for some ordinal $< \tau$.

Now, if τ is a limit ordinal, then R can be regarded as the union of an ascending chain of subrings R_σ $(\sigma < \tau)$ of R, where R_σ is generated by all $\varepsilon_\rho \in E$ with $\rho < \sigma$. In this case, the above rule of selecting a basis will obviously lead to a basis of R^+.

If $\tau = \rho + 1$, then let R_0 be the subring generated by all $\varepsilon_\sigma \in E$ with $\sigma < \rho$. The induction hypothesis, of course, applies to R_0, and we intend to show that the basis of R_0^+ extends, in the way stated above, to a basis of R^+. Write $\varepsilon_\rho = \varepsilon$. Clearly, $\varepsilon\mathsf{R}_0 = \varepsilon\mathsf{R}$ can be regarded as a ring with unit ε, where $\varepsilon\varepsilon_0, \cdots, \varepsilon\varepsilon_\sigma, \cdots$ $(\sigma < \rho)$ is a generating set of idempotents. From the unique form of elements α of R mentioned earlier, it is readily seen that both R_0 and $\varepsilon\mathsf{R}_0$ are pure in R, hence $\mathsf{R}_0 \cap \varepsilon\mathsf{R}_0$ is a pure ideal in $\varepsilon\mathsf{R}_0$. It follows that $\overline{\mathsf{R}} = \varepsilon\mathsf{R}_0/(\mathsf{R}_0 \cap \varepsilon\mathsf{R}_0)$ has torsion-free additive group, so the induction hypothesis applies to $\overline{\mathsf{R}}$. Consequently, those products of images of $\varepsilon\varepsilon_\sigma$ $(\sigma < \rho)$ which are not equal in $\overline{\mathsf{R}}$ to linear combinations of lexicographically smaller products will constitute a basis for $\overline{\mathsf{R}}^+$. Going back to $\varepsilon\mathsf{R}_0$, these products of $\varepsilon\varepsilon_\sigma$ will exactly be those members of E^* which involve $\varepsilon = \varepsilon_\rho$ and fail to be equal to linear combinations of lexicographically smaller products. Obviously, a basis for R_0^+ together with all these members of E^* will form a basis for R^+. □

We are now ready to prove the main theorem on Specker groups over Z a very special case of which was proved by Specker [1].

Theorem 97.3 (Nöbeling [1]). *Let S and T be Specker groups over Z such that $S < T$. Then there exists a free subgroup F of T with a characteristic basis such that $T = S \oplus F$.*

There is no loss of generality in assuming $h_I \in S$. For, if h_I is not in S, but $h_I \in T$, then applying the theorem to the Specker groups $S \oplus \langle h_I \rangle$ and T, the result will follow at once for S and T. If $h_I \notin T$, too, then the application of the theorem to $S \oplus \langle h_I \rangle$ and $T \oplus \langle h_I \rangle$ will give $T \oplus \langle h_I \rangle = S \oplus \langle h_I \rangle \oplus F$ for some free group $F = \bigoplus \langle h_j \rangle$. It is readily seen that for each h_j, either h_j or $h_I - h_j$ is an element of T, thus $T = S \oplus F'$, where F' is freely generated by the h_j' with $h_j' = h_j$ or $h_I - h_j$, whichever belongs to T.

By virtue of (97.1), both S and T can be viewed as rings generated by idempotents and having torsion-free additive groups. By the preceding paragraph, they may be assumed to contain an identity, thus (97.2) is applicable. If the well-ordering is chosen such that the idempotent generators of S will precede the idempotent generators of T not in S, then the assertion becomes evident from (97.2). □

Corollary 97.4. *Specker groups over the infinite cyclic group are free and have characteristic bases.* □

The following generalization of (97.3) is now easily established.

Corollary 97.5 (Kaup and Keane [1]). *If S and T are Specker groups over any group A such that $S < T$, then T contains a subgroup F with a characteristic A-basis such that $T = S \oplus F$.*

If A is a cyclic group of order 2, then all subgroups of A^I are both Specker groups and vector spaces over the prime field of characteristic 2; in this case the assertion is trivial.

So let $|A| > 2$. If Ah_X and $Ah_Y \leqq S$ for some subsets X and Y of I, then choosing different nonzero elements a, b of A,

$$ah_X + bh_Y = ah_{X \setminus Y} + (a + b)h_{X \cap Y} + bh_{Y \setminus X}$$

implies the inclusions $Ah_{X\setminus Y}, Ah_{X \cap Y}, Ah_{Y\setminus X} \leqq S$. An appeal to (97.1) shows that the subgroup S' of Z^I, generated by those h_X for which $Ah_X \leqq S$, is a Specker group over Z. Specker groups over Z being free, a routine computation shows that $S = A \otimes S'$ holds. Similarly, $T = A \otimes T'$ for a Specker group T' over Z. If F' is a free group with a characteristic basis such that $T' = S' \oplus F'$, then $F = A \otimes F'$ will satisfy $T = S \oplus F$.□

Corollary 97.6. *Specker groups over any group A have characteristic A-bases; in particular, they are direct sums of copies of A.*□

A significant special case is when I is a topological space and the group S consists of all continuous, finite-valued functions $f: I \to A$ [A can carry any Hausdorff topology]. In the canonical form (1), all X_i are now both open and closed, thus Ah_{X_i} also belong to S. In other words, S is a Specker group. Consequently,

Corollary 97.7. *The group of all continuous and finite-valued functions from a topological space I into any group A is a direct sum of copies of A and has a characteristic A-basis $\{h_{X_j}\}$, where the X_j are open and closed subspaces of I.*□

If I is compact and A is discrete, then all continuous functions $f: I \to A$ are finite-valued, and hence the group of *all* continuous functions from I to A is isomorphic to a direct sum of copies of A.

EXERCISES

1. Show that any group of bounded [possibly transfinite] sequences of integers is free.
2. For any index set I, the direct sum $\bigoplus_{i \in I} A$ is a summand of the group $\bar{A}$ ($\leqq A^I$).
3. (a) The intersection of Specker groups is a Specker group. Conclude that there is a smallest Specker subgroup Sp(H) containing a subset H of $\bar{A}$.

(b) The minimal Specker group containing a finite subset of $\bar{Z}$ is a finitely generated free group.

4. If S is a Specker group and if h_X is a characteristic function, then
$$S^X = Sh_X = \{fh_X | f \in S\}$$
is again a Specker group.
5. (Nöbeling [1]) Let $\bar{X} = I \backslash X$. Then for any Specker group S, $S^X \oplus S^{\bar{X}}$ is a Specker group and S is a subdirect sum of S^X and $S^{\bar{X}}$.
6. (Nöbeling [1]) If S is a Specker group over a group A of cardinality >2, and if $Ah_I \leqq S$, then the minimal Specker group $\mathrm{Sp}(S, h_X)$ containing S and a characteristic function h_X is equal to $S^X \oplus S^{\bar{X}}$.
7. (Nöbeling [1]) Using transfinite induction, show that it suffices to prove (97.3) only for $T = \mathrm{Sp}(S, h_X)$ with some characteristic function h_X.
8. (Nöbeling [1]) If S is a Specker group over A such that $Ah_I \leqq S$, and if h_X and h_Y are any two characteristic functions, then
$$\mathrm{Sp}(S, h_X, h_Y) = [\mathrm{Sp}(S, h_X) + \mathrm{Sp}(S, h_Y)] \oplus F$$
holds for some group F with a characteristic A-basis. [*Hint*: putting $T = \mathrm{Sp}(S, h_Y)$, show that $U = S^X + (T^X \cap T)$ is a Specker group, thus $T = U \oplus F$, and verify claim for this F.]

98.* HOMOGENEOUS AND HOMOGENEOUSLY DECOMPOSABLE GROUPS

By a homogeneous group was meant a torsion-free group all of whose elements $\neq 0$ are of the same type $\mathbf{t}$. We have seen several examples of homogeneous groups among the completely decomposable and separable groups, and even among indecomposable groups. Here we discuss a few more results on homogeneous groups and on their direct sums.

The first result answers the question as to when finite rank homogeneous groups are completely decomposable.

Proposition 98.1 (Baer [6]). *A homogeneous group A of finite rank n is completely decomposable exactly if A/C is finite for each* [*or some*] *subgroup C which is a direct sum of n pure subgroups of rank* 1.

If $A = A_1 \oplus \cdots \oplus A_n$ with A_i of rank 1 and if C is as formulated, then all the elements of C have the same type in C as in A, thus $A_i \cap C$ is of finite index in A_i. Since $\oplus_i (A_i \cap C) \leqq C$, the finiteness of A/C is evident.

Conversely, assume $C = Rc_1 \oplus \cdots \oplus Rc_n$ with Rc_i pure in A and with R a subgroup of Q of the type of A. It suffices to consider the case when $|A : C|$ is a prime p. If $a \in A \backslash C$, then write $pa = m^{-1}(m_1 c_1 + \cdots + m_n c_n)$ with $m^{-1} m_i \in R$. Here $(m_1, \cdots, m_n) = d = 1$ may be assumed, for if $(d, m) = 1$, then

$d^{-1}pa \in C$ and we can replace a by $d^{-1}a$. In view of an obvious generalization of (15.3), there is a decomposition $C = Rb_1 \oplus \cdots \oplus Rb_n$ with $b_1 = m_1c_1 + \cdots + m_nc_n$. Then $A = Rma \oplus Rb_2 \oplus \cdots \oplus Rb_n$.□

For countable groups, the following analog of (19.1) holds.

Theorem 98.2. *A countable, homogeneous torsion-free group A is completely decomposable if and only if every finite rank subgroup C which is the direct sum of pure subgroups of rank 1 is of finite index in its pure closure $\langle C\rangle_*$.*

If A is completely decomposable, then (86.6) implies $\langle C\rangle_*$ is completely decomposable. Thus necessity follows from (98.1).

Conversely, let A satisfy the stated condition. If A is of finite rank, then in view of (98.1) there is nothing to prove. Suppose $a_1, \cdots, a_n, \cdots$ is a maximal independent set in A, and write $C_n = \langle a_1\rangle_* \oplus \cdots \oplus \langle a_n\rangle_*$, $B_n = \langle C_n\rangle_*$. The complete decomposability of B_n follows from (98.1), while (86.8) shows that B_n is a summand of B_{n+1}, say $B_{n+1} = B_n \oplus A_{n+1}$ $(n = 1, 2, \cdots)$. Clearly, the subgroups $A_1 = B_1, A_2, \cdots, A_n, \cdots$ generate their direct sum in A which must be the entire group A.□

We proceed to *homogeneously decomposable* groups. These are defined as groups that are direct sums of homogeneous groups, i.e., they are of the form $A = \bigoplus_{j \in J} G_j$ with homogeneous groups G_j. We can collect the components G_j which are of the same type and take their sum in order to get a smallest homogeneous decomposition

$$A = \bigoplus_{\mathbf{t}} H_{\mathbf{t}}, \tag{1}$$

where the $H_{\mathbf{t}}$ are homogeneous of different types $\mathbf{t}$. It is easy to see that smallest homogeneous decompositions are unique up to isomorphism; in fact, $H_{\mathbf{t}} \cong A(\mathbf{t})/A^*(\mathbf{t})$.

Theorem 98.3 (Baer [6], Erdös [1]). *Let A be a torsion-free group such that the types of its elements satisfy the maximum condition. Then A is homogeneously decomposable if and only if, for each type $\mathbf{t}$, the following two conditions hold:*

(a) *$A^*(\mathbf{t})$ is a summand of $A(\mathbf{t})$;*

(b) *$A^*(\mathbf{t}) = A(\mathbf{t}) \cap A^{**}(\mathbf{t})$, where $A^{**}(\mathbf{t})$ is generated by all $a \in A$ whose types are not $\leqq \mathbf{t}$.*

Evidently, (a) is a necessary condition. To prove this for (b), write A in the form (1) with homogeneous $H_{\mathbf{t}}$ of different types $\mathbf{t}$ and notice that [the sign $\|$ denotes incomparability]

$$\begin{aligned} A(\mathbf{t}) \cap A^{**}(\mathbf{t}) &= \bigoplus_{\mathbf{s} \geqq \mathbf{t}} H_{\mathbf{s}} \cap \bigoplus_{\mathbf{s} \not\leqq \mathbf{t}} H_{\mathbf{s}} = \Big(\bigoplus_{\mathbf{s} > \mathbf{t}} H_{\mathbf{s}} \oplus H_{\mathbf{t}}\Big) \cap \Big(\bigoplus_{\mathbf{s} > \mathbf{t}} H_{\mathbf{s}} \oplus \bigoplus_{\mathbf{s} \| \mathbf{t}} H_{\mathbf{s}}\Big) \\ &= \bigoplus_{\mathbf{s} > \mathbf{t}} H_{\mathbf{s}} = A^*(\mathbf{t}). \end{aligned}$$

To verify sufficiency, for any type $\mathbf{t}$, let $H_\mathbf{t}$ be defined by $A^*(\mathbf{t}) \oplus H_\mathbf{t} = A(\mathbf{t})$. If $H_\mathbf{t} \neq 0$, then it is homogeneous of type $\mathbf{t}$. These $H_\mathbf{t}$ generate their direct sum in A. For, if $h_1 + \cdots + h_n = 0$ with $h_i \in H_{\mathbf{t}_i}$ and different $\mathbf{t}_i$, and if $\mathbf{t}_1$ is minimal among $\mathbf{t}_1, \cdots, \mathbf{t}_n$, then

$$h_2 + \cdots + h_n = -h_1 \in A^{**}(\mathbf{t}_1) \cap A(\mathbf{t}_1) = A^*(\mathbf{t}_1)$$

and $h_1 \in A^*(\mathbf{t}_1) \cap H_{\mathbf{t}_1} = 0$. To show that $\bigoplus_\mathbf{t} H_\mathbf{t} = H$ coincides with A, let $a \in A$ be of type $\mathbf{t}$. Because the types in A satisfy the maximum condition, we can suppose that every element of A of type $>\mathbf{t}$ belongs to H, i.e., $A^*(\mathbf{t}) \leqq H$. But then $a \in A(\mathbf{t}) = A^*(\mathbf{t}) \oplus H_\mathbf{t} \leqq H$ establishes sufficiency.$\square$

EXERCISES

1. Show that homogeneity is essential in (98.1).
2. (Kolettis [3]) Let A be a torsion-free group satisfying: (i) for every type $\mathbf{t}$, both $A(\mathbf{t})$ and $A^*(\mathbf{t})$ are summands of A; (ii) every element of A lies in a direct summand of A which is a (finite) direct sum of homogeneous groups. Show that properties (i) and (ii) are inherited by direct summands of A. [*Hint*: argue as in the proof of (87.5).]
3. (Kolettis [3]) (a) A countable torsion-free group A is homogeneously decomposable if and only if it satisfies (i) and (ii) in Ex. 2.
   (b) Extend (a) to direct sums of countable groups.
4. If A is a direct sum of countable groups and homogeneously decomposable, then the summands of A have the same property.
5. Prove that countability is essential in (98.2).
6. Show that (98.3) need not hold if the maximum condition on the types is dropped. [*Hint*: product of rational groups with idempotent types $\mathbf{t}_1 < \mathbf{t}_2 < \cdots$.]

99. WHITEHEAD'S PROBLEM

A rather difficult problem proposed by J. H. C. Whitehead asks for the characterization of groups G satisfying $\operatorname{Ext}(G, Z) = 0$. Though a great deal of information has been obtained about such groups G, a complete solution to this problem is known only in the countable case. Except for free groups, no groups have been found so far with this property.

Following Rotman [3], we call a group G a *Whitehead group*, or simply a *W-group* if it satisfies

$$\operatorname{Ext}(G, Z) = 0.$$

A few elementary properties of W-groups are listed as follows.

(a) *Free groups are W-groups.*

(b) *Subgroups of W-groups are again W-groups.* In fact, by (51.3), a monomorphism $H \to G$ implies an epimorphism $\operatorname{Ext}(G, Z) \to \operatorname{Ext}(H, Z)$.

(c) *Direct sums of W-groups are again W-groups.* This follows at once from (52.2).

(d) *W-groups are torsion-free.* In view of (b), it suffices to show that, for any prime p, $Z(p)$ is not a W-group. But this is evident because of $\operatorname{Ext}(Z(p), Z) \cong Z(p)$ [cf. **52**(D)].

(e) *A torsion-free W-group G of finite rank is free.* For if not, then it is not finitely generated, and so for an essential free subgroup F of G, G/F is an infinite torsion group. The exact sequence $0 \to F \to G \to G/F \to 0$ implies that the induced sequence

$$\operatorname{Hom}(F, Z) \to \operatorname{Ext}(G/F, Z) \to \operatorname{Ext}(G, Z) = 0$$

is exact. This is absurd, since $\operatorname{Hom}(F, Z)$ is countable, while—as it is readily seen [Ex. 1]—$\operatorname{Ext}(G/F, Z)$ is of the power of the continuum.

Hence we can easily derive our first result on W-groups.

Theorem 99.1 (Stein [1], Rotman [3]). *W-groups are $\aleph_1$-free and separable.*

To prove $\aleph_1$-freeness, by Pontryagin's criterion (19.1) it suffices to show that a pure subgroup F of finite rank in a W-group G is free. This is an immediate consequence of (b) and (e).

To establish separability, let F be a pure subgroup of finite rank in G. As in (e), one obtains an epimorphism $\operatorname{Hom}(F, Z) \to \operatorname{Ext}(G/F, Z)$. Here the first group is $\cong F$, thus finitely generated, while the second is divisible [see **52**(I)] and hence 0. Consequently, $\operatorname{Ext}(G/F, F) = 0$; in other words, F must be a summand of G. The separability of G now follows at once from (87.2).□

We infer that *countable W-groups are free*, and all W-groups are subgroups in products of infinite cyclic groups. But they cannot themselves be products:

Proposition 99.2 (Rotman [1], Nunke [2]). *W-groups are slender.*

By (95.3), an $\aleph_1$-free group is slender exactly if it does not contain a product P of infinitely many cyclic groups $\langle e_n \rangle$. Therefore, we need only show that $\operatorname{Ext}(P, Z) \neq 0$. If S is the direct sum of the $\langle e_n \rangle$, then by (42.2), P/S is algebraically compact, hence $\operatorname{Hom}(P/S, Z) = 0$. Thus the exact sequence $0 \to S \to P \to P/S \to 0$ induces an exact sequence

$$0 \to \operatorname{Hom}(P, Z) \to \operatorname{Hom}(S, Z) \xrightarrow{\partial} \operatorname{Ext}(P/S, Z) \to \operatorname{Ext}(P, Z) \to 0.$$

Here $\operatorname{Hom}(P, Z) \cong S$, $\operatorname{Hom}(S, Z) \cong P$, and the naturality of the maps implies $\operatorname{Im} \partial \cong P/S$. This group is not divisible, but $\operatorname{Ext}(P/S, Z)$ must be divisible [cf. **52**(I)], hence $\operatorname{Coker} \partial \neq 0$.□

A variety of subgroups of products of copies of Z are separable and slender, so more information is needed concerning W-groups.

Lemma 99.3 (Chase [2]). *For every prime p, p-basic subgroups B of a W-group G satisfy*

$$2^{r(B)} = 2^{r(G)}.$$

The obvious exact sequence $0 \to B \to G \to G/B \to 0$ implies the exactness of $\operatorname{Hom}(B, Z) \to \operatorname{Ext}(G/B, Z) \to 0$. If $r(B)$ is finite, then by (99.1), $B = G$ and there is nothing to prove. If $r(B)$ is infinite, then $\operatorname{Hom}(B, Z) \cong Z^{r(B)}$ has cardinality $2^{r(B)}$. Furthermore, G/B contains a subgroup isomorphic to the direct sum of $r_0(G/B)$ copies of $Q^{(p)}$, thus

$$|\operatorname{Ext}(G/B, Z)| \geqq |\operatorname{Ext}(Q^{(p)}, Z)|^{r_0(G/B)} \geqq 2^{r_0(G/B)},$$

because the exactness of

$$Z \cong \operatorname{Hom}(Z, Z) \to \operatorname{Ext}(Z(p^\infty), Z) \cong J_p \to \operatorname{Ext}(Q^{(p)}, Z) \to 0$$

[cf. **52**(M)] implies that the last Ext is not 0 [it has cardinality $2^{\aleph_0}$]. Hence $2^{r(B)} \geqq 2^{r_0(G/B)}$. From $r(G) = r(B) + r_0(G/B)$ we conclude $2^{r(B)} \geqq 2^{r(G)}$. Strict inequality being absurd, the assertion follows. Assuming the generalized continuum hypothesis, it is equivalent to $r(B) = r(G)$. □

Proposition 99.4 (Chase [2]). *For a W-group G of infinite rank,*

$$|\operatorname{Hom}(G, Z)| = 2^{|G|}.$$

It is enough to verify the inequality $\geqq$. For a W-group G, $\operatorname{Hom}(G, Z) \to \operatorname{Hom}(G, Z/pZ)$ is an epimorphism. Evidently, the last group is isomorphic to $\operatorname{Hom}(G/pG, Z/pZ) \cong \operatorname{Hom}(B/pB, Z/pZ)$, where B is p-basic in G. The cardinality of the last Hom is $2^{r(B)} = 2^{|B|}$ for infinite $r(B)$. By (99.3), $2^{|B|} = 2^{|G|}$, completing the proof. □

Exercises

1. Complete the proof of (e) by showing that

   $$|\operatorname{Ext}(T, Z)| \geqq 2^{\aleph_0}$$

   for every infinite torsion group T. [*Hint*: if T has a finite socle, it contains a $Z(p^\infty)$.]
2. (Nunke [2]) Prove that $\operatorname{Ext}(P, Z)$ is the direct product of continuously many copies of Q/Z. [*Hint*: use the exact sequence in (99.2) and the fact that P/S is the extension of a direct sum of J_ps by a torsion-free divisible group; cf. **51**, Ex. 7 and **52**, Ex. 16.]

3. (Rotman [3]) If B is a pure subgroup of a W-group G such that G/B is divisible, then $2^{r(B)} = 2^{r(G)}$.
4. (Procházka [20]) A W-group belonging to some class Γ_σ [see **86**, Ex. 14] is free.
5. (Chase [2]) If $|\mathrm{Ext}(G, Z)| < 2^{\aleph_0}$, then G is a direct sum of a finite group and an $\aleph_1$-free group. [*Hint*: as in (e) show that for every finitely generated free subgroup F of G, the torsion part of G/F must be finite.]

6*. (Griffith [10]) Let F be a free group of infinite rank $\mathfrak{m}$. A group G satisfies $\mathrm{Ext}(G, F) = 0$ if and only if (i) every subgroup of G of rank $\leqq \mathfrak{m}$ is free; (ii) every subgroup of G of index $\leqq \mathfrak{m}$ contains a summand of G of index $\mathfrak{m}$. [*Hint*: necessity: if $|G| \leqq \mathfrak{m}$, then $0 \to H \to F \to G \to 0$ with F free of rank $\mathfrak{m}$ splits; if $A, B < G$, $A + B = G$, and $|B| \leqq \mathfrak{m}$, then B is free, and $0 \to C \to A \oplus B \to G \to 0$ (outer direct sum), with C isomorphic to a subgroup of B, is splitting; sufficiency: if $|G| > \mathfrak{m}$ and $A/F = G$, then there is an $H < A$ with $H \cap F = 0$, mapping upon a summand of index $\mathfrak{m}$ in G; show that $A/(H \oplus F)$ is free.]

NOTES

Except for finitely generated groups, virtually nothing was known until the 1930s about torsion-free groups. The first important development was Pontryagin's paper [1] in which the famous criterion (19.1) for the freeness of countable groups appeared, together with an example of an indecomposable group of rank 2. [In view of his duality theory, this example served to establish the existence of a two-dimensional compact connected abelian group which does not decompose into the direct product of two one-dimensional groups.] Subsequently, the theory of torsion-free groups of finite rank was developed by Derry [1], Kurosh [2], and Malcev [1]. [Cf. also Kaloujnine [1].] This theory has failed to live up to expectations, and no relevant applications of the theory have been known. [Only recently, D. M. Arnold has discovered some interesting applications.] The results have been extended by Szekeres [1], in a slightly different form, to cover the countable rank case. For a different approach, see Campbell [1].

The general theory of torsion-free groups has its origin in R. Baer's fundamental paper [6]. By establishing the indecomposability of pure subgroups of the p-adic integers, he at once produced indecomposable groups up to the continuum. Several basic results on completely decomposable and separable groups were also proved in this paper, which is still an important source of ideas. Lyapin [1–5] devotes attention to related problems.

Actually, the structure theory of torsion-free groups has not gotten far since R. Baer's paper. An apparent setback was the discovery of rather obscure direct decompositions by Jónsson [1], and later by Corner [1], which left the theory of torsion-free groups of finite rank in a chaos. Jónsson [2] himself succeeded in restoring some order out of this confusion, in an unexpected way [see (92.5)], shaking our belief in isomorphism as the exclusive principle of classification. By now we know that the correct way of viewing quasi-isomorphism is to interpret it as an isomorphism in an adequate quotient category [Walker [7]]. Unfortunately, quasi-isomorphism has not contributed much to the structure problem: the investigation by Beaumont and Pierce [4] on the rank 2 groups is a convincing evidence that there is still a long way to go to classify groups up to quasi-isomorphism.

There is an extensive literature on torsion-free groups of finite rank, see, e.g., Rotman [2, 4], Butler [1], Richman [2], several papers by Procházka, etc.

The existence of large indecomposable groups has been, for a while, a central question. Several more or less explicit examples have been given up to the continuum, and it was suspected [e.g., by the late T. Szele] that the continuum was an upper bound. Ideas of de Groot [2] and Łoś [on slender groups] helped step across the magic continuum, and indecomposable groups of power $2^{2^{\aleph_0}}$ were discovered by Hulanicki [3], Fuchs [11], and Sąsiada [3]. The claim in Fuchs [18], *viz.* the existence of indecomposable groups of arbitrary cardinality, has not been substantiated by the proof, which contained a set-theoretical gap. This gap has been eliminated by Corner [8], for cardinals less than the first strongly inaccessible cardinal. The proof of (89.2) follows the method developed in Fuchs [18]. A. L. S. Corner has announced a different proof of a stronger version, establishing the existence of homogeneous indecomposable groups, for the same cardinalities.

Indecomposability raises intriguing questions for modules. There are no indecomposable torsion-free modules of rank > 1 over the p-adic integers, and as is pointed out by Kaplansky [3], a valuation ring shares this property if and only if it is a maximal valuation ring. It is not yet known exactly which commutative domains retain this property, but in several important cases a complete answer is known, thanks to E. Matlis [*Trans. Amer. Math. Soc.* **134** (1968), 315–324]. In this connection, a surprising result by C. Faith and E. A. Walker [*J. Algebra* **5** (1967), 203–221] can be mentioned: a ring R has to be Noetherian if there is a cardinal number $\mathfrak{m}$ such that all R-injective modules are direct sums of modules of cardinalities $\leq \mathfrak{m}$. The commutative rings R over which all modules are direct sums of indecomposable modules are precisely the Artinian principal ideal rings, see R. B. Warfield [*Math. Z.* **125** (1972), 187–192].

Over Dedekind domains, torsion-free modules of rank 1 can be classified, as shown by P. Ribenboim [*Summa Brasil. Math.* **3** (1952), 21–36] and Kolettis [3]. Torsion-free modules of finite rank behave, in many respects, like their group counterparts, cf. I. Kaplansky [*Trans. Amer. Math. Soc.* **72** (1952), 327–340]. Several theorems on completely decomposable and separable groups extend to modules over Dedekind domains, see Kolettis [3]; and the rank 1 torsion-free modules with the cancellation property can almost completely be described, as pointed out by Fuchs and Loonstra [2].

The theory of slender groups has a short history; they were developed within a few years, mainly by J. Łoś, Sąsiada [4], and Nunke [2]. For a generalization to modules, see D. Allouch [Thèse, Univ. Montpellier, 1969–70].

Direct products of rank 1 groups and modules have also attracted attention. If P is the direct product of countably many infinite cyclic groups, equipped with the product topology [the components are discrete], then any two pure and dense subgroups of countable rank of P can be carried onto each other by automorphisms of P [Chase [4]]. It is somewhat surprising that while closed subgroups of P are again products of infinite cyclic groups, this is no longer true for products of uncountably many factors [Nunke [4]]. Chase [1] has shown that direct products of copies of a ring R are always projective as left R-modules exactly if R is left perfect and finitely generated right ideals of R are finitely presented.

Whitehead's problem seems to be one of the hardest open questions on torsion-free groups. The uncountable case was first discussed by Rotman [3], and more extensively by Chase [2, 3]; cf. also Viljoen [1]. A nice survey was given by G. A. Reid [1]. Griffith [5] has obtained additional results. If the framework is broadened to modules, new difficulties arise already in the countably generated case. For modules of countable rank over principal ideal rings, the problem has been settled by O. Gerstner, L. Kaup, and H. G. Weidner [*Arch. Math.* **20** (1969), 503–514].

Flat modules are perhaps the most relevant generalizations of torsion-freeness, for arbitrary rings. They enjoy the property that the finitely presented ones are projective. A certain amount of simple facts carry over to flatness, e.g., the class of flat modules is closed under direct sums. However, direct products of flat left R-modules need not be flat: they are if and only if finitely generated right ideals of R are finitely presented [cf. Chase [1]].

An interesting observation was made by G. Kolettis [*Canad. J. Math.* **12** (1960), 482–486] concerning commutative domains R with the property that pure submodules of finite rank in completely decomposable torsion-free R-modules are likewise completely decomposable: this holds if R has at most two primes, while if R has three or more primes, this need not be true.

Griffith [8] noticed the rather surprising fact that a torsion-free group A which is an extension of a free subgroup F by a totally projective p-group A/F is necessarily free. Hill [22] noted that this fails to hold for arbitrary reduced p-groups A/F. Procházka [11] and Bican [2] proved similar results on homogeneous completely decomposable F under stronger hypotheses.

Problem 66. Characterize torsion-free groups of rank 2 by invariants.

Problem 67. Given the integer $r \geqq 3$, find all sequences $n_1 < \cdots < n_s$ of integers, for which a group of rank r exists having decompositions into $n_1, \cdots, n_s$ indecomposable summands, respectively.

Problem 68. Given the positive integers $r_1, \cdots, r_k, r'_1, \cdots, r'_l$ such that $r_1 + \cdots + r_k = r'_1 + \cdots + r'_l$, under what conditions does a torsion-free group exist which has direct decompositions into indecomposable summands of ranks $r_1, \cdots, r_k$ and $r'_1, \cdots, r'_l$, respectively?

Problem 69. (Corner [6]) Do there exist finite rank groups with infinitely many pairwise nonisomorphic direct decompositions?

Recently, E. L. Lady has shown that a torsion-free group of rank 3 cannot have infinitely many nonisomorphic direct decompositions.

Problem 70. Characterize the finite rank torsion-free groups which have the cancellation property.

The rank one case is settled in Fuchs and Loonstra [2].

Problem 71. Define an equivalence relation for torsion-free groups A, B of finite rank by declaring them equivalent whenever $A \oplus C \cong B \oplus C$ for some torsion-free group C of finite rank. Relate this equivalence relation to quasi-isomorphism [it is stronger, but how stronger is it?].

Problem 72. (A. L. S. Corner) Does there exist a group such that every summand $\neq 0$ is a direct sum of infinitely many subgroups $\neq 0$?

Problem 73. Can a direct sum of a finite number of copies of A have an indecomposable summand if A does not have any?

Problem 74. Which summands of a vector group are again vector groups?

Problem 75. (a) Investigate Hopfian groups.

(b) (G. Baumslag) Are indecomposable torsion-free groups Hopfian? Are those with finite automorphism groups?

For Hopfian abelian groups, see Baumslag [1, 2] and Corner [5].

Problem 76. (G. A. Reid [1]) Are the following classes of groups all different: $\oplus Z$, $\prod Z$, $\prod(\oplus Z)$, $\oplus(\prod Z)$, $\oplus(\prod(\oplus Z))$, etc., where the number of summands can be arbitrary?

Problem 77. Investigate the structure of separable torsion-free groups, in particular, the homogeneous case.

Problem 78. (a) Which slender groups can be embedded in a direct sum of countable reduced groups?

(b) Is there any set of groups such that all slender groups can be obtained from them by using the operations of forming direct sums, subgroups, and extensions?

Problem 79. (J. H. C. Whitehead) Characterize the groups A satisfying $\operatorname{Ext}(A, Z) = 0$.

XIV

MIXED GROUPS

The mixed groups A form the most general class of abelian groups, and a satisfactory structure theory must obviously give full information about their torsion parts T and the corresponding torsion-free group A/T and at the same time describe the way in which these groups are put together to form A. Therefore, such a theory can be expected only for those classes of mixed groups A for which both T and A/T can be characterized in a satisfactory way. Then the problem reduces to finding tools by means of which the nonisomorphic extensions of T by A/T can be described. The height-matrices to be discussed in **103** seem to be adequate for several groups of torsion-free rank 1.

A great deal of work has been done on the splitting problem, i.e., when the mixed group splits into the direct sum of its torsion part and a torsion-free group. In **100** and in **101** we shall completely describe all torsion groups T and all torsion-free groups G, respectively, such that every mixed group splits whose torsion part is isomorphic to T and whose quotient mod its torsion part is isomorphic to G, respectively. The most general splitting problem, which consists in describing all pairs T, G of torsion and torsion-free groups satisfying $\operatorname{Ext}(G, T) = 0$, is, however, left unanswered.

The final section of this chapter is devoted to the construction of groups with given Ulm sequences. A rather intricate procedure leads to an existence theorem which is useful in establishing the existence of groups with certain properties.

100. SPLITTING MIXED GROUPS

We begin the theory of mixed groups with the discussion of conditions under which the mixed group A splits, i.e., it is the direct sum of its torsion part T and some torsion-free group G. Needless to say, this G is unique up to isomorphism only.

We start with examples of nonsplitting mixed groups.

Example 1. Let T be a reduced unbounded torsion group. From (55.1) and (55.4) we infer that $A = \mathrm{Ext}(Q/Z, T)$ is a cotorsion group whose torsion part is T and which has no torsion-free summand $\neq 0$. By (54.4), $A \neq T$, and so A is a nonsplitting mixed group with torsion part T.

Example 2. Let $p_1, \cdots, p_i, \cdots$ be different primes, and define

$$T_1 = \bigoplus_{i=1}^{\infty} \langle a_i \rangle \qquad \text{with} \quad o(a_i) = p_i.$$

Then T_1 is the torsion part of $\prod_{i=1}^{\infty} \langle a_i \rangle$. Consider $b_0 = (a_1, \cdots, a_i, \cdots) \in \prod \langle a_i \rangle$. For $i \neq j$, the equation $p_j x = a_i$ is uniquely solvable in $\langle a_i \rangle$, thus $\prod \langle a_i \rangle$ contains unique elements b_i ($i = 1, 2, \cdots$) such that b_i has 0 for its ith coordinate and satisfies

$$p_i b_i = (a_1, \cdots, a_{i-1}, 0, a_{i+1}, \cdots) = b_0 - a_i.$$

Then T_1 is the torsion part of $A_1 = \langle T_1, b_1, \cdots, b_i, \cdots \rangle$. Suppose A_1 is splitting, $A_1 = T_1 \oplus G$ for some $G < A_1$. Then $b_i = h_i + g_i$ ($h_i \in T_1$, $g_i \in G$) implies

$$p_i h_i + p_i g_i = p_i b_i = b_0 - a_i = (h_0 - a_i) + g_0.$$

Equating the T_1-coordinates, we get $p_i h_i = h_0 - a_i$ for $i = 1, 2, \cdots$. One of our primes, p_j, satisfies $p_j h = h_0$ for some $h \in T_1$. This leads to the contradictory equality $p_j(h - h_j) = a_j$. Consequently, A_1 does not split.

Example 3. For some prime p, let

$$T_2 = \bigoplus_{i=1}^{\infty} \langle a_i \rangle \qquad \text{with} \quad o(a_i) = p^{2i}.$$

For $i = 1, 2, \cdots$, define

$$b_i = (0, \cdots, 0, a_i, pa_{i+1}, p^2 a_{i+2}, \cdots) \in \prod_{i=1}^{\infty} \langle a_i \rangle.$$

These b_i are of infinite order and satisfy $pb_{i+1} = b_i - a_i$ ($i = 1, 2, \cdots$). Using these relations, it is readily checked that T_2 is the torsion part of $A_2 = \langle T_2, b_1, \cdots, b_i, \cdots \rangle$. If $A_2 = T_2 \oplus G$ held for some $G < A_2$, then G would be, in view of $pb_{i+1} \equiv b_i \bmod T_2$, a p-divisible subgroup of A_2, contrary to the fact that $\prod \langle a_i \rangle$ has no p-divisible subgroups $\neq 0$. Thus A_2 is not splitting.

The main question we wish to investigate in this section is that of characterizing the torsion groups T such that all mixed groups with torsion part T are splitting. This amounts to saying that T satisfies $\mathrm{Ext}(G, T) = 0$ for all torsion-free groups G; in other words, T is cotorsion. A full characterization of torsion cotorsion groups is known to us, and the following is just a restatement of (54.4).

Theorem 100.1 (Baer [4], Fomin [1]). *A torsion group T has the property that every mixed group with torsion part T splits if and only if it is a direct sum of a divisible and a bounded group.*

The following proof is independent of the theory of cotorsion groups. It is convenient to start with a simple observation: Let T be the torsion part of A, and let S, B be subgroups of A such that $S \leqq T \leqq B \leqq A$. If A splits, then so does B/S. In fact, if $A = T \oplus G$, then $B = T \oplus (B \cap G)$ and $B/S \cong T/S \oplus (B \cap G)$.

Given a torsion group T, write $T = D \oplus T'$ with D divisible and T' reduced. If T' is not bounded, then it either has an infinity of nonzero p_i-components T_{p_i} or contains an unbounded p-component T_p. In the first case, there are epimorphisms $T_{p_i} \to Z(p_i)$, yielding an epimorphism $T' \to T_1$, where T_1 is defined in Example 2. In the second case, the basic subgroup of T_p is unbounded, and thus has an epimorphism onto the group T_2 of Example 3. By virtue of (36.1), there is an epimorphism $T_p \to T_2$. Thus if T' is unbounded, then T has an epimorphism η onto T_1 or T_2 with kernel, say, S. By (24.6), there is a group H fitting in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & T & \xrightarrow{\eta} & T_k & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & H & \longrightarrow & A_k & \longrightarrow & 0 \end{array} \qquad (k = 1 \text{ or } 2)$$

with exact rows. Since the vertical maps are monic, the torsion part of H is evidently T. Since A_k does not split, from what has been said at the outset we conclude that H is not splitting. This proves the necessity part.

Suppose next that $T = D \oplus B$, where D is a divisible torsion group and B is bounded, is the torsion part of some group A. Then $A = D \oplus A'$ and $T = D \oplus B'$ for some subgroup B' of A' such that $B' \cong B$. By (27.5), B' is a summand of A', and so T is a summand of A.$\square$

The sufficiency part of (100.1) has an immediate generalization.

Proposition 100.2 (Fuchs [16]). *If A is a mixed group such that, for some positive integer n, nA splits, then A splits too.*

It suffices to prove this if n is a prime p. Let $pA = T' \oplus G'$, where T' is torsion and G' is torsion-free. Clearly, $T' = pT$ for the torsion part T of A. Define G as a T-high subgroup of A, containing G'. If $a \in A$ and $pa = b + c$ ($b \in T$, $c \in G$), then $b \in T' = pT$, and so a reference to (9.9) establishes $A = T \oplus G$.$\square$

General criteria under which a mixed group splits are scarce. Here we present one which has been used in the study of group algebras. First we prove a simple lemma.

Lemma 100.3. *Let A be a mixed group with bounded torsion part T, and let $C = T \oplus H$ be a pure subgroup of A. Then $A = T \oplus G$ for some G containing H.*

Let $nT = 0$. Factoring out nC, from (27.10) we obtain $A/nC = C/nC \oplus K/nC$ for some $K \geqq nC$. Define $G = H + K$; then $T + G = A$ is obvious. If $x = u + v$ ($u \in H$, $v \in K$) belongs to T, then $x - u = v \in C \cap K = nC = nH$ implies $x = 0$. Consequently, $A = T \oplus G$.□

Proposition 100.4 (May [4]). *A mixed group A splits if and only if there is an ascending sequence $A_1 \leqq \cdots \leqq A_n \leqq \cdots$ of subgroups in A such that*

(i) $\bigcup_{n=1}^{\infty} A_n = A$;
(ii) $T(A_n)$ *is bounded for every* n;
(iii) $T(A/A_n) = (T(A) + A_n)/A_n$ *for every* n.

If $A = T \oplus G$ with T torsion and G torsion-free, then $A_n = T[n!] \oplus G$ $(n = 1, 2, \cdots)$ satisfies (i)–(iii).

Conversely, let A satisfy (i)–(iii). By (ii), $A_1 = T(A_1) \oplus G_1$ with torsion-free G_1. Suppose that for all $i \leqq n$ we have decompositions $A_i = T(A_i) \oplus G_i$ satisfying $G_1 \leqq \cdots \leqq G_n$. Define $A'_{n+1} = T(A_{n+1}) \oplus G_n$, and notice that $(T(A) + A_n) \cap A_{n+1} = A'_{n+1}$, whence by the isomorphism theorems and (iii)

$$A_{n+1}/A'_{n+1} \cong (T(A) + A_{n+1})/(T(A) + A_n) \cong [(T(A) + A_{n+1})/A_n]/T(A/A_n)$$

is torsion-free. From (100.3) we infer that $A_{n+1} = T(A_{n+1}) \oplus G_{n+1}$ for some $G_{n+1} \geqq G_n$. Clearly, $A = T(A) \oplus G$ holds for $G = \bigcup G_n$.□

Exercises

1. Summands of splitting groups are splitting.
2. (Procházka [1]) A subgroup of finite index in A splits if and only if A splits.
3. A mixed group A with torsion part T need not split even if all of its subgroups of torsion-free rank 1 containing T are splitting. [*Hint*: non-splitting extension by $\prod Z$.]
4. (a) (Fomin [1]) The group
$$\langle a_1, \cdots, a_n, \cdots; pa_1 = \cdots = p^n a_n = \cdots \rangle$$
does not split.
(b) (Kulikov [5]) The group
$$\langle a_1, \cdots, a_n, \cdots; p^2(a_1 - pa_2) = \cdots = p^{2n}(a_n - pa_{n+1}) = \cdots = 0 \rangle$$
is not splitting.
5. (Oppelt [3]) Let $T = T_1 \oplus \cdots \oplus T_n$, where T_i is a p_i-group, and let A be a mixed group with torsion part T. Then A splits if and only if, for every i, the group $A/(T_1 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus \cdots \oplus T_n)$ is splitting.

6. (Kulikov [4]) Any two direct decompositions of a splitting mixed group $A = T \oplus G$ have isomorphic refinements if and only if both T and G have this property.
7. (de Groot [1]) Let A and B be mixed groups that are direct sums of rank 1 torsion and torsion-free groups. If each is isomorphic to a pure subgroup of the other, then $A \cong B$. [*Hint*: reduce to p-groups and torsion-free groups; **27**, Ex. 9 and **86**, Ex. 4.]
8. Show that there exist epimorphisms $A_1 \to T_1$ (Example 2) and $A_2 \to T_2$ (Example 3). [*Hint*: $a_i \mapsto 0$, $b_0 \mapsto 0$, $b_i \mapsto a_i$, and $a_{4i} \mapsto a_i$, $a_{4i+k} \mapsto 0$ $(k = 1, 2, 3)$.]
9. Give an example of a mixed group such that the torsion part is not an endomorphic image. [*Hint*: use Example 1, or in Example 2 choose C such that $A_1 \leqq C < \prod\langle a_i\rangle$ and $C/T_1 \cong Q$, and look up the alleged image of b_0.]
10. (a) Let A be a mixed group whose torsion part T has but a finite number of p-components $T_i \neq 0$. Then T is an endomorphic image of A if and only if each T_i is an endomorphic image of A.
    (b) Give a counterexample in case A has an infinity of reduced p-components $\neq 0$.
11. To every torsion group T there exists a mixed group A with torsion part T such that every torsion epic image of A is the direct sum of a divisible and a bounded group. [*Hint*: $A = \mathrm{Ext}(Q/Z, T) \oplus D$ with D the divisible part of T.]
12. (A. L. S. Corner) The following properties are equivalent for a torsion group T:
    (a) T is an endomorphic image of every mixed group in which T is the torsion part;
    (b) a basic subgroup of T is an endomorphic image of every mixed group whose torsion part is T;
    (c) T is a direct sum of a divisible and a bounded group. [*Hint*: Ex. 11.]
13. (Mishina [2]) Let the mixed group A have an automorphism which acts as multiplication by -1 on the torsion part T and induces the identity map on A/T. If $A[2] = 0$, then A is splitting.

101. BAER GROUPS ARE FREE

The next question that concerns us was formulated by Baer [4], and is dual to the one settled in (100.1): which are the torsion-free groups G such that every mixed group A whose torsion part T satisfies $A/T \cong G$ is splitting? Following Rotman's terminology [3], we call an arbitrary group G with this property a *Baer group*; that is to say, a Baer group is a group G such that

$$\mathrm{Ext}(G, T) = 0 \qquad \text{for all torsion groups} \quad T.$$

Trivially, free groups are Baer groups, and our goal is to show that there are no other Baer groups.

Theorem 101.1 (Griffith [7]). *Baer groups are free.*

The proof of this result is based on an existence theorem which is of independent interest. This reads as follows:

Theorem 101.2 (Griffith [7]). *For every cardinal* $\mathfrak{m}$, *there exists a mixed group* M *satisfying the following two conditions*:

(i) M/T *is divisible of rank* $\mathfrak{m}$, *where* T *is the torsion part of* M;
(ii) *every torsion-free subgroup of* M *is free.*

We shall first show how (101.1) follows from (101.2), deferring the more complicated proof of (101.2) for a while.

Therefore, suppose we have proved (101.2), and let G be a Baer group. It is straightforward to see that subgroups of Baer groups are again Baer groups. From **52**(D) it follows then that G must be torsion-free. Choose $\mathfrak{m} = r(G)$ so that by (i) there exists a monomorphism $\phi: G \to M/T$. Owing to (51.3), from the exact sequence $0 \to T \to M \xrightarrow{\alpha} M/T \to 0$, we obtain the exact sequence

$$\mathrm{Hom}(G, M) \xrightarrow{\alpha_*} \mathrm{Hom}(G, M/T) \to \mathrm{Ext}(G, T) = 0.$$

Thus α_* is epic, hence ϕ comes from some $\psi: G \to M$, that is, $\alpha\psi = \phi$. Since ϕ is monic, ψ is monic; in other words, G is isomorphic to a subgroup of M. The freeness of G is immediate from (ii).□

We break down the proof of (101.2) into two lemmas.

Lemma 101.3 (Griffith [7]). *There exists a countable mixed group* N *which satisfies*:

(a) $N/T \cong Q$, *where* T *is the torsion part of* N; *and*
(b) *every torsion-free subgroup* $\neq 0$ *of* N *is an infinite cyclic group.*

For every prime p, take cyclic groups $\langle a_{pn} \rangle \cong Z(p^{2n+1})$ $(n = 0, 1, \cdots)$ and form the groups

$$T_p = \bigoplus_{n=0}^{\infty} \langle a_{pn} \rangle \qquad \text{and} \qquad U_p = \prod_{n=0}^{\infty} \langle a_{pn} \rangle.$$

Then the element

$$u_p = (a_{p0}, pa_{p1}, \cdots, p^n a_{pn}, \cdots) \in U_p$$

is of infinite order and $p \nmid u_p$. We set $T = \bigoplus_p T_p$ and $U = \prod_p U_p$, and notice that T is in the torsion part of U and

$$u = (u_2, \cdots, u_p, \cdots) \in U$$

is not divisible by any prime p. It is readily seen that there is a subgroup $N/T \cong Q$ of U/T which contains $u + T$. To show that this N satisfies (b) too, suppose $F \neq 0$ is a subgroup of N and $F \cap T = 0$. Then $F \cong (F + T)/T \leqq N/T$ implies F is of rank 1. Given $a \neq 0$ in F, there are nonzero integers m, n such that $ma = nu$. Almost all primes p satisfy: $(p, n) = 1$ and hence $p \nmid ma$. Consequently, the p-height of a in F equals 0 for almost all primes p, and since N contains no elements of infinite order of infinite q-height for any prime q, $F \cong Z$ follows.□

Lemma 101.4 (Griffith [7]). *Let $\{M_i\}_{i \in I}$ be groups of finite torsion-free rank such that, for each i, all torsion-free subgroups of M_i are free. Then $M = \bigoplus_{i \in I} M_i$ has the same property.*

Let T_i denote the torsion part of M_i. Then $T = \bigoplus T_i$ is the torsion part of M, and we can write $M/T = \bigoplus_{i \in I} M_i/T_i$. The canonical maps $M_i \to M_i/T_i$, $M \to M/T$ will be denoted by θ_i and θ, respectively.

(A) We first settle the case of two summands, $M = M_1 \oplus M_2$ [now M_1 need not even be of finite torsion-free rank]. Let $\pi_i: M \to M_i$ ($i = 1, 2$) denote the obvious projections, and let F be a torsion-free subgroup of M. Define

$$H = \{a \in F \mid \pi_1 a \in T_1\} = \{a \in F \mid \theta a \in \theta_2 M_2\}.$$

Clearly, H is a torsion-free subgroup of M such that $H \cap \operatorname{Ker} \pi_2 = H \cap M_1 = 0$, because $H \cap M_1$ is contained in T_1. Hence $\pi_2 \mid H$ is monic, and thus $\pi_2 H$ is free, as a torsion-free subgroup of M_2. Since $\pi_2 H$ must be of finite rank, there is an integer $m > 0$ such that $mH \leqq M_2$. Setting $\phi = m\pi_1$, it is readily checked that $F \cap \operatorname{Ker} \phi = H$ and $\phi F[\leqq M_1]$ is torsion-free. Hence ϕF is free, which implies that $F \cong H \oplus \phi F$ is free.

(B) We claim that, for arbitrary I, every torsion-free subgroup F of $M = \bigoplus_{i \in I} M_i$ is $\aleph_1$-free. By virtue of (19.1), we need only examine the finite rank subgroups K of F. Such a K is contained in the direct sum of a finite number of M_i, thus a trivial induction from (A) shows that K is free. Consequently, F is $\aleph_1$-free.

(C) Given a torsion-free subgroup $F \neq 0$ of M, we establish a well-ordered increasing sequence

$$0 = F_0 < F_1 < \cdots < F_\sigma < \cdots < F_\tau = F$$

of subgroups of F, for some ordinal τ, and a sequence

$$\varnothing = I_0 \subset I_1 \subset \cdots \subset I_\tau = I$$

of subsets of I satisfying the following conditions for every σ:

(α) $F_\sigma = F \cap (\bigoplus_{i\in I_\sigma} M_i)$;
(β) $\bigcup_{\rho<\sigma} F_\rho = F_\sigma$ and $\bigcup_{\rho<\sigma} I_\rho = I_\sigma$, if σ is a limit ordinal;
(γ) $|I_\sigma \backslash I_{\sigma-1}| \leqq \aleph_0$, if $\sigma \geqq 1$ is not a limit ordinal;
(δ) $H_\sigma = \{a \in F \mid \theta a \in \bigoplus_{i\in I_\sigma} \theta_i M_i\}$ is equal to F_σ.

For $\sigma = 0$, there is nothing to prove. So let $\sigma \geqq 1$ and assume the subgroups $F_\rho \leqq F$ and the subsets $I_\rho \subseteq I$ have been constructed for all ordinals $\rho < \sigma$. If σ is a limit ordinal, F_σ and I_σ will be defined as required by (β). Then (α) and (δ) are obvious. If $\sigma - 1$ exists and if still $F_{\sigma-1} < F$, then select any $b \in F\backslash F_{\sigma-1}$ and consider $B_1 = \langle F_{\sigma-1}, b\rangle$, $J_0 = I_{\sigma-1}$. Let J_1 be the smallest subset of I containing J_0 such that $B_1 \leqq \bigoplus_{i\in J_1} M_i$. If $B_1 \subseteq \cdots \subseteq B_k$ and $J_1 \subseteq \cdots \subseteq J_k$ have been defined, let J_{k+1} be the smallest subset of I containing J_k and satisfying

$$B_{k+1} = \{a \in F \mid \theta a \in \bigoplus_{i\in J_k} \theta_i M_i\} \leqq \bigoplus_{i\in J_{k+1}} M_i \qquad (k \geqq 1).$$

Defining

$$I_\sigma = \bigcup_{k=1}^{\infty} J_k \qquad \text{and} \qquad F_\sigma = \bigcup_{k=1}^{\infty} B_k,$$

we want to verify (α), (γ), and (δ) for σ. To establish (γ), it suffices to show that $|J_{k+1}\backslash J_0| \leqq \aleph_0$ for every k. This is obvious for $k = 1$. If $\eta\colon M \to \bigoplus_{i\in J_{k+1}\backslash J_0} M_i$ is the obvious projection, then manifestly $F_{\sigma-1} \leqq \operatorname{Ker}\eta \cap B_{k+1}$. If $x \in B_{k+1}$ satisfies $mx \in \operatorname{Ker}\eta \cap B_{k+1}$ for an integer $m > 0$, then $m\eta x = 0$ and so $\theta x \in \bigoplus_{i\in I_{\sigma-1}} \theta_i M_i$. By ($\delta$), $x \in H_{\sigma-1} = F_{\sigma-1}$. We conclude that $F_{\sigma-1} = \operatorname{Ker}\eta \cap B_{k+1}$ and ηB_{k+1} is torsion-free. From $|J_k\backslash J_0| \leqq \aleph_0$ and the definition of the M_i, it follows that $|\eta B_{k+1}| \leqq \aleph_0$ too, whence the countability of $J_{k+1}\backslash J_0$ is evident. To prove (α), let $x \in F \cap (\bigoplus_{i\in I_\sigma} M_i)$. There is a k such that $x \in F \cap (\bigoplus_{i\in J_{k+1}} M_i)$, i.e., $\theta x \in \bigoplus_{i\in J_k} \theta_i M_i$ and so $x \in B_{k+1} \leqq F_\sigma$. Finally, ($\delta$) follows similarly: if $x \in H_\sigma$, then $\theta x \in \bigoplus_{i\in J_k} \theta_i M_i$ for some k, and so $x \in B_{k+1} \leqq F_\sigma$. This completes the proof of (C).

(D) Keeping the notations of (C), we show that

$$F_{\sigma+1} = F_\sigma \oplus A_{\sigma+1}$$

for some free group $A_{\sigma+1}$. Let $\xi\colon M \to \bigoplus_{i\in I_{\sigma+1}\backslash I_\sigma} M_i$ be the obvious projection; then evidently,

$$F_{\sigma+1} \cap \operatorname{Ker}\xi = F_{\sigma+1} \cap \left(\bigoplus_{i\in I_\sigma} M_i\right) = F \cap \left(\bigoplus_{i\in I_\sigma} M_i\right) = F_\sigma.$$

If for some $x \in F_{\sigma+1}$, ξx is of finite order, then $x \in H_\sigma = F_\sigma$, and so $\xi F_{\sigma+1}$ is a torsion-free subgroup of $\bigoplus_{i\in I_{\sigma+1}\backslash I_\sigma} M_i$. The torsion-free rank of the last group is at most $\aleph_0$, so $|\xi F_{\sigma+1}| \leqq \aleph_0$, and (B) implies $\xi F_{\sigma+1}$ is free. Hence F_σ is a summand of $F_{\sigma+1}$, $F_{\sigma+1} = F_\sigma \oplus A_{\sigma+1}$ with $A_{\sigma+1} \cong \xi F_{\sigma+1}$ free.

Now, if F is any torsion-free subgroup of the group M in (101.4), then F is the union of the F_σ, so it is generated by the $A_{\sigma+1}$ with $\sigma + 1 < \tau$. These groups generate their direct sum in F, hence F is free.□

Resuming *the proof of Theorem* 101.2, choose an index set I of the given cardinality $\mathfrak{m}$, and for each $i \in I$, set $M_i \cong N$, where N is as in (101.3). By (101.4), $M = \bigoplus_{i \in I} M_i$ is as desired.□

It should be noticed that, surprisingly, there do exist torsion-free groups G, which are not free, but $\mathrm{Ext}(G, T) = 0$ for all p-groups T, for all primes p [cf. Ex. 3]. Moreover, for every nonempty proper subset Π of primes, there exists a torsion-free group G that is not free, but $\mathrm{Ext}(G, T) = 0$ for all torsion groups T which have trivial p-components for primes $p \notin \Pi$ (Griffith [7]).

Exercises

1. (Sąsiada [2]) Let T be a torsion group, B a basic subgroup of T, and let G be a torsion-free group.
   (a) Show that $\mathrm{Ext}(G, T) = 0$ if and only if $\mathrm{Ext}(G, B) = 0$.
   (b) Conclude that in investigating pairs T, G for which $\mathrm{Ext}(G, T) = 0$, one can restrict himself to direct sums T of cyclic p-groups.
2. A group G is a Baer group exactly if $\mathrm{Ext}(G, T) = 0$ for all direct sums T of finite cyclic groups such that $|T| \leqq |G|$.
3. (Chase [4]) There is a torsion-free group G which is not free, but $\mathrm{Ext}(G, T) = 0$ for all primary groups T. [*Hint*: in **95**, Ex. 11 take all X_σ to be the group of rationals with square-free denominators and from **95**, Ex. 12 conclude that G is not free; show that $Q_p \otimes G$ is a free $\mathbf{Q}_p$-module and use the epimorphism $\mathrm{Ext}(Q_p \otimes G, T) \to \mathrm{Ext}(G, T)$.]
4. (Griffith [7]) If G satisfies $\mathrm{Ext}(G, T) = 0$ for all primary groups T, then $\mathrm{Ext}(G, T)$ is torsion-free for all torsion groups T. [*Hint*: if T has no p-component, then multiplication by p is an automorphism on T, and hence on $\mathrm{Ext}(*, T)$.]
5. Let A be a mixed group with torsion part T such that A/T is a separable torsion-free group. A need not split even if all of its subgroups split which contain T and have finite torsion-free rank.
6. (Baer [4]) Let T and G be torsion and torsion-free groups, respectively, such that $\mathrm{Ext}(G, T) = 0$. Then:
   (a) if $p_1, \cdots, p_i, \cdots$ is an infinite set of different primes for which $p_i T < T$, then G contains no pure subgroup S of finite rank such that G/S has elements $\neq 0$ divisible by all p_i;
   (b) if for some prime p, the reduced part of the p-component of T is unbounded, then G contains no pure subgroup S of finite rank such that G/S has elements $\neq 0$ divisible by all powers of p.

7. (Baer [4]) If T is a torsion group and G a countable torsion-free group, then conditions (a) and (b) in Ex. 6 are necessary and sufficient for $\operatorname{Ext}(G, T) = 0$. [*Hint*: write G as the union of an ascending chain of pure subgroups G_n of finite rank n, and use induction.]
8. (Žuravskiĭ [1]) Let A be a mixed group with separable torsion part T such that A/T is p-divisible for every prime p, for which T has a non-trivial p-component. If for every $a \in A\backslash T$, the coset $a + T$ contains an element b such that $h_p(b) = h_p(a + T)$ for every prime p, then A is splitting. [*Hint*: these b form a complement.]
9. (Žuravskiĭ [1]) Let A be a mixed group of torsion-free rank 1 such that the p-component of its torsion part is separable for every prime p for which A/T is p-divisible. If the cosets $a + T$ satisfy the condition of Ex. 8, then A splits. [*Hint*: starting from a b, construct a complement as the union of infinite cyclic groups.]
10. (Lyapin [2]) Let A be a mixed group such that its torsion part T is separable and A/T is completely decomposable. If the cosets $a + T$ satisfy the condition of Ex. 8, then A is splitting.

