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Uniform Spaces

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PREFACE

The subject matter of this book might be labelled fairly accurately *Intrinsic geometry of uniform spaces.* {For an impatient reader, this means elements (25%), dimension theory (40%), function spaces ($12\frac{1}{2}$%), and special topics in topology.} As the term "geometry" suggests, we shall not be concerned with applications to functional analysis and topological algebra. However, applications to topology and specializations to metric spaces are of central concern; in fact, these are the two pillars on which the general theory stands. This dictum brings up a second exclusion: the book is not much concerned with restatements of the basic definitions or generalizations of the fundamental concepts. These exclusions are matters of principle. A third exclusion is dictated mainly by the ignorance of the author, excused perhaps by the poverty of the literature, and at any rate violated in several places in the book: this is extrinsic (combinatorial and differential) geometry or topology.

More than 80% of the material is taken from published papers. The purpose of the notes and bibliography is not to itemize sources but to guide further reading, especially in connection with the exercises; so the following historical sketch serves also as the principal acknowledgement of sources.

The theory of uniform spaces was created in 1936 by Weil [**W**]. All the basic results, especially the existence of sufficiently many pseudometrics, are in Weil's monograph. However, Weil's original axiomatization is not at all convenient, and was soon succeeded by two other versions: the orthodox (Bourbaki [**Bo**]) and the heretical (Tukey [**T**]). The present author is a notorious heretic, and here advances the claim that in this book each system is used where it is most convenient, with the result that Tukey's system of uniform coverings is used nine-tenths of the time.

In the 1940's nothing of interest happened in uniform spaces. But three interesting things happened. Dieudonné [1] invented paracompactness and crystallized certain important metric methods in general topology, mainly the partition of unity. Stone [1] showed that all metrizable spaces are paracompact, and in doing so, established two important covering theorems whose effects are still spreading through uniform geometry. Working in another area, Eilenberg and MacLane defined the notions of category, functor, and naturality, and pointed out that their spirit is the spirit of Klein's Erlanger Programm and their reach is greater.

The organization of this book is largely assisted by a rudimentary version of the Klein-Eilenberg-MacLane program (outlined in a foreword to this book). We are interested in the single category of uniform spaces, two or three of its subcategories, and a handful of functors; but to consider them as instances of more general notions gives us a platform to stand on that is often welcome.

In 1952 Shirota [1] established the first deep theorem in uniform spaces, depending on theorems of Stone [1] and Ulam [1]. Except for reservations involving the axioms of set theory, the theorem is that every topological space admitting a complete uniformity is a closed subspace of a product of real lines. A more influential step was taken in 1952 by Efremovič [1] in creating proximity spaces. This initiated numerous significant Soviet contributions to uniform and proximity geometry (which are different but coincide in the all-important metric case), central among which is Smirnov's creation of uniform dimension theory (1956; Smirnov [4]). The methods of dimension theory for uniform and uniformizable spaces are of course mainly taken over from the classical dimension theory epitomized in the 1941 book of Hurewicz and Wallman [HW]. Classical methods were pushed a long way in our direction (1942-1955) by at least two authors not interested in uniform spaces: Lefschetz [L], Dowker [1; 2; 4; 5]. These methods—infinite coverings, sequential constructions—were brought into uniform spaces mainly by Isbell [1; 2; 3; 4] (from 1955).

Other developments in our subject in the 1950's do not really fall into a coherent pattern. What has been described above corresponds to Chapters I, II, IV and V of the book. Chapter III treats function spaces. The material is largely classical, with additions on injectivity and functorial questions from Isbell [5], and some new results of the same sort. The main results of Chapters VI (compactifications) and VIII (topological dimension theory) are no more recent than 1952 (the theorem Ind = dim of Katětov [2]).

The subject in Chapters VII and VIII is special features of fine spaces, i.e., spaces having the finest uniformity compatible with the topology. Chapter VII is as systematic a treatment of this topic as our present ignorance permits. Central results are Shirota's theorem (already mentioned) and Glicksberg's [2] 1959 theorem which determines in almost satisfactory terms when a product of fine spaces is fine. There is a connecting thread, a functor invented by Ginsburg—Isbell [1] to clarify Shirota's theorem, which serves at least to make the material look more like uniform geometry rather than plain topology. There are several new results in the chapter (VII. 1-2, 23, 25, 27-29, 32-35, 39); and a hitherto unpublished result of A. M. Gleason appears here for the first time. Gleason's theorem (VII. 19) extends previous results due mainly to Marczewski [1; 2] and Bokštein [1]. He communicated it to me after I had completed a draft of this book including the Marczewski and Bokštein theorems; I am grateful for his permission to use it in place of them.

Most chapters are followed by exercises adding details to the theory (in some cases doing duty for proofs omitted in the text), starred exercises whose results will be used later in the text, occasional unsolved problems, and a major unsolved problem. The major unsolved problems are Problems A, B_1, B_2, B_3, C, D. Not all are precisely posed, but all describe areas in which there seems to be good reason to expect interesting results although the results now known are quite unsatisfactory. The appendix might reasonably be counted as another such problem, for it gives several characterizations of the line and draws attention to the plane.

A preliminary version of this book was prepared as a set of lecture notes at Purdue University in 1960. The work of writing it has been supported at Purdue by the Office of Naval Research and at the University of Washington by the National Science Foundation. I am indebted to Professors M. Henriksen and M. Jerison for helpful criticisms of the Purdue lecture notes. Professors E. Alfsen, H. Corson, J. de Groot, E. Hewitt, E. Michael, D. Scott, and J. Segal have contributed some criticisms and suggestions during the writing of the final version. Many blemishes surviving that far were caught and exposed by Professor P. E. Conner for the Editorial Committee. But none of my distinguished colleagues has assumed responsibility for the remaining errors, which are mine.

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FOREWORD

Categories. A *concrete category* $\mathscr{C}$ is defined by defining a class $\mathfrak{O}$ of sets, called *objects* of $\mathscr{C}$, and for each ordered pair of objects (X, Y) a set $\mathrm{Map}(X, Y)$ of functions $f: X \to Y$, called *mappings*, such that

(a) The identity function on each object is a mapping;

(b) Every function which is a composition of mappings is a mapping.

The analysis of this definition presents some peculiar difficulties, because the class $\mathfrak{O}$ may be larger than any cardinal number, and is larger in all cases arising in this book. But the difficulties need not concern us here. All we need is an indication of what is superfluous in the definition, i.e., of when two concrete categories determine the same abstract category.

A *covariant functor* F: $\mathscr{C} \to \mathscr{D}$ is given when we are given two functions, F_0 and F_1, as follows. F_0 assigns to each object X of $\mathscr{C}$ an object $F_0(X)$ of $\mathscr{D}$. F_1 assigns to each mapping f: $X \to Y$ of $\mathscr{C}$ a mapping $F_1(f)$: $F_0(X) \to F_0(Y)$ of $\mathscr{D}$. Further,

(A) For each identity mapping 1_X of $\mathscr{C}$, $F_1(1_X) = 1_{F_0(X)}$;

(B) For every composed mapping gf of $\mathscr{C}$, $F_1(gf) = F_1(g)F_1(f)$.

Having noted the distinction between F_0 and F_1, we can ignore it for applications, writing $F_0(X)$ as $F(X)$, $F_1(f)$ as $F(f)$. The short notation defines a *composition* of covariant functors F: $\mathscr{C} \to \mathscr{D}$, G: $\mathscr{D} \to \mathscr{E}$ for us: $GF(X) = G(F(X))$, $GF(f) = G(F(f))$. Then an *isomorphism* is a covariant functor F: $\mathscr{C} \to \mathscr{D}$ for which there exists a covariant functor F^{-1}: $\mathscr{D} \to \mathscr{C}$ such that both FF^{-1} and $F^{-1}F$ are identity functors.

A *categorical property* is a predicate of categories $\mathfrak{P}$ such that if $\mathfrak{P}(\mathscr{C})$ and $\mathscr{C}$ is isomorphic with $\mathscr{D}$ then $\mathfrak{P}(\mathscr{D})$. Similarly we speak of categorical definitions, ideas, and so on.

The notion of a mapping f: $X \to Y$ having an inverse f^{-1}: $Y \to X$ is categorical. The defining conditions are just $ff^{-1} = 1_Y$, $f^{-1}f = 1_X$. A mapping having an inverse in $\mathscr{C}$ is called an *isomorphism* in $\mathscr{C}$.

The notion of a mapping f: $X \to Y$ being one-to-one is not categorical. However, it has an important categorical consequence. If f: $X \to Y$ is one-to-one then for any two mappings d: $W \to X$, e: $W \to X$, $fd = fe$ implies $d = e$. A mapping having this left cancellation property is called a *monomorphism*. Similarly a mapping f such that $gf = hf$ implies $g = h$ is called an *epimorphism*. A mapping $f: X \to X$ satisfying $ff = f$ is a *retraction*.

REMARK. If a retraction is either a monomorphism or an epimorphism then it is an identity.

A *contravariant functor* $F: \mathscr{C} \to \mathscr{D}$ assigns to each object X of $\mathscr{C}$ an object

$F(X)$ of $\mathscr{D}$ or to each mapping $f: X \to Y$ of $\mathscr{C}$ a mapping in the opposite direction $F(f): F(Y) \to F(X)$ of $\mathscr{D}$ satisfying condition (A) above and

(B*) For every composed mapping gf of $\mathscr{C}$, $F(gf) = F(f)F(g)$. Contravariant functors can be composed; but the composition of two contravariant functors is a covariant functor. In fact, functors of mixed variances can be composed, with an obvious rule for the variance of the composition.

A *duality* $F: \mathscr{C} \to \mathscr{D}$ is a contravariant functor admitting an inverse, which is a contravariant functor $F^{-1}: \mathscr{D} \to \mathscr{C}$ such that both FF^{-1} and $F^{-1}F$ are identity functors. It is a theorem that:

Every concrete category is the domain of a duality.

An interested reader may prove this, letting $F(X)$ be the set of all subsets of X and $F(f) = f^{-1}$.

The *principle of duality* says roughly that any categorical theorem θ for arbitrary categories implies another theorem θ^* for arbitrary categories. For example, if θ is a theorem about a single category $\mathscr{C}$, the statement of θ for $\mathscr{C}$ is equivalent to a statement about a category $\mathscr{D}$ related to $\mathscr{C}$ by a duality $F: \mathscr{C} \to \mathscr{D}$; that statement about D is θ^*, and it is true for arbitrary categories because every category is the range of a duality.

The theorem θ that a retraction f which is a monomorphism is an identity can illustrate duality. The statement θ^* is that if f is a mapping in $\mathscr{C}$, $F: \mathscr{C} \to \mathscr{D}$ is a duality, and $F(f)$ is a retraction and a monomorphism, then $F(f)$ is an identity. Using several translation lemmas we can simplify θ^* to the equivalent form: if f is a retraction and an epimorphism then f is an identity.

Note that we may have "dual problems" which are not equivalent to each other. A typical problem in a category $\mathscr{C}$ is, does $\mathscr{C}$ have the property $\mathfrak{P}$? If $\mathfrak{P}$ is categorical, there is a *dual property* $\mathfrak{P}^*$, and the given problem is equivalent to the problem does a category dual to $\mathscr{C}$ have the property $\mathfrak{P}^*$? By the *dual problem* we mean: Does $\mathscr{C}$ have the property $\mathfrak{P}^*$?

Finally, we need definitions of *subcategory, full subcategory,* and *functor of several variables.* A subcategory $\mathscr{D}$ of $\mathscr{C}$ is a category such that every object of $\mathscr{D}$ is an object of $\mathscr{C}$ and every mapping of $\mathscr{D}$ is a mapping of $\mathscr{C}$. $\mathscr{D}$ is a full subcategory if further, every mapping of $\mathscr{C}$ whose domain and range are objects of $\mathscr{D}$ is a mapping of $\mathscr{D}$.

For several variables we want the notion of a *product* of finitely many categories $\mathscr{C}_1, \cdots, \mathscr{C}_n$. The product is a category whose objects may be described as n-tuples $(X_1, \cdots, X_n)$, each X_i an object of $\mathscr{C}_i$. To represent these as sets the union $X_1 \cup \cdots \cup X_n$ would serve, if some care is taken about disjointness. The set $\mathrm{Map}((X_1, \cdots, X_n), (Y_1, \cdots, Y_n))$ is the product set $\mathrm{Map}(X_1, Y_1)$

$\times \cdots \times$ Map (X_n, Y_n). Again, the mappings $(f_1, \cdots, f_n)$ can be represented as functions on $X_1 \cup \cdots \cup X_n$, with $(f_1, \cdots, f_n) | X_i = f_i$.

A *pure covariant functor* on $\mathscr{C}_1, \cdots, \mathscr{C}_n$ is a covariant functor defined on the product $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$. A functor on $\mathscr{C}_1, \cdots, \mathscr{C}_n$, *covariant in* the set I of indices and *contravariant in* the remaining indices, is a function F on $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ to a category $\mathscr{D}$, taking objects to objects, mappings to mappings, identities to identities; taking mappings $f = \{f_i\} : \{X_i\} \rightarrow \{Y_i\}$ to mappings $F(f) : F(\{Z_i\}) \rightarrow F(\{W_i\})$, where for $i \in I$, $Z_i = X_i$ and $W_i = Y_i$, but for $i \notin I$, $Z_i = Y_i$ and $W_i = X_i$; and preserving the composition operation defined in the product category by $g \circ f = h$, where $h_i = g_i f_i$ for $i \in I$, $h_i = f_i g_i$ otherwise.

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CHAPTER I

FUNDAMENTAL CONCEPTS

Some of the notions of the theory of uniform spaces are familiar from metric spaces. It is possible to found the entire theory on the notion of a set with a family of pseudometrics called a gage. We shall not do this here, but we shall lead in gradually from metric notions to the idea which will be the foundation of the development: the uniform coverings.

Metric uniform spaces. Recall that a *pseudometric* d on a set X is a real-valued function on $X \times X$ satisfying $d(x, y) = d(y, x) \geqq 0$ and $d(x, z) + d(z, y) \geqq d(x, y)$, for all x, y, z in X. It is called a *metric* if it separates points, i.e., $x \neq y$ implies $d(x, y) > 0$. A metric space consists of a set X with a metric d on X. Commonly we refer to "the metric space X" and use the letter d freely for the distance in any metric space.

Recall that a function $f: X \rightarrow Y$, where X and Y are metric spaces, is called *uniformly continuous* if for each $\epsilon > 0$ there is $\delta > 0$ such that whenever $d(x, x') < \delta$ in X, $d(f(x), f(x')) < \epsilon$. Every uniformly continuous function is continuous, but the converse is not true.

A covering $\mathscr{U}$ of X is called a *uniform covering* provided there is a positive number ϵ such that every subset of X of diameter less than ϵ is a subset of some element of $\mathscr{U}$. Such an ϵ is called a *Lebesgue number* for $\mathscr{U}$.

1. *A covering $\mathscr{U}$ of a metric space X is uniform if and only if there is $\delta > 0$ such that for each point x in X, the δ-neighborhood of x is contained in some element of $\mathscr{U}$.*

PROOF. If $\mathscr{U}$ is uniform with Lebesgue number ϵ, then every $(\epsilon/2)$-neighborhood of a point is contained in an element of $\mathscr{U}$. Conversely, if elements of $\mathscr{U}$ contain all δ-neighborhoods then δ is a Lebesgue number for $\mathscr{U}$.

2. *A function $f: X \rightarrow Y$ is uniformly continuous if and only if for every uniform covering $\mathscr{V}$ of Y there is a uniform covering $\mathscr{U}$ of X such that, for each element U of $\mathscr{U}$, $f(U)$ is contained in some element of $\mathscr{V}$.*

PROOF. Suppose f is uniformly continuous and $\mathscr{V}$ is a covering of Y having Lebesgue number ϵ. If δ is such that any two points of X at distance less than δ have images at distance less than ϵ, then the collection of $(\delta/2)$-neighborhoods of all points of X is the required uniform covering $\mathscr{U}$.

Conversely, suppose the condition on coverings is satisfied and $\epsilon > 0$. Let $\mathscr{V}$ be the covering of Y consisting of all sets of diameter $< \epsilon$, and $\mathscr{U}$ a uniform covering of X each element of which has its image contained in a single element of $\mathscr{V}$. If δ is a Lebesgue number for $\mathscr{V}$, then any two points of X at distance $< \delta$ have images at distance $< \epsilon$.

A uniformly continuous function $f: X \to Y$ is called a *uniform equivalence* if f is one-to-one and onto and the inverse function $f^{-1}: Y \to X$ is also uniformly continuous.

3. *If $f: X \to Y$ is a uniform equivalence, then a collection $\{U_\alpha\}$ of subsets of X is a uniform covering if and only if the collection $\{f(U_\alpha)\}$ is a uniform covering of Y. The converse is also true.*

The proof is left as an exercise.

Finally, we may define a *metric uniform space* as a set X together with a family μ of coverings of X such that for at least one distance function d on X, μ is precisely the family of all uniform coverings of the metric space (X, d). The preceding remarks and results show that every metric space determines uniquely a metric uniform space; that two different distance functions d, e, on the same set X, determine the same uniform space if and only if the identity mapping is a uniform equivalence between (X, d) and (X, e); and further, if we are given the metric uniform spaces (X, μ) and (Y, ν) and a function $f: X \to Y$, we can determine whether f is uniformly continuous by I.2, without knowing or constructing any specific distance functions.

This summary treatment of metric uniform spaces will not be used in developing the general theory. It is included in justification; we propose to define such terms as "uniformly continuous" and "completion", and we ought to show that the notions are true generalizations of the familiar notions for metric spaces. The remaining details in this showing will be swept into exercises or omitted.

Another important point is that, while we lose the distance function in passing from metric space to metric uniform space, we do not lose the topology. For example,

4. *For metric spaces X and Y, a function f: $X \to Y$ is continuous if and only if for each point x in X, for each uniform covering $\mathscr{V}$ of Y, there exist an element V of $\mathscr{V}$ and a uniform covering $\mathscr{U}$ of X such that for every element U of $\mathscr{U}$ which contains x, $f(U) \subset V$.*

Another important point: the metric uniform space has *more* structure than the topological space. There may be two homeomorphic metric uniform spaces which are not uniformly equivalent; and more.

5. *There is an uncountable family of countable discrete metric spaces, no two of which are uniformly equivalent to each other.*

PROOF. From topology we know that there is an uncountable family of compact subspaces of the plane, no two of which are homeomorphic with each other.

(Recall the construction. For any increasing sequence of positive integers, $n_1 < n_2 < \cdots$, begin with the segment from $(0,0)$ to $(0,1)$ in the plane and at each point $(0, 2^{-i})$ attach n_i short whiskers.)

Let $\{C_\alpha\}$ be such a family of spaces in the plane. In each C_α select a countable dense subset $\{p_n^\alpha\}$. Let the coordinates of p_n^α be (x_n^α, y_n^α). Let X_α be the subset of three-space consisting of all points $(x_n^\alpha, y_n^\alpha, 1/m)$, with $m \geqq n$. Then X_α is a countable metric space. It is discrete since each of its points is above the horizontal coordinate plane and for each $\epsilon > 0$ there are only finitely many points of X_α with third coordinate greater than ϵ. Of course, X_α is not closed in three-space; its closure Y_α consists of X_α and a copy of C_α. Moreover, Y_α is the completion of X_α. Now suppose $f: X_\alpha \to X_\beta$ is a uniform equivalence. In particular, f maps X_α uniformly continuously into the complete space Y_β. Hence f has a unique uniformly continuous extension $g: Y_\alpha \to Y_\beta$. The same reasoning applies to $f^{-1}: X_\beta \to X_\alpha$, which must have a uniformly continuous extension $h: Y_\beta \to Y_\alpha$.