102. QUASI-SPLITTING MIXED GROUPS

We turn our attention to a class of mixed groups which is close to being splitting in the following sense. Call the group A *quasi-splitting* if it has a splitting subgroup B such that

$$nA \leqq B \leqq A \qquad \text{for some integer} \quad n > 0.$$

If this B splits, i.e., $B = S \oplus G$, where S is torsion and G is torsion-free, then $B + T = T \oplus G$ also splits for the torsion part T of A. Consequently, in the definition of quasi-splitting, B may be assumed to contain T.

The following is an example for a quasi-splitting mixed group that does not split.

Example. Put $T = \oplus_{n=1}^{\infty} Z(p^n)$, and let G be the p-adic closure of $\oplus_{\aleph_0} J_p$. Then G is a torsion-free algebraically compact group such that $G/pG \cong T/pT$, so there is an epimorphism $\eta\colon G \to T/pT$ whose kernel is pG. Let the group A be defined by the diagram

$$\begin{tikzcd}
0 \arrow[r] & pT \arrow[r] & T \arrow[r] & T/pT \arrow[r] & 0 \\
0 \arrow[r] & pT \arrow[r] \arrow[u, equal] & A \arrow[r] \arrow[u] & G \arrow[r] \arrow[u, "\eta"'] \arrow[ul, dashed, "\xi"'] & 0
\end{tikzcd}$$

with exact rows and commutative squares. Here A can be thought of as

$$A = \{(g, h) \mid g \in G, h \in T \quad \text{with} \quad \eta g = h + pT\}.$$

It is then clear that $pA \leq pG \oplus pT \leq A$, thus A is quasi-splitting. If A were splitting, then there would exist a $\xi\colon G \to T$ such that $\eta g = \xi g + pT$ for all $g \in G$. But by (54.4), ξG must be of bounded order, and so the image of ξG in T/pT would be finite, which is absurd.

Before entering into the discussion of quasi-splitting groups, we prove a lemma in a more general situation than needed; an even more general case will appear in Ex. 5.

Let C be a group and n a positive integer. The multiplication by n in C factors as $C \xrightarrow{\mu} nC \xrightarrow{\iota} C$, where $\mu\colon c \mapsto nc$ and ι is the inclusion map. Given the top exact sequence in the following diagram, what has been said in **50** implies that there is a commutative diagram with exact rows as follows:

(1)
$$\begin{tikzcd}
E\colon & 0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\
E\iota\colon & 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[u, equal] & nB + A \arrow[r, "\beta"] \arrow[u] & nC \arrow[r] \arrow[u, "\iota"'] & 0 \\
E\iota\mu\colon & 0 \arrow[r] & A \arrow[r] \arrow[u, equal] & B' \arrow[r, "\beta'"] \arrow[u] & C \arrow[r] \arrow[u, "\mu"'] & 0
\end{tikzcd}$$

Lemma 102.1 (C. Walker [1]). *If $E\iota$ splits, then E represents an element of* $\mathrm{Ext}(C, A)[n]$. *If $C[n] = 0$, then the converse is also true.*

Notice that $E\iota\mu = nE$ [cf. (52.1)]. Now, if $E\iota$ splits, so does nE, and therefore E belongs to $\mathrm{Ext}(C, A)[n]$. Conversely, E belongs to $\mathrm{Ext}(C, A)[n]$ means that E is contained in the kernel of the endomorphism of $\mathrm{Ext}(C, A)$ induced by the multiplication by n in C. If $C[n] = 0$, we refer to (53.2)(ii) to conclude that this is equivalent to the splitting of $E\iota$. $\square$

The main characterization of quasi-splitting groups can now easily be proved.

Theorem 102.2 (C. Walker [1]). *A mixed group A with torsion part T is quasi-splitting exactly if the exact sequence*

(2)
$$0 \to T \to A \to A/T \to 0$$

represents an element of finite order in $\mathrm{Ext}(A/T, T)$.

If (2) represents an element of finite order n, then the second part of (102.1) implies that the exact sequence

(3)
$$0 \to T \to nA + T \to (nA + T)/T \to 0$$

is splitting. Since $nA \leqq nA + T \leqq A$, this means A is quasi-splitting.

Conversely, let A be quasi-splitting, say, some B with $T \leqq B$ and $nA \leqq B \leqq A$ is splitting for some integer n. Then T is a summand of $nA + T$ and (3) is a splitting sequence. From (102.1) it follows that (2) represents an element in $\operatorname{Ext}(A/T, T)[n]$.□

We can now derive a result showing that countable quasi-splitting groups necessarily split.

Proposition 102.3 (C. Walker [1]). *If A is a quasi-splitting mixed group with countable A/T, then A is splitting.*

Set $G = A/T$ and consider the exact sequence $0 \to G \xrightarrow{\mu} G \to G/pG \to 0$, where p is a prime and $\mu: g \mapsto pg$ for $g \in G$. In the induced exact sequence

$$\operatorname{Hom}(G, T) \xrightarrow{\delta} \operatorname{Ext}(G/pG, T) \to \operatorname{Ext}(G, T) \xrightarrow{\mu^*} \operatorname{Ext}(G, T) \to 0,$$

the map μ^* is again multiplication by p. If we can show that $\operatorname{Ker} \mu^* = 0$, then $\operatorname{Ext}(G, T)[p] = 0$, i.e., $\operatorname{Ext}(G, T)$ is torsion-free, and our claim will follow at once from (102.2). It suffices to show that δ is epic; the proof of this is based on Baer's paper [11].

Since G is countable, it is the union of a finite or infinite chain $G_1 < \cdots < G_n < \cdots$ of pure subgroups such that $r(G_n) = n$ for every n. Let B_n be a p-basic subgroup of G_n such that $B_1 \leqq \cdots \leqq B_n \leqq \cdots$. Given a p-group S and $\eta: G \to S/pS$, we want to show that η lifts to a $\xi: G \to S$. Since $pG \leqq \operatorname{Ker} \eta$ and $G_n/pG_n \cong B_n/pB_n$, the restriction $\eta_n = \eta | B_n$ lifts to a homomorphism $B_n \to S$ whose image is finite, so it might be regarded as a homomorphism $\xi_n: G_n \to S$. Moreover, B_n being a summand of B_{n+1}, ξ_{n+1} can be chosen so as to extend ξ_n. Consequently, there is a unique $\xi: G \to S$ such that $\xi | G_n = \xi_n$ for all n, and ξ is clearly as wanted.

We can now finish the proof that δ is epic. Let an element of $\operatorname{Ext}(G/pG, T)$ be represented by the exact sequence in the top row of the diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] & S \arrow[r] & G/pG \arrow[r] & 0 \\
0 \arrow[r] & G \arrow[r, "\mu"] \arrow[u, "\psi"] & G \arrow[r, "\eta"] \arrow[u, dashed, "\xi"] & G/pG \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Since S is a torsion group, what has been proved in the preceding paragraph implies the existence of a ξ making the right square commute. Hence there is a ψ making the left square commute. This ψ is mapped by δ upon the given element of $\operatorname{Ext}(G/pG, T)$.□

Exercises

1. For every integer $n > 0$, A and nA are simultaneously quasi-splitting.
2. Let A and A' contain isomorphic subgroups $B \cong B'$ such that $nA \leqq B \leqq A$ and $mA' \leqq B' \leqq A'$ for some positive $m, n \in \mathbb{Z}$. Then A is quasi-splitting if and only if A' is.

3. Show that a mixed group A is quasi-splitting if and only if it is a pullback in a diagram

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, dashed] & G \arrow[d] \\ T \arrow[r] & B \end{tikzcd}$$

where T is a torsion group, G is torsion-free, and B is bounded.

4. Let A be a mixed group with torsion part T such that $A/(pA + T)$ is finite for each prime p. If A is quasi-splitting, then A splits. [*Hint*: proof of (102.3).]

5. (C. Walker [1]) An exact sequence $E\colon 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ represents an element of finite order in $\operatorname{Ext}(C, A)$ if and only if there exists an integer $n > 0$ for which

$$(4) \qquad 0 \to A/(A[n]) \xrightarrow{\bar{\alpha}} (\alpha A + nB)/(\alpha A[n]) \xrightarrow{\bar{\beta}} nC \to 0$$

is splitting exact. Here $\bar{\alpha}\colon a + A[n] \mapsto \alpha a + \alpha A[n]$ and $\bar{\beta}\colon b + \alpha A[n] \mapsto \beta b$. [*Hint*: if the bottom row in (1) splits, use a right inverse to β to obtain a right inverse to $\bar{\beta}$; if (4) splits, then E belongs to $\operatorname{Ext}(C, A)[n^2]$.]

6. (C. Walker [1]) Prove the second part of (102.1) by replacing the hypothesis $C[n] = 0$ by $A[n] = 0$.

7. Show that A need not split if it contains a splitting subgroup B such that A/B is countable and bounded.

8. (Griffith [7]) Let G be a torsion-free group such that every extension of every torsion group by G is quasi-splitting. Show that G must be free. [*Hint*: argue as in (101.1), now $\alpha\psi = n\phi$ for some integer $n > 0$.]

9. (Griffith [7]) Let G satisfy $\operatorname{Ext}(G, T) = 0$ for all primary groups T. Show that every quasi-splitting extension of a torsion group by G is necessarily splitting. [*Hint*: **101**, Ex. 4.]

103. HEIGHT-MATRICES

Having discussed the splitting and quasi-splitting mixed groups, we have come to the structure problem of mixed groups in general. As indicated, a structure theorem can be hoped for only concerning mixed groups whose torsion and torsion-free parts can be characterized by invariants in a satisfactory way. So far not much is known except for the case of groups of torsion-free rank 1. We devote this section and the next one to these groups.

In investigating the structure of p-groups and torsion-free groups, we have found it of extraordinary importance to have an invariant associated with the elements, describing their behavior relative to divisibility by integers: the indicator [in **65**] and the characteristic [in **85**], respectively. We combine the information given by them into the height-matrix $\mathbb{H}(a)$ which will now be defined for elements a of arbitrary groups A [cf. Rotman [2], Megibben [6], Myshkin [5]].

Let $p_1, \cdots, p_n, \cdots$ be the sequence of primes in the order of magnitude. Given an element a in the group A, $h_p^*(a)$ will denote the generalized p-height of a in A, as defined in **37**, i.e.,

$$h_p^*(a) = \sigma \qquad \text{if} \quad a \in p^\sigma A \backslash p^{\sigma+1} A \quad \text{for the ordinal} \quad \sigma.$$

In case $a \in p^\tau A = p^{\tau+1} A$, we set as usual $h_p^*(a) = \infty$ and consider ∞ larger than every occurring ordinal. With the element a, we associate the *height-matrix* $\mathbb{H}(a)$, an infinite matrix with ordinal numbers for entries, as follows:

$$\mathbb{H}(a) = \begin{bmatrix} h_{p_1}^*(a) & h_{p_1}^*(p_1 a) & \cdots & h_{p_1}^*(p_1^k a) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_{p_n}^*(a) & h_{p_n}^*(p_n a) & \cdots & h_{p_n}^*(p_n^k a) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = [\sigma_{nk}].$$

Thus the element σ_{nk} in the (n, k)-position of $\mathbb{H}(a)$ records the generalized p_n-height of $p_n^k a$, for all $n = 1, 2, \cdots$ and $k = 0, 1, 2, \cdots$. The nth row of $\mathbb{H}(a)$ will be called the *p_n-indicator* of a.

The reader will immediately recognize that for p_n-groups A, the nth row of $\mathbb{H}(a)$ is precisely the indicator of a, while the other rows are constantly ∞ and thus give no additional information about the element a. On the other hand, in torsion-free groups A, we always have $h_p(pa) = h_p(a) + 1$, so in this case the first column of $\mathbb{H}(a)$ gives already the same amount of information as the entire matrix $\mathbb{H}(a)$.

The following observations are immediate consequences of the definition:

(a) $\mathbb{H}(-a) = \mathbb{H}(a)$ for all $a \in A$;

(b) $\mathbb{H}(p_n a)$ is obtained from $\mathbb{H}(a)$ by replacing the nth row $\sigma_{n0}, \cdots, \sigma_{nk}, \cdots$ of $\mathbb{H}(a)$ by $\sigma_{n1}, \cdots, \sigma_{n,k+1}, \cdots$;

(c) $p_1^{k_1} \cdots p_n^{k_n} a$ is divisible by the integer $m = p_1^{l_1} \cdots p_n^{l_n}$ exactly if $l_i \leqq \sigma_{ik_i}$ for $i = 1, \cdots, n$;

(d) almost all entries of $\mathbb{H}(a)$ are ∞ whenever a is of finite order;

(e) every entry of $\mathbb{H}(a)$ is ∞ exactly if a belongs to the divisible part of A;

(f) a belongs to the σth Ulm subgroup A^σ of A if and only if $\sigma_{n0} \geqq \omega\sigma$ for all n;

(g) if $A = B \oplus C$ and $a = b + c$ ($b \in B$, $c \in C$), then $\mathbb{H}(a) = \min(\mathbb{H}(b), \mathbb{H}(c))$, where "min" means pointwise minimum.

Example. Let T be an unbounded separable p-group and B its basic subgroup. We write $B = \bigoplus_{i=1}^{\infty} B_{m_i}$, where $B_{m_i} \neq 0$ is a direct sum of cyclic groups of the same order p^{m_i} and $m_1 < m_2 < \cdots$. We consider T as being embedded [as a pure subgroup] in the p-adic completion $\hat{B}$ of B. Let $(l_0, \cdots, l_k, \cdots)$ be an increasing sequence of nonnegative integers. Then *$\hat{B}$ contains an element a of infinite order such that $h_p^*(p^k a) = l_k$ for every k exactly if $l_k + 1 < l_{k+1}$ implies that $l_k + 1$ is one of the m_i*. For the necessity we can argue as in (65.3) [cf. also (ii) *infra*]. To prove sufficiency, we may clearly restrict ourselves to the case when every m_i is equal to some $l_k + 1$. Let $b_i \in B_{m_i}$ be of height 0, for every i. Comparing the l_ks with the m_is, we can write

$$l_0 < \cdots < l_{k_1 - 1} < m_1 = l_{k_1 - 1} + 1 \leqq l_{k_1} < \cdots < l_{k_i - 1}$$
$$< m_i = l_{k_i - 1} + 1 \leqq l_{k_i} < \cdots,$$

where $k_1 < \cdots < k_i < \cdots$ and between m_i and m_{i+1} the l_ks increase exactly by 1. Thus $l_0 \leqq l_{k_1} - k_1 \leqq \cdots \leqq l_{k_i} - k_i \leqq \cdots$, and hence the element

$$a = (p^{l_0} b_1, p^{l_{k_1} - k_1} b_2, \cdots, p^{l_{k_i} - k_i} b_{i+1}, \cdots) \in \hat{B}$$

has height l_0. Since $m_j \leqq l_{k_j} - k_j + k_i$ for $j \leqq i$, it is clear that the first i coordinates of $p^{k_i} a$ vanish and $h_p^*(p^{k_i} a) = l_{k_i}$. It follows readily that $h_p^*(p^k a) = l_k$ for every k.

Let A be an arbitrary group. If a, b are elements of infinite order in A and if they satisfy $ra = sb$ for some integers $r, s \neq 0$, then it is easily seen from (b) that the nth rows of $\mathbb{H}(a) = [\sigma_{nk}]$ and $\mathbb{H}(b) = [\rho_{nk}]$ can be different only if $p_n | rs$, and for such a p_n there must exist integers $l, m \geqq 0$ satisfying

$$\sigma_{n, k+l} = \rho_{n, k+m} \qquad \text{for all} \quad k. \tag{1}$$

In view of this, one defines two $\omega \times \omega$ matrices $[\sigma_{nk}]$ and $[\rho_{nk}]$ *equivalent* if, for almost all n, the nth rows of the two matrices are identical and for every remaining n there exist nonnegative integers l, m (depending on n) such that (1) holds.

Note that the height-matrices of elements in the same coset mod the torsion part of A are equivalent.

If A has torsion-free rank 1, then any two elements $a, b \in A$ of infinite order are dependent, and so their height-matrices $\mathbb{H}(a)$ and $\mathbb{H}(b)$ are equivalent in the sense defined above. Consequently, there is a uniquely determined equivalence class of matrices associated with A which we shall denote by $\mathbb{H}(A)$.

Our next purpose is to find out what matrices can occur as height-matrices for elements of infinite order in mixed groups with a given reduced torsion group T. Restriction to groups A of torsion-free rank 1 does not mean any loss of generality. In fact, let G be a mixed group with torsion part T and let $a \in G$ be of infinite order; then there is a unique smallest pure subgroup A of G containing both T and a, namely $A/T = \langle a + T \rangle_*$ in G/T. The height-matrices of elements in A are the same both in G and in A, as follows at once from the following simple observation.

Lemma 103.1 (Megibben [6]). *If A is a subgroup of G with G/A torsion-free, then*

(2) $$p^\sigma A = A \cap p^\sigma G \qquad \textit{for all ordinals} \quad \sigma \quad \textit{and primes} \quad p.$$

If (2) holds, we say that A is an *isotype* subgroup of G [cf. **80**], while if it holds only for some prime p, then A is said to be *p-isotype* in G.

To prove that A is isotype in G whenever G/A is torsion-free, we use transfinite induction on σ. In view of the definition of $p^\sigma G$ for limit ordinals σ, it suffices to show that if (2) holds for σ, then it holds for $\sigma + 1$, too. If $a \in A \cap p^{\sigma+1}G$ and if $g \in p^\sigma G$ satisfies $pg = a$, then $g \in A$, since G/A is torsion-free. Therefore $g \in A \cap p^\sigma G = p^\sigma A$ and $a \in p^{\sigma+1}A$.□

Let A be a mixed group of torsion-free rank 1. Let T_n denote its p_n-component and a an element of infinite order in A. From the definition it is evident that for the height-matrix $\mathbb{H}(a) = [\sigma_{nk}]$ we have:

(i) *for every n,*

$$\sigma_{n0} < \sigma_{n1} < \cdots < \sigma_{nk} < \cdots,$$

equality can occur only when for some k on, $\sigma_{nk} = \sigma_{n,k+1} = \cdots = \infty$.

In accordance with **65**, we say that there is a *gap* between σ_{nk} and $\sigma_{n,k+1}$ if $\sigma_{nk} + 1 < \sigma_{n,k+1}$. The proof of (65.3) applies to verify:

(ii) *if there is a gap between σ_{nk} and $\sigma_{n,k+1}$, then the σ_{nk}th Ulm–Kaplansky invariant of T_n is different from* 0.

From (37.2) we readily deduce that the p_n-length of A cannot exceed $\lambda_n + \omega$ where λ_n is the p_n-length of T_n. It cannot exceed λ_n if $p_n^{\lambda_n}A$ is p_n-divisible. Consequently, we have for every n:

(iii) *if $\sigma_{nk} \neq \infty$ for all k, then*

$$\sigma_{nk} < \lambda_n + \omega \qquad \textit{for every} \quad k;$$

while if $\sigma_{nl} = \infty$ for some l, then $\sigma_{nk} < \lambda_n$ for all $\sigma_{nk} \neq \infty$.

There is one more condition which $\mathbb{H}(a)$ must satisfy. To derive this, we first prove an easy lemma.

Lemma 103.2. *Let A be a reduced nonsplitting mixed group of torsion-free rank* 1, *and let T be its torsion part. There is an isomorphism ϕ of A with a subgroup of* $\mathrm{Ext}(Q/Z, T)$ *such that $\phi|T$ is the canonical embedding $T \to$* $\mathrm{Ext}(Q/Z, T)$.

Starting with the natural map $T \to \mathrm{Ext}(Q/Z, T)$ [cf. (55.1)], (58.2) enables us to extend this map to a $\phi\colon A \to \mathrm{Ext}(Q/Z, T)$. Clearly, Ker ϕ is torsion-free, hence if Ker $\phi \neq 0$, then Im ϕ is torsion. This means that ϕ maps A onto T, and T is a summand of A, in contradiction to hypothesis. Therefore, ϕ is monic. □

For every ordinal σ, the σth Ulm subgroup $\mathrm{Ext}(Q/Z, T)^\sigma$ of $\mathrm{Ext}(Q/Z, T)$ is isomorphic to $\mathrm{Ext}(Q/Z, T^\sigma) \oplus \mathrm{Hom}(Q/Z, H_\sigma)$, where H_σ is as defined in (56.5). Suppose $p_n^m T_n^\sigma = 0$ for some n; then $p_n^m\ \mathrm{Ext}(Q/Z, T^\sigma)$ will be p_n-divisible. Let $\sigma \geqq 1$. If $\sigma - 1$ exists and $T_n^{\sigma-1}$ is torsion-complete, then H_σ has zero p_n-component, so $\mathrm{Hom}(Q/Z, H_\sigma)$ will also be p_n-divisible. If σ is a limit ordinal and T_n/T_n^σ is the torsion part of $L_{n\sigma} = \varprojlim T_n/T_n^\rho$ $(\rho \to \sigma)$, then the same situation occurs. This leads to the following necessary condition:

(iv) *if for some n, there is an integer $m \geqq 0$ such that $p_n^m T_n^\sigma = 0$, and if $T_n^{\sigma-1}$ is torsion-complete or T_n/T_n^σ is the torsion part of $\varprojlim T_n/T_n^\rho$ $(\rho \to \sigma)$ [according as σ is an isolated or a limit ordinal], then $\sigma_{nk} \neq \infty$ implies*

$$\sigma_{nk} < \max(\lambda_n, \omega).$$

The last inequality includes ω to take care of the case when λ_n is a finite ordinal [and T_n is a summand of A].

Our next goal is to verify that conditions (i)–(iv) yield sufficient amount of information about the height-matrix.

Theorem 103.3. *Let T be a reduced torsion group and $\mathbb{M} = [\sigma_{nk}]$ an $\omega \times \omega$-matrix whose entries are ordinals or symbols ∞. There exists a mixed group A of torsion-free rank 1 whose torsion part is T and which contains an element a of infinite order with*

$$\mathbb{H}(a) = \mathbb{M}$$

if and only if $\mathbb{M}$ satisfies (i)–(iv).

First, we localize the proof of sufficiency by showing that it will be enough to establish, for each n, the existence of a mixed group A_n of torsion-free rank 1 having the p_n-component T_n of T as its torsion part and containing an $a_n \in A_n$ of infinite order such that the nth row of $\mathbb{H}(a_n)$ is the same as the nth row of $\mathbb{M}$ and all other entries of $\mathbb{H}(a_n)$ are ∞. Taking the existence of such A_ns for granted, we can form their direct product $G = \prod_n A_n$. The torsion part of G is precisely T. For every ordinal ρ and integer n, we have

$$p_n^\rho G = p_n^\rho A_n \oplus \prod_{i \neq n} A_i.$$

Hence, it is readily checked that $\mathbb{M}$ will be the height-matrix of $a = (a_1, \cdots, a_n, \cdots)$ in G. If we define A as the unique pure subgroup of G containing both T and a, which is of torsion-free rank 1, then in view of (103.1) we can claim that the height-matrices of a in G and in A are identical.

Therefore, we must be concerned with one prime only; so for the rest of the proof suppose that T is a reduced p-group of p-length λ and $(\sigma_0, \cdots, \sigma_k, \cdots)$ is a sequence of ordinals satisfying (i)–(iv). We wish to construct a mixed group A of torsion-free rank 1 whose torsion part is T such that A contains an element a of infinite order with $(\sigma_0, \cdots, \sigma_k, \cdots)$ as p-indicator and satisfies $qA = A$ for all primes $q \neq p$. We distinguish several cases.

(a) If there is a k such that $\sigma_k = \infty$, then let k be the smallest such integer. Because of (i), we have

$$\sigma_0 < \sigma_1 < \cdots < \sigma_{k-1} < \sigma_k = \infty = \sigma_{k+1} = \cdots.$$

From (ii) and (65.3) we infer the existence of an $x \in T$ whose indicator is $(\sigma_0, \sigma_1, \cdots, \sigma_{k-1}, \infty)$. Putting $A = T \oplus Q$ and selecting any $y \neq 0$ in Q, $a = x + y$ is as desired.

From now on, we may assume that $\sigma_0, \cdots, \sigma_k, \cdots$ are all different from ∞.

(b) If there is a k such that $\sigma_k \geqq \lambda$, then again let k be the smallest such integer. In this case (ii) implies that $\sigma_{k+j} = \sigma_k + j$ for $j = 0, 1, \cdots$. Thus if $\lambda < \omega$, then $A = T \oplus Z$ will contain an element with the given indicator. If $\lambda \geqq \omega$, then consider the group $C = \operatorname{Ext}(Z(p^\infty), T)$. Condition (iv) guarantees, in view of (56.5), that C has a last Ulm subgroup and this contains an element y with the indicator $(\lambda, \lambda + 1, \cdots)$. Now (iii) implies that the σ_j $(j = k, k+1, \cdots)$ are $< \lambda + \omega$, so by making use of (65.3), it is easy to find an element $a = x + p^r y$ [for suitable $x \in T$ and $r \geqq k$] with the required property.

(c) Henceforth, $\sigma_k < \lambda$ will be assumed. Suppose there are but a finite number of gaps in the sequence $\sigma_0, \sigma_1, \cdots$, and say, the last gap is between σ_{k-1} and σ_k. In view of (ii) and (65.3), there is an $x \in T$ whose indicator is $(\sigma_0, \cdots, \sigma_{k-1}, \infty)$. There is an ordinal μ such that $\omega\mu \leqq \sigma_j < \omega(\mu + 1)$ for $j \geqq k$, and using Example (b) shows that we can find an element $g \in C^\mu$ [of infinite order] with indicator $(\sigma_k, \sigma_k + 1, \sigma_k + 2, \cdots)$. Choosing $y \in C$ satisfying $p^k y = g$ with $h_p^*(y) + k \leqq \sigma_k$, $a = x + y$ will have the given indicator.

(d) If there are infinitely many gaps in the sequence $\sigma_0, \sigma_1, \cdots$, then let these occur before $\sigma_{k_1}, \sigma_{k_2}, \cdots$, where $0 < k_1 < k_2 < \cdots$. It is readily seen that $\rho = \sup \sigma_k$ is a limit ordinal. Invoking (65.3) again, by (ii) we can find, for every i, an element $x_i \in T$ whose indicator is $(\sigma_0, \cdots, \sigma_{k_i - 1}, \infty)$; moreover, these x_i may be assumed to satisfy $x_i - x_{i+1} \in p^{\sigma_{k_i}} T$. Therefore, there is an element b in $E = \varprojlim T/p^\sigma T$ (when $\sigma \to \rho$) which is mapped upon x_i by the canonical map $E \to T/p^{\sigma_i} T$, for every i. Hence the indicator of b is $\leqq (\sigma_0, \cdots, \sigma_k, \cdots)$, and from (37.1) by a routine transfinite induction the reverse inequality follows. It is easily seen that there is a subgroup H of E such that $b \in H$, $T^* = T/p^\rho T$ is the torsion part of H and $H/T^* \cong Q$. Define G by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\psi"] & G \arrow[r] \arrow[d] & H/T^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^* \arrow[r] & H \arrow[r] & H/T^* \arrow[r] & 0
\end{tikzcd}$$

where $\psi: T \to T^* = T/p^\rho T$ is the canonical map and the rows are exact. If $a \in G$ maps upon $b \in H$ under $G \to H$, then a will have the same indicator in G as b has in H.

This completes the proof of the theorem.□

EXERCISES

1. Prove the triangle inequality for height-matrices:

$$\mathbb{H}(b+c) \geqq \min(\mathbb{H}(b), \mathbb{H}(c)).$$

2. Investigate the behavior of the height-matrices under passage to subgroups, pure or isotype subgroups, epimorphic images.
3. Given the height-matrix $\mathbb{H}(a)$, find the characteristic $\chi(a+T)$.
4. Define $A = \langle a_0, a_1, \cdots, a_n, \cdots \rangle$, where $p^{k_n}a_0 = p^n a_n$ with $0 \leqq k_1 \leqq k_2 \leqq \cdots$ and $k_n < n$. What is the height-matrix of a_0?
5. Show that any matrix $[\sigma_{nk}]$ of ordinals is the height-matrix of some element of infinite order in a suitable mixed group if and only if (i) holds.
6. (Megibben [6], Myshkin [5]) If T is countable, then condition (iv) can be dropped from (103.3).
7. Let T be an unbounded, countable reduced p-group. Show that there are continuously many nonisomorphic extensions of T by Q.
8. (Stratton [1]) Show that (103.2) is no longer true if A has torsion-free rank 2, even if A is assumed not to have any torsion-free summand $\neq 0$. [*Hint*: $A = \langle B \oplus C, q^{-1}(b+c) \rangle$ such that B is torsion-free of rank 1, C is mixed of torsion-free rank 1, nonsplitting, and has a p-group T for torsion part, B and C/T are p-divisible, and A/T is indecomposable.]
9. Given a torsion group T, describe the height-matrices of elements in $\operatorname{Ext}(Q/Z, T)$.

104. MIXED GROUPS OF TORSION-FREE RANK 1

In this section we prove a structure theorem on mixed groups A of torsion-free rank 1. We shall show that if its torsion subgroup T is countable, or more generally, a direct sum of countable groups, then the invariants of T along with the equivalence class $\mathbb{H}(A)$, form a complete system of invariants for A. Our procedure is similar to the one used in **77** and in **83**.

In accordance with the definition "proper" in **77** and **86**, an element $a \in A$ is called *p-proper with respect to a subgroup G of A* if it has maximal p-height in the coset $a + G$. Clearly, this amounts to the equality of $h_p^*(a)$ and $h_p^*(a+G)$ [the first taken in A, the second in A/G].

The proof of the main theorem is based on the following two lemmas.

Lemma 104.1 (Rotman [2], Megibben [6]). *Let G be a finitely generated subgroup of the reduced mixed group A of torsion-free rank 1. For every $a \in A \backslash G$ with $p^m a \in G$ $(m \geqq 1)$, $a + G$ contains an element p-proper with respect to G.*

We need to prove this only for $G = \langle g \rangle \oplus E$, where $o(g) = \infty$ and E is a finite p-group. We can write $p^t a = p^r sg$ with $(p, s) = 1$ and $t \geqq m$.

By way of contradiction, suppose there is an infinite sequence $l_n g + e_n$ $(l_n \in \mathbf{Z}, e_n \in E)$ in G such that

$$h_p^*(a) < h_p^*(a + l_n g + e_n) < h_p^*(a + l_{n+1} g + e_{n+1}) \qquad \text{for} \quad n = 1, 2, \cdots.$$

Without loss of generality, $e_1 = \cdots = e_n = \cdots$ may be assumed, and since $a + e_1$ can play the role of a, only sequences of the form $l_n g$ $(n = 1, 2, \cdots)$ must be dealt with. By the triangle inequality, $h_p^*(l_n g) = h_p^*(a)$ for all n, hence there is a $k \geqq 0$ such that $l_n = p^k s_n$ with $(s_n, p) = 1$ for all n. Notice that $h_p^*(l_n g) = h_p^*(p^k g)$.

If $r \neq k + t$, then

$$h_p^*(a + p^k s_n g) < h_p^*(p^t a + p^{k+t} s_n g) = h_p^*(p^r sg + p^{k+t} s_n g) = h_p^*(p^u g)$$

with $u = \min(r, k + t)$. This contradicts the fact that, because of

$$h_p^*((l_{n+1} - l_n)g) = h_p^*(a + l_n g),$$

the highest power of p dividing $l_{n+1} - l_n$ increases with increasing n. If $r = k + t$, then a can be replaced by $a - p^k sg$, so $p^t a = 0$ may be assumed. Now $h_p^*(a + p^k s_n g) < h_p^*(p^{k+t} s_n g) = h_p^*(p^{k+t} g)$ leads to a similar contradiction.□

An isomorphism $\phi: G \to H$ between $G \leqq A$ and $H \leqq C$ is said to be *height-preserving* if

$$h_p^*(\phi g) = h_p^*(g),$$

for all elements $g \in G$ and all primes p, where the heights are taken in C and A, respectively.

Lemma 104.2 (Rotman [2]). *Let A and C be reduced groups with the same Ulm–Kaplansky invariants for some prime p, and let G and H be finitely generated subgroups of torsion-free rank 1 in A and C, respectively.*

If $\phi: G \to H$ is a height-preserving isomorphism and if $a \in A \backslash G$, $pa \in G$, then ϕ can be extended to a height-preserving isomorphism $\phi': \langle G, a \rangle \to \langle H, c \rangle$ for a suitably chosen $c \in C$.

Owing to (104.1), no generality is lost in assuming that a is p-proper with respect to G. Just as in (77.1), one can find a $c \in C$ subject to the conditions:

$c \notin H$, $pc \in H$, $\phi(pa) = pc$, $h_p^*(a) = h_p^*(c)$, and c has maximal p-height in $c + H$. We claim: if $(k, p) = 1$, then $h_q^*(ka + g) = h_q^*(kc + \phi g)$ for all primes q. For $q = p$, this immediately follows from the fact that both a and c are p-proper with respect to G and H, respectively. For $q \neq p$, we have

$$h_q^*(ka + g) = h_q^*(pka + pg) = h_q^*(pkc + p\phi g) = h_q^*(kc + \phi g).$$

Now ϕ' is defined by mapping a to c.□

The structure theorem on countable mixed groups of torsion-free rank 1 now follows easily.

Theorem 104.3 (Rotman [2], Megibben [6], Myshkin [1]). *Let A and C be countable mixed groups of torsion-free rank 1. The groups A, C are isomorphic if and only if:*

1. *the torsion subgroups $T(A)$ and $T(C)$ are isomorphic; and*
2. *the height-matrices $\mathbb{H}(A)$ and $\mathbb{H}(C)$ are equivalent.*

In view of our discussion in **103**, we need only verify the sufficiency. Let us assume that conditions 1 and 2 are satisfied, and in addition, $T(A)$ and $T(C)$ are reduced. If A is not reduced, then $A \cong T(A) \oplus Q$ and from the equivalence of the height-matrices one easily derives $C \cong T(C) \oplus Q$. So let A and C be reduced. By the equivalence of height-matrices, there are elements $g \in A$ and $h \in C$ of infinite order with the same height-matrix. This means, $g \mapsto h$ induces a height-preserving isomorphism $\phi\colon \langle g \rangle \to \langle h \rangle$. Since A, C are countable and have the same Ulm–Kaplansky invariants for all primes, the isomorphism ϕ can be extended [*via* stepwise adjunction of elements, alternately from A and C] to an isomorphism of A with C by making use of (104.2) [cf. the proof of Ulm's theorem (77.3)].□

The last result easily extends to the case when the torsion subgroups are direct sums of countable groups.

Corollary 104.4 (Megibben [6]). *Mixed groups of torsion-free rank 1 are isomorphic if their torsion parts are isomorphic direct sums of countable groups and their height-matrices are equivalent.*

Let A and C be as stated. Suppose G is a countable pure subgroup of A containing an element of infinite order [cf. (26.2)]. Then $G \cap T(A)$ is contained in a countable direct summand U' of $T(A)$, say, $T(A) = U' \oplus S$. From $A = \langle G, T(A) \rangle$ we derive $A = U \oplus S$ with $U = \langle G, U' \rangle$ countable and S torsion. Similary, $C = V \oplus T$ with V countable and T torsion. In view of the structure theorem (78.4) on direct sums of countable p-groups, no generality is lost in assuming the isomorphisms $T(U) \cong T(V)$ and $S \cong T$. As the height-matrices for any element of infinite order are the same in A and U [in C and V], (104.3) implies the isomorphy of A and C. $\square$

Wallace [1] has extended the structure theorem to the case when the torsion subgroups are totally projective. Megibben [6] proved the result for groups with torsion-complete torsion parts.

In general, the isomorphy of torsion subgroups and the equivalence of height-matrices fail to imply that the groups themselves are isomorphic. This is illustrated by the following example.

Example (Megibben [6]). Let B be an unbounded countable direct sum of cyclic p-groups and $\bar{B}$ the corresponding torsion-complete p-group. There is a pure subgroup T of $\bar{B}$ such that $B < T$ and $\bar{B}/T \cong Z(p^\infty)$. By (56.5), $\mathrm{Ext}(Z(p^\infty), T)^1 \cong \mathrm{Hom}(Z(p^\infty), Z(p^\infty)) \cong J_p$, so there is a subgroup H of $\mathrm{Ext}(Z(p^\infty), T)$ such that $T < H$, $H/T \cong Q$, and $H^1 \neq 0$. In a similar way, we get a subgroup G of $\mathrm{Ext}(Z(p^\infty), B)$ such that $B < G$, $G/B \cong Q$, and $G^1 \neq 0$. It is readily seen that there are elements $g \in G$, $h \in H$ of infinite order such that $h_p^*(p^k g) = \omega + k = h_p^*(p^k h)$, for $k = 0, 1, \cdots$, so that $\mathbb{H}(A)$ and $\mathbb{H}(C)$ are equivalent for $A = G \oplus T$ and $C = H \oplus B$. Furthermore, $T(A) = B \oplus T \cong T(C)$. In order to show that A and C are not isomorphic, we prove:

$$\text{(i)}\quad H/H^1 \cong \bar{B}; \qquad \text{(ii)}\quad G/G^1 \cong B; \qquad \text{(iii)}\quad B \oplus T \not\cong \bar{B} \oplus B.$$

Now, (i) follows from the fact that if T is pure in a separable p-group S such that $S/T \cong Z(p^\infty)$, then $S \cong \bar{B}$ [see **68**, Ex. 15]. Since G/G^1 is a countable, separable p-group and B is isomorphic to a pure subgroup with $Z(p^\infty)$ as quotient, (ii) follows from (68.3). Finally, (73.7) implies that if $B \oplus T \cong \bar{B} \oplus B$, then T would be a direct sum of torsion-complete p-groups, which is clearly impossible.

Exercises

1. Two countable mixed groups of torsion-free rank 1 are necessarily isomorphic if each is isomorphic to an isotype subgroup of the other.
2. (Megibben [6]) Let A, C be mixed groups of torsion-free rank 1, and let ϕ be an isomorphism between a pure mixed subgroup G of A and a pure subgroup H of C. If $\phi \mid T(G)$ extends to an isomorphism of $T(A)$ with $T(C)$, then ϕ can be extended to an isomorphism of A with C. [*Hint*: the obvious map.]
3. (Megibben [6]) Let A, C be mixed groups of torsion-free rank 1 such that $T(A) \cong T(C)$ is torsion-complete and $\mathbb{H}(A)$, $\mathbb{H}(C)$ are equivalent. Prove that $A \cong C$. [*Hint*: apply Ex. 2 with $T(G)$ basic in $T(A)$.]
4. (Rotman and Yen [1]) Let T be a reduced torsion group with finite Ulm–Kaplansky invariants and let A, C be countable mixed groups of torsion-free rank 1. Show that $A \oplus T \cong C \oplus T$ implies $A \cong C$.
5. (Rotman [2]) Let A and C be countable mixed groups of torsion-free rank 1 such that $A \oplus A \cong C \oplus C$. Prove that $A \cong C$.

6. (Megibben [6]) Let A be a countable mixed group of torsion-free rank 1. A is splitting if and only if almost every row in a height-matrix is free of gaps, no row has an infinity of gaps, and if a row contains entries other than integers, then it contains an ∞.

105. GROUPS WITH PRESCRIBED ULM SEQUENCES

One is often confronted with the problem of constructing groups from certain constituents. In this section, we investigate such a construction for groups with given Ulm sequences. In **37** we have studied Ulm sequences in general and established a number of their properties. But we have left some basic questions unanswered, e.g.: Do there exist groups of arbitrarily large Ulm type? Are the conditions listed in (37.6) sufficient to guarantee the existence of a group with a given sequence of groups as Ulm sequence? Zippin's theorem (76.2) answered these questions for countable p-groups, and we will give a full solution in (105.3) in the general case.

Thus, our problem consists in finding necessary and sufficient conditions for a well-ordered sequence of groups

$$A_0, A_1, \cdots, A_\sigma, \cdots \qquad (\sigma < \tau), \tag{1}$$

in order that a reduced group A of cardinality $\mathfrak{m}$ exists with (1) as its Ulm sequence. We intend to show that the following necessary conditions listed in (37.6) are sufficient:

(a) *the first Ulm subgroup of A_σ vanishes for every $\sigma < \tau$;*
(b) $\sum_{0 \leqq \sigma < \tau} |A_\sigma| \leqq \mathfrak{m} \leqq \prod_{0 \leqq n < \min(\omega, \tau)} |A_n|$;
(c) $r(B_{\sigma+1,p}) \leqq \operatorname{fin} r(A_{\sigma,p})$ *for every $\sigma + 1 < \tau$ and prime p;*
(d) $\sum_{\rho < \sigma < \tau} |B_{\sigma,p}| \leqq |B_{\rho,p} \cap A_{\rho,p}|^{\aleph_0}$ *for every $0 \leqq \rho < \tau$ and prime p.*

Recall that here $A_{\sigma,p}$ and $B_{\sigma,p}$ denote the p-component and a p-basic subgroup of A, respectively.

One of the basic steps in the construction is the content of the next lemma.

Lemma 105.1. *Let G and H be arbitrary reduced groups, B_p a p-basic subgroup of G, and τ the Ulm type of H. If τ is not a limit ordinal and if*

$$r(B_p) \leqq \operatorname{fin} r(H_p^{\tau-1}) \qquad \textit{for every prime } p, \tag{2}$$

then there exists a [necessarily reduced] group A such that

$$A^\tau \cong G \qquad \textit{and} \qquad A/A^\tau \cong H.$$

First of all, we select for each prime p a quasibasis in $H_p^{\tau-1}$ relative to a lower basic subgroup: $\{a_i(p), c_{jn}(p)\}$ connected by equations like

$$\begin{cases} p^{e_i} a_i(p) = 0, \\ pc_{j1}(p) = 0, \quad pc_{j,n+1}(p) = c_{jn}(p) + s_1 a_{i_1}(p) + \cdots + s_t a_{i_t}(p) \qquad (n \geqq 1). \end{cases} \tag{3}$$

Here the s are integers, while i and j run over suitable index sets $I(p)$ and $J(p)$, respectively, where $|J(p)| = \text{fin}\ r(H_p^{\tau-1})$.

Next we pick a complete set of representatives $\{d_k\}_{k \in K}$ of H modulo the direct sum $\bigoplus_p H_p^{\tau-1}$. Then every $h \in H$ may be written in the form

$$(4)\quad h = d_k + \sum_p [m_1 a_{i_1}(p) + \cdots + m_r a_{i_r}(p) + q_1 c_{j_1 n_1}(p) + \cdots + q_t c_{j_t n_t}(p)],$$

with uniquely determined summands whenever $(q, p) = 1$ and no ma vanishes. Also, we choose some basis $\{b_{pl}\}_{l \in L(p)}$ for B_p where, obviously, $|L(p)| = r(B_p)$.

Define A as the group generated by G and elements $u_i(p)$, $v_{jn}(p)$ [for all primes p] and w_k, in one-to-one correspondence with $a_i(p)$, $c_{jn}(p)$, d_k, respectively, subject exactly to the same defining relations as the corresponding elements of H with the sole exception that $pc_{j1}(p) = 0$ is replaced by

$$(5)\qquad pv_{j1}(p) = \begin{cases} 0, & \text{or} \\ \text{some}\ \bar{b}_{pl}, & \text{such that every } \bar{b}_{pl}\ \ (l \in L(p)) \\ & \text{occurs exactly once.} \end{cases}$$

Here, obviously, $\bar{b}_{pl}$ denotes the element of A corresponding to $b_{pl} \in G$. Note that (5) makes sense in view of (2). It will be convenient to write the elements of A in the form $\bar{g} + \bar{h}$ with $g \in G$ and h a linear combination of u_i, v_{jn}, and w_k; operations with them are performed as in G and as prescribed by the defining relations.

Equating all $\bar{g}$ to 0, we obtain an epimorphism of A onto H whose kernel $\bar{G}$ is the set of all $\bar{g}$ with $g \in G$. To verify $\bar{G} \cong G$, take the divisible hull D of G. Starting with the correspondence $\bar{g} \mapsto g$, we can assign successively suitable images in D to $u_i(p)$, $v_{jn}(p)$, and w_k in such a manner that all the defining relations will be preserved. That this can be done proves that the map $g \mapsto \bar{g}$ of G into A is monic, $\bar{G} \cong G$, and so $A/\bar{G} \cong H$.

Visibly, $c_{j1}(p) \in H_p^{\tau-1}$ and (5) imply that $\bar{b}_{pl} \in A^\tau$. Let $\bar{B}_p$ correspond to B_p in $\bar{G}$, and let $C/\sum_p \bar{B}_p$ be the divisible part of $A/\sum_p \bar{B}_p$. From the reducedness of A/A^τ, we obtain $C \leq A^\tau$, and so the divisibility of $\bar{G}/\sum_p \bar{B}_p$ implies $\bar{G} \leq C$; therefore, $\bar{G} \leq A^\tau$. The converse inclusion is a simple consequence of $(A/\bar{G})^\tau \cong H^\tau = 0$. Hence $A^\tau \cong G$.$\square$

It is clear how to construct, with the aid of (105.1), a group of finite Ulm type if its Ulm factors are prescribed. The case of Ulm type ω is settled in the next lemma.

Lemma 105.2. *Let $A_0, A_1, \cdots, A_m, \cdots$ be a sequence of groups satisfying* (a) *and* (c). *If* $\mathfrak{m}$ *is a cardinal satisfying* (b), *then there exists a reduced group G of cardinality* $\mathfrak{m}$ *and of type ω whose Ulm sequence is $A_0, A_1, \cdots, A_m, \cdots$.*

As in the proof of (105.1), a quasibasis $\{a_i^m(p), c_{jn}^m(p)\}$ is selected, for each prime p, in each A_{mp}, relative to a lower basic subgroup B'_{mp}, and then we choose a complete set of representatives $\{d_k^m\}$ of A_m mod its torsion subgroup. A group C is now defined as being generated by all elements $u_i^m(p)$, $v_{jn}^m(p)$, and w_k^m [for all p and m] which are, in a one-to-one correspondence with $a_i^m(p)$, $c_{jn}^m(p)$, and d_k^m, respectively, subject exactly to the same defining relations as the corresponding a, c, and d, with a single exception: $pc_{j1}^m(p) = 0$ is replaced by

$$(6)\quad pv_{j1}^m(p) = \begin{cases} 0 & \text{for as many as fin } r(A_{mp}) \text{ indices } j, \\ \text{some} \quad u_i^{m+1}(p) & \text{or some} \quad \bar{u}_i^{m+1}(p) \text{ such that each} \\ & \bar{u}_i^{m+1}(p), u_i^{m+1}(p) \text{ occurs exactly once.} \end{cases}$$

Here $\bar{u}_i^{m+1}(p)$ denote elements of A_{m+1} such that $\{u_i^{m+1}(p), \bar{u}_i^{m+1}(p)\}$ corresponds to a basis of a p-basic subgroup $B_{m+1,p}$ whose torsion part is $B'_{m+1,p}$.

Let $C^{(m)}$ denote the subgroup of C which is generated by all $u_i^s(p)$, $v_{jn}^s(p)$, and w_k^s with $s \geqq m$ $(m = 0, 1, 2, \cdots)$. Then $C^{(0)} = C$ and, by induction, we deduce from (105.1) and from (6) that $C^{(0)}/C^{(m)}, \cdots, C^{(m-1)}/C^{(m)}$ are the Ulm subgroups of $C/C^{(m)}$ with Ulm factors $A_0, A_1, \cdots, A_{m-1}$. Consequently, $C^{(0)}, C^{(1)}, \cdots, C^{(m)}, \cdots$ are the Ulm subgroups and $A_0, A_1, \cdots, A_m, \cdots$ is the Ulm sequence of C. It also follows that $|C| = \mathfrak{n}$, where $\mathfrak{n} = \sum_{m<\omega} |A_m|$.

We proceed to define a group D with the same Ulm sequence as C, but of cardinality $\mathfrak{p} = \prod_{m=0}^{\infty} |A_m|$. For every m, let $\pi_m^{m+1} : C/C^{m+1} \to C/C^m$ be the canonical map, and let D be the inverse limit of the system $\{C/C^m, \pi_m^{m+1}\}$. Since $c \mapsto (c + C^1, \cdots, c + C^{m+1}, \cdots)$ is a monomorphism of C into D, C can be identified with its image in D. Let $\pi_m : D \to C/C^m$ be the canonical map; it is readily seen that π_m is epic for $m = 0, 1, \cdots$. By an easy induction we can convince ourselves that $(x_0, \cdots, x_m, \cdots)$ (with $x_m \in C/C^{m+1}$) belongs to D^m if and only if $x_0 = \cdots = x_{m-1} = 0$. We conclude that $C \cap D^m = C^m$, and moreover, C is isotype in D. Since $C + D^m = D$ for every m, we see that D/C is divisible and $D/D^m \cong C/(C \cap D^m) = C/C^m$. Hence D has the same Ulm sequence as C. That D is of cardinality $\mathfrak{p}$ is clear.

Now let G/C be a divisible subgroup of D/C such that $|G| = \mathfrak{m}$. By a straightforward induction, G must be isotype in D, thus $G^m = D^m \cap G$ for every m. Hence $G/G^m = (G + D^m)/D^m = D/D^m$, where $G + D^m \geqq C + D^m = D$. It follows at once that G has the same Ulm sequence as D, so G is as desired.□

In connection with the last proof, two remarks should be made.

Remark 1. The p-socle of C is generated by all the elements of the form $p^{e_i - 1}u_i^m(p)$ and those $v_{j1}^m(p)$ which satisfy $pv_{j1}^m(p) = 0$ in (6). If these with a fixed m generate P_m, then clearly

$$C^m[p] = P_m \oplus C^{m+1}[p] \qquad \text{and} \qquad C[p] = \bigoplus_{m=0}^{\infty} P_m.$$

Moreover, from the definitions it is easy to verify that

$$D^m[p] = P_m \oplus D^{m+1}[p] \qquad \text{and} \qquad D[p] = \prod_{m=0}^{\infty} P_m .$$

Remark 2. To find an estimate for the p-rank of D/C, we estimate the dimension of $D[p]/C[p]$ [cf. (28.1)]. If P_m is defined as before, then in view of (6) manifestly $|P_m| = |A_{m,p}|$. This, together with the direct sum and product representations of $C[p]$ and $D[p]$, yields

$$\text{(7)} \qquad \dim D[p]/C[p] \geqq \min_t \prod_{t \leqq m} |A_{m,p}| \geqq \min_m |A_{m,p}|^{\aleph_0} = \mathfrak{r}_p .$$

Consequently, G in the proof of (105.2) can always be chosen such that $r_p(G/C) = \min(\mathfrak{m}, \mathfrak{r}_p)$ for every p.

We are ready to prove our main result on the construction of groups with given Ulm sequences.

Theorem 105.3. *Let* $\mathfrak{m}$ *be a cardinal,* τ *an ordinal number, and* (1) *a sequence of nontrivial groups. There exists a reduced group A of cardinality* $\mathfrak{m}$, *of Ulm type* τ *with* (1) *as Ulm sequence if and only if conditions* (a)–(d) *are fulfilled.*

Because of (37.6), we need only prove that if $\mathfrak{m}$, τ, and (1) satisfy (a)–(d), then such an A does exist.

(A) We start the proof with writing the ordinal τ in the [unique] form $\tau = \omega\mu + r$, where μ is an ordinal and r a nonnegative integer. The given Ulm sequence can be broken up into pairwise disjoint subsequences

$$\text{(8)} \qquad A_{\omega\nu}, \quad A_{\omega\nu+1}, \cdots, A_{\omega\nu+n}, \cdots \qquad (n < \omega)$$

of type ω, where $0 \leqq \nu < \mu$, and a finite sequence $A_{\omega\mu}, \cdots, A_{\omega\mu+r-1}$ which disappears whenever $r = 0$.

(B) For each $\nu < \mu$, we define a group G_ν of Ulm type ω with the Ulm sequence (8). Set $\mathfrak{m}_0 = \mathfrak{m}$ and

$$\mathfrak{m}_\nu = \sum_{\omega\nu \leqq \sigma < \tau} |A_\sigma| \qquad \text{for} \quad \nu > 0.$$

Note that for arbitrary cardinals $\mathfrak{m}_i$ ($i \in I$), the inequality

$$\sum \mathfrak{m}_i^{\aleph_0} \leqq \left(\sum \mathfrak{m}_i\right)^{\aleph_0}$$

holds. Hence (34.3) and (d) imply

$$\sum_{\rho<\sigma<\tau} |A_\sigma| \leqq \sum_{\rho<\sigma<\tau} \left(\sum_p |B_{\sigma,p}|\right)^{\aleph_0} \leqq \left(\sum_p \sum_{\rho<\sigma<\tau} |B_{\sigma,p}|\right)^{\aleph_0}$$

$$\leqq \left(\sum_p |B_{\rho,p} \cap A_{\rho,p}|^{\aleph_0}\right)^{\aleph_0} \leqq \left(\sum_p |B_{\rho,p}|\right)^{\aleph_0} \leqq |A_\rho|^{\aleph_0}.$$

Now if $|A_\rho| = \min_\sigma |A_\sigma|$ with $\omega\nu \leqq \sigma < \omega(\nu+1)$, then

$$\mathfrak{m}_\nu = \sum_{\omega\nu \leqq \sigma \leqq \rho} |A_\sigma| + \sum_{\rho < \sigma < \tau} |A_\sigma| \leqq \sum_{\omega\nu \leqq \sigma \leqq \rho} |A_\sigma| + |A_\rho|^{\aleph_0}$$

$$\leqq \prod_{\omega\nu \leqq \sigma < \omega(\nu+1)} |A_\sigma|.$$

In view of this and (b), conditions (a) and (c) imply the existence of a group G_ν of type ω and of cardinality $\mathfrak{m}_\nu$ whose Ulm sequence is (8) [see (105.2)]. This holds for all $\nu < \mu$; and if $r > 0$, G_μ exists with the Ulm sequence $A_{\omega\mu}, \cdots, A_{\omega\mu+r-1}$.

In each group G_ν, we choose a quasibasis $\{a_i^\nu(p), c_{jn}^\nu(p)\}$ for every prime p, and then select a complete set $\{d_k^\nu\}$ of representatives of G_ν mod its torsion group.

(C) Our group A will now be defined as follows. For every $\nu \leqq \mu$, we select generators $u_i^\nu(p), v_{jn}^\nu(p), w_k^\nu$ corresponding bijectively to the generators of G_ν in the obvious fashion. They are subject to the following defining relations:

1. $p^e u_i^\nu(p) = 0$ if $e = e(a_i^\nu(p))$;
2. $pv_{j1}^\nu(p) = 0$ or some $u_i^\kappa(p)$ or some $\bar{u}_i^\kappa(p)$ for $\kappa > \nu$.

Here the $\bar{u}_i^\kappa(p)$ are the elements corresponding to $\bar{a}_i^\kappa(p)$, where $\{a_i^\kappa(p), \bar{a}_i^\kappa(p)\}$ is a basis of $B_{\kappa,p}$.

3. $pv_{j,n+1}^\nu(p) = v_{jn}^\nu(p) + s_1 u_{i_1}^\nu(p) + \cdots + s_t u_{i_t}^\nu(p)$ if and only if

$$pc_{j,n+1}^\nu(p) = c_{jn}^\nu(p) + s_1 a_{i_1}^\nu(p) + \cdots + s_t a_{i_t}^\nu(p)$$

holds in G_ν;

4. $w_{k_1}^\nu + w_{k_2}^\nu = w_k^\nu + \bar{f}$, where $\bar{f}$ is a linear combination of generators $u_i^\nu(p), v_{jn}^\nu(p)$ with fixed ν [and varying p] exactly if $d_{k_1}^\nu + d_{k_2}^\nu = d_k^\nu + f$ holds in G_ν, where f is the corresponding expression for $a_i^\nu(p), c_{jn}^\nu(p)$.

The following will be assumed:

(α) Because of Remark 2, the group G_ν ($\nu < \mu$) may be supposed to have been constructed in such a way that the p-rank of G_ν/C_ν is equal to $\min(\mathfrak{m}_\nu, \mathfrak{r}_p^\nu)$, where

$$\mathfrak{r}_p^\nu = \min_{\omega\nu \leqq \rho < \omega(\nu+1)} |A_{\rho,p}|^{\aleph_0}$$

and C_ν corresponds to the group C in (105.2).

(β) For any p, the set of all $\{u_i^\kappa(p), \bar{u}_i^\kappa(p)\}$ with $\kappa > \nu$ is of cardinality $\sum_{\nu<\kappa} r(B_{\omega\kappa,p})$. This is $\leqq \sum_{\nu<\kappa} |A_{\omega\kappa}| \leqq \mathfrak{m}_\nu$. Furthermore, by (d), for every ρ between $\omega\nu$ and $\omega(\nu+1)$ we have

$$\sum_{\nu<\kappa} r(B_{\omega\kappa,p}) \leqq \min_\rho \sum_{\rho<\sigma} |B_{\sigma,p}| \leqq \min_\rho |A_{\rho,p}|^{\aleph_0} = \mathfrak{r}_p^\nu.$$

Consequently, we can assume that in 2, we put $pv_{j1}^{\nu}(p) = 0$ whenever $c_{j1}^{\nu}(p) \in C_{\nu}$, and for every fixed ν, all $u_i^{\kappa}(p)$ and $\bar{u}_i^{\kappa}(p)$ for $\kappa > \nu$ occur at least once.

(D) We prove that A as defined in (C) satisfies all the required conditions.

It is clear that every $a \in A$ can be written as a sum $a = x_{\nu_1} + \cdots + x_{\nu_t}$ with $\nu_1 < \cdots < \nu_t$, where each x_{ν} corresponds to a nonzero element g_{ν} of G_{ν} which is of the form (4). We claim that $a \neq 0$ if $t \geqq 1$. Equating all generators with upper index $> \nu_1$ to 0, it suffices to show that $x_{\nu_1} \neq 0$ if $g_{\nu_1} \neq 0$. This can be proved, as in the proof of (105.1), by extending the embedding $G_{\nu_1} \to E$ of G_{ν_1} in its divisible hull E to a homomorphism $A \to E$.

From (b) we infer that $\mathfrak{m}_{\nu} \leqq \mathfrak{m}$ for every ν and $|\tau| \leqq \mathfrak{m}$. Hence $\mathfrak{m} = \mathfrak{m}_0 \leqq |A| = \sum_{\nu} \mathfrak{m}_{\nu} \leqq \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$, and A is of cardinality $\mathfrak{m}$.

Next we show that all $u_i^{\nu}(p)$, $v_{jn}^{\nu}(p)$, w_k^{ν} with $\nu \geqq 1$ belong to the first Ulm subgroup A^1 of A. For the same reason as in the proof of (105.1), it suffices to verify this claim for $u_i^{\nu}(p)$ and $\bar{u}_i^{\nu}(p)$ only. But this follows at once from 2 and (β).

In view of (β), an element v^{ν} of A corresponding to an element c^{ν} of $C_{\nu}[p]$ belongs to $A[p]$. Thus if $pv_{jn}^{\nu}(p) = u_i^{\kappa}(p)$, then also $p(v_{jn}^{\nu}(p) + v^{\nu}) = u_i^{\nu}(p)$ for all such v^{ν}. Given an integer m, v^{ν} can be chosen such that $c_{jn}^{\nu}(p) + c^{\nu} \in G_{\nu}^{m}$; therefore, $u_i^{\nu}(p)$ and $\bar{u}_i^{\nu}(p)$ with $\nu \geqq 1$ all belong to A^{ω}, and the same is true for all $u_i^{\nu}(p)$, $v_{jn}^{\nu}(p)$, w_k^{ν} with $\nu \geqq 1$. We are led at once to $A/A^{\omega} \cong G_0$; therefore, the first ω Ulm factors of A are exactly those of G_0. An obvious transfinite induction, with repeated reference to (β), will complete the proof that the Ulm sequence of A is, in fact, the given one.□

Exercises

1. Give a detailed proof for the first inequality in (7).
2. Formulate and prove the analog of (105.1) for limit ordinals τ.
3. Prove (105.1) in the following way:

   (i) Using Prüfer's example in **35**, define a group D_p such that $D_p^1 = B_p$ and $K_p = D_p/D_p^1$ is a direct sum of cyclic p-groups. Call $E: 0 \to B_p \to D_p \to K_p \to 0$.

   (ii) For a monomorphism $\kappa_p: K_p \to H_p^{\tau-1}$, define $E\kappa_p^{-1}: 0 \to B_p \to E_p \to H_p^{\tau-1} \to 0$ and show $B_p = E_p^1$.

   (iii) For the natural embedding $\nu: \bigoplus_p H_p^{\tau-1} \to H$ and the obvious homomorphism $\mu: \bigoplus_p B_p \to G$, define

   $$\mu(\bigoplus_p E\kappa_p^{-1})\nu: 0 \to G \to A \to H \to 0,$$

   and verify that A is as desired.
4. Replacing κ_p in Ex. 3 by suitable monomorphisms $\lambda_p: K_p \to H_p$, outline a similar proof in case τ is a limit ordinal.

5. Let the Ulm type τ of A be a limit ordinal, and let $G = \varprojlim A/A^\sigma$ with $\sigma \to \tau$, where the maps $A/A^\sigma \to A/A^\rho$ $(\rho \leqq \sigma)$ are the natural ones. Show that the Ulm factors of G are not necessarily the same as those of A. [*Hint*: make $|G|$ bigger than $|B|^{\aleph_0}$, where B is basic in A.]
6. Improve on (105.3) by putting conditions on the cardinalities of $|A^{\omega\nu}/A^{\omega(\nu+1)}|$, too.
7*. How does (105.3) read if A is restricted to the class of cotorsion groups? [*Hint*: cf. (54.3).]
8*. Formulate (105.3) for the existence of the pair A, T, where T is the torsion part of A.
9. Give examples of groups of different cardinalities which have the same Ulm sequence.
10. For every ordinal $\tau \geqq \omega$, there exist $2^{|\tau|}$ nonisomorphic reduced groups of Ulm type τ and of cardinality $\leqq |\tau|$.
11. Given a group G with $G^1 = 0$, determine the cardinality of the set of nonisomorphic reduced groups A satisfying $A_0 \cong G$.

NOTES

The first nonsplitting mixed group was constructed by Levi [1]. Nearly two decades later, Fomin [1] and Baer [4] discovered the sufficient condition and the precise criterion (100.1), respectively, for the splitting of mixed groups. The dual problem for torsion-free groups was open for thirty years, until it was settled by Griffith [7]; see (101.1).

The tensor product was used by Irwin, Khabbaz, and Rayna [1, 2] to introduce a kind of measure indicating how far a mixed group is from being splitting. Some of their results have been improved by Lawver and Toubassi [1].

The splitting problem has been investigated for modules by various authors. J. Rotman [*Anais Acad. Brasil. Ci.* **32** (1960), 193–194] proved that the commutative domains over which all mixed modules split are exactly the fields. The same problem in the noncommutative case was considered by M. L. Teply [*J. London Math. Soc.* **4** (1971), 157–164]. I. Kaplansky [*Trans. Amer. Math. Soc.* **72** (1952), 327–340] observed that mixed modules with bounded torsion parts necessarily split over Dedekind domains. That among the commutative domains only the Dedekind rings enjoy this property was shown by Chase [1]. For the splitting of modules over commutative domains, see also I. Kaplansky [*Arch. Math.* **13** (1962), 341–343].

For modules M over arbitrary rings R, torsion submodules can be defined in a variety of ways, and for each definition, a splitting criterion can be asked for. A very useful generalization of the torsion part is the *singular submodule* $Z(M)$ of M, defined as the collection of elements in M whose annihilator left ideals are essential in R; for the corresponding splitting problem see V. C. Cateforis and F. L. Sandomierski [*J. Algebra* **10** (1968), 149–165; *Pacific J. Math.* **31** (1969), 289–292]. There is an extensive literature on torsion theories, cf. J. Lambek [Lecture Notes in Mathematics, Nr. 177. Springer Verlag, 1971].

Since the splitting problem plays a central role in the theory of mixed groups, it is no wonder that various generalizations have attracted attention. Oppelt [1] considered completely decomposable torsion-free groups G such that all extensions of torsion groups by G are direct sums of *p-mixed groups*, for various primes p [a p-mixed group is one whose torsion part is a p-group]. Griffith [6] settled this question for arbitrary torsion-free G. About quasi-splitting mixed groups, relevant information was furnished by C. Walker [1].

A satisfactory classification of mixed groups has been beyond our reach, and will probably remain so for a while. As a matter of fact, mixed groups exist in such abundance that there is little hope of soon finding the glue that binds their torsion and torsion-free parts together. A happy exception is when the torsion-free group is of rank 1, and intuitively it is clear that we can learn a great deal about mixed groups, in general, by investigating those of torsion-free rank 1. With this motivation in mind, in [16] the author called attention to the question of mixed groups of torsion-free rank 1. Today, we are in a possession of a nice structure theory for those with totally projective torsion parts. The key concept is the height-matrix which was introduced by Rotman [2], and utilized extensively by Megibben [6] and Myshkin [5]. Our (103.3) was established for countable groups in the two latter papers; the general case seems to be new. The structure theorem (104.3) is due [to Rotman [2] in a special case] to Megibben [6] and Myshkin [5]. Wallace [1] succeeded in extending it to groups whose torsion parts are totally projective.

For mixed modules over complete discrete valuation rings the situation is more favorable: here we have to work with one prime only, and in effect, they constitute a more tractable class. Cf. J. Rotman [*Pacific J. Math.* **10** (1960), 607–623], Rotman and Yen [1], C. M. Bang [*Bull. Amer. Math. Soc.* **76** (1970), 380–383; *J. Algebra* **14** (1970), 552–560; *Proc. Amer. Math. Soc.* **28** (1971), 381-388]. The most recent results of Warfield [5] yield a significant generalization of the theory of totally projective p-groups.

The construction of mixed groups with given Ulm sequence has been given independently by R. B. Warfield, Jr. and the author [unpublished]. The discussion in **105** is modeled on the author's paper [2] where the case of primary groups was dealt with. For mixed modules over Q_p and Q_p^*, the problem has been solved by Kulikov [3]; his result extends easily to mixed modules over arbitrary discrete valuation rings.

Problem 80. (Baer [4]) Give necessary and sufficient conditions for pairs (T, G) of torsion T and torsion-free groups G such that $\operatorname{Ext}(G, T) = 0$.

A special case has been discussed by Oppelt [2]. For countable groups, see **101**, Ex. 7.

Problem 81. Using the theory of totally projective p-groups and height-matrices, develop a theory for mixed groups A such that $T(A)$ is totally projective and $A/T(A)$ is divisible [or, more generally, completely decomposable].

Problem 82. Investigate groups A with the following property: if A is contained in a direct sum of reduced groups A_i $(i \in I)$, then for some integer $n > 0$, an essential subgroup of nA is contained in the direct sum of a finite number of the A_i.

Note that all algebraically compact and cotorsion groups, all torsion-complete p-groups, and $P - Z^{\aleph_0}$ share this property. Cf. S. U. Chase [*Proc. Amer. Math Soc.* **13** (1962), 214–216].

Problem 83. Combine the theories of totally projective p-groups and completely decomposable torsion-free groups to find a more general theory for arbitrary groups.

For modules over discrete valuation rings, this has been done by R. E. Warfield.

XV

ENDOMORPHISM RINGS

With an abelian group A, one can associate the ring $\mathrm{E}(A)$ of all endomorphisms of A. This is an associative ring with 1 which reflects certain properties of A to some extent. It is an obvious attempt to find out the precise relations between group properties of A and ring properties of $\mathrm{E}(A)$.

Examples for nonisomorphic groups with isomorphic endomorphism rings are abundant. Consequently, the endomorphism rings do not determine, in general, the groups. In the important case of torsion groups A, however, $\mathrm{E}(A)$ does characterize the group A [see **108**].

One of the fundamental problems on endomorphism rings is to determine criteria under which a ring is the endomorphism ring of some abelian group. Such criteria are not known as yet, in general, except for some more or less restrictive classes. For instance, the endomorphism rings of separable torsion groups can be characterized in a fairly satisfactory manner [see **109**] and a rather general sufficient condition can be established in the torsion-free case under countability hypothesis [see **110**]. Let us point out that the endomorphism rings also throw a light on the basic difference between torsion and torsion-free groups: while the endomorphism rings of torsion groups belong to a restrictive class of rings, all countable reduced torsion-free rings with 1 turn out to be endomorphism rings.

The problem as to when a ring is an endomorphism ring may become more tractable if one confines himself to distinguished classes of rings. Full or nearly full answers are available if $\mathrm{E}(A)$ is simple, Artinian, regular, or π-regular [see **111** and **112**].

Matrix representations of endomorphism rings and some topologies on endomorphism rings will also be discussed briefly.

106. ENDOMORPHISM RINGS

It is a familiar fact that the endomorphisms $\alpha, \beta, \cdots$ of an abelian group A form a ring under the addition and multiplication of homomorphisms:

$$(\alpha + \beta)a = \alpha a + \beta a \qquad \text{and} \qquad (\alpha\beta)a = \alpha(\beta a) \qquad \text{for all} \quad a \in A.$$

In this way one obtains an associative ring with identity, called the *endomorphism ring* E(A) *of* A.

Evidently, the additive group of E(A) is nothing else than the group End A which was introduced in **43**.

Example 1. If $A = Z$, then from Example 1 of **43** we know that every $\alpha: Z \to Z$ is completely determined by $\alpha 1$. It is readily seen that the correspondence $\alpha \mapsto \alpha 1$ between the endomorphisms of Z and the integers is an isomorphism not only in the group- but also in the ring-theoretical sense. In other words,

$$\mathsf{E}(Z) \cong \mathsf{Z}.$$

Example 2. An analogous argument, with a reference to Example 2 of **43**, gives

$$\mathsf{E}(Z(m)) \cong \mathsf{Z}/(m).$$

Example 3. From Example 3 of **43**, we are led to the isomorphism

$$\mathsf{E}(Z(p^\infty)) \cong \mathsf{Q}_p^*.$$

Example 4. Let R be a rational group, and say, $1 \in R$. Here again the endomorphisms α are fully determined by $\alpha 1$, thus the endomorphisms are simply multiplications by rationals. Necessarily, an endomorphism preserves divisibility of elements by integers; therefore, a rational number represents an endomorphism of R if and only if every prime factor p of its denominator satisfies $pR = R$. Hence E(R) is isomorphic to the subring of Q, generated by 1 and all p^{-1} with $pR = R$. In particular,

$$\mathsf{E}(Q) \cong \mathsf{Q}.$$

Example 5. From Example 5 in **43** it follows readily that

$$\mathsf{E}(J_p) \cong \mathsf{Q}_p^*.$$

Example 6. If M is a left R-module, then by an R-*endomorphism* is meant an endomorphism α of the additive group of M that commutes with multiplications by ring elements, i.e.,

$$\alpha(\rho a) = \rho\alpha(a) \qquad \text{for all} \quad a \in M, \quad \rho \in \mathsf{R}.$$

These form a subring $\mathsf{E}_{\mathsf{R}}(M)$ of E(M). If R contains an identity, then the R-endomorphisms of R as an R-module form a ring, antiisomorphic to R. In particular, the isomorphism in Example 5 above continues to hold if E(J_p) is replaced by $\mathsf{E}_{\mathsf{Q}_p^*}(J_p)$.

Example 7. Let M be a module over $\hat{\mathsf{Z}} = \prod_p \mathsf{Q}_p^*$ such that $M^1 = 0$. Then every Z-endomorphism η of M is a $\hat{\mathsf{Z}}$-endomorphism. To verify this, we show that the endomorphism $\bar{\pi}$ which is the multiplication by $\pi \in \hat{\mathsf{Z}}$ is in the center of E(M). If k_m $(m = 1, 2, \cdots)$ are integers such that $k_m \to \pi$ in the Z-adic topology of $\hat{\mathsf{Z}}$, then both $\pi\eta$ and $\eta\pi$ are the limit of $k_m\eta = \eta k_m$ in the Z-adic topology of E(M). Since E$(M)^1 = 0$, we get $\pi\eta = \eta\pi$.

We have made frequent use of the fact that the direct decompositions of a group correspond to idempotent endomorphisms. In our study of endomorphism rings, this interplay between decompositions and endomorphisms will be of considerable importance. We begin with the following simple observations.

(a) A group-isomorphism $\phi: A \to C$ induces a ring-isomorphism $\phi^*: \mathsf{E}(A) \to \mathsf{E}(C)$ as follows:

$$\phi^*: \alpha \mapsto \phi\alpha\phi^{-1}.$$

(b) If $A = B \oplus C$, then an endomorphism of B can be regarded as an endomorphism of A which annihilates C. Thus, if convenient, $\mathsf{E}(B)$ will be considered as a subring of $\mathsf{E}(A)$. More precisely:

(c) *Suppose $A = B \oplus C$ and let $\varepsilon: A \to B$ be the corresponding projection. Then we can make the indentification*

$$\mathsf{E}(B) = \varepsilon \mathsf{E}(A)\varepsilon.$$

For $\alpha \in \mathsf{E}(A)$, $\varepsilon\alpha\varepsilon$ is an endomorphism of B. On the other hand, if θ is an endomorphism of B, then after the indicated identification of θ with an endomorphism of A, $\theta = \varepsilon\theta\varepsilon$.

(d) *Suppose $A = B \oplus C$ and A' have isomorphic endomorphism rings, and $\psi: \mathsf{E}(A) \to \mathsf{E}(A')$ is an isomorphism between them. Then $A' = B' \oplus C'$ such that ψ induces isomorphisms $\mathsf{E}(B) \to \mathsf{E}(B')$ and $\mathsf{E}(C) \to \mathsf{E}(C')$.*

Denoting again by ε the projection $A \to B$ with kernel C, let $\psi: \varepsilon \mapsto \varepsilon'$. Manifestly, ε' is again idempotent, thus $A' = B' \oplus C'$ with $B' = \operatorname{Im} \varepsilon'$ and $C' = \operatorname{Ker} \varepsilon'$. From (c) it is clear that $\mathsf{E}(B) = \varepsilon\mathsf{E}(A)\varepsilon$ is carried by ψ into $\mathsf{E}(B') = \varepsilon'\mathsf{E}(A')\varepsilon'$, and $\psi \mid \mathsf{E}(B)$ must be an isomorphism, since it has an inverse.

(e) *There is a one-to-one correspondence between the finite direct decompositions*

$$A = A_1 \oplus \cdots \oplus A_n$$

of A and the decompositions of $\mathsf{E}(A)$ into finite direct sums of left ideals

$$\mathsf{E}(A) = \mathsf{L}_1 \oplus \cdots \oplus \mathsf{L}_n,$$

namely, if $A_i = \varepsilon_i A$ with pairwise orthogonal idempotents ε_i, then $\mathsf{L}_i = \mathsf{E}(A)\varepsilon_i$.

Given $A = \varepsilon_1 A \oplus \cdots \oplus \varepsilon_n A$ with orthogonal idempotents ε_i, the well-known Peirce-decomposition of $\mathsf{E}(A)$ yields $\mathsf{E}(A) = \mathsf{E}(A)\varepsilon_1 \oplus \cdots \oplus \mathsf{E}(A)\varepsilon_n$. Conversely, given $\mathsf{E}(A) = \mathsf{L}_1 \oplus \cdots \oplus \mathsf{L}_n$ with left ideals L_i of $\mathsf{E}(A)$, it is known [and readily checked] that $\mathsf{L}_i = \mathsf{E}(A)\varepsilon_i$, where ε_i is the ith component of the identity of $\mathsf{E}(A)$. These ε_i are orthogonal idempotents, hence $A = \varepsilon_1 A \oplus \cdots \oplus \varepsilon_n A$ follows. It is readily seen that the correspondence is one-to-one.

An idempotent $\varepsilon \neq 0$ is said to be *primitive* if it can not be written as a sum of two nonzero, orthogonal idempotents. We have obviously:

(f) *For an idempotent $\varepsilon \neq 0$ of $\mathsf{E}(A)$, εA is an indecomposable summand of A if and only if ε is a primitive idempotent.*

Our next remark shows how the isomorphism of two summands can be determined in terms of endomorphisms.

(g) (Corner [4]) *Let* $A = B \oplus C = B' \oplus C'$ *be direct decompositions of* A, *and* $\varepsilon\colon A \to B$, $\varepsilon'\colon A \to B'$ *the corresponding projections. Then* $B \cong B'$ *if and only if there exist* $\alpha, \beta \in \mathsf{E}(A)$ *such that*

$$\alpha\beta = \varepsilon \quad \textit{and} \quad \beta\alpha = \varepsilon'. \tag{1}$$

If $\alpha, \beta \in \mathsf{E}(A)$ satisfy these equalities, then $\beta\alpha\beta = \beta\varepsilon = \varepsilon'\beta$ and $\alpha\beta\alpha = \varepsilon\alpha = \alpha\varepsilon'$ show that $\beta^* = \beta\alpha\beta | B$ and $\alpha^* = \alpha\beta\alpha | B'$ are homomorphisms $\beta^*\colon B \to B'$ and $\alpha^*\colon B' \to B$. Now $(\alpha\beta\alpha)(\beta\alpha\beta) = \varepsilon$ and $(\beta\alpha\beta)(\alpha\beta\alpha) = \varepsilon'$ imply that β^* and α^* are inverse to each other, thus $B \cong B'$. Conversely, if $\beta^*\colon B \to B'$ and $\alpha^*\colon B' \to B$ are inverse isomorphisms, then $\beta = \beta^*\varepsilon$ and $\alpha = \alpha^*\varepsilon'$ satisfy (1).

For the sake of later reference, we prove the following, purely technical result [Hallett and Hirsch [1]].

(h) *Let* A *be a torsion-free group and* $\alpha, \beta \in \mathsf{E}(A)$. *Assume that*:

(i) $\alpha\beta = 0$;

(ii) *the left ideal of* $\mathsf{E}(A)$ *generated by* α *and* β *contains* $m\mathsf{E}(A)$ *for some integer* $m > 0$;

(iii) $\mathsf{E}(A)$ *has no nilpotent elements* $\neq 0$.

Then Ker α *and* Ker β *are disjoint fully invariant subgroups of* A *such that the following inclusion holds*:

$$mA \leqq \operatorname{Ker} \alpha \oplus \operatorname{Ker} \beta.$$

For every $\eta \in \mathsf{E}(A)$, by (i) we have $(\beta\eta\alpha)^2 = 0$, whence (iii) implies $\beta\eta\alpha = 0$; in particular, $\beta\alpha = 0$ and thus $\alpha\eta\beta = 0$, too. By (ii), there are $\gamma, \delta \in \mathsf{E}(A)$ such that $\gamma\alpha + \delta\beta = m1_A$. Therefore, $\gamma\alpha^2 = m\alpha = \alpha\gamma\alpha$, and $(\gamma\alpha - \alpha\gamma)\alpha = 0$. We obtain $\alpha(\gamma\alpha - \alpha\gamma) = 0$, which implies $m(\gamma\alpha - \alpha\gamma) = (\gamma\alpha + \delta\beta)(\gamma\alpha - \alpha\gamma) = 0$. By torsion-freeness, $\gamma\alpha - \alpha\gamma = 0$. This proves that α and γ, and similarly, β and δ commute. Consequently, $ma = \alpha\gamma a + \beta\delta a \in \operatorname{Ker}\beta + \operatorname{Ker}\alpha$ for every $a \in A$; hence $mA \leqq \operatorname{Ker}\alpha + \operatorname{Ker}\beta$. If $x \in A$ satisfies $\alpha x = \beta x = 0$, then $mx = \gamma\alpha x + \delta\beta x = 0$ shows $x = 0$; thus the last sum is direct.

For $a \in \operatorname{Ker}\alpha$ and $\eta \in \mathsf{E}(A)$, $\eta m a = \eta(\delta\beta)a$ and $\alpha(\eta\delta)\beta = 0$ imply $m\eta a \in \operatorname{Ker}\alpha$. Since Ker α is pure in A, $\eta a \in \operatorname{Ker}\alpha$, establishing the full invariance of Ker α. By symmetry, Ker β is likewise fully invariant.

(j) *Let* $\hat{A}$ *be the Z-adic completion of a group* A *with* $A^1 = 0$. *For every* $\eta \in \mathsf{E}(A)$ *there is exactly one* $\hat{\eta} \in \mathsf{E}(\hat{A})$ *such that* $\hat{\eta} | A = \eta$.

The hypothesis $A^1 = 0$ implies that A can be viewed as a pure subgroup of $\hat{A}$. By the pure-injectivity of $\hat{A}$, η can be extended to an $\hat{\eta}\colon \hat{A} \to \hat{A}$. This $\hat{\eta}$ must be unique, because there is no other endomorphism of A which agrees with $\hat{\eta}$ on a dense subgroup. [Example 7 shows that $\hat{\eta}$ is moreover a $\hat{\mathbf{Z}}$-endomorphism.]

With motivations from linear algebra in mind, it is natural to study the matrix representations of endomorphism rings. These are extracted from direct decompositions of the groups. Infinite decompositions yield representations by means of infinite matrices, and in order to select the types of infinite matrices occurring in such representations, we shall need the concept of finite topology; this will be introduced and discussed more intensively in the next section.

A matrix $[\alpha_{ji}]$ with entries in $\mathrm{E}(A)$ is said to be *column-convergent* if, for each column i, the sum $\sum_j \alpha_{ji}$ exists in the finite topology of $\mathrm{E}(A)$.

Let $A = \bigoplus_{i \in I} A_i$ be a direct sum and ε_i $(i \in I)$ the associated projections, considered as idempotents in $\mathrm{E}(A)$. Every $a \in A$ can be written as $a = \sum_i \varepsilon_i a$, where almost all $\varepsilon_i a$ vanish. For $\alpha \in \mathrm{E}(A)$, we have $\alpha a = \sum_i \alpha\varepsilon_i a = \sum_{i,j} (\varepsilon_j \alpha \varepsilon_i) a$. In this way, with every $\alpha \in \mathrm{E}(A)$ there is associated an $I \times I$-matrix:

$$\phi: \alpha \mapsto [\alpha_{ji}]_{j,\, i \in I} \qquad \text{where} \quad \alpha_{ji} = \varepsilon_j \alpha \varepsilon_i.$$

If $\beta \in \mathrm{E}(A)$ and if $[\beta_{ji}]$ with $\beta_{ji} = \varepsilon_j \beta \varepsilon_i$ is the corresponding matrix, then the matrices associated with $\alpha - \beta$ and $\alpha\beta$ are precisely the difference $[\alpha_{ji} - \beta_{ji}]$ and the product $[\sum_k \alpha_{jk} \beta_{ki}]$ of the matrices $[\alpha_{ji}]$ and $[\beta_{ji}]$, respectively, i.e., ϕ is a ring-homomorphism.

From the definition it is clear that the zero matrix can arise only from the zero endomorphism. For each index i, $\alpha\varepsilon_i a = \sum_j \alpha_{ji} a$ exists for all $a \in A$, i.e., the matrices $[\alpha_{ji}]$ are column-convergent. Conversely, if $[\alpha_{ji}]_{i,\, j \in I}$ is a column-convergent matrix with $\alpha_{ji} \in \varepsilon_j \mathrm{E}(A) \varepsilon_i$, then it comes from an $\alpha \in \mathrm{E}(A)$, namely,

$$\alpha a = \sum_{i,\, j} \alpha_{ji} a.$$

If, analogously to (b), $\mathrm{Hom}(A_i, A_j)$ is identified with the subgroup $\varepsilon_j \mathrm{E}(A) \varepsilon_i$ of $\mathrm{E}(A)$ in the obvious way, then we may state our findings as follows:

Theorem 106.1. *Let $A = \bigoplus_{i \in I} A_i$ be a direct decomposition of A. Then $\mathrm{E}(A)$ is isomorphic to the ring of all column-convergent $I \times I$ matrices*

$$[\alpha_{ji}]_{j,\, i \in I} \qquad \textit{where} \quad \alpha_{ji} \in \mathrm{Hom}(A_i, A_j). \square$$

In the remainder of this section, we give some applications.

Example 8. Let $A = \bigoplus_{i \in I} \langle a_i \rangle$ be a free group. In the matrix representation of (106.1), the entries α_{ji} are integers and the columns are finite.

Example 9. If $A = \bigoplus A_i$ is a torsion-free divisible group, $A_i \cong Q$, then we are in the same situation as in Example 8, except that the α_{ji} can be arbitrary rationals.

Example 10. Let $A = A_0 \oplus \bigoplus_p A_p$, where A_0 is torsion-free and A_p are p-groups with distinct primes p. Then the corresponding representation of endomorphisms of A is given by the matrices

$$\begin{bmatrix} \alpha_{00} & 0 & 0 & \cdots & 0 & \cdots \\ \alpha_{20} & \alpha_{22} & 0 & \cdots & 0 & \cdots \\ \alpha_{30} & 0 & \alpha_{33} & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{p0} & 0 & 0 & \cdots & \alpha_{pp} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

where $\alpha_{p0} \in \mathrm{Hom}(A_0, A_p)$ and $\alpha_{pp} \in \mathsf{E}(A_p)$. Here the first column is necessarily finite if A_0 is finitely generated.

Exercises

1. The subring of $\mathsf{E}(A)$, generated by the identity, is isomorphic to $\mathbf{Z}$ or to a quotient of $\mathbf{Z}$.
2. Show that $\mathrm{Hom}(A, C)$ with $C \leqq A$ can be made into a ring; actually, it is a right ideal of $\mathsf{E}(A)$.
3. Extend (e) to infinite direct decompositions $A = \bigoplus A_i$. [*Hint*: $\sum \lambda_i$ with $\lambda_i \in \mathsf{L}_i = \mathsf{E}(A)\varepsilon_i$ exists in the finite topology.]
4. (a) If $A = \bigoplus A_i$ and if the A_i are all fully invariant in A, then $\mathsf{E}(A) \cong \prod \mathsf{E}(A_i)$.
   (b) Apply (a) to the p-components A_p of a torsion group A.
   (c) In general, if $A = \bigoplus A_i$, then $\mathsf{E}(A)$ contains a subring isomorphic to $\prod \mathsf{E}(A_i)$.
5. Show that the endomorphism ring of an infinite group must be infinite.
6. (a) Using (106.1), determine the order of the endomorphism ring of a finite p-group.
   (b) Show that if A is a finite p-group of order p^n, then its endomorphism ring is at most of order p^{n^2}.
7. (Dlab [2]) Use (106.1) to represent the endomorphism ring of a divisible p-group as a matrix ring over the p-adic integers. Show that in every column, almost all entries are divisible by p^k, for any integer $k \geqq 0$.
8. Prove that the endomorphism ring of a torsion-free group of rank $\mathfrak{m}$ can be represented by column-finite $\mathfrak{m} \times \mathfrak{m}$ matrices with rational entries.
9. If the coordinates of $a \in A = \bigoplus A_i$ are arranged in the form of a column vector $\mathbf{a}$, then for an endomorphism α of A, $\alpha\mathbf{a}$ is the column vector $[\alpha_{ji}]\mathbf{a}$.
10. Show that the matrices in (106.1) are necessarily row-convergent; here the entries are viewed as endomorphisms of A.
11. If $A = \bigoplus A_i$ with countable components A_i, then there are not more

than countably many nonzero entries in every column of the matrices representing endomorphisms of A, in the given direct decomposition of A.

12. If C is a dense fully invariant subgroup of a reduced group A, then $\mathrm{E}(A)$ is isomorphic to a subring of $\mathrm{E}(C)$.

107. TOPOLOGIES OF ENDOMORPHISM RINGS

Endomorphism rings admit various topologies which are defined mostly in terms of the underlying groups. One of these topologies is playing an increasingly important role in certain investigations on rings of endomorphisms, and therefore, we give a survey of a few results on this topology.

This topology is the so-called *finite topology* of $\mathrm{E}(A)$. For a finite subset X of A, the *X-neighborhood* of $\alpha \in \mathrm{E}(A)$ is defined as

$$U_X(\alpha) = \{\eta \in \mathrm{E}(A) \mid \eta x = \alpha x \text{ for all } x \in X\}.$$

In this definition, $U_X(\alpha) = \bigcap_{x \in X} U_x(\alpha)$ and $U_X(\alpha) = \alpha + U_X(0)$. Thus the finite topology can more conveniently be defined with the aid of a subbase of neighborhoods of 0:

$$U_x = \{\eta \in \mathrm{E}(A) \mid \eta x = 0\} \qquad \text{for all} \quad x \in A.$$

This topology is evidently Hausdorff. U_x are left ideals of $\mathrm{E}(A)$, hence the continuity of addition and subtraction in $\mathrm{E}(A)$ is immediate. Moreover:

Theorem 107.1. *The endomorphism ring* $\mathrm{E}(A)$ *of an abelian group* A *is a complete topological ring in the finite topology.*

To prove ring multiplication continuous, let $\alpha, \beta \in \mathrm{E}(A)$, and let $\alpha\beta + U_x$ be a neighborhood of $\alpha\beta$. Since U_x is a left ideal and $U_{\beta x}\beta \leqq U_x$, from

$$(\alpha + U_{\beta x})(\beta + U_x) \leqq \alpha\beta + U_{\beta x}\beta + U_x \leqq \alpha\beta + U_x,$$

we obtain the desired continuity. Thus $\mathrm{E}(A)$ is a topological ring. To verify its completeness, let $\{\alpha_i\}_{i \in I}$ be a Cauchy net; in the present case, the index set I is partially ordered inversely to the finite subsets of A. The Cauchy net satisfies: given $x \in A$, $\alpha_i - \alpha_j \in U_x$ holds for all i, j larger than some $i_0 \in I$. In other words, $\alpha_i x$ is the same element of A for large indices. Therefore, if we define αx as the common value of all these $\alpha_i x$, then $\alpha \in \mathrm{E}(A)$ and $\alpha - \alpha_i \in U_x$ for all these i.□

Naturally, the finite topology is discrete if the group is finitely generated. It is discrete for rigid groups, too.

An infinite series $\sum_{j \in J} \alpha_j$ in $\mathrm{E}(A)$ converges to α if the net $\{\sum_{j \in J_0} \alpha_j\}$ with J_0 running over all finite subsets of J, ordered by inclusion, has the limit α. This means that, for each $a \in A$, almost all $\alpha_j a = 0$ and $\alpha a = \sum \alpha_j a$ [cf. Szele [17]].

It should be pointed out that in the special case when A is a reduced torsion group, the finite topology of $\mathsf{E}(A) = \mathsf{E}$ can be defined without any reference whatsoever to the underlying group A. If A is separable, then, because of (27.9), a subbase of neighborhoods of 0 for the finite topology can be given as the sets U_x with those $x \in A$ only for which $\langle x \rangle$ is a summand of prime power order of A. Here $U_x = \mathsf{E}(1 - \varepsilon)$ is just the left annihilator of the projection $\varepsilon: A \to \langle x \rangle$ in E. If A is any reduced torsion group, then to each $x \in A$ there are a projection $\varepsilon: A \to \langle y \rangle$ onto a summand $\langle y \rangle$ and an endomorphism η of A such that $x = \eta y$ and U_x is the left annihilator of $\eta\varepsilon$ in E. To sum up:

Proposition 107.2. *The finite topology of the endomorphism ring* E *of a reduced torsion group can be defined by taking the left annihilators of the elements* $\eta\varepsilon$ *as a subbase of neighborhoods about* 0, *where* $\eta \in \mathsf{E}$ *and* ε *is a primitive idempotent.*

If the group is separable, then it suffices to take the left annihilators of the primitive idempotents only.□.

Let A be a p-group and $\mathsf{E} = \mathsf{E}(A)$ its endomorphism ring. E_0 will denote the left ideal of E which is generated by the primitive idempotents of finite order in E.

Proposition 107.3 (Liebert [4]). *If A is a separable p-group, then, in the finite topology,* E_0 *is dense in* E *and* E *is the completion of* E_0.

Let $\sigma \in \mathsf{E}$ and $a_1, \cdots, a_n$ a finite subset of A. We can embed this set in a finite summand G of A. For the density of E_0 in E, it suffices to show that $\mathsf{E}_0 \cap (\sigma + U_G)$ is not empty. If $\pi: A \to G$ is a projection, then $1 - \pi \in U_G$. Since U_G is a left ideal of E, $\sigma(1 - \pi) \in U_G$, and so $\sigma\pi = \sigma - \sigma(1 - \pi)$ lies in the intersection of E_0 and $\sigma + U_G$. That E is the completion of E_0 in the finite topology is evident because of (107.1).□

If A is a separable torsion-free group, then—analogously to the torsion case—we can prove that the finite topology of $\mathsf{E}(A)$ can be defined by taking the left annihilators of the primitive idempotents, and if E_0 denotes the left ideal generated by the primitive idempotents of E, then E will be the completion of E_0.

Compactness being always of particular interest, let us turn our attention to the question as to when $\mathsf{E}(A)$ is compact in the finite topology.

Before stating the relevant result, we introduce the following concept. For $x \in A$, the fully invariant subgroup

$$O_x = \{\eta x \mid \eta \in \mathsf{E}(A)\}$$

will be called the *orbit of* x. Now, the evaluation map $\eta \mapsto \eta x$ is a group-homomorphism of $\mathsf{E}(A)$ onto O_x with kernel U_x; consequently, the group-isomorphism

$$\mathsf{E}(A)/U_x \cong O_x$$

holds true.

Proposition 107.4. *The endomorphism ring* $\mathrm{E}(A)$ *of a group* A *is compact in the finite topology if and only if* A *is a torsion group whose* p*-components are finite direct sums of cocyclic groups.*

In view of the completeness of $\mathrm{E}(A)$, it is readily checked that for the compactness of $\mathrm{E}(A)$, it is necessary and sufficient that all U_x be of finite index in $\mathrm{E}(A)$; or, equivalently, every element of A have a finite orbit.

First, suppose $\mathrm{E}(A)$ compact. As $\langle x\rangle$ is a subgroup of O_x, A must be a torsion group. By (27.3), every nonzero p-component A_p of A contains a cocyclic summand; let C_p be one of minimal order. Then the orbit of its cogenerator is the socle of A_p, thus A_p has a finite socle; that is, owing to (25.1), it is a finite direct sum of cocyclic groups.

Conversely, if $A = \bigoplus_p A_p$ with finitely cogenerated p-components A_p, then for every integer n, $A[n]$ is finite. Note that $O_x \leqq A[n]$ if $nx = 0$, thus all U_x are of finite index in $\mathrm{E}(A)$, and $\mathrm{E}(A)$ is compact in the finite topology.□

Recall that in (46.1), the completeness of $\mathrm{Hom}(A, C)$ has been established in the Z-adic topology, for torsion groups A. Any ring multiplication being trivially continuous in the Z-adic topology, we are led at once to:

Proposition 107.5. *The endomorphism ring* $\mathrm{E}(A)$ *of a torsion group* A *is a complete topological ring in its* Z*-adic topology.*□

Let us stop for a moment to compare the finite and the Z-adic topologies of a torsion group A. Given $x \in A$, there is an integer $n > 0$ such that $nx = 0$. Therefore, $n\mathrm{E}(A) \leqq U_x$, and the open sets U_x in the finite topology are open in the Z-adic topology; in other words, *the* Z*-adic topology is finer than the finite topology* [for torsion groups only!].

Exercises

1. What are the topologies of the p-adic integers obtained *via* finite topologies on the endomorphism rings of $Z(p^\infty)$ and J_p?
2. The finite topology of the endomorphism ring of a torsion-free group of finite rank is always discrete.
3. Let A be an infinite group such that $|\mathrm{E}(A)| > |A|$. Show that $\mathrm{E}(A)$ can not be discrete in the finite topology. [*Hint*: the index of U_x in $\mathrm{E}(A)$ is at most $|A|$.]
4. (a) If A is complete in the Z-adic topology, then $\mathrm{E}(A)$ is a complete topological ring in the Z-adic topology.
   (b) The same holds if A is torsion-complete.
5. Let A be a separable p-group with basic subgroup B. Show that $\mathrm{E}(A)$ is a closed subring of $\mathrm{E}(\bar{B})$ in the finite topology.
6. (a) Let $\rho_1, \cdots, \rho_n, \cdots \in \mathrm{E} = \mathrm{E}(A)$ be such that:
   (i) $\mathrm{E}\rho_n \geqq \cdots \geqq \mathrm{E}\rho_n \geqq \cdots$ is a descending chain;
   (ii) the union of the chain $\mathrm{Ker}\,\rho_1 \leqq \cdots \leqq \mathrm{Ker}\,\rho_n \leqq \cdots$ is A.

Show that E is a complete group in the topology defined by $\mathsf{E}\rho_n$ as a neighborhood base about 0. If the ρ_n are in the center of E, then E is a topological ring in the same topology. [*Hint*: follow (46.1).]

(b) Generalize (a) to the case when $\{\mathsf{E}\rho_i\}$ is a system directed downwards under inclusion.

(c) Apply (a) and (b) to special cases, e.g., $\rho_n =$ multiplication by p^n, $\rho_j =$ projection of $\bigoplus_i A_i$ to the direct sum of almost all A_i, etc.

7. Let $\{A_i\}_{i \in I}$ be a system of subgroups of A which is directed upwards under inclusion such that the union of all A_i is equal to A. Define a topology in E with
$$\mathsf{L}_i = \{\eta \in \mathsf{E} \mid \eta A_i = 0\}$$
as a base of neighborhoods about 0. Show that E is a complete group in this topology, and if the A_i are fully invariant in A, then E is a topological ring. [This generalizes Ex. 6.]
8. (Pierce [3]) Let A be a p-group and set $A_n = A[p^n]$. Then E is a complete topological ring in the topology of Ex. 7.
9. (Liebert [4]) Let A be a p-group and E, E_0 as defined in the text, furnished with the finite topology. Then:
(a) E_0 is dense in the torsion subring of E;
(b) E_0 is dense in E if and only if $A^1 = 0$.

108. ENDOMORPHISM RINGS OF TORSION GROUPS

A distinguished feature of torsion groups is that they have many endomorphisms; moreover, they have an adequate supply of idempotent endomorphisms. It is reasonable to wonder to what extent the endomorphisms determine the group, or, more explicitly, whether the isomorphy of endomorphism rings implies that the groups themselves are isomorphic. We will show that this is in fact the case. Actually, a little more will be proved:

Theorem 108.1 (Baer [9], Kaplansky [2]). *If A and C are torsion groups whose endomorphism rings are isomorphic, then every isomorphism ψ between $\mathsf{E}(A)$ and $\mathsf{E}(C)$ is induced by a group-isomorphism ϕ: $A \to C$, i.e., $\psi: \eta \mapsto \phi\eta\phi^{-1}$.*

The proof can be reduced at once to p-groups; in fact, $\mathsf{E}(A) = \prod \mathsf{E}(A_p)$ and $\mathsf{E}(C) = \prod \mathsf{E}(C_p)$ for the p-components A_p, C_p, and any ring-isomorphism between $\mathsf{E}(A)$ and $\mathsf{E}(C)$ must carry $\bigcap_{(n,p)=1} n\mathsf{E}(A) = \mathsf{E}(A_p)$ into $\mathsf{E}(C_p)$. We may, therefore, suppose A and C are p-groups. We write simply $\psi(\eta) = \eta^*$ for $\eta \in \mathsf{E}(A)$.

Before entering into the proof, we note that if A is cocyclic, then from **106**(f) it follows at once that C, too, is indecomposable, and thus cocyclic. Examples 2 and 3 in **106** show that only $C \cong A$ is possible.

In the proof, we distinguish three cases.

1. If A is bounded, then it contains an element g of maximal order p^k, and from (15.1) we know that $\langle g\rangle$ is a summand of A. If $\varepsilon: A \to \langle g\rangle$ is a projection, then ε^* maps C onto a summand which must again be a cyclic group $\langle h\rangle$ of order p^k [cf. **106**(d) and the preceding remark]. Given any $a \in A$, pick an endomorphism η of A such that $a = \eta g$, and define $\phi: A \to C$ as $\phi: a \mapsto \eta^* h$. This definition is independent of the choice of η, for if $a = \eta_1 g$, too, then $(\eta - \eta_1)g = 0$ and $(\eta - \eta_1)\varepsilon = 0$, whence $(\eta^* - \eta_1^*)\varepsilon^* = 0$ and $(\eta^* - \eta_1^*)h = 0$. It is readily checked that ϕ is one-to-one, preserves addition, and is onto C, in other words, is an isomorphism. Now if $\xi \in \mathrm{E}(A)$, then writing $c = \phi a$ as $c = \eta^* h$ for some $\eta \in \mathrm{E}(A)$, we see $\xi^* c = \xi^* \eta^* h = \phi(\xi\eta g) = \phi\xi a = \phi\xi\phi^{-1}c$, that is, $\xi^* = \phi\xi\phi^{-1}$, and ϕ induces ψ.

2. If $A = B \oplus D$, where B is bounded and D is divisible $\neq 0$, then let $\langle g\rangle$ be a cyclic summand of maximal order p^k in B and $D' = \langle d_1, \cdots, d_n, \cdots\rangle$, with $pd_1 = 0$, $pd_{n+1} = d_n$ for $n \geqq 1$, a quasicyclic summand of D. Let $\varepsilon: A \to \langle g\rangle$ and $\pi: A \to D'$ be projections, and let $\varepsilon^*: C \to \langle h\rangle = Z(p^k)$, $\pi^*: C \to E' = \langle e_1, \cdots, e_n, \cdots\rangle$ with $pe_1 = 0$, $pe_{n+1} = e_n$. Write $a \in A$ as $a = a_1 + a_2$ ($a_1 \in B$, $a_2 \in D$) and choose an endomorphism η of A such that $\eta g = a_1$, $\eta d_n = a_2$ for some n. Then define $\phi a = \eta^*(h + e_n)$. To show ϕa independent of η, n, pick $\eta_1 \in \mathrm{E}(A)$ such that $\eta_1 g = a_1$ and $\eta_1 d_m = a_2$ with $m \geqq n$. Then $(\eta - \eta_1)\varepsilon = 0$ and $(p^{m-n}\eta - \eta_1)d_m = 0$, thus $\zeta = (p^{m-n}\eta - \eta_1)\pi$ annihilates $D'[p^m]$. But for quasicyclic groups D', this implies that the homomorphism ζ is divisible by p^m, and therefore $p^m | \zeta^*$, too; thus $\zeta^* e_m = 0$. We find $\eta^* h = \eta_1^* h$ and $\eta^* e_n = p^{m-n}\eta^* e_m = \eta_1^* e_m$, establishing $\eta^*(h + e_n) = \eta_1^*(h + e_m)$. That ϕ is bijective and preserves addition is straightforward, hence it is an isomorphism $A \to C$. As in 1, we can verify that ϕ induces ψ.

3. In the remaining case, A has an unbounded basic subgroup, thus there are decompositions

$$A = \langle a_1\rangle \oplus \cdots \oplus \langle a_k\rangle \oplus A_k \qquad (k = 1, 2, \cdots)$$

such that $A_k = \langle a_{k+1}\rangle \oplus A_{k+1}$ and $o(a_k) = p^{n_k}$ satisfy $1 \leqq n_1 < \cdots < n_k < \cdots$. Let ε_k denote the projection $A \to \langle a_k\rangle$, and define ξ_{jk} $(j \neq k)$ as the endomorphism of A which maps a_k upon a_j or $p^{n_j - n_k}a_j$ according as $j < k$ or $j > k$, and sends the complement of $\langle a_j\rangle$ to 0 [in the above decomposition of A]. In this case,

(i) the ε_k are pairwise orthogonal idempotents;
(ii) $\xi_{jk}\varepsilon_k = \xi_{jk} = \varepsilon_j \xi_{jk}$ for all $j \neq k$;
(iii) $\xi_{kj}\xi_{jk} = p^{|n_j - n_k|}\varepsilon_k$ for all $j \neq k$;
(iv) $\xi_{ij}\xi_{jk} = \xi_{ik}$ if $i < j < k$ or $i > j > k$.

The endomorphisms ε_k^* and ξ_{jk}^* of C also satisfy (i)–(iv). By virtue of **106**(d), $\varepsilon_k^* C$ are cyclic summands of C, of the same orders as $\varepsilon_k A$. Owing to (ii), $\xi_{k,k+1}^*$

will map ε_{k+1}^*C into ε_k^*C. We set $\varepsilon_k^*C = \langle c_k \rangle$ and show that generators c_k can be selected so as to have $\xi_{k,k+1}^* c_{k+1} = c_k$ for all k. In fact, if $c_1, \cdots, c_k$ have been chosen in this way and if c'_{k+1} generates ε_{k+1}^*C, then $\xi_{k,k+1}^* c'_{k+1} = tc_k$ for some $t \in \mathbf{Z}$. Hence (iii) implies $\xi_{k+1,k}^* tc_k = p^{n_{k+1}-n_k} c'_{k+1}$, and the comparison of orders shows $(p, t) = 1$. This means $c_{k+1} = sc'_{k+1}$, with $st \equiv 1 \bmod p^{n_{k+1}}$ will satisfy $\xi_{k,k+1}^* c_{k+1} = c_k$. Then, by (iv), $\xi_{jk}^* c_k = c_j$ for all $j < k$.

To complete the proof of case 3, let $a \in A$ and pick an $\eta \in \mathrm{E}(A)$ such that $\eta a_k = a$ for some k. Define $\phi: a \mapsto \eta^* c_k$. This is a good definition, for if $\eta_1 a_j = a$ with $j \geqq k$, then $(\eta \xi_{kj} - \eta_1)\varepsilon_j = 0$ implies $(\eta^* \xi_{kj}^* - \eta_1^*)\varepsilon_j^* = 0$, whence $\eta^* c_k = \eta_1^* c_j$. The proof that ϕ is an isomorphism inducing ψ is again straightforward.□

Kaplansky [3] points out that the last theorem can be extended to primary modules over complete discrete valuation rings. If, however, the modules are not assumed to be primary, the result fails [see Ex. 1]. Some generalization is expected to hold if the endomorphism rings are furnished with the finite topology.

An immediate corollary to (108.1) is the following remarkable fact:

Corollary 108.2 (Baer [9]). *Every automorphism of the endomorphism ring of a torsion group is inner.*

In fact, if $\alpha: \mathrm{E}(A) \to \mathrm{E}(A)$ is an automorphism, then by (108.1) it must be of the form $\alpha: \eta \mapsto \phi\eta\phi^{-1}$ for some automorphism ϕ of A [=unit of $\mathrm{E}(A)$].□

Now we go on to find the center of the endomorphism ring of torsion groups. The general case immediately reduces to p-groups, and it is more elegant to formulate the result for p-groups.

Theorem 108.3 (Charles [1], Kaplansky [3]). *The center of the endomorphism ring $\mathrm{E}(A)$ of a p-group A consists of multiplications by p-adic integers or by rational integers* mod *p^k according as A is unbounded or p^k is the least upper bound for the orders of its elements.*

Multiplications by p-adic integers are trivially in the center of $\mathrm{E}(A)$.

Let γ belong to the center of $\mathrm{E}(A)$. If B is a summand of A and $\varepsilon: A \to B$ is a projection, then $\gamma B = \gamma\varepsilon B = \varepsilon\gamma B \leqq B$ indicates that γ maps every summand of A into itself. In particular, γ acts as a multiplication by a p-adic integer ρ and by an integer m mod p^k on $Z(p^\infty)$ and a summand $\langle g \rangle$ of order p^k, respectively. From now on, we divide the proof as in (108.1) and keep the notations used there.

1. Let $g \in A$ be of maximal order p^k, and $\gamma g = mg$. If $a = \eta g$, then $\gamma a = \gamma\eta g = \eta\gamma g = \eta m g = ma$ implies that γ is precisely the multiplication by m mod p^k.

2. Let γ act as multiplication by the p-adic integer ρ on D' and by the integer m on B. For any $d \in D$, there is an $\eta \in \mathrm{E}(A)$ such that $d = \eta d_n$ for

some n. Hence $\gamma d = \gamma\eta d_n = \eta\gamma d_n = \eta\rho d_n = \rho d$ implies that γ is the multiplication by ρ on all of D. Some $\xi \in \mathrm{E}(A)$ maps $g \in B$ onto $d_k \in D$, whence $\gamma d_k = \gamma\xi g = \xi\gamma g = md_k$ and $\rho \equiv m \bmod p^k$. Therefore, γ is the multiplication by ρ on A.

3. We know that $\gamma a_k = m_k a_k$ for some $m_k \in \mathbf{Z}$, and for all k. For $j > k$, from $m_k a_k = \gamma a_k = \gamma\xi_{kj} a_j = \xi_{kj}\gamma a_j = m_j a_k$ one concludes $m_k \equiv m_j \bmod o(a_k)$. Thus the m_k converge to a p-adic integer ρ such that $\gamma a_k = \rho a_k$ for all k. For $a = \eta a_k$, we find $\gamma a = \gamma\eta a_k = \rho\eta a_k = \rho a$ and the assertion follows.$\square$

EXERCISES

1. Show that (108.1) does not extend to p-adic modules. [*Hint*: compare J_p and $Z(p^\infty)$.]
2. (a) The endomorphisms α of a torsion group A for which $\operatorname{Im}\alpha$ is finitely cogenerated form an ideal $\mathrm{F}(A)$ in $\mathrm{E}(A)$.
   (b) $\mathrm{F}(A)$ can be recognized in $\mathrm{E}(A)$ as the ideal generated by the primitive idempotents of $\mathrm{E}(A)$.
3. (a) Analyzing the proof of (108.1), verify that two torsion groups A and C are necessarily isomorphic if $\mathrm{F}(A) \cong \mathrm{F}(C)$.
   (b) (E. Pogány) Extend (a) to other ideals of the endomorphism rings [e.g., consisting of endomorphisms of countable images].
4. (a) If A is a torsion group and A_p are its p-components, then the center of $\mathrm{E}(A)$ is the product of the centers of $\mathrm{E}(A_p)$.
   (b) Extend (a) to the case when $A = \bigoplus_{i \in I} A_i$ with A_i fully invariant in A.
5. (Levy [2]) Describe the endomorphisms which map every subgroup into itself.
6. (Szele and Szendrei [1]) The endomorphism ring of a torsion group A is commutative exactly if A is a subgroup of Q/Z.
7. Let A be an adjusted cotorsion group and T its torsion part. Show that $\mathrm{E}(A)$ and $\mathrm{E}(T)$ can be identified in a natural fashion. [*Hint*: every $\eta \in \mathrm{E}(T)$ extends uniquely to an $\bar{\eta} \in \mathrm{E}(A)$.]
8. Using Ex. 7, prove (108.1) for adjusted cotorsion groups A, C.
9. (a) A torsion group A is finitely generated as an $\mathrm{E}(A)$-module if and only if it is bounded. Otherwise it is countably generated.
   (b) For a p-group A, $\mathrm{E}(A)$ is locally cyclic as a left $\mathrm{E}(A)$-module in the sense that for all $a, b \in A$ there is a $c \in A$ such that $a, b \in \mathrm{E}(A)c$.
10. (a) Prove that if, for a group A, the automorphisms generate $\mathrm{E}(A)$ as a ring, then every characteristic subgroup of A must be fully invariant.
    (b) Show that for the group in Example of **67**, the automorphism group does not generate the endomorphism ring.

109. ENDOMORPHISM RINGS OF SEPARABLE p-GROUPS

In investigations on endomorphism rings, a general trend is to find conditions on an abstract ring to be the endomorphism ring of some abelian group. It seems to be a rather difficult problem to obtain necessary and sufficient conditions in general, but in a few special cases fairly satisfactory answer is known. In this section we discuss this problem for separable p-groups, while the next section will be devoted to the case of torsion-free groups.

Our program is, naturally, to collect a sufficient amount of information about the endomorphism ring $\mathsf{E}(A)$ of separable p-groups A to the extent that these properties can collectively be satisfied by an endomorphism ring only. Needless to say, our conditions must be ring-theoretical in nature: though they are consequences of certain properties of the underlying group A, they ought to be recaptured solely from the ring structure of $\mathsf{E}(A)$. Not unexpectedly, the projections onto cyclic summands are the main tools in our discussion: they can be identified in $\mathsf{E}(A)$ as primitive idempotents.

Let A be a separable p-group and $\mathsf{E} = \mathsf{E}(A)$ its ring of endomorphisms. We denote by E_0 the left ideal of E, generated by the primitive idempotents in E. Since primitive idempotents correspond to indecomposable summands, these idempotents are now of finite order.

(i) *The right annihilator of* E_0 *in* E_0 *is* 0.

If η is a right annihilator of E_0, then Im η has trivial projection in every cyclic summand of A. Therefore Im $\eta \leqq A^1 = 0$.

(ii) *For primitive idempotents* π, $\rho \in \mathsf{E}$, *the group* $\pi\mathsf{E}\rho$ *is cyclic of order* p^k *for some* k [p *being fixed*].

As in **106**(b) we can see at once that the group $\pi\mathsf{E}\rho$ can be regarded as Hom$(\rho A, \pi A)$. Here both ρA and πA are cyclic of prime power orders whence the same holds for $\pi\mathsf{E}\rho$.

(iii) *If* π, ρ *are primitive idempotents in* E *such that* $o(\pi) \leqq o(\rho)$, *then the left annihilator of* $\mathsf{E}\rho$ *is contained in the left annihilator of* $\mathsf{E}\pi$ *and we have* $\mathsf{E}\pi\mathsf{E}\rho = \mathsf{E}\rho[o(\pi)]$.

Write $\pi A = \langle a \rangle$ and $\rho A = \langle b \rangle$, and select an $\eta \in \mathsf{E}$ such that $\eta b = a$. If $\xi \in \mathsf{E}$ satisfies $\xi\mathsf{E}\rho = 0$, then $(\xi\mathsf{E})b = 0$, thus $\xi\mathsf{E}a = \xi\mathsf{E}(\eta b) = 0$, and ξ annihilates E, too. The inclusion $\mathsf{E}\pi\mathsf{E}\rho \subseteq \mathsf{E}\rho[o(\pi)]$ being obvious, let $\chi \in \mathsf{E}\rho[o(\pi)]$. Then $\chi b = A[o(\pi)]$ and $\lambda a = \chi b$ for some $\lambda \in \mathsf{E}\pi$. Now $\chi b = \lambda\eta b$ implies $\chi = \lambda\eta \in \mathsf{E}\pi\mathsf{E}\rho$.

(iv) *If* $\mathsf{E}_0 = \mathsf{K} \oplus \mathsf{L}$ *with right ideals* K, L, *and if* $\mathsf{K} \neq 0$, *then* $\tau\mathsf{L} = 0$ *for some primitive idempotent* $\tau \in \mathsf{E}_0$.

Clearly, $A = \mathsf{E}_0 A = \mathsf{K}A + \mathsf{L}A$ (where $\mathsf{K}A$ denotes the set of all finite sums of the form $\sum \kappa_i a_i$ with $\kappa_i \in \mathsf{K}$, $a_i \in A$). Let $x \in \mathsf{K}A \cap \mathsf{L}A$ and write $x = \sum \kappa_i a_i = \sum \lambda_j b_j$ ($\lambda_j \in \mathsf{L}$, $b_j \in A$). If $A = \langle c \rangle \oplus C$ where $o(c) \geqq \max\{o(a_i), o(b_j)\}$

and if $\xi_i, \eta_j \in \mathsf{E}_0$ satisfy $a_i = \xi_i c$, $b_j = \eta_j c$, $\xi_i C = \eta_j C = 0$, then $\sum \kappa_i \xi_i c = x = \sum \lambda_j \eta_j c$ implies $\sum \kappa_i \xi_i = \sum \lambda_j \eta_j \in \mathsf{K} \cap \mathsf{L} = 0$. Thus $x = 0$ and $A = \mathsf{K}A \oplus \mathsf{L}A$. Here $\mathsf{K}A \neq 0$, so we can choose τ as a projection onto a cyclic summand of $\mathsf{K}A$.

(v) E *is the completion of* E_0 *in the topology obtained by taking the left annihilators of the primitive idempotents of* E_0 *as a subbase of neighborhoods of* 0 *in* E_0.

In view of (107.2) and (107.3), there is nothing to prove. [Recall that in our definition of completeness, the Hausdorff property was included; see **13**.]

We want to show that (i)–(v) together characterize the endomorphism rings of separable p-groups. The next result was obtained in a slightly different form by Liebert [4].

Theorem 109.1. *An associative ring* E *with* 1 *is isomorphic to the endomorphism ring* $\mathsf{E}(A)$ *of some separable p-group* A *if and only if it satisfies conditions* (i)–(v).

(a) Suppose $\mathsf{E} \neq 0$ satisfies (i)–(v). Then $\mathsf{E}_0 \neq 0$ and E must contain primitive idempotents. If $\pi \in \mathsf{E}$ is one, then from (ii) and $\pi \in \pi\mathsf{E}\pi$ we see that $o(\pi) = p^k$ for some k. Thus the primitive idempotents of E are of finite order.

(b) Let π, ρ be primitive idempotents of E such that $o(\pi) = p^k \leqq o(\rho)$. Because of (ii), we may write $\pi\mathsf{E}\rho = \langle \xi \rangle$ for some $\xi \in \mathsf{E}$. Using this ξ, define $\phi\colon \mathsf{E}\pi \to \mathsf{E}\rho$ as $\eta \mapsto \eta\xi$ ($\eta \in \mathsf{E}\pi$). This is manifestly an E-homomorphism between the left ideals $\mathsf{E}\pi$ and $\mathsf{E}\rho$. It is monic, for if $\eta\xi = 0$, then $\eta\mathsf{E}\rho = 0$ and, by (iii), $\eta\mathsf{E}\pi = 0$ too; in particular, $\eta = \eta\pi = 0$.

(c) Now we choose a sequence $\pi_1, \cdots, \pi_n, \cdots$ of primitive idempotents in E such that $o(\pi_1) < \cdots < o(\pi_n) < \cdots$; this sequence can be chosen infinite, unless the orders of primitive idempotents in E are bounded. Just as in the preceding paragraph, we set $\pi_n \mathsf{E} \pi_{n+1} = \langle \xi_n \rangle$ and define the monomorphisms

$$\phi_n\colon \mathsf{E}\pi_n \to \mathsf{E}\pi_{n+1} \qquad \text{as} \qquad \eta \mapsto \eta\xi_n$$

for each n. With the aid of the direct system of the E-modules $\mathsf{E}\pi_n$ and E-monomorphisms ϕ_n, we are able to define a group A as the E-module

$$A = \lim_{n\to\infty} \mathsf{E}\pi_n .$$

$\mathsf{E}\pi_n$ being annihilated by $o(\pi_n)$, A is a p-group; it is bounded if and only if the sequence $\{\pi_n\}$ is finite. The canonical homomorphism of $\mathsf{E}\pi_n$ into the limit group is monic, thus it identifies $\mathsf{E}\pi_n$ with a submodule of A, and A is simply their set-union. Since the ϕ_n are E-homomorphisms, every $\xi \in \mathsf{E}$ gives rise to an endomorphism of A. Here $\xi A = 0$ only if $\xi\mathsf{E}\pi_n = 0$ for all n. But then, by (iii), $\xi\mathsf{E}\rho = 0$ for every primitive idempotent $\rho \in \mathsf{E}$, and the Hausdorff property in (v) implies $\xi = 0$. Consequently, E is isomorphic to a subring of the endomorphism ring $\mathsf{E}(A)$ of the group A.