Finally, consider the composed mapping hg of Y_α into itself. On the dense set X_α, hg coincides with the identity mapping i. But for any two continuous mappings $p: A \to B, q: A \to B$, where A and B are Hausdorff spaces, the set of all a in A such that $p(a) = q(a)$ is a closed set. Hence $hg: Y_\alpha \to Y_\alpha$ is the identity. Similarly $gh: Y_\beta \to Y_\beta$ is the identity. But then g maps C_α homeomorphically onto C_β, which is absurd. It follows that X_α and X_β cannot be uniformly equivalent.

This result illustrates a useful rule of thumb in the theory of uniform spaces: *All counterexamples are discrete.*

Uniformities and preuniformities. We shall need a little of the terminology of the theory of quasi-ordered sets. Moreover, it will be convenient to use slightly nonstandard terminology; so every reader should note carefully the following definitions.

A set S is said to be *quasi-ordered* by a relation $<$ if $<$ is transitive. A subset Q of S is *cofinal* in S if for each element s of S there exists an element q of Q such that $q < s$. Q is *residual* in S if whenever $q \in Q$ and $r < q$ in S, $r \in Q$. Q is *antiresidual* if for $q \in Q$ and $r > q$ in S, r is in Q. The quasi-ordered set S is *directed* if for any p and q in S there exists s in S satisfying $s < p$ and $s < q$. Such an s may be called a *common successor* of p and q.

One important example of a quasi-ordered set is any family S of subsets of a given set X, ordered by inclusion $\subset$. In this case a common successor of p and q is a subset of their intersection.

For another example consider coverings of X. A covering $\{U_\alpha\}$ is called a *refinement* of a covering $\{V_\beta\}$ if each U_α is a subset of at least one V_β. We write $\{U_\alpha\} < \{V_\beta\}$, and note that $<$ is a quasi-ordering.

Inclusion is actually a partial ordering, i.e., besides being transitive it is reflexive and anti-symmetric. Note that refinement is also reflexive, but not anti-symmetric. Two coverings $\mathscr{U}$, $\mathscr{V}$, may be equivalent in the sense that $\mathscr{U} < \mathscr{V}$ and $\mathscr{V} < \mathscr{U}$.

Any two coverings $\{U_\alpha\}$, $\{V_\beta\}$, of a set have a coarsest common refinement, the covering $\{U_\alpha \cap V_\beta\}$. It may be denoted by $\{U_\alpha\} \wedge \{V_\beta\}$. Of course, there are usually other coverings equivalent to this one.

If $\mathscr{U}$ is a covering of X and A is a subset of X, the *star* $\mathrm{St}(A,\mathscr{U})$ of A with respect to $\mathscr{U}$ is the union of all elements of $\mathscr{U}$ which have a nonempty intersection with A. The collection $\{\mathrm{St}(U,\mathscr{U}) : U \in \mathscr{U}\}$ is a covering and is called $\mathscr{U}^*$, the *star* of $\mathscr{U}$. If $\mathscr{U}^*$ is a refinement of $\mathscr{V}$, $\mathscr{U}$ is called a *star-refinement* of $\mathscr{V}$, and one writes $\mathscr{U} <^* \mathscr{V}$. The relation $<^*$ is again a quasi-ordering, generally not reflexive.

In any quasi-ordered set, a *filter* is a directed antiresidual subset. A *filter base* is a cofinal subset of a filter, i.e., a directed set. In a partially ordered family of sets, ordered by inclusion, a *proper filter* is a filter which does not have the empty set as an element.

Now we come to the main definitions. A *preuniformity* μ on a set X is a family of coverings of X which forms a filter with respect to $<^*$. A *uniformity* μ on X is a preuniformity such that for any two points, x, y, of X, there is a covering $\mathscr{U}$ in μ, no element of which contains both x and y. A *uniform space* μX is a set X with a uniformity μ on X. The elements of μ are called *uniform coverings*.

6. *A family μ of coverings is a preuniformity if and only if* (i) *for $\mathscr{U}$ and $\mathscr{V}$ in μ, $\mathscr{U} \wedge \mathscr{V}$ is in μ*; (ii) *for $\mathscr{U} < \mathscr{V}$ and $\mathscr{U}$ in μ, $\mathscr{V}$ is in μ*; *and* (iii) *every element of μ has a star-refinement in μ.*

7. *For any two points, x, y, of a uniform space, there is a uniform covering $\mathscr{U}$ such that* $\mathrm{St}(x,\mathscr{U})$ *and* $\mathrm{St}(y,\mathscr{U})$ *are disjoint.*

In some of the literature, what we call a uniform space is called a separated uniform space, and a set with any preuniformity on it is called a uniform space. We shall use the term "separate" mainly in the following (customary) sense: a family $\{f_\alpha\}$ of functions with the same domain X but possibly different ranges separates points provided $x \neq y$ in X implies that for some $\alpha, f_\alpha(x) \neq f_\alpha(y)$. Other uses of the word are introduced in places, particularly in Chapter VI.

The uniformities or preuniformities on any set form a partially ordered set under inclusion. The preuniformities, like the topologies, form a complete lattice. (This is a corollary of Proposition 9 below.) Evidently, any preuniformity which contains a uniformity is a uniformity. As with topologies, a preuniformity μ containing a preuniformity ν is said to be *finer* than ν. The usage of the terms "strong" and "weak" is not standardized, and we shall avoid them as far as possible.

The weak uniformity induced by a family of functions is too useful to be avoided, and fortunately there has been little or no terminological confusion here. We have

8. THEOREM. *For any family $\{f_\alpha\}$ of functions on a set X into various uniform spaces, there is a coarsest preuniformity on X including all the inverse images of uniform coverings under these functions. If the functions separate points, then this preuniformity is a uniformity.*

This uniformity is the *weak uniformity* induced by the family $\{f_\alpha\}$. The proof of Theorem 8 is not difficult, but we shall take some time marking out important ideas in it.

A *basis* for a uniformity μ is a filter base for μ considered as a filter of coverings; and similarly for a preuniformity. A *sub-basis* for a uniformity or preuniformity is a family of coverings whose finite intersections form a basis. Now a family ν of coverings which satisfies condition (iii) of Proposition 6, every covering in ν has a starrefinement in ν, is called a *normal family*. It is convenient and customary to use the term *normal sequence* for something more special than a sequence which is a normal family: specifically, for a sequence of coverings $\mathscr{U}^n$ such that $\mathscr{U}^{n+1} <^* \mathscr{U}^n$ for each n.

9. *Every normal family of coverings is a sub-basis for a preuniformity.*

PROOF. The required preuniformity is the family of all coverings which can be refined by finite intersections of coverings from the given family, which automatically satisfies (i) and (ii) of I.6. For (iii) we need only observe that if $\mathscr{V}^i <^* \mathscr{U}^i$ for $i = 1, \cdots, n$, then $\mathscr{V}^1 \wedge \cdots \wedge \mathscr{V}^n <^* \mathscr{U}^1 \wedge \cdots \wedge \mathscr{U}^n$.

PROOF OF THEOREM 8. The operation $f_\alpha^{\cdot}$ on coverings preserves starrefinements; so the inverse images of uniform coverings form a normal family. Then this is a sub-basis for a preuniformity μ, which is the coarsest possible.

We should note that a sub-basis need not be a normal family; of course, a basis must.

There is a complement to Proposition 9. Every union of normal families of coverings is a normal family; so every family λ of coverings contains a largest normal subfamily μ. The coverings in μ are called *normal in* λ. A slightly broader notion is generally more useful. A covering $\mathscr{U}$ is said to be *normal with respect to* λ provided $\mathscr{U}$ belongs to the smallest anti-residual family containing λ and is normal in it; that is, provided $\mathscr{U}$ is the first term of some normal sequence of coverings each of which has a refinement in λ. We note that if $\mathscr{U}$ is known to belong to some preuniformity contained in λ, it must be normal with respect to λ. The converse is true if λ is a filter with respect to $<$.

10. *If λ is a family of coverings of a set which forms a filter under refinement, then there is a finest preuniformity contained in λ, and it consists of all coverings normal with respect to λ.*

Proof. It suffices to show that the family μ of all coverings normal with respect to λ is a preuniformity, for we have already noted that every preuniformity contained in λ is contained in μ. We use Proposition 6. Conditions (ii) and (iii) are obviously satisfied. It remains only to note that if $\mathscr{U}$ and $\mathscr{V}$ are in μ, there are normal sequences $\{\mathscr{U}^n\}$ and $\{\mathscr{V}^n\}$ in the filter λ with $\mathscr{U} = \mathscr{U}^1$, $\mathscr{V} = \mathscr{V}^1$; and $\{\mathscr{U}^n \wedge \mathscr{V}^n\}$ is a normal sequence.

Uniform topology and uniform continuity. The *uniform topology* of a uniform space X is defined as follows. A subset N of X is a neighborhood of a point x of N if for some uniform covering $\mathscr{U}$, N contains $\mathrm{St}(x, \mathscr{U})$. N is open if it is a neighborhood of each of its points.

We wish to prove

11. Theorem. *Every uniform space is a completely regular Hausdorff space in the uniform topology.*

The construction required for this proof can be made to yield another important theorem; so we put it off for a while. Observe here that the uniform space X is at least a T_1 space; for X is an open set (obvious), any union of open sets is open (obvious), the intersection of any two open sets is open (easy exercise), and a point is closed (proof follows). For any point x, for any point $y \neq x$, by definition there must be a uniform covering $\mathscr{U}$, no element of which contains both x and y. Then $\mathrm{St}(y, \mathscr{U}) \subset X - \{x\}$. Thus $X - \{x\}$ is a neighborhood of each of its points, and the point x is closed.

For the rest of the proof we shall need real-valued continuous functions. We may as well construct uniformly continuous functions, since it is no harder. A function f on a uniform space X to a uniform space Y is called *uniformly continuous* if for every uniform covering $\mathscr{V}$ of Y there is a uniform covering $\mathscr{U}$ of X such that, for each element U of $\mathscr{U}$, $f(U)$ is contained in some element of $\mathscr{V}$. We can restate this in terms of the convenient notion of the *inverse image* $f^{-1}(\mathscr{V})$ of a covering $\mathscr{V}$, which is the set of all $f^{-1}(V)$, $V \in \mathscr{V}$; f is uniformly continuous if and only if the inverse image of every uniform covering is uniform. (This is equivalent to the definition since a covering having a uniform refinement is uniform.)

12. *Every uniformly continuous function is continuous.*

PROOF. Suppose $f: X \to Y$ is uniformly continuous. In Y, let V be a neighborhood of $f(x)$. Then V contains $\mathrm{St}(f(x), \mathscr{V})$ for some uniform covering $\mathscr{V}$. It follows that $f^{-1}(V)$ contains $\mathrm{St}(x, f^{-1}(\mathscr{V}))$; thus $f^{-1}(V)$ is a neighborhood of x, and f is continuous.

Further, clearly, a composition of uniformly continuous functions is uniformly continuous. Thus the uniform spaces and the uniformly continuous functions form a category. This is *the category of uniform spaces.*

Now that we are into the category of uniform spaces, we shall use the short term *isomorphism* in preference to *uniform equivalence* for a reversibly uniformly continuous mapping. When the context makes the meaning clear, we may simply say *mapping* for a uniformly continuous mapping.

To complete the proof of Theorem I.11, it will suffice to prove

13. *In a uniform space X, for any point x and any closed set S not containing x, there is a real-valued uniformly continuous function f which vanishes on S but not at x.*

Here, and wherever we speak of real-valued functions, the real line is taken as a metric uniform space with the uniformity induced by the usual distance function. To see that Proposition 13 implies Theorem 11, observe that these functions are continuous and can therefore be used as the coordinate functions for a homeomorphic embedding of X in a product of real lines.

We are still not ready for the proof, for there is a stronger result.

14. THEOREM. *For every uniform covering $\mathscr{U}$ of a uniform space X, there exists a uniformly continuous function g mapping X upon a metric space, so that the inverse image of every set of diameter less than 1 is a subset of an element of $\mathscr{U}$.*

This fundamental theorem is worth restating in language which, though less precise, may be more memorable. *As far as single coverings are concerned, all uniform spaces are like metric spaces.*

To deduce I.13 from I.14, for x and S, take a uniform covering $\mathscr{U}$, no element of which simultaneously contains x and meets S. Map into a metric space so that the image of x is distant by at least 1 from the image of S. For the function f which is wanted, use distance from the image of S.

To prove I.14, write $\mathscr{U}=\mathscr{U}_0$, and take a normal sequence of uniform coverings $\mathscr{U}_n$ beginning with $\mathscr{U}_0$. For any x, y, in X, define $t(x, y)$ to be 0 if $\mathrm{St}(x, \mathscr{U}_n)$ contains y for all n, 2 if $\mathrm{St}(x, \mathscr{U}_0)$ does not contain y, and otherwise 2^{1-n}, where n is the largest index for which y is in $\mathrm{St}(x, \mathscr{U}_n)$. Now consider the set of all real numbers r which can be expressed as finite sums $\sum t(p_i, p_{i+1})$, where both x and y occur among the p_i's. Let $d(x, y)$ be the infimum of this set, or 1 if the infimum exceeds 1.

From the form of the definition, $d(x, y) \geqq 0$ and $d(x, y) = d(y, x)$. The triangle inequality also follows (in a few lines) from the form of the definition. Consequently, X is split up into equivalence classes of d-diameter 0 and d defines a metric on the set M of equivalence classes. Let $g: X \to M$ take each point to its equivalence class.

For any set $W \in \mathscr{U}_n$, for x, y, in W, $d(x, y) \leqq t(x, y) = 2^{1-n}$; hence g is uniformly continuous.

To show that the inverse image of any set of diameter less than 1 is contained in some element of $\mathscr{U}$, it suffices to show that any two points at d-distance less than 1 lie together in some element of the star-refinement $\mathscr{U}_1$. We shall prove that if $d(x, y) < 2^{1-n}$ then x and y are in some element of $\mathscr{U}_n$. The hypothesis implies that some particular finite sum $t(p_1, p_2) + \cdots + t(p_{k-1}, p_k) \leqq 2^{1-n}$, where $p_1 = x$, $p_k = y$. For $k=2$, this implies x and y are in an element of $\mathscr{U}_n$ (in fact, of $\mathscr{U}_{n+1}$). If $k>2$, choose p and q so that

(1) p is the last p_i such that $t(p_1, p_2) + \cdots + t(p_{i-1}, p_i) \leqq 2^{-n}$, and

(2) q is the last p_j such that $t(p_i, p_{i+1}) + \cdots + t(p_{j-1}, p_j) \leqq 2^{-n}$.

Computation shows that the rest of the sum, from q to y, is also at most 2^{-n}. By induction on k, each of the pairs (x, p), (p, q), (q, y) is contained in some element of $\mathscr{U}_{n+1}$; therefore (x, y) is in some element of $\mathscr{U}_n$.

This completely establishes the connection of uniformities with metrics. Later we shall have other language to describe the connection, but hardly any real additional information.

The connection of uniformity and topology involves more various aspects of the spaces. First, the *uniformizable* spaces—i.e., the topological spaces homeomorphic with uniform spaces—are completely determined by I.11. For if X is a completely regular Hausdorff space, I.8 shows that the continuous real-valued functions on X induce a weak uniformity; this uniformity is sometimes designated as μ_C. It remains to check that the uniform topology of $\mu_C X$ is the correct one. Since all the given continuous real-valued functions are uniformly continuous on $\mu_C X$, hence still continuous, the uniform topology contains the given topology. On the other hand, if a set N is a neighborhood of a point p in $\mu_C X$, this means by definition that N contains $\mathrm{St}(p, \mathscr{U})$ for some $\mathscr{U} \in \mu_C$. By I.8, $\mathscr{U} \in \mu_C$ means that for some continuous $f_1, \cdots, f_n$ on X to the real line, for some $\epsilon > 0$, every subset of X whose image under each f_i has diameter $< \epsilon$ is contained in an element of $\mathscr{U}$. But then N is a neighborhood of p in the original topology, and we are done.

15. *The uniformizable spaces are precisely the completely regular Hausdorff spaces.*

The proof of Proposition 15 establishes also

16. *Let X be a set and $\{f_\alpha\}$ a family of functions on X into various uniform spaces, separating points. Then the uniform topology of the weak uniformity induced by $\{f_\alpha\}$ coincides with the weak topology induced by $\{f_\alpha\}$.*

We now have two functors connecting uniform and uniformizable spaces. First, we can associate to each uniform space μX the topological space X, and to a uniformly continuous function $f: \mu X \rightarrow \nu Y$ the same function $f: X \rightarrow Y$. By Proposition 12, f is continuous; and obviously this association is functorial. Further, to every uniformizable space X we may associate the uniform space $\mu_C X$, and again, to $f: X \rightarrow Y$ we associate $f: \mu_C X \rightarrow \mu_C Y$. Here, f is uniformly continuous. In fact, there is a general criterion as follows.

17. *Let μX be a uniform space, g a function from X to a set Y, and $\{f_\alpha\}$ a family of functions (separating points) from Y to uniform spaces Z_α. For g to be uniformly continuous into Y in the weak uniformity induced by $\{f_\alpha\}$, it is necessary and sufficient that every composition $f_\alpha g: \mu X \rightarrow Z_\alpha$ be uniformly continuous.*

PROOF. *Necessity.* Let ν denote the weak uniformity on Y. If $g: \mu X \rightarrow \nu Y$ is uniformly continuous, then since $f_\alpha: \nu Y \rightarrow Z_\alpha$ is uniformly continuous, so is $f_\alpha g$.

Sufficiency. By Proposition 10, there is a finest preuniformity σ on Y relative to which g is uniformly continuous. Specifically σ is the set of all coverings of Y lying in normal sequences $\{\mathscr{U}_n\}$ for which each $g^{-1}(\mathscr{U}_n)$ is uniform. If $f_\alpha g$ is uniformly continuous, then σ includes all $f_\alpha^{-1}(\mathscr{V})$ where $\mathscr{V}$ is a uniform covering of Z_α. But the weak uniformity on Y is the coarsest preuniformity containing all these coverings; hence σ contains it, and g is uniformly continuous.

This establishes two important functors, the "topologizing" functor T on uniform spaces to topological spaces ($T(\mu X) = X$) and the "weak" functor W on uniformizable spaces X to uniform spaces $\mu_C X$. The composition WT is evidently idempotent, retracting the category of all uniform spaces upon the full subcategory of spaces $\mu_C X$. The composition TW is the identity.

18. THEOREM. *The category of uniformizable spaces is a retract of the category of uniform spaces.*

The proof that every completely regular Hausdorff space is uniformizable led us naturally to the functor W; but W is not the most convenient such functor. It is often easier to use the finest uniformity on a space X inducing a given topology on X. The proof of existence (and some sort of description) for this uniformity is now easy to get.

19. *In a uniform space X, every uniform covering has a uniform refinement consisting of open sets.*

PROOF. For each uniform covering $\mathscr{U}$ there is a uniformly continuous mapping $g: X \to M$, M metric, such that $\mathscr{U}$ is refined by the inverse image of the covering of M by all sets of diameter less than 1. Then the same is true for the covering $\mathscr{V}$ of M by all open sets of diameter less than 1. $\mathscr{V}$ is uniform, so $g^{-1}(\mathscr{V})$ is uniform. $\mathscr{V}$ is open, so $g^{-1}(\mathscr{V})$ is open.

20. THEOREM. *Among the uniformities on a set X inducing a given uniformizable topology, there is a finest; it consists of all coverings normal with respect to the family of all open coverings.*

PROOF. Propositions 10 and 19.

It is easy to check that putting the finest uniformity on each uniformizable space again gives us a functor F. TF, like TW, is the identity.

A uniformity inducing a given topology is sometimes called *compatible* with the topology. A space whose uniformity is the finest compatible with its topology is a *fine* space (and the uniformity, a *fine* uniformity). The open uniform coverings of a fine space are called (simply) *normal coverings*.

21. *Every continuous function on a fine space into any uniform space is uniformly continuous.*

PROOF. The inverse image of an open covering is an open covering and the inverse image of a star-refinement is a star-refinement. Thus the inverse image of any normal covering is a normal covering, and with I.19 the proof is complete.

We should note that the embedding of uniformizable spaces in uniform spaces does not reduce all of topology to uniform geometry. There are significant structural features and classificatory notions which do not agree for the two kinds of spaces. The fine spaces form one of the more important subcategories of the category of uniform spaces.

Exercises.

1. Prove Propositions 3, 4, and 6.

2. Prove Proposition 7. (Hint: Use star-refinements.)

*3. STAR-REFINEMENT ORDERING.

(a) A *partition* is a covering whose elements are pairwise disjoint. Show that a covering $\mathscr{U}$ which is equivalent to a partition $\mathscr{P}$ (i.e., $\mathscr{U} < \mathscr{P} < \mathscr{U}$) is a star-refinement of itself.

(b) For any covering $\mathscr{U}$, there is a finest partition $\mathscr{P}(\mathscr{U})$ coarser than $\mathscr{U}$. Each element U of $\mathscr{U}$ lies in a unique element $P(U)$ of $\mathscr{P}(\mathscr{U})$.

(c) If $\mathscr{U}$ is a star-refinement of itself, then for each $U \in \mathscr{U}$, $\mathrm{St}(U, \mathscr{U})$ contains $P(U)$.

(d) $\mathscr{U}$ is a star-refinement of itself if and only if $\mathscr{U}$ is equivalent to a partition.

4. Under what conditions on a covering $\mathscr{U}$ is there a coarsest preuniformity containing $\mathscr{U}$?

5. NONHOMEOMORPHIC SPACES.

(a) Show that the spaces in the proof of Proposition 5 (the Y_α) can be topologically embedded in the plane.

(b) Show that there are uncountably many nonhomeomorphic compact subsets of the line. (Hint: Use ordinal numbers.) There are in fact c nonhomeomorphic compact subsets of the line. The proof (Reichbach [1]) again uses ordinals, but in a fairly complicated way.

6. Show that we actually have two different embeddings of the category of uniformizable spaces in the category of uniform spaces, by exhibiting a space X whose fine uniformity cannot be induced by real-valued functions. (Hint: All counterexamples are discrete.)

*7. STRONG UNIFORMITIES. Given a family $\{f_\alpha\}$ of functions on various uniform spaces into a set X, Proposition 10 shows that there is a finest preuniformity μ on X such that every $f_\alpha^{-1}(\mathscr{U})$, $\mathscr{U}$ in μ, is uniform. When is μ a uniformity?

(a) Suppose all f_α have disjoint images.

(b) Suppose the functions $f_\alpha: X_\alpha \to X$ are all one-to-one. Suppose the sets $f_\alpha(X_\alpha)$ form a star-finite collection, i.e., each one intersects only finitely many others. Suppose for each finite union U of sets $f_\alpha(X_\alpha)$ there is a uniformity on U making these f_α uniformly continuous. Suppose finally that the uniformity on each such finite union U can be so chosen that the sets $f_\alpha(X_\alpha)$ form a uniform covering of U. Then prove that there is a suitable uniformity μ on all of X.

(c) The final assumption in (b) cannot be omitted. (Let X consist of the rational numbers in $[0, 1]$. Choose all X_α to be dense subsets of X in the usual topology. Select a sequence of irrationals t_α converging to 0, and modify the uniformity and topology on X_α so that some sequence in it which in the usual topology converges to t_α now converges to 1.)

*8. ENTOURAGES. Let $\mathscr{U}$ be a uniform covering of a space X. Two points x, y are said to be *within* $\mathscr{U}$ of each other, or *near of order* $\mathscr{U}$, if some element of $\mathscr{U}$ contains the set $\{x, y\}$. This defines a reflexive symmetric relation $R(\mathscr{U})$ on X. On the other hand, given a relation S on X, we may define a covering $\mathscr{V}(S)$ indexed by the points x of X: $V_x = \{y: (x, y) \in S\}$. ($\mathscr{V}(S)$ covers X if S is reflexive.)

(a) For any covering $\mathscr{U}$, $\mathscr{V}(R(\mathscr{U})) < \mathscr{U}^*$.

(b) For any symmetric relation S, $R(\mathscr{V}(S))$ is a subset of the relation $S \circ S = \{(x, y): (\exists z)\{(x, z), (z, y)\} \subset S\}$.

(c) In view of (a), knowledge of all the relations $R(\mathscr{U})$ for $\mathscr{U}$ in a uniformity μ determines μ. All the relations (symmetric or not) containing relations $R(\mathscr{U})$ are called *entourages* of the uniform space. Give necessary and sufficient conditions on a family $\mathscr{R}$ of relations for $\mathscr{R}$ to be the set of entourages of a uniform space.

(d) A function $f: X \to Y$ is uniformly continuous if and only if for every entourage V of Y, the relation $(f(x), f(x')) \in V$ is an entourage of X.

Notes. There has been no general investigation of the topic of Exercise 7. 7(a) is solved in Ginsburg-Isbell [1].

The topic of Exercise 8 has been extensively investigated. In fact, entourages are the basis of the definition of uniform spaces in Bourbaki [**Bo**]. The approach by means of coverings was invented by Tukey [**T**], who showed the equivalence of this approach and Bourbaki's.

The two rules of thumb stated in this chapter ("All counterexamples are discrete"; "As far as single coverings are concerned, all uniform spaces are like metric spaces") actually represent theorems of similar tendency, to the effect that a suitably restricted property $\mathfrak{P}$ possessed by (1) all topologically discrete uniform spaces, or (2) all metric uniform spaces, will be possessed by all uniform spaces. Rule (2) is already partially formalized in I.14. Exercises IV.5 and IV.8 show how to formalize fully a weaker result: *a continuous property possessed by all metric spaces is possessed by all spaces*. But the full formalization of rule (2), and any useful formalization of rule (1), stand as unsolved problems.

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CHAPTER II

FUNDAMENTAL CONSTRUCTIONS

As in many familiar categories, so in uniform spaces we have the notions of free sum, direct product, subspace, quotient space. Completion, though not paralleled in topology, is much like several algebraic completion constructions. Ideas more peculiar to our present subject emerge in connection with compactification—a sort of overcompletion—and the convergence of subsets.

Sum, product, subspace, quotient. In any category, an object Σ is said to be a *sum* of a family $\{X_\alpha\}$ of objects if there is a family $\{i_\alpha\}$ of mappings $i_\alpha: X_\alpha \to \Sigma$ such that for every object Y in the category (1) if $f: \Sigma \to Y$ and $g: \Sigma \to Y$ are two different mappings, then for at least one α the compositions fi_α and gi_α are different, and (2) for every family $\{f_\alpha\}$ of mappings $f_\alpha: X_\alpha \to Y$ there exists a mapping $f: \Sigma \to Y$ such that $fi_\alpha = f_\alpha$ for each α. The mappings i_α are called *canonical injections*. If they fail to be one-to-one functions, one might quarrel with the terminology. One will not find a better sum; for

1. *Any two sums of a given set of objects are isomorphic.*

PROOF. Let Σ and Σ' be sums of $\{X_\alpha\}$, with canonical injections i_α, i'_α, respectively. Since Σ is a sum, there is a mapping $f: \Sigma \to \Sigma'$ such that $fi_\alpha = i'_\alpha$ for all α. Similarly, there is $g: \Sigma' \to \Sigma$ such that $gi'_\alpha = i_\alpha$ for all α. Then the composition $gf: \Sigma \to \Sigma$ has the property $gfi_\alpha = i_\alpha$ for all α. This is the same as $1i_\alpha$, where 1 is the identity map of Σ; hence $gf = 1$. Similarly, fg is an identity map; so g and f are isomorphisms.

Moreover, the modest assumption that each X_α admits at least one mapping to each X_β suffices to assure that each i_α embeds X_α as a retract of Σ. (Proof omitted.) In uniform spaces, this assumption may fail if X_β is empty; but it is easy to show that even so, canonical injections are always one-to-one.

2. *Every set of uniform spaces has a sum.*

PROOF. Given the uniform spaces X_α $(\alpha \in A)$, define the uniform space Σ as follows. The points of Σ are the ordered pairs (x, α), where $\alpha \in A$ and

$x \in X_\alpha$. The canonical injections i_α are defined by $i_\alpha(x) = (x, \alpha)$. A covering of Σ is uniform if and only if its inverse image under i_α is uniform for each α.

The verification that Σ is a uniform space is a part of Exercise I.7(a). Then of the defining conditions for a sum, (1) is clear because different mappings must differ on some point of Σ, and every point of Σ is $i_\alpha(x)$ for some x and α. For (2), given $\{f_\alpha : X_\alpha \to Y : a \in A\}$, we define $f\colon \Sigma \to Y$ by $f(x, \alpha) = f_\alpha(x)$. Since each $fi_\alpha = f_\alpha$ is uniformly continuous, the inverse image under f of a uniform covering of Y is a uniform covering of Σ.

The general definition of a *product* is precisely dual to the definition of a sum. For a space Π to be a product of $\{X_\alpha\}$, we must have uniformly continuous mappings $p_\alpha : \Pi \to X_\alpha$ (*canonical projections* or, more usually, *coordinate projections*) such that (1) if $f\colon W \to \Pi$ and $g\colon W \to \Pi$ are two different mappings, then for at least one α the compositions $p_\alpha f$ and $p_\alpha g$ are different, and (2) for every family of mappings $f_\alpha\colon W \to X_\alpha$, there exists a mapping $f\colon W \to \Pi$ such that $p_\alpha f = f_\alpha$ for each α. The proof of uniqueness of a product need not be given, because of the principle of duality.

1*. *Any two products of a given set of objects are isomorphic.*

The construction of a product is roughly dual to the construction of a sum. For Π we take the Cartesian product set of the factors X_α; for canonical projections p_α, the set-theoretic projections. In place of the finest uniformity making the i_α uniformly continuous we take the coarsest uniformity making the p_α uniformly continuous. This is a uniformity by I.8. The proposed product space Π satisfies defining condition (1), since a point is determined by its coordinates; for condition (2), the definition of the required mapping is obvious and the uniform continuity follows from I.17.

3. *Every set of uniform spaces has a product.*

As a general rule, canonical projections are onto mappings; there is an exception if some of the factors are empty.

The abstract definitions of sum and product are standard (though the terminology is not. It is almost as common to call them the *free product* and *direct product*, respectively. The safe way, suggested by P. R. Halmos, is to say *free sum* and direct product). We shall discuss abstract definitions of subspace and quotient too. Here the ground is slippery, but not impassable.

We need the notions of "one-to-one" and "onto", which of course are not categorical notions. In uniform spaces they are equivalent to rather simple categorical notions. This point is brought up here, not because we shall use the categorical notions, but because THEY ARE NOT DUAL.

4. *In uniform spaces, the one-to-one mappings are precisely the monomorphisms.*

THE PROOF. A one-to-one mapping is always monomorphic, since $hf=hg$ requires $h(x)=h(y)$ whenever $x=f(p)$, $y=g(p)$, and if this means $x=y$ it means $f=g$. On the other hand, if h is not one-to-one, $h(x)=h(y)$ for some distinct x, y, there are constant mappings to show h is not monomorphic.

Similarly, an onto mapping is epimorphic; but this is also true of any mapping whose image is dense in the range. In Exercise II.3 we shall see that the onto mappings are characterized as the so-called "pure epimorphisms". The pure monomorphisms are not so easy to describe. The reader may refer to the notes, p. 35; at any rate, remember that duality is not with us any more.

The usual idea of an embedding mapping $e: S \to X$ is, certainly, a one-to-one mapping such that the structure of S coincides with the structure relativized from X to $e(S)$. In the same spirit as the definitions of sum and product we may say: if e has the form gf, where $f: S \to S'$ is a one-to-one onto mapping, then f is an isomorphism. That is, no other structure on the set S can be interposed between the domain and range of e. It is immediate that, for uniform spaces, this means that the uniformity of S is the weak uniformity induced by the single mapping e. It is also immediate that, in any category of sets and functions, two embeddings with the same image are related by an isomorphism. This concludes the treatment of existence and uniqueness for subspaces.

There are no less than three distinguishable usual ideas of a quotient mapping. They coincide, in uniform spaces and in all the individual categories usually studied. It is not hard to manufacture examples of categories where they differ, but that is not relevant to our subject. What is relevant is that the three definitions suggest three different formulations of the existence and uniqueness problems for quotient mappings in uniform spaces.

First, the reason for treating quotients together with subspaces. Employing the pseudo-duality "one-to-one"$\longleftrightarrow$"onto", we translate the definition of an embedding to this: a *quotient mapping* $q: X \to Q$ is an onto mapping such that whenever $q=gf$, where $g: Q' \to Q$ is one-to-one onto, then g is an isomorphism. Employing no formal procedure, one naturally thinks of the question of existence and uniqueness in such a way that the following proposition answers it.

5. *Every uniformly continuous function f: $X \to Y$ has the form f_0q where q $X \to Q$ is a quotient mapping and f_0: $Q \to Y$ is one-to-one. Any two such factorizations of f are related by an isomorphism of the middle spaces.*

Each f also has the form ef_1 where e: $S \to Y$ is an embedding and f_1 is onto; and this expression is unique up to isomorphism. Moreover, the relation between the factors e and q is that there is a one-to-one onto mapping m: $Q \to S$ such that $mq=f_1$, $em=f_0$.

PROOF. Consider all uniformities on the set of points $f(X)$. For at least one of them, the one induced by the function $e: f(X) \to Y$ coincident with the identity, f is uniformly continuous. Hence the finest preuniformity making f uniformly continuous is a uniformity. Let Q be the set $f(X)$ with this uniformity; let $q: X \to Q$ be the mapping coincident with f. Let $f_0: Q \to Y$ coincide with the identity.

If $q: X \to Q$ factors across a set Q' in one-to-one correspondence with Q, $X \to Q' \to Q$, then we have a finer uniformity on Q making f uniformly continuous; that is the same uniformity and $Q' \to Q$ is an isomorphism. This shows that q is a quotient mapping. Further, another such factorization $X \to Q'' \to Y$ gives a one-to-one correspondence $Q \to Q''$, uniformly continuous because of the choice of finest uniformity for Q; since Q'' is also a quotient, $Q \to Q''$ is an isomorphism.

We have already constructed S and proved its uniqueness. The construction makes S the same point set as Q, and the function $m: Q \to S$ coincident with the identity is uniformly continuous. This completes the proof.

The second idea of a quotient is of interest, here, mainly as a bridge to the third. An onto mapping $q: X \to Q$ determines an equivalence relation on the points of X by $q(x) = q(x')$. It is a fact (we shall indicate how it follows from Proposition 7 below), and one might make it a definition, that q is a quotient mapping if and only if every mapping $f: X \to Y$ determining a coarser equivalence relation is a left multiple of q, $f = f_0 q$.

REMARKS. This is a stronger requirement on q than the one adopted above as a definition—superficially stronger. The associated uniqueness problem, is Q determined by the equivalence relation, is quite trivial. The existence problem, to characterize the equivalence relations induced on X by various mappings q, is part of Exercise I.7(a).

The third idea is a categorical (point-free) variation of the second. Instead of the equivalence relation $q(x) = q(x')$ on points of X, we consider the equivalence relation $qf = qg$ on mappings into X. In any category, a *right polar class* is the class of all mappings h with a given domain X satisfying some class of equations $hf = hg$. We shorten this term to *polar class* here. (For the dual notion, see Exercise 4.) A *polar epimorphism* is an epimorphism $q: X \to Q$ such that the class of all its left multiples is a polar class.

REMARK. Considering the constant mappings, it is obvious that polar epimorphisms in uniform spaces coincide with quotient mappings in the second proposed sense—once we show they are onto mappings. Hence, every polar epimorphism is a quotient mapping (first sense).

6. *In the category of uniform spaces, every polar epimorphism is an onto mapping.*

PROOF. Let $p: X \to P$ be an epimorphism which is not onto. By I.13, $p(X)$ is a dense subset of P. Consider the mapping $f: X \to p(X)$ coincident with p. Clearly it is in every polar class containing p. If f had the form $f_0 p$, we should have f_0: $P \to p(X)$ coinciding with the identity on the dense set $p(X)$ but not on its closure, which is absurd. Hence p is not a polar epimorphism.

7. *The quotient mappings are precisely the polar epimorphisms. Every polar class consists of the left multiples of some polar epimorphism.*

PROOF. As we noted before, every onto mapping whose left multiples form a polar class is a quotient mapping (by specializing the definition). Conversely, suppose $q: X \to Q$ is a quotient mapping and $f: X \to Y$ satisfies $f(x) = f(x')$ whenever $q(x) = q(x')$. Then there is a unique single-valued function f_0 from Q to Y such that $f_0 q = f$. The weak preuniformity induced on the set Q by f_0 consists of coverings whose inverse images under q are uniform; so by construction of the quotient space, f_0 is uniformly continuous.

Given a polar class $\mathscr{P}$ of mappings with domain X, consider the equivalence relation on the points of $X: x \equiv x'$ if for every p in $\mathscr{P}$, $p(x) = p(x')$. There is a set $\{f_\alpha\}$ of mappings in $\mathscr{P}$ which suffices to induce the same equivalence relation. Let the f_α be the coordinates of a mapping into a product space, $f: X \to Y$, and factor f as $f_0 q$ according to II.5. The quotient mapping $q: X \to Q$ induces the same equivalence relation as $\mathscr{P}$. As was just proved, q must be a polar epimorphism; so $\mathscr{P}$ consists of the left multiples of q.

This concludes the proofs of existence and uniqueness for the four "universal" constructions. Some of the properties of sums and subspaces are obvious. For products and quotients, almost nothing is obvious. We do have

8. *The uniform topology of a sum, product, or subspace is the sum, product, subspace topology, respectively.*

PROOF. This is among the obvious properties of sums and subspaces mentioned above. For products, it follows from I.16.

For quotients, Proposition 8 is horribly false; this will be brought out in the exercises.

9. *If S is a subspace of a uniform space X and $\mathscr{U}$ a uniform covering of S, the closures of the elements of $\mathscr{U}$ form a uniform covering of S^-.*

PROOF. The uniformity of the subspace S evidently consists of coverings $\{U_\alpha \cap S\}$, where $\{U_\alpha\}$ is a uniform covering of X. By I.19, $\{U_\alpha\}$ has an open uniform refinement $\{V_\beta\}$. If V_β is open, then $(V_\beta \cap S)^-$ contains $V_\beta \cap S^-$; thus the uniform covering $\{V_\beta \cap S^-\}$ of S^- is a refinement of $\{(U_\alpha \cap S)^-\}$.