(d) The next step is to verify that E and E(A) share their primitive idempotents. If ρ is a primitive idempotent in E, then it must be primitive in E(A), for $\rho A = \varinjlim \rho \mathsf{E}\pi_n$, being a direct limit of cyclic p-groups of orders $\leqq o(\rho)$, is itself cyclic. On the other hand, if σ is a primitive idempotent of order p^m in E(A), then $A = \sigma A \oplus B$ where $B = \text{Ker } \sigma$. If H(σA), H(B) denote the set of all $\eta \in \mathsf{E}_0$ with $\eta A \leqq \sigma A$ and $\leqq B$, respectively, then they are right ideals of E_0 such that $\mathsf{E}_0 = \mathsf{H}(\sigma A) \oplus \mathsf{H}(B)$. By (iv), $\tau \cdot \mathsf{H}(B) = 0$ for some primitive idempotent $\tau \in \mathsf{E}_0$. Using the fact that the orders of primitive idempotents in E(A) cannot be larger than those in E_0, we infer $\tau B = 0$. Clearly, $A = \tau A \oplus \text{Ker } \tau$ and $B \leqq \text{Ker } \tau$ imply $B = \text{Ker } \tau$. By (9.5), $\sigma = \tau - (1 - \tau)\phi\tau$ for some $\phi \in \mathsf{E}(A)$. But $\mathsf{E}(A)\tau = \mathsf{E}\tau$, since by (iii), $\mathsf{E}\tau\mathsf{E}\pi_n = \mathsf{E}\pi_n[o(\tau)]$ for $o(\tau) \leqq o(\pi_n)$, and the latter group is essentially $A[o(\tau)]$. Hence $\sigma \in \mathsf{E}\tau$, and σ belongs to E.

(e) To verify that E(A) does not have other primitive idempotents [i.e., A is reduced] and, moreover, A is separable, we observe that if $a \in A^1$ is represented by $\eta\pi_n \in \mathsf{E}\pi_n$, then $\rho a = \rho\eta\pi_n = 0$ must hold for all primitive idempotents $\rho \in \mathsf{E}(A)$ of finite order. From $\rho \in \mathsf{E}_0$ and (i) we get $\eta\pi_n = 0$, i.e., $A^1 = 0$.

(f) To conclude the proof, we refer to (v), which tells us that E is the completion of E_0 in the finite topology of E_0. On the other hand, because of the separability of A, the same holds for E(A), and then $\mathsf{E} \leqq \mathsf{E}(A)$ implies $\mathsf{E}(A) = \mathsf{E}$.□

EXERCISES

1. (Shoda [1], Baer [9]) (a) For a subset S of A, define $\Lambda(S) = \{\eta \in \mathsf{E}(A) \mid \eta S = 0\}$ and $P(S) = \{\eta \in \mathsf{E}(A) \mid \eta A \leqq S\}$. Show that they are one-sided ideals such that $\Lambda(S) \cdot P(S) = 0$.

   (b) For a subset Σ of E(A), $N(\Sigma) = \{a \in A \mid \Sigma a = 0\}$ is a subgroup of A such that $\Lambda(\Sigma A)$ is the left and $P(N(\Sigma))$ is the right annihilator of Σ in E(A).
2. (Liebert [1]) Let E be a finite ring with 1 such that $p^k\mathsf{E} = 0$ but $p^{k-1}\mathsf{E} \neq 0$. Then E is isomorphic to the endomorphism ring of a finite group A if and only if the following conditions are satisfied:
   (i) E has an antiautomorphism;
   (ii) every primitive idempotent ρ of E satisfies $\rho\mathsf{E}\rho = \langle\rho\rangle$;
   (iii) $p^i\mathsf{E} \cap \Lambda(\mathsf{E}[p^{k-1}])$ with $i = 0, \cdots, k$ are the only ideals of E contained in the right socle [=union of all minimal right ideals] of E.
3. (Liebert [2]) (a) If A is the direct sum of cyclic groups of the same order p^k and if $A = B \oplus C$ with $r(B) \leqq r(C)$, then every endomorphism ϕ of A with $\phi C = 0$ is contained in the subring $\mathsf{E}_0(A)$ of E(A) generated by the idempotents of E(A).

   (b) If $A = A_1 \oplus \cdots \oplus A_n$ and if, for all i, $\mathsf{E}_0(A_i) = \mathsf{E}(A_i)$, then $\mathsf{E}_0(A) = \mathsf{E}(A)$, too.

(c) Conclude that the endomorphism ring of a bounded group is generated by its idempotent elements.

4. Characterize the endomorphism ring of a divisible p-group of finite rank.
5. For a reduced p-group A, the set of all left annihilators of primitive idempotents is $P(A^1)$ [cf. Ex. 1].
6. Let $A = C \oplus D$ be a p-group where C is reduced and D is divisible.
(a) The primitive idempotents of infinite order in $\mathsf{E}(A)$ generate a left ideal isomorphic to $\mathsf{E}(D)$.
(b) The quotient of $\mathsf{E}(A)$ modulo the ideal generated by the primitive idempotents of infinite order is isomorphic to $\mathsf{E}(C)$.

110. COUNTABLE TORSION-FREE ENDOMORPHISM RINGS

Turning our attention to the endomorphism rings of torsion-free groups the first thing we observe is that nonisomorphic torsion-free groups may very well have isomorphic endomorphism rings. Illustrative examples are abundant; e.g., it is easy to find such examples in rigid systems or among groups of different cardinalities, or even among groups of rank 1.

Another major difference in the behavior of endomorphism rings between torsion and torsion-free groups is that, while in the torsion case they belong to a rather special class of rings, far less restriction is displayed in the torsion-free case. This statement is fully justified in view of the following striking theorem.

Theorem 110.1 (Corner [3]). *Every countable reduced torsion-free ring* R *with* 1 *is isomorphic to the endomorphism ring* $\mathsf{E}(A)$ *of a countable reduced torsion-free group* A.

We devote this section to the proof of this powerful result. The idea of localization in the proof is due to Orsatti [6], so Corner's method will be needed merely in the more tractable local case.

(a) First, we localize the problem, and in addition assume that our ring R_p is an algebra over Q_p for some prime p; this amounts to saying that $q\mathsf{R}_p = \mathsf{R}_p$ for all primes $q \neq p$. We furnish R_p with the p-adic topology; the reducedness of R_p guarantees that this is Hausdorff. Then we form the completion $\hat{\mathsf{R}}_p$ in the p-adic topology; this is a Q_p^*-algebra containing R_p as a (pure) subring [cf. (119.4)]. R_p being countable, there is a finite or a countably infinite set $\{\xi_n\}$ of elements of R_p which is maximal independent over Q_p^*. That is, for every $\alpha \in \mathsf{R}_p$ there is a dependence relation

$$p^n\alpha = \pi_1\xi_1 + \cdots + \pi_k\xi_k \qquad (\pi_i \in \mathsf{Q}_p^*).$$

Here the π_i are uniquely determined up to factors p^m. Thus it is meaningful to speak of the pure subring S_p of Q_p^* generated by Q_p and the π_i, taken for all $\alpha \in \mathsf{R}_p$. Trivially, S_p is still countable.

(b) Next we prove that if $\gamma_1\alpha_1 + \cdots + \gamma_m\alpha_m = 0$, for some $\alpha_1, \cdots, \alpha_m \in \mathsf{R}_p$, where $\gamma_1, \cdots, \gamma_m \in \mathsf{Q}_p^*$ are linearly independent over S_p, then $\alpha_1 = \cdots = \alpha_m = 0$. In fact, for a sufficiently large n, $p^n\alpha_j = \sum_i \pi_{ji}\xi_i$ with $\pi_{ji} \in \mathsf{S}_p$, thus $\sum_{i,j} \gamma_j \pi_{ji} \xi_i = 0$ and, by the independence of the ξ_i, $\sum_j \gamma_j \pi_{ji} = 0$. Hence all $\alpha_j = 0$.

(c) For every α in R_p, we choose p-adic integers ρ_α and σ_α such that the set $\{\rho_\alpha, \sigma_\alpha | \alpha \in \mathsf{R}_p\}$ is algebraically independent over S_p. This is possible, for S_p is countable, Q_p^* is continuum, and thus the transcendence degree of Q_p^* over S_p is likewise continuum. With these ρ_α, σ_α, we set

$$\varepsilon_\alpha = \rho_\alpha 1 + \sigma_\alpha \alpha \in \hat{\mathsf{R}}_p. \tag{1}$$

Define the group A as the pure subgroup

$$A = \langle \mathsf{R}_p, \mathsf{R}_p \varepsilon_\alpha \text{ for all } \alpha \in \mathsf{R}_p \rangle_*$$

in $\hat{\mathsf{R}}_p$. Obviously, A is countable, reduced, and torsion-free.

(d) In view of this definition, it is evident that the elements of R_p act from the left on A as given by the multiplication in $\hat{\mathsf{R}}_p$. Different elements of R_p act differently because of $1 \in A$. Hence R_p is [isomorphic to] a subring of $\mathsf{E}(A)$.

(e) To show the equality $\mathsf{R}_p = \mathsf{E}(A)$, take some $\eta \in \mathsf{E}(A)$. As R_p is pure in A which is pure in $\hat{\mathsf{R}}_p$, the p-adic completion $\hat{A}$ of A is again $\hat{\mathsf{R}}_p$. Therefore, **106**(j) guarantees the extensibility of η to a unique Q_p^*-endomorphism $\hat{\eta}$ of $\hat{\mathsf{R}}_p$. Using this $\hat{\eta}$, we obtain

$$\eta\varepsilon_\alpha = \hat{\eta}(\rho_\alpha 1 + \sigma_\alpha \alpha) = \rho_\alpha(\hat{\eta}1) + \sigma_\alpha(\hat{\eta}\alpha) = \rho_\alpha(\eta 1) + \sigma_\alpha(\eta\alpha).$$

By the definition of A, we can write more explicitly

$$p^k(\eta\varepsilon_\alpha) = \beta_0 + \sum_{i=1}^n \beta_i \varepsilon_{\alpha_i},$$

$$p^k(\eta 1) = \gamma_0 + \sum_{i=1}^n \gamma_i \varepsilon_{\alpha_i},$$

$$p^k(\eta\alpha) = \delta_0 + \sum_{i=1}^n \delta_i \varepsilon_{\alpha_i}$$

for α, β, γ, $\delta \in \mathsf{R}_p$ and for some fixed k and n; for simplicity, we suppose $\alpha = \alpha_1$. Substitution gives

$$\begin{aligned}\beta_0 &+ \sum_{i=1}^n \beta_i(\rho_{\alpha_i} 1 + \sigma_{\alpha_i}\alpha_i) \\ &= \rho_\alpha\left[\gamma_0 + \sum_{i=1}^n \gamma_i(\rho_{\alpha_i} 1 + \sigma_{\alpha_i}\alpha_i)\right] + \sigma_\alpha\left[\delta_0 + \sum_{i=1}^n \delta_i(\rho_{\alpha_i} 1 + \sigma_{\alpha_i}\alpha_i)\right].\end{aligned}$$

By the choice of the ρ_α and σ_α, the ρ_α, σ_α and their products are linearly independent over S_p, thus by (b) comparison of coefficients on both sides yields $\beta_1 = \gamma_0$ and $\beta_1 \alpha = \delta_0$, and all the other β, γ, and δ vanish. We find $p^k(\eta 1) = \gamma_0$ and $p^k(\eta\alpha) = \gamma_0 \alpha$, whence with the notation $\eta 1 = \gamma$, the equality $\eta\alpha = \gamma\alpha$ follows for all $\alpha \in R_p$. In consequence, η acts on R_p as multiplication by γ from the left. The same holds for $\hat{\eta}$ and for $\eta = \hat{\eta} | A$. This completes the proof in the local case.

(f) Proceeding to the global case, suppose R is as formulated in the theorem. The Z-adic completion $\hat{R}$ of R will contain R as a pure subring, due to the reducedness and torsion-freeness of R. (40.1) gives a representation

$$\hat{R} = \prod_p \hat{R}_p,$$

where $\hat{R}_p$ is the p-adic completion of the reduced part R_p of $Q_p \otimes R$. Notice that the canonical embedding $R \to \hat{R}$ makes R into a subring of $\prod_p R_p$; for $\alpha \in R$, we may write $\alpha = (\cdots, \alpha_p, \cdots)$ with $\alpha_p \in R_p$.

(g) Just as in (c), for every α in R, we choose ρ_α, σ_α as $\rho_\alpha = (\cdots, \rho_{\alpha_p}, \cdots)$, $\sigma_\alpha = (\cdots, \sigma_{\alpha_p}, \cdots)$ with p-adic integers ρ_{α_p}, σ_{α_p} subject to the condition of algebraic independence over S_p [if $R_p = 0$ for some p, then $\rho_{\alpha_p} = \sigma_{\alpha_p} = 0$ may be chosen]. With the aid of ε_α in (1), the group A is defined as

(2) $$A = \langle R, R\varepsilon_\alpha \text{ for all } \alpha \in R\rangle_*,$$

which is again a countable reduced subgroup of $\hat{R}$. That R is a subring of $E(A)$ is trivial. Given $\eta \in E(A)$, it extends uniquely to an $\hat{\eta} \in E(\hat{R})$ which must act coordinate-wise, $\hat{R}_p$ being fully invariant in $\hat{R}$. The local case (e) shows that $\hat{\eta}$ acts on $\hat{R}_p$ as left multiplication by the R_p-component of $\eta 1 = \gamma \in R$, thus $\hat{\eta}$ agrees with the left multiplication by γ on all of $\hat{R}$, and hence on A. This establishes $E(A) \cong R$.

To summarize, the group A as given in (2) is a countable reduced torsion-free group whose endomorphism ring is isomorphic to the ring R given in (110.1).□

Applications of (110.1) are immediate to verifying the existence of countable torsion-free groups with certain properties expressible in terms of endomorphisms [see, e.g., (91.5) and (91.6)].

It should be pointed out that the endomorphism rings of several countable torsion-free groups are of the power of the continuum. These rings, if furnished with the finite topology, can also be characterized [cf. Corner [6]].

If the ring R in the theorem is of countable rank, then the group A must obviously be at least of countable rank. However, for rings of finite rank, (110.1) can be improved:

Theorem 110.2 (Corner [3]). *Every reduced torsion-free ring* R *of finite rank n, with* $1 \in$ R, *is isomorphic to the endomorphism ring of a reduced torsion-free group A of rank* $\leqq 2n$.

Let $1 = \alpha_1, \cdots, \alpha_n$ be linearly independent elements of R [over Z] and choose $\rho_i = (\cdots, \rho_{ip}, \cdots)$ $(i = 1, \cdots, n)$ such that, for each prime p, the p-adic integers $\rho_{1p}, \cdots, \rho_{np}$ are algebraically independent over S_p. In the Z-adic completion $\hat{R}$ of R, take

$$\varepsilon = \rho_1 \alpha_1 + \cdots + \rho_n \alpha_n$$

and form

$$A = \langle R, R\varepsilon \rangle_* . \tag{3}$$

Then A is obviously reduced, torsion-free, and of rank $\leqq 2n$, such that R is a subring of E(A). Pick again an $\eta \in$ E(A) and extend it to an endomorphism $\hat{\eta}$ of $\hat{R}$. Then $\eta\varepsilon = \sum_{i=1}^n \rho_i \eta \alpha_i$ where, for some positive integer m, and for $\beta_i, \gamma_i \in$ R, we have

$$m(\eta\varepsilon) = \beta_0 + \gamma_0 \varepsilon, \qquad m(\eta\alpha_i) = \beta_i + \gamma_i \varepsilon \qquad (i = 1, \cdots, n).$$

Substitution gives

$$\beta_0 + \gamma_0 \Big(\sum_i \rho_i \alpha_i\Big) = \sum_i \rho_i \Big(\beta_i + \gamma_i \sum_j \rho_j \alpha_j\Big),$$

and the analog of (b) forces the equalities

$$\beta_0 = 0, \quad \gamma_0 \alpha_i = \beta_i \quad (i = 1, \cdots, n), \qquad 0 = \gamma_i \alpha_j + \gamma_j \alpha_i \quad (i, j = 1, \cdots, n).$$

The last equation with $i = j = 1$ gives $\gamma_1 = 0$, so setting $j = 1$ in the same equation we obtain $\gamma_i = 0$ $(i = 1, \cdots, n)$. Consequently, $m(\eta\varepsilon) = \gamma_0 \varepsilon$ and $m(\eta\alpha_i) = \beta_i = \gamma_0 \alpha_i$. So, writing $\eta 1 = \gamma \in$ R, we find $\eta\alpha_i = \gamma\alpha_i$. This shows that η agrees with the left multiplication by γ, and E(A) = R, in fact.□

Corner [3] points out that, for $n \geq 2$, the last result can not be sharpened: there is a torsion-free ring of rank n that is not isomorphic to the endomorphism ring of any group of rank $< 2n$. Zassenhaus [1] has given conditions for a ring of rank n to be the endomorphism ring of a torsion-free group of the same rank. For an extension of this result, see Butler [2].

EXERCISES

1. (Corner [3]) If E(A) is countable, reduced, and torsion-free, then A must be reduced and torsion-free.
2. (Corner [3]) Show that if we drop either of the three conditions in (110.1): (i) countable, (ii) reduced, (iii) torsion-free, then the ring R need not be an endomorphism ring. [*Hint*: $Q_p^* \oplus Q_p^*$, $Q \oplus Q$, $Z/(p) \oplus Z/(p)$.]
3. Given any countable semigroup S with 1, the semigroup-ring ZS of S over the integers can always be realized as an endomorphism ring.

4. (Sąsiada [6], Corner [3]) There exist torsion-free groups of finite rank with isomorphic endomorphism groups whose endomorphism rings are not isomorphic. [*Hint*: choose $\mathsf{Z} \oplus \mathsf{Z}$ and the Gaussian integers as endomorphism rings.]
5. (Orsatti [6]) Replace countability by the condition that the reduced part R_p of $\mathsf{Q}_p \otimes \mathsf{R}$ is countable for every prime p, and extend (110.1) in this way for $|A| = |\mathsf{R}|$.
6. Every torsion-free ring of rank 1, with identity, is the endomorphism ring of a torsion-free group of rank 1.
7. Show that the group in (3) is precisely of rank $2n$.
8. Let R be the ring of algebraic integers in an extension of degree 2 of the rationals such that ± 1 are the only units in R [e.g., choose $\mathsf{R} = \mathsf{Z}[\sqrt{-5}]$]. Let A be a torsion-free group of rank 4 with $\mathsf{E}(A) \cong \mathsf{R}$. Verify that:
   (a) all subgroups of A are characteristic;
   (b) all endomorphisms of A are monic;
   (c) A has no fully invariant subgroups of rank 1.
   Conclude that torsion-free groups of finite rank can have characteristic subgroups which are not fully invariant.
9. Given a cardinal $\mathfrak{m}$ less than the first strongly inaccessible one, there exist torsion-free groups A of cardinality $\mathfrak{m}$ which can not be generated as $\mathsf{E}(A)$-modules by less than $\mathfrak{m}$ elements. [*Hint*: rigid groups.]
10. For every cardinal $\mathfrak{m}$ as in Ex. 10, there exist commutative endomorphism rings of cardinality $2^{\mathfrak{m}}$. [*Hint*: direct sum of a rigid system.]
11. (J. D. Reid [5], Orsatti [2]) A ring R is said to be *subcommutative* if for all $\alpha, \beta \in \mathsf{R}$ there exists $\gamma \in \mathsf{R}$ such that $\alpha\gamma = \beta\alpha$.
    (a) If $\mathsf{E}(A)$ is subcommutative, then endomorphic images of A are fully invariant.
    (b) Using the fact that the rational quaternions contain a subcommutative [but not commutative] ring with 1 whose additive group is reduced, conclude that the full invariance of endomorphic images does not imply the commutativity of the endomorphism ring.
12. The center of the endomorphism ring of a homogeneous separable torsion-free group is isomorphic to a subring of Q.
13. A homogeneous separable torsion-free group A has no fully invariant subgroups other than nA ($n = 0, 1, 2, \ldots$).

111. ENDOMORPHISM RINGS WITH SPECIAL PROPERTIES

So far our investigations of the interrelation between groups and their endomorphism rings have chiefly been concerned with the question as to how the group structure is reflected in its endomorphism ring. Turning things around, we can now ask what influence the ring structure of the endomorphism rings has on the underlying groups. Naturally, our main interest lies in the conventional ring-theoretical properties.

First, we pause to make a few simple observations of a technical nature.

(a) *If* $\alpha \in \mathsf{E}(A)$, *then* $n|\alpha$ *implies* $\alpha A \leqq nA$, *and* $n\alpha = 0$ *implies* $\alpha A \leqq A[n]$.

If $\beta \in \mathsf{E}(A)$ satisfies $n\beta = \alpha$, then $\alpha A = n\beta A \leqq nA$. If $n\alpha = 0$, then $n\alpha A = 0$ and $\alpha A \leqq A[n]$.

(b) *Suppose the group* A *has a decomposition*

$$A = A_1 \oplus \cdots \oplus A_n \oplus C_n \qquad \text{with} \quad C_n = A_{n+1} \oplus C_{n+1}$$

and $A_n \neq 0$, *for every* n. *Then neither the right nor the left* [*principal*] *ideals of* $\mathsf{E} = \mathsf{E}(A)$ *satisfy the minimum condition.*

Let $\pi_n \colon A \to C_n$ be the obvious projection. Then $\pi_n \pi_{n+1} = \pi_{n+1}$, but no $\alpha \in \mathsf{E}$ exists with $\pi_{n+1}\alpha = \pi_n$, because $\operatorname{Im} \pi_{n+1}\alpha \leqq \operatorname{Im} \pi_{n+1} < \operatorname{Im} \pi_n$. Hence the proper inclusions $\pi_n \mathsf{E} \supset \pi_{n+1}\mathsf{E}$ $(n = 1, 2, \cdots)$ hold. Also, $\pi_{n+1}\pi_n = \pi_{n+1}$, and there is no $\beta \in \mathsf{E}$ with $\beta\pi_{n+1} = \pi_n$, since $\operatorname{Ker} \pi_n < \operatorname{Ker} \pi_{n+1} \leqq \operatorname{Ker} \beta\pi_{n+1}$. This shows $\mathsf{E}\pi_n \supset \mathsf{E}\pi_{n+1}$.

From $\mathsf{E} = (1 - \pi_n)\mathsf{E} \oplus \pi_n \mathsf{E} = \mathsf{E}(1 - \pi_n) \oplus \mathsf{E}\pi_n$ and from (b) we are at once led to the following conclusion [which is, as a matter of fact, stronger than (b) by virtue of (123.3)]:

(c) *If* A *is as in* (b), *then* $\mathsf{E}(A)$ *has an infinite, properly ascending chain of right* [*left*] *ideals.*

Our discussion starts with the case of [not necessarily commutative] fields. It is surprising to learn how rarely they occur among endomorphism rings.

Theorem 111.1 (Szele [5]). *The endomorphism ring* $\mathsf{E}(A)$ *of* A *is a field if and only if* A *is isomorphic to* Q *or to* $Z(p)$ *for some* p [*i.e.*, A *is the additive group of some prime field*].

A field is the endomorphism ring of an abelian group exactly if it is a prime field.

If $\mathsf{E}(A)$ is a field, then every nonzero $\alpha \in \mathsf{E}(A)$ is an automorphism, thus for every prime p, either $pA = 0$ or $pA = A$. If $pA = 0$ for some p, then A is an elementary p-group. Projections onto nonzero summands are now automorphisms, hence A is indecomposable and $A \cong Z(p)$. On the other hand, if $pA = A$ for all p, then A is divisible, and—as in the first alternative—indecomposable. Multiplication by p has 0 kernel, thus A is torsion-free, which leaves only Q as a possibility.

Conversely, the endomorphism rings of Q and $Z(p)$ are the prime fields of characteristic 0 and p, respectively.□

Next we examine simple rings, i.e., rings having no nontrivial ideals. The simple endomorphism rings can be listed easily:

Theorem 111.2. $\mathsf{E}(A)$ *is a simple ring if and only if A is a finite direct sum of full rational groups Q or a finite direct sum of cyclic groups of fixed prime order p.*

A simple ring is the endomorphism ring of an abelian group exactly if it is a complete matrix ring of finite order over a prime field.

If $\mathsf{E} = \mathsf{E}(A)$ is simple, then for every prime p, either $p\mathsf{E} = 0$ or $p\mathsf{E} = \mathsf{E}$. In the first alternative, from (a) it follows readily that $A \leqq A[p]$, thus A is an elementary p-group. If $p\mathsf{E} = \mathsf{E}$ for every prime p, then E is divisible, and so is A, owing to (a). The socle of E is an ideal [cf. **117**(A)], hence E must be torsion-free, and multiplication by p^{-1} is an endomorphism of A. We deduce that A, too, is torsion-free. Thus either $A = \bigoplus Z(p)$ or $A = \bigoplus Q$. The endomorphisms of A mapping A onto a subgroup of finite rank form a nonzero ideal in E which ought to be equal to E. Hence A is a finite direct sum, and necessity follows.

The sufficiency is, owing to (106.1), clear, and so is the second assertion of the theorem.□

In particular, (111.2) shows that a simple endomorphism ring must satisfy the minimum condition on left and right ideals. Passing to Artinian rings in general, we find:

Theorem 111.3. *The endomorphism ring* $\mathsf{E}(A)$ *of a group A is left [or right] Artinian exactly if $A = B \oplus D$, where B is a finite group and D is a torsion-free divisible group of finite rank.*

Supposing $\mathsf{E} = \mathsf{E}(A)$ left [or right] Artinian, there is an integer $m > 0$ such that $m\mathsf{E}$ is divisible [see, e.g., **122**]. By (a), $nm \mid m1_A$ implies $mA \leqq nmA$, i.e., mA is divisible. Thus $A = B \oplus D$ with $mB = 0$ and D divisible. (b) implies that both B and D are of finite rank; in particular, B is a finite group. We can rule out the presence of a $Z(p^\infty)$ in D, since $m1_A$ has to be divisible by the powers of p. So A must be of the form stated in the theorem.

Conversely, if A has this structure, then $\mathsf{E}(A) = \mathsf{E}(B) \oplus \mathsf{E}(D)$, for both B and D are fully invariant in A. Here $\mathsf{E}(B)$ is finite, while $\mathsf{E}(D)$ is a complete matrix ring of finite order over Q. The left and right Artinian property of $\mathsf{E}(A)$ is now immediate.□

Notice that if A is as in (111.3), then its endomorphism ring is the ring direct sum of the finite ring $\mathsf{E}(B)$ and $\mathsf{E}(D)$, where the latter ring is isomorphic to a complete matrix ring of finite order over Q.

Reviewing the last proof, one may notice that the same conclusion can be reached under the milder hypothesis that $\mathsf{E}(A)$ satisfies the minimum condition on left [or right] principal ideals [see Szász [1]].

In contrast to minimum condition, the maximum condition in the endomorphism ring does not restrict so strongly the group structure, as is shown, e.g., by large indecomposable groups with a subring of Q as endomorphism ring. For the torsion case, however, the maximum condition turns out to be a heavy restriction:

Proposition 111.4. *Let A be a torsion group.* $\mathsf{E}(A)$ *is left* [*or right*] *Noetherian if and only if A is the direct sum of a finite number of cocyclic groups.*

From (c), the finiteness of the basic subgroup B of A is readily checked [see (32.4)], thus by (27.5), $A = B \oplus C$ with C divisible, necessarily of finite rank.

To prove the converse, let $A = B \oplus D$ with B finite and D divisible of finite rank, and let $\pi: A \to D$ be the obvious projection. Then $\mathsf{E} = \mathsf{E}\pi \oplus \mathsf{E}(1 - \pi)$ where the second summand is finite, while the first summand $\mathsf{E}\pi = \pi\mathsf{E}\pi$ is, in view of (106.1), a complete matrix ring over Q_p^* and thus again a Noetherian ring [on both sides]. Consequently, both $\mathsf{E}\pi$ and $\mathsf{E}(1 - \pi)$ are Noetherian left E-modules, and so is E. A similar argument applies to the other side.□

More conditions on endomorphism rings will be discussed in the next section.

EXERCISES

1. (a) Prove that $\mathsf{E}(A)$ is a semisimple ring with minimum condition on left [or, equivalently, right] ideals exactly if $A = B \oplus D$, where B is a finite elementary group and D is torsion-free divisible of finite rank.
   (b) Which semisimple rings with minimum condition are endomorphism rings?
2. (A. Kertész) The Jacobson radical of $\mathsf{E}(A)$, for a p-group A, vanishes exactly if A is elementary. [*Hint*: $\sum_{n=1}^{\infty}(-1)^n p^n \theta^n$ is quasi-inverse to $p\theta$.]
3. Prove the following symmetry for torsion groups A: the ring $\mathsf{E}(A)$ satisfies the minimum [maximum] condition on left (or right) ideals if and only if the group A satisfies the maximum [minimum] condition on subgroups.
4. The radical of an Artinian endomorphism ring is finite.
5. Describe the p-groups whose endomorphism rings satisfy the minimum [maximum] condition on left or right annihilators of elements.
6. If $\mathsf{E}(A)$ has the maximum condition on left [or right] ideals, then the torsion part of A is finitely cogenerated.
7. (a) If $\mathsf{E}(A)$ is a domain, then A is indecomposable.
   (b) Show that not every endomorphism ring which is a domain can be embedded in a field. [*Hint*: use (110.1).]

8. (Szélpál [3]) (a) $\mathrm{E}(A)$ is a torsion ring if and only if A is a bounded group. (b) $\mathrm{E}(A)$ is torsion-free exactly if $A = D \oplus C$, where D is a divisible torsion group and C is torsion-free such that $pC = C$ for primes p with $D[p] \neq 0$.
9. (Orsatti [1]) (a) Every group A with local endomorphism ring must be indecomposable.
   (b) A nontorsion-free group has a local endomorphism ring exactly if it is cocyclic.
   (c) Pure subgroups of finite rank in J_p have local endomorphism rings.
10. List the groups in which every endomorphism is either an automorphism or nilpotent.
11. The endomorphism ring of a separable torsion-free group A is left [right] Noetherian if and only if A is of finite rank.
12. (Szele and Szendrei [1]) (a) If $\mathrm{E}(A)$ is commutative, then A has cocyclic p-components T_p and A/T is p-divisible for every prime p with $T_p \neq 0$.
    (b) A splitting group A has commutative endomorphism ring exactly if both $\mathrm{E}(T)$ and $\mathrm{E}(A/T)$ are commutative and if A satisfies the condition in (a).

112. REGULAR AND GENERALIZED REGULAR ENDOMORPHISM RINGS

We continue our search for groups whose endomorphism rings belong to certain interesting classes of rings. In this section, we focus our attention on endomorphism rings which are regular or generalized regular in a suitable sense.

To begin with, we state the relevant definitions.

An element α of the ring R is said to be *regular* if

$$\alpha\beta\alpha = \alpha \qquad \text{for some} \quad \beta \in \mathrm{R},$$

and *m-regular* for an integer $m \geqq 1$ if α^m is a regular element. α is called *left* [*right*] *regular* if

$$\alpha = \gamma\alpha^2 \qquad [\alpha = \alpha^2\gamma] \qquad \text{for some} \quad \gamma \in \mathrm{R}.$$

If α^m is left [right] regular, we then say α is *left* [*right*] *m-regular.* A ring R is *regular*, *m-regular*, etc. if each of its elements has the same property, while it is called [left, right] *π-regular* if every element of R is [left, right] m-regular for some integer m, depending on the element.

Our results lean heavily on the following simple, but crucial lemma.

Lemma 112.1 (Rangaswamy [8]). *An endomorphism α of A is a regular element in* $\mathrm{E}(A)$ *if and only if both* Im α *and* Ker α *are direct summands of A.*

Let $\alpha \in \mathsf{E}(A)$ satisfy $\alpha\beta\alpha = \alpha$ for some $\beta \in \mathsf{E}(A)$. Since $\alpha\beta$ and $\beta\alpha$ are idempotents in $\mathsf{E}(A)$, they are projections of A, and therefore their images and kernels are summands of A. Evidently,

$$\operatorname{Im} \alpha\beta\alpha \leqq \operatorname{Im} \alpha\beta \leqq \operatorname{Im} \alpha \qquad \text{and} \qquad \operatorname{Ker} \alpha \leqq \operatorname{Ker} \beta\alpha \leqq \operatorname{Ker} \alpha\beta\alpha,$$

whence $\operatorname{Im} \alpha = \operatorname{Im} \alpha\beta$ and $\operatorname{Ker} \alpha = \operatorname{Ker} \beta\alpha$ are summands of A.

Conversely, suppose $\operatorname{Im} \alpha = G$ and $\operatorname{Ker} \alpha = K$ are summands of A, $A = G \oplus H = K \oplus L$ for some $H, L \leqq A$. Owing to $L \cap K = 0$, we can assert that $\alpha | L$ maps L isomorphically into, and hence onto G. Consequently, there is a $\beta \in \mathsf{E}(A)$ which acts trivially on H and is inverse to $\alpha | L$ on G. Writing $a \in A$ in the form $a = a_1 + a_2$ $(a_1 \in K, a_2 \in L)$, we obtain $(\alpha\beta\alpha)a = \alpha[\beta(\alpha a_2)] = \alpha a_2 = \alpha a$, whence $\alpha\beta\alpha = \alpha$.□

The following result is an immediate consequence of our lemma; it is merely an intrinsic restatement of the condition of having regular [π-regular] endomorphism rings.

Proposition 112.2. *The endomorphism ring of a group A is regular [π-regular] if and only if both the images and the kernels of [suitable powers of] the endomorphisms of A are summands of A.*□

Before becoming involved with the discussion of the structures of groups whose endomorphism rings are regular or π-regular, let us exhibit a few examples and prove subsequently a preliminary result which we shall need later on.

Example 1. The endomorphism ring of an elementary group is regular. In fact, every subgroup is a summand, thus (112.2) implies the regularity of the endomorphism ring.

Example 2. Every torsion-free divisible group has regular endomorphism ring—this is obvious from (112.2).

Example 3. The groups covered by (111.3) have m-regular endomorphism rings. To verify this, we show that a left Artinian ring R with 1 is m-regular for some m. From (123.4) it follows at once that R has a finite composition series as a left R-module, say of length l. Given $\alpha \in \mathsf{R}$, the descending chain of left ideals generated by $\alpha, \alpha^2, \cdots$, respectively, becomes stationary after at most l steps, i.e., $\alpha^l = \beta\alpha^{2l}$ for some $\beta \in \mathsf{R}$. Hence $(1 - \alpha^l\beta)\alpha^{2l} = 0$. For the same reason, the left annihilators of $\alpha, \alpha^2, \cdots$ are equal from at most the lth term on, thus $1 - \alpha^l\beta$ annihilates α^l too, $(1 - \alpha^l\beta)\alpha^l = 0$. Hence R is l-regular. As the proof shows, it is left l-regular, too.

Example 4. Let $A = \bigoplus_p Z(p^m)$ for a fixed m, with p varying over different primes. Since every endomorphism of $Z(p^m)$ is either an automorphism or nilpotent of index $\leq m$, the m-regularity of $\mathsf{E}(A)$ follows readily. [Now $\mathsf{E}(A)$ is commutative.]

Example 5. The endomorphism ring of the additive group of the commutative π-regular ring N of (125.3) is isomorphic to N. It is π-regular, but not m-regular for any m unless the l_i are bounded.

Lemma 112.3. *If A has regular [π-regular] endomorphism ring, then the same holds for every summand C of A.*

Let $\varepsilon: A \to C$ be a projection. Every $\alpha \in \mathrm{E}(C)$ may be viewed as the endomorphism of A, acting trivially on $\operatorname{Ker} \varepsilon$. If it is m-regular for some $m \geqq 1$ in $\mathrm{E}(A)$, i.e., $\alpha^m \beta \alpha^m = \alpha^m$ for a suitable $\beta \in \mathrm{E}(A)$, then $\beta' = \varepsilon\beta\varepsilon \in \mathrm{E}(C)$ satisfies $\alpha^m \beta' \alpha^m = \alpha^m$.□

Our first task is to obtain basic information about the influence the π-regularity of $\mathrm{E}(A)$ has on the structure of A.

Proposition 112.4 (Fuchs and Rangaswamy [1]). *If $\mathrm{E}(A)$ is π-regular, then $A = C \oplus D$, where:*

(i) *C is a reduced group such that its p-components C_p are finite or elementary and $C/T(C)$ is divisible; moreover,*

$$\bigoplus_p C_p \leqq C \leqq \prod_p C_p;$$

(ii) *D is a torsion-free divisible group.*

Multiplication by a prime p is an endomorphism of A which without fear of ambiguity may be denoted simply by p. By π-regularity, $p^m = p^m \beta p^m$ for some $m \geqq 1$ and $\beta \in \mathrm{E}(A)$. Thus for all $a \in A$, $p^m a = p^{2m} \beta a$, which implies that $p^m A = p^{2m} A$ is p-divisible and $p^m a = 0$ for all a in the p-component A_p of A. Consequently, A_p is a summand of A, $A = A_p \oplus B$ for some $B \leqq A$. Hence $p^m A = p^m B$, thus $p^m B$ is p-divisible. Divisibility by p in B is unique, therefore $p^m B = B$. Thus we have $A = A_p \oplus p^m A$, where $p^m A_p = 0$. It is now clear that the maximal divisible subgroup D of A satisfies (ii). Also, $C/T(C)$ is an epic image of every $A/A_p \cong p^m A$, where the latter group is p-divisible; thus $C/T(C)$ is divisible.

We still have to show that $C_p = A_p$ is finite whenever it fails to be elementary. By boundedness, C_p is a direct sum of cyclic groups of orders $\leqq p^m$. If C_p is not finite, then it has a summand $G = G_0 \oplus G_1 \oplus \cdots \oplus G_n \oplus \cdots$ such that $G_0 \cong Z(p^m)$ and $G_n \cong Z(p^i)$ for a fixed $i \leqq m$, for all $n \geqq 1$. There is an endomorphism α of A which acts trivially on a complement of G in A and satisfies $\alpha G_0 = 0$, $\alpha G_1 = p^{m-1} G_0$, while $\alpha G_n = G_{n-1}$ for $n \geqq 2$. Visibly, $p^{m-1} G_0 \leqq \operatorname{Im} \alpha^k$ for all $k \geqq 1$, and thus $\operatorname{Im} \alpha^k$ can never be a summand of G unless $m = 1$. In view of (112.1), this establishes the claim concerning C_p.

Finally, from $C = C_p \oplus p^m C$ we obtain epimorphisms $C \to C_p$ for all p. They induce a homomorphism $C \to \prod_p C_p$, and to verify $C \leqq \prod C_p$, it suffices to show that $K = \bigcap_p p^m C$ vanishes. If q is a prime $\neq p$, then C_q is a summand of $p^m C$ and $C = C_p \oplus C_q \oplus (p^m C \cap q^n C)$. Notice that every $a \in p^m C \cap q^n C$ is uniquely divisible by p in $p^m C \cap q^n C$, so that every $a \in K = \bigcap_q (p^m C \cap q^n C)$ is divisible by p in K. This amounts to the divisibility of K, whence $K = 0$.□

We are now in a position to settle the case of torsion groups with π-regular endomorphism rings.

Theorem 112.5 (Fuchs and Rangaswamy [1]). *Let A be a torsion group with p-components A_p. For the π-regularity of* $\mathsf{E}(A)$ *it is necessary and sufficient that there exists an integer $e \geqq 0$ such that for each p, A_p is either elementary or it satisfies $|A_p| \leqq p^e$.*

Suppose $\mathsf{E}(A)$ is π-regular and A_p is not elementary for some p. By (112.4)(i) there is a smallest integer $k_p \geqq 2$ such that $p^{k_p}A_p = 0$. Then (112.1) shows that multiplication by p in A_p is k_p-regular, but not k-regular for any k with $1 \leqq k < k_p$. The endomorphism of A which acts as the multiplication by p on A_p, for every p, can thus be m-regular for some m only if the set $\{k_p\}$, taken for all primes p with $pA_p \neq 0$, is bounded. We next assume A_p has a direct summand $B = B_1 \oplus \cdots \oplus B_{l_p}$, where all B_j are cyclic such that $|B_1| \geqq p^2$ and $|B_1| \geqq \cdots \geqq |B_{l_p}|$. There is an endomorphism β of B mapping B_j upon B_{j+1} ($j = 1, \cdots, l_p - 1$) and B_{l_p} upon $B_1[p]$. Because of (112.1), β is not l-regular for any $l \leqq l_p$. Just as for the k_p we infer the existence of an upper bound for the l_p. Combining the bounds for k_p and for l_p, it follows that there is an integer e such that $|A_p| \leqq p^e$ for all nonelementary p-components A_p of A.

Conversely, let A have the indicated property. If A_p is of order $\leqq p^e$, then it is readily checked that $\mathsf{E}(A_p)$ is at most of order p^{e^2}. Hence $\mathsf{E}(A_p)$ is of length $\leqq l = e^2$ as a left module over itself, and Example 4 implies the l-regularity of $\mathsf{E}(A_p)$. If A_p is an elementary p-group, then Example 1 shows $\mathsf{E}(A_p)$ is regular. It is now clear that $\mathsf{E}(A) = \prod_p \mathsf{E}(A_p)$ is l-regular.□

In the general case we have failed to obtain a full characterization of groups with π-regular endomorphism rings. The most we can do is to reduce the problem to the case of reduced groups—even this is far from being straightforward.

Proposition 112.6 (Fuchs and Rangaswamy [1]). *Let $A = C \oplus D$, where C is reduced and D divisible.* $\mathsf{E}(A)$ *is π-regular if and only if*:

(a) $\mathsf{E}(C)$ *is π-regular*;
(b) *D is torsion-free*;
(c) *C is a torsion group whenever D is of infinite rank.*

If $\mathsf{E}(A)$ is π-regular, then (a) and (b) follow from (112.3) and (112.4), respectively. To establish (c), let D have infinite rank and write $D = D_0 \oplus \bigoplus_{n=1}^{\infty} D_n$ with $D_n \cong Q$ for $n \geqq 1$. If C is not torsion, then by (112.4) it has an epimorphism onto D_1, so A has an endomorphism α such that $\alpha | D_0 = 1_{D_0}$, $\alpha D_n = D_{n+1}$, for $n \geqq 1$ and $\alpha C = D_1$. A moment's reflection shows that $\operatorname{Ker} \alpha = \operatorname{Ker} \alpha^k$ for all $k \geqq 1$ and $T(C) \leqq \operatorname{Ker} \alpha < C$, thus $\operatorname{Ker} \alpha^k$ is never a summand of A. This contradiction proves (c).

To prove sufficiency, let $A = C \oplus D$ satisfy (a)–(c). If D is of infinite rank, then by (c), C is fully invariant in A. Therefore, $\mathsf{E}(A) = \mathsf{E}(C) \oplus \mathsf{E}(D)$, and (b) together with Example 2 establishes the π-regularity of $\mathsf{E}(A)$. If D is of finite rank, then a more sophisticated argument is necessary.

For $\alpha \in \mathsf{E}(A)$ write $A_k = \alpha^k A$ $(k \geqq 1)$, and decompose $A_k = C_k \oplus D_k$, where C_k is reduced and D_k is divisible. Since $A_{k+1} \leqq A_k$ and $D_{k+1} \leqq D_k$, the finiteness of the rank of D implies $D_n = D_{n+1} = \cdots$ for some n. Then we may choose $C_{n+i} = C_n \cap A_{n+i}$ for $i \geqq 1$, i.e., $C_n \geqq C_{n+1} \geqq \cdots$. It is evident that every C_k is an epimorphic image of C. We deduce that $C_k/T(C_k)$ is divisible. Plainly, C_k is a subdirect sum of a subgroup C' of C and a subgroup D' of D such that $T(C') = T(C_k)$. Hence $C_k/T(C_k)$ is a subdirect sum of the torsion-free, necessarily divisible groups $C'/T(C')$ and D'. The kernels of this subdirect sum are pure in the subdirect sum, as the components are torsion-free [see p. 43, Vol. I], so the divisibility of $C_k/T(C_k)$ implies that $C_k/T(C_k)$, and hence C_k intersects D' in a divisible subgroup. This can happen only if $C_k \cap D' = 0$, whence $C_k \cap D = 0$. Without loss of generality, $C_k \leqq C$ may be assumed for all $k \geqq n$; in other words: if $\varepsilon\colon A \to C$ is the obvious projection, then $\varepsilon A_k = C_k$ for all $k \geqq n$. Observe that the endomorphism $\varepsilon\alpha^k\varepsilon$ satisfies $\varepsilon\alpha^k\varepsilon C = \varepsilon\alpha^k C = C_k$, and from (a) the existence of an m follows such that both $\operatorname{Im}(\varepsilon\alpha^n\varepsilon)^m$ and $\operatorname{Ker}(\varepsilon\alpha^n\varepsilon)^m \cap C$ are summands of C. Since $(\varepsilon\alpha^n\varepsilon)^m = \varepsilon\alpha^{nm}\varepsilon$, we find that C_{nm} and hence A_{nm} is a summand of A. To complete the proof, we need only verify $\operatorname{Ker}\alpha^{nm} + D = \operatorname{Ker}\varepsilon\alpha^{nm}\varepsilon$, because then the reduced parts of these two kernels are the same. The inclusion $\leqq$ being straightforward, let $\varepsilon\alpha^{nm}\varepsilon a = 0$ and write $a = c + d\,(c \in C, d \in D)$. Now $\varepsilon\alpha^{nm}c = 0$ means $\alpha^{nm}c \in D$, and thus $\alpha^{nm}c \in D_n$. By the choice of n, $\alpha^{nm}c = \alpha^{nm}g$ for some $g \in D$; therefore, $a = (c - g) + (d + g) \in \operatorname{Ker}\alpha^{nm} + D$, as claimed.□

Let us summarize what our theorems yield in the special case of regular endomorphism rings.

Proposition 112.7. (a) *If A is not reduced, then $\mathsf{E}(A)$ is regular if and only if A is a direct sum of a torsion-free divisible group and an elementary group.*

(b) *If A is torsion, then $\mathsf{E}(A)$ is regular exactly if A is elementary.*

(c) *If A is reduced and $\mathsf{E}(A)$ is regular, then $T(A)$ is elementary, $A/T(A)$ is divisible and $\bigoplus_p A_p \leqq A \leqq \prod_p A_p$.*

From the first part of the proof of (112.4) we infer that the p-components A_p are all elementary whenever $\mathsf{E}(A)$ is regular. This combined with Example 1 proves (b). In view of (112.4)(i) this also establishes (c). Finally, to prove (a), write $A = C \oplus D$ as in (112.4) and notice that the kernel of a nontrivial homomorphism $C \to D$ cannot be a summand; now the necessity of (a) is immediate. Its sufficiency has been established in the proof of (112.6).□

Left and right π-regular endomorphism rings can be treated similarly; in fact, virtually all we have found for the π-regular case carry over to both left and right π-regular endomorphism rings, though the proofs have to be modified. It is by no means *a priori* obvious that for endomorphism rings, left and right π-regularity are equivalent [and either implies π-regularity]; this will, however, follow from (112.9).

We begin with the analog of (112.4).

Proposition 112.8 (Fuchs and Rangaswamy [1]). *If* $\mathrm{E}(A)$ *is left or right* π*-regular, then* $A = C \oplus D$, *where*

(i) C *is reduced with finite p-components* C_p *such that* $C/T(C)$ *is divisible, and* $\bigoplus_p C_p \leqq C \leqq \prod_p C_p$;

(ii) D *is torsion-free divisible of finite rank.*

The first and the last paragraphs of the proof of (112.4) hold *verbatim* for left and right π-regularity. To complete the proof it will thus be enough to show that A does not have any summand G which is the direct sum of infinitely many isomorphic groups G_n, $G = \bigoplus_{n=1}^{\infty} G_n$. If A does have such a summand G, then there are $\alpha, \beta \in \mathrm{E}(A)$ such that $\alpha G_n = G_{n+1}$ and $\beta G_1 = 0$, $\beta G_{n+1} = G_n$, for $n = 1, 2, \cdots$ and they are trivial on a complement of G in A. We conclude that $\mathrm{Im}\, \alpha^k \neq \mathrm{Im}\, \alpha^{k+1}$ and $\mathrm{Ker}\, \beta^k \neq \mathrm{Ker}\, \beta^{k+1}$, for all $k = 1, 2, \cdots$. But if α is right m-regular, then $\mathrm{Im}\, \alpha^m = \mathrm{Im}\, \alpha^{m+1}$, and if β is left m-regular, then $\mathrm{Ker}\, \beta^m = \mathrm{Ker}\, \beta^{m+1}$.□

What we are about to prove is a more intricate analog of (112.2) which at the same time reveals the equivalence of left and right π-regularity for endomorphism rings.

Proposition 112.9 (Fuchs and Rangaswamy [1]). *For the endomorphism ring* $\mathrm{E}(A)$ *of a group A the following are equivalent*:

(i) $\mathrm{E}(A)$ *is left* π*-regular*;

(ii) $\mathrm{E}(A)$ *is right* π*-regular*;

(iii) *for every* $\alpha \in \mathrm{E}(A)$ *there is an integer* $m \geqq 1$ *such that*

$$A = \mathrm{Im}\, \alpha^m \oplus \mathrm{Ker}\, \alpha^m.$$

Assuming $A = \mathrm{Im}\, \alpha^m \oplus \mathrm{Ker}\, \alpha^m$, it is clear that α^m induces a monomorphism on $\mathrm{Im}\, \alpha^m = G$. It must be an automorphism, since $G = \alpha^m A = \alpha^m G$. If $\gamma \in \mathrm{E}(A)$ is inverse to α^m on G and annihilates $\mathrm{Ker}\, \alpha^m$, then $\gamma\alpha^{2m} = \alpha^m = \alpha^{2m}\gamma$, that is to say, α is both left and right m-regular. We conclude that (iii) implies (i) and (ii).

In proving the reverse implications, we first suppose A reduced.

From (i) we infer that for every $\alpha \in \mathrm{E}(A)$, $\alpha^m = \gamma\alpha^{2m}$ holds with suitable $\gamma \in \mathrm{E}(A)$ and integer $m \geqq 1$. Evidently, $\alpha^m = \gamma\alpha^{2m}$ implies $\mathrm{Im}\, \alpha^m \cap \mathrm{Ker}\, \alpha^m = 0$, so to prove (iii), it remains only to show $A = \mathrm{Im}\, \alpha^m + \mathrm{Ker}\, \alpha^m$. By (112.8)(i), the p-components A_p of A are finite, and therefore, for each prime p, comparison of orders yields $A_p = \mathrm{Im}(\alpha^m | A_p) + \mathrm{Ker}(\alpha^m | A_p)$, or more explicitly: every $a \in A_p$ can be written in the form $a = \alpha^m a_1 + a_2$, where $a_1, a_2 \in A_p$ and $\alpha^m a_2 = 0$. Hence $a - \gamma\alpha^m a = a_2$ and $\alpha^m a - \alpha^m\gamma\alpha^m a = 0$. Thus representing A as a subdirect sum of the A_p like in (112.8)(i), for any $a \in A$, all the coordinates of $(\alpha^m - \alpha^m\gamma\alpha^m)a$ must vanish. It follows that $\alpha^m\gamma\alpha^m = \alpha^m$ and $A = \mathrm{Im}\ \gamma\alpha^m + \mathrm{Im}(1 - \gamma\alpha^m) \leqq \mathrm{Im}\ \alpha^m + \mathrm{Ker}\ \alpha^m$.

Assume (ii) and write $\alpha^m = \alpha^{2m}\gamma$ with $\alpha, \gamma \in \mathrm{E}(A)$. Since $\alpha^m(a - \alpha^m\gamma a) = 0$ for all $a \in A$, we have $a = a_1 + a_2$ with $a_1 = \alpha^m\gamma a \in \mathrm{Im}\ \alpha^m$ and $a_2 \in \mathrm{Ker}\ \alpha^m$. In other words, $A = \mathrm{Im}\ \alpha^m + \mathrm{Ker}\ \alpha^m$. Now we argue as in the preceding paragraph. The p-components A_p being finite, $\mathrm{Im}(\alpha^m | A_p) \cap \mathrm{Ker}(\alpha^m | A_p) = 0$ holds for every prime p, whence the elements in $\mathrm{Im}\ \alpha^m \cap \mathrm{Ker}\ \alpha^m$ have 0 coordinates in each A_p. This establishes $\mathrm{Im}\ \alpha^m \cap \mathrm{Ker}\ \alpha^m = 0$, and hence (iii).

If A is not reduced, then we have to refer to the next lemma.□

Lemma 112.10. *For a group $A = C \oplus D$ with reduced C and divisible D, $\mathrm{E}(A)$ is left [right] π-regular exactly if*

(a) *$\mathrm{E}(C)$ is left [right] π-regular;*
(b) *D is torsion-free of finite rank.*

The necessity is a direct consequence of (112.8) and the obvious analog of (112.3). To establish sufficiency, let $\varepsilon: A \to C$ be a projection. Quoting the proven part of (112.9), we can write $C = \mathrm{Im}\ \varepsilon\alpha^m\varepsilon \oplus (C \cap \mathrm{Ker}\ \varepsilon\alpha^m\varepsilon)$ for some m; moreover, the proof in (112.9) shows that the same decomposition holds for all larger integers. Hence $D \leqq \mathrm{Ker}\ \varepsilon\alpha^k\varepsilon$ implies $A = \mathrm{Im}\ \varepsilon\alpha^k\varepsilon \oplus \mathrm{Ker}\ \varepsilon\alpha^k\varepsilon$ for all $k \geqq m$. If D_n is chosen as in the proof of (112.6), then $\mathrm{Im}\ \alpha^k = \mathrm{Im}\ \varepsilon\alpha^k\varepsilon \oplus D_k$ and $\mathrm{Ker}\ \alpha^k \oplus D_k = \mathrm{Ker}\ \varepsilon\alpha^k\varepsilon$ for every $k \geqq n$. Consequently, $A = \mathrm{Im}\ \alpha^k \oplus \mathrm{Ker}\ \alpha^k$ for some k, and (112.9) implies $\mathrm{E}(A)$ is left [right] π-regular.□

A satisfactory, more or less explicit description of reduced groups with regular, π-regular or left π-regular endomorphism rings seems to be a hard problem. Manifestly, the difficulty lies in singling out the suitable mixed groups between the direct sum and the direct product of their p-components.

EXERCISES

1. Show that if $\mathrm{E}(A)$ is m-regular, then in (112.4)(i) the nonelementary p-components C_p satisfy $|C_p| \leqq p^{m^2}$.
2. Prove (112.5) for $A = \prod_p A_p$ with p-groups A_p.
3. Find nonisomorphic groups whose endomorphism rings are isomorphic regular rings. [*Hint*: direct sum and product of $Z(p)$.]

4. (a) If $\mathrm{E}(C)$ is m-regular and D is of rank r in (112.6), then $\mathrm{E}(A)$ is $(r+1)m$-regular.
   (b) The endomorphism ring of $Q \oplus \prod_p Z(p)$ is 2-regular, but not regular.
5. Prove that $\bigoplus_p Z(p) \oplus \prod_p Z(p)$ is the additive group of a regular ring and satisfies the conclusion of (112.7)(c), but its endomorphism ring is not regular.
6. (Kertész and Szele [1]) Show that if every endomorphic image of A is a summand of A, then every p-component of A is elementary or divisible, and $A/T(A)$ is divisible. [*Hint*: pA.]
7. (Rangaswamy [8]) Kernels and images of endomorphisms of A are pure in A if and only if $T(A)$ is elementary and $A/T(A)$ is divisible.
8. If $\mathrm{E}(A)$ is a commutative π-regular ring, then A is either a pure subgroup of $\prod Z(p^{k_p})$ containing $\bigoplus_p Z(p^{k_p})$ or isomorphic to $Q \oplus \bigoplus_p Z(p^{k_p})$, where p runs over a set of different primes and k_p are integers.
9. Characterize the algebraically compact [cotorsion] groups with π-regular endomorphism rings. Show that the rings are then m-regular for some m.
10. A left or right π-regular endomorphism ring is necessarily π-regular.
11. Describe the structure of torsion groups with left π-regular endomorphism rings.
12. An endomorphism ring is left 1-regular if and only if it is commutative and regular.
13. (Rangaswamy [9]) A ring R with 1 is called a *Baer ring* if the left [or, equivalently, the right] annihilators of nonempty subsets of R are generated by idempotents.
    (a) For a torsion group A, $\mathrm{E}(A)$ is a Baer ring exactly if for every prime p, the p-component A_p of A is either divisible or elementary.
    (b) If A satisfies $A = A_p \oplus pA$ for every prime p, then $\mathrm{E}(A)$ is a Baer ring if and only if each subgroup of A closed in the Z-adic topology of A is a summand of A. [*Hint*: in this case, "closed" is equivalent to being an intersection of kernels of endomorphisms.]
14*. (a) (Wolfson [1]) For a free group F, $\mathrm{E}(F)$ is a Baer ring if and only if intersections of kernels of endomorphisms are summands of F.
    (b) (Rangaswamy [9]) F has the property in (a) if and only if it is countable. [*Hint*: cf. (19.2).]

NOTES

The study of endomorphism rings has its origin in the theory of linear transformations of a finite-dimensional vector space; it has long been known that they form a simple ring isomorphic to a matrix ring. Shoda [1] initiated the theory of endomorphism rings of finite abelian groups, and the matrix representation given in (106.1) is due to him in the finite case.

Baer [9] was the first to focus attention on endomorphism rings *qua* special rings. He succeeded in giving a complete characterization of endomorphism rings of bounded groups, in terms of their ideal theory. Different approach was followed by Liebert [1, 2 and 4], winding up with a ring-theoretical characterization of $\mathsf{E}(A)$ for separable p-groups A. Various aspects of endomorphism rings of p-groups have been investigated by Pierce [2, 3] and Corner [7]. For the divisible case, see Liebert [5].

That the endomorphism rings of torsion groups determine the groups themselves, up to isomorphism, was discovered by Baer [9] in the bounded case and proved by Kaplansky [2] in general. No analogous result holds for torsion-free groups, unless we limit our considerations to groups with an adequate supply of endomorphisms. E.g., the following is true, as was shown by G. Hauptfleisch: if A and C are homogeneous separable torsion-free groups of types $\mathbf{s}$ and $\mathbf{t}$, respectively, then $\mathsf{E}(A) \cong \mathsf{E}(C)$ implies $A \otimes T \cong B \otimes S$, where S, T are rational groups of types $\mathbf{s}$ and $\mathbf{t}$, respectively. There are more special cases when isomorphism can be derived from the isomorphy of endomorphism rings, cf. K. G. Wolfson [*Proc. Amer. Math. Soc.* **13** (1962), 712–714, and **14** (1963), 589–594; *Michigan Math. J.* **9** (1962), 69–75].

In the torsion-free case, undoubtedly (110.1) is the most significant result known so far. It has been greatly sharpened by Corner [6] to larger cardinalities at the expense of looking at $\mathsf{E}(A)$ as a topological ring endowed with the finite topology. For further results on endomorphisms of torsion-free groups, see Brenner and Butler [1], Corner [8], and Kishkina [1], Król [2].

Comparatively little attention has been paid so far to the endomorphism rings of mixed groups. An interesting special case is when the endomorphisms are fully determined by their actions on the torsion part, like in adjusted cotorsion groups [see Ex. 7 in **108**].

An interesting generalization of (108.3) was found by Nunke [8].

In [23], the author has raised the question of groups A for which the subring generated by the units of $\mathsf{E}(A)$ coincides with $\mathsf{E}(A)$. This question seems to be more tractable for torsion groups A. While it is easy to find even a finite 2-group which lacks the indicated property [see Ex. 10 in **108**; cf. also Pierce [2]], all totally projective p-groups with $p > 2$ and all torsion-complete p-groups share the stronger property that every endomorphism is the sum of two automorphisms [cf. Hill [14] and Castagna [1]; see also Freedman [2] and Stringall [3]].

Every ring with identity is trivially an endomorphism ring of some module. However, the problem of endomorphism rings becomes very hard as soon as the ring is fixed over which modules are to be considered or conditions are imposed on the modules. One of the most informative results is concerned with quasi-injective modules M [see C. Faith and Y. Utumi, *Arch. Math.* **15** (1964), 166–174]: the Jacobson radical J of $\mathsf{E}_\mathsf{R}(M)$ consists now of all endomorphisms whose kernels are essential submodules of M and the quotient $\mathsf{E}_\mathsf{R}(M)/\mathsf{J}$ is a regular ring. For radicals of endomorphism rings, cf. also Haimo [3, 4].

There are several classes of rings whose structure is better known, and it is a plausible program to examine their roles as endomorphism rings, in particular, to single out the endomorphism rings among them. This program was initiated by Szele [5], and since that it has received a great deal of attention [cf. **111** and **112**, Rangaswamy [9 and 10]], but still there are numerous open questions. The initial hope that these investigations will provide a good insight into the group structure has not realized, since the traditional ring properties have turned out to be severe limitations for endomorphism rings. The situation is more complex for modules, and so far as we are aware, only sporadic results have been obtained. R. Ware and J. Zelmanowitz [*Amer. Math. Monthly* **77** (1970), 987–989] gave an account of modules M over commutative rings R for which $\mathsf{E}_\mathsf{R}(M)$ is simple, and of torsion-free modules M over commutative domains such that $\mathsf{E}_\mathsf{R}(M)$ is regular. In addition, we note that commutative endomorphism rings have been investigated by J. Zelmanowitz [*Canad. J. Math.* **23** (1971), 69–76] and semihereditary endomorphism rings by H. Lenzing [*Math. Z.* **118** (1970), 219–240].

Problem 84. Find criteria for various types of rings to be an endomorphism ring; in particular:

(a) principal ideal domains;
(b) local and semilocal rings;
(c) Noetherian rings;
(d) self-injective rings.

Problem 85. Characterize the endomorphism rings of p-groups in general. Which of them are endomorphism rings of totally projective p-groups?

Problem 86. The endomorphism rings of which p-groups admit a compact ring topology?

Problem 87. Are two mixed groups of torsion-free rank 1 necessarily isomorphic if their endomorphism rings [automorphism groups] are and their quotients mod torsion subgroups are isomorphic to Q?

XVI

AUTOMORPHISM GROUPS

Every group A determines a group, Aut A, consisting of all automorphisms of A. This is commutative only in exceptional cases, so for the study of automorphism groups, noncommutative methods are inevitable.

Our main concern will be the relationship between a group and its automorphism group. Since Aut A is nothing else than the group of units in $\mathsf{E}(A)$, it is evident that Aut A can give us less information only about A than we can collect from $\mathsf{E}(A)$. On the other hand, concentration on Aut A means that we are able to make use of powerful group-theoretical methods; consequently, our approach must be different and we can expect to obtain new data in this way.

The first section, **113**, is introductory, and its main purpose is to get acquainted with the simple relations that exist between a group, its direct decompositions, and the automorphisms. A more comprehensive study of automorphism groups begins with the investigation of several normal subgroups in Aut A, closely related to A. More relevant results will be obtained in **115** and **116**, where Aut A is thoroughly examined for torsion and torsion-free groups A, respectively.

113. GROUPS OF AUTOMORPHISMS

The automorphisms of a group A form a group under composition: the product $\alpha\beta$ of two automorphisms α, β of A acts as

$$(\alpha\beta)a = \alpha(\beta a) \qquad \text{for all} \quad a \in A.$$

This group Aut A is called the *automorphism group* of A. Clearly, Aut A is exactly the group of units in $\mathsf{E}(A)$.

The cyclic groups of order 1 and 2 have no automorphisms other than the identity. But every group A of order >2 has a nontrivial automorphism. In fact, $a \mapsto -a$ is an automorphism which is different from 1_A except when A is an elementary 2-group. For an elementary 2-group A of order $\geqq 4$, a permutation of the elements of a basis gives rise to an automorphism $\neq 1_A$.

The automorphism groups of some important groups can easily be described:

Example 1. Evidently, Z has exactly two automorphisms, and Aut $Z \cong Z(2)$.

Example 2. An automorphism of $Z(n)$ maps a generator a onto another generator b; clearly, b must be of the form $b = ka$ with $(k, n) = 1$. Conversely, any such k gives rise to an automorphism mapping a upon ka. It follows at once that the automorphism group of $Z(n)$ is isomorphic to the multiplicative group of those residue classes mod n which are prime to n. Thus Aut $Z(n)$ is commutative and its order is given by Euler's function $\phi(n) = n(1 - p_1^{-1}) \cdots (1 - p_r^{-1})$, where $p_1, \cdots, p_r$ are the different prime divisors of n.

Example 3. From Example 3 in **106** it follows that the automorphism group of $Z(p^\infty)$ is isomorphic to the multiplicative group of p-adic units. [This group will be described explicitly in **127**.]

Example 4. Similarly, the automorphism group of J_p is isomorphic to the multiplicative group of p-adic units.

Example 5. Let R be a rational group of type $\mathbf{t} = (k_1, \cdots, k_n, \cdots)$. Then $a \mapsto p_n a$ ($a \in R$, p_n is the nth prime) is an automorphism of R if and only if $k_n = \infty$. Hence we conclude that Aut R is isomorphic to the multiplicative group of all rational numbers whose numerators and denominators are divisible by primes p_n only for which $k_n = \infty$. Thus Aut R is isomorphic to the discrete direct product of $Z(2)$ and as many copies of Z as many k_n are ∞.

Example 6. Let A be an elementary p-group of rank $\mathfrak{m}$. Then A is a vector space over the prime field F_p of characteristic p, and dim $A = \mathfrak{m}$. Since every automorphism of A is a linear transformation of the vector space, we see that Aut A is now the general linear group $\mathrm{GL}(\mathfrak{m}, p)$.

We continue with a few simple observations.

(a) If C is a characteristic subgroup of A and $\alpha \in$ Aut A, then $\alpha | C$ is an automorphism of C and $a + C \mapsto \alpha a + C$ is an automorphism of A/C.

(b) An isomorphism $\phi: A \to C$ induces an isomorphism $\phi^*:$ Aut $A \to$ Aut C between the automorphism groups *via*

$$\phi^* : \alpha \mapsto \phi\alpha\phi^{-1}.$$

(c) If A is a direct sum, and if we represent the endomorphisms of A by matrices as described in (106.1), then the automorphisms will correspond precisely to the invertible matrices $[\alpha_{ji}]$.

(d) If $A = B \oplus C$, then Aut B may be viewed as a subgroup of Aut A, identified with the set of all $\alpha \in$ Aut A satisfying $\alpha | C = 1$. [This identification, however, depends on the choice of C.]

(e) If $A = \bigoplus_{i \in I} A_i$, then under the identification indicated in (d), all Aut A_i may be regarded as subgroups of Aut A. Moreover, the cartesian product $\prod_i \operatorname{Aut} A_i$ of all the $\operatorname{Aut} A_i$ is a subgroup of $\operatorname{Aut} A$: it consists of all $\alpha \in \operatorname{Aut} A$ which carry every A_i into itself. In the matrix representation of α, $\prod_i \operatorname{Aut} A_i$ is exactly the set of all diagonal matrices.

(f) *If $A = \bigoplus A_i$, where each A_i is fully invariant in A, then* $\operatorname{Aut} A = \prod_i \operatorname{Aut} A_i$. Since $\operatorname{Hom}(A_i, A_j) = 0$ for $i \neq j$, the nondiagonal elements in the matrix representation of $\alpha \in \operatorname{Aut} A$ must vanish [cf. (106.1)]. Hence we infer:

(g) *If $A = \bigoplus_p A_p$ is a torsion group and A_p is its p-component, then* $\operatorname{Aut} A$ *is the cartesian product of the* $\operatorname{Aut} A_p$ *with p running over all primes.*

Let B be a subgroup of A. By the *stabilizer* of the chain $0 \leqq B \leqq A$ is meant the subgroup of $\operatorname{Aut} A$ that consists of all $\alpha \in \operatorname{Aut} A$ such that $\alpha b = b$ and $\alpha a - a \in B$ for all $b \in B$ and $a \in A$; i.e., α induces the identity both on B and on A/B.

Lemma 113.1. *The stabilizer of* $0 \leqq B \leqq A$ *is isomorphic to* $\operatorname{Hom}(A/B, B)$. *It is a normal subgroup of* $\operatorname{Aut} A$ *whenever B is characteristic in A.*

Let $\alpha, \beta \in \operatorname{Aut} A$ belong to the stabilizer Σ of $0 \leqq B \leqq A$. Then, for $a \in A$, $\alpha a = a + b$ and $\beta a = a + c$ hold for some $b, c \in B$. We obtain $\alpha\beta a = \alpha(a + c) = a + b + c$. Note that the mappings $\bar{\alpha}\colon a + B \mapsto (\alpha - 1)a = b$, $\bar{\beta}\colon a + B \mapsto (\beta - 1)a = c$ are homomorphisms of A/B into B such that $\overline{\alpha\beta}\colon a + B \mapsto b + c$. Hence $\alpha \mapsto \bar{\alpha}$ is a homomorphism $\Sigma \to \operatorname{Hom}(A/B, B)$ whose kernel is trivial. Given $\eta \in \operatorname{Hom}(A/B, B)$, the map $\alpha\colon a \mapsto a + \eta(a + B)$ is easily seen to belong to Σ, hence $\Sigma \cong \operatorname{Hom}(A/B, B)$.

The second assertion is straightforward to check.□

Recall that a group Γ is the *semidirect product* of a normal subgroup Δ and a subgroup Ξ of Γ if $\Delta \cap \Xi = 1$ and Δ, Ξ generate Γ.

(h) *Let $A = B \oplus C$, where B is fully invariant in A. Then* $\operatorname{Aut} A$ *is the semidirect product of the stabilizer* $\Sigma \cong \operatorname{Hom}(C, B)$ *of* $0 \leqq B \leqq A$ *and the subgroup* $\operatorname{Aut} B \times \operatorname{Aut} C$.

By (a), every $\alpha \in \operatorname{Aut} A$ induces an $\alpha_B \in \operatorname{Aut} B$ and an $\alpha_C \in \operatorname{Aut} A/B = \operatorname{Aut} C$. The correspondence $\alpha \mapsto (\alpha_B, \alpha_C)$ is a homomorphism $\phi\colon \operatorname{Aut} A \to \operatorname{Aut} B \times \operatorname{Aut} C$ whose kernel is obviously Σ. Since $\Sigma \cap \operatorname{Im} \phi = 1$, it follows that $\operatorname{Aut} A$ is the semidirect product of Σ and $\operatorname{Im} \phi$. The indicated isomorphism is a consequence of (113.1).

We leave it to the reader to state (h) in the special case when B is divisible and C is reduced.

Recall that direct decompositions of a group are recognizable in terms of endomorphisms. There is a similar, though not so effective, tool of recognizing direct decompositions *via* automorphisms; these are the so-called *involutions*, i.e., automorphisms ε of A with $\varepsilon^2 = 1$. As we shall see, they do not provide as much information as the projections, and in addition, to make it work and to avoid unnecessary nuisance, one has to assume that multiplication by 2 is an automorphism of A. Evidently, this means no restriction at all for p-groups A with $p \geqq 3$.

Throughout (j)–(n) we shall assume that A is a group such that $a \mapsto 2a$ is an automorphism of A. Thus, for every $a \in A$, $\frac{1}{2}a$ is a well-defined element of A.

(j) A direct decomposition

$$A = C_1 \oplus \cdots \oplus C_k \quad \text{with} \quad C_i \neq 0 \tag{1}$$

determines k commuting involutions: ε_i is defined as the automorphism of A satisfying $\varepsilon_i | C_i = -1$ and $\varepsilon_i | C_j = 1$ for $i \neq j$. The system $\{\varepsilon_1, \cdots, \varepsilon_k\}$ is said *to belong to* (1).

(k) For an involution ε of A, define

$$A_\varepsilon^+ = \{a \in A \,|\, \varepsilon a = a\} \qquad \text{and} \qquad A_\varepsilon^- = \{a \in A \,|\, \varepsilon a = -a\}.$$

Then $A = A_\varepsilon^+ \oplus A_\varepsilon^-$; thus involutions $\varepsilon \neq \pm 1$ give rise to nontrivial direct decompositions of A. The projections associated with this decomposition are $\frac{1}{2}(1 + \varepsilon)$ and $\frac{1}{2}(1 - \varepsilon)$.

(l) Two involutions ε, ζ of A commute if and only if

$$A = (A_\varepsilon^+ \cap A_\zeta^+) \oplus (A_\varepsilon^+ \cap A_\zeta^-) \oplus (A_\varepsilon^- \cap A_\zeta^+) \oplus (A_\varepsilon^- \cap A_\zeta^-)$$

holds. If ε and ζ commute, then clearly $A_\varepsilon^+ = (A_\varepsilon^+ \cap A_\zeta^+) \oplus (A_\varepsilon^+ \cap A_\zeta^-)$, and analogously for A_ε^-. The converse follows from the simple fact that ε and ζ commute on all four summands, and hence on A.

(m) Commuting involutions $\zeta_1, \cdots, \zeta_n$ of A determine a unique decomposition (1) of A such that (i) $\zeta_l | C_i = \pm 1$ for all i, l, and (ii) given $i \neq j$, there is a ζ_l such that one of $\zeta_l | C_i$ and $\zeta_l | C_j$ is $+1$ and the other is -1. In fact, a repeated application of (l) yields such a decomposition [after cancelling zero components]. If (1) satisfies (i), then $C_i \leqq A_{\zeta_1}^\pm \cap \cdots \cap A_{\zeta_n}^\pm$ for an appropriate choice of signs $\pm$, and (ii) excludes that for different C_i and C_j the same signs occur.

(n) Let the system $\{\varepsilon_1, \cdots, \varepsilon_k\}$ of involutions belong to the direct decomposition (1). Then the *centralizer*

$$\mathfrak{c}\{\varepsilon_1, \cdots, \varepsilon_k\} = \{\alpha \in \operatorname{Aut} A \,|\, \alpha\varepsilon_i = \varepsilon_i\alpha \quad \text{for} \quad i = 1, \cdots, k\}$$

of $\{\varepsilon_1, \cdots, \varepsilon_k\}$ satisfies

$$\mathfrak{c}\{\varepsilon_1, \cdots, \varepsilon_k\} = \operatorname{Aut} C_1 \times \cdots \times \operatorname{Aut} C_k. \tag{2}$$

That under the identification, mentioned in (d), all $\operatorname{Aut} C_i$ belong to the centralizer is evident. Conversely, if α commutes with each of $\varepsilon_1, \cdots, \varepsilon_k$, then $\alpha C_i = \alpha A_{\varepsilon_i}^- \leqq A_{\varepsilon_i}^- = C_i$ for every i, whence $\alpha C_i = C_i$ and α belongs to the direct product of the $\operatorname{Aut} C_i$.

The following two results have been proved for p-groups by Leptin [4] and Fuchs [20], respectively.

Proposition 113.2. *An automorphism α of a group A satisfies:*

(i) *α induces the identity on A/nA,*
(ii) *α leaves the elements of $A[n]$ fixed,*

if and only if $\alpha = 1 - n\eta$ for some endomorphism η of A.

Every automorphism of the form $\alpha = 1 - n\eta$ satisfies (i) and (ii), so suppose $\alpha \in \operatorname{Aut} A$ satisfies (i) and (ii). It is readily seen that it suffices to consider the case when n is a prime power p^k. Writing $\xi = 1 - \alpha \in \mathrm{E}(A)$, we have $\xi A \leqq p^k A$ and $\xi A[p^k] = 0$. Let $B = \bigoplus \langle b_i \rangle$ be a p-basic subgroup of A. For each b_i, we select a $c_i \in A$ such that $\xi b_i = p^k c_i$ with the proviso that $c_i = b_i$ if $o(b_i) \leqq p^k$ [and thus $\xi b_i = 0$]. The correspondence $b_i \mapsto c_i$ extends to a well-defined homomorphism $\bar{\eta}\colon B \to A$.

With the aid of $\bar{\eta}$, define η as follows: if $a = b + p^k x$ with $b \in B$, $x \in A$, then let $\eta a = \bar{\eta} b + \xi x$. Noting that $p^k \bar{\eta} = \xi$ on B, we can show that this definition is correct. In fact, if $a = b' + p^k x'$ with $b' \in B$, $x' \in A$, then $p^k(x' - x) = b - b' \in B$ implies that $b - b' = p^k b''$ for some $b'' \in B$. Hence $p^k(x' - x - b'') = 0$ and $\xi x' = \xi x + \xi b''$, thus $\bar{\eta} b' + \xi x' = \bar{\eta}(b - p^k b'') + \xi x + \xi b'' = \bar{\eta} b + \xi x$. Consequently, η is an endomorphism of A which satisfies $p^k \eta = \xi$, whence $\alpha = 1 - p^k \eta$, indeed.□

Proposition 113.3. *For any group A and for any integer $n > 0$, every automorphism of nA is induced by some automorphism of A.*

Evidently, the proof can, without loss of generality, be restricted to the case when n is a prime p. Let α be an automorphism of pA and $\{a_i\}_{i \in I}$ a p-basis of A. Every $a \in A$ has the form

$$a = k_1 a_{i_1} + \cdots + k_r a_{i_r} + pb \qquad (1 \leqq k_i \leqq p - 1, \quad b \in A),$$

where the terms $k_1 a_{i_1}, \cdots, k_r a_{i_r}$ and pb are uniquely determined by a. To every a_i of order $\geqq p^2$, we pick a $c_i \in A$ such that $pc_i = \alpha(pa_i)$, and set $c_i = a_i$ whenever a_i is of order p. Writing $a \in A$ as above, we define a mapping $\beta\colon A \to A$ as follows:

$$\beta a = k_1 c_{i_1} + \cdots + k_r c_{i_r} + \alpha(pb).$$

Clearly, β is a well-defined endomorphism of A such that $\beta | pA = \alpha$. To argue that it is an automorphism of A, let $\beta a = 0$ for some $a \in A$. Then $\beta(pa) = 0$ and $pa = 0$, thus a may be written in the form $a = k_1 a_{i_1} + \cdots + k_r a_{i_r} + pb$ with $o(a_{i_1}) = \cdots = o(a_{i_r}) = p$. Hence $\beta a = k_1 a_{i_1} + \cdots + k_r a_{i_r} + \alpha(pb) = 0$ implies $r = 0$, i.e., $a = pb \in pA$. Now $\alpha a = \beta a = 0$ implies $a = 0$, and β is monic. Given any $x \in A$, let $c \in A$ satisfy $\alpha(pc) = px$. Then $x - \beta c$ is of order p, so we may write $x - \beta c = k_1 a_{i_1} + \cdots + k_r a_{i_r} + pb$ with $a_{i_1}, \cdots, a_{i_r}$ of order p. There exists a $d \in A$ with $pd = \alpha(pb)$, and it follows that $y = c + k_1 a_{i_1} + \cdots + k_r a_{i_r} + pd$ is mapped by β upon x. We see that β is epic and so an automorphism.□

EXERCISES

1. Prove that every automorphism of a group A can be extended to an automorphism of its divisible hull. The extension is unique whenever A is torsion-free.
2. (a) Let A satisfy $A^1 = 0$. Every automorphism of A extends to a unique automorphism of its Z-adic completion $\hat{A}$.
   (b) Every automorphism of a reduced group A can be uniquely extended to an automorphism of its cotorsion hull $A^\bullet$.
3. (Baer [3]) Let Γ be the automorphism group of A. Show that:
   (a) if C is a characteristic subgroup of A, then $\Delta = \{\alpha \in \Gamma \mid \alpha c = c \text{ for all } c \in C\}$ is a normal subgroup of Γ;
   (b) if Δ is a normal subgroup of Γ, then $C = \{a \in A \mid \delta a = a \text{ for all } a \in A\}$ is a characteristic subgroup of A.
4. The automorphism group of a locally cyclic group is commutative.
5. Let A be an elementary p-group of rank $\mathfrak{m}$. Then
$$|\mathrm{Aut}\, A| = (p^m - 1)(p^m - p) \cdots (p^m - p^{m-1}) \qquad \text{or} \qquad 2^{\mathfrak{m}},$$
   according as $\mathfrak{m}$ is finite $(= m)$ or infinite. [*Hint*: count the ways the image of a basis can be chosen.]
6. (a) Let A be the direct sum of m copies of $Z(p^k)$. Show that the order of Aut A is divisible by p^{mk-1}. [*Hint*: induct on m; if $A = \langle a \rangle \oplus B$, then the order of the subgroup of Aut A consisting of lower triangular matrices is divisible by p^{mk-1}.]
   (b) If A is a p-group of order p^n, then the order of Aut A is divisible by p^{n-1}.
7. A summand of A which is a characteristic subgroup of A is necessarily fully invariant in A.
8. Verify the following converse of (f): if $A = \bigoplus A_i$ and Aut $A = \prod$ Aut A_i, then every A_i is fully invariant in A.
9. If A is complete in its Z-adic topology and A_p is its p-adic component, then Aut $A = \prod_p$ Aut A_p.
10. Let ε, ζ be commuting involutions of A as in (1). Find the four projections associated with the decomposition of A stated there.
11. Every automorphism of A and every automorphism of C induce commuting automorphisms on Hom(C, A), Ext(C, A), $C \otimes A$, and Tor(C, A).

12. Let $\langle a_n \rangle$ be a cyclic group of order p_n, and let $T = \bigoplus_n \langle a_n \rangle$, $A = \prod_n \langle a_n \rangle$ for infinitely many different primes p_n.
(a) Every automorphism (endomorphism) of T extends uniquely to one of A, thus A has continuously many automorphisms.
(b) Two pure subgroups of A containing T are isomorphic if and only if some automorphism of A carries one onto the other.
(c) Conclude that there are $2^{2^{\aleph_0}}$ nonisomorphic pure subgroups of A with T as torsion part.
(d) All groups in (c) have commutative automorphism groups.
13. (Mishina [6]) A group A has the property that every automorphism of every subgroup is induced by some automorphism of A if and only if A is one of the following groups: 1. divisible groups; 2. direct sums of a torsion divisible group and a torsion-free group of rank 1; 3. torsion groups whose p-components are direct sums of isomorphic cocylic groups. [*Hint*: consider subgroups which are direct sums of cyclic groups.]

114.* NORMAL SUBGROUPS IN AUTOMORPHISM GROUPS

At the present time, our knowledge of the automorphism groups of abelian groups is extremely limited, even in very special cases. Systematic analysis is only just beginning. Whatever the outcome of such a study, it is clear that even a partial list of normal subgroups is helpful in getting a better understanding of the structure of automorphism groups. This justifies our special interest in search for normal subgroups of Aut A.

Our avenue of approach will be through the group A itself—in this way one can expect to obtain normal subgroups of Aut A more intimately related to A.

We start with the so-called stabilizers. Let $\{X_i\}_{i \in I}$ be a chain of subgroups in a group A; that is to say, given X_i and X_j, either $X_i \leqq X_j$ or $X_j \leqq X_i$ holds. There is a *jump* between X_i and X_j if $X_i \neq X_j$ and if there is no subgroup in the chain between X_i and X_j other than X_i and X_j. The *stabilizer* of the chain $\{X_i\}$ is defined to consist of all $\alpha \in$ Aut A such that, for every jump $X_i > X_j$ in the chain, α leaves the cosets of X_i mod X_j invariant: if $x \in X_i$, then $\alpha x = x + u$ for some $u \in X_j$.

It is readily checked that if the groups X_i in the chain are characteristic subgroups of A, then the stabilizer is a normal subgroup of Aut A.

Let $\{X_i\}$ be a chain of characteristic subgroups of a group A and Σ the stabilizer of the chain. Every $\alpha \in$ Aut A induces, in an obvious fashion, an automorphism α_i of X_i/X_j, where $X_i > X_j$ is a jump in the chain:

$$\alpha_i : a + X_j \mapsto \alpha a + X_j \qquad (a \in X_i).$$

In this way, one obtains a homomorphism $\phi: \alpha \mapsto (\cdots, \alpha_i, \cdots)$ of Aut A into the cartesian product $\prod$ Aut X_i/X_j of the automorphism groups of quotients corresponding to jumps. Evidently, Ker $\phi = \Sigma$. Not much can be said about ϕ in the general case, so let us specialize the chain.

Let A be a p-group of length τ. We focus our attention on the stabilizer of the well-ordered chain

$$(1) \qquad P = P_0 > P_1 > \cdots > P_\omega > \cdots > P_\sigma > \cdots > P_\tau,$$

where $P_\sigma = p^\sigma A[p]$. In this case, $P_\sigma/P_{\sigma+1}$ is a vector space over the prime field of characteristic p whose dimension is the σth Ulm–Kaplansky invariant $f_\sigma(A)$ of A. Hence Aut $P_\sigma/P_{\sigma+1}$ is the general linear group $\mathrm{GL}(f_\sigma(A), p) = \Lambda_\sigma$, and we conclude that there is a homomorphism

$$(2) \qquad \phi: \alpha \mapsto (\alpha_0, \cdots, \alpha_\sigma, \cdots) \qquad (\sigma < \tau)$$

of Aut A into the cartesian product $\Lambda = \prod_{\sigma<\tau} \Lambda_\sigma$. Evidently, Ker ϕ is the stabilizer Δ of (1).

It is of interest to know for which groups A the mapping ϕ is surjective. In this context, we can prove two results. The first one is concerned with torsion-complete p-groups.

Proposition 114.1. *For any p-group A, the composite* Aut $A \to \Lambda \to \Lambda_n$ *of ϕ and the natural projection of Λ onto Λ_n is an epimorphism for every integer n. If A is a torsion-complete p-group, then ϕ in* (2) *is onto.*

We can write $p^n A = U \oplus V$, where $V[p] = P_{n+1}$ and $P_n = U \oplus P_{n+1}$. Any $\alpha_n \in \Lambda_n$ acts as an automorphism on U, and $p^n A$ has an automorphism β which coincides with α_n on U and induces the identity on V. By (113.3), this β can be extended to an automorphism α of A. It is clear that α is mapped upon α_n by the mapping in question.

Let A be a separable p-group and $B = \bigoplus_{n=1}^{\infty} B_n$ with $B_n = \bigoplus Z(p^n)$ a basic subgroup of A. What has been verified shows that Im ϕ is a subdirect product of the Λ_n $(n = 0, 1, \cdots)$. Given any $(\alpha_0, \cdots, \alpha_n, \cdots)$ with $\alpha_n \in \Lambda_n$, we may regard α_n as an automorphism of $p^n B_{n+1}$. It follows from (113.3) that α_n is induced by an automorphism β_n of B_{n+1}. It is clear that there is a unique automorphism β of B such that $\beta | B_{n+1} = \beta_n$ for every $n \geqq 0$. In view of (69.1), β can be extended to a unique $\alpha \in$ Aut $\bar{B}$. This α is mapped by ϕ upon $(\alpha_0, \cdots, \alpha_n, \cdots)$ whenever $A = \bar{B}$.□

There is another important special case in which ϕ is surjective.

Theorem 114.2 (Freedman [1], Hill [23]). *For totally projective p-groups A, the mapping ϕ in* (2) *is onto.*

As in **84**, we set $p^\sigma A[p] = p^{\sigma+1} A[p] \oplus S_\sigma$ for every ordinal σ less than the length τ of A. For each τ, we are given an automorphism α_σ of S_σ, and what amounts to the surjectivity of ϕ is the existence of an $\alpha \in \operatorname{Aut} A$ inducing α_σ on S_σ. This follows from (83.4) and its proof.□

We return our attention to the group Δ, the stabilizer of the chain (1). The subset Δ^* of all elements of finite order in Δ can be described in a satisfactory way.

The following observation is merely technical.

Lemma 114.3 (Leptin [4]). *Let A be a p-group with $p \geqq 5$ and $\alpha \in \operatorname{Aut} A$ such that α fixes the cosets of P_0 mod P_1 and the cosets of P_1 mod P_2. If $\alpha^p = 1$ and $p^2(1-\alpha) = 0$, then $p(1-\alpha) = 0$, too.*

Write $A = B_1 \oplus B_2 \oplus A_2$, where B_i is a maximal direct sum of cyclic groups of orders p^i $(i = 1, 2)$ in a basic subgroup of A. Setting $\eta = 1 - \alpha$, we obtain $\eta P_0 \leqq P_1$, $\eta P_1 \leqq P_2$, and $\eta P_2 = 0$, whence $\eta^3 P_0 = 0$. From $p\eta A = p\eta B_2 + p\eta A_2 \leqq P_2$, $p\eta^2 A = 0$ follows. Hence $\eta^2 A \leqq P_0$ and $\eta^5 A = 0$. Since $p \geqq 5$, $\eta^p = 0$. Clearly $1 = \alpha^p = 1 - p\eta + \binom{p}{2}\eta^2 + \cdots - \eta^p$ implies $p\eta = -\eta^p = 0$.□

We now state and prove the following result.

Theorem 114.4 (Freedman [1], Leptin [4], Hill [23], Hausen [4]). *Let A be a p-group with $p \geqq 5$. The set Δ^* of elements of finite order in the stabilizer Δ of (1) is a p-group. For $\alpha \in \Delta^*$, the following are equivalent;*

(i) $\alpha^{p^n} = 1$;
(ii) $p^n(1-\alpha) = 0$;
(iii) *α induces the identity on $p^n A$.*

If A is a totally projective p-group or if A is separable, then Δ^* is the union of all normal p-subgroups of Aut A.

Let $\alpha \in \Delta^*$ be of finite order $k > 1$. Suppose there is an $a \in P$ such that $\alpha a = a + b$ with $b \neq 0$. If $\alpha b = b + c$, then $h^*(a) < h^*(b) < h^*(c)$. We have $a = \alpha^k a = a + kb + \beta c$ [for some $\beta \in \mathsf{E}(A)$], whence $kb = 0$ follows; thus $p \mid k$ and k is a power of p. If α leaves every element of P fixed, then choose an $a \in A$ of smallest order satisfying $\alpha a \neq a$. Now $\alpha a = a + b$ with $o(b) = p$, whence $\alpha^k a = a + kb$ again implies $p \mid k$. It follows that the orders of the elements of Δ^* are powers of p.

To prove the equivalence of the stated conditions, note that $p^n(1-\alpha) = 0$ is equivalent to $\alpha(p^n a) = p^n a$ for all $a \in A$, thus (ii) and (iii) are trivially equivalent. We next show that $\alpha^p = 1$ implies $\alpha \mid pA$ is the identity. We induct on the length τ on A. If $\tau = 1$, there is nothing to prove. Suppose the length of A is $\tau + 1$ and the implication holds for $A/p^\tau A$. Since α is in the stabilizer of the chain (1) for $A/p^\tau A$, by induction, α fixes the cosets of pA mod $p^\tau A$, i.e., $(1-\alpha)pA \leqq p^\tau A$. From $p(p^\tau A) = 0$ we get $p^2(1-\alpha) = 0$, and (114.3) implies $p(1-\alpha) = 0$. If the length τ of A is a limit ordinal, then $(1-\alpha)pA \leqq p^\sigma A$ for all $\sigma < \tau$, and hence again $p(1-\alpha) = 0$. Now if $\alpha^{p^n} = 1$ for $n > 1$, then $\alpha^{p^{n-1}}|pA$ is the identity, and so α can be viewed as an automorphism of pA whose p^{n-1}st power is the identity. By an obvious induction, $\alpha|p^nA$ is the identity.

Conversely, suppose $\alpha \in \Delta^*$ induces the identity on pA. Then for any $a \in A$, $\alpha a = a + b$ with $pb = 0$. Since $\alpha \in \Delta^*$, $\alpha b = b + c$ for some $c \in A$ of height $> h^*(b)$, and hence $c \in pA$, $\alpha c = c$. From $\alpha^k b = b + kc$ we derive that

$$b + \alpha b + \cdots + \alpha^{p-1} b = pb + c + 2c + \cdots + (p-1)c = pb + \binom{p}{2}c = 0,$$

thus $\alpha^p a = a + b + \alpha b + \cdots + \alpha^{p-1}b = a$ and $\alpha^p = 1$. Now a simple induction on n shows that if $\alpha \in \Delta^*$ induces the identity on p^nA, then $\alpha^{p^n} = 1$.

Using this information, it is easy to check that Δ^* is a subgroup of Δ: if $\alpha, \beta \in \Delta^*$, then they leave some p^nA elementwise fixed, so does $\alpha\beta^{-1}$, and thus $\alpha\beta^{-1} \in \Delta^*$. Consequently, Δ^* is the maximal torsion subgroup of Δ, and thus it is normal in Aut A.

To prove the final assertion, let Θ be a normal p-subgroup of Aut A. Evidently, ϕ in (2) maps Θ onto a normal subgroup of Λ whenever ϕ is epic. But the general linear groups Λ_σ contain no nontrivial normal p-subgroups [see e.g. Freedman [1]], hence $\phi(\Theta)$ is the trivial subgroup of Λ, and $\Theta \subseteq \Delta$. Invoking (114.2), the proof is completed for totally projective groups A. For separable p-groups A, the components of $\phi(\Theta)$ in Λ_n are, because of (114.1), normal, and so again $\phi(\Theta) = 1$ and $\Theta \subseteq \Delta^*$.$\square$

Actually, the group Δ^* is p-nilpotent in the following generalized sense. A group N is called *generalized p-nilpotent* if it has a well-ordered descending chain

$$N = N_0 \rhd N_1 \rhd \cdots \rhd N_\sigma \rhd \cdots \rhd N_\tau = 1 \tag{3}$$

of normal subgroups N_σ such that:

(i) $N_\sigma/N_{\sigma+1}$ is an elementary abelian p-group, for every ordinal $\sigma < \tau$;
(ii) $N_\sigma = \bigcap_{\rho<\sigma} N_\rho$ if σ is a limit ordinal.

Proposition 114.5 (Leptin [4]). *For a p-group A, the group Δ^* is generalized p-nilpotent.*

For every ordinal σ less than the length τ of A, define $\Delta_\sigma = \{\alpha \in \Delta^* | \alpha$ induces the identity on $P/P_\sigma\}$. Then $\Delta_\sigma \lhd$ Aut A for every σ, $\Delta_0 = \Delta^*$, and $\Delta_\sigma = \bigcap_{\rho<\sigma} \Delta_\rho$ for limit ordinals σ. Let $\alpha, \beta \in \Delta_\sigma$; then for $a \in P$, $\alpha a = a + u$ and $\beta a = a + v$ for some $u, v \in P_\sigma$, and $\beta u = u + u'$, $\alpha v = v + v'$ for u', $v' \in P_{\sigma+1}$. Hence

$$\alpha\beta a = a + u + v + v' \equiv a + v + u + u' = \beta\alpha a \mod P_{\sigma+1},$$

thus $\Delta_\sigma/\Delta_{\sigma+1}$ is commutative. Moreover, $\alpha^k a \equiv a + ku \mod P_{\sigma+1}$ implies $\Delta_\sigma/\Delta_{\sigma+1}$ is an elementary p-group. Clearly, $\Delta_\tau = \{\alpha \in \Delta^* | \alpha a = a$ for all $a \in P\}$.

Next define $\Delta_{\tau+n-1} = \{\alpha \in \Delta_\tau \mid \alpha$ induces the identity on $(A[p^n] + pA)/pA\}$ for $n = 1, 2, \cdots$. These are again normal subgroups of Aut A, and we claim that, for each n, $\Delta_{\tau+n-1}/\Delta_{\tau+n}$ is an elementary abelian p-group. Suppose $\alpha, \beta \in \Delta_{\tau+n-1}$. For $a \in A[p^{n+1}]$ we must have $\alpha a = a + u$, $\beta a = a + v$ for some $u, v \in A[p^n]$, since $\alpha \in \Delta_\tau$. Hence

$$\alpha\beta a = a + u + \alpha v \equiv a + u + v \equiv \beta\alpha a \bmod pA \qquad \text{and} \qquad \alpha^p a \equiv a \bmod pA$$

imply what has been asserted.

The group $\Delta_{\tau+\omega} = \bigcap_{n=1}^{\infty} \Delta_{\tau+n-1}$ is normal in Aut A and consists of all $\alpha \in$ Aut A which leave the elements of P fixed and induce the identity on A/pA. By (113.2), $A_{\tau+\omega} = \{1 - p\eta \mid \eta \in \mathsf{E}(A)\}$ [notice that for all $\eta \in \mathsf{E}(A) = \mathsf{E}$, $1 - p\eta$ has an inverse $\sum_{n=0}^{\infty}(p\eta)^n$ in $\mathsf{E}(A)$]. It is clear that $\Delta_{\tau+\omega+n-1} = \{1 - p^n\eta \mid \eta \in \mathsf{E}(A)\}$ are normal subgroups of Aut A. Because of

$$(1 - p^n\eta)(1 - p^n\xi) \equiv 1 - p^n(\eta + \xi) \bmod p^{n+1}\mathsf{E},$$

the correspondence $1 - p^n\eta \mapsto p^n\eta + p^{n+1}\mathsf{E}$ is a group epimorphism of $\Delta_{\tau+\omega+n-1}$ onto $p^n\mathsf{E}/p^{n+1}\mathsf{E}$ whose kernel is $\Delta_{\tau+\omega+n}$. Since $p^n\mathsf{E}/p^{n+1}\mathsf{E}$ is an elementary abelian p-group, so is $\Delta_{\tau+\omega+n-1}/\Delta_{\tau+\omega+n}$ for $n = 1, 2, \cdots$.

In order to complete the proof of (114.5), it only remains to note that $\bigcap_{n=1}^{\infty} \Delta_{\tau+\omega+n} = 1$, since by (46.1), $p^\omega\mathsf{E} = 0$.□

The stabilizer Ω of the chain

$$A > pA > \cdots > p^\sigma A > \cdots > p^\tau A = 0 \tag{4}$$

of a reduced p-group A is also worthwhile investigating. Notice that if an automorphism α of A leaves the cosets of $p^\sigma A \bmod p^{\sigma+1}A$ invariant, then it also leaves the cosets of $p^\sigma A[p] = P_\sigma \bmod p^{\sigma+1}A[p] = P_{\sigma+1}$ invariant. In other words, Ω is a subgroup of the stabilizer Δ of (1). It is routine to check that subgroups of generalized p-nilpotent groups are likewise generalized p-nilpotent; thus (114.5) implies

Corollary 114.6. *The group of elements of finite order in the stabilizer of the chain* (4) *for a p-group A is generalized p-nilpotent.*□

We proceed to another interesting stabilizer for p-groups A; this is the stabilizer Σ of the ascending chain

$$0 < A[p] < A[p^2] < \cdots < A[p^n] < \cdots < A$$

[this chain is finite if A is bounded]. Define

$$\Sigma_n = \{\alpha \in \Sigma \mid \alpha a = a \text{ for all } a \in A[p^n]\};$$

then $\Sigma_n \lhd$ Aut Γ, and we obtain a descending chain

$$\Sigma = \Sigma_1 \rhd \cdots \rhd \Sigma_n \rhd \cdots \rhd 1 \tag{5}$$

of normal subgroups with $\bigcap_{n=1}^{\infty} \Sigma_n = 1$. Note that $A[p^n] < A[p^{n+1}]$ implies $\Sigma_n < \Sigma_{n+1}$ because multiplication by $1 + p^n$ is in Σ_n, but not in Σ_{n+1}. Evidently, Σ_n/Σ_{n+1} is isomorphic to a subgroup of the stabilizer of the chain $0 < A[p^n] < A[p^{n+1}]$ which is, by virtue of (113.1) an elementary abelian p-group. This completes the proof of

Proposition 114.7. *For a p-group A, the chain* (5) *is a properly descending chain of normal subgroups of* Aut A *where the factors are elementary p-groups. It terminates with* $\Sigma_n = 1$ *exactly if* p^n *is the maximal order of elements in* A. □

Hitherto we have been primarily concerned with stabilizers of chains. There are other methods of getting chains of normal subgroups of Aut A.

We return to (4) and define

$$\Gamma_\sigma = \{\alpha \in \operatorname{Aut} A \mid \alpha a = a \text{ for all } a \in p^\sigma A\}$$

to obtain the following well-ordered ascending chain of normal subgroups of $\Gamma = \operatorname{Aut} A$:

$$1 = \Gamma_0 \leqq \Gamma_1 \leqq \cdots \leqq \Gamma_\sigma \leqq \Gamma_{\sigma+1} \leqq \cdots \leqq \Gamma_\tau = \Gamma. \tag{6}$$

The following result gives some information about this chain.

Proposition 114.8. *Let A be a reduced p-group. If n is a natural integer* $< \tau$, *then* $\Gamma_n < \Gamma_{n+1}$ *and* $\Gamma/\Gamma_n \cong \operatorname{Aut} p^n A$.

If A is a totally projective p-group, then $\Gamma_\sigma < \Gamma_{\sigma+1}$ *and* $\Gamma/\Gamma_\sigma \cong \operatorname{Aut} p^\sigma A$, *for every ordinal* $\sigma < \tau$.

In view of (113.3) [(83.4)], it suffices to establish the proper inclusion $\Gamma_n < \Gamma_{n+1}$ [$\Gamma_\sigma < \Gamma_{\sigma+1}$] for $n = 0$ [$\sigma = 0$] only. A has a cyclic summand $\langle a \rangle$, say of order p^k, $A = \langle a \rangle \oplus C$. Then the automorphism of A which is the identity on C and acts as a multiplication by $1 + p^{k-1}$ on $\langle a \rangle$ belongs to Γ_1 but not to Γ_0, provided that $k \geqq 2$. In the remaining case, A is an elementary p-group and $\Gamma_1 = \Gamma$.

The correspondence $\alpha \mapsto \alpha_\sigma = \alpha \mid p^\sigma A$ is a homomorphism $\operatorname{Aut} A \to \operatorname{Aut} p^\sigma A$ with kernel Γ_σ. In view of (113.3) and (83.4) [or just (83.4) alone], it is surjective in the two mentioned cases. □

Our main concern will be to obtain information about the quotients $\Gamma_{\sigma+1}/\Gamma_\sigma$ in (6). To this end we intercalate two normal subgroups of Γ between Γ_σ and $\Gamma_{\sigma+1}$. Set

$$\Gamma_\sigma^* = \{\alpha \in \Gamma_{\sigma+1} \mid \alpha a = a \text{ for all } a \in p^\sigma A[p]\}$$

and

$$\Gamma_\sigma^{**} = \{\alpha \in \Gamma_{\sigma+1} \mid \alpha a - a \in p^{\sigma+1} A \text{ for all } a \in p^\sigma A[p]\}.$$

These are normal subgroups of Γ and obviously, $\Gamma_\sigma \leqq \Gamma_\sigma^* \leqq \Gamma_\sigma^{**} \leqq \Gamma_{\sigma+1}$. Using the notations $f_\sigma(A)$ for the σth Ulm–Kaplansky invariant of A and P_σ for $p^\sigma A[p]$, we have:

Theorem 114.9 (Fuchs [20]). *The quotient groups* $\Gamma_\sigma^*/\Gamma_\sigma$, $\Gamma_\sigma^{**}/\Gamma_\sigma^*$, *and* $\Gamma_{\sigma+1}/\Gamma_\sigma^{**}$ *are isomorphic to subgroups of the following groups, respectively:* $\prod_\mathfrak{r} P_\sigma$ *with* $\mathfrak{r} = r(p^{\sigma+1}A/p^{\sigma+2}A)$; $\prod_\mathfrak{r} P_{\sigma+1}$ *with* $\mathfrak{r} = f_\sigma(A)$; $GL(f_\sigma(A), p)$.

If σ *is a natural integer or if* A *is totally projective, then the quotient groups are isomorphic to the indicated groups themselves.*

Write $p^\sigma A = S_\sigma \oplus C$, where $p^{\sigma+1}A$ is an essential subgroup of C. Every $\alpha \in \Gamma_\sigma^*$ induces an automorphism of $p^\sigma A$, and $1 - \alpha$ induces a map of $C/p^{\sigma+1}A$ into P_σ, namely, $\chi_\alpha: c + p^{\sigma+1}A \mapsto (1-\alpha)c$. It is easily verified that the correspondence $\alpha \mapsto \chi_\alpha$ is a homomorphism of Γ_σ^* into $\mathrm{Hom}(C/p^{\sigma+1}A, P_\sigma)$ whose kernel is the set of all α such that $(1-\alpha)C = 0$, i.e., the kernel is Γ_σ. Since $C/p^{\sigma+1}A \cong p^{\sigma+1}A/p^{\sigma+2}A$, $\Gamma_\sigma^*/\Gamma_\sigma$ is, in fact, isomorphic to a subgroup of the cartesian product of $r(p^{\sigma+1}A/p^{\sigma+2}A)$ copies of P_σ.

Every $\alpha \in \Gamma_\sigma^{**}$ gives rise to a map $\psi_\alpha: x \mapsto (1-\alpha)x$ of S_σ into $P_{\sigma+1}$. The correspondence $\alpha \mapsto \psi_\alpha$ is a homomorphism of Γ_σ^{**} into $\mathrm{Hom}(S_\sigma, P_{\sigma+1})$ with Γ_σ^* as kernel. Now the second assertion follows from $r(S_\sigma) = f_\sigma(A)$.

Every $\alpha \in \Gamma_{\sigma+1}$ induces an automorphism $\bar\alpha$ on S_σ as given by $\alpha s = \bar\alpha s + x$, where $s, \bar\alpha s \in S_\sigma$, $x \in P_{\sigma+1}$. The mapping $\alpha \mapsto \bar\alpha$ is a homomorphism of $\Gamma_{\sigma+1}$ into $GL(f_\sigma(A), p)$ whose kernel is exactly Γ_σ^{**}. Hence the third assertion is immediate.

The final claim of the theorem can be proved *via* an easy application of (113.3) and (83.4). The proof is straightforward and may be left to the reader. □

Exercises

1. If A is torsion-free and G is an essential subgroup of A, then the stabilizer of $0 < G \leqq A$ is the one-element subgroup of Aut A.
2. For a torsion-free group A, the stabilizer of any chain in A is a torsion-free group.
3. Let Θ be the stabilizer of a chain $\{X_i\}$ of subgroups of A. Relate Θ to the stabilizer of a subchain and to the stabilizer of the chain $\{X_i \cap Y\}$, where Y is a characteristic subgroup of A.
4. Using the method of (114.1) show that if A is a direct sum of cyclic p-groups, then ϕ in (2) is surjective.
5. Show that for a bounded p-group A, say $p^kA = 0$, the group Δ^* is a solvable p-group. Give an upper bound, in terms of k, for the class of solvability. [*Hint*: proof of (114.5).]
6. If $\alpha \in$ Aut A induces the identity on $p^\sigma A/p^{\sigma+1}A$, then it induces the identity on $p^{\sigma+n}A/p^{\sigma+n+1}A$ for every integer $n \geqq 1$.
7. Show that in the chain (6), $\bigcup_{\rho<\sigma} \Gamma_\rho$ need not be equal to Γ_σ for limit ordinals σ. [*Hint*: unbounded direct sum of cyclic p-groups.]

8. It is known that $GL(\mathfrak{m}, p)$ is a solvable group if and only if 1. $\mathfrak{m} = 1$, or 2. $p = 2, 3$ and $\mathfrak{m} = 2$. Using this fact, show that for a bounded p-group A, Aut A is solvable exactly if A is finite and its invariants are $\leqq 2$ in case $p = 2$ or 3, and are $\leqq 1$ in case $p \geqq 5$.
9. (Hill [23]) Let A be a direct sum of cyclic p-groups. Every automorphism of $A[p^n]$ that preserves heights [taken in A] can be extended to an automorphism of A. [*Hint*: extend it to $A[p^{n+1}]$.]
10*. Give an example where ϕ in (2) is not epic.
11. (Mader [2]) Let A be a divisible p-group and Σ_n the normal subgroup of Aut A which consists of all automorphisms of A leaving $A[p^n]$ elementwise fixed. Show that Aut $A/\Sigma_n \cong$ Aut $A[p^n]$.

115. AUTOMORPHISM GROUPS OF TORSION GROUPS

We can now attempt to answer the basic question as to what extent the groups are determined by their automorphism groups. The analog of (108.1) holds true at least for p-groups with $p \geqq 5$: they ought to be isomorphic if their automorphism groups are isomorphic.

The proof of this result is more sophisticated, since the powerful machinery furnished by the projections in the course of the proof of (108.1) is now unavailable. One has to use, instead, involutions, which are less sensitive against direct summands. We shall be content with the proof of a weaker result.

Our first concern is the center of the automorphism group of torsion groups; we shall use the notation $\mathfrak{z}\Gamma$ for the center of the group Γ. Because of **113**(g), there is no loss of generality in restricting our considerations to p-groups.

Theorem 115.1 (Baer [5]). *The center of the automorphism group of a p-group A consists of multiplications by p-adic units if A is unbounded, and multiplications by integers k with $(k, p) = 1$, $1 \leqq k < p^r$, if p^r is the smallest bound for orders of elements of A.*

There is, however, one exception: if A is a 2-group of the form

$$A = Z(2^\infty) \oplus Z(2^r) \oplus G \qquad \textit{with} \quad 2^{r-1}G = 0, \quad r \geqq 1, \tag{1}$$

then $\mathfrak{z}$ Aut *A is the direct product of the multiplicative group of dyadic units and a cyclic group of order 2. The latter group can, e.g., be generated by the automorphism δ of A which acts trivially on $Z(2^\infty)$ and G and carries any generator b of $Z(2^r)$ into $b + a_0$, where a_0 is the unique element of order 2 and of infinite height in A.*

It is straightforward to check [and it follows from (108.1) at once] that multiplications by p-adic units are necessarily contained in $\mathfrak{z}$ Aut A. In the exceptional case, a_0 is left fixed by all automorphisms, and hence it follows readily that the automorphism δ described in the theorem commutes with all automorphisms of A.

Let now γ be a central element of Aut A, and suppose $A = B \oplus C$, where we denote by ε the involution $\varepsilon | B = 1_B$, $\varepsilon | C = -1_C$. For any $b \in B$, we can write $\gamma b = b_0 + c_0$ ($b_0 \in B$, $c_0 \in C$). Hence $b_0 + c_0 = \gamma\varepsilon b = \varepsilon\gamma b = b_0 - c_0$ implies $2c_0 = 0$. This means γ carries $2B$ into [and hence onto] itself. Consequently, quasicyclic subgroups and, in case $p \neq 2$, all summands of A are carried by γ onto themselves.

If $\eta: B \to C$ is a homomorphism, then

$$\alpha: b \mapsto b + \eta b, \qquad c \mapsto c \qquad (b \in B, \quad c \in C)$$

is an automorphism of A. The equality $\alpha\gamma = \gamma\alpha$ yields $\eta b_0 = \gamma\eta b$, showing that γ carries Im η into itself. If $\chi: C \to B$ is a homomorphism, then

$$\beta: b \mapsto b, \qquad c \mapsto c + \chi c \qquad (b \in B, \quad c \in C)$$

is an automorphism of A, and from $\beta\gamma = \gamma\beta$ we obtain $\chi c_0 = 0$. Consequently, c_0 lies in the kernel of all homomorphisms of C into B.

First specialize $B = \langle b \rangle$, say, of order p^n. Then $\gamma b = kb + c_0$ for some integer k with $1 \leqq k < p^n$. Now $\eta kb = \gamma\eta b$ implies that γ acts as multiplication by k on Im η. Letting η run over all $\eta: B \to C$, we see that γ acts as multiplication by k on $C[p^n]$, and so necessarily $(k, p) = 1$. If C has a cyclic summand of order $\geqq p^n$, then we can infer that $\gamma b = kb$, too, thus γ acts as multiplication by k on $A[p^n]$. If A has an unbounded basic subgroup, then it is readily seen that there must exist a p-adic unit ρ such that γ is the multiplication by ρ [on each $A[p^n]$ and hence] on A.

Next we specialize $B = Z(p^\infty)$. We know that γ induces an automorphism on B which is, in view of Example 3 in **113**, a multiplication by a p-adic unit ρ. If C is not reduced, then using a monomorphism $\eta: B \to C$ we obtain $\eta\rho = \gamma\eta$, whence γ is likewise multiplication by ρ on Im η. Our conclusion is that γ always acts as a multiplication by a p-adic unit ρ in the divisible part of A.

It remains to consider the case $A = D \oplus E$, where D is divisible and $E \neq 0$ is bounded, say $p^r E = 0$ but $p^{r-1}E \neq 0$. If $p \geqq 3$ or if E has at least two independent elements of order p^r, then it follows in the same way as above that γ carries every cyclic subgroup of A into itself and it must be a multiplication by a p-adic unit ρ or an integer k prime to p, $1 \leqq k < p^r$, according as $D \neq 0$ or $D = 0$. Suppose $p = 2$ and $E = \langle b \rangle \oplus E'$ with $o(b) = 2^r$, $2^{r-1}E' = 0$. By what has been shown we know for sure that γ is a multiplication by a dyadic unit ρ on D and by an integer $\equiv \rho \bmod 2^{r-1}$ on $\langle 2b \rangle \oplus E'$. Concentrating on γb, we write $\gamma b = kb + a_0$, where $a_0 \in D \oplus E'$, $2a_0 = 0$. If a_0 is of finite height, then there is a $\chi: D \oplus E' \to \langle b \rangle$ with $\chi a_0 \neq 0$, in contradiction to what has been observed above. If $0 \neq a_0 \in D_0 \cong Z(2^\infty)$ and if $D = D_0 \oplus D_1$ with $D_1 \neq 0$, then applying the same remark to the decomposition $A = (\langle b \rangle \oplus D_1) \oplus (E' \oplus D_0)$, the existence of a monomorphism $\chi\colon D_0 \to D_1$ shows that $a_0 \neq 0$ implies $D \cong Z(2^\infty)$ and a_0 is the unique element of $A[2]$ of infinite height. Hence A has form (1) and γ is either a multiplication by a dyadic unit or the product of a multiplication by a dyadic unit and the automorphism δ. This completes the proof.□

From Examples 1 and 2 in **128** we conclude that for a p-group A with $p \neq 2$,

(2) $$\mathfrak{z}\,\mathrm{Aut}\,A \cong Z(p-1) \times J_p \quad \text{or} \quad Z(p-1) \times Z(p^{k-1}).$$

Since a group is commutative exactly if it coincides with its center, from the preceding theorem it is easy to derive:

Corollary 115.2 (Châtelet [1], Baer [5]). *The automorphism group of a p-group A is commutative if and only if A is either cocyclic or isomorphic to* $Z(2^\infty) \oplus Z(2)$.□

Before taking up the question of isomorphic automorphism groups, we make a few comments. In the sequel, A always stands for a p-group, where $p \geqq 3$.

(a) From (115.1) it follows that A is bounded and p^n is the l.u.b. for the orders of its elements exactly if $\mathfrak{z}\,\mathrm{Aut}\,A \cong Z(p^{n-1}(p-1))$.

(b) If ε is any involution of A, $\varepsilon \neq \pm 1$, then it is easy to tell whether or not one or both summands in $A = A_\varepsilon^+ \oplus A_\varepsilon^-$ are bounded, and if so, what the bounds are. In fact, $\mathfrak{z}\mathfrak{c}\{\varepsilon\} = \mathfrak{z}\,\mathrm{Aut}\,A_\varepsilon^+ \times \mathfrak{z}\,\mathrm{Aut}\,A_\varepsilon^-$ and (115.1) show that $\mathfrak{z}\mathfrak{c}\{\varepsilon\}$ contains the direct product of 0, 1, or 2 copies of J_p according as none, one, or both of A_ε^+, A_ε^- are unbounded. Utilizing (a), the bounds for the bounded components can easily be determined.

(c) If $A = A_1 \oplus \cdots \oplus A_k$ with $A_i \neq 0$, and if $\{\varepsilon_1, \cdots, \varepsilon_k\}$ is the system of involutions belonging to this decomposition, then $\mathfrak{z}\mathfrak{c}\{\varepsilon_1, \cdots, \varepsilon_k\}$ contains exactly 2^k involutions. This follows from **113**(n) and from the fact that $\mathfrak{z}\,\mathrm{Aut}\,A_k$ contains no involutions other than ± 1.

An involution ε is said to be *extremal* if either A_ε^+ or A_ε^- is indecomposable $\neq 0$. If ζ is any involution commuting with an extremal ε, then the decomposition of A given in **113**(l) can contain at most three nonzero summands. On the other hand, if ε is not extremal, then it is easy to find an involution ζ that commutes with ε and yields four nonzero summands in **113**(l). Consequently:

(d) An involution ε is extremal exactly if the number of involutions in $\mathfrak{z}\mathfrak{c}\{\varepsilon, \zeta\}$ is at most 8, for every involution $\zeta \in \mathfrak{c}\{\varepsilon\}$. Hence extremal involutions are carried into extremal involutions under any isomorphism between automorphism groups.

Leptin [2] has proved that *if* $p \geqq 5$ *and if* A, C *are p-groups with isomorphic automorphism groups, then* A *and* C *are isomorphic.* We intend to prove here a weaker result whose proof is easier.

Theorem 115.3 (Leptin [2]). *If* A *and* C *are p-groups with* $p \geqq 3$, *and if* $\operatorname{Aut} A \cong \operatorname{Aut} C$, *then* A *and* C *have isomorphic basic subgroups and isomorphic divisible parts.*

Let ϕ be an isomorphism $\operatorname{Aut} A \to \operatorname{Aut} C$. Let $B = \bigoplus B_n$, $B_n = \bigoplus Z(p^n)$, be a basic subgroup of A, and write $A = B_1 \oplus \cdots \oplus B_n \oplus A_n^*$, where $A_n^* = p^nA + B_{n+1} + B_{n+2} + \cdots$ [cf. (32.4)]. Suppose B unbounded. Replacing $B_1 \oplus \cdots \oplus B_n$ by a more explicit $\bigoplus_{i \in I}\langle a_i \rangle$, where $o(a_i) \leqq p^n$, we consider the involutions ε_i, ε belonging to the decomposition

$$A = \bigoplus_{i \in I}\langle a_i \rangle \oplus A_n^*. \tag{3}$$

Our first purpose is to establish properties of the system $\{\varepsilon_i, \varepsilon\}$ which enable us to recognize that it belongs to a decomposition where $\bigoplus\langle a_i \rangle$ is a maximal p^n-bounded summand of A. Clearly, $\{\varepsilon_i, \varepsilon\}$ is a commutative system of involutions, where each ε_i is extremal. If η is any extremal involution in $\mathfrak{c}\{\varepsilon_i, \varepsilon\}$, different from the ε_i, and if say A_η^- is indecomposable, then **113**(l) implies that A_η^- is a subgroup of every $\bigoplus_{j \neq i}\langle a_j \rangle \oplus A_n^*$, thus $A_\eta^- \leqq A_n^*$ and A_η^- is an indecomposable summand of A_n^*, $|A_\eta^-| \geqq p^{n+1}$. If ε is not extremal, then $\{\varepsilon_i, \varepsilon, \eta\}$ gives rise to a refinement of (3) such that

$$\mathfrak{z}\mathfrak{c}\{\varepsilon_i, \varepsilon, \eta\}/\mathfrak{z}\mathfrak{c}\{\varepsilon_i, \varepsilon\} \tag{4}$$

will be isomorphic to a cyclic group of order $p^k(p-1)$ with $k \geqq n$ or to $Z(p-1) \times J_p$. Evidently, if A_n^* had a cyclic summand of order $\leqq p^n$, then we could find an η for which (4) would be cyclic of order $p^k(p-1)$ with $k \leqq n-1$.