Completeness and completion. A proper filter $\mathscr{F}$ of subsets of a topological space is *convergent*, to the point p, if every neighborhood of p is an element of $\mathscr{F}$. A proper filter in a uniform space is *Cauchy* if it includes at least one element of each uniform covering. A convergent filter (in the uniform topology) is Cauchy by I.19. If every Cauchy filter is convergent, the space is called *complete.*

10. **Theorem.** *Let A be a subspace of the uniform space X and f a uniformly continuous mapping of A into a complete uniform space. Then there is a unique uniformly continuous extension of f over the closure of A.*

Proof. Since A is dense in its closure A^-, there can be at most one extension of f over A^- which is even continuous. On the other hand, for any point p of A^-, the neighborhoods of p form a convergent filter whose trace on A is still a Cauchy filter $\mathscr{F}$. From the definitions, the image of $\mathscr{F}$ under f is a base of a Cauchy filter $f(\mathscr{F})$. Since Y is complete, $f(\mathscr{F})$ converges to a unique limit $f^*(p)$.

This defines a function f^*: $A^- \to Y$. For p in A, since the inverse image under f of a neighborhood of $f(p)$ is a neighborhood of p, $f^*(p)$ is $f(p)$. Finally, consider any uniform covering $\mathscr{V}$ of Y. Let $\mathscr{W}$ be a uniform star-refinement of $\mathscr{V}$. Then $f^{-1}(\mathscr{W})$ is a uniform covering of A, and by Proposition 9, the closures of the elements of $f^{-1}(\mathscr{W})$ form a uniform covering of A^-. From the definition of f^*, this covering is finer than $f^{*-1}(\mathscr{V})$.

11. **Corollary.** *If two complete uniform spaces have isomorphic dense subspaces they are isomorphic.*

A *completion* of a uniform space X is defined as a complete space which contains X as a dense subspace. The corollary shows that completions are unique in the same sense that sums and products are.

The existence of a completion can be proved by a direct construction, but the argument can be simplified by using some facts about complete metric spaces; also, we can get extra information. Consider the chain of propositions:

12. *A closed subspace of a complete space is complete.*

13. *Every metric space can be embedded (isometrically) in a complete metric space.*

14. *Every uniform space can be embedded in a product of metric spaces.*

15. *A product of complete spaces is complete.*

Once we have these, we have

16. **Theorem.** *Every uniform space has a completion.*

To prove the theorem, we embed in a product of metric spaces by 14; embed the metric spaces in complete spaces by 13; observe that the product of subspaces is a subspace of the product, from the definitions (the two uniformities are weak uniformities induced by the same functions); and finally take the closure.

Now Proposition 12 is trivial (a Cauchy filter in the subspace must converge in the big space and cannot converge to a point in the complement of the subspace, which is an open set). Probably Proposition 13 is known to the reader; if not, Propositions 19 and 22 below provide a proof. In any case we should note at this point that a sequence of points $\{x_n\}$ in a uniform space is called a *Cauchy sequence* if in any uniform covering, one element contains all but finitely many x_n; and

17. *A metric space is complete if and only if every Cauchy sequence is convergent.*

The proof is left as an exercise.

It remains to prove 14 and 15. There is an important embedding lemma involved here. A uniform covering $\mathscr{U}$ of a space X is said to be *realized* by a mapping $f: X \to Y$ if for some uniform covering $\mathscr{V}$ of Y, $f^{-1}(\mathscr{V}) < \mathscr{U}$. Thus I.14 says that every uniform covering can be realized by a mapping into a metric space.

18. *If a family of uniformly continuous functions $f_\alpha: X \to Y_\alpha$ realizes every uniform covering of X, then the mapping f of X into the product of all Y_α whose coordinate projections are the f_α is an embedding.*

PROOF. For an embedding we require that the given uniformity is the weak uniformity. It is at least as fine, since the f_α are uniformly continuous; and it is the same, since every covering is realized.

Note that the condition here is not necessary for f to be an embedding. However, it is good enough. The application to prove Proposition 14 requires only the most elementary set-theoretic precautions. Fix some set S whose cardinal number is sufficiently large (say $S = X$) and consider the set of all mappings f_α of X onto metric spaces Y_α whose set of points is a subset of S. Then I.14 and II.18 achieve the proof.

PROOF OF PROPOSITION 15. Let $\mathscr{F}$ be a Cauchy filter in a product P of complete spaces X_α. As in Theorem 10, each projection p_α takes $\mathscr{F}$ to a Cauchy filter $\mathscr{F}^\alpha$, which must converge to a limit x_α. Consider the point x in P defined by $p_\alpha(x) = x_\alpha$ for all α. In the product topology, the filter $\mathscr{F}$ includes every sub-basic neighborhood $p_\alpha^{-1}(N)$, N a neighborhood of x_α in X_α; being a filter, it includes every finite intersection of these and therefore converges to x.

While we are embedding, we may as well go on and get into elementary spaces. The *ell-infinity space* $l_\infty(S)$, for any set S, is the metric space whose points are all the bounded real-valued functions on S with the distance $d(f, g) = \sup|f-g|$. We remark that it is a Banach space. (The capital ell L_∞ is used in connection with measure spaces.)

19. *Every metric space can be isometrically embedded in an ell-infinity space.*

PROOF. Let M be a metric space and p a point of M. (The case that M is empty is left as an exercise.) To each point x of M we associate the function x^* in $l_\infty(M)$ whose value at each point y is $\min(d(x, y), 2d(x, p))$. For any x and y, $d(x^*, y^*)$ is the supremum of $|x^*-y^*|$, which is $|\min(d(x,z), 2d(x, p)) - \min(d(y, z), 2d(y, p))|$ at each z in M. Now $d(x, y)$ does not exceed $d(x, p)+d(y, p)$, and therefore does not exceed twice the larger of these. If $d(x,p) \geqq d(y,p)$, then $x^*(y) = d(x,y)$; since $y^*(y)=0$, $|x^*(y)-y^*(y)| = d(x, y)$. On the other hand, for any $z, |x^*(z)-y^*(z)| \leqq |d(x, z)-d(y, z)| \leqq d(x, y)$. Hence we have an isometric embedding.

20. *Every metric space is isomorphic (uniformly equivalent) with a bounded metric space, and hence can be embedded in the unit ball of an ell-infinity space*

PROOF. In the metric space (M,d) the uniform coverings will not be changed if we replace d by $d' = \min(d, 1)$.

21. *Every uniform space can be embedded in a product of unit balls of ell-infinity spaces.*

PROOF. Propositions 14 and 20.

22. *Ell-infinity spaces (and their unit balls) are complete.*

PROOF. We use Proposition 17 for convenience. In $l_\infty(S)$ let $\{f_n\}$ be a Cauchy sequence of functions. Then for each $s \in S$, $\{f_n(s)\}$ is a Cauchy sequence of real numbers, hence convergent to a limit $f(s)$. This defines a real-valued function f on S. For any $\epsilon > 0$, finally $\sup|f_m - f_n| < \epsilon$; hence for sufficiently large m, $|f_m - f| \leqq \epsilon$. This shows that $f \in l_\infty(S)$ and that $\{f_n\}$ converges to f.

Since the unit ball is closed, it is also complete.

Compactness and compactification. The reader is assumed to be acquainted with general compact (Hausdorff) spaces and compactification. In particular, a space is compact if and only if every ultrafilter (maximal proper filter) is convergent.

23. **Theorem.** *Every open covering of a compact uniform space is uniform.*

Proof. To show that an open covering $\mathscr{U}$ is uniform it suffices to show $\mathscr{U}$ has a uniform refinement. Now for each point x there is a uniform covering $\mathscr{V}_x$ such that $\mathrm{St}(x, \mathscr{V}_x)$ is contained in some element of U; if $\mathscr{W}_x$ is a star-refinement of $\mathscr{V}_x$, there is an open neighborhood N_x of x such that each element of $\mathscr{W}_x$ which meets N_x is contained in some element of $\mathscr{U}$. Finitely many N_x's cover the space. Then if $\mathscr{W}$ is the intersection of the corresponding $\mathscr{W}_x$'s, every nonempty element of $\mathscr{W}$ meets some N_x and is contained in an element of $\mathscr{W}_x$; hence $\mathscr{W}$ refines $\mathscr{U}$.

24. **Corollary.** *Every continuous function on a compact uniform space to a uniform space is uniformly continuous. Two homeomorphic compact uniform spaces are isomorphic.*

The process of compactifying a uniform space does not involve embedding it in a compact space. Indeed, if that is possible, completion does it. However, we can consider all the uniformly continuous mappings from X to compact spaces Y_α, and use them as coordinates of a mapping into the product of the Y_α; the closure of the image is compact, and it is in a sense the "nearest" compact space to X. This procedure is familiar from topology as the construction of the Stone-Čech compactification. Moreover, it is familiar from the last few pages as the construction of the completion. Let us formalize it.

We are given a space X and a class $\mathscr{L}$ of spaces. ($\mathscr{L}$ determines a full subcategory which will also be called $\mathscr{L}$.) We expect to find a space $X|\mathscr{L}$ and a mapping $r: X \to X|\mathscr{L}$ such that every mapping $f: X \to Y$, Y in $\mathscr{L}$, has the form $f_0 r$; $X|\mathscr{L}$ itself should be in $\mathscr{L}$, and f_0 should be unique. (Commonly the requirement that f_0 is always unique will make r an epimorphism, but this is by the way.) Such a space $X|\mathscr{L}$ is called a *reflection of X in $\mathscr{L}$, or a $\mathscr{L}$-reflection of X.*

25. *The completion of a uniform space X is a complete reflection of X.*

This is a corollary of Theorem 10. Further, the construction of a completion can be imitated to make a compactification, or a $\mathscr{L}$-reflection for many other classes of spaces $\mathscr{L}$. (Cf. Exercise 8.)

26. *A $\mathscr{L}$-reflection of X is unique up to isomorphism.*

The proof is like the uniqueness proofs of Propositions 1, 1*, 5.

We adopt the simplifying convention that if X is already in $\mathscr{L}$, the only $\mathscr{L}$-reflection of X is X itself and the only reflection mapping $r: X \to X$ is the identity. Then we have the following:

27. THEOREM. *Suppose the uniform spaces X, Y have $\mathscr{L}$-reflections $X \mid \mathscr{L}$, $Y \mid \mathscr{L}$. Then any uniformly continuous mapping f: $X \to Y$ determines a unique mapping $f \mid \mathscr{L} : X \mid \mathscr{L} \to Y \mid \mathscr{L}$ such that $rf = (f \mid \mathscr{L})r$.*

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[d, "r"] \\
X \mid \mathscr{L} \arrow[r, "f \mid \mathscr{L}"] & Y \mid \mathscr{L}
\end{tikzcd}
$$

The correspondence $f \to f \mid \mathscr{L}$ is functorial. Thus if every space has a reflection in $\mathscr{L}$, then reflection defines a functor retracting the category of uniform spaces upon the full subcategory $\mathscr{L}$.

PROOF. $f \mid \mathscr{L}$ is the unique f_0 such that the mapping rf is $f_0 r$. If we have also $g : Y \to Z$ and $g \mid \mathscr{L} : Y \mid \mathscr{L} \to Z \mid \mathscr{L}$, the composition $h_0 = (g \mid \mathscr{L})(f \mid \mathscr{L})$ satisfies $h_0 r = (g \mid \mathscr{L}) rf = rgf$; so h_0 is $(gf) \mid \mathscr{L}$. Hence the functor, and it is the identity on $\mathscr{L}$ by convention.

All general propositions about reflections (and clearly 26 and 27 are general) apply dually to *coreflections* $\mathscr{L} \mid X$, determined by coreflection mappings $l : \mathscr{L} \mid X \to X$ over which all maps from $\mathscr{L}$ to X factor uniquely. Coreflections are less familiar (not only in uniform spaces, but in algebra, where commutator quotient groups and ring/radical are perhaps the best known reflections); but one may check that the functor FT of Chapter I is a coreflection.

A uniform space is called *precompact* if every uniform covering has a finite uniform refinement.

28. *A uniform space is compact if and only if it is precompact and complete.*

PROOF. From II.23 and I.19, a compact space is precompact; from II.16 and II.12 (since a compact space is closed wherever it lies), it is complete. Conversely, in a precompact space every ultrafilter is Cauchy; hence, in a precompact complete space every ultrafilter is convergent.

29. *The completion of a precompact space is compact.*

Proof. We must show the completion Y of a precompact space X is precompact. Let $\mathscr{U}$ be any uniform covering of Y, $\mathscr{V}$ a star-refinement—which we may take to be open—$\mathscr{W}$ a finite uniform covering of X finer than the trace of $\mathscr{V}$ on X. Let $\mathscr{Y}$ be the collection of closures in Y of elements of $\mathscr{W}$. By Proposition 9, $\mathscr{Y}$ is a uniform covering of Y. Since $\mathscr{V}$ is open it is clear that $\mathscr{Y} < \mathscr{V}^* < \mathscr{U}$.

30. *Every finite uniform covering of a uniform space has a finite uniform star-refinement.*

Proof. Let $\mathscr{U}$ be any finite uniform covering and $\mathscr{W}$ a uniform star-refinement of $\mathscr{U}$. Define an equivalence relation on $\mathscr{W}$: two elements are equivalent if they are contained in the same elements of $\mathscr{U}$ and their stars are contained in the same elements of $\mathscr{U}$. Then there are only finitely many equivalence classes, and their unions form a uniform covering that is still a star-refinement of $\mathscr{U}$.

31. Theorem. *Every uniform space X has a precompact reflection pX. pX has the same points as X; the finite uniform coverings of X form a basis of uniform coverings for pX.*

Proof. The finite uniform coverings form a basis for a uniformity, from Proposition 30 and the fact that the intersection of two finite coverings is finite. Then pX is a reflection since the inverse image of a finite covering under a mapping is finite.

The completion of pX is called the *Samuel compactification* or *uniform compactification* of X. When a uniform space is named μX, the usual notation for the Samuel compactification is $\beta\mu X$. If the uniformity is not named, the notation βX should be avoided since that is the standard notation for the Stone-Čech compactification βFTX. (The proof that βFTX *is* the Stone-Čech compactification is an easy exercise.)

32. Theorem. *The Samuel compactification of a uniform space X is a compact reflection of X.*

Proof. It is compact by Proposition 29, and any mapping to a compact space factors over the reflection pX and then over its completion.

Let us think a moment about discrete spaces, which stand in a very rough duality with compact spaces. The fine discrete spaces are usually called *uniformly discrete*. A disjoint collection of subsets X_α of a uniform space is called *uniformly discrete* if the collection $\{X_\alpha\}$ is a uniform covering of its union. In referring to a *discrete collection* the modifier "uniformly" can usually be omitted. Note that for a discrete collection $\{X_\alpha\}$ in a space X, there is a uniform covering $\mathscr{U}$ of X such that the sets $\mathrm{St}(X_\alpha, \mathscr{U})$ are pairwise disjoint (or even form a uniformly discrete collection. Proof by star-refinements).

33. *For any cardinal number m, the following properties of a uniform space are equivalent*:

(1) *Every uniform covering has a uniform refinement of power less than m*;
(2) *Every uniform covering has a subcovering of power less than m*;
(3) *There is no uniformly discrete subspace of power m*.

Proof. Trivially (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Suppose (1) is false, $\mathcal{U}$ being a counterexample. Let f be a mapping onto a metric space M such that $\mathcal{U}$ is refined by the inverse images of sets of diameter 1. Let D be a maximal set of points of M such that the distance between any two points of D is at least $1/4$. The $(1/3)$-neighborhoods of the points of D form a uniform covering whose inverse image refines $\mathcal{U}$; hence D has at least m points. But there is a subset E of the given space which f maps one-to-one onto D, and E is uniformly discrete.

Proximity. The notions of "near" and "far" in a uniform space are of considerable importance. Two sets A, B are said to be *far*, or *uniformly distal*, if the pair $\{A, B\}$ is a discrete collection. Sets are *near* or *proximal* if they are not far. A set U is a *uniform neighborhood* of A if the complement of U is far from A. Of course this implies that U is a neighborhood of A.

A function f from a uniform space to a uniform space is called a *δ-mapping* if whenever A and B are near, $f(A)$ and $f(B)$ are near.

34. *Every uniformly continuous function is a δ-mapping and every δ-mapping is continuous*.

Proof. Suppose $f: X \to Y$ is uniformly continuous, A and B are subsets of X, and $f(A)$ is not near $f(B)$. Then $\{f(A), f(B)\}$ is a uniformly discrete collection; so $\{f^{-1}(f(A)),\ f^{-1}(f(B))\}$ is discrete, and $\{A, B\}$ is also. That is, f is a δ-mapping. For the other assertion, observe that a uniform neighborhood of a single point is just a neighborhood.

A one-to-one function f from X onto Y such that both f and f^{-1} are δ-mappings is called a *δ-isomorphism*. In the category whose objects are the uniform spaces and whose mappings are the δ-mappings, the objects are called *proximity spaces* or *δ-spaces*. There is an intrinsic theory of proximity spaces, which we shall examine briefly here; but we are mainly interested in the fundamental concepts of near, far, and uniform neighborhood, and in the notion of a *δ-invariant* property of uniform spaces—that is, a property shared by all δ-isomorphic spaces.

In the intrinsic theory we are given a set X and a symmetric relation on subsets of X; the relation δ is written $A\delta B$, which we read "A and B are near", or "$\cdots$ are proximal". Further,

(i) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$.

(ii) Two one-point sets are proximal if and only if they are identical.

(iii) No set is proximal to the empty set.

(iv) If A is not proximal to B then X is the union of two sets C, D, such that A is not proximal to D and B is not proximal to C.

(The axioms would be a little simpler if we based them on "far" instead of "near", i.e., on the relation complementary to δ. However, the present axioms and notation and the term "δ-space" are well established. The term "infinitesimal space" is also used.)

For our purposes the crucial theorem is

35. THEOREM. *Every δ-space given by a relation δ satisfying* (i)-(iv) *above is δ-isomorphic with a uniform space. Two uniform spaces X, Y, are δ-isomorphic if and only if their precompact reflections pX, pY are uniformly equivalent. In fact, a function $f: X\to Y$ is a δ-mapping if and only if $f: X\to pY$ is uniformly continuous.*

In short, the category of δ-spaces is nothing more nor less than the category of precompact uniform spaces. To prove the last part of Theorem 35, we note that the identity mapping from Y to pY is a δ-isomorphism. To prove this we must show that if A, B, are far in Y—that is, for some uniform covering $\mathscr{U}$, $\mathrm{St}(A,\mathscr{U})$ and $\mathrm{St}(B,\mathscr{U})$ are disjoint — then for some finite uniform covering $\mathscr{V}$, $\mathrm{St}(A, \mathscr{V})$ and $\mathrm{St}(B, \mathscr{V})$ are disjoint. Let $\mathscr{V}$ consist of three sets, $\mathrm{St}(A, \mathscr{U})$, $\mathrm{St}(B, \mathscr{U})$, and the union of all elements of $\mathscr{U}$ which meet neither A nor B.