Now we pass to $\operatorname{Aut} C$ *via* ϕ. Evidently, $\phi(\varepsilon_i)$ is extremal for every $i \in I$, thus it gives rise to a direct decomposition of C, $C = C_{\phi(\varepsilon_i)}^+ \oplus C_{\phi(\varepsilon_i)}^-$, where one of the summands, say C_i, is indecomposable, and denote the other summand by C_i'. By (b), we know the bounds for the summands; since we have assumed B unbounded, it follows at once that $C_i \cong \langle a_i \rangle$. Now **113**(m) implies that for every finite number of indices, $C = C_{i_1} \oplus \cdots \oplus C_{i_k} \oplus (C_{i_1}' \cap \cdots \cap C_{i_k}')$ holds. Consequently, $C' = \sum_i C_i = \bigoplus_i C_i$ is a pure subgroup of C. It is bounded, therefore it is a summand of C. We also have $C = C_{\phi(\varepsilon)}^- \oplus C_{\phi(\varepsilon)}^+$, where, say, the first summand is p^n-bounded. $\phi(\varepsilon_i)$ commutes with $\phi(\varepsilon)$, so **113**(l) implies that each C_i is contained in one of the summands. $C_i \leqq C_{\phi(\varepsilon)}^+$ would lead to the existence of an involution $\eta' \in \mathfrak{c}\{\phi(\varepsilon)\}$ such that $\mathfrak{z}\mathfrak{c}\{\phi(\varepsilon), \eta'\}$ is never isomorphic to any $\mathfrak{z}\mathfrak{c}\{\varepsilon, \eta\}$, for any involution $\eta \in \mathfrak{c}\{\varepsilon\}$. Hence $C' \leqq C_{\phi(\varepsilon)}^-$, and C' is a summand of $C_{\phi(\varepsilon)}^-$. If it was a proper summand, then there would exist an involution $\eta' \in \mathfrak{c}\mathfrak{z}\{\phi(\varepsilon_i), \phi(\varepsilon)\}_{i \in I}$ such that the factor group corresponding to (4) would be $Z(p^k(p-1))$ with $k \leqq n-1$, which is absurd.

Hence $C = \bigoplus_i C_i \oplus C^+_{\phi(\varepsilon)}$, where $p^n C_i = 0$ for all i. From what has been said in the last paragraph about $\{\varepsilon_i, \varepsilon\}$, it follows that $C^+_{\phi(\varepsilon)}$ fails to have any cyclic summand of order $\leqq p^n$. Consequently, $\bigoplus C_i$ is a maximal p^n-bounded summand of C.

Notice that ε_i and $\phi(\varepsilon_i)$ determine isomorphic indecomposable summands of A and C, respectively. Hence, for every n, maximal p^n-bounded summands of A and C are isomorphic. The isomorphism of the basic subgroups of A and C follows at once from (33.2).

If we replace (3) by $A = \bigoplus_j D_j \oplus G$, where $D_j \cong Z(p^\infty)$ and G is reduced, then a similar argument applies. It is obvious how to formulate the absence of quasicyclic summands in G in terms of involutions. This completes the proof of (115.3) for A with unbounded basic subgroups.

If A happens to have a bounded basic subgroup B, say $p^m B = 0$, then we can write $A = \bigoplus_i \langle a_i \rangle \oplus \bigoplus_j D_j$ with $D_j \cong Z(p^\infty)$ and consider the corresponding system $\{\varepsilon_i, \eta_j\}$ of involutions. Then $\{\phi(\varepsilon_i), \phi(\eta_j)\}$ will be a maximal commutative system of extremal involutions such that the corresponding indecomposable components are $Z(p^\infty)$ or cyclic of order $\leqq p^m$. As above, we conclude that the indecomposable components generate a pure subgroup C' which must be a summand of C, thus $C' = C$ and C is a direct sum of cocyclic p-groups. Since $\mathfrak{c}\{\phi(\varepsilon_i), \phi(\eta_j)\}$ is now the cartesian product of the automorphism groups of these cocyclic groups, the isomorphism of A and C is evident whenever A is a finite direct sum of cocyclic groups. If A is not such, then one first establishes that $\phi(\eta_j)$ will correspond to quasicyclic summands, and proceeds with decompositions (3) for $n = 1, \cdots, m-1$.□

Leptin's proof of the isomorphy of A and C is based on a delicate analysis of stabilizers of direct decompositions. The main difficulty in the proof arises from the fact that if ε and $\phi(\varepsilon)$ are the involutions belonging to the decompositions $A = A_1 \oplus A_2$ and $C = C_1 \oplus C_2$, then the stabilizer of $0 < A_1 < A$ is carried by ϕ into the stabilizer of $0 < C_1 < C$ or of $0 < C_2 < C$. This ambiguity can not always be eliminated even if we knew that A_1, C_1 were bounded and A_2, C_2 were unbounded. It would be nice to find a shorter proof for Leptin's important theorem, preferably including the primes 2 and 3, too.

Exercises

1. (Baer [5]) (a) The set of elements in a p-group A, left fixed under all automorphisms of A is not 0 if and only if A is a 2-group which has a unique element $g \neq 0$ of maximal height.
   (b) The set mentioned in (a) is $\langle g \rangle$ in the exceptional case.

2. (Baer [5]) (a) Let A be a separable p-group, $p \geqq 3$, and C a subgroup of A. The set of all elements of A left fixed under all automorphisms of A leaving C elementwise fixed is equal to C^-, the closure of C in the p-adic topology.
(b) For $p = 2$, the exceptional subgroup $\langle g \rangle$ in Ex. 1 must be adjoined to C^-.
3. (Boyer [1], E. A. Walker [4], Khabbaz [1], Mader [1], Hill [4]) (a) If A is a nonreduced p-group, then $|\mathrm{Aut}\, A| = 2^{|A|}$.
(b) If A is an infinite reduced torsion group, then $|\mathrm{Aut}\, A| = 2^{|B|}$, where B is a basic subgroup of A.
4. (Leptin [2]) For an involution ε of a p-group A ($p \geqq 3$), the summands A_ε^+ and A_ε^- have isomorphic summands $\neq 0$ if and only if there exist a system $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ of commuting involutions in $\mathfrak{c}\{\varepsilon\}$ and an involution $\eta \in \mathfrak{c}\{\varepsilon_3, \varepsilon_4\}$ satisfying $\varepsilon = \varepsilon_3 \varepsilon_4$ and $\eta\varepsilon_2 = \varepsilon_3 \eta$. [*Hint*: $A_\varepsilon^+ = A_1 \oplus A_2$, $A_\varepsilon^- = A_3 \oplus A_4$, and if $A_2 \cong A_3$, then choose η to switch A_2 with A_3; for the converse, use matrix representation.]
5. Apply the last part of the proof of (115.3) to show that if A is a direct sum of cyclic p-groups and if $\mathrm{Aut}\, A \cong \mathrm{Aut}\, C$ for a p-group C, then $A \cong C$.
6. (Leptin [2]) Let A be a p-group and C a q-group, where $3 \leqq p < q$ are primes. Then for $\mathrm{Aut}\, A \cong \mathrm{Aut}\, C$, it is necessary and sufficient that $A = Z(p^k)$, $C = Z(q)$, and $p^{k-1}(p-1) = q - 1$.
7. (R. Baer) A p-group A ($p \geqq 3$) satisfies the minimum condition exactly if all torsion subgroups of $\mathrm{Aut}\, A$ are finite. [*Hint*: an infinity of independent summands yields an infinite torsion group in the centralizer of the system of involutions; conversely, reduce to divisible groups and consider the matrix representation.]
8. (Hausen [2]) The automorphism group of a group with minimum condition is residually finite. [*Hint*: the normal subgroups, consisting of all automorphisms fixing $A[p^k]$ elementwise, are of finite index and have trivial intersection.]
9. (Tarwater [3]) Let A be the direct sum of cocyclic groups of the same order. There is an automorphism of A carrying a subgroup G onto a subgroup H if and only if

$$G \cong H \qquad \text{and} \qquad A[p]/G[p] \cong A[p]/H[p].$$

10. (Megibben [5]) (a) Let ϕ be a monomorphism of a large subgroup $G = A(r_0, \cdots, r_n, \cdots)$ of a p-group A into a p-group C. It can be extended to an isomorphism from A onto C if and only if: (i) $f_n(A) = f_n(C)$ for $n \leqq r_0$; (ii) ϕ is height-preserving; (iii) $\mathrm{Im}\, \phi$ is a large subgroup of C.
(b) Every automorphism of a large subgroup of a p-group A which preserves heights in A can be extended to an automorphism of A.

116. AUTOMORPHISM GROUPS OF TORSION-FREE GROUPS

While p-groups for $p \geqq 5$ are completely determined up to isomorphism by their automorphism groups, the automorphism groups of torsion-free groups give little information about the torsion-free groups themselves. In fact, there are large torsion-free groups with the same automorphism group as Z, and not even the rank 1 groups can be recaptured from their automorphism groups [cf. Ex. 1]. Also, in contrast to the torsion case, the automorphism group of an infinite group A need not be infinite when A is torsion-free.

Our knowledge of groups which can be realized as automorphism groups of torsion-free groups does not go much further than the finite case. We give here an account of the results obtained by Hallett and Hirsch [1, 2].

Throughout this section, A will denote a torsion-free group and

$$\Gamma = \text{Aut } A.$$

(a) *If Γ is a torsion group, then $\mathsf{E}(A)$ has no nilpotent elements.*

If $\eta \in \mathsf{E}(A)$ satisfies $\eta^2 = 0$, then the endomorphisms $1 + \eta$ and $1 - \eta$ are inverse to each other, thus they are in Γ. But $(1 + \eta)^n = 1 + n\eta \neq 1$ unless $\eta = 0$, because of the torsion-freeness of End A.

(b) *If Γ is torsion, then every involution $\alpha \in \Gamma$ belongs to the center of Γ.*

For any $\beta \in \mathsf{E}(A)$, $\phi = (1 + \alpha)\beta(1 - \alpha)$ and $\psi = (1 - \alpha)\beta(1 + \alpha)$ are nilpotent endomorphisms, hence $\phi = 0 = \psi$ by (a). Thus $2(\alpha\beta - \beta\alpha) = \phi - \psi = 0$, and $\alpha\beta = \beta\alpha$ follows.

(c) *If Γ is torsion and $\alpha \in \Gamma$ is of odd order $n > 1$, then $n = 3$.*

Manifestly, it suffices to prove this under the additional hypothesis that $n = p^k$, where p is a prime $\geqq 3$. The endomorphism $\beta = 1 - \alpha + \alpha^2 - + \cdots + \alpha^{n-3}$ is inverse to

$$\gamma = \begin{cases} \alpha^2 - \alpha^4 + - \cdots + \alpha^{n+1} & \text{if} \quad n \equiv 1 \bmod 4, \\ \alpha^3 - \alpha^5 + - \cdots + \alpha^n & \text{if} \quad n \equiv -1 \bmod 4, \end{cases}$$

thus $\beta \in \Gamma$. To show that β is of finite order, observe that $\alpha^n - 1$ annihilates A, thus there is a minimal polynomial $g(\alpha)$ with integral coefficients annihilating A. Evidently, $g(x)$ divides $x^{p^k} - 1$, but not $x^{p^{k-1}} - 1$; from the irreducibility of the p^kth cyclotomic polynomial $\Phi_n(x) = (x^{p^k} - 1)/(x^{p^{k-1}} - 1)$ we deduce the divisibility relation $\Phi_n(x) \mid g(x)$. Consequently, the mapping $\alpha \mapsto \zeta$, where ζ is a primitive nth root of unity, extends to a ring homomorphism $\mathsf{S}(\alpha) \to \mathsf{Q}(\zeta)$ of the subring of $\mathsf{E}(A)$ generated by α into the extension field of Q by ζ. This map carries β into the complex number $1 - \zeta + \zeta^2 - + \cdots + \zeta^{n-3} = (1 + \zeta^{n-2})(1 + \zeta)^{-1}$ whose absolute value is $\neq 1$ unless $\zeta^{n-2} = \zeta$ or $\bar{\zeta}$, in which case $n = 3$. Therefore, the image of β, and hence β too, is of infinite order for $n > 3$.

(d) *If Γ is torsion, it does not contain elements of order* 8.

In fact, if $\alpha \in \Gamma$ is of order 8, then $\beta = 1 + (1 - \alpha^4)(1 + \alpha - \alpha^3)$ and $\gamma = 1 + (1 - \alpha^4)(1 - \alpha + \alpha^3)$ are inverse to each other in $\mathsf{E}(A)$. As in (c), we can show that the subring $\mathsf{S}(\alpha)$ of $\mathsf{E}(A)$ generated by α has a homomorphism into $\mathsf{Q}(\zeta)$ such that $\alpha \mapsto \zeta = (1 + i)/\sqrt{2}$ = primitive 8th root of unity. Under this map, β is sent into $3 + 2\sqrt{2}$ which is a unit of infinite order in $\mathsf{Q}(\zeta)$, showing that β can not be of finite order.

(e) *If Γ is torsion, then not every involution in Γ is contained in a cyclic subgroup of order* 12.

We claim that no $\alpha \in \Gamma$ of order 12 can satisfy $\alpha^6 = -1$. Otherwise $\beta = 1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4$ is inverse to $\gamma = 1 - \alpha^2 - \alpha^3 - \alpha^4 - \alpha^5$, and it only remains to show that β is not of finite order. We again examine the polynomial $g(x)$ with integral coefficients minimal with respect to the property $g(\alpha)A = 0$. Notice that $x^6 + 1 = (x^2 + 1)(x^4 - x^2 + 1)$ with irreducible factors and $(\alpha^2 + 1)A = 0$ is impossible, $\alpha^2 = -1$ being absurd; thus we see that $\Phi_{12}(x) = x^4 - x^2 + 1$ divides $g(x)$. As above we get a homomorphism carrying β into $1 - \zeta + \zeta^2 - \zeta^3 + \zeta^4 = (1 + \zeta^5)(1 + \zeta)^{-1}$, where ζ is a primitive 12th root of unity. Since $|1 + \zeta^5| \neq |1 + \zeta|$, β must be of infinite order.

(f) *Still assuming Γ torsion, the Sylow 3-subgroups of Γ are commutative.*

By (c), the Sylow 3-subgroups S are of exponent 3, thus from a known result on noncommutative groups we conclude that any two elements $\alpha, \beta \in S$ and their commutator $\gamma = \alpha^{-1}\beta^{-1}\alpha\beta = [\alpha, \beta]$ satisfy $\alpha^3 = \beta^3 = \gamma^3 = 1$ and $\alpha\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta$. Hence we can apply **106**(h) with $1 + \gamma + \gamma^2$, $1 - \gamma$ and $m = 3$ [note that $(1 + \gamma + \gamma^2) + (2 + \gamma)(1 - \gamma) = 3$] to obtain $3A \leqq B \oplus C$, where $B = \operatorname{Ker}(1 + \gamma + \gamma^2)$, $C = \operatorname{Ker}(1 - \gamma)$ are fully invariant in A. In a similar way, we obtain $3B \leqq [\operatorname{Ker}(1 + \alpha + \alpha^2) \cap B] \oplus [\operatorname{Ker}(1 - \alpha) \cap B]$. Here α, and thus γ too, induces the identity on the second summand; this summand is therefore contained in C, i.e., it vanishes. We conclude that $(1 + \alpha + \alpha^2)(1 - \gamma) = 0$, i.e., $\gamma + \gamma\alpha + \gamma\alpha^2 = 1 + \alpha + \alpha^2$. Repeated conjugation by β gives $\gamma + \gamma^2\alpha + \alpha^2 = 1 + \gamma\alpha + \gamma^2\alpha^2$ and $\gamma + \alpha + \gamma^2\alpha^2 = 1 + \gamma^2\alpha + \gamma\alpha^2$ [using $\beta^{-1}\alpha\beta = \gamma\alpha$ and $\gamma^3 = 1$], whence addition and cancellation yield $3\gamma = 3$. Hence $\gamma = 1$, establishing commutativity.

(g) *If $\alpha \in \Gamma$ is an involution, then A has a characteristic subgroup B such that if $\phi\colon \Gamma \to \operatorname{Aut} B$ is the restriction homomorphism $\gamma \mapsto \gamma | B$, then $\phi(\alpha) = -1$ and $|\phi(\Gamma)|$ divides* 24.

Since $\alpha^2 = 1$, we can apply **106**(h) with $1 + \alpha$, $1 - \alpha$, and $m = 2$ to get $2A \leqq B_1 \oplus C_1$, where α is -1 on $B_1 \neq 0$ and the identity on C_1. Let $\phi_1 : \Gamma \to \operatorname{Aut} B_1$ be the restriction homomorphism. Evidently, (c), (d), and (f) hold for $\phi_1(\Gamma)$. To verify that every element $\beta \in \phi_1(\Gamma)$ of order 2 belongs to the center of Aut B_1, assume by way of contradiction that some $\gamma \in \operatorname{Aut} B_1$ satisfies $[\beta, \gamma] \neq 1$. Now, $\phi = (1+\beta)\gamma(1-\beta)$ and $\psi = (1-\beta)\gamma(1+\beta)$ are nilpotent endomorphisms of B_1, and from the argument in (b) and from $[\beta, \gamma] \neq 1$ we infer that $\phi \neq 0$ or $\psi \neq 0$, i.e., $\mathsf{E}(B_1)$ contains nonzero nilpotent elements. The same is true for $\mathsf{E}(A)$, contrary to (a).

Now, if $\phi_1(\Gamma)$ contains, apart from $\phi_1(\alpha)$, another element β of order 2, then we get analogously, $2B_1 \leqq B_2 \oplus C_2$, where $\beta | B_2 = -1$, $\beta | C_2 = 1$, and B_2, C_2 are characteristic subgroups of [B_1 and hence of] A. Thus proceeding, we are finally led to a characteristic subgroup B_0 of A such that $\phi_0(\alpha) = -1$ is the only element of order 2 in $\phi_0(\Gamma)$. Since a finite 2-group with a single element of order 2 is cyclic or generalized quaternion, from (d) we conclude that the Sylow 2-groups of $\phi_0(\Gamma)$ are cyclic of order 2 or 4, or quaternion groups.

Consequently, $|\phi_0(\Gamma)| = 2^k 3^l$ with $k \leqq 3$. If $l > 1$, we continue our process and select an element $\delta \in \phi_0(\Gamma)$ of order 3. Applying **106**(h) with $1 + \delta + \delta^2$, $1 - \delta$, and $m = 3$, there results $3B_0 \leqq B' \oplus C'$, where B', C' are characteristic in A such that $1 + \delta + \delta^2 | B' = 0$ and $\delta | C' = 1$. If the restriction of $\phi_0(\Gamma)$ to B' contains another element ε of order 3 with the property $(1 + \varepsilon + \varepsilon^2)B' \neq 0$, then we continue this process until we get a characteristic subgroup B such that $1 + \beta + \beta^2 = 1 + \gamma + \gamma^2 = 0$ on B for any two elements $\beta, \gamma \neq 1$ of a Sylow 3-subgroup of $\phi(\Gamma)$. If $\gamma \neq \beta^{-1}$, then $1 + \beta\gamma + (\beta\gamma)^2 = 0$ on B, too, thus

$$-\beta\gamma = 1 + \beta^2\gamma^2 = 1 + (1+\beta)(1+\gamma) = 2 + \beta + \gamma + \beta\gamma,$$

and from

$$\begin{aligned}(\beta - \gamma)^2 = \beta^2 - 2\beta\gamma + \gamma^2 &= \beta^2 + (2 + \beta + \gamma) + \gamma^2 \\ &= (1 + \beta + \beta^2) + (1 + \gamma + \gamma^2) = 0\end{aligned}$$

and (a) we infer $\beta = \gamma$. Therefore, the Sylow 3-subgroups of $\phi(\Gamma)$ are of order 3 and $|\phi(\Gamma)|$ divides 24. This concludes the proof of (g).

After these preparations, we can now prove:

Theorem 116.1 (Hallett and Hirsch [1]). *If the finite group Γ is the automorphism group of a torsion-free group A, then Γ is isomorphic to a subgroup of a finite direct product of groups of the following types:*

(i) *cyclic groups of orders* 2, 4, *or* 6;
(ii) *the quaternion group* $\mathrm{Q}_8 = \langle \alpha, \beta | \alpha^2 = \beta^2 = (\alpha\beta)^2 \rangle$ *of order* 8;
(iii) *the dicyclic group* $\mathrm{DC}_{12} = \langle \alpha, \beta | \alpha^3 = \beta^2 = (\alpha\beta)^2 \rangle$ *of order* 12;
(iv) *the binary tetrahedral group* $\mathrm{BT}_{24} = \langle \alpha, \beta | \alpha^3 = \beta^3 = (\alpha\beta)^2 \rangle$ *of order* 24.

A repeated application of the procedure in (g) yields that for some integer m,

$$mA \leqq C_1 \oplus \cdots \oplus C_k,$$

where $C_1, \cdots, C_k$ are characteristic subgroups of A such that, for the restriction map $\phi_i: \Gamma \to \mathrm{Aut}\, C_i$, $|\phi_i(\Gamma)|$ divides 24, for every i. The correspondence: $\alpha \mapsto (\phi_1(\alpha), \cdots, \phi_k(\alpha))$ is a homomorphism $\psi: \Gamma \to \phi_1(\Gamma) \times \cdots \times \phi_k(\Gamma)$ such that $\psi(\alpha) = 1$ only if $\phi_i(\alpha) = 1$ for $i = 1, \cdots, k$, that is, α is the identity on $C_1 \oplus \cdots \oplus C_k$ and hence on A. Thus ψ is monic.

The groups whose orders divide 24 and satisfy the conditions in (b), (d) are easily listed: they are (i)–(iv), and in addition, $Z(12)$ and $\mathrm{DC}_{24} = \langle \alpha, \beta \mid \alpha^6 = \beta^2 = (\alpha\beta)^2 \rangle$. The last two groups can not occur as automorphism groups, as they violate (e). We now show how to eliminate these groups from our considerations.

Let $\alpha \in \Gamma$ be of order 12. Then $\alpha^{12} - 1$ annihilates A, and as in (c) we infer that there exists a minimal polynomial $g(\alpha)$ with integral coefficients annihilating A. The 12th cyclotomic polynomial $\Phi_{12}(x) = x^4 - x^2 + 1$ can not divide $g(x)$, because then under the homomorphism $\mathsf{S}(\alpha) \to \mathsf{Q}(\zeta)$ [where $\alpha \mapsto \zeta$] of the subring of $\mathsf{E}(A)$ generated by α into the extension field of Q by a primitive 12th root ζ of unity, the unit $1 + \alpha$ [with inverse $\alpha^2 - \alpha^3$] would be mapped upon a complex number with absolute value $\neq 1$. It is clear that α can be of order 12 only if the subgroups annihilated by $\alpha^6 - 1$ and $\alpha^4 - 1$ generate A, i.e., $(\alpha^6 - 1)(\alpha^2 + 1) = 0$. Noting that $-(x^6 - 1) + (x^4 - x^2 + 1)(x^2 + 1) = 2$, **106**(h) implies $2A \leqq B \oplus C$, where $\alpha^6 - 1$ annihilates B and $\alpha^2 + 1$ annihilates C. If the image of Γ in Aut B or in Aut C still contains elements of order 12, then this process is continued until all the elements of order 12 are eliminated from the $\phi(\Gamma)$.$\square$

We now turn our attention to the converse problem and wish to show, first of all, that all the six groups listed in (116.1) are in fact automorphism groups of suitable torsion-free groups. The case of $Z(2)$ being trivial, we consider only the other five groups.

Example 1. $Z(4)$ as automorphism group. Let $p_1, \cdots, p_i, \cdots$ be an infinite set of primes with $p_i \equiv 1 \bmod 4$. Then -1 is a quadratic residue $\bmod\, p_i$, and let the integers k_i satisfy $k_i^2 \equiv -1 \bmod p_i$. Define

$$A = \langle a, b, p_i^{-1}(a + k_i b) \text{ for all } i \rangle.$$

Then $a \mapsto -b$, $b \mapsto a$ induce an automorphism β of order 4 in A. We show that there are no automorphisms of A other than the powers of β. For, every $\alpha \in \mathrm{Aut}\, A$ must act as

$$(1) \qquad \alpha a = ra + sb, \qquad \alpha b = ta + ub$$

with integers r, s, t, u satisfying $ru - st = \pm 1$. Now $\alpha(a + k_i b) = (r + k_i t)a + (s + k_i u)b$ ought to be divisible by p_i, hence it is a linear combination of $a + k_i b$, $p_i a$, and $p_i b$. In view of the linear independence of a, b, this leads to a relation $s + k_i u \equiv k_i(r + k_i t) \bmod p_i$, whence $s + t \equiv k_i(r - u) \bmod p_i$, and on squaring, $(s+t)^2 \equiv -(r-u)^2 \bmod p_i$. If p_i is large enough, congruence can be replaced by equality, whence $r = u$ and $t = -s$. This together with $ru - st = 1$ leaves only four possibilities: $r = \pm 1, s = 0$; and $r = 0, s = \pm 1$. Hence $\operatorname{Aut} A \cong Z(4)$.

Example 2. $Z(6)$ as automorphism group. Let $q_1, \cdots, q_i \cdots$ be an infinite set of primes with $q_i \equiv 1 \bmod 6$. Then there are integers l_i such that $l_i^2 + l_i \equiv -1 \bmod q_i$. Consider the group

$$A = \langle a, b, q_i^{-1}(a + l_i b) \text{ for all } i\rangle.$$

The correspondence $a \mapsto b$, $b \mapsto -a + b$ induces an automorphism of order 6 of A. If α is any automorphism acting as in (1), with integers r, s, t, u, then from $q_i | \alpha(a + l_i b) = (r + l_i t)a + (s + l_i u)b$ we obtain $l_i(r + l_i t) \equiv s + l_i u \bmod q_i$. Writing $1 \equiv -l_i - l_i^2$ and cancelling by l_i, $r + l_i t \equiv -(1 + l_i)s + u$, that is to say, $r + s - u \equiv -l_i(s + t) \bmod q_i$. Eliminating l_i, it follows that $(r + s - u)^2 - (r + s - u)(s + t) + (s + t)^2 \equiv 0 \bmod q_i$, and here again, congruence mod a large q_i can be replaced by equality. Observing that $x^2 - xy + y^2 = 0$ has only one real solution, namely, $x = y = 0$, we conclude that $t = -s$ and $u = r + s$. Because of $ru - st = \pm 1$ we find $r^2 + rs + s^2 = \pm 1$ whose solutions are $r = \pm 1$, $s = 0$; $r = 1, s = -1$ and those obtained by changing the roles of r and s. Consequently, A has at most six automorphisms.

Example 3. Q_8 as automorphism group. Bearing Example 1 in mind, we select two disjoint infinite sets $\{p_i\}$ and $\{q_i\}$ of primes such that there are integers k_i, l_i satisfying $k_i^2 \equiv -1 \bmod p_i$ and $l_i^2 \equiv -1 \bmod q_i$. Our group will now be

$$A = \langle a, b, c, d, p_i^{-1}(a + k_i b), p_i^{-1}(d + k_i c), \\ q_i^{-1}(a + l_i c), q_i^{-1}(b + l_i d) \text{ for all } i\rangle.$$

If $\alpha \in \operatorname{Aut} A$, then from the divisibility relations $p_i | \alpha(a + k_i b)$, $q_i | \alpha(a + l_i c)$, and $p_i | \alpha(d + k_i c)$ it follows, in the same way as in Example 1, that α acts as follows:

$$\alpha a = ra + sb + tc + ud, \qquad \alpha b = -sa + rb + uc - td,$$
$$\alpha c = -ta - ub + rc + sd, \qquad \alpha d = -ua + tb - sc + rd,$$

where r, s, t, u are integers such that the arising determinant—which is easily seen to be equal to $(r^2 + s^2 + t^2 + u^2)^2$—is a unit in Z and hence 1. The equation $r^2 + s^2 + t^2 + u^2 = 1$ has exactly eight solutions, and the mappings

$$\alpha\colon\ a \mapsto b, \quad b \mapsto -a, \quad c \mapsto d, \quad d \mapsto -c,$$
$$\beta\colon\ a \mapsto c, \quad b \mapsto -d, \quad c \mapsto -a, \quad d \mapsto b,$$

corresponding to $s = 1, r = t = u = 0$, and $t = 1, r = s = u = 0$, respectively, are easily seen to induce automorphisms of A such that $\operatorname{Aut} A \cong Q_8$.

Example 4. DC_{12} *as automorphism group.* We select two disjoint, infinite sets of primes $\{p_i\}$ and $\{q_i\}$ such that $p_i \equiv 1 \bmod 4$ and $q_i \equiv 1 \bmod 6$. Choose k_i, l_i so as to satisfy $k_i^2 \equiv -1 \bmod p_i$ and $l_i^2 + l_i \equiv -1 \bmod q_i$. This time we define our group as

$$A = \langle a, b, c, d, p_i^{-1}(a + k_i b), p_i^{-1}(c + k_i d),$$
$$q_i^{-1}(a + l_i c), q_i^{-1}(d + l_i b) \text{ for all } i\rangle.$$

The correspondences

$$\begin{array}{lllll} \alpha\colon & a \mapsto c, & b \mapsto d, & c \mapsto -a + c, & d \mapsto -b + d, \\ \beta\colon & a \mapsto d, & b \mapsto -c, & c \mapsto b, & d \mapsto -a \end{array}$$

are easily seen to induce automorphisms of A such that $\alpha^3 = \beta^2 = (\alpha\beta)^2 = -1$. The same technique as above leads to the following equations:

$$\alpha a = ra + sb + tc + ud, \qquad \alpha b = -sa + rb - uc + td,$$
$$\alpha c = -ta + (u + s)b + (r + t)c - sd, \quad \alpha d = -(u + s)a - tb + sc + (r + t)d.$$

The determinant of the matrix of α will be $(r^2 + rt + t^2 + s^2 + su + u^2)^2 = 1$ [a unit in Z], whose twelve solutions are $r = \pm 1, s = t = u = 0$; $r = -t = 1$, $s = u = 0$, and those obtained from these by obvious permutations.

Example 5. BT_{24} *as automorphism group.* Let p_i, q_i, k_i, and l_i have the same meaning as in Example 4. Define

$$A = \langle a, b, c, d, p_i^{-1}(a + k_i b), p_i^{-1}(a - c + k_i d),$$
$$q_i^{-1}(a + l_i d), q_i^{-1}(c + l_i b) \text{ for all } i\rangle.$$

It is straightforward to check that the maps

$$\begin{array}{lllll} \alpha\colon & a \mapsto d, & b \mapsto -a + c, & c \mapsto -a - b + c + d, & d \mapsto -a + d, \\ \beta\colon & a \mapsto -c, & b \mapsto b - d, & c \mapsto c - d, & d \mapsto b \end{array}$$

induce automorphisms of A such that $\alpha^3 = \beta^3 = (\alpha\beta)^2 = -1$. Thus BT_{24} is a subgroup of Aut A. We leave it as an exercise to the reader to show that $|\text{Aut } A| = 24$.

These examples enable us to prove:

Theorem 116.2 (Hallett and Hirsch [2]). *If* Γ *is a finite direct product of groups each of which is isomorphic to one of the groups listed in* (116.1), *then there exists a torsion-free group* A *such that* Aut $A \cong \Gamma$.

We start off with the observation that our Examples 1–5 can easily be modified by replacing A by $A \otimes R$, where R is a rational group whose type is of the form $(k_1, \cdots, k_n, \cdots)$, where each k_n is finite and $k_n = 0$ whenever the nth prime occurs among the primes used in the example. Only the case when infinitely many $k_n \neq 0$ is of interest.

Now let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_s$, where each Γ_j is one of the types in (116.1). For each Γ_j, select a torsion-free A_j such that $\mathrm{Aut}\, A_j \cong \Gamma_j$. It is clear that the primes $\not\equiv 1 \bmod 4$ or 6 are still infinite in number, so we can select a rigid system $R_1, \cdots, R_s$ of rational groups of the types indicated above such that $k_n > 0$ only if the nth prime $\not\equiv 1 \bmod 4$ or 6. Then the groups $B_j = A_j \otimes R_j$ $(j = 1, \cdots, s)$ are readily seen to form again a rigid system. Hence

$$\mathrm{Aut}(B_1 \oplus \cdots \oplus B_s) = \mathrm{Aut}\, B_1 \times \cdots \times \mathrm{Aut}\, B_s \cong \Gamma_1 \times \cdots \times \Gamma_s,$$

as desired.□

It is a more delicate question to single out those subgroups of direct products of the groups in (116.1) which actually occur as automorphism groups.

EXERCISES

1. (a) Let A be a rank 1 torsion-free group whose type is $(k_1, \cdots, k_n, \cdots)$ with all k_n finite. Show that $\mathrm{Aut}\, A \cong Z(2)$.
   (b) There is a rigid system of cardinality of the continuum consisting of rank 1 groups with automorphism groups of order 2.
2. (a) Prove that for every cardinal $\mathfrak{m}$, less than the first strongly inaccessible aleph, there exist groups of rank $\mathfrak{m}$ with only two automorphisms.
   (b) Extend this to a rigid system.
3. Every elementary 2-group, whose cardinality is $2^{\mathfrak{m}}$, with $\mathfrak{m}$ less than the first strongly inaccessible cardinal, is the automorphism group of a torsion-free group.
4. Prove that $Z(p)$ with an odd prime p is not isomorphic to the automorphism group of any abelian group.
5. (de Vries and de Miranda [1]) Show that the automorphism group of
$$A = \langle p_1^{-\infty} a_1, p_1^{-\infty} b_1, p_2^{-\infty} a_2, p_2^{-\infty} b_2, q_1^{-\infty}(a_1 + a_2),$$
$$q_1^{-\infty}(b_1 + b_2), q_2^{-\infty}(a_1 + b_2), q_2^{-\infty}(b_1 - a_2 + b_2)\rangle,$$
   where p_1, p_2, q_1, q_2 are different primes, is isomorphic to $Z(6)$. [*Hint*: $a_1 \mapsto b_1, b_1 \mapsto -a_1 + b_1, a_2 \mapsto b_2, b_2 \mapsto -a_2 + b_2$ is a generator.]
6. Prove that a torsion-free group A whose automorphism group is isomorphic to one of six groups in (116.1) is necessarily indecomposable. [*Hint*: center of $\mathrm{Aut}\, A$.]
7. By making use of the techniques of Examples 1–4, show that $|\mathrm{Aut}\, A| = 24$ in Example 5.
8. (Hallett and Hirsch [2]) Let A be a torsion-free group with $\mathrm{Aut}\, A \cong Q_8 = \langle \alpha, \beta \mid \alpha^2 = \beta^2 = (\alpha\beta)^2 \rangle$.

(a) For every nonzero $a \in A$, the elements a, αa, βa, and $\alpha\beta a$ are independent.

(b) If $r(A)$ is finite, then it is a multiple of 4.

9. (Hallett and Hirsch [2]) Let A be torsion-free of finite rank. If Aut $A \cong DC_{12}$, then $r(A)$ is even, and if Aut $A \cong BT_{24}$, then $r(A)$ is divisible by 4.
10. Show that the groups in Examples 1–5 can be chosen to have infinite ranks.
11. (Hallett and Hirsch [2]) A finite abelian group Γ is isomorphic to Aut A for some torsion-free A if and only if:
    (i) $|\Gamma|$ is even;
    (ii) $\alpha^{12} = 1$ for all $\alpha \in \Gamma$;
    (iii) not every $\alpha \in \Gamma$ of order 2 is contained in a cyclic group of order 12.

NOTES

Automorphism groups of finite groups have been the subjects of numerous investigations, but no satisfactory description has been obtained so far. The abelian case is no exception: our present knowledge of automorphism groups of abelian groups is very limited. Several examples of groups are known which can never be automorphism groups, and it seems hard to visualize what sort of condition will make a group into an automorphism group. [The situation is rather mysterious, especially in comparison with some other algebraic structures, like commutative rings where every group can be an automorphism group, as was pointed out by J. Sichler.]

In view of the abundance of automorphisms in p-groups, more information is expected in this case. For infinite p-groups A, the normal structure of Aut A was first studied by Baer [3]; he investigated the correspondence between characteristic subgroups of A and normal subgroups of Aut A. That the results gave special status to the prime number 2 was no surprise. In the last decade or so, the study of the normal structure of Aut A has advanced considerably [see Freedman [1], Fuchs [20], Leptin [4], Mader [1, 2], Hill [23]]. In spite of all efforts, very little is known in effect about Aut A, we do not even know in which way Aut A reveals the length of A, not to mention the Ulm–Kaplansky invariants. The intuitive belief that Aut A faithfully reflects the properties of the p-group A has actually been verified by Leptin [2]: p-groups for $p \geq 5$ are determined by their automorphism groups, and consequently, all properties of A must be recaptured from Aut A. Apparently, the essence of the problem lies in that the transition between A and Aut A has been so far practically a one-way street.

For more results on the automorphism groups of p-groups, we refer to Hausen [2, 3] and Faltings [1].

For torsion-free groups A, automorphism groups behave differently: on the one hand, Aut A no longer determines A, and on the other hand, there seems to be a stronger restriction on Aut A, as shown by the results of Hallett and Hirsch [1, 2]. The problem of countable Aut A for countable reduced torsion-free A is, in view of (110.1), equivalent to the question of groups of units in countable reduced torsion-free rings. Even the rank 2 case is very complicated; see Król [2]. For automorphisms of indecomposable groups, see de Groot and de Vries [1].

Automorphism groups of infinitely generated mixed groups have not been investigated so far.

Problem 88. Characterize Aut A for [separable] p-groups A.

Problem 89. Find a simple proof of Leptin's theorem that p-groups ($p \geqq 5$) are isomorphic if their automorphism groups are, and settle the cases $p = 2$ and 3. Are all isomorphisms between the automorphism groups induced by isomorphisms of the groups?

Problem 90. Study the automorphism groups of torsion-free groups of finite rank.

Problem 91. Investigate the equivalence classes of subgroups of a group under the automorphism group of the group.

For basic subgroups of countable p-groups, see Hill [20]. Cf. also Hill [28], and Tarwater and Walker [1].

Problem 92. How much information can be obtained for a p-group A from the normal structure of Aut A? Does it determine the Ulm–Kaplansky invariants of A?

XVII

ADDITIVE GROUPS OF RINGS

We wish to study the relation between the structure of a ring and the structure of its additive group. The principal problems in our study can be easily formulated:

1. given a group A, find all rings R whose additive group is isomorphic to A;
2. given a class of rings, find criteria for a group A such that there is a ring R in the given class whose additive group is isomorphic to A.

To the first problem, no satisfactory answer has been given. The [not necessarily associative] rings on a given group A are associated with bilinear functions $\mu: A \times A \to A$ which form an additive group, Mult A. The study of this group gives a certain amount of information about the rings on A, but one of the basic questions, namely when two rings on A are isomorphic, is left unanswered, except for a very few special cases.

It is a remarkable feature of torsion groups that the ring structures on them are completely determined by multiplications of elements in a p-basis. In view of this, the torsion case is more tractable, and for instance, one can get a full description of groups admitting but a finite number of nonisomorphic ring structures.

As far as the second problem is concerned, we shall deal mainly with ring classes where the additive groups are rather restrictive: the Artinian and (generalized) regular rings. A full characterization of those groups can be obtained which are additive groups of Artinian rings.

117. SUBGROUPS THAT ARE ALWAYS IDEALS

In this section we present an introductory account of the relationship that exists between a ring and its additive group. We examine some elementary implications of the ring structure and describe the subgroups of a group A which are necessarily ideals in every ring with A as additive group.

Let us start with a remark on terminology. By a *ring* will be meant a not necessarily associative or commutative ring [but multiplication is always assumed to be distributive on both sides over addition]. We attribute to a ring properties of its underlying additive group in cases where no confusion can arise. Consequently, the terms like p-ring, torsion or torsion-free ring, divisible or reduced ring, pure ideal, etc., will make perfectly good sense without any further comment. If desirable, distinction will be made between a ring R and its additive group R^+; caution is necessary especially in the context of direct decompositions. A ring R whose additive group is [isomorphic to] A is called a *ring on* A.

We shall have many occasions to use the simple facts collected in the next lemma.

Lemma 117.1. *For all* $a, c \in \mathsf{R}$ *we have*:

(a) $m \mid a$ *and* $n \mid c$ *imply* $mn \mid ac$;
(b) *if* $ma = 0$ *and* $nc = 0$, *then* $(m, n)ac = 0$;
(c) *if* $m \mid a$ *and* $mc = 0$, *then* $ac = 0$.
The proof is elementary; the same as in (59.2).□

The lemma has several noteworthy implications; the proofs are straightforward applications of (117.1).

(A) In every ring R, the following are ideals: $n\mathsf{R}$ and $\mathsf{R}[n]$ for every n, the torsion part $T(\mathsf{R})$ and its p-components, the socle, the maximal divisible subgroup, the Ulm subgroups R^σ, and $p^\sigma\mathsf{R}$ for every ordinal σ. More generally, for every left [right] ideal L of R, the subgroups $n\mathsf{L}$, $\mathsf{L}[n]$ etc. are likewise left [right] ideals of R.

(B) If R is a torsion ring, then for each prime p

$$h_p(ac) \geqq h_p(a) + h_p(c) \qquad \text{for all} \quad a, c \in \mathsf{R}.$$

Furthermore, $ac = 0$ if $h_p(a) \geqq k$ and $p^k c = 0$. Thus R^1 is an annihilator of R, and the p-component R_p annihilates the q-component R_q for different primes p, q. Consequently, for a torsion ring R,

$$\mathsf{R} = \bigoplus_p \mathsf{R}_p$$

holds in the ring-theoretical sense, too.

(C) In a torsion-free ring R,

$$\chi(ac) \geqq \chi(a)\chi(c) \qquad \text{for all} \quad a, c \in \mathsf{R}.$$

Therefore, for every left ideal L of R and for every type $\mathbf{t}$, both $\mathsf{L}(\mathbf{t})$ and $\mathsf{L}^*(\mathbf{t})$ are left ideals.

Given any ring R, the left and right multiplications

$$x \mapsto ax \qquad \text{and} \qquad x \mapsto xa \qquad (x \in \mathsf{R})$$

for any fixed $a \in \mathsf{R}$ are endomorphisms of R^+. A direct consequence is that *fully invariant subgroups of* R^+ *are necessarily ideals in* R, no matter how multiplication is defined in R. This suggests the problem of describing those subgroups of a group A which are ideals in every ring R such that $\mathsf{R}^+ = A$.

In solving this problem, we require an ideal of the endomorphism ring $\mathsf{E}(A)$ of A. Define

$$\mathsf{I}(A) = \langle \varphi A \mid \varphi \in \mathrm{Hom}(A, \mathsf{E}(A)) \rangle,$$

i.e., $\mathsf{I}(A)$ is the subgroup of $\mathsf{E}(A)$ generated by all homomorphic images of A in $\mathsf{E}(A)$. To show that $\mathsf{I}(A)$ is an ideal, notice that, for every $\eta \in \mathsf{E}(A)$, the mappings $a \mapsto \eta(\varphi a)$ and $a \mapsto (\varphi a)\eta$ are homomorphisms $A \to \mathsf{E}(A)$, as is clear from $\eta(\varphi(a + b)) = \eta(\varphi a + \varphi b) = \eta(\varphi a) + \eta(\varphi b)$ and the analogous equations. Thus $\eta(\varphi a)$ and $(\varphi a)\eta$ are among the generators of $\mathsf{I}(A)$ for all $a \in A$, i.e., $\mathsf{I}(A)$ is an ideal of $\mathsf{E}(A)$.

Theorem 117.2 (Fried [1]). *A subgroup C of group A is an ideal in every ring on A if and only if C is $\mathsf{I}(A)$-admissible, i.e., $\mathsf{I}(A)C \leqq C$.*

Let R be a ring on A. We associate with $a \in A$ the left multiplication $\lambda_a\colon x \mapsto ax$. This correspondence $\varphi\colon a \mapsto \lambda_a$ is a homomorphism of A into $\mathsf{E}(A)$, and thus into $\mathsf{I}(A)$. The subgroup C is a left ideal in R exactly if every λ_a carries C into itself; the same argument applies to right multiplication. The conclusion is that if $\mathsf{I}(A)C \leqq C$, then C is an ideal in every ring R on A.

Conversely, every $\varphi \in \mathrm{Hom}(A, \mathsf{E}(A))$ gives rise to a ring multiplication by defining the product of $a, c \in A$ as $ac = (\varphi a)c$. Therefore, if C is an ideal in every ring on A, then $(\varphi a)c \in C$ for $c \in C$, i.e., $\mathsf{I}(A)C \leqq C$.□

For reduced torsion groups A, it is easy to characterize $\mathsf{I}(A)$:

Proposition 117.3. *If A is a reduced torsion group, then $\mathsf{I}(A)$ is the torsion part of* $\mathsf{E}(A)$.

A reduced p-group A has cyclic summands of orders $\geqq p^k$ for every large k, unless A is bounded when it has a cyclic summand of maximal order. In both cases the images of its cyclic summands in $\mathsf{E}(A)$ will generate the torsion part of $\mathsf{E}(A)$, so this must be $\mathsf{I}(A)$.□

EXERCISES

1. In a torsion-free ring R with identity 1, $\chi(1) \leqq \chi(a)$ holds for every $a \in \mathsf{R}$.
2. Prove the inequality in (C) for height-matrices in mixed groups.
3. In any ring on A, two disjoint fully invariant subgroups of A annihilate each other.
4. (O. Steinfeld) If the two-sided annihilator of every nonzero element of R vanishes, then either $p\mathsf{R} = 0$ for some prime p or R is torsion-free such that the types of its elements $\neq 0$ form a set directed upward.

5. Let M be an R-module [we keep supposing that module means a unital left module over an associative ring with 1].
   (a) Verify the analog of (117.1) for $a \in \mathsf{R}$, $c \in M$.
   (b) Prove that nM, $M[n]$, etc. [as in (A)], are submodules of M.
   (c) R^1 annihilates the torsion part of the group M.
6. If $C \leqq A$ is a left ideal in every ring on A, then it is necessarily two-sided.
7. (Fried [1]) For $C \leqq A$, define

$$C^\circ = \langle a \in A \mid \psi a \in C \text{ for all } \psi \in \mathsf{I}(A)\rangle$$

   and

$$C_\circ = \langle \psi C \mid \text{for all } \psi \in \mathsf{I}(A)\rangle.$$

   Prove that:
   (a) C° and $C_\circ$ are fully invariant in A;
   (b) $(C^\circ)_\circ \leqq C \leqq (C_\circ)^\circ$;
   (c) $B \leqq C$ implies $B^\circ \leqq C^\circ$ and $B_\circ \leqq C_\circ$.
8. (Fried [1]) 0° is the intersection of the annihilators of all rings on A.
9. (Fried [1]) In the notation of Ex. 7, verify that the following conditions on $C \leqq A$ are equivalent:
   (i) C is an ideal in every ring on A;
   (ii) $C_\circ \leqq C$;
   (iii) $G \leqq C \leqq G^\circ$ for some fully invariant subgroup G of A;
   (iv) $H_\circ \leqq C \leqq H$ for some fully invariant $H \leqq A$;
   (v) $C \leqq C^\circ$.
10. (Fried [1]) A group A has the property that only its fully invariant subgroups are ideals in every ring on A, if and only if for every $a \in A$, $\mathsf{E}(A)$ as a group is generated by 1, $\mathsf{I}(A)$, and $\{\eta \in \mathsf{E}(A) \mid \eta a = 0\}$.
11. Given a ring R, a *universal* R-*module* on a group A is an R-module U together with a group-homomorphism $\phi: A \to U$ such that for any group-homomorphism $\alpha: A \to M$ between A and an R-module M, there exists a unique R-homomorphism $\psi: U \to M$ satisfying $\psi\phi = \alpha$. Show that there exists a universal R-module on any A and it is unique up to isomorphism. [*Hint*: $U = \mathsf{R} \otimes_{\mathbf{Z}} A$ and $\phi: a \mapsto 1 \otimes a$.]
12. Given a ring R and a group A, A admits a unital R-module structure exactly if 1_A factors through ϕ of Ex. 11.

118. MULTIPLICATIONS ON A GROUP

Every group A can be provided with a ring structure in a trivial way by defining all products to be 0; such a ring is called a *zero-ring*. However, this is not the only ring on A, in general, and so we are looking for a general method of finding all ring structures on A.

Essential to our purpose is the following concept. Call a function $\mu: A \times A \to A$ a *multiplication on A* if it satisfies

$$\mu(a, b + c) = \mu(a, b) + \mu(a, c),$$

$$\mu(b + c, a) = \mu(b, a) + \mu(c, a)$$

for all $a, b, c \in A$. Evidently, every ring R on A gives rise to a multiplication μ, namely, $\mu(a, b) = ab$, and this correspondence between ring structures and multiplications on A is bijective. Thus, we may think of R as a pair (A, μ).

It is a trivial observation that $\mu(na, b) = n\mu(a, b) = \mu(a, nb)$ for all $a, b \in A$ and $n \in \mathbf{Z}$.

If both μ and ν are multiplications on A, then their *sum* $\mu + \nu$ is defined by the rule

$$(\mu + \nu)(a, b) = \mu(a, b) + \nu(a, b) \qquad \text{for all} \quad a, b \in A,$$

which is again a multiplication on A. Under this rule of composition, the multiplications on A form an abelian group, the *group of multiplications on A*, Mult A. The 0 of Mult A is the multiplication corresponding to the zero-ring on A.

It is this group Mult A that is mainly responsible for the admission of nonassociative rings in our discussions. As a matter of fact, if we wish to ignore totally the nonassociative rings, then we can not even introduce this group Mult A, since the associative multiplications form a group only in a few exceptional cases [see Hardy [1]].

In order to get more information about the rings on A, we investigate the group Mult A.

Theorem 118.1 (Fuchs [16]). *The following isomorphisms hold*:

(1) $$\text{Mult } A \cong \text{Hom}(A \otimes A, A),$$

(2) $$\text{Mult } A \cong \text{Hom}(A, \text{End } A).$$

As μ is a bilinear function $A \times A \to A$, (59.1) ensures the existence of a unique homomorphism $\varphi: A \otimes A \to A$ such that $\varphi(a \otimes b) = \mu(a, b)$. In this way, we get a map $\mu \mapsto \varphi$ which is evidently a homomorphism of Mult A into $\text{Hom}(A \otimes A, A)$. It is readily inferred that it is, moreover, an isomorphism. The second isomorphism follows from the first by a simple reference to **59**(J).$\square$

We shall find it useful to fix the isomorphisms in (1) and (2) in the natural way:

$$\mu \mapsto \varphi \qquad \text{if} \quad \varphi(a \otimes b) = \mu(a, b),$$

and

$$\mu \mapsto \psi \qquad \text{if} \qquad (\psi a)b = \mu(a, b).$$

The last result adapts itself to many uses. We list the following consequences.

(A) If A is a cyclic group, then Mult $A \cong A$. This follows from the natural isomorphisms $A \otimes A \cong A \cong \operatorname{End} A$, valid for cyclic A.

(B) $mA = 0$ implies m Mult $A = 0$.

(C) $pA = A$ implies that Mult A contains no elements of order p. This follows from **59**(A) and **43**(E). In particular, Mult A is torsion-free whenever A is divisible.

(D) If A has no elements of order p, then the same holds for Mult A. This is a trivial consequence of **43**(B). Thus Mult A is torsion-free for torsion-free groups A.

(E) If A is torsion-free divisible, then so is Mult A. This results from **43**(D).

(F) *If $A = \bigoplus_{i \in I} A_i$ with all A_i fully invariant in A, then*

$$\text{Mult } A = \prod_{i \in I} \text{Mult } A_i.$$

By **59**(I) and (43.1), Mult A is the product of all $\operatorname{Hom}(A_i \otimes A_j, A)$ for $i, j \in I$. For $\varphi \in \operatorname{Hom}(A_i \otimes A_j, A)$ and fixed $a_j \in A_j$, $a_i \mapsto \varphi(a_i \otimes a_j)$ is a homomorphism of A_i into A. The full invariance of A_i in A implies that the image is in A_i, and, for reasons of symmetry, in A_j; thus $\operatorname{Hom}(A_i \otimes A_j, A)$ vanishes for $i \neq j$, and equals $\operatorname{Hom}(A_i \otimes A_i, A_i)$ if $i = j$.

(G) If A is torsion and A_p is its p-component, then Mult $A = \prod_p$ Mult A_p.

(H) Mult $J_p = J_p$. This is a consequence of Example 5 in **43** and (2).

Naturally, two rings on A are regarded as essentially the same whenever they are isomorphic. It is easy to see that the multiplications μ and ν define isomorphic rings on A exactly if there is an automorphism α of A which preserves products, i.e.,

$$\mu(a, b) = \alpha^{-1}\nu(\alpha a, \alpha b) \qquad \text{for all} \quad a, b \in A.$$

It is not an easy problem to decide if two given multiplications define isomorphic rings.

The multiplications ν which are commutative in the obvious sense that $\nu(a, b) = \nu(b, a)$ for all $a, b \in A$, form a subgroup $\text{Mult}_c A$ of Mult A. Under the isomorphism (1), a homomorphism $\varphi \in \operatorname{Hom}(A \otimes A, A)$ corresponds to a commutative multiplication exactly if $\varphi(a \otimes b) = \varphi(b \otimes a)$ for all $a, b \in A$. This amounts to saying that φ annihilates the "commutator subgroup" $C = \langle a \otimes b - b \otimes a \mid \text{for all } a, b \in A\rangle$ of $A \otimes A$. It is readily seen that

$$\text{Mult}_c A \cong \operatorname{Hom}((A \otimes A)/C, A).$$

It is more difficult to identify within Mult A the associative multiplications. Using a different approach, it is clear that a multiplication μ: $A \times A \to A$ is associative if and only if the corresponding homomorphism φ: $A \otimes A \to A$ induces a commutative diagram

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "1 \otimes \varphi"] \arrow[d, "\varphi \otimes 1"'] & A \otimes A \arrow[d, "\varphi"] \\ A \otimes A \arrow[r, "\varphi"] & A \end{tikzcd}$$

It is not much known how the associative multiplications are located in Mult A. The isomorphism (2) can give us some information. Since (2) associates with $\mu \in$ Mult A the homomorphism $\psi: a \mapsto \lambda_a \in \mathrm{Hom}(A, \mathrm{End}\, A)$ such that $\mu(a, b) = \lambda_a b$ for all $a, b \in A$, the associativity $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c)$ is equivalent to $\lambda_a \lambda_b c = \lambda_{\mu(a, b)} c$, or simply $\lambda_a \lambda_b = \lambda_{\mu(a, b)}$. Therefore, *$\mu$ is associative if and only if the corresponding $\psi: A \to \mathrm{End}\, A$ is a ring-homomorphism of (A, μ) into* $\mathrm{E}(A)$.

Let us illustrate the use of Mult. Notice that $\mu \in$ Mult A is completely determined by the values of $\mu(a, b)$ for a, b in a generating system of A.

Example 1. Let $A = \langle g \rangle$ be cyclic. We know from (A) that Mult $A \cong A$, and it is a triviality to check that Mult $A = \langle \lambda \rangle$, where $\lambda(g, g) = g$. Hence all multiplications on A are given by $\mu = k\lambda$, where $k = 0, 1, \cdots, m-1$ if $o(g) = m$ is finite, and $k = 0, \pm 1, \pm 2, \cdots$ otherwise. All these multiplications are associative and commutative. [For their isomorphisms, see Exs. 5 and 6.]

Example 2. Let $A = \bigoplus_{i \in I} \langle g_i \rangle$ be a direct sum of indecomposable cyclic groups. Then $A \otimes A = \bigoplus_{i, j} (\langle g_i \rangle \otimes \langle g_j \rangle)$, where the group in parentheses is a cyclic group of order $(o(g_i), o(g_j))$, generated by $g_i \otimes g_j$ [set $(m, \infty) = m$ and $(\infty, \infty) = \infty$]. A multiplication μ on A is completely determined by the $\varphi(g_i \otimes g_j) \in A$ which can be chosen arbitrarily subject to the sole condition that the order of $\varphi(g_i \otimes g_j)$ be a divisor of the order of $g_i \otimes g_j$. [For a more explicit description, with emphasis on the associative case, cf. Beaumont [2] or Toskey [1].]

Example 3. Let A be a torsion-free group $\ncong Z$ such that $\mathrm{E}(A) \cong \mathrm{Z}$. It is easy to check that $A \otimes A$ cannot have an infinite cyclic group as a summand whence (2) implies Mult $A = 0$.

EXERCISES

1. Give a direct proof for (2).
2. Show that Mult A is always an $\mathrm{E}(A)$-module in a natural way.
3. Give an example to show that the associative multiplications need not form a subgroup of Mult A. [*Hint*: $Z(n) \oplus Z(n)$.]
4. If $\theta_1, \theta_2, \theta_3$ are endomorphisms of A, then $(\theta_1, \theta_2, \theta_3): \mu(a, b) \mapsto \theta_1 \mu(\theta_2 a, \theta_3 b)$ is an endomorphism of Mult A. Prove that $(\theta_1, 1, 1)$, $(1, \theta_2, 1)$, and $(1, 1, \theta_3)$ pairwise permute.
5. (a) Two multiplications μ, ν on Z define isomorphic rings exactly if $\mu = \pm \nu$.
   (b) There are infinitely many nonisomorphic rings on Z, and these are isomorphic to $n\mathrm{Z}$ $(n > 0)$ or to the zero-ring on Z.
6. (a) Two multiplications μ, ν define isomorphic rings on $Z(m)$ if and only if $\mu = k\nu$ for some $(k, m) = 1$.
   (b) Any ring on $Z(p^n)$ is isomorphic to precisely one of the following rings: $p^k\mathrm{Z}/p^{n+k}\mathrm{Z}$ $(k = 0, 1, \cdots, n)$.
7. Any ring on J_p is isomorphic to one of the rings $p^k\mathrm{Q}_p^*$ $(k = 0, 1, 2, \cdots)$ or to the zero-ring on J_p.

8. Let $\alpha: A \to B$. Then α respects the ring structures corresponding to $\varphi: A \otimes A \to A$ and to $\psi: B \otimes B \to B$ if and only if the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\varphi} & A \\ {\scriptstyle \alpha\otimes\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ B \otimes B & \xrightarrow{\psi} & B \end{array}$$

9. (Beaumont [2]) Let $A = \bigoplus \langle a_i \rangle$, where $o(a_i) = n_i$ is an integer or ∞. Show that $\mu(a_i, a_j) = \sum_k t_{ijk} a_k$ with integers t_{ijk} defines an associative ring on A if and only if:
   (i) for all fixed i, j, almost all t_{ijk} vanish;
   (ii) $t_{ijk} \equiv 0 \bmod n_k (n_i, n_j, n_k)^{-1}$ for all i, j, k;
   (iii) $\sum_k t_{ijk} t_{klm} = \sum_k t_{ikm} t_{jlk}$ for all i, j, l, m.

119. EXTENSIONS OF PARTIAL MULTIPLICATIONS

Next we take up a question which is a natural generalization of the problem raised in the preceding section. Suppose C is a subgroup of A and we are given a *partial multiplication*, i.e., a bilinear function $\nu: C \times C \to A$. The problem consists in extending ν to a bilinear function $\mu: A \times A \to A$. We are confronted with such a situation if, for instance, some ring structure on a subgroup is to be extended to the entire group.

The first question which comes to mind is whether or not a ring structure on A can be extended to the divisible hull D of A. From (118.1) it is readily seen that on a divisible torsion group there is no multiplication other than the trivial one [see also (120.3)]; hence, the real interest lies in the torsion-free case.