It follows, using II.34, that if $f: X\to pY$ is uniformly continuous then $f: X\to Y$ is a δ-mapping. Now suppose $f: X\to Y$ is a δ-mapping, or what is the same thing, $f: X\to pY$ is a δ-mapping. First consider a two-element uniform covering $\{U, V\}$ of Y. To say that $\{U, V\}$ is a uniform covering means exactly that $\{Y-U. Y-V\}$ is a discrete collection. Then $Y-U$ and $Y-V$ are far; so $f^{-1}(Y-U)$ and $f^{-1}(Y-V)$ cannot be near. Hence $f^{-1}(\{U, V\})$ is uniform.

The proof that f is uniformly continuous into pY will be completed by the following interesting lemma.

36. LEMMA. *Every finite uniform covering is refined by a finite intersection of two-element uniform coverings.*

PROOF. Let $\mathscr{U}$ be a finite uniform covering and $\mathscr{V}$ a finite uniform star-refinement (Proposition 30). For each $V\in\mathscr{V}$, form the two-element covering consisting of its complement and its star $\mathrm{St}(V, \mathscr{V})$. Now any member of the intersection of all these coverings is contained either in the star of some element of $\mathscr{V}$ or in the complement of every element of $\mathscr{V}$. In the latter case, it is empty. Hence the intersection refines $\mathscr{V}^*$ and also $\mathscr{U}$, as was to be shown.

For the rest of Theorem 35 we must construct a uniformity, working from the proximity axioms. It will suffice to construct a basis, or even a sub-basis. Proposition 36 suggests a simple sub-basis; but clearly two-element coverings are not enough for the construction of star-refinements. The combinatorial complications in general finite coverings are not unmanageable, but they are decidedly tedious. Therefore we proceed as follows.

A *δ-chain* of sets is a finite sequence of sets $U_1, \cdots, U_n$, covering the δ-space X, so that each U_i is far from all U_j for which $|i-j|>1$, and $U_i - U_{i+1}$ is far from $U_{i+1}-U_i$. A *standard refinement* of a δ-chain $\{U_1, \cdots, U_n\}$ is a δ-chain $\{V_1, \cdots, V_{3n-2}\}$ with each V_{3i-2} contained in U_i and both V_{3i-1} and V_{3i} contained in $U_i \cap U_{i+1}$. In particular, it is a star-refinement.

37. LEMMA. *Every δ-chain has a standard refinement.*

PROOF. Applying axiom (iv) to the far sets $U_1 - U_2$, $X - U_1$, we get in particular a uniform neighborhood V_1 of $U_1 - U_2$ which is still far from $X - U_1$. Inductively, having V_{3i-2}, so that $I_{3i-2} = \bigcup [V_j : j \leqq 3i-2]$ is a uniform neighborhood of $U_1 \cup \cdots \cup U_i - U_{i+1}$ and far from $U_n \cup \cdots \cup U_{i+1} - U_i$, we construct successively I_{3i-1} and I_{3i}, larger uniform neighborhoods which are still far from $U_n \cup \cdots \cup U_{i+1} - U_i$, by axiom (iv). Let V_{3i-1} be $I_{3i-1} - (U_1 \cup \cdots \cup U_i)$, $V'_{3i} = I_{3i} - I_{3i-2}$. One applies axiom (iv) again to trim a little off V'_{3i} to get it far from V_{3i-2}. Similarly V_{3i+1} must be past V_{3i-1}, still large enough so that I_{3i} is a uniform neighborhood of $V_{3i} - V_{3i+1}$, and also large enough so that $I_{3i} \cup V_{3i+1}$ is a uniform neighborhood of $U_1 \cup \cdots \cup U_{i+1} - U_{i+2}$. Axiom (iv) does all this, and completes the construction.

Thus the δ-chains form a normal family of coverings, a sub-basis for a uniformity μ. Is the identity mapping from the δ-space X to the uniform space μX a δ-isomorphism? If A and B are not δ-related, then $\{X-A, X-B\}$ is a δ-chain; so A and B are far in μX. On the other hand, if $A \delta B$, consider any δ-chain $\{U_1, \cdots, U_n\}$. By axiom (i), some $(A \cap U_i)\delta(B \cap U_j)$. Then the difference between i and j does not exceed 1. At worst we have $(A \cap U_i)\delta (B \cap U_{i+1})$. Since $A \cap U_i - U_{i+1}$ and $B \cap U_{i+1} - U_i$ are far, we conclude that $(A \cap U_k)\delta(B \cap U_k)$ either for $k = i$ or for $k = i+1$. Now for any δ-chain $\{V_1, \cdots, V_m\}$, there must be l such that $(A \cap U_k \cap V_l)\delta(B \cap U_k \cap V_l)$; and the argument extends to any finite intersection of δ-chains. Hence A and B are near in μX. and Theorem 35 is completely proved.

38. *Every δ-mapping from a metric space to a uniform space is uniformly continuous.*

Proof. Since every uniform space can be embedded in a product of metric spaces, and the coordinate projections are δ-mappings, we may assume both the domain and range are metric. If $f: X \to Y$ is not uniformly continuous then for some positive ϵ there are points a_n, b_n in X with distance approaching zero while $f(a_n)$, $f(b_n)$ are at least ϵ apart. If we can find an infinite set of indices I such that $\{f(a_n) : n \in I\}$ and $\{f(b_n) : n \in I\}$ are far from each other in Y, we can conclude f is not a δ-mapping.

Consider the set S of all $f(a_n)$ and all $f(b_m)$. If S has an infinite subset of diameter at most $\epsilon/2$, there is such a set consisting entirely of $f(a_n)$'s or entirely of $f(b_m)$'s. If all $f(a_n)$, $n \in I$, are within $\epsilon/2$ of each other, then every $f(b_m)$, $m \in I$, is distant at least $\epsilon/2$ from all these $f(a_n)$, and we have the proof. Similarly for b's. In the contrary case the $(\epsilon/4)$-neighborhood of any of these points contains only finitely many others; so by induction we can find an infinite set I such that all $f(a_n)$ and $f(b_m)$, m and n in I, are separated by distances at least $\epsilon/4$. The cases are exhausted and the proof is complete.

This shows that each metric uniformity μ is finer than any other δ-isomorphic uniformity. (Two uniformities on the same set of points are called δ-isomorphic if the identity mapping is a δ-isomorphism between them.) We shall conclude the treatment of proximity spaces with a list of assertions whose proofs are outlined in the exercises. First

39. *For some precompact uniformities there is no finest δ-isomorphic uniformity. In fact. this is true for any product space $X \times Y$ each of whose factors is δ-isomorphic with some nonprecompact space.*

40. *Products of δ-isomorphic spaces need not be δ-isomorphic. In fact, whenever X and Y are not precompact, $X \times Y$ is not δ-isomorphic with $pX \times pY$.*

On the other hand

41. *The category of proximity spaces and δ-mappings does have sums, products, subspaces, and quotients. Specifically, two sums of families of uniform spaces whose summands are δ-isomorphic are themselves δ-isomorphic. Every family of proximity spaces has a product. δ-isomorphic uniformities on a set induce δ-isomorphic uniformities on its subsets. If $f: \mu X \to \sigma Y$ is a quotient mapping of uniform spaces, and ν is a uniformity on X δ-isomorphic with μ, then there is a uniformity τ on Y, δ-isomorphic with σ, such that $f: \nu X \to \tau Y$ is a quotient mapping.*

Hyperspace. There is another important notion of nearness for sets according to which A and B are near if they almost coincide. More precisely, for a uniform covering $\mathscr{U}$, A and B are *near of order* $\mathscr{U}$ if $\mathrm{St}(A, \mathscr{U})$ contains B and $\mathrm{St}(B, \mathscr{U})$ contains A.

Note that if A and B have the same closure they are near of order $\mathscr{U}$ for all $\mathscr{U}$. For closed sets, however, nearness of this sort is a finer screen than proximity, and it suffices to determine a uniformity on the *hyperspace* $H(X)$ of all nonempty closed subsets of a uniform space X. (The empty set could be included but is customarily left out; it would be an isolated point, as it is not near of order $\mathscr{U}$ to any nonempty set, for any $\mathscr{U}$.)

Here is one place (there is another at the beginning of the next chapter) where we pay a price for having defined uniformity by means of coverings. For metric spaces (both here and in the next chapter) there is a shortcut; cf. Theorem 48. In general, nearness of order $\mathscr{U}$ is a binary relation or entourage $H(\mathscr{U})$ on $H(X) : H(\mathscr{U})$ is precisely the set of all ordered pairs $(A, B) \in H(X) \times H(X)$ such that A and B are near of order $\mathscr{U}$. We can simplify the general method of entourages outlined in Exercise I.8., but the reader should recall, or consult, that exercise for the constructions $S \circ S$ and $\mathscr{V}(S)$ and perhaps for background.

A *symmetric entourage basis* $\mathscr{E}$ on a set X is a set of reflexive symmetric relations $R, S, \cdots$, on X satisfying

(i) for any R, S, in $\mathscr{E}$ there is T in E contained in $R \cap S$;

(ii) for any R in $\mathscr{E}$ there is S in $\mathscr{E}$ such that $S \circ S \subset R$;

(iii) for any two distinct points x, y, of X there is R in E such that $(x, y) \notin R$.

As a supplement to Exercise I.8 we have

42. *If $\mathscr{E}$ is a symmetric entourage basis on X then the coverings $\mathscr{V}(S), S \in \mathscr{E}$, form a basis for a uniformity on X.*

43. *For any uniform space μX, the relations $H(\mathscr{U}), \mathscr{U} \in \mu$, form a symmetric entourage basis on $H(\mu X)$. Thus $H(\mu X)$ is a uniform space.*

PROOF. $H(\mathscr{U} \wedge \mathscr{V}) \subset H(\mathscr{U}) \cap H(\mathscr{V})$; $H(\mathscr{U}) \circ H(\mathscr{U}) \subset H(\mathscr{U}^*)$; if $A \neq B$ there is a point x of $(A - B) \cup (B - A)$ and (since A and B are closed) a uniform covering $\mathscr{U}$ such that A and B are not near of order $\mathscr{U}$.

This gives us a notion of convergence for subsets: the convergence of a filter of families of closed sets to a limiting closed set. This is awkward to work with, and we shall restate it, beginning with nets. A *net* is a function whose domain is a directed set and whose range is a topological or uniform space. A net is often denoted by such a symbol as $\{x_\alpha\}$, where the subscript α ranges over a directed set and x_α is the value of the net on the index α. The net $\{x_\alpha\}$ is *convergent* to the point x if for every neighborhood N of x, the set $\{\alpha : x_\alpha \in N\}$ is nonempty and residual (hence cofinal). This condition is also described by the language "x_α is *finally* in N". More generally, x is a *cluster point* of $\{x_\alpha\}$ if each $\{\alpha : x_\alpha \in N\}$ (N a neighborhood of x) is cofinal. Moreover, a net in a uniform space is *Cauchy* if for each uniform covering $\mathscr{U}$ the net is finally in some element of $\mathscr{U}$. We defer to Exercise 16 the proof of

44. *A uniform space is complete if and only if every Cauchy net is convergent.*

A *cluster point* of a net of subsets A_α of X is a point of X each of whose neighborhoods meets A_α for a cofinal set of indices α. Strictly, this should be called an *X-cluster point;* but confusion is rare and can always be removed by referring to "a cluster point $x \in X$".

45. *The set C of cluster points of a net of subsets of X is closed. If the net converges in $H(X)$, it converges to C.*

Proof. Since every neighborhood of a limit point of C is a neighborhood of a point of C, C is closed. If the net converges to $A \in H(X)$, it is finally in each neighborhood of A; hence its members finally fail to meet a neighborhood in X of any point not in A but must finally meet each open set meeting A.

A proper filter $\mathscr{F}$ in a uniform space X may be regarded as a net $\{F_\alpha\}$ of subsets of X; the subscripts are arbitrary indices, ordered by the inclusion ordering of the members F_α of $\mathscr{F}$. If we confine attention to the closed sets $F_\alpha \in \mathscr{F}$, we have a net $\{F_\alpha\}$ in the uniform space $H(X)$. The filter $\mathscr{F}$ is said to be *uniformly convergent to a set,* or *hyperconvergent,* if this net $\{F_\alpha\}$ is convergent. The filter F is *stable* if $\{F_\alpha\}$ is a Cauchy net. It is time we noted explicitly that every convergent net is Cauchy; so every hyperconvergent filter is stable.

46. *A stable filter in a uniform space X is precisely a proper filter $\mathscr{F}$ in X such that for every uniform covering $\mathscr{U}$, there is $A \in \mathscr{F}$ such that for every $B \in \mathscr{F}$,* $\mathrm{St}(B, \mathscr{U}) \supset A$.

Proof. By definition of a stable filter $\mathscr{F}$, finally $\mathrm{St}(B, \mathscr{U}) \supset A$; that is, every $F \in \mathscr{F}$ has a subset B satisfying this condition, which implies $\mathrm{St}(F, \mathscr{U}) \supset A$. Conversely, if $\mathscr{F}$ satisfies the given condition, then for any $\mathscr{U}$ we have $A \in \mathscr{F}$ such that (in particular) for subsets B of A in $\mathscr{F}$, $\mathrm{St}(B, \mathscr{U}) \supset A$. Since $\mathrm{St}(A, \mathscr{U}) \supset A \supset B$, all subsets of A in $\mathscr{F}$ are within $\mathscr{U}$ of A. Clearly, this makes $\{F_\alpha\}$ Cauchy and $\mathscr{F}$ stable.

47. *The hyperspace $H(X)$ is complete if and only if every stable filter in X is hyperconvergent.*

Proof. If $H(X)$ is complete then from II.44 and definitions, every stable filter is hyperconvergent. For the converse, let $\{G_\alpha\}$ be a Cauchy net in $H(X)$. Let $\mathscr{F}$ be the set of all subsets F of X which contain the union $\bigcup[G_\alpha : \alpha\in R]$ for some nonempty residual set of indices R. For any uniform covering $\mathscr{U}$ of X, there is a residual set R of indices α such that all $G_\alpha, \alpha\in R$, lie in one element of $\mathscr{V}(H(\mathscr{U}))$. That is, there is $K\in H(X)$ such that all these G_α are within $\mathscr{U}$ of K. Then if A is $\mathrm{St}(K,\mathscr{U})$, $A\in\mathscr{F}$; and for any $B\in\mathscr{F}$, B contains some G_α $(\alpha\in R)$, so $\mathrm{St}(B, \mathscr{U}^*)$ contains A. Thus $\mathscr{F}$ is stable. From the hypothesis, $\mathscr{F}$ is hyperconvergent to its set of cluster points C. This means that for any $\mathscr{U}$, for some residual set R, $\bigcup[G_\alpha : \alpha\in R]$ is within $\mathscr{U}$ of C; and the same is true for each residual subset of R. For some residual $S\subset R$, all G_α $(\alpha\in S)$ are within $\mathscr{U}$ of each other; hence they are within $\mathscr{U}$ of their union and within $\mathscr{U}^*$ of C.

Two nets or filters in a uniform space are called *equivalent* if they converge to the same point in the completion. What we just proved is that each Cauchy net in $H(X)$ is equivalent to a stable filter in X.

48. THEOREM. *The hyperspace of a metric space is metric. The hyperspace of a complete metric space is complete.*

PROOF. The *Hausdorff distance* between two (closed) sets A, B in a metric space is defined as the maximum of $\sup[d(a, B) : a\in A]$ and $\sup[d(A,b) : b\in B]$. (As usual, $d(x,S)$ is $\inf[d(x,s) : s\in S]$.) It may be infinite. For some geometric purposes the value ∞ should be admitted, but here we apply II.20 and consider only bounded metrics. Then the Hausdorff distance $\rho(A,B)$ is a pseudometric; for, first, it is non-negative by definition. Second, for sets A, B, C, with arbitrarily small error ϵ, the distance $\rho(A,C)$ has the form $d(a, C)$. (Interchange A and C if necessary.) Were $\rho(A,B)+\rho(B,C)$ smaller than this we could find b in B so close to a, and c in C so close to b, that $d(a,b)+d(b,c)<d(a,c)$; a contradiction. Further, $\rho(A,B)$ separates points, since the sets are closed.

If $\mathscr{U}$ is a uniform covering of the metric space X, having Lebesgue number ϵ, then $\rho(A,B)<\epsilon$ implies that A and B are near of order $\mathscr{U}$. In the other direction, if A and B are near of order $\mathscr{V}$, where every element of V has diameter at most ϵ, then $\rho(A,B)\leqq\epsilon$. Hence the hyperspace is the metric uniform space determined by ρ on $H(X)$.

Suppose X is complete metric, and consider a Cauchy sequence $\{A_n\}$ in $H(X)$. For any positive number ϵ, if n is an index after which the A's are within ϵ of each other, then within 2ϵ of any point a of A_n we can find a cluster point c of the sequence. For this, put $n_0=n$, and select increasing indices n_i after which the A's are within $\epsilon/2^i$ of each other. Put $a_0=a$, and having a_i, select a_{i+1} from $A_{n_{i+1}}$ within $\epsilon/2^i$ of a_i. We get a Cauchy sequence of points a_i, which must converge to c. Thus the 2ϵ-neighborhood of the set C of cluster points contains A_n. But the ϵ-neighborhood of A_n must contain C; for a point distant by $\epsilon+\delta$ from A_n is distant by δ from all succeeding A_m, and so not a point of C. Hence $\{A_n\}$ is convergent and the proof is complete.

The hyperspace of a compact space is complete, too. A mild generalization of this fact is occasionally useful. A proper filter $\mathcal{F}$ in a uniform space is called *semi-Cauchy* if for every uniform covering $\mathcal{U}$, some finite union of elements of $\mathcal{U}$ belongs to $\mathcal{F}$. In a precompact space, every nonempty filter is semi-Cauchy.

49. *Every semi-Cauchy filter is stable, and, in a complete space, hyperconvergent to a compact set. The hyperspace of a compact space is compact.*

PROOF. Let $\mathcal{F}$ be a semi-Cauchy filter and $\mathcal{U}$ a uniform covering. The union of some $U_1, \cdots, U_n$ in U belongs to $\mathcal{F}$, and we can choose them so that if any U_i were omitted the union would fail to belong to $\mathcal{F}$. Then since $\mathcal{F}$ is a filter, no element B of $\mathcal{F}$ can fail to meet every U_i; thus $\mathrm{St}(B, \mathcal{U})$ contains $U_1 \cup \cdots \cup U_n$ for all B, and $\mathcal{F}$ is stable.

Let C be the intersection of the closed sets belonging to $\mathcal{F}$. If C were not precompact, there would be a uniform covering $\mathcal{U}$ for which infinitely many elements meeting C had disjoint stars (II.33), and $\mathcal{F}$ would not be semi-Cauchy. Thus if the space is complete, C is compact. To see that every $\mathrm{St}(C, \mathcal{U})$ contains an element of $\mathcal{F}$, take a uniform star-refinement $\mathcal{V}$ and a minimal finite union $V_1 \cup \cdots \cup V_n \in \mathcal{F}$. It suffices to show that each V_i^- meets C. The sets $A \cap V_i^-$, $A \in \mathcal{F}$, still form a proper filter since $\{V_i\}$ is minimal. Let $\mathcal{G}$ be an ultrafilter containing it. Since $\mathcal{G} \supset \mathcal{F}$, $\mathcal{G}$ is semi-Cauchy; since $\mathcal{G}$ is an ultrafilter, this means $\mathcal{G}$ is Cauchy, and therefore convergent to a point c. But $c \in C \cap V_i^-$. Conversely, since every A^- in $\mathcal{F}$ contains C, so does every $\mathrm{St}(A, \mathcal{U})$; and $\mathcal{F}$ converges to C.

This shows that the hyperspace of a compact space is complete. For precompactness, for every finite uniform covering $\mathcal{U} = \{U_i : i \in I\}$, select a nonempty closed set $A(J)$ meeting exactly those U_i for which $i \in J$, for each subset J of I for which such a selection is possible. Evidently every point of the hyperspace is within $\mathcal{U}$ of one of these. This completes the proof.

We remark that the hyperspace of a uniformly discrete space is uniformly discrete.

A space whose hyperspace is complete is called *supercomplete*. Every supercomplete space is complete, since the one-point sets form a subspace of $H(X)$ isomorphic with X.

50. *A discrete space of the power of the continuum, with the uniformity determined by all countable coverings, is complete but not supercomplete.*

PROOF. For the proof of completeness it is convenient to identify the points with real numbers in $[0, 1]$. Let $\mathcal{F}$ be a Cauchy filter. For any descending sequence of elements F_n of $\mathscr{F}$, the sets $F_n - F_{n+1}$ with the complement of F_1 and the intersection $\bigcap F_n$ form a countable covering; hence $\bigcap F_n$ is in $\mathscr{F}$. Then $\mathscr{F}$ has an element of minimum diameter, F_0. If the diameter of F_0 were not 0 we could split it into two sets of smaller diameter, one of which would have to be in the Cauchy filter $\mathscr{F}$. Hence F_0 is a single point and the space is complete.

Consider the filter $\mathscr{G}$ of all sets whose complements are countable. $\mathscr{G}$ has no cluster points so is not hyperconvergent. But for any countable covering $\{U_i\}$, let G be the union of all U_i which are uncountable. Then $G \in \mathscr{G}$; and for any $B \in \mathscr{G}$, B meets every uncountable set and $\mathrm{St}(B, \{U_i\})$ contains G. Thus the space is not supercomplete.

The characterization of supercomplete spaces requires more concepts; we reach it in Chapter VII.

Exercises.

1. Prove Propositions 17, 26 and 42.

2. The uniformity of a product of factors X_α has a basis consisting of finite intersections of coverings $p_\alpha^{-1}(\mathscr{U})$, $\mathscr{U}$ a uniform covering of X_α.

3. A *pure epimorphism* is an epimorphism f such that whenever $f = gh$ with g a monomorphism, h is an epimorphism.

(a) Every polar epimorphism is pure.

(b) In uniform spaces the pure epimorphisms are precisely the onto mappings.

4. A *polar monomorphism* is the precise dual of a polar epimorphism: a monomorphism $e: S \rightarrow X$ such that for every $f: W \rightarrow X$ not of the form et, there exist g, h on X to some Y such that $ge = he$, $gf \neq hf$.

(a) An embedding of a closed subspace is a polar monomorphism.

(b) A monomorphism which is not an embedding is not polar.

(c) A mapping with image not closed is not a polar monomorphism.

5. Quotient topology.

(a) There exist quotient mappings $f: X \rightarrow Y$ of uniform spaces such that the topological quotient of X by f is not uniformizable. In fact, for every non-normal uniformizable space X, in the fine uniformity, there is such a mapping.

(b) In the category of uniformizable spaces and continuous mappings, there are quotients; every mapping f is $f_0 q$, where q is a polar epimorphism and f_0 is one-to-one. The topology of the quotient space is called the *uniformizable quotient topology.*

(c) There exist quotient mappings $f: X \rightarrow Y$ of uniform spaces such that the uniformizable quotient topology differs from the uniform topology of Y. X and Y may be complete metric spaces, even plane curves: X a hyperbola plus asymptotes.

6. A metric quotient space of a complete metric space (e.g. of the set of all (x, y) in the plane with $xy \geqq 1$) need not be complete.

*7. METRIC UNIFORMITIES. A uniform space is metric if and only if it has a countable basis. There are at least two ways to prove this, a direct argument modeled on I.14 and an argument using the results of I.14 and II.18.

In the next two exercises we simplify the category by removing all but one member of each class of mutually isomorphic objects. The results could be stated in another way, along the lines of II.27.

8. REFLECTIVE SUBCATEGORIES. We define a *reflective* subcategory as a full subcategory $\mathscr{C}$ of a given category in which every object has a $\mathscr{C}$-reflection. First note that Theorem 27 applies; there exists a reflection functor upon $\mathscr{C}$, perhaps more than one.

It is an open question whether every intersection of reflective subcategories of uniform spaces is reflective. We can prove that every set of spaces is contained in a smallest reflective subcategory, as follows.

Suppose $\mathscr{C}$ is a reflective subcategory of uniform spaces.

(a) Every product of spaces in $\mathscr{C}$ is isomorphic with its reflection, and therefore belongs to $\mathscr{C}$ (by the simplifying assumption above).

(b) For any $f: X \to Y$ and $g: X \to Y$ in $\mathscr{C}$, the subspace $\{x \in X : f(x) = g(x)\}$ is in $\mathscr{C}$. The same holds for any intersection of subspaces of X of this type.

(c) For any set $\mathscr{S}$ of spaces, there exists a smallest full subcategory $\mathscr{C}_0$ containing $\mathscr{S}$ and satisfying (a) and (b).

(d) The category $\mathscr{C}_0$ of step (c) is reflective.

9. ARITHMETIC OF REFLECTIONS. Let $\mathscr{C}$ and $\mathscr{D}$ be reflective subcategories of any category, with reflection functors C, D (idempotent).

(a) Even if $\mathscr{C} \cap \mathscr{D}$ is reflective, CD need not be a reflection functor. (Hint: try the reflections in this chapter.)

(b) Suppose all the reflection mappings $X \to X|\mathscr{C}$ are epimorphic, and every epimorphism whose domain is in $\mathscr{D}$ has its range also in $\mathscr{D}$. Then CD is a reflection functor; and $(DC)^2 = CD$.

10. COARSEST UNIFORMITY. If X is a locally compact uniformizable space and Y any compactification of X, there is a (unique) continuous mapping from Y to the one-point compactification αX which coincides on X with the identity. Use this and precompact reflection to prove:

(a) A locally compact topology has a coarsest compatible uniformity.

On the other hand, a coarsest uniformity must evidently be precompact; hence it is determined by a embedding in a compact space, whose uniformity is determined by the topology.

(b) If Y is a compactification of X and $Y - X$ contains two or more points, then the uniformity induced on X by embedding in Y is not the coarsest compatible one.

(c) If X is not locally compact, it has no coarsest compatible uniformity.

11. UNIQUE UNIFORMITY.

(a) If there is an unbounded continuous real valued function on a topological space X, there is a continuous function f such that neither $f^{-1}(0)$ nor $f^{-1}(1)$ is compact.

(b) A topological space admitting a compatible uniformity which is not precompact has unbounded continuous real-valued functions.

(c) A uniformizable space X has a unique compatible uniformity if and only if the Stone-Čech compactification βX contains at most one point not in X.

A proximity space has a unique compatible uniformity if and only if it has no nonprecompact compatible uniformity. These spaces have not been investigated. The question of a finest compatible uniformity is considered in Alfsen and Njåstad [1]. See Notes, p. 35.

12. FINEST UNIFORMITY. Let X and Y be precompact spaces which are δ-isomorphic with nonprecompact spaces X_0, Y_0, respectively.

(a) $p(X\times Y_0) = X\times Y$. To prove this, one must show that a finite covering $\mathscr{W}$ of $X\times Y_0$ having a refinement $\{U_i\times V_\alpha\}$, i running through a finite set and α, an arbitrary set, has a refinement of the form $\{U_i\times T_j\}$, where the T_j's are a finite number of unions of V_α's exhausting $\{V_\alpha\}$. Do it by considering equivalence relations R_i on the indices α: $(\alpha, \alpha')\in R_i$ if the same elements of $\mathscr{W}$ contain $U_i\times V_\alpha$ and $U_i\times V_{\alpha'}$.

(b) $p(X_0\times Y_0)$ is not $X\times Y$. In fact, if $\{x_n\}$, $\{y_n\}$ are countably infinite uniformly discrete sets in X_0 and Y_0, the set of all (x_n, y_n) in $X_0\times Y_0$ is far from the set of all (x_m, y_n), $m\neq n$; but the images of these sets in $X\times Y$ are near (Proposition 40).

(c) The proximity of $X\times Y$ is induced by various uniformities among which none is finest (Proposition 39).

*13. FAR AND SEPARATED.

(a) The following conditions on a pair of sets A, B in a uniform space μX are equivalent:

(i) A and B are far in μX;

(ii) A and B are far in $p\mu X$;

(iii) A and B have disjoint closures in $\beta\mu X$;

(iv) A and B are separated by a bounded uniformly continuous real-valued function on μX.

(b) For a normal space X in the uniformity induced by embedding in βX (the *Čech uniformity*), every finite open covering is uniform.

(c) A uniformizable space is normal if and only if every finite open covering is normal.

*14. ENTOURAGES AND TOPOLOGY.

(a) Every entourage in $X\times X$ is a neighborhood of the diagonal.

(b) If every neighborhood of the diagonal is an entourage, the uniformity is fine.

(c) For a fine paracompact space, every neighborhood of the diagonal is an entourage.

15. PROXIMITY CONSTRUCTIONS.

(a) Sums of δ-isomorphic uniform spaces are δ-isomorphic; subspaces of δ-isomorphic spaces, with corresponding images, are δ-isomorphic; and these constructs are sums and subspaces in the category of δ-spaces and δ-mappings.

(b) The isomorphism of δ-spaces with precompact uniform spaces gives us the construction of products in the category of δ-spaces; they correspond to uniform products.

(c) Suppose $f: \mu X \to \sigma Y$ is a quotient mapping and ν is a uniformity on X δ-isomorphic with μ. Then $f: \nu X \to p\sigma Y$ is uniformly continuous.

(d) The quotient space τY obtained by factoring $f: \nu X \to p\sigma Y$ from part (c) is δ-isomorphic with σY. (Hint: Use Exercise 13.) This proves Proposition 41.

16. NETS.

(a) A convergent net is Cauchy; and a hyperconvergent filter is stable.

(b) If a point x is in the closure of a set S, there exist a net in S and a filter containing S which converge to x.

(c) Every Cauchy net is equivalent to a filter and every Cauchy filter is equivalent to a net.

(d) A uniform space is complete if and only if every Cauchy net is convergent (Proposition 44).

17. HYPERSPACE TOPOLOGY. Let μ, ν be two distinct uniformities on X. If they are homeomorphic we may identify the point sets $H(\mu X)$, $H(\nu X)$. But the identity mapping between $H(\mu X)$ and $H(\nu X)$ is never a homeomorphism. In fact, the identity from $H(\mu X)$ to $H(\nu X)$ is not continuous at X unless μ is finer than ν. For if $\mathscr{V} \in \nu$ and $\mathscr{V}^* \notin \mu$, then for every $\mathscr{U}$ in μ there is $A \subset X$ such that $\mathrm{St}(A, \mathscr{U}) = X \neq \mathrm{St}(A, \mathscr{V})$.

Notes. Concerning quotient mappings, pure and polar epi- and monomorphisms, and embeddings, see Isbell [7]. Cf. Semadeni [1].

The theory of reflections and coreflections was developed by Freyd [1]. It is important in many contexts that a reflective subcategory need not be full.

Proximity theory is extensively developed in the works of Smirnov [1; 2; 4]. Much of it will be covered in this book; instead of speaking of properties of proximity spaces we speak of δ-invariant properties of uniform spaces.

In connection with Exercise 14, spaces X for which every neighborhood of the diagonal in $X \times X$ is an entourage need not be paracompact. These spaces were characterized by Cohen [1] and named *almost 2-fully normal* by Mansfield [1]. (*Fully normal* means every open covering is normal; the fact that this is equivalent to *paracompact* should be known to the reader from topology, e.g. from Kelley [K].) The conjecture that a complete almost 2-fully normal space must be paracompact was raised by Kelley [K] and buried by Corson [1].

Sandberg [1] has axiomatized the notion of families of sets (not necessarily Cauchy but) containing arbitrarily small sets, and shown that these families determine the uniformity. Exercise 17 shows that a uniformity on X is also determined by what we may call arbitrarily large sets; specifically, by the families of sets whose families of closures have X as a limit point in $H(X)$. The exercise is due to H. Kenyon.

Added in proof. The proximity spaces admitting a unique uniformity are characterized in my paper *Spaces without large projective subspaces* submitted to Compositio Mathematica.

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CHAPTER III

FUNCTION SPACES

This chapter treats several topics in and around function spaces. For use later in the book are Ascoli's theorem (III.37), the category argument for existence of an embedding (III.34), and some definitions and preliminary results at the beginning of each quarter of the chapter (Propositions 1-2, 8-12, 19-21, 31-32). Each of the first three quarters wanders off into other interesting questions.

The functor U. Let $U(X, Y)$ denote the set of all uniformly continuous functions from X to Y. For each uniform covering $\mathscr{V}$ of Y, an entourage $U(\mathscr{V})$ is defined on $U(X, Y)$ as follows: $(f, g) \in U(\mathscr{V})$, or f and g are *near of order* $\mathscr{V}$, if for each $x \in X$, $f(x)$ and $g(x)$ are near of order $\mathscr{V}$ in Y. A routine check shows that these relations form a symmetric entourage basis; hence they define a uniformity which makes $U(X, Y)$ a uniform space, the *function space*.

As a functor, $U(X, Y)$ is contravariant in X and covariant in Y. For a pair of mappings $f: X \to X'$, $g: Y' \to Y$, the definition of $U(f, g): U(X', Y') \to U(X, Y)$ is $U(f, g)(h) = ghf$.

1. LEMMA. *Let X and Y be uniform spaces with uniformities defined by symmetric entourage bases $\mathscr{A}$, $\mathscr{B}$, respectively. For a function $f: X \to Y$ to be uniformly continuous it is necessary and sufficient that for each $S \in \mathscr{B}$ there is $R \in \mathscr{A}$ such that $(p, q) \in R$ implies $(f(p), f(q)) \in S$.*

PROOF. Suppose f is uniformly continuous and $S \in \mathscr{B}$. Take $T \in \mathscr{B}$ such that $T \circ T \subset S$, and consider the uniform covering $\mathscr{V}(T)$. Its inverse image is refined by some $\mathscr{V}(R)$, $R \in \mathscr{A}$. Then $(p, q) \in R$ implies $\{f(p), f(q)\}$ is contained in some $V_y \in \mathscr{V}(T)$, and $(f(p), f(q)) \in S$.

For the converse, if $(p, q) \in R$ implies $(f(p), f(q)) \in S$ then the image of each set V_x in $\mathscr{V}(R)$ is contained in the set $V_{f(x)} \in \mathscr{V}(S)$.

2. *U is a functor.*

PROOF. To see that $U(f, g)$ is a mapping, consider any basic entourage

$U(\mathscr{V})$ of $U(X, Y)$. We have gh_1f and gh_2f near of order $\mathscr{V}$ if h_1 and h_2 are near of order $g^{-1}(\mathscr{V})$. By Lemma 1, it is a mapping. The functorial property is clear from the defining formula.

There are some natural questions to ask about any functor; we have not considered them in Chapter II because wherever they were nontrivial, we were not ready for them. Now we ask first, when is $U(f, g)$ an isomorphism?

3. *If $U(f, g) : U(X', Y') \to U(X, Y)$ is an isomorphism then it is the product of isomorphisms $U(f, 1) : U(X', Y') \to U(X, Y')$ and $U(1, g) : U(X, Y') \to U(X, Y)$. The necessary and sufficient condition for $U(1, g)$ to be an isomorphism is that X is empty or g is an isomorphism.*

PROOF. First, if both X and X' are empty, everything is true. If X is empty and X' not, then since $U(X, Y)$ is a one-point space, so is $U(X', Y')$, so also Y'; and everything is true. Next, if g is an isomorphism then, clearly, so is $U(1, g)$. It remains to prove that if X is nonempty and $U(f, g)$ is an isomorphism then g and $U(f, 1)$ are isomorphisms. Consider the subspaces of constant mappings, Y_0 in $U(X, Y)$, Y_0' in $U(X', Y')$. Since X is nonempty, Y_0 is clearly isomorphic with Y. Since f maps X into X', X' is nonempty and Y_0' is isomorphic with Y'. Since $U(f, g)$ is onto, g is onto; so $U(f, g)$ maps Y_0' isomorphically onto Y_0. It follows that g is an isomorphism. Then $U(f, 1)$ is the product of two isomorphisms, $U(f, g)$ and $U(1, g^{-1})$.

The problem of $U(f, 1)$ remains open. Reference to the corresponding problem in abelian groups is interesting and suggests a restatement of the results on $U(1, g)$ (valid for both abelian groups and uniform spaces) and a conjecture on $U(f, 1)$ (valid at least for abelian groups). We omit details concerning groups, but note without proof that the factorization of Proposition 3 fails there; for example, it fails for the natural isomorphism $U(0, 0)$ between $\mathrm{Hom}(Z_2, Z_3)$ and $\mathrm{Hom}(Z_3, Z_2)$.

Motivated by the facts for abelian groups, we define a construct $\mathrm{Im}(X, Y)$. It is a very simple construct: Y, unless X is empty, and in that case $\mathrm{Im}(X, Y)$ is the empty space. However, the natural definition is longer. Form the sum Σ of a family of copies X_f of X, indexed by $f \in U(X, Y)$. Let $m : \Sigma \to Y$ be the mapping which on each summand $X_f = X$ is f. Then $\mathrm{Im}(X, Y)$ is the subspace $m(\Sigma)$ of Y.

Note that the definition shows how to define a covariant functor of one variable, $\mathrm{Im}(X, \ )$, for fixed X; any mapping from Y to Y' takes $m(\Sigma) \subset Y$ into $m'(\Sigma') \subset Y'$. There is no functor of X involved; X has been "summed out".

From Proposition 3:

4. *A mapping $U(1, g) : U(X, Y') \to U(X, Y)$ is an isomorphism if and only*

if $\mathrm{Im}(X, g) : \mathrm{Im}(X, Y') \to \mathrm{Im}(X, Y)$ *is an isomorphism.*

On the other side we define a construct $\mathrm{Ref}(X, Y)$. Begin with the product Π of a family of copies Y_f of Y, indexed (again) by $f \in U(X, Y)$. Let $r : X \to \Pi$ be the mapping whose fth coordinate projection $r_f : X \to Y$ is f. Now $\mathrm{Ref}(X, Y)$ is the subspace of Π which is the union of all subspaces E containing $r(X)$ such that any two mappings from E to Y which agree on $r(X)$ are identical.

Evidently $\mathrm{Ref}(X, Y)$ itself has the property that any two mappings to Y which agree on $r(X)$ are identical. Moreover, every mapping of X to Y, being represented by a coordinate projection on Π to Y, is factorable over $r : X \to \mathrm{Ref}(X, Y)$. Thus $U(r, 1) : U(\mathrm{Ref}(X, Y), Y) \to U(X, Y)$ is one-to-one onto. It is an open question whether $U(r, 1)$ is always an isomorphism. If this were true, a simple argument on a diagram would prove the analog of Proposition 4: a map $U(f, 1)$ is isomorphic if and only if $\mathrm{Ref}(f, Y)$ is an isomorphism. (The proof that $\mathrm{Ref}(\ , Y)$ is a functor is left for an exercise. It is very easy, but a little more care is needed to show that it is a reflection—the reason for the name.)