Theorem 119.1. *Every torsion-free ring* R *can be embedded as a subring in a minimal divisible torsion-free ring* D *which is unique up to isomorphism.*

The ring D will be defined on the divisible hull D of R^+. It is readily checked that there is only one way to extend the multiplication ν on R to D; namely, if $x, y \in D$ and if m, n are integers $\neq 0$ such that $mx, ny \in \mathsf{R}$, then the extension must be

$$\mu(x, y) = (mn)^{-1} \nu(mx, ny).$$

This definition is independent of the choice of m, n and makes D into a ring D which is associative [commutative] if and only if R was associative [commutative]. By a routine argument the reader can convince himself that every divisible torsion-free ring that contains a subring isomorphic to R also contains a subring isomorphic to D. $\square$

There are two other most important embeddings of groups, namely, in algebraically compact and cotorsion groups, respectively. Before analyzing when and in how many ways the ring structure can be extended in these cases, we prove an easy lemma. [Recall that C is dense in A means A/C is divisible.]

Lemma 119.2. *Let C be a pure and dense subgroup of the reduced group A. Then a partial multiplication $\nu: C \times C \to A$ can at most in one way be extended to a multiplication $\mu: A \times A \to A$.*

From the pure-exact sequence $0 \to C \to A \to A/C \to 0$, a repeated application of (60.4) yields the exact sequences

$$0 \to C \otimes C \to A \otimes C \to (A/C) \otimes C \to 0$$

and

$$0 \to A \otimes C \to A \otimes A \to A \otimes (A/C) \to 0.$$

By **59**(A), both $(A/C) \otimes C$ and $A \otimes (A/C)$ are divisible, and we are led to the exact sequence

$$0 \to C \otimes C \to A \otimes A \to [(A/C) \otimes C] \oplus [A \otimes A/C] \to 0.$$

In view of (44.4) and the reducedness of A, the exactness of $0 \to \mathrm{Hom}(A \otimes A, A) \to \mathrm{Hom}(C \otimes C, A)$ is evident. That is, any $\nu: C \times C \to A$ can have at most one extension $\mu: A \times A \to A$.□

Rephrasing, the lemma states that any multiplication on a reduced group is fully determined by its restriction to a pure and dense subgroup.

Now we are ready to examine the embeddings in algebraically compact and cotorsion groups [cf. **41** and **58**]. Our main objective is to show that ring structures on groups can always be extended to their pure-injective and cotorsion hulls, and in certain cases these extensions are unique.

Theorem 119.3. *Let G be the pure-injective hull of A. Every partial multiplication $\nu: A \times A \to G$ can be extended to a multiplication $\mu: G \times G \to G$. If G is reduced, then this extension is unique.*

From the pure-exact sequence $0 \to A \to G \to G/A \to 0$ we obtain, in the same way as in (119.2), the pure-exact sequence $0 \to A \otimes A \to G \otimes G$. Because of the pure-injectivity of G, every homomorphism $A \otimes A \to G$ extends to a homomorphism $G \otimes G \to G$. The second assertion follows from (119.2).□

Our theorem shows that the situation is particularly interesting if the pure-injective hull is reduced. Moreover, in this case we can slightly improve on (119.3) and show:

Corollary 119.4. *Suppose R is a ring such that $\mathsf{R}^1 = 0$. Then there is one and only one ring structure on the Z-adic completion $\hat{\mathsf{R}}$ of R which extends that* *of* R, *and this preserves the polynomial identities* [*in particular, associativity, commutativity*] *in* R. *In addition,* $\hat{\mathsf{R}}$ *becomes a* $\hat{\mathsf{Z}}$*-algebra.*

If the first Ulm subgroup R^1 of R vanishes, then R^+ can be regarded as being canonically embedded in $\hat{\mathsf{R}} = \varprojlim_n \mathsf{R}/n\mathsf{R}$ in the way specified in (39.5). Since R is a ring and $n\mathsf{R}$ is an ideal of R, the natural map $\mathsf{R} \to \mathsf{R}/n\mathsf{R}$ furnishes $\mathsf{R}/n\mathsf{R}$ with a ring structure. It is now clear that the maps $\mathsf{R}/nm\mathsf{R} \to \mathsf{R}/n\mathsf{R}$ in the inverse system are ring-homomorphisms, so the inverse limit will again be a ring $\hat{\mathsf{R}}$ containing R as a pure and dense subring. That this is the only ring structure on $\hat{\mathsf{R}}$ extending that of R follows from (119.2), while the preservation of polynomial identities is clear, since they are preserved under homomorphisms and inverse limits. The final statement follows at once from Example 7 in **106**.□

In the cotorsion case, we have the exact analog of (119.3):

Theorem 119.5. *Every partial multiplication* $\nu\colon A \times A \to G$, *where* G *is the cotorsion hull of* A, *can be extended to a multiplication* $\mu\colon G \times G \to G$. *This* μ *is unique whenever* G *is reduced.*

Notice that by (58.1), G/A is torsion-free and divisible. The same is true for $(G/A) \otimes A$ and $G \otimes (G/A)$, and we obtain the exact sequence

$$0 \to A \otimes A \to G \otimes G \to [(G/A) \otimes A] \oplus [G \otimes (G/A)] \to 0.$$

From (58.2), the claimed extensibility follows.□

EXERCISES

1. Find the divisible hull of the ring of integers in a finite algebraic extension of Q.
2. Let R and D be as in (119.1).
   (a) Establish a one-to-one correspondence between the pure left ideals of R and those of D.
   (b) Show that this correspondence preserves primeness.
   (c) D can have an identity even if R does not have any.
   (d) If D has an identity, then every left ideal of D is pure.
3. Let again R and D be as in (119.1). D contains a nonzero nilpotent left ideal exactly if R has one.
4. A torsion-free ring R is an ideal in its divisible hull D if and only if the square R^2 of R is contained in the maximal divisible ideal of R.
5. Apply (119.2) to a reduced torsion group A and its basic subgroup C.
6. If A is a torsion group with $A^1 = 0$, then there is a natural isomorphism Mult $\hat{A} \cong$ Mult A.
7. (Fuchs [23]) If A is a reduced torsion group and $A^\bullet$ is its cotorsion hull, then there is a natural isomorphism Mult $A^\bullet \cong$ Mult A.

120. TORSION RINGS

The theory of torsion rings reduces immediately to p-rings; in fact, **117**(B) implies that the p-components R_p of a torsion ring R are ideals and R is their ring-direct sum: $\mathsf{R} = \bigoplus_p \mathsf{R}_p$.

The starting result in our discussion of torsion rings is the following theorem which reveals an intimate connection between the ring structure and basic subgroups and is a handy tool in the construction of p-rings.

Theorem 120.1 (Fuchs [6]). *A multiplication μ on a p-group A is completely determined by the values $\mu(a_i, a_j)$ with a_i, a_j running over a p-basis of A.*

Moreover, any choice of $\mu(a_i, a_j) \in A$ with a_i, a_j from a p-basis of A—subject to the sole condition that $o(\mu(a_i, a_j)) \leqq \min(o(a_i), o(a_j))$—extends to a multiplication on A.

There are several proofs to choose from; the following is most elementary and direct. If $\mu(a_i, a_j)$ is known for all a_i, a_j in a p-basis of A, then by distributivity, $\mu(b, c)$ is uniquely determined for all b, c in the basic subgroup B generated by the a_i. Given any $g, h \in A$, let $o(h) = p^k$, and write $g = b + p^k x$ with $b \in B$ and $x \in A$ to obtain

$$\mu(g, h) = \mu(b, h) + \mu(p^k x, h) = \mu(b, h) + \mu(x, p^k h) = \mu(b, h).$$

In a similar fashion, $\mu(b, h) = \mu(b, c)$ for some $c \in B$, establishing the first part of the theorem.

For the second part, note that, every $b \in B$ being a unique linear combination of the a_i, the ring postulates for the multiplication $\mu(b, c)$ $(b, c \in B)$ are easily checked. If μ is extended to the whole of A [in the way shown in the preceding paragraph], then just a routine verification of the postulates is needed to complete the proof. □

Before we pursue further developments of p-rings, let us make a few comments on the foregoing theorem and its proof.

The equality $\mu(g, h) = \mu(b, c)$ in the proof shows that *μ is associative* [*commutative*] *if it is associative* [*commutative*] *on a p-basis of A*. We also see that *a subgroup L of a p-ring* R *is a left ideal of* R *if and only if $a_i L \leqq L$ for all a_i in a p-basis of* R^+. In particular, the subring B generated by a basic subgroup of R^+ is always an ideal of R such that $\mathsf{R}^2 \leqq \mathsf{B}$.

Example 1. Let A be a p-group and $B = \bigoplus_{i \in I} \langle a_i \rangle$ a basic subgroup of A. In accordance with (120.1), a multiplication μ on A is uniquely determined if we put

$$\mu(a_i, a_j) = \begin{cases} 0 & \text{if } i \neq j, \\ a_i & \text{if } i = j. \end{cases}$$

The arising ring (A, μ) is associative and commutative; its square is the direct sum of the rings $\mathsf{Z}/(o(a_i))$ on the $\langle a_i \rangle$.

Example 2. Other associative and commutative rings can be defined if we change the second part of our definition in Example 1 and let $\mu(a_i, a_i) = a_i$ or 0 at random.

From (61.1) we infer that there is a natural isomorphism $B \otimes B \cong A \otimes A$, where B is a basic subgroup of the p-group A. This, in conjunction with (118.1), gives at once:

Proposition 120.2 (Fuchs [16]). *For a p-group A and its basic subgroup B, there is an isomorphism* Mult $A \cong \text{Hom}(B \otimes B, A)$.□

Consequently, if A is torsion, Mult A is always an algebraically compact group [see (46.1)]. Its invariants can be easily determined by means of (61.3) and (43.3).

A particular case immediately presents itself: the one in which $B = 0$; then A admits the trivial multiplication only. In general, we call a group A a *nil group* if there is no ring on A other than the zero-ring, i.e., if Mult $A = 0$; and call A a *quasi-nil* group if it admits but a finite number of nonisomorphic rings. In the torsion case, both the nil and the quasi-nil groups can be described without difficulty; moreover, with a little extra effort, the problem of mixed nil groups can also be settled [while for torsion-free nil groups we refer to (121.2) and to Fuchs [9]]:

Theorem 120.3 (Szele [3]). *A torsion group is nil if and only if it is divisible. There exist no mixed nil groups.*

A direct summand of a nil group is again nil, and since $Z(p^k)$ ($k = 1, 2, \cdots$) is not nil, the torsion part of a nil group must be divisible. On the other hand, for a divisible torsion group A we have $A \otimes A = 0$, and A is nil.

If A is mixed and if its torsion part $T \neq 0$ is divisible, then $A \otimes A \neq 0$ has a nontrivial homomorphism into T, and so Mult A can not vanish.□

Theorem 120.4 (Fuchs [9]). *A torsion group A is quasi-nil exactly if $A = B \oplus D$, where B is finite and D is divisible.*

The basic subgroup B of a quasi-nil A must be finite. Otherwise, it is possible to define on A infinitely many rings R such that R^2 are finite and of different orders [see Example 2 above]. Thus A must have the indicated form.

Conversely, it suffices to consider p-groups $A = B \oplus D$ with finite basic subgroup B. Let $p^nB = 0$ and $r = r(B)$. Then $B \otimes B$ satisfies $p^n(B \otimes B) = 0$ and is of rank r^2. We can assert that any $\varphi \in \text{Hom}(B \otimes B, B \oplus D)$ maps $B \otimes B$ into $B \oplus D_\varphi[p^n]$, where D_φ is a divisible subgroup of rank $\leq r^2$ of D. Though D_φ does depend on φ, it can be chosen independently up to automorphisms of D, and therefore, in our search for nonisomorphic rings on A, just one D_φ need to be considered, namely, one of rank $\min(r^2, r(D))$. Now $B \otimes B$ has but finitely many homomorphisms into $B \oplus D_\varphi[p^n]$, and thus A is quasi-nil.□

Our next objective is to point out some interesting connections between torsion rings and their additive structures. Though we do not ignore non-associative rings, we concentrate on the associative case in the rest of this section.

The *annihilator* of a ring R is the set of all $a \in \mathrm{R}$ such that $a\mathrm{R} = \mathrm{R}a = 0$. If R is a torsion ring, then from **117(B)** it follows that the first Ulm subgroup R^1 of R must be contained in the annihilator of R. But no larger subgroup ought to belong to the annihilator:

Theorem 120.5 (Fuchs [6]). *The annihilator of every ring on the torsion group A contains A^1. There exists an associative and commutative ring on A whose annihilator is exactly A^1.*

To construct a ring (A, μ) whose annihilator is A^1, we refer to Example 1. If a is of finite height and of order p^k, then according to (32.4), with the standard decomposition $B = \bigoplus_n B_n$ of the basic subgroup $B = \bigoplus \langle a_i \rangle$ of A, we decompose

$$A = B_1 \oplus \cdots \oplus B_n \oplus A_n, \qquad \text{where} \quad A_n = \langle B_{n+1} \oplus \cdots, p^n A \rangle.$$

Here we choose n such that $n \geqq k + l$, where l is the index of the first nonzero coordinate of a in the given decomposition of A. Thus $a = b_l + \cdots + b_n + g_n$ ($b_l \neq 0$) with obvious notation. If $b_l = n_1 a_{j_1} + \cdots + n_s a_{j_s}$ with nonzero terms, then $\mu(a, a_{j_1}) \neq 0$, for $\mu(b_l, a_{j_1}) = n_1 a_{j_1} \neq 0$, while $\mu(b_{l+1}, a_{j_1}) = \cdots = \mu(b_n, a_{j_1}) = \mu(g_n, a_{j_1}) = 0$, the last equality because of $h(g_n) \geqq n - k \geqq l$ and $o(a_{j_1}) \leqq p^l$. We see that no element of finite height belongs to the annihilator of the ring (A, μ) of Example 1.□

Certain elements of a p-group A turn out to generate nilpotent ideals in every ring on A. As a matter of fact, given $a \in A$ of order p^k, every element of the ideal (pa) generated by pa is divisible by p in A and is of order $\leqq p^{k-1}$. Hence we deduce that the product of any k elements of (pa) vanishes, i.e., the kth power of (pa) vanishes. [This holds even for nonassociative multiplications, for the product of k elements from (pa) is always zero, no matter how parentheses are inserted.] The precise result on nilpotency is as follows.

Theorem 120.6 (Fuchs [6]). *Let A be a torsion group and $F = \bigcap_p pA$, the Frattini subgroup of A. Every element of F generates a nilpotent ideal in every ring on A. There exists an associative and commutative ring on A such that every nilpotent element is contained in F.*

In the proof of the second assertion we may restrict ourselves to p-groups A. We go back to the ring (A, μ) in Example 1. Since every element of pA is nilpotent, it will suffice to prove that the quotient ring over $A/pA \cong B/pB$ does not contain any nilpotent element $\neq 0$. But this ring is exactly the ring-direct sum of the rings $\langle a_i \rangle / \langle pa_i \rangle \cong \mathbf{Z}/(p)$, and in a direct sum of fields no nonzero nilpotent element exists.□

Confining our attention to the associative case, we can ask the question: which is the largest subgroup of a torsion group A that necessarily belongs to the Jacobson radical of a ring on A? The preceding theorem and its proof settle this problem at once: since nilpotent ideals are necessarily contained in the Jacobson radical and since a direct sum of fields is radical-free, we find:

Corollary 120.7. *The Frattini subgroup F of a torsion group A is contained in the Jacobson radical of every associative ring on A, and there is an associative and commutative ring on A whose radical coincides with F.*□

The final result of this section is nothing more than a simple remark. It shows that the existence of an identity element in torsion rings is a surprisingly strong restriction.

Proposition 120.8. *A ring with [left] identity element exists on a torsion group A if and only if A is bounded.*

If n is the order of a left identity e, then $na = nea = 0$ for all $a \in A$, and A is bounded. Conversely, if n is the smallest integer with $nA = 0$, then A has a cyclic summand $\langle e \rangle$ of order n, say $A = \langle e \rangle \oplus C$. A ring with e as identity element can be defined over A by putting the trivial multiplication on C and letting e act as multiplication by 1.□

Exercises

1. Construct a p-ring which is not nilpotent, but each of its elements generates a nilpotent ideal.
2. Determine the invariants of the algebraically compact group Mult A, for a p-group A.
3. For a p-group A, Mult A is a finite group $\neq 0$ if and only if A is finitely cogenerated.
4. Show that there are continuously many nonisomorphic associative rings on a countable elementary p-group. [*Hint*: direct sums of finite fields.]
5. (Fuchs [9]) If at most a countable set of nonisomorphic rings exist on the torsion group A, then A must be quasi-nil. [*Hint*: basic subgroup is bounded and of finite rank.]
6. Every prime ideal and every maximal ideal of a p-ring R contains pR.
7. (a) There exists a semisimple ring [i.e., with zero Jacobson radical] on a torsion group if and only if it is an elementary group.

   (b) A ring with zero annihilator exists on a torsion group A exactly if $A^1 = 0$.

8. Let R be a p-ring. If every element of a basic subgroup B of R^+ is nilpotent, then every element of R is nilpotent. [*Hint*: divide by $o(a)$ mod B and show $a^2 = b^2$ for some $b \in B$.]
9. The Jacobson radical of any ring on a mixed group necessarily contains the Frattini subgroup of its torsion part.

121. TORSION-FREE RINGS

In this section, we take up the problem of rings whose additive groups are torsion-free. Unfortunately, our limited knowledge of the structure of torsion-free groups prevents us from getting a satisfactory theory for torsion-free rings.

The class of torsion-free divisible rings is easily tractable. In order to construct rings on a torsion-free divisible group D, let $\{a_i\}_{i \in I}$ be a maximal independent system in D. The definition

$$\mu(a_i, a_j) = \sum_k t_{ijk} a_k \qquad (i, j, k \in I)$$

with arbitrary rational numbers t_{ijk} [for all fixed i, j, almost all t_{ijk} vanish] gives rise to a multiplication μ on D, and (D, μ) becomes a ring on D. Every ring on D arises in this way. It is easy to formulate the postulates for associativity: for any fixed i, j, l, m, the following equality holds:

$$\sum_k t_{ijk} t_{klm} = \sum_k t_{ikm} t_{jlk}.$$

As we see, torsion-free divisible rings form an immense class, and it is no wonder that not much can be said about them in general.

As soon as the condition of divisibility is dropped, the situation changes. It is no longer possible to pick the t_{ijk} freely: they ought to be chosen so that the products $\mu(x, y)$ for all x, y in the group lie again in the group. This can be done whenever we are in a possession of an adequate knowledge of the given group.

In the very special case when the torsion-free group is of rank 1, it is possible to give a complete survey of all rings on it. Ignoring the zero-rings, let R be a torsion-free ring of rank 1 such that $ab \neq 0$ for some $a, b \in$ R. Here b is a rational multiple of a, thus $a^2 \neq 0$, too. For nonzero elements $c, d \in$ R, there are nonzero rational numbers r, s such that $c = ra$ and $d = sa$, whence $cd = (rs)a^2$. We conclude that a^2 completely determines the multiplication in R, and R is of necessity commutative, associative, and has no divisors of zero.

Let us find out when a non-zero-ring exists on a torsion-free group A of rank 1. Writing $\mathbf{t} = \mathbf{t}(A)$, from (85.3) and (85.4) we infer that for Mult $A \cong$ Hom$(A \otimes A, A) \neq 0$, it is necessary and sufficient that $\mathbf{t}^2 \leqq \mathbf{t}$, or equivalently, $\mathbf{t}$ be idempotent. Thus, A is a nil group if and only if $\mathbf{t}$ is not idempotent.

Next, we want to list all rings on A, under the hypothesis that $\mathbf{t}$ is idempotent. Choose $a \neq 0$ in A such that $\chi(a)$ consists of 0s and ∞s only. If R is a non-zero-ring on A, then $a^2 = ma$ for some rational number $m \neq 0$. Without loss of generality, it can be supposed that m is a positive integer not divisible by any prime q at which $\chi(a)$ is ∞; otherwise a could be replaced by a suitable rational multiple of a with the same characteristic for which $m > 0$ is an integer of the stated kind. If $\{q_j\}_{j \in J}$ is the set of primes at which $\chi(a)$ is infinity [i.e., $q_j A = A$], and if $\mathsf{Z}(q_j^{-1}; j \in J)$ denotes the subring of Q, generated by all the q_j^{-1}, then there is a ring-isomorphism $\mathsf{R} \cong m\mathsf{Z}(q_j^{-1}; j \in J)$. In fact, it is readily seen that the map $ra \mapsto mr$ for $r \in \mathsf{Z}(q_j^{-1}; j \in J)$ is bijective and preserves both addition and multiplication. This completes the proof of the following theorem.

Theorem 121.1 (Rédei and Szele [1], Beaumont and Zuckerman [1]). *A torsion-free ring of rank* 1 *is either a zero-ring or isomorphic to a subring of the rational number field of the form*

$$(1) \qquad m\mathsf{Z}(q_j^{-1}; j \in J) \quad \text{with} \quad (m, q_j) = 1.$$

A torsion-free group of rank 1 *is not a nil group if and only if its type is idempotent.*□

The isomorphy problem for rings on a rank 1 torsion-free group A can easily be settled. In fact, *two rings of the form* (1) *are isomorphic exactly if the set* $\{q_j\}_{j \in J}$ *of primes and the integer* $m > 0$ *are the same for both rings.* To prove this, it suffices to observe that the q_j are determined by A alone [viz., $q_j A = A$], while m can be described as the positive integer which is prime to all q_j and for which $\mathsf{R}^2 = m\mathsf{R}$.

Whereas the nil groups can be characterized explicitly in the torsion case, their description in the torsion-free case is too difficult a problem to tackle. The preceding theorem can be extended to completely decomposable groups [see Ex. 6], but again, our insufficient information of the general structure of torsion-free groups makes a satisfactory classification of nil groups virtually impossible.

All rigid groups of rank $\geqq 2$ are nil groups. Their endomorphism rings are subrings of Q, so our assertion is an immediate corollary to

Proposition 121.2. *Let A be a torsion-free group such that all nonzero endomorphisms of A are monomorphisms and A as an $\mathsf{E}(A)$-module contains at least two independent elements. Then A is a nil group.*

Let $a, b \in A$ be independent over $\mathsf{E}(A)$, i.e., $\alpha a = \beta b$ for $\alpha, \beta \in \mathsf{E}(A)$ implies $\alpha = 0 = \beta$. In any ring on A, the left multiplication λ_a by a and the right multiplication ρ_b by b are endomorphisms of A such that $\rho_b a = \lambda_a b$; by hypothesis $\lambda_a = 0 = \rho_b$. We conclude that $0 = \lambda_a c = ac = \rho_c a$ for all $c \in A$; therefore ρ_c is not monic, and $\rho_c = 0$ for all $c \in A$. In other words, A is a nil group.□

Turning our interest to the ideals, the following two simple results may be mentioned.

Proposition 121.3. *For any maximal [left] ideal* M *of a torsion-free ring* R, *either* R/M *is torsion-free divisible or* M *contains* pR *for some prime* p.

Let M be a maximal [left] ideal of R such that $p\mathrm{R} \leqq \mathrm{M}$ for no prime p. Then $p\mathrm{R} + \mathrm{M} = \mathrm{R}$ for every prime p, i.e., R/M is divisible. R/M must be torsion-free, for otherwise $p^{-1}\mathrm{M}$ would properly contain M for some p.□

Proposition 121.4. *The union* N *of all nilpotent [or nil] left ideals of a torsion-free ring* R *is pure.*

If L is a nilpotent left ideal of R, then the pure subgroup L_* generated by L is again a nilpotent left ideal of R. Since every $a \in \mathrm{N}$ generates a nilpotent left ideal, the first assertion follows. For nil left ideals, the proof is even more trivial.□

More relevant results can be obtained if the rings are of finite rank. The rest of this section is devoted to the finite rank case. In addition, associativity will be assumed throughout.

From (119.1) it follows readily that a torsion-free ring R of finite rank is a subring of a uniquely determined finite-dimensional Q-algebra A of the same rank. In the opposite direction, one can prove:

Proposition 121.5 (Beaumont and Pierce [3]). *A torsion-free ring* R *of finite rank contains a subring* B *whose additive group is free and of the same rank as* R. *If* R *has an identity* e, *then* B *may be chosen so that* $e \in \mathrm{B}$.

Let $a_1, \cdots, a_m$ be a maximal independent system in R, and let $a_i a_j = \sum_k t_{ijk} a_k$ with $t_{ijk} \in \mathrm{Q}$. If n is a positive integer such that nt_{ijk} are integers for all i, j, k, then the subgroup $\langle b_1, \cdots, b_m \rangle$ where $b_i = na_i$ is clearly closed under multiplication, thus it is the additive group of a subring B. If $e \in \mathrm{R}$, then the elements of the subring B_0 generated by B and e are of the form $ke + b$ with $k \in \mathrm{Z}$ and $b \in \mathrm{B}$. Hence B^+, B_0^+ are finitely generated.□

Let R be a torsion-free ring of finite rank. From (119.1) we infer that R^+ is an essential subgroup in a finite-dimensional Q-algebra D. By the well-known Wedderburn principal theorem on separable finite-dimensional algebras, there is a vectorspace decomposition $\mathrm{D} = \mathrm{S} \oplus \mathrm{N}$, where S is a semisimple subalgebra [not necessarily an ideal] of D and N is the radical of D, necessarily nilpotent. S can be decomposed, in view of the Wedderburn structure theorem, into the direct sum of a finite number of simple algebras each of which is a full matrix ring over some [in general, noncommutative] field. Our considerations will be confined to the case when $\mathrm{N} = 0$, i.e., D is semisimple, while for the general case [when more sophisticated arguments are required] we refer to Beaumont and Pierce [3].

Following Beaumont and Pierce [3], a torsion-free ring R will be called *of simple* or *of semisimple algebra type* if its divisible hull D is a simple or a semisimple Q-algebra. Analogously, R is said to be *of field type* if D is a [not necessarily commutative] field.

Lemma 121.6. *Let* R *be a torsion-free ring of finite rank and of simple algebra type. Then* R^+ *is homogeneous of idempotent type and the nonzero ideals of* R *are of finite index in* R.

For any $a \neq 0$ in R, the types of the elements in the ideal A generated by a are $\geqq \mathbf{t}(a)$. Because of the simplicity of D, there is an integer $m > 0$ such that $me \in A$, where e is the identity of D. Thus $\mathbf{t}(me) \geqq \mathbf{t}(a)$, and since the converse inequality is obvious, we conclude that R^+ is homogeneous as a torsion-free group. Its type is idempotent, since $(me)^2$ and me are of the same type in R. Our argument implies $mR \leqq A$, and since R/mR is finite for all $m \neq 0$, every [principal] ideal is of finite index in R.□

We now verify the main result of this section.

Theorem 121.7 (Beaumont and Pierce [3]). *Let* R *be a torsion-free ring of finite rank and of semisimple algebra type. Then* R *contains a subring* S *of finite index such that*

$$S = S_1 \oplus \cdots \oplus S_n \qquad (\textit{ring-direct sum}),$$

where each S_i *is a full matrix ring over a ring* F_i *of field type.*

Without loss of generality, the identity e of the semisimple algebra D may be assumed to belong to R, since R is of finite index in the subring generated by R and e. Write $D = D_1 \oplus \cdots \oplus D_n$ with simple algebras D_i, and let $e = e_1 + \cdots + e_n$ with $e_i \in D_i$ orthogonal idempotents. If $m > 0$ is chosen such that $me_i \in R$ for all i, and if $a = a_1 + \cdots + a_n$ ($a_i \in D_i$), then $ma_i = me_i a_i = (me_i)a \in R$ implies that $R_1 \oplus \cdots \oplus R_n$ with $R_i = D_i \cap R$ is of finite index in R. This reduces the proof to the case $n = 1$.

Thus, assume R is of simple algebra type, i.e., D is a full matrix ring over a field K [containing Qe]. Let e_{jk} ($j, k = 1, \cdots, s$) be the matrix units in D; then $\bar{K} = e_{11}De_{11}$ is algebra-isomorphic to K and $R \cap \bar{K}$ is a subring of R with $\bar{K}$ as divisible hull. Let $m > 0$ be such that $me_{jk} \in R$ for all j, k, and define $K_{jk} = (me_{1j})R(me_{k1})$; these subrings are of $\bar{K}$, and we set $F = \bigcap_{j,k} K_{jk}$. For any $a \in \bar{K} \cap R$, $m^4 a = (me_{1j})(me_{j1})a(me_{1k})(me_{k1}) \in K_{jk}$ shows that F is of finite index in $\bar{K} \cap R$, so that F is of field type. Given a matrix $[x_{jk}]$ of order s over F, we can write $x_{jk} = (me_{1j})a_{jk}(me_{k1})$ for some $a_{jk} \in R$, and so $e_{j1}x_{jk}e_{1k} = (me_{jj})a_{jk}(me_{kk}) \in R$. The correspondence $[x_{jk}] \mapsto \sum_{j,k} e_{j1}x_{jk}e_{1k}$ is an isomorphism of the full matrix ring $F(s)$ of order s over F with a subring S of R. For every $a \in R$, $m^6 a = m^6 \sum_{j,k} e_{jj}ae_{kk} = \sum e_{j1}[m^4(me_{1j})a(me_{k1})]e_{1k} \in S$ holds; therefore S is of finite index in R.□

EXERCISES

1. Let F be a free group and G a subgroup of F. There is a ring R on F such that the additive group of R^2 is G.
2. (a) A free group of finite rank admits countably many [associative] pairwise nonisomorphic rings.
   (b) There are $2^{\mathfrak{n}}$ nonisomorphic rings on a free group of infinite rank $\mathfrak{n}$. [*Hint*: Ex. 1.]
3. In torsion-free rings, left annihilators of elements are pure left ideals.
4. Give an example for a torsion-free ring of finite rank where the Jacobson radical is not pure. Can the Jacobson radical of a torsion-free ring be of finite index?
5. Describe the construction of rings on completely decomposable homogeneous groups of idempotent types.
6. (Ree and Wisner [1]) (a) Let R, S, T be rational groups containing the integers, and define $(R, S, T) = \{q \in Q \mid qRS \leqq T\}$. Show that (R, S, T) is a subgroup of Q.
   (b) If $A = \bigoplus_{i \in I} R_i$, where R_i are rational groups such that $(R_i, R_j, R_k) = 0$ for all choices of indices $i, j, k \in I$, then A is a nil group.
7. Torsion-free nil groups of any rank exist.
8. If A and C are nil groups, $A \oplus C$ need not be one.
9. (Szele [3]) If the group A has an endomorphic image which is not nil, then A is not a nil group.
10. (A. L. S. Corner) (a) Show that the group of Ex. 8 in **88** is a nil group.
    (b) There exist homogeneous torsion-free nil groups of type $(0, \cdots, 0, \cdots)$.
11. (Szele [7]) (a) Give an example of a group A on which an associative ring R with $\mathsf{R}^n \neq 0$, but no ring with $\mathsf{R}^{n+1} \neq 0$ can be defined, where n is a preassigned positive integer. [*Hint*: $A = R_1 \oplus \cdots \oplus R_n$, where R_i is a rational group of type $(i, \cdots, i, \cdots)$.]
    (b) If $n > 1$, no such torsion group A exists.
    (c) If $n = 1$, A can not be mixed.

122. ADDITIVE GROUPS OF ARTINIAN RINGS

We proceed to investigate the additive structures of some important types of rings. Our study opens with the illustrious class of [*left*] *Artinian rings*, i.e., associative rings in which the left ideals satisfy the minimum condition. [Actually, in no place do we need associativity, except for (122.7).] The principal result will present a necessary and sufficient condition for a group to be the additive group of some Artinian ring.

To begin with, we consider nilpotent Artinian rings R, that is, $\mathsf{R}^k = 0$ for some integer k. Their additive structures are easily characterized:

Proposition 122.1 (Szele [15]). *A group A is the additive group of some nilpotent Artinian ring if and only if A satisfies the minimum condition on subgroups.*

Let R be a nilpotent Artinian ring and $\mathsf{R}^k = 0$. Every subgroup C of R^+ which contains R^{i+1} and is contained in R^i $(i = 1, 2, \cdots, k-1)$ is an ideal of R. Thus the minimum condition is satisfied by the subgroups of $\mathsf{R}^i/\mathsf{R}^{i+1}$ $(i = 1, \cdots, k-1)$, and hence by the subgroups of R, since the minimum condition on subgroups is preserved under extensions.

Conversely, if A satisfies the minimum condition on subgroups, then the zero-ring on A is nilpotent and Artinian.□

Removing the hypothesis of nilpotency, we turn our attention to Artinian rings in general, and start with two preliminary lemmas.

Lemma 122.2. *Every torsion-free divisible group $\neq 0$ is the additive group of some commutative field.*

The additive group of an algebraic extension field, of finite degree n, of the rational field Q is a torsion-free divisible group of rank n. If $\mathfrak{n}$ is an infinite cardinal, then the field adjunction of $\mathfrak{n}$ indeterminates to Q yields a field whose additive group is torsion-free, divisible, and of rank $\mathfrak{n}$.□

Lemma 122.3. *Let $\mathfrak{m}$ be an infinite cardinal and A the direct sum of $\mathfrak{m}$ copies of $Z(p^k)$. Then there exists an associative and commutative ring R with 1 on A whose only ideals are R, $p\mathsf{R}$, $\cdots$, $p^k\mathsf{R} = 0$.*

Supposing we have at hand such a ring R, we show how to construct a larger ring S with the same properties. Let S be the ring $\mathsf{R}[[x]]$ of all formal Laurent series in the indeterminate x with coefficients in R; i.e., the elements of S are of the form

$$f(x) = a_{-m}x^{-m} + \cdots + a_{-1}x^{-1} + a_0 + a_1x + \cdots + a_nx^n + \cdots \qquad (a_n \in \mathsf{R}).$$

Thus, additively,

$$\mathsf{S}^+ = \bigoplus_{n=-1}^{-\infty} \mathsf{R}^+ \oplus \prod_{n=0}^{\infty} \mathsf{R}^+,$$

and so S^+ is again a direct sum of copies of $Z(p^k)$. We will show that every principal ideal $\mathsf{L} = (f(x))$ of S is equal to one of S, $p\mathsf{S}$, $\cdots$, $p^k\mathsf{S} = 0$; then the same will hold true for every ideal.

Collecting the terms whose coefficients are divisible by the same power of p, we write the generator of $\mathsf{L} \neq 0$ in the form

$$(1) \quad f(x) = p^{k-1}f_{k-1}(x) + \cdots + p^s f_s(x) \qquad (0 \leqq s \leqq k-1, \;\; f_s(x) \neq 0),$$

where each of $f_i(x)$ is a Laurent series whose coefficients $\neq 0$ are not divisible by p. Thus L contains $p^{k-s-1}f(x) = p^{k-1}f_s(x) \neq 0$. A direct calculation [or a reference to the fact that $\mathsf{S}/p\mathsf{S}$ is a formal Laurent series field over the field

$\mathsf{R}/p\mathsf{R}]$ leads us to the existence of $g_s(x)$, $h_s(x) \in \mathsf{S}$ satisfying $f_s(x)g_s(x) = 1 + ph_s(x)$. Hence $p^{k-1}f_s(x)g_s(x) = p^{k-1}$ belongs to L, i.e., $p^{k-1}\mathsf{S} \leqq \mathsf{L}$. If $s \leqq k-2$, then $p^{k-1}f_{s+1}(x) + p^{k-2}f_s(x) = p^{k-s-2}f(x) \in \mathsf{L}$ and what has already been proved implies $p^{k-2}f_s(x) \in \mathsf{L}$, whence $p^{k-2}\mathsf{S} \leqq \mathsf{L}$. Thus proceeding, we obtain $p^s\mathsf{S} \leqq \mathsf{L}$. From $f(x) \in p^s\mathsf{S}$, the desired equality $\mathsf{L} = p^s\mathsf{S}$ follows.

To verify (122.3), first we dispose of the general case $\mathfrak{m} \geqq 2^{\aleph_0}$. Let $\{x_i\}_{i \in I}$ be a set of commuting indeterminates such that $|I| = \mathfrak{m}$. Let $\mathsf{R}_0 = \mathsf{Z}/(p^k)$ and define R as the union [or the direct limit] of the formal Laurent series rings $\mathsf{R}_0[[x_{i_1}, \cdots, x_{i_n}]]$, taken for all finite subsets $\{i_1, \cdots, i_n\}$ of I. Then $\mathsf{R}^+ = \bigoplus_{\mathfrak{m}} Z(p^k)$, and from what has been shown in the preceding paragraph it follows that every $f(x_{i_1}, \cdots, x_{i_n})$ generates one of R, $p\mathsf{R}, \cdots, p^k\mathsf{R} = 0$.

To fill the gap in the case $\aleph_0 \leqq \mathfrak{m} < 2^{\aleph_0}$, we select in $\mathsf{R}_0[[x]] = \mathsf{S}$ a subring S_0 that contains 1 and is of cardinality $\mathfrak{m}$. Let S_1 be the subring of S generated by the $f_{k-1}(x), \cdots, f_s(x)$ occurring in (1), together with the suitable $g_s(x)$, $h_s(x)$, for all $f(x) \in \mathsf{S}_0$. Repeating this process with S_0 replaced by S_1, we get a ring S_2 containing S_1, etc., and let $\mathsf{R} = \bigcup_{n=1}^{\infty} \mathsf{S}_n$. This ring is of cardinality $\mathfrak{m}$, and is pure in S [if $p^l g(x) \in \mathsf{S}_n$, then $g(x) \in \mathsf{S}_{n+1}$], so it has the desired additive structure. It has no ideals other than R, $p\mathsf{R}, \cdots, p^k\mathsf{R} = 0$, as shown by the proof above. □

Now we are about to prove the theorem we referred to in the opening paragraph of this section: a perfect structure theorem on the additive groups of Artinian rings.

Theorem 122.4 (Szele and Fuchs [1]). *In order that a group A be the additive group of an Artinian ring it is necessary and sufficient that it have the form*

$$(2) \qquad A = \bigoplus_{\mathfrak{m}} Q \oplus \bigoplus_{\text{finite}} Z(p_i^{\infty}) \oplus \bigoplus_{\mathfrak{n}} Z(p_j^{k_j}) \qquad \text{with} \quad p_j^{k_j} \mid m,$$

where $\mathfrak{m}$, $\mathfrak{n}$ *are arbitrary cardinals*, p_i, p_j *are primes, and m is a fixed integer.*

Suppose that R is an Artinian ring. In view of the minimum condition, the set of ideals $n\mathsf{R}$ ($n = 1, 2, \cdots$) contains a minimal one, say $m\mathsf{R} = \mathsf{L}$. This satisfies $p\mathsf{L} = \mathsf{L}$ for all primes p, showing that L is divisible. As such, it is a group-direct summand of R, $\mathsf{R}^+ = \mathsf{L}^+ \oplus T$ for some subgroup T of R. From $\mathsf{L}^+ = m\mathsf{R}^+ = m\mathsf{L}^+ \oplus mT$ we infer that $mT = 0$, and R^+ is the direct sum of a divisible group L^+ and a bounded group T. To complete the proof of the necessity, we need only show that the torsion part of L^+ is of finite rank. The proof of this is based on a simple fact which is interesting enough to be formulated as a separate lemma.

Lemma 122.5. *In an Artinian ring, quasicyclic subgroups belong to the annihilator of the ring.*

In any ring, elements of finite order are certainly annihilated by elements in the maximal divisible subgroup. If R is Artinian, then from what has been verified so far in (122.4) we conclude that the torsion part of L^+ is annihilated by both L^+ and T.□

Resuming the proof of (122.4), the lemma implies that the subgroups of the torsion part of L^+ are ideals of R. Consequently, they satisfy the minimum condition and the torsion part of L^+ must have finite rank. Thus R^+ has the structure (2).

To prove sufficiency, suppose A is a group of the form (2). Collecting isomorphic summands, we may write $A = D \oplus C \oplus A_1 \oplus \cdots \oplus A_k$, where D is torsion-free divisible, C is torsion and divisible, and each A_i is the direct sum of cyclic groups of the same prime power order. If $D \neq 0$, (122.2) shows that there is a field D over D. The zero-ring C over C is Artinian, and, by (122.3), each A_i admits a ring structure A_i with only finitely many left ideals. Their ring-direct sum, $\mathsf{R} = \mathsf{D} \oplus \mathsf{C} \oplus \mathsf{A}_1 \oplus \cdots \oplus \mathsf{A}_k$, is an Artinian ring, as desired. [Moreover, we see that even a commutative Artinian ring exists on A in (2).]□

The first part of the proof above holds true if R is replaced by a left ideal. For emphasis and future reference we record this as

Corollary 122.6. *The additive group of any left ideal of an Artinian ring has the form* (2).□

The remainder of this section is devoted to the proof of a ring-theoretical result. Not only its proof, but its very formulation involves its additive group.

Let us recall a simple-minded result which has been a part of the folklore of Artinian rings and reproved by several authors: *A torsion-free Artinian ring* R *has a left identity element*. We give a quick proof. By (122.1), R is not nilpotent, hence there is an idempotent $e \neq 0$ in R such that $e + \mathsf{N}$ is the identity element of the semisimple Artinian ring R/N, where N is the union of all nilpotent left ideals of R. For any $a \in \mathsf{R}$, the left ideal generated by $a' = a - ea$ is divisible, so $\frac{1}{2}a' = na' + ba'$ for some $n \in \mathbb{Z}$, $b \in \mathsf{R}$, whence $(2n-1)a' = -ba'$ and $a' = ca'$, for $c = -(2n-1)^{-1}b \in \mathsf{R}$. From $a' = ca'$ we deduce $eca' = ea' = 0$, which implies $a' = (c - ec)a' = \cdots = (c - ec)^m a'$ for every $m \geq 1$. By $c - ec \in \mathsf{N}$, we obtain $a - ea = a' = 0$.

Theorem 122.7 (Szele and Fuchs [1], Szász [2]). *Every Artinian ring* R *is the ring-theoretical direct sum of a torsion-free Artinian ring* S *and a finite number of Artinian p-rings* T_{p_i}, *belonging to different primes* p_i:

$$\mathsf{R} = \mathsf{S} \oplus \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_k}.$$

The torsion part T of R is an ideal. It is the direct sum of its p-components T_p, and in view of (122.4), only a finite number of them are different from 0. Again by (122.4), we can write $\mathsf{R}^+ = D \oplus \mathsf{T}^+$, for a torsion-free divisible subgroup D of R^+. The quotient R/T is a torsion-free Artinian ring; let $e + T$ be a left identity of R/T, where $e \in D$ may be assumed. Set $S = eR$. Now, e being a left identity mod T, every $a \in R$ is of the form $a = ea + (a - ea)$ with $ea \in S$, $a - ea \in T$; further, $ea \in T$ only if $a \in T$ in which case $ea = 0$, since every element of the divisible group D annihilates the torsion part T. Consequently, $R = S \oplus T$, and it only remains to show that S is a left ideal. But this follows directly from $TS = TeR = 0$.□

An immediate consequence of (122.7) is that the structure theory of Artinian rings can be reduced to torsion-free Artinian rings and Artinian p-rings. Since quasicyclic groups do not contribute much to the structure of Artinian rings, among the Artinian p-rings only the bounded ones are of real interest.

EXERCISES

1. (Szele [15]) Describe the construction of all nilpotent Artinian rings.
2. Quasicyclic subgroups need not belong to the annihilator of the ring if the ring is not Artinian.
3. Show that (122.4)–(122.6) hold *verbatim* for rings with minimum condition on two-sided ideals.
4. If M is a [not necessarily unital] left module over a ring R, and if the submodules of M satisfy the minimum condition, then M is a group-direct sum of copies of Q, $Z(p^\infty)$ and $Z(p^k)$, where p^k is a divisor of a fixed integer m.
5. (Szász [2]) (a) A is the additive group of a ring with minimum condition on principal [left] ideals if and only if it is a direct sum of a torsion-free divisible group and a torsion group.
   (b) For rings with minimum condition on principal [left] ideals, (122.5) holds true.
6. For Artinian rings with no quasicyclic subgroups, prove (122.7) by showing that R is the ring-direct sum of T and the maximal divisible ideal of R.
7. Generalize (122.7) in the following way: If R is a ring such that the quotient ring R/T mod its torsion part T is a divisible ring with one-sided identity element and if T is a group-direct summand of R, then $R = S \oplus T$ for some ideal S of R.

123. ARTINIAN RINGS WITHOUT QUASICYCLIC SUBGROUPS

In this section we propose to discuss two ring-theoretical questions concerning Artinian rings: *when can an Artinian ring* R *be embedded in an Artinian ring with identity element*? And, *when do the left ideals of* R *satisfy the maximum condition, too*? Naturally, we are not so ambitious as to undertake the analysis of ring-theoretical problems in this volume, but the discussion of the two questions is justified in view of the fact that the only full solutions known to us rely upon properties of the additive structure of R.

It is a familiar result that every ring can be embedded as an ideal in a ring with identity element. Starting off with an Artinian ring R, the standard procedure [adjunction of the integers] does not produce an Artinian ring again. As a matter of fact, not every R can be embedded in an Artinian ring with identity. A necessary condition is readily obtainable from (122.5): An Artinian ring with identity has no annihilators $\neq 0$, thus it contains no quasicyclic subgroups, and the "only if" part of the next theorem becomes evident.

Theorem 123.1 (Szele and Fuchs [1]). *An Artinian ring can be embedded [as an ideal] in an Artinian ring with identity if and only if it does not contain any $Z(p^\infty)$.*

To prove the "if" part, suppose R is Artinian and has no quasicyclic subgroups. (122.7) implies that R is the ring-direct sum of a torsion-free Artinian ring S and a finite number of Artinian p-rings T_p. Therefore, it suffices to show that each of S and T_p can be embedded as an ideal in an Artinian ring with identity.

In order to avoid the repetition of arguments, we separate off one piece of the proof for future reference in

Lemma 123.2. *Let* A *be an associative and commutative ring with identity* ε *and* R *an associative ring which is at the same time a unital* A*-algebra. The set of pairs* (α, a) $(\alpha \in A,\ a \in \mathrm{R})$ *under the rules*:

$$(\alpha, a) + (\beta, b) = (\alpha + \beta, a + b),$$

$$(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab)$$

for all $\alpha, \beta \in \mathrm{A}$, $a, b \in \mathrm{R}$, *is an associative ring* R_A *with identity* $(\varepsilon, 0)$. R *can be identified with an ideal of* R_A *under the injection* $a \mapsto (0, a)$, *and the quotient of* R_A *modulo this ideal is isomorphic to* A.

The proof consists of a straightforward verification of the ring postulates and is highly reminiscent of the customary adjunction of identity when $\mathrm{A} = \mathrm{Z}$ is used.□

Before we continue with the proof, notice that R_A can be Artinian only if A is Artinian, since the property "Artinian" is hereditary under epimorphisms. This observation will guide us in the selection of a good A needed for the proof of (123.1).

A torsion-free Artinian ring S is divisible by (122.4), thus it may be regarded as an algebra over Q. A simple appeal to (123.2), and it remains only to show that the arising ring S_Q is likewise Artinian. But a ring S_Q is necessarily Artinian if both its ideal S and the quotient $\mathrm{S}_\mathrm{Q}/\mathrm{S} \cong \mathrm{Q}$ are Artinian rings.

The absence of subgroups $Z(p^\infty)$ in R implies, owing to (122.4), that the p-components T_p are bounded, say $p^k\mathsf{T}_p = 0$. Then T_p is an algebra over $\mathsf{A} = \mathsf{Z}/(p^k)$, so (123.2) is applicable. The new ring is Artinian, being an extension of an Artinian ring by a finite ring.□

Passing to the second question, raised at the beginning of this section, an answer is given in the following theorem.

Theorem 123.3 (Szele and Fuchs [1], Fuchs [13]). *The left ideals of an Artinian ring satisfy the maximum condition exactly if the ring contains no quasicyclic subgroups.*

Each quasicyclic subgroup of an Artinian ring R is contained in the annihilator of R, and every subgroup of the annihilator is an ideal of R. Quasicyclic groups violate the maximum condition on subgroups; as a result, R cannot have quasicyclic subgroups if its left ideals satisfy the maximum condition, too.

Conversely, let R be Artinian and contain no $Z(p^\infty)$. To begin with, observe that none of the quotients R/L with L an ideal of R can contain any $Z(p^\infty)$. For, if R is decomposed as in (122.7), then L is the direct sum of its intersections with the summands, and (122.6) assures that R/L has no subgroup of type p^∞.

It is an elementary result that the left ideals of a ring R satisfy both the minimum and the maximum conditions exactly if R as a [not necessarily unital] left R-module has a finite length. Hence, we work for establishing a finite composition series of left ideals of R.

From the minimum condition in R, it follows that $\mathsf{N}^k = 0$ holds for some k, where N is the [Jacobson] radical of R; here $\mathsf{N}^{k-1} \neq 0$ may be assumed. If $\mathsf{N} = \mathsf{R}$, then (122.1) and the absence of $Z(p^\infty)$ force the finiteness of R. Our attention now narrows to the case when

$$\mathsf{R} = \mathsf{N}^0 > \mathsf{N} > \mathsf{N}^2 > \cdots > \mathsf{N}^{k-1} > \mathsf{N}^k = 0$$

is a properly descending chain of ideals. The quotients $M_i = \mathsf{N}^{i-1}/\mathsf{N}^i$ $(i = 1, \cdots, k)$ are annihilated by N, they can therefore be viewed as left modules over the semisimple Artinian ring $\mathsf{R}/\mathsf{N} = \mathsf{R}_0$. It is a well-known result that a [not necessarily unital] module over a semisimple Artinian ring R_0 decomposes into the direct sum of simple [unital] R_0-modules and a submodule annihilated by R_0. In the present case, this direct sum is finite, $M_i = S_{i1} \oplus \cdots \oplus S_{il_i} \oplus U_i$, where the S_{ij} are simple R_0-modules and $\mathsf{R}_0 U_i = 0$ for all i. The minimum condition on submodules of M_i implies the same on subgroups of U_i, and since, as noticed above, $U_i \leqq \mathsf{R}/\mathsf{N}^i$ has no quasicyclic subgroups, U_i must be finite. For this reason, M_i has a finite composition series. These composition series can be put together to form one for R.□

Curiously enough, the failure of subgroups of type p^∞ is equivalent to a simple condition of purely ring-theoretical nature: *an Artinian ring has no quasicyclic subgroups if and only if its annihilator is finite.* This is immediately seen by virtue of (122.5) and the minimum condition on the subgroups of the annihilator.

We immediately make use of this idea in the following ring-theoretical reformulation of (123.3):

Corollary 123.4. *In an associative ring, the minimum condition on left ideals implies the maximum condition on left ideals if and only if the annihilator of the ring is finite.* □

A trivial corollary is a frequently cited result by C. Hopkins: an Artinian ring with one-sided identity is Noetherian [*Ann. Math.* **40** (1939), 712–736]. Though (123.4) does not indicate explicitly any connection with the additive structure, no purely ring-theoretical proof has been found.

EXERCISES

1. A nilpotent Artinian ring is embeddable in an Artinian ring with identity if and only if it is finite.
2. (a) For an Artinian ring R to be a unital algebra over a suitable commutative Artinian ring A, it is necessary and sufficient that R contain no $Z(p^\infty)$.
   (b) The same holds for any ring R whose additive group is of the form (2) in **122**.
3. Let P be a ring property such that if an ideal L of a ring R has it, then R and R/L simultaneously do or do not have P. Prove that if R is a ring with P and at the same time a unital algebra over some commutative ring A with identity which also has P, then R can be embedded as an ideal in a ring with identity with preservation of property P.
4. In (123.4), "annihilator" can be replaced by "right annihilator."
5. Let M be a [not necessarily unital] module with minimum condition on submodules over a semisimple Artinian ring. M has the maximum condition on submodules exactly if it contains no quasicyclic subgroup.

124. ADDITIVE GROUPS OF REGULAR AND π-REGULAR RINGS

Our program for this section is to study the additive structures of regular and generalized regular rings. So far, no full characterization is known for the groups which can be additive groups of regular rings, but a fair amount of information can be obtained. [In this section, associativity will not be made use of.]

Let R be a regular ring [in the sense of von Neumann; for definition see **112**]. The union of all quasicyclic subgroups of R is an ideal of R, with trivial multiplication. Hence, for an a in this ideal, there can exist an $x \in \mathsf{R}$ such that $axa = a$ only if $a = 0$; in consequence, R has no quasicyclic subgroups. Now, if $p^k | a$, then $p^{2k} | axa = a$, which shows that an $a \in \mathsf{R}$ is either not divisible by a prime p or it is divisible by all powers of p. It follows that a torsion-free regular ring is divisible; furthermore, the p-component T_p of R is an elementary p-group. As such, it is a summand of R:

$$\mathsf{R} = \mathsf{T}_p \oplus \mathsf{R}_p. \tag{1}$$

This is moreover a ring-direct sum, because R_p is p-divisible: for every $b \in \mathsf{R}_p$, pb is divisible by p^2, and since R_p has no elements of order p, b itself has to be divisible by p.

Note that if q is a prime $\neq p$, then T_q is contained in R_p, and a repeated application of (1) yields the ring-direct sum

$$\mathsf{R} = \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_k} \oplus \mathsf{R}_0, \tag{2}$$

where $p_1, \cdots, p_k$ are distinct primes and multiplications by these primes are automorphisms of R_0.

Next consider $\mathsf{D} = \bigcap_p \mathsf{R}_p$; it is obviously a torsion-free ideal. Its divisibility follows in the same way as in the proof of (112.4). Write $\mathsf{R}^+ = \mathsf{D}^+ \oplus C$, where C is a reduced group containing $\mathsf{T} = \bigoplus_p \mathsf{T}_p$. For each p, (1) defines a projection $\varepsilon_p \colon \mathsf{R} \to \mathsf{T}_p$ such that $\bigcap_p \operatorname{Ker} \varepsilon_p = \bigcap_p \mathsf{R}_p = \mathsf{D}$. As a result, we obtain that R/D is isomorphic to a ring-subdirect sum of the $\operatorname{Im} \varepsilon_p = \mathsf{T}_p$. Now R/T—as a torsion-free regular ring—is divisible and so is its pure subgroup C/T^+. Thus we arrive at

Theorem 124.1 (Fuchs [6]). *The additive group of a regular ring is the direct sum of a torsion-free divisible group and a reduced group C such that*

$$T = \bigoplus_p T_p \leqq C \leqq \prod_p T_p,$$

where T_p are elementary p-groups and C/T is torsion-free and divisible. □

It is not known which groups with the indicated additive structure do admit a regular ring. For $C = \bigoplus T_p$ or $C = \prod T_p$, the answer is in the affirmative, simply by putting fields over each T_p and the torsion-free divisible part.

In sharp contrast to regular rings, the additive groups of [left or right] m-regular ($m \geqq 2$) or π-regular rings can be arbitrary, since zero-rings have all these properties. However, the situation changes drastically if the rings are supposed to have an identity element, or even if they are just left ideals in m-regular etc. rings with identity.

Theorem 124.2 (Fuchs and Rangaswamy [1]). *If* S *is a* [*left*] *ideal in a π-regular ring* R *with identity* 1, *then*:

(i) *for every prime p, the p-component* S_p *of* S *is bounded, and* S *is a ring-direct sum* $S = S_p \oplus p^m S$ *for some* $m > 0$;

(ii) *for the torsion part* T *of* S, S/T *is a divisible ring.*

The π-regularity of R implies $(p \cdot 1)^m x (p \cdot 1)^m = (p \cdot 1)^m$, that is, $p^{2m}x = p^m \cdot 1$ for some $x \in R$ and integer $m > 0$. Hence, for every $a \in S$, $p^{2m}xa = p^m a$; thus every $p^m a$ is divisible by p^{2m} in the left ideal generated by a. We conclude $p^{m+1}S = p^m S$ and $p^m S$ is p-divisible. In particular, if $a \in S_p$ is of order p^k, then, for some $y \in R$, $p^m a = p^k(ya) = y(p^k a) = 0$, and so $p^m S_p = 0$. We infer that $S^+ = S_p^+ \oplus C$ for some subgroup C of S. Notice that $p^m S^+ = p^m C$ is p-divisible and division by p in C is unique. This shows that $C = p^m C$ and $S = S_p \oplus p^m S$ is a ring-theoretical direct sum. The proof of (i) is complete.

To verify (ii), all that we have to do is observe that S/T is an epic image of the p-divisible rings $p^m S$.□

In the next section we shall prove a theorem which will show that the converse of (124.2) also holds.

If some other kind of generalized regularity is substituted for π-regularity, (124.2) does not change [see Ex. 4].

An argument, very much in the spirit of the one carried out in (124.1) proves, if combined with (124.2), one half of the following theorem, while the other half can be derived from (125.4):

Corollary 124.3. *A group is the additive group of a π-regular ring that is a* [*left*] *ideal in some π-regular ring with* 1 *if and only if it is the direct sum of a torsion-free divisible group and a reduced group C such that* $T = \bigoplus_p T_p \leqq C \leqq \prod_p T_p$, *where* T_p *are bounded p-groups and* C/T *is torsion-free divisible.*

Anticipating (125.4), it is enough to endow the group of the stated kind with the trivial multiplication.□

Exercises

1. A group is the additive group of a Boolean ring [every element is idempotent] if and only if it is an elementary 2-group.
2. (a) A group is isomorphic to the additive group of a left Noetherian regular ring exactly if it is a finite direct sum of elementary p-groups and a torsion-free divisible group.

   (b) Prove the analogous result for left Noetherian π-regular rings.
3. If a group A is the additive group of a regular ring, then there is a regular ring on A which is the ring-direct sum of a divisible and reduced regular ring.

4. (a) Show that (124.2) holds *verbatim* if the π-regularity of R is replaced by left or right π-regularity.
   (b) If R is [left, right] m-regular in (124.2), then (i) holds with this m.
5. Prove that (124.2) holds for the left ideal S generated by a in a π-regular ring R whenever there is an idempotent $e \in R$ such that $ea = a$.
6. The additive group of an ideal of an m-regular ring with identity is not necessarily the additive group of any π-regular ring with identity.
7. How does (124.3) read for m-regular rings?
8. The torsion part of a Baer ring [see **112**, Ex. 13] is an elementary group.

125. EMBEDDINGS IN REGULAR AND π-REGULAR RINGS WITH IDENTITY

Our knowledge of the additive structures of regular rings and generalizations will enable us to solve the problem of embedding a ring in a ring with 1 with preservation of regularity or π-regularity. In our solution, an important device is the use of (123.2), exploiting an idea formulated explicitly in **123**, Ex. 3. The associativity of the rings will be supposed throughout.

We start with the case of regular rings; we prefer to give independent attention to them, partly because of their great importance and partly because they can be handled without any additional hypothesis.

First of all, we construct a commutative regular ring M with 1 as follows. For every prime p, take the prime field F_p of characteristic p; let ε_p be the identity of F_p. Form $F = \prod_p F_p$ which is a commutative regular ring with the identity $\varepsilon = (\cdots, \varepsilon_p, \cdots)$. The quotient $F/\bigoplus_p F_p$ is a torsion-free divisible ring in which the pure subgroup generated by the coset of ε is a ring $M/\bigoplus_p F_p$ isomorphic to Q. In this way, we obtain a subring M of F which contains ε and contains every F_p. This M is regular: it contains the regular ring $\bigoplus_p F_p$ as an ideal modulo which M is regular.