We note without proof that among the few categories *having* function spaces there is at least one in which the following analysis is correct: "$U(f, g)$ is an isomorphism from $U(X', Y')$ to $U(X, Y)$ if and only if its factors $U(f, 1) : U(X', Y') \to U(X, Y')$ and $U(1, g) : U(X, Y') \to U(X, Y)$ are isomorphisms, which means that $\mathrm{Ref}(f, Y')$ and $\mathrm{Im}(X, g)$ are isomorphisms". This is true in the category of locally convex linear topological spaces.

Some worthwhile conclusions are suggested by the incomplete results above. If the reflection mapping $r : X \to \mathrm{Ref}(X, Y)$ is an epimorphism then $U(r, 1)$ is an isomorphism. Moreover,

5. *If Y lies in a category $\mathscr{L}$ in which X has a reflection, and the reflection mapping $r : X \to X | \mathscr{L}$ is an epimorphism, then $U(r, 1) : U(X | \mathscr{L}, Y) \to U(X, Y)$ is an isomorphism.*

Proof. Since r is an epimorphism and a reflection mapping, $U(r, 1)$ is one-to-one onto. For any open covering $\mathscr{V}$ of Y, if two mappings rf_1, rf_2 in $U(X, Y)$ are near of order $\mathscr{V}$, then f_1 and f_2, restricted to the dense set $r(X)$, are near of order $\mathscr{V}$; so f_1 and f_2 on $X | \mathscr{L}$ are near of order $\mathscr{V}^*$.

Another natural question is whether there is any space Y such that isomorphism of $U(X, Y)$ and $U(X', Y)$ implies isomorphism of X and X'. (From Proposition 3, any nonempty space X has the property that if $U(X, Y)$ and $U(X, Y')$ are isomorphic by a mapping $U(1, g)$, then Y and Y' are isomorphic. But the qualification "by a mapping $U(1, g)$" can be omitted if we take X to consist of one point.) There is no such space. One can use Proposition 5 and the fact that for any cardinal number m (infinite), every space maps onto a reflection in the category of all spaces having no uniformly discrete subset of power m. This is left as an exercise.

The systematic examination of a batch of further questions of this type gives us some easy exercises, some more difficult exercises, one or two interesting details, and one of the fundamental geometric problems of the theory of uniform spaces. The next proposition may be classified with the exercises.

6. *If $g: Y' \to Y$ is a monomorphism, embedding, or polar monomorphism, then so is $U(1, g)$ for every identity mapping $1: X \to X$. If g has a right inverse $g_0: Y \to Y'$, then so does $U(1, g)$; hence $U(1, g)$ is a polar epimorphism. If g has no right inverse, then $U(1, g)$ is not even an onto mapping for $1: Y \to Y$. For every $U(1, g)$ to be epimorphic, the necessary and sufficient condition on g is that $1: Y \to Y$ is a limit of mappings gg_α.*

For $1: X \to X$ to make $U(1, g)$ onto for every polar epimorphism g, X must be uniformly discrete; to make $U(1, g)$ epimorphic for every polar epimorphism g, $1: X \to X$ must be a limit of mappings into uniformly discrete subspaces of X.

7. *If $f: X \to X'$ is an epimorphism then for every identity mapping $1: Y \to Y$, $U(f, 1)$ is a polar monomorphism. If f has a left inverse then $U(f, 1)$ has a right inverse; if not, then $U(f, 1)$ need not be onto.*

Proposition 7 is like Proposition 6 except for the interesting detail that in one place we get out more than we put in; from a mere epimorphism we get a polar monomorphism. The reasons for this are worth thinking about; but we defer the matter to the exercises.

The "other side" of the second paragraph of Proposition 6 motivates much of the rest of this chapter; but the subject is far from being worked out.

Injective spaces. A uniform space Y is called *injective* if whenever A is a subspace of X, every mapping from A to Y can be extended to a mapping from X to Y. Equivalently, for every embedding $f: A \to X$, the map $U(f, 1): U(X, Y) \to U(A, Y)$ is onto.

"Injective" is a sort of semi-standard notion in categories. The present definition is consistent with definitions usually given in algebra, but superficially inconsistent with the usual definition in topology; for the latter, one demands only that $U(f, 1)$ be onto whenever f is a polar monomorphism. Actually, there is no inconsistency.

8. *Suppose a uniform space Y has the property that whenever A is a closed subspace of X, every mapping from A to Y can be extended to a mapping from*

X to Y. Then Y is complete; consequently, Y is injective.

PROOF. Let Y be any incomplete space, and Z its completion. We put $A = Y$, and construct X as follows. W will be the set of all ordinal numbers less than or equal to w, where w is some large regular initial ordinal; in the order topology, W is a compact Hausdorff space. Recall that any continuous function f on W or even on $W - w$ into a metric space is finally constant. Since Y is embeddable in a product of metric spaces, choice of a sufficiently large ordinal w will make every continuous function on $W - w$ into Y finally constant. Further, make sure that w exceeds the cardinal of some subset of Y having a limit point in $Z - Y$; let S be such a set, and p such a limit point of S. Then in the product space $Z \times W$, let X consist of $Y \times w$, $S \times W$, and $p \times (W - w)$; identify Y with $Y \times w \subset X$. Here Y is closed in X; but the sizes of S and the regular initial ordinal w assure that any continuous extension over $S \times W$ of the identity $1: Y \to Y$ must agree with coordinate projection on a uniform neighborhood of $S \times w$, and cannot be extended over $p \times (W - w)$.

Thus if Y has the property indicated, Y is complete. For any $A \subset X$ and $f: A \to Y$, f can be extended over the closure of A by II.10 and then over X; so Y is injective.

There are some injective spaces. Recall that in the category of compact spaces, an interval I is injective. The following important theorem comes out at once.

9. THEOREM. *A closed interval I is an injective uniform space.*

PROOF. Let $e: A \to X$ be an embedding and $f: A \to I$ a mapping. Apply Samuel compactification β; and observe that $\beta(e): \beta(A) \to \beta(X)$ is again an embedding. Hence the mapping $\beta(f)$ can be extended to $g: \beta(X) \to I$; and the reflection $r: X \to \beta(X)$ gives gr, extending f over X.

The real line is also an injective space in topology, but not in uniform spaces. We are not prepared to prove that this is true no matter what compatible uniformity one puts on the line (see Notes, p. 55), but for the usual metric we need only this: the function on the integers which takes n to n^2 is uniformly continuous (since the integers are uniformly discrete), but no extension over the line is uniformly continuous. The reason for the last assertion is worth noting, in a fairly general setting. A *convex metric space* is a metric space (X, d) in which every two points lie in a subspace isometric with a closed interval.

10. *For any uniformly continuous function f from a convex metric space X to a metric space Y, there is a constant γ such that for any two points p, q of X, $d(f(p), f(q)) \leqq \max(\gamma d(p, q), \gamma)$.*

PROOF. On the contrary assumption we could find intervals from p_n to q_n which are stretched by f by factors at least n $(n \to \infty)$ and to length at least n; and in these, intervals of lengths $\leqq 1/n$ stretched to lengths at least 1 —a violation of uniform continuity.

On the other hand, in metric geometry, the line is injective.

11. *If M is a metric space and R a subspace isometric with the real line, then there is a retraction $r: M \to R$ such that $d(r(x), r(y)) \leqq d(x, y)$ for all x, y in M.*

PROOF. Exercise 5.

12. *Every product of injective spaces is injective. Every retract of an injective space is injective.*

PROOF. If Π is a product of injective spaces Y_α and the subspace A of X is mapped into Π by f, then each coordinate function $f_\alpha: A \to Y_\alpha$ can be extended to $g_\alpha: X \to Y_\alpha$. This defines a mapping $g: X \to \Pi$ extending f. If Z is a retract of an injective space Y, and $A \subset X$, then $f: A \to Z$ can be considered as a mapping into Y. There is an extension $h: X \to Y$, and a retraction $r: Y \to Z$ gives $rh: X \to Z$ extending f.

It is an open question whether $U(M, Y)$ is injective whenever Y is injective. First suppose M is discrete.

13. *A function $f: X \to U(M, Y)$ is uniformly continuous (when M is uniformly discrete) if and only if the function $f^*: X \times M \to Y$ defined $f^*(x, m) = f(x)(m)$ is uniformly continuous.*

PROOF. Suppose f is uniformly continuous and $\mathscr{V}$ is a uniform covering of Y. There is a uniform covering $\mathscr{U}$ of X such that any two points near of order $\mathscr{U}$ are mapped by f to functions near of order $\mathscr{V}$. Then the product of $\mathscr{U}$ and the covering of M by all single points is finer than $f^{*-1}(\mathscr{V}^*)$. Conversely, if f^* is uniformly continuous, then each $f^{*-1}(\mathscr{V})$ is refined by a product covering, and $f^{-1}(\mathscr{V}(U(\mathscr{V})))$ is uniform.

14. *If M is discrete and Y is injective then $U(M, Y)$ is injective.*

PROOF. For $A \subset X$, $A \times M$ is contained in $X \times M$. Then a mapping $f: A \to U(M, Y)$ corresponds to a mapping $f^*: A \times M \to Y$, which has an extension $g^*: X \times M \to Y$, which yields an extension g of f.

15. THEOREM. *Every uniform space can be embedded in an injective uniform space.*

Proof. The unit ball of a space $l_\infty(S)$ is the set of all functions on S to $I=[-1, 1]$; and two functions are near of order ϵ in the metric if and only if they are near of order ϵ as functions. Thus this space is $U(M, I)$, where M is the set S with the fine discrete uniformity. Such spaces, and products of them, are injective by III.9, III.14, III.12; so II.21 completes the proof.

16. **Corollary.** *Every bounded uniformly continuous pseudometric on a subspace of a uniform space X can be extended to a (bounded) uniformly continuous pseudometric on X.*

Proof. Such a pseudometric on $A \subset X$ amounts to a mapping of A onto a bounded metric space Z. By II.19, Z can be isometrically embedded in an ell-infinity space; and since Z is bounded it will be embedded in an injective subspace $U(M, I)$. Then extension of the indicated mapping $f: A \to U(M, I)$ will yield an extension of the given pseudometric.

This corollary does not hold for unbounded pseudometrics, e.g. for $|x^2-y^2|$ on the subspace of integers in the real line.

17. **Corollary.** *A uniform space is injective if and only if it is a retract of every space containing it. A bounded metric space is injective if and only if it is a retract of every metric space containing it isometrically.*

Proof. If a space is injective then of course every embedding of it extends to a retraction. If it is a retract of some injective space it is injective; so the embeddings of III.15, II.19 complete the proof.

18. **Theorem.** *For every metric space M, the function space $U(M, I)$ is injective.*

Proof. It suffices to show that $U(M, I)$ is a retract of the unit ball F of $l_\infty(M)$. Write U for $U(M, I)$. Let d be a bounded metric on M equivalent to the given metric; let D be the diameter of M.

For f in F, let $\lambda(f)$ be the distance of f from the subspace U. If $\lambda(f) \neq 0$ then, since f is within $3\lambda(f)/2$ of a uniformly continuous function, there is $\epsilon > 0$ such that the diameter in I of the image under f of any subset of M of diameter ϵ is at most $3\lambda(f)$. Let $\mu(f)$ be the least upper bound of all such ϵ in $[0, D]$. On U, where $\lambda(f)=0$, let $\mu(f)=0$. Now for any real number t in $[0, 1]$, for any x in M, f in F, define $h(f, x, t)$ as the least upper bound of the set

$$\{g(y)\colon g \in F,\ y \in M,\ d(g, f) \leqq t\lambda(f),\ d(y, x) \leqq t\mu(g)\}.$$

Of course, $h(f, x, t)\in I$. Moreover, $g(x)\leqq f(x)+t\lambda(f)$, $g(y)\leqq g(x)+3\lambda(g)$, $f(x)+7\lambda(f)$ is an upper bound for $h(f, x, t)$ and $f(x)$ is a lower bound. If $f\in U$ then for all x and t, $h(f, x, t)=f(x)$. In any case $j(f, x)=\int h(f, x, t)\,dt$ exists (since the integrand is bounded and monotonic in t) and is uniformly continuous in x; for if f is not in U, then $\mu(f)>0$, and for x and y within $\delta\mu(f)$ of each other we have $|j(f, x)-j(f, y)|<\delta d(f, 0)+7\delta\lambda(f)$. Thus the formula $r(f)(x)=j(f, x)$ defines a retraction function r from F upon U.

To verify uniform continuity of r, suppose $d(f, g)<\delta^2<1$. If $\lambda(f)$ or $\lambda(g)$ is less than δ, we find that $r(f)$ is within $7\delta+7\delta^2$ of f, $r(g)$ is within the same distance from g, and $d(r(f), r(g))<14\delta+15\delta^2$. In the contrary case, the smaller of $\lambda(f)$ and $\lambda(g)$ is $k\delta$ for some $k\geqq 1$. Then $(1+\delta/k)\lambda(f)\geqq\lambda(g)$, and symmetrically. It follows that $h(f, x, t+\delta/k)\geqq h(g, x, t)$ for $t<1-\delta/k$, and symmetrically. On the other hand, for any u, v in $[0, 1]$, the difference between $h(f, x, u)$ and $h(g, x, v)$ does not exceed the supremum of the expressions $f'(y)-g'(z)$, where the conditions on f', g', y and z are as follows. The distance $d(f', f)$ is at most $\lambda(f)$; so $\lambda(f')\leqq 2\lambda(f)$. $d(x, y)\leqq\mu(f')$; so $|f'(y)-f'(x)|\leqq 3\lambda(f')\leqq 6\lambda(f)$. The like conditions hold for g' and z. Consequently, $|h(f, x, u)-h(g, x, v)|\leqq 14k\delta+8\delta^2$, and $|r(f)(x)-r(g)(x)|<22\delta^2$. This completes the proof.

Equiuniform continuity and semi-uniform products. What got us off the ground in the matter of injective spaces was Proposition 13. Further thought about it would naturally lead to the following definition. A family $\{f_\alpha\}$ of functions on a uniform space X (or even on various subspaces S_α of X) to a uniform space Y is called *equiuniformly continuous* (with respect to X) if for each uniform covering $\mathscr{V}$ of Y there is a uniform covering $\mathscr{U}$ of X such that every $f_\alpha^{-1}(\mathscr{V})$ is refined by $\mathscr{U}$ (or by the trace of $\mathscr{U}$ on S_α).

19. *A family $\{f_\alpha:\alpha\in A\}$ of functions on X to Y is equiuniformly continuous if and only if, when A is considered as a fine discrete space, the function $f^*: X\times A\to Y$ defined $f^*(x,\alpha)=f_\alpha(x)$ is uniformly continuous.*

PROOF. Either condition means that for x_1, x_2 sufficiently near (near of order $\mathscr{U}$, for some $\mathscr{U}$), $f_\alpha(x_1)$ and $f_\alpha(x_2)$ are arbitrarily near (of order $\mathscr{V}$); α is fixed but the order of nearness needed in X does not depend on α.

In terms of equiuniform continuity we can explain why the theory of uniform spaces lacks some mappings which are present in algebra and topology. If we associate to each x in X a function x^{**} on $U(X, Y)$ to Y, defined $x^{**}(f)=f(x)$, we have an equiuniformly continuous family $\{x^{**}: x\in X\}$, but the function taking x to x^{**} is not uniformly continuous.

20. *The uniformity of a function space $U(X, Y)$ makes all the evaluation functions x^{**} form an equiuniformly continuous family, and is the coarsest*

uniformity having this property. Therefore a mapping $h: T \rightarrow U(X, Y)$ is precisely a function whose evaluations $h_x(t) = h(t)(x)$ form an equiuniformly continuous family in $U(T, Y)$.

PROOF. Since the basic coverings $\mathscr{V}(U(\mathscr{V}))$ refine $x^{**-1}(\mathscr{V}^*)$ for all x, the evaluations are equiuniformly continuous. But $\{x^{**-1}(\mathscr{V}) : x \in X\}$ has no common refinement which fails to refine $\mathscr{V}(U(\mathscr{V}))$. The second sentence of the proposition follows immediately.

Now apply this to the function on X to $U(U(X, Y), Y)$ taking x to x^{**}. For it to be uniformly continuous, its evaluations must be equiuniformly continuous; but this means all of $U(X, Y)$ is an equiuniformly continuous family on X. This is false, e.g. for $X = Y = I$.

Similarly the pairing function $\pi : U(X, Y) \times X \rightarrow Y$, $\pi(f, x) = f(x)$, is not generally uniformly continuous. In fact, the condition for uniform continuity of π is identical with the condition for uniform continuity of $x \rightarrow x^{**}$.

21. *On a product space $A \times B$, a function f into a uniform space is uniformly continuous if and only if the functions $f(a, \ )$ on B form an equiuniformly continuous family and the functions $f(\ , b)$ on A form an equiuniformly continuous family.*

PROOF. If f: $A \times B \rightarrow Y$ is uniformly continuous, then for every uniform covering $\mathscr{V}$ of Y there is a product covering $\{T_\alpha \times U_\beta\}$ finer than $f^{-1}(\mathscr{V})$. This implies that $\{T_\alpha\}$ refines every $f(\ , b)^{-1}(\mathscr{V})$ and $\{U_\beta\}$ refines every $f(a, \ )^{-1}(\mathscr{V})$. Conversely, if $\{T_\alpha\}$ and $\{U_\beta\}$ satisfy the latter conditions, then each set $f(T_\alpha \times U_\beta)$ is contained in one element of $\mathscr{V}^*$.

The loss of two such mappings as $**$ and π is not to be taken lightly. We have a remnant of π, for uniformly discrete spaces X; we have used it in Proposition 13, and it has already shown its worth. In trying to recover more of π, we should first think of changing the definition of the function space. But Proposition 21 shows that this will not do it. As long as we have a direct product, equiuniform continuity is necessary; and as a matter of fact, Proposition 20 shows that everything that can be done to the function space to save π has been done.

Therefore we turn to the product set $A \times B$, and try to define a uniformity on it so that the uniformly continuous functions f will be just those for which $f(\ , b)$ is equiuniformly continuous and $f(a, \ )$ is separately uniformly continuous. Provisionally we define a *semi-uniform function* on $A \times B$ as a function f satisfying these conditions. We define the *semi-uniform product* $A * B$ to be the product set $A \times B$ with the weak uniformity induced by all semi-uniform functions from $A \times B$ onto uniform spaces whose points form subsets of $A \times B$. This set-theoretic precaution assures us of a well-defined preuniformity; it is a uniformity finer than the product uniformity, in view of III.21. It is clear, further, that every semi-uniform function is uniformly continuous on A^*B.

22. THEOREM. *For any uniform spaces A, B, Y, a function on A^*B into Y is uniformly continuous if and only if it is semi-uniform. A^*B is homeomorphic with $A\times B$; so on the direct product, every semi-uniform function is continuous.*

PROOF. Suppose $g: A^*B\to Y$ is uniformly continuous, and $\mathscr{V}$ is a uniform covering of Y. By I.9, there are finitely many uniform coverings $f_\alpha^{-1}(\mathscr{U}_\alpha)$, f_α semi-uniform on A^*B, whose intersection refines $g^{-1}(\mathscr{V})$. Then there are uniform coverings $\mathscr{S}_\alpha$ of A all of whose members S satisfy $f_\alpha(S\times b)\subset U\in\mathscr{U}_\alpha$ for every $b\in B$; by finite intersection, there is a uniform covering $\mathscr{S}$ of A whose members satisfy these conditions for all α. Then $g(S\times b)\subset V\in\mathscr{V}$; and $g(\ \ ,b)$ is equiuniformly continuous. For any fixed a, the same argument shows that $g(a,\ \ )$ is a uniformly continuous function, q.e.d.