Theorem 125.1 (Fuchs and Halperin [1]). *Every regular ring is a unital* M-*algebra.*

Let R be a regular ring. To define the product μa of $\mu \in M$ and $a \in R$, we write $\mu = (\cdots, \mu_p, \cdots)$ with $\mu_p \in F_p$, and notice that, by construction, there is a rational number mn^{-1} (m, n integers, $n \neq 0$) such that $n\mu_p \equiv m \bmod p$ for almost all primes p. We select a finite set $\{p_1, \cdots, p_k\}$ of primes which includes all prime divisors of m and n, and, in addition, the primes for which the last congruence fails to hold. With such a set of primes, a ring-direct sum

$$M = F_{p_1} \oplus \cdots \oplus F_{p_k} \oplus M_0 \tag{1}$$

is obtained, where M_0 is an ideal of M such that multiplication by p_i ($i = 1, \cdots, k$) is an automorphism on M_0; hence $\mu = \mu_{p_1} + \cdots + \mu_{p_k} + \mu_0$, where $\mu_{p_i} \in \mathsf{F}_{p_i}$, $\mu_0 \in \mathsf{M}_0$. Corresponding to (1), R has a decomposition (2) in **124**, and we can write $a = a_{p_1} + \cdots + a_{p_k} + a_0$ with $a_{p_i} \in \mathsf{T}_{p_i}$, $a_0 \in \mathsf{R}_0$.

Now we are in a position to define

$$(2) \qquad \mu a = \mu_{p_1} a_{p_1} + \cdots + \mu_{p_k} a_{p_k} + mn^{-1} a_0.$$

Since T_{p_i} is a vector space over F_{p_i}, $\mu_{p_i} a_{p_i}$ makes sense, and so does $mn^{-1}a_0$, since multiplication by n is an automorphism on R_0. Naturally, we have to convince ourselves that μa in (2) does not change if a larger set of primes or a different form of mn^{-1} is used. But this follows at once from our selection of primes which guarantees that each μ_p with $p \notin \{p_1, \cdots, p_k\}$ acts on F_p as a multiplication by mn^{-1}. We leave it to the reader to check the algebra postulates: $\mu a \in \mathsf{R}$, $\mu(a + b) = \mu a + \mu b$, $(\mu + \nu)a = \mu a + \nu a$, $(\mu\nu)a = \mu(\nu a)$, $\mu(ab) = (\mu a)b = a(\mu b)$, and $\varepsilon a = a$ for all $\mu, \nu \in \mathsf{M}$ and $a, b \in \mathsf{R}$.□

It is now easy to prove the following theorem.

Theorem 125.2 (Fuchs and Halperin [1]). *Every regular ring can be embedded as an ideal in a regular ring with identity.*

Given a regular ring R, we know from (125.1) that it is a unital M-algebra, where M is the commutative regular ring defined above. (123.2) yields an embedding of R as an ideal in R_M. Since both R and $\mathsf{R}_\mathsf{M}/\mathsf{R} \cong \mathsf{M}$ are regular, R_M too is regular.□

Proceeding to π-regular rings, the key step in the embedding theorem is, analogously, to establish the existence of a commutative π-regular ring with 1 over which the given ring is a unital algebra. Such a ring fails to exist, in general; as a matter of fact, conditions (i) and (ii) of (124.2) are necessary for the existence of such a ring. On the other hand, they enable us to prove the analog of (125.1).

Theorem 125.3 (Fuchs and Rangaswamy [1]). *Let* R *be a π-regular ring whose p-components* T_p *are bounded and for which the quotient ring* R/T (*where* $\mathsf{T} = \bigoplus_p \mathsf{T}_p$) *is divisible. Then there exists a commutative π-regular ring* N *with* 1 *such that* R *is a unital* N*-algebra.*

The π-regular ring N is now not universal [like M was in (125.1)], it depends on R. If R has but a finite number of nonzero p-components T_p, then by their boundedness we obtain $\mathsf{R} = \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_k} \oplus \mathsf{R}_0$ with a torsion-free divisible R_0. Obviously, the choice $\mathsf{N} = \mathsf{Z}/(p)_1^{l_1} \oplus \cdots \oplus \mathsf{Z}/(p_k^{l_k}) \oplus \mathsf{Q}$ is suitable if the exponents l_i are chosen so as to satisfy $p_i^{l_i}\mathsf{T}_{p_i} = 0$. [We can drop Q if $\mathsf{R}_0 = 0$.]

If R has infinitely many nonzero p-components T_p, then N will be defined as follows. Choosing the integers l_i such that $p_i^{l_i}\mathsf{T}_{p_i} = 0$, we form the direct product N′ of the rings $\mathsf{Z}/(p_i^{l_i})$ whose identities will be denoted by ε_{p_i}. Now N is defined to be the subring of N′ which contains the direct sum of the rings

$\mathsf{Z}/(p_i^{l_i})$ and, mod this direct sum, maps upon the pure subring isomorphic to Q generated by the identity $\varepsilon = (\cdots, \varepsilon_{p_i}, \cdots)$. From the commutativity of N and from the π-regularity of the direct sum and Q, it is readily verified that N is, in fact, π-regular. The product va for $v \in \mathsf{N}$ and $a \in \mathsf{R}$ can be defined in the same way as in (2), by using a decomposition like (1) for N and a corresponding decomposition for R [by separating off all p_i-components which are exceptional for some reason or another]. One should add that our hypotheses guarantee not only that for each finite set $\{p_1, \cdots, p_k\}$ of primes we can write $\mathsf{R} = \mathsf{T}_{p_1} \oplus \cdots \oplus \mathsf{T}_{p_k} \oplus \mathsf{R}_0$, but also that R_0 will be p_i-divisible for $i = 1, \cdots, k$.□

The following theorem gives a full answer to our embedding problem for π-regular rings.

Theorem 125.4 (Fuchs and Rangaswamy [1]). *A π-regular ring* R *can be embedded as an ideal in a π-regular ring with identity exactly if*:

(i) *for every prime p, the p-component T_p of R is bounded*;
(ii) *for the torsion part $\mathsf{T} = \bigoplus_p \mathsf{T}_p$ of R, R/T is divisible.*

The necessity of conditions (i) and (ii) follows from (124.2). To prove the converse, let the π-regular ring R satisfy (i) and (ii). To begin with, we pick a ring N as described in (125.3) and form R_N according to (123.2). This R_N contains [an isomorphic copy of] R as an ideal such that $\mathsf{R}_\mathsf{N}/\mathsf{R} \cong \mathsf{N}$. The proof will be completed as soon as the π-regularity of R_N will be established. We need an additional argument to obtain this.

We claim it will suffice to prove that for every $u \in \mathsf{R}_\mathsf{N}$ there exists an integer m and a $v \in \mathsf{R}_\mathsf{N}$ such that $u^m v = vu^m$ and $u^m vu^m - u^m \in \mathsf{R}$. In fact, by the π-regularity of R, we can then find an integer $n > 0$ and an $x \in \mathsf{R}$ such that

$$(u^m vu^m - u^m)^n x(u^m vu^m - u^m)^n = (u^m vu^m - u^m)^n,$$

whence we obtain $u^{mn}wu^{mn}(vu^m - 1)^n = u^{mn}(vu^m - 1)^n$ for some $w \in \mathsf{R}_\mathsf{N}$. Multiplication by $[(vu^m)^{n-1} + (vu^m)^{n-2} + \cdots + vu^m + 1]^n$ from the right gives $u^{mn}[(vu^m)^n - 1]^n$ on the right side, in whose expansion every term with the exception of the last one contains u^{mn} at least twice as a factor. Hence we obtain an equality of the form $u^{mn}yu^{mn} = u^{mn}$ for some $y \in \mathsf{R}_\mathsf{N}$.

The elements of R_N are of the form $u = (v, a)$ with $v \in \mathsf{N}$ and $a \in \mathsf{R}$. By the π-regularity of N, we can find an $m > 0$ and $\mu \in \mathsf{N}$ such that $v^m \mu v^m = v^m$. Choosing $v = (\mu, 0)$, we have $uv = vu$ and $u^m vu^m - u^m \in \mathsf{R}$. Therefore, what has been proved in the preceding paragraph assures the π-regularity of R_N.□

EXERCISES

1. Show that the ring M in (125.1) is the best choice in the following sense: If M^* is any commutative regular ring with identity such that every regular ring R is a unital M^*-algebra, then there is an identity-preserving epimorphism $\varphi: M^* \to M$ such that $\mu^* a = \varphi(\mu^*)a$ for all $\mu^* \in M^*$ and $a \in R$.
2. Let M^* be a regular ring with 1 which has an identity-preserving homomorphism onto every prime field, but no proper, regular subring of M^* containing 1 has this property. Then M^* has such a homomorphism onto M.
3. Select a ring N in (125.3) which is the best choice for a given R in the sense of Ex. 1.
4. (a) Check that (i) and (ii) in (125.4) are automatically satisfied by regular rings R.
   (b) Show that (i) and (ii) are fulfilled by an Artinian ring if and only if it has no quasicyclic subgroups.
5. Let R be m-regular and $p^n T_p = 0$ for every prime p, with the same n, such that R/T is divisible.
   (a) N in (125.3) can be chosen to be n-regular.
   (b) R can be embedded as an ideal in an mn-regular ring with identity.
   (c) Give an example to show that not every m-regular ring R with (i) and (ii) is embeddable as an ideal in an m-regular ring with 1.
6. Every torsion-free, divisible, m-regular ring is an ideal in an m-regular ring with 1.
7. Show that (125.3) and (125.4) carry over to left [and right] π-regularity.

126. ADDITIVE GROUPS OF NOETHERIAN RINGS AND RINGS WITH RESTRICTED MINIMUM CONDITION

Recall that a ring is called *[left] Noetherian* if its left ideals satisfy the maximum condition. We shall obtain some information about the additive groups of Noetherian rings; as a matter of fact, full description will be given modulo the torsion-free case.

We start with nilpotent rings. The following is the exact dual to (122.1):

Proposition 126.1 (Szele [15]). *A group is the additive group of a nilpotent Noetherian ring if and only if it is finitely generated.*

Let R be a Noetherian ring such that $R^k = 0$ for some $k \geqq 1$. The subgroups between R^{i+1} and R^i ($i = 1, \cdots, k-1$) are ideals of R, thus they satisfy the maximum condition, or, equivalently, R^i/R^{i+1} is finitely generated as a group. The same holds for R.

Conversely, the zero-ring over a finitely generated group is Noetherian. □

Returning to the general situation, we focus our attention on the torsion part T of the Noetherian ring R. We know that R$[n]$ are ideals of R, for all $n > 0$; thus there exists a maximal one, say R$[m]$, among them. Then necessarily T = R$[m]$, mT $= 0$, thus R$^+$ = T$^+ \oplus C$ for some torsion-free subgroup C of R. Since C is isomorphic to the additive group of the Noetherian ring R/T, we are led to the following result:

Proposition 126.2. *A group A is the additive group of a Noetherian ring if and only if $A = T \oplus C$, where T is bounded and C is a torsion-free group admitting a Noetherian ring structure.*

That there does exist a Noetherian ring on a bounded group is an immediate corollary to (122.3).□

In view of (126.2), a study of additive groups of Noetherian rings may be restricted to the torsion-free case. A further reduction to the reduced case is evident as soon as we notice that there is a Noetherian ring [namely, a field] on every torsion-free divisible group.

Our present knowledge of Noetherian rings R on reduced torsion-free groups A does not go beyond a few elementary remarks.

(A) For every left ideal L of R, nL$_*$ $\leqq$ L for some $n > 0$, where L$_*^+$ is the pure subgroup generated by L$^+$.

This follows at once from the fact that L$_*$/L must satisfy the ascending chain condition on R-submodules, so it is bounded.

(B) The types of elements in A satisfy the minimum condition.

In fact, to a descending chain $\mathbf{t}_1 > \cdots > \mathbf{t}_n > \cdots$ of types of elements in A, there corresponds an ascending chain $A(\mathbf{t}_1) < \cdots < A(\mathbf{t}_n) < \cdots$ of ideals in R.

(C) The minimal types of elements in A are idempotent.

Let $a \in A$ be of a minimal type $\mathbf{t}$. Ignoring the trivial case, suppose $\mathbf{t} \neq (0, \cdots, 0, \cdots)$. Because of (A), the left ideal L generated by a satisfies: L $\cap \langle a \rangle_*$ is of finite index in $\langle a \rangle_*$. Consequently, L $\neq \langle a \rangle$ and a does not belong to the right annihilator of R. Hence some $r \in$ R satisfies: $0 \neq ra \in \langle a \rangle_*$. We obtain $\mathbf{t} = \mathbf{t}(ra) \geqq \mathbf{t}(r)$, whence $\mathbf{t}(r) = \mathbf{t}$ and $\mathbf{t}(ra) \geqq \mathbf{t}^2$.

In commutative ideal theory, the so-called restricted minimum condition is used to characterize, along with other conditions, the Dedekind domains. Rings with the mentioned condition will be the next subjects of our analysis of additive groups.

We recall that a ring R satisfies the *restricted minimum condition* on left ideals if the minimum condition holds modulo every nonzero ideal of R. We want to get information about the additive groups of these rings R.

If R is not torsion-free, then for some prime p, R$[p]$ is a nonzero ideal of R, and thus R/R$[p]$ is an Artinian ring. From (122.4) it is clear that mR is divisible for some integer m, hence the additive group of R must have the form (2) of **122**. This establishes the first half of

Theorem 126.3 (Fuchs [16]). *A not torsion-free group is the additive group of a ring with restricted minimum condition if and only if it is of the form* (2) *of* **122**.

The additive group of a torsion-free ring with restricted minimum condition is homogeneous and its type is idempotent.

Let R be torsion-free with restricted minimum condition. If its maximal divisible ideal is not 0, then R is, by (122.4), divisible modulo this ideal, and hence R itself is divisible. Suppose R reduced, and consider ideals of the form $L = \bigcap p_i^{k_i} R$ with $p_i^{k_i}$ running over an infinite set of prime powers. Either $L = 0$ or R/L is Artinian. This excludes the existence of elements in R whose characteristics are $(k_1, \cdots, k_n, \cdots)$ with infinitely many positive integers k_n, i.e., all types in R are idempotent. Furthermore, if some nonzero $a \in R$ is divisible by all powers of a prime p, then the same is true for all other elements of R, for otherwise $R > pR > \cdots > p^k R > \cdots$ would be a properly descending chain with nonzero intersection. The homogeneity of R is now evident.□

It is an open question which homogeneous torsion-free groups of idempotent types admit rings with restricted minimum condition.

EXERCISES

1. Give an example for a Noetherian ring R whose additive group is a free group of rank $\aleph_0$. [*Hint*: $Z[x]$.]
2. Use the method of (122.3) to obtain a Noetherian ring of the power of the continuum with torsion-free additive group homogeneous of type $(0, \cdots, 0, \cdots)$.
3. The additive group of the annihilator of a Noetherian ring is finitely generated.
4. Let R be a torsion-free Noetherian ring. Show that R contains an ideal A such that A^+ is a finitely generated free group and R/A is a Noetherian ring with zero annihilator. [*Hint*: take the annihilator A_1 of R, then that of R/A_1, etc.]
5. Let R be a torsion-free Noetherian ring.
   (a) If P is a rational group of idempotent type and $1 \in P$, then $R^+ \otimes P$ carries a Noetherian ring structure S such that the natural map $a \to a \otimes 1$ makes R into a subring of S.
   (b) The Q-algebra over the divisible hull of R^+, extending the ring structure of R, is likewise Noetherian.
6. (a) Every Noetherian ring can be embedded as an ideal in a Noetherian ring with identity.
   (b) There is such an embedding which preserves torsion-freeness and homogeneity.
7. A torsion-free group of rank 1 is the additive group of a ring with restricted minimum condition exactly if its type is idempotent.

8. Give an example for a homogeneous, torsion-free group with idempotent type which is not the additive group of any ring with restricted minimum condition. [*Hint*: show that Ex. 8 in **88** is a nil group.]
9. Let R be a not torsion-free ring with restricted minimum condition. Every left ideal of R has an additive group of the form (2) of **122**.

NOTES

The problem of defining ring structures on an additive group was raised by Beaumont [2] who considered rings on direct sums of cyclic groups. Nearly at the same time, Szele [3] investigated zero-rings, Rédei and Szele [1], and Beaumont and Zuckerman [1] described the rings on subgroups of the rationals. A more systematic study of constructing rings on a group appeared in Fuchs [6] where the fundamental role of the basic subgroup was pointed out. More satisfactory results have been obtained for torsion-free groups of finite rank by Beaumont and Pierce [3, 5] where an interesting analog of the Wedderburn principal theorem on finite-dimensional algebras has been established.

It was recognized only twenty years ago that the additive group of a ring can provide some interesting information about the ring. The program of investigating systematically the additive structures of rings was formulated by Szele who started the study with nilpotent rings [15]. His premature death prevented him from finishing his research on the additive groups of Artinian rings; his paper on this subject was completed by the author [see Szele and Fuchs [1]]. (122.4) gives a satisfactory description of the additive groups of Artinian rings [for another discussion, see A. Kertész, "Vorlesungen über Artinsche Ringe." Akadémiai Kiadó Budapest, 1968]. The results on regular and π-regular rings are not so satisfactory [Fuchs [16], Fuchs and Rangaswamy [1]].

It would be a serious mistake to expect too much from a study of the additive structures of rings, as far as ring theory is concerned. In many important cases the additive structures are too trivial [e.g., torsion-free divisible or an elementary p-group] to give any real information about the ring structure. This especially applies to the torsion-free case, where a close interrelation between the additive and the multiplicative structures can be expected only if the additive group is more complicated. One should, however, remember that there are intriguing questions even if the additive group is too easy to describe; for instance, we do not know of any uncountable Noetherian ring whose additive group is free.

Problem 93. On which groups can a ring be defined whose only ideals are those described in (117.2)?

Problem 94. Investigate the "absolute" annihilators, the "absolute" Jacobson radical, etc., for groups A. [Here we mean by absolute annihilator the set of elements in A which belong to the annihilator of every ring on A.]

Problem 95. Give a survey of rings on strongly indecomposable torsion-free groups of finite rank.

Problem 96. Characterize the additive groups of regular rings, local rings, etc.

Problem 97. Do Noetherian rings exist whose additive groups are free groups of large cardinalities?

XVIII

GROUPS OF UNITS IN RINGS

An important field of application of abelian groups is the theory of multiplicative groups of commutative fields and, more generally, the groups of units in commutative and associative rings with identity. [In this chapter, both associativity and commutativity will be supposed of all occurring rings.]

Until recently, only a little attention has been given to this area. In our study, we shall mainly be concerned with two aspects of the theory: to find information about the multiplicative groups of special fields and groups of units in some rings, and to establish conditions on a group to be the unit group of an appropriate ring.

We intend to characterize the multiplicative groups of prime fields and their finite algebraic extensions, algebraically and real closed fields, and the p-adic number fields. For the unit groups, we do not have much to say, in general, except for some remarks. It will be relatively easy to verify that, for every group A, $Z(2) \times A$ is the unit group of some ring.

127. MULTIPLICATIVE GROUPS OF FIELDS

Our study of the multiplicative groups of units opens with the important special case when the ring is a field. Intuitively, it is expected that the characteristic of the field will play a decisive role in the structure of its multiplicative group; we shall learn, however, that so far just a minor difference has been discovered, with the exception of the absolute algebraic extensions.

From now on we assume that all fields are commutative. The notation $\mathsf{K}^\times$ will be used for the multiplicative group of the field K; as a set, $\mathsf{K}^\times = \mathsf{K}\backslash 0$. For obvious reasons, we have to adhere to the multiplicative notation in $\mathsf{K}^\times$; thus, 1 will be the neutral element of $\mathsf{K}^\times$ and the symbol $\times$ will stand for restricted direct products.

It is not hard to describe the torsion part of $\mathsf{K}^\times$:

Theorem 127.1. *A torsion group is isomorphic to the torsion part of* $\mathsf{K}^\times$ *for some field* K *of characteristic* 0 *if and only if it is isomorphic to a subgroup of Q/Z with nontrivial* 2*-component.*

Let $u \in \mathsf{K}^\times$ satisfy $u^p = 1$ for some prime p. The equation $x^p - 1 = 0$ can not have more than p different roots in any commutative field; therefore, the socle of the p-component of $\mathsf{K}^\times$ is of order $\leqq p$. From (25.1) we deduce that the p-component of $\mathsf{K}^\times$ is cocyclic. The isomorphism of the torsion part T of $\mathsf{K}^\times$ with a subgroup of Q/Z is now obvious. Since $-1 \in \mathsf{K}^\times$, T contains an element of order 2.

Let T be a subgroup of Q/Z with nontrivial 2-component. We realize T as a multiplicative group of complex roots of unity and select a finite or infinite ascending chain of cyclic groups $\langle \zeta_k \rangle$ of even orders m_k such that T is their union. Define K as the union of the tower $\mathsf{Q}(\zeta_1) \subseteqq \cdots \subseteqq \mathsf{Q}(\zeta_k) \subseteqq \cdots$, where $\mathsf{Q}(\zeta_k)$ is the subfield of complex numbers generated by Q and ζ_k. By construction, the torsion part of $\mathsf{K}^\times$ will contain T. Since the torsion part of $\mathsf{K}^\times$ is the union of the torsion parts of the $\mathsf{Q}(\zeta_k)^\times$, the converse inclusion will follow if we can show that the torsion part of $\mathsf{Q}(\zeta_k)$ is exactly $\langle \zeta_k \rangle$. But this is simply a consequence of the fact that the degree of $\mathsf{Q}(\zeta_k)$ over Q is given by Euler's function $\phi(m_k)$, and for an even m_k, a primitive nth root of 1 with $m_k \mid n$ is of degree $\leqq \phi(m_k)$ only if $n = m_k$. $\square$

If the torsion part has trivial 2-component, then the situation is more difficult. First of all, let us point out that if K is of characteristic p, then 1 is the only root of $x^p - 1 = (x-1)^p = 0$, hence the p-component of $\mathsf{K}^\times$ is missing. As a result, a torsion group with trivial 2-component can be realized only in case of characteristic 2. For more information, see Ex. 4.

We now turn to the most important classes of fields and try to discover their multiplicative structures.

1. *The prime fields.* These are: Q and, for each prime p, $\mathsf{F}_p = \mathsf{Z}/(p)$.

The fundamental theorem of arithmetic tells us that every rational number $\neq 0$ may be written uniquely in the form $\pm p_1^{k_1} \cdots p_r^{k_r}$, where p_i are different primes and $k_i \neq 0$ are integers. Hence we obtain

$$\mathsf{Q}^\times = \langle -1 \rangle \times \mathop{\mathsf{X}}_{p} \langle p \rangle \cong Z(2) \times \mathop{\mathsf{X}}_{\aleph_0} Z$$

with p running over all primes [one should not forget that $\langle p \rangle$ is now the multiplicative group generated by p, and so it consists of all p^k ($k \in \mathsf{Z}$)].

For a prime p, the existence of primitive roots mod p implies that $\mathsf{F}_p^\times$ is a cyclic group:

$$\mathsf{F}_p^\times \cong Z(p-1).$$

2. *Finite algebraic extensions of prime fields.* In describing the structure of $K^\times$ for a finite algebraic extension K of Q, we shall make use of two standard results from algebraic number theory.

First, *the fundamental theorem of ideal theory* asserts that if R is the ring of algebraic integers in K, then every integral and fractional ideal $A \neq 0$ of R is equal to a uniquely determined product of prime ideals P_i, that is, $A = P_1^{k_1} \cdots P_r^{k_r}$, where $k_i \neq 0$ are integers. Consequently, the ideals $\neq 0$ form a free group under multiplication.

The other result is *Dirichlet's theorem on units*: the multiplicative group of units in R is the direct product of a finite cyclic group and $r_1 + r_2 - 1$ infinite cyclic groups. Here r_1 is the number of real and r_2 the number of pairs of complex conjugate roots of the defining equation of K over Q.

It is clear that two elements a, b of $K^\times$ generate the same (possibly fractional) ideal if and only if their quotient ab^{-1} is a unit in the ring R of algebraic integers in K. Hence the mapping $\phi\colon a \mapsto Ra$ is a multiplicative homomorphism of $K^\times$ into the group of ideals such that Ker ϕ is the group U of units of R. It follows that $K^\times/U$ is isomorphic to a subgroup of the multiplicative group of nonzero ideals of R, hence $K^\times/U$ itself is free, and it is readily checked [e.g., by looking up the rationals] that $K^\times/U$ is of countable rank. From (14.4) we obtain that $K^\times$ is isomorphic to the direct product of U and countably many Z, i.e., $K^\times \cong Z(m) \times \mathsf{X}_{\aleph_0} Z$. Here m is even because of $-1 \in K^\times$.

The situation is basically different in case the characteristic is a prime p. A finite algebraic extension K of F_p, of degree n, is a Galois field with p^n elements, hence $K^\times$ is cyclic of order $p^n - 1$. We have thus proved the first half of

Theorem 127.2 (Skolem [1]). *The multiplicative group of a finite algebraic extension* K *of a prime field has the form*

$$K^\times \cong Z(m) \times \mathop{\mathsf{X}}_{\aleph_0} Z \qquad (m \text{ is even}), \tag{1}$$

or

$$K^\times \cong Z(p^n - 1), \tag{2}$$

according as the characteristic of K *is* 0 *or a prime* p.

Conversely, every group of the form (1) *or* (2) *can be realized by some finite algebraic extension* K *of a prime field.*

To verify the second part, notice that by the existence of a Galois field of any prime power order, case (2) is settled at once. For (1), it suffices to observe that in the proof of (127.1) it has been shown that if ζ is a complex, primitive mth root of 1 for even m, then the torsion part of $Q(\zeta)$ is isomorphic to $Z(m)$.□

3. *Algebraically closed fields.* In an algebraically closed field A, for every $a \in \mathsf{A}$ and for every prime p, the equation $x^p - a = 0$ is solvable. Consequently, $\mathsf{A}^\times$ is divisible, and hence the direct product of groups isomorphic to Q and $Z(p^\infty)$. By (127.1), at most one $Z(p^\infty)$ can occur, for every prime p.

If A has zero characteristic, then every cyclotomic polynomial splits into linear factors over A; therefore, A contains, for every n, exactly n nth roots of unity. This means, $\mathsf{A}^\times$ contains Q/Z as a subgroup. If the characteristic of A is a prime p, then—as we noticed earlier—the p-component of $\mathsf{A}^\times$ is missing, but otherwise the same holds. The algebraic closure of any field of infinite cardinality $\mathfrak{n}$ being again of cardinality $\mathfrak{n}$, we are led to the following conclusion:

Theorem 127.3. *A group is isomorphic to the multiplicative group of an algebraically closed field if and only if it has the form*

$$Q/Z \times \mathop{\times}_{\mathfrak{n}} Q, \qquad \textit{or} \qquad \mathop{\times}_{p_i \neq p} Z(p_i^\infty) \times \mathop{\times}_{\mathfrak{n}} Q,$$

according as the characteristic is 0 *or a prime* p. *Here* $\mathfrak{n}$ *is any infinite cardinal or, in the second alternative only,* $\mathfrak{n} = 0$.□

4. *Real closed fields.* A real closed field B [i.e., a field that can be linearly ordered, but no proper algebraic extension admits a linear order] has characteristic 0. If ζ is a primitive nth root of unity in the algebraic closure A of B, then for $n \neq 2$,

$$1 + \zeta^2 + \cdots + \zeta^{2(n-1)} = \frac{\zeta^{2n} - 1}{\zeta^2 - 1} = 0, \qquad \zeta^2 + \cdots + \zeta^{2(n-1)} = -1,$$

and since -1 cannot be a sum of squares in a linearly ordered field, $\zeta \notin \mathsf{B}$. Consequently, ± 1 are the only roots of unity in B. From the theory of real closed fields it is well known that $\mathsf{A} = \mathsf{B}(\sqrt{-1})$ is a quadratic extension of B; hence all polynomials $x^p - a$ ($a \in \mathsf{B}$), with odd primes p, are reducible and must have irreducible factors of degree 1. In other words, extraction of pth roots for odd primes p is possible in B. As far as square roots are concerned, we know that B admits a linear order in which positive elements are complete squares, that is, for every $a \in \mathsf{B}$, either a or $-a$ has a square root in B. The following theorem summarizes the facts.

Theorem 127.4 (Fuchs [16]). *A group is isomorphic to the multiplicative group of a real closed field exactly if it is of the form*

$$Z(2) \times \mathop{\times}_{\mathfrak{n}} Q,$$

where $\mathfrak{n}$ *is an infinite cardinal.*□

5. *p-adic number fields*. Next we determine the multiplicative groups of the p-adic number fields. Since every p-adic number $\neq 0$ can be written uniquely as $p^k\pi$ with an integer k and a p-adic unit π, it is clear that the group is a direct product $Z \times U_p$, where U_p is the group of units in Q_p^*.

Consider a complete reduced residue system $t_1, \cdots, t_{p-1} \in Z \bmod p$. By Fermat's congruence, $t_j^{p^k - p^{k-1}} \equiv 1$, i.e., $t_j^{p^k} \equiv t_j^{p^{k-1}} \bmod p^k$ for every $k \geqq 1$, and therefore the sequence $t_j^{p^k}$ $(k = 0, 1, \cdots)$ converges to a p-adic unit ε_j. From $\varepsilon_j \equiv t_j^{p^k} \bmod p^{k+1}$, we obtain $\varepsilon_j^{p-1} \equiv (t_j^{p^k})^{p-1} \equiv 1 \bmod p^{k+1}$, thus $\varepsilon_j^{p-1} = 1$ for $j = 1, \cdots, p-1$, and $\varepsilon_1, \cdots, \varepsilon_{p-1}$ are distinct $p-1$st roots of unity. They form a subgroup E_p of U_p which must be cyclic by virtue of the existence of a primitive root mod p. Evidently, every $\pi \in U_p$ can be factored uniquely as $\pi = \varepsilon_j \rho$ for some $\varepsilon_j \in E_p$ and some $\rho \in U_p$ with $\rho \equiv 1 \bmod p$. The elements of U_p which are $\equiv 1 \bmod p$ form a subgroup V_p of U_p, and from what has been said it follows that $U_p = E_p \times V_p$.

To find the structure of V_p, we first consider odd primes p. Then there is a primitive root $t \bmod p$ which is a primitive root modulo every power of p. Setting $l = t^{p-1}$, $l^m \equiv 1 \bmod p^k$ will be equivalent to $p^{k-1} \mid m$. This shows that to every $\pi \in V_p$ and every p^k $(k \geqq 1)$ there exists a unique integer s_{k-1} such that $0 \leqq s_{k-1} < p^{k-1}$ and $l^{s_{k-1}} \equiv \pi \bmod p^k$. In view of $s_{k-1} \equiv s_k \bmod p^{k-1}$, the s_k converge to a p-adic integer σ such that $\sigma \equiv s_k \bmod p^k$, where $\sigma \equiv s_0 = 0 \bmod p$. Formally, we can write $\pi = l^\sigma$, and it is readily checked that $\pi_1 = l^{\sigma_1}$ and $\pi_2 = l^{\sigma_2}$ imply $\pi_1\pi_2 = l^{\sigma_1+\sigma_2}$. Conversely, given any p-adic integer $\sigma = r_1 p + r_2 p^2 + \cdots$ $(r_i \in Z)$ divisible by p, the sequence $l^0, l^{r_1 p}$, $l^{r_1 p + r_2 p^2}, \cdots$ converges to a p-adic integer π such that $\pi \equiv 1 \bmod p$ and $\pi = l^\sigma$. We can now assert that V_p is isomorphic to the additive group $pJ_p \cong J_p$, whence $U_p \cong Z(p-1) \times J_p$.

If $p = 2$, then $l = 5$ has the property that $l^m \equiv 1 \bmod 2^k$ is equivalent to $2^{k-2} \mid m$. We can establish a similar "logarithmic" correspondence as above between the elements of V_2 and the 2-adic integers divisible by 4. All in all we have:

Theorem 127.5 (K. Hensel). *The multiplicative group of the p-adic number field is isomorphic to the direct product*

$$Z \times Z(p-1) \times J_p. \square$$

EXERCISES

1. Show that the additive and multiplicative groups of a field can not be isomorphic.
2. A finite group is isomorphic to the multiplicative group of a field exactly if it is cyclic of order $p^n - 1$ for some prime p and integer $n \geqq 1$.

3. A finite group is the torsion part of $K^\times$ for some field K if and only if it is cyclic and its order is either even or of the form $2^n - 1$ for some integer $n \geqq 1$.
4. (a) If a subgroup of Q/Z with trivial 2-component is the torsion part of $L^\times$ for some field L, then it is isomorphic to $K^\times$ for some subfield K of L.
(b) A subgroup T of Q/Z with trivial 2-component is the multiplicative group of some field if and only if T is the union of an ascending chain of cyclic groups whose orders are of the form $2^n - 1$.
5. (Čarin [1]) A multiplicative group of algebraic numbers of degree $\leqq n$, with fixed n, is a direct product of cyclic groups. [*Hint*: subgroups with a fixed number of generators satisfy the maximum condition.]
6*. (Schenkman [1]) Let M be the algebraic field over Q, generated by all algebraic numbers of degree $\leqq n$, for fixed n. Then $M^\times$ is the direct product of cyclic groups. [*Hint*: M does not contain mth root of 1 for $m > 4n!$; use (19.1).]
7. If K is a field closed with respect to extracting roots, then its multiplicative group $K^\times$ is isomorphic to one of the groups in (127.3).
8. (a) For a field K of prime characteristic p to be perfect [i.e., irreducible polynomials over K have simple roots], it is necessary and sufficient that $K^\times$ be p-divisible.
(b) Prove in this way that algebraic extensions of F_p are perfect. [*Hint*: (127.2).]
9. If x is transcendental over the field K, then the multiplicative group of $K(x)$ is the direct product of $K^\times$ and as many as $\max(|K|, \aleph_0)$ copies of Z. [*Hint*: unique factorization in $K[x]$.]

128. UNITS OF COMMUTATIVE RINGS

In every associative ring R with identity 1, the elements which have (two-sided) multiplicative inverses with respect to 1, form a group under multiplication; we call this the *unit group of* R, and denote it by $U(R)$. Our intention is to investigate the groups $U(R)$ in the commutative case. Notice that if $U(R)$ is commutative, then the subring R′ of R generated by $U(R)$ is likewise commutative and $U(R') = U(R)$. For this reason, we restrict ourselves to associative and commutative rings with identity when studying commutative unit groups. [Such reduction is, however, not possible if only rings with a certain ring property are considered.]

Example 1. The units in $R = Z/(p^k)$, for a prime power p^k, are obviously the cosets prime to p. Hence $U(R) \cong Z(p^k - p^{k-1})$; any primitive root mod p^k can be chosen as a generator.

Example 2. The units of the ring Q_p^* of the p-adic integers have been described in the proof of (127.5). We conclude:

$$U(\mathsf{Q}_p^*) \cong Z(p-1) \times J_p.$$

Example 3. If R is the ring of integers in a finite algebraic extension of Q, then by Dirichlet's theorem on units, $U(\mathsf{R})$ is the direct product of a finite cyclic group and a finite number of infinite cyclic groups.

Before listing some useful facts about unit groups, we repeat: all occurring rings are associative, commutative and have an identity 1 even if these hypotheses are not stated explicitly.

(A) If S is a subring of R such that $1 \in \mathsf{S}$, then $U(\mathsf{S})$ is a subgroup of $U(\mathsf{R})$. Every identity-preserving ring-homomorphism $\mathsf{R} \to \mathsf{T}$ induces a group-homomorphism $U(\mathsf{R}) \to U(\mathsf{T})$.

(B) The unit group of the cartesian product $\prod \mathsf{R}_i$ of rings is the cartesian product of the unit groups $U(\mathsf{R}_i)$. In fact, $(\cdots, u_i, \cdots) \in \prod \mathsf{R}_i$ is a unit exactly if each u_i is a unit in R_i. In particular, for finite direct sums one has

$$U(\mathsf{R}_1 \oplus \cdots \oplus \mathsf{R}_n) = U(\mathsf{R}_1) \times \cdots \times U(\mathsf{R}_n).$$

(C) The polynomial ring $\mathsf{R}[\cdots, x_i, \cdots]$ over a commutative domain R in the commuting indeterminates x_i $(i \in I)$ satisfies

$$U(\mathsf{R}[\cdots, x_i, \cdots]) \cong U(\mathsf{R}).$$

In fact, only a constant polynomial can be a unit in a polynomial ring.

(D) In the formal power series ring $\mathsf{R}[[x]]$, the power series of the form $1 + c_1 x + \cdots + c_n x^n + \cdots$ $(c_n \in \mathsf{R})$ are units, and they form a subgroup E of $U(\mathsf{R}[[x]])$. It is a straightforward matter to verify that the units of $U(\mathsf{R}[[x]])$ with R a commutative domain are precisely the elements $u\rho$ with $u \in U(\mathsf{R})$ and $\rho \in E$, whence

$$U(\mathsf{R}[[x]]) \cong U(\mathsf{R}) \times E.$$

(E) Let R be a cummutative ring with 1 and S a multiplicative subsemigroup of R, consisting exclusively of nonzero-divisors. Let G be the group of quotients of S. For the ring R_s of quotients of R with respect to S we have

$$U(\mathsf{R}_s) = (U(\mathsf{R}) \times G)/H$$

where H is the subgroup of all (u, u^{-1}) with $u \in U(\mathsf{R}) \cap G$.

(F) Let A be a unital R-module with commutative R. Let $\mathsf{R}(A)$ consist of all pairs (r, a), where $r \in \mathsf{R}$, $a \in A$, with the following rules of operations:

$$(r, a) + (s, b) = (r + s, a + b) \qquad \text{and} \qquad (r, a)(s, b) = (rs, rb + sa)$$

for all $r, s \in \mathsf{R}$ and $a, b \in A$. This $\mathsf{R}(A)$ is a ring with the identity $(1, 0)$, where $r \in U(\mathsf{R})$ is necessary and sufficient for (r, a) to be a unit in $\mathsf{R}(A)$. From $(r, a) = (r, 0)(1, r^{-1}a)$ we conclude that $U(\mathsf{R}(A)) \cong U(\mathsf{R}) \times A$.

(G) Of special interest is the role the Jacobson radical J of R plays in connection with the unit group of R. For all $a \in \mathsf{J}$, $1 + a \in U(\mathsf{R})$, and if J is considered as a group under the "circle" composition $a \circ b = a + b + ab$, then the correspondence

$$\delta \colon a \mapsto 1 + a \in U(\mathsf{R}) \qquad (a \in \mathsf{J})$$

is an isomorphism of $(\mathsf{J}, \circ)$ with a subgroup of $U(\mathsf{R})$ which can be written as $1 + \mathsf{J}$. Moreover, δ induces an isomorphism of $(\mathsf{K}, \circ)$ with $1 + \mathsf{K}$ for every ideal K of R, contained in J. We can use this observation and proceed to prove:

(H) Let R be an associative and commutative ring with 1 and K an ideal of R, contained in the Jacobson radical J of R. Then:

(a) if the coset $a + \mathsf{K}$ contains a unit of R, then every element of $a + \mathsf{K}$ is a unit;

(b) the units of R/K are the cosets $u + \mathsf{K}$ for units $u \in \mathsf{R}$, such that

$$U(\mathsf{R}/\mathsf{K}) \cong U(\mathsf{R})/(1 + \mathsf{K}).$$

If u is a unit of R and $x \in \mathsf{K}$, then $u^{-1}x \in \mathsf{K}$ implies that $(1 + u^{-1}x)(1 + b) = 1$ for some $b \in \mathsf{R}$. Hence $(u + x)(u^{-1} + bu^{-1}) = 1$, $u + x \in u + \mathsf{K}$ is a unit in R, and (a) follows. To prove (b), it is clear that $u \mapsto u + \mathsf{K}$ is a homomorphism of $U(\mathsf{R})$ into $U(\mathsf{R}/\mathsf{K})$ whose kernel is $1 + \mathsf{K}$. It is epic, for if $a + \mathsf{K}$ is a unit in R/K, i.e., $(a + \mathsf{K})(b + \mathsf{K}) = 1 + \mathsf{K}$ for some $b \in \mathsf{R}$, then $ab \in 1 + \mathsf{K}$ implies by (a) that ab and hence a is a unit in R.

In the particular case when $\mathsf{K}^2 = 0$, the circle composition in K is the same as addition, and we infer that $U(\mathsf{R})$ contains a subgroup isomorphic to K^+.

EXERCISES

1. Let φ be a ring-homomorphism $\mathsf{R} \to \mathsf{S}$ such that $\varphi 1 = 1$. Then $\varphi \mid U(\mathsf{R})$ is a group-homomorphism $U(\mathsf{R}) \to U(\mathsf{S})$ whose kernel is $U(\mathsf{R}) \cap (1 + \operatorname{Ker} \varphi)$.
2. If $\{\mathsf{R}_i\ (i \in I);\ \pi_i^j\}$ is a direct system of rings R_i such that the π_i^j preserve the identity elements, then
$$U(\varinjlim \mathsf{R}_i) = \varinjlim U(\mathsf{R}_i).$$
3. The unit group of the quotient ring $\prod \mathsf{R}_i / \bigoplus \mathsf{R}_i$ is isomorphic to the cartesian product of the unit groups $U(\mathsf{R}_i)$ modulo their (restricted) direct product $\bigtimes_i U(\mathsf{R}_i)$.

4. A torsion ring R with 1 has but a finite number of p-components $R_{p_1}, \cdots, R_{p_n}$, and satisfies

$$U(R) = U(R_{p_1}) \times \cdots \times U(R_{p_n}).$$

5. Let R be a subring of an algebraic number field of finite degree over Q. Then $U(R)$ is a direct product of cyclic groups.
6. Extend (E) to the case when $0 \notin S$, but S has zero-divisors.
7. An ideal K of R satisfies (H) (a) if and only if it is contained in the Jacobson radical J of R.
8. Suppose J is the Jacobson radical of R and K, L ideals of R such that $J \supseteq K \supseteq L \supseteq K^2$. Then the additive group of K/L is isomorphic to the multiplicative group $(1 + K)/(1 + L)$ under $x + L \mapsto (1 + x)(1 + L)$.

129. GROUPS THAT ARE UNIT GROUPS

We shall proceed to the problem of describing the (abelian) groups that can be groups of units in some ring, and give an account of the main results which have been obtained on this problem. It is important to note that we do not make any assumption on the rings other than associativity, commutativity, and possession of 1.

It is easy to see that every group A [we continue using the multiplicative notation] can at least be embedded in the unit group of some ring; in fact, in the group ring ZA of A with integral coefficients, the basis elements $a \in A$ are manifestly units. The canonical embedding $\kappa\colon A \to U(ZA)$—where every $a \in A$ corresponds to itself—has notably the "universal" property: if R is any ring with 1 and if $\lambda\colon A \to U(R)$ is any homomorphism, then there exists an identity-preserving ring-homomorphism $\psi\colon ZA \to R$ such that $\psi\kappa = \lambda$. The commutative diagram below may be of assistance in visualizing the situation:

$$\begin{tikzcd} A \arrow[r, "\kappa"] \arrow[d, "\lambda"'] & ZA \arrow[dl, dashed, "\psi"] \\ R & \end{tikzcd}$$

For the proof it suffices to observe that there is one and only one way of extending λ to ZA, namely: $\psi(\sum n_i a_i) = \sum n_i \lambda(a_i)$. However, $U(ZA)$ is, in general, larger than A. As a matter of fact, there exist groups [e.g., the cyclic group of order 5] which can not be unit groups of any ring whatsoever.

We shall prove that certain types of groups are necessarily unit groups of some rings.

The following theorem is extremely simple, but it almost settles the entire problem.

Theorem 129.1. *For every group A, there is a ring whose unit group is isomorphic to $Z(2) \times A$.*

We glance back at **128**(F) and use our given A along with $\mathsf{R} = \mathsf{Z}$, to construct a desired ring.□

It is a rather delicate problem to get rid of the factor $Z(2)$. The difficulty lies, of course, in the fact that -1 is always an element of order 2 in $U(\mathsf{R})$ unless $-1 = 1$, i.e., $2\mathsf{R} = 0$.

To proceed, we shall require the following lemma, generalizing a result by Cohn [2].

Lemma 129.2. *Let U be the unit group of some commutative domain, and assume A is a group containing U such that A/U is torsion-free. Then A, too, is the unit group of some domain.*

Following the pattern of **49**, we select a representative $a_g \in A$ in each coset $g, h, \cdots \in A/U$ with 1 in the coset of U, and consider the corresponding factor set $\{u_{g,h}\} \subseteqq U$ as given by

$$(1) \qquad a_g a_h = u_{g,h} a_{gh} \qquad \text{for all} \quad g, h \in A/U.$$

Let R be a domain with $U(\mathsf{R}) = U$. Define S as the algebra over R with the basis $\{a_g\}_{g \in A/U}$ such that the basis elements multiply as prescribed by (1) where the $u_{g,h}$ are now regarded as belonging to the coefficient domain R. Since the $u_{g,h}$ come from A, they satisfy the associativity and commutativity conditions, and guarantee that S will, in fact, be an associative and commutative unital R-algebra with identity. Moreover, the multiples ua_g of the basis elements a_g (where $u \in U$) form a subgroup of $U(\mathsf{S})$ which is evidently isomorphic to A in a canonical way.

We need only show that S is a domain and $U(\mathsf{S})$ has no elements other than ua_g. Recall that every torsion-free abelian group, in particular, our A/U, admits a linear order such that $g \leqq h$ and $g' \leqq h'$ imply $gg' \leqq hh'$. Choose a fixed, but otherwise arbitrary linear order of A/U, and write the nonzero elements $\sigma \in \mathsf{S}$ in the form $\sigma = \sum_{i=1}^{m} s_i a_{g_i}$, with $s_i \neq 0$ in R and a_{g_i} in the above set of representatives, such that $g_1 < \cdots < g_m$. If $\tau = \sum_{j=1}^{n} t_j a_{h_j} \in \mathsf{S}$ with $t_j \neq 0$ in R and $h_1 < \cdots < h_n$, then obviously the basis element $a_{g_1 h_1}$ will have the smallest and $a_{g_m h_n}$ the largest index occurring in the product $\sigma\tau$, and their coefficients are $s_1 t_1 u_{g_1, h_1}$ and $s_m t_n u_{g_m, h_n}$, respectively. These coefficients are certainly different from zero, for R is a domain. It is thus clear that $\sigma\tau \neq 0$ unless $\sigma = 0$ or $\tau = 0$, and $\sigma\tau = 1$ can occur only if $g_1 h_1 = g_m h_n$, that is to say, only if $m = 1 = n$, and in addition, s_1, t_1 are units in R.□

We are now ready to establish our next result on unit groups.

Theorem 129.3. *Every torsion-free group is the unit group of some domain [of characteristic 2].*

The trivial group $\langle 1 \rangle$ is the unit group of F_2. The assertion follows at once from (129.2).□

Our earlier remarks indicate that unit groups without elements of order 2 are, in a sense, exceptional. This suggests the following question: what can be derived from the mere assumption that the 2-component of $U(\mathsf{R})$ is trivial? Some information can be gathered from our next results.

Proposition 129.4. *Let* R *be a commutative and associative ring with* 1. *The unit group* $U(\mathsf{R})$ *has trivial 2-component if and only if* R *is a subdirect sum of domains of characteristic* 2.

If $U(\mathsf{R})$ has trivial 2-component, then $-1 = 1$ and R is an algebra over F_2. If $a^2 = 0$ for some $a \in \mathsf{R}$, then $(1 + a)^2 = 1$ in R, thus $1 + a = 1$ and $a = 0$. Consequently, R has no nilpotent elements. By a well-known result in ring theory, R is a subdirect sum of domains which are now of characteristic 2.

Conversely, in any domain of characteristic 2, and hence in their subdirect sums too, 1 is the only square root of itself.□

We specialize our considerations, for a moment, to the case in which R—in addition to the absence of the 2-component of $U(\mathsf{R})$—is a domain. R can be viewed as an F_2-algebra and F_2 as a subring of R. The elements of R that are algebraic over F_2 form a subring K of R. This K must be a field, because subrings of absolute algebraic fields of prime characteristics are themselves fields. The group $U(\mathsf{K}) = \mathsf{K}^\times$ is torsion, and since roots of unity contained in R have to belong already to K, we deduce that $\mathsf{K}^\times$ is identical with the torsion part of $U(\mathsf{R})$.

Actually, our primary interest lies in this torsion group $\mathsf{K}^\times$, because with the aid of (129.2), every mixed group with torsion part $\mathsf{K}^\times$ can be realized as $U(\mathsf{R})$ for some, necessarily transcendental, ring-extension R of K.

Returning to the general situation, we drop the hypothesis of R being a domain. We wish to concentrate on the torsion part T of $U(\mathsf{R})$. First of all, we show that T is the unit group of some subring S of R. Representing R as a subdirect sum of domains R_i of characteristic 2, it is clear that $u = (\cdots, u_i, \cdots) \in \mathsf{R}$ satisfies $u^m = 1$ (for $m \in \mathsf{Z}$) exactly if $u_i^m = 1$ for all i. By virtue of the preceding paragraph, u_i is contained in a finite subfield S_i of R_i. Moreover, for some n, $|\mathsf{S}_i| \leqq 2^n$ may be chosen for all i, because an odd m divides $2^{\phi(m)} - 1$ with Euler's function ϕ and $u^{2^k - 1} = 1$ holds for all $u \neq 0$ in a Galois field of order 2^k. We can now claim that every element x of the subring generated by an arbitrary $u \in T$ satisfies an equation of the form $x^{2^k} = x$ with fixed k. From well-known properties of Galois fields it follows immediately that the same holds for subrings generated by finitely many elements of T.

Moreover,

$$S = \{x \in R \mid x^{2^k} = x \text{ for some } k = k(x)\}$$

is a subring of R such that $T \subset S$. It is readily seen that S is equal to the "algebraic closure" of F_2 in R, i.e., the set of all $x \in R$ which satisfy an algebraic equation over F_2. Since every unit $u \in R$ with $u^{2^k} = u$ must belong to T, we conclude that $U(S) = T$.

For the ring S introduced above, we can establish a structure theorem:

Theorem 129.5. *Let* R *be a commutative and associative ring with* 1 *such that the* 2*-component of the torsion part* T *of* $U(R)$ *is missing. Then the "algebraic elements" of* R *form a subring* S *such that* $U(S) = T$ *and* S *is a subdirect sum of absolute algebraic fields of characteristic* 2.

In view of the preceding considerations, the proof will be completed as soon as we notice that the projection of S on R_i must be an absolute algebraic field.□

EXERCISES

1. Prove that the canonical embedding $\kappa_A: A \to ZA$ is functorial, i.e., every group-homomorphism $\alpha: A \to B$ induces a ring-homomorphism $\bar{\alpha}: ZA \to ZB$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\kappa_A} & ZA \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}} \\ B & \xrightarrow{\kappa_B} & ZB \end{array}$$

commute.
2. If R is a domain and A is torsion free, then

$$U(RA) \cong U(R) \times A.$$

3. (a) If $U_1, \cdots, U_n$ are unit groups for some rings, then their direct product $U_1 \times \cdots \times U_n$ is likewise a unit group.
(b) A direct factor of a unit group need not be a unit group.
4. List all cyclic groups of prime orders which are unit groups. [*Hint*: 2 and $p = 2^n - 1$.]
5. (Cohn [2]) Give an example of a field whose multiplicative group is a nonsplitting mixed group.
6. Let R be a torsion-free domain such that $U(R)$ is a torsion group. Then $T \cong Z(2)$ or $Z(4)$.
7. If A is a group with an automorphism of order 4, then $Z(4) \times A$ is a unit group. [*Hint*: Gaussian integers and **128**(F).]

8. Let R be a 2-divisible torsion-free associative and commutative ring such that there are at least two units of order 2. Then R is a direct sum of two proper ideals. [*Hint*: $\frac{1}{2}(1 - \varepsilon)$ with $\varepsilon^2 = 1$, $\varepsilon \neq -1$.]

NOTES

Some of the results in this chapter are classical: L. Dirichlet's theorem on the groups of units in absolute algebraic number fields, K. Hensel's description of the unit group of the p-adic integers, and of course, the multiplicative groups of Galois fields. Surprisingly, Skolem's theorem [1] on the multiplicative groups of absolute algebraic number fields is not more than a quarter of a century old. It was just a simple exercise in abelian groups to characterize the multiplicative groups of algebraically closed and real closed fields [Fuchs [16]].

The big question is, naturally, to describe those groups which can be multiplicative groups of fields. The problem has been reformulated by Dicker [1] [without giving any real insight into the essence of the problem] in terms of the existence of a certain function on the group with 0 adjoined. It is a sad fact that no characterization can be given in terms of what an algebraist would regard satisfactory, namely by a set of sentences of the first-order language of group theory. In fact, S. R. Kogalowski [*Dokl. Akad. Nauk SSSR* **140** (1961), 1005–1007] has shown that the class of multiplicative groups of fields is not arithmetically closed in the sense of A. I. Malcev, and so not axiomatizable; cf. also Sabbagh [1]. In view of this, a recent result by May [3] sounds most satisfactory: Let A be a group whose torsion subgroup is a subgroup of Q/Z and has a nontrivial 2-component; then there exists a field K such that

$$\mathsf{K}^\times \cong A \times F,$$

where F is a free abelian group. The proof of this beautiful result goes beyond the scope of this book, so regretfully, it could not be included in this chapter.

The first systematic study of unit groups is due to G. Higman [*Proc. London Math. Soc.* **46** (1938–39), 231–248]; he investigated the groups of units of group rings over finite algebraic extensions of the integers. Rings with cyclic groups of units were described by R. W. Gilmer [*Amer. J. Math.* **85** (1963), 247–252] for finite rings and by K. R. Pearson and J. E. Schneider [*J. Algebra* **16** (1970), 243–251] for infinite rings.

There is an intensified interest in noncommutative groups of units. Of the numerous results, let us mention a simple-minded but very informative result by S. Z. Ditor [*Amer. Math. Monthly* **78** (1971), 722–723]: if G is a not necessarily commutative group of odd order which is the group of units of some ring R, then the subring of R generated by G is isomorphic to a finite direct sum of Galois fields of characteristic 2, and hence G is a finite direct product of cyclic groups of orders of the form $2^k - 1$. Recently, K. E. Eldridge has extended some of the results in **129** to the noncommutative case.

Problem 98. Study the change of groups of units under ring [field] extensions.

Problem 99. Describe the structure of the groups of units in power series rings.

Problem 100. Give conditions on a group to be the group of units in various important types of ring [regular, Noetherian etc.].

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TABLE OF NOTATIONS

$A, B, C, G, H, \cdots$	groups or subsets of groups
$I, J, K, \cdots$	index sets
$a, b, c, d, \cdots, x, y, \cdots$	elements of groups
$f, g, h, \cdots$	functions
$i, j, \cdots$	indices
$k, l, m, n, p, q, \cdots$	integers (p, q primes)
$\alpha, \beta, \gamma, \eta, \phi, \psi, \cdots$	maps, homomorphisms
$\rho, \sigma, \tau, \cdots$	ordinal numbers
$\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \cdots$	cardinal numbers
$\mathbf{a}, \mathbf{v}, \mathbf{w}, \cdots$	vectors
$\mathbf{t}, \mathbf{s}, \cdots$	types of torsion-free groups
$\mathsf{R}, \mathsf{S}, \mathsf{K}, \mathsf{L}, \cdots$	rings, ideals (K a field)
$\mathbf{I}, \mathbf{K}, \mathbf{D}, \cdots$	ideals or dual ideals in a lattice
$\mathbb{B}, \mathbb{C}, \mathbb{H}, \mathbb{M}, \cdots$	matrices
$\Gamma, \Delta, \Xi, \Sigma, \cdots$	noncommutative groups

SET THEORY

$\in$	is a member of
$\subseteq, \subset$	is contained, properly contained in
$\cup, \cap$	set union, intersection
$\setminus$	difference set
$\varnothing$	empty set
$\times$	cartesian product
$\{a \in A \mid \cdots\}$	the set of all $a \in A$ with $\cdots$
$\{a_i\}_{i \in I}$	the set of all a_i with $i \in I$

$\lvert A\rvert$	the cardinality of A
$\aleph_\sigma$	the σth infinite cardinal
ω_σ ($\omega_0 = \omega$)	the smallest ordinal of cardinality $\aleph_\sigma$

MAPS

$\mapsto$	correspondence
$\to$	mapping between sets or classes
$\alpha \mid A$	restriction of α to A
1_A	the identity map of A
Ker α	the kernel of the map α
Im α	the image of α
$\oplus\alpha_i$, $\prod\alpha_i$	direct sum, direct product of maps α_i
Δ, ∇	diagonal, codiagonal map

GROUP THEORY

$o(a)$	order of a
$e(a)$	exponent of a
$h_p(a)$, $h(a)$, $h^*(a)$	height, generalized height of a
$H(a)$	indicator (Ulm sequence) of a
$\chi(a)$	characteristic (height-sequence) of a
$\mathbf{t}(a)$, $\mathbf{t}(R)$	type of a (of a subgroup R of Q)
$\mathbb{H}(a)$	height-matrix of a
$n \mid a$	n divides a
$\leqq$, $<$	is a subgroup, proper subgroup of
$\prec$	is quasi-contained in
$\langle\cdots\rangle$, $\langle\cdots\rangle_*$	subgroup, pure subgroup generated by $\cdots$
$\lvert A : B\rvert$	index of B in A
A/B	quotient group
$A + B$, $\sum B_i$	subgroup generated by B and C, by the B_i
$\oplus$, $\bigoplus_{\mathfrak{m}}$	direct sum, direct sum of $\mathfrak{m}$ copies of
$\hat{\oplus}$	admissible quasi-direct sum
Π, $\prod_{\mathfrak{m}}$	direct product, direct product of $\mathfrak{m}$ copies of
$\oplus_{\mathbf{K}}$	K-direct sum
$\cong$	isomorphism
$\approx$, $\sim$	quasi-equality, quasi-isomorphism
nA	the set of all na with $a \in A$
$A[n]$	the set of all $a \in A$ with $na = 0$
$n^{-1}B$	the set of all $a \in A$ with $na \in B$
$T(A)$	torsion part of A
A^σ, A_σ	the σth Ulm subgroup, σth Ulm factor of A
$p^\sigma A$	certain subgroups of A formed by using multiplication by p and intersections
$A(\mathbf{u})$	fully invariant subgroup associated with $\mathbf{u} = (\sigma_0, \ldots, \sigma_n, \ldots)$
$A(\mathbf{t})$, $A^*(\mathbf{t})$	fully invariant subgroups associated with the type $\mathbf{t}$

$f_\sigma(A), f_\sigma(A, G)$	the σth Ulm—Kaplansky invariant of A (relative to G)
$r(A), r_0(A), r_p(A)$	the ranks of A
fin $r(A)$	final rank of A
$\hat{A}$	completion of A
$A^\bullet$	cotorsion completion of A
$\bar{A}$	torsion-completion of A
S^-	topological closure of subgroup S
R^+	additive group of ring R
$\mathsf{K}^\times$	multiplicative group of field K
$U(\mathsf{R})$	group of units of ring R
$\mathfrak{c}\{\cdots\}, \mathfrak{z}\{\cdots\}$	centralizer, center of $\cdots$

PARTICULAR GROUPS, RINGS

Z	group of integers, infinite cyclic group
$Z(m)$	cyclic group of order m
$Z(p^\infty)$	quasicyclic group
H_σ	Prüfer group of length σ
Q	group of rationals
Q_p	group of rationals with denominators prime to p
$Q^{(p)}$	group of rationals with denominators powers of p
J_p	group of p-adic integers
Z	ring of integers
Q	field of rationals
Q_p	ring of rationals with denominators prime to p
Q_p^*	ring of p-adic integers
K_p	field of p-adic numbers
$\mathsf{E}(A), \tilde{\mathsf{E}}(A)$	endomorphism, quasi-endomorphism ring of A
$\mathsf{Z}A$	group ring of A over Z
$\varinjlim, \varprojlim$	direct, inverse limit
Hom	group of homomorphisms
Ext, Pext	group of extensions, pure extensions
$\otimes$	tensor product
Tor	torsion product
Aut	automorphism group
Mult	group of multiplications

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