We have noted that the uniformity of A^*B is finer than the direct product uniformity. Consequently, a neighborhood of a point (a,b) in $A\times B$ is a neighborhood in A^*B. Suppose N is a neighborhood in A^*B. There is a semi-uniform real-valued function f which is 1 at (a,b) and 0 outside N. Since $f(a,\ \ )$ is continuous, there is a neighborhood U of b such that $f(a,u)>1/2$ for u in U. Since $f(\ \ ,u)$ is equiuniformly continuous, there is a neighborhood T of a such that $f(t,u)>0$ for $t\in T$, $u\in U$. Thus $T\times U\subset N$ and the proof is complete.

23. *Every uniform covering of A^*B has a refinement of the form $\{U_\alpha\times V_\beta^\alpha\}$, where $\{U_\alpha\}$ is a uniform covering of A and for each α, $\{V_\beta^\alpha\}$ is a uniform covering of B. Every covering which is normal with respect to the coverings of this form is uniform.*

PROOF. By I.14 and III.22, each uniform covering of A^*B is realized by a semi-uniform function $h: A^*B\to M$, M a metric space; it suffices to find a covering of the given form, the images of whose elements in M have diameters less than 1. By equiuniform continuity there is a uniform covering $\mathscr{W}$ of A such that when a and a' are near of order $\mathscr{W}$, $d(h(a,b),h(a',b))<1/3$ for all b. For each a in A there is a uniform covering $\{V_\beta^\alpha\}$ of B, the images of whose elements under $h(a,\ \ )$ have diameters at most $1/3$. Let $U_a=\mathrm{St}(a,\mathscr{W})$. Then for any two points both in a set $U_a\times V_\beta^a$, their images are at distance $<1/3+1/3+1/3$, as required.

Let μ be the uniformity (I.10) consisting of all coverings normal with respect to the family λ of all coverings of the form we are considering, $\{U_\alpha\times V_\beta^\alpha\}$. We have just shown that the uniformity of A^*B is contained in λ, and therefore contained in μ. That is, the identity function $1: \mu(A\times B)\to A^*B$

is a mapping. It remains to check that $1: A^*B \to \mu(A \times B)$ is semi-uniform. For each $\{U_\alpha \times V_\beta^\alpha\}$, all the inverse images under functions $1(\ \ , b): A \to A \times B$ coincide with the uniform covering $\{U_\alpha\}$; and each inverse image under a function $1(a,\ \ )$ is refined by one of the uniform coverings $\{V_\beta^\alpha\}$. Thus 1 is an isomorphism, and the proof is complete.

This restores the pairing mapping π on the space $U(X, Y)^*X$ to Y(since, by III.20, π is semi-uniform), and we know, as far as III.23 tells us, what the space $U(X, Y)^*X$ is. Now our previous results give us some conclusions, and suggest some more, concerning semi-uniform products. First, if A and B are both compact, then $A^*B = A \times B$. This follows from Theorem 22 with II.24 (homeomorphic compact spaces are isomorphic). Second, if B is discrete (uniformly) then $A^*B = A \times B$. This follows from III.21 and III.22, since all uniformly continuous functions on B are equiuniformly continuous. These facts and Theorem 18 suggest at least the following.

24. **Theorem.** *The semi-uniform product A^*B is the same as the direct product $A \times B$ if A is compact or B is discrete.*

25. *If B is metric, then whenever A is a subspace of X, A^*B is a subspace of X^*B. Consequently, for every injective space Y, $U(B, Y)$ is injective.*

It is not difficult to prove Theorem 24 by going right in and looking at the coverings. However, we have already proved the discrete half of it by looking at the functions, and we can profit from handling the rest in the same spirit.

26. *For any three spaces X, M, Y, $U(X, U(M, Y))$ and $U(X^*M, Y)$ are isomorphic; the correspondence which associates to each mapping $f: X \to U(M, Y)$ the mapping f^* defined $f^*(x, m) = f(x)(m)$ is an isomorphism.*

Proof. By III.20, uniform continuity of f into $U(M, Y)$ means that while each value $f(x) = f^*(x,\ \ )$ is uniformly continuous, the family of all evaluations $f^*(\ \ , m)$ is equiuniformly continuous; i.e., it is equivalent to uniform continuity of f^* on X^*M. Thus we have a one-to-one correspondence. If f_1, f_2 on X to $U(M, Y)$ are near of order $\mathscr{V}(U(\mathscr{V}))$, then f_1^* and f_2^* are near of order $\mathscr{V}^*$; if f_1^* and f_2^* are near of order $\mathscr{V}$, then f_1 and f_2 are near of order $\mathscr{V}(U(\mathscr{V}))$.

27. *Every precompact subset of a function space $U(X, Y)$ is equiuniformly continuous.*

Proof. If $\{f_\alpha\}$ is a precompact set of functions then for each uniform covering $\mathscr{V}$ of Y there is a finite set F of f_α's such that every f_α is within $\mathscr{V}$ of a member of F. By finite intersection, X has a uniform covering $\mathscr{U}$ the images of whose elements under any function in F are contained in elements of $\mathscr{V}$. The image of an element of $\mathscr{U}$ under any f_α is then contained in an element of $\mathscr{V}^*$; so $\{f_\alpha\}$ is equiuniformly continuous.

To conclude Theorem 24, suppose A is precompact and f^* is a semi-uniform function on $A \times B$ to some space Y. Then $f: A \to U(B, Y)$ defined as in Proposition 26 is uniformly continuous. Thus $f(A)$ is precompact and hence equiuniformly continuous. That is, besides $f^*(\ \ , b)$ forming an equi-uniformly continuous family, $f^*(a,\ \ )$ does also. By Proposition 21, $f^*: A \times B \to Y$ is uniformly continuous. Taking $Y = A^*B$, $f^* = 1$, we have the theorem.

For Proposition 25, the method will be quite direct. There is (again) a complementary result involving a restriction on A. A family of sets is called *σ-disjoint* if it is a union of countably many subfamilies each of which is a disjoint family.

28. THEOREM. *Suppose either that B is metric or that A has a basis of σ-disjoint uniform coverings. Then every covering of A^*B of the form $\{U_\alpha \times V_\beta^\alpha\}$, where $\{U_\alpha\}$ is a uniform covering of A and, for each α, $\{V_\beta^\alpha\}$ is a uniform covering of B, is uniform.*

PROOF. By III.23, it suffices to show that $\{U_\alpha \times V_\beta^\alpha\}$ has a star-refinement of the same form. Suppose B is metric. Then B has a countable basis of uniform coverings $\mathscr{V}^n$, which we may take to be a normal sequence. For each α, choose $n = n(\alpha)$ such that $\mathscr{V}^n$ refines $\{V_\beta^\alpha\}$. Let $\mathscr{T} = \{T_\lambda\}$ be a uniform covering of A satisfying $\mathscr{T}^{**} < \{U_\alpha\}$. For each λ, let $p(\lambda)$ be the least integer i such that for some U_α containing $\mathrm{St}(\mathrm{St}(T_\lambda, \mathscr{T}), \mathscr{T})$, $n(\alpha) = i$. Let $q = q(\lambda) = p(\lambda) + 1$. Form the covering $\mathscr{Z}$ consisting of all sets $Z_{\lambda\mu} = T_\lambda \times W_\mu^q$, where $q = q(\lambda)$ and $W_\mu^q \in \mathscr{V}^q$. To show that $\mathscr{Z}^* < \{U_\alpha \times V_\beta^\alpha\}$, consider any $Z_{\lambda\mu}$; consider the following conditions on indices α, β.

(i) U_α contains $\mathrm{St}(T_\lambda, \mathscr{T})$; and subject to this condition, $n(\alpha)$ is a minimum, n.

(ii) V_β^α contains $\mathrm{St}(W_\mu^q, \mathscr{V}^{n+1})$.

Since $n \leq p(\lambda)$, $\mathrm{St}(W_\mu^q, \mathscr{V}^{n+1})$ is contained in an element of $\mathscr{V}^n$, hence in some V_β^α; that is, (i) and (ii) can be satisfied. When they are satisfied, $U_\alpha \times V_\beta^\alpha$ contains every $T_\sigma \times W_\tau^r \in \mathscr{Z}$ which meets $Z_{\lambda\mu}$. For, T_σ is in $\mathrm{St}(T_\lambda, \mathscr{T}) \subset U_\alpha$. Also $W_\tau^r \subset \mathrm{St}(W_\mu^q, \mathscr{V}^{n+1})$; to see this, note that $r = p(\sigma) + 1$ is $n(\gamma) + 1$ for some U_γ which contains the star of $\mathrm{St}(T_\sigma, \mathscr{T}) \supset T_\lambda$, so that $n(\gamma)$ is not less than the minimum, n. By (ii), $W_\tau^r \subset V_\beta^\alpha$, and this case is finished.

In the other case we may suppose $\{U_\alpha\}$ to be re-indexed as $\{U_{n\alpha}\}$, where for fixed n the different sets $U_{n\alpha}$ are disjoint. Again let $\mathscr{T} = \{T_\lambda\}$ be a uniform covering, $\mathscr{T}^{**} < \{U_{n\alpha}\}$. Define $p(\lambda)$ as the least n such that some $U_{n\alpha}$ contains the iterated star $\mathrm{St}(\mathrm{St}(T_\lambda, \mathscr{T}), \mathscr{T})$. For each n, α, take a uniform covering $\mathscr{V}^{n\alpha} <^* \{V_\beta^{n\alpha}\}$. For each λ, for each $i \leq p(\lambda)$, there is at most one α such that $U_{i\alpha} \supset T_\lambda$; let $\mathscr{W}^\lambda = \{W_\mu^\lambda\}$ be a uniform covering of B refining all $\mathscr{V}^{i\alpha}$

for this finite collection of indices i, α. Form $\mathscr{G}$ consisting of all $T_\lambda \times W_\mu^\lambda$. Every $T_\sigma \times W_\tau^\sigma$ in $\mathscr{G}$ which meets $T_\lambda \times W_\mu^\lambda$ is contained in $U_{n\alpha} \times V_\beta^{n\alpha}$ selected as follows. (i) $U_{n\alpha}$ contains $\mathrm{St}(T_\lambda, \mathscr{T})$, hence contains T_σ; subject to this condition, n is a minimum. (ii) $V_\beta^{n\alpha}$ contains $\mathrm{St}(W_\mu^\lambda, \mathscr{V}^{n\alpha})$; this is possible for the same reason as before, $n \leqq p(\lambda)$ (with $T_\lambda \subset U_{n\alpha}$). Again $n \leqq p(\sigma)$ too. Since $T_\sigma \subset U_{n\alpha}$, $\mathscr{W}^\sigma < \mathscr{V}^{n\alpha}$; so $W_\tau^\sigma \subset V_\beta^{n\alpha}$, completing the proof.

Whenever A is a subspace of X, every uniform covering $\{U_\alpha\}$ of A is the trace on A of a uniform covering of X; hence the same is true for coverings $\{U_\alpha \times V_\beta^\alpha\}$ of a product set $A \times B \subset X \times B$. If B is metric, or X (not just A) has a basis of σ-disjoint coverings, this implies that $A*B$ is a subspace of $X*B$. Proposition 25 follows in view of Proposition 26.

On the other side:

29. Corollary. *If C is metric and B a subspace of C then $A*B$ is a subspace of $A*C$ for every uniform space A. Hence if Y is injective and $e: B \to C$ an embedding, the mapping $U(e, 1): U(C, Y) \to U(B, Y)$ has a right inverse.*

Proof. The coverings $\{U_\alpha \times V_\beta^\alpha\}$ of $A*B$ extend to coverings of the same form on $A*C$, by extending each $\{V_\beta^\alpha\}$. Then put $A = U(B, Y)$. The pairing mapping π from $A*B$ into the injective space Y extends to a mapping $f^*: A*C \to Y$. III.26 turns this into $f: A \to U(C, Y)$, i.e., $f: U(B, Y) \to U(C, Y)$. Since $f^*(a, b) = \pi(a, b) = a(b)$, $f(a)$ agrees with a on B; that is, $U(e, 1)f(a) = f(a)e = a$.

The mapping f, which takes each function a to an extension of it, is called a *simultaneous extension* operator. The corollary shows that as far as metric spaces are concerned, if Y always permits extension then it always permits simultaneous extension. (Recall that the question whether a metric space is injective can be settled by examining its relations with other metric spaces, III.17.)

The problem whether Theorem 28 (hence its corollaries, 25 and 29) generalizes to arbitrary uniform spaces is one of the principal currently unsolved problems of the theory; see below, Research Problem B_1. As things stand now, the σ-disjoint side of the theorem has corollaries like 25 and 29, but they are not very satisfying.

We can conclude one piece of unfinished business with

30. *If $e: B \to C$ is an embedding in any uniform space and Y is an injective space, then $U(e, 1): U(C, Y) \to U(B, Y)$ is a polar epimorphism.*

Proof. $U(e, 1) = r$ takes each $f: C \to Y$ to its restriction $f|B$. From II.7, it suffices (since we know r is onto) to show that no finer uniformity on $U(B, Y)$ leaves r uniformly continuous. We shall deduce this from (*): For every uniform covering $\mathscr{V}$ of Y there is a uniform covering $\mathscr{W}$ of Y such that for any mapping $f: C \to Y$, and any $g_0: B \to Y$ within $\mathscr{W}$ of $f \mid B$, g_0 has an extension $g: C \to Y$ within $\mathscr{V}$ of f.

Suppose (*) established, and $\{U_\alpha\}$ a nonuniform covering of $U(B, Y)$. For each uniform covering $\mathscr{V}$ of Y, we construct a set in $\mathscr{V}(U(\mathscr{V}))$ on $U(C, Y)$ which is not contained in any $r^{-1}(U_\alpha)$. Take $\mathscr{W}$ as described in (*), and note that $\mathscr{V}(U(\mathscr{W}))$ on $U(B, Y)$ does not refine $\{U_\alpha\}$. Thus some f_0: $B \to Y$ has the set of all mappings within $\mathscr{W}$ of it contained in no U_α. Let f: $C \to Y$ be an extension of f_0, and extend all these mappings within $\mathscr{V}$ of f.

For (*), we embed Y in a product Π of factors U_α which are unit balls of ell-infinity spaces; let q be a retraction of Π upon Y. The covering $q^{-1}(\mathscr{V})$ is a uniform covering of the product Π ; so it has a refinement $\mathscr{U}$ which is a finite intersection of coverings $p_\alpha^{-1}(\mathscr{U}'_\alpha)$, $\mathscr{U}'_\alpha$ the covering of U_α by all sets of diameter ϵ. Now given $f: C \to Y$ and $g_0: B \to Y$, to get an extension $g: C \to Y$ within $\mathscr{V}$ of f it suffices to get an extension $g_1: C \to \Pi$ within $\mathscr{U}$ of f, and put $g = qg_1$. For all coordinate indices α except finitely many, the coordinate functions $p_\alpha g_0$ may be extended in an arbitrary manner. For the remaining ones, if $p_\alpha g_0$ is within ϵ of $p_\alpha f \mid B$, the difference $p_\alpha(g_0 - f \mid B)$ is a mapping d_α into the ϵ-ball of an ell-infinity space; let e_α be an extension over C with values in the ϵ-ball, and define $p_\alpha g_1 = e_\alpha - p_\alpha f$. The covering $\mathscr{W}$ that we need is just the trace of $\mathscr{U}$ on Y.

Closure properties. The theorem that a uniform limit of uniformly continuous functions is uniformly continuous provides one of the most basic tools for the theory. It amply justifies the introduction of a space of discontinuous functions: the space $F(X, Y)$ of all functions defined on X with values in Y. The symbol $F(X, Y)$ will be used for this set of functions, no uniformity being involved unless specified. The *function space* uniformity, or uniformity of *uniform convergence*, is defined for $F(X, Y)$ just as for $U(X, Y)$. Inspection shows that the relations $U(\mathscr{V})$ on $F(X, Y)$ still form a symmetric entourage basis; so $F(X, Y)$ as a function space is a uniform space. Inspection also shows that $U(X, Y)$ is a subspace. The *product* or *pointwise* uniformity, or uniformity of *pointwise convergence*, on $F(X, Y)$, is the product uniformity; $F(X, Y)$ is, we note, logically identical with the product set of a family of copies of Y indexed by the points of X. These terms are used also for the uniformity induced on $U(X, Y)$ considered as a subset of the product $F(X, Y)$. The notation $U(X, Y)$ signifies the function space unless a different uniformity is specified.

31. THEOREM. *$U(X, Y)$ is complete if Y is complete. In any case $U(X, Y)$ is closed in the function space $F(X, Y)$.*

PROOF. Suppose Y is complete and $\{f_\alpha\}$ is a Cauchy net in the function space $F(X, Y)$. Then each $\{f_\alpha(x)\}$ is a Cauchy net in Y, so converges to a point $f(x)$. For any uniform covering $\mathcal{V}$ of Y, and a residual set R of indices α for which all f_α are in some element of $\mathcal{V}(U(\mathcal{V}))$, all $f_\alpha(x)$ are within $\mathcal{V}^{**}$ of $f(x)$; thus $\{f_\alpha\}$ converges to f. Now suppose $f \in F(X, Y) - U(X, Y)$. There is a uniform covering $\mathcal{V}$ of Y such that in X one can find points p, q, arbitrarily near, such that $f(p)$ and $f(q)$ are not near of order $\mathcal{V}^*$. Then for any $g \in F(X, Y)$ within $\mathcal{V}$ of f, these $g(p)$ and $g(q)$ are not near of order $\mathcal{V}$. This shows that $F(X, Y) - U(X, Y)$ is open, and completes the proof.

32. Corollary. *A uniform limit of continuous functions is continuous.*

33. *In any function space $U(X, Y)$, for any uniform covering $\mathcal{U}$ of X, the mappings realizing $\mathcal{U}$ form an open set.*

Proof. If f realizes $\mathcal{U}$, with $f^{-1}(\mathcal{V}^*)$ refining $\mathcal{U}$, then for any g within $\mathcal{V}$ of f, $g^{-1}(\mathcal{V}) < f^{-1}(\mathcal{V}^*) < \mathcal{U}$.

34. *Let Y be a complete metric space and X a metric space such that, for each uniform covering $\mathcal{U}$ of X, every mapping $f: X \to Y$ can be approximated arbitrarily closely by mappings realizing $\mathcal{U}$. Then X can be embedded in Y. In fact, every mapping $f: X \to Y$ can be approximated by embeddings.*

Proof. $U(X, Y)$ is complete by Theorem 31 and metric by Exercise II.7. From topology, any countable family of dense open subsets of $U(X, Y)$ has a dense intersection. Hence there is a mapping in $U(X, Y)$ which realizes every covering in some basis for X and therefore embeds X in Y.

Proposition 34 gives us what may be called the random method of embedding. It is a strikingly successful method in some cases; and hardly anything has yet been done on embedding general spaces X in a fixed space Y beyond obvious refinements of the random method.

35. *A function $f: X \to Y$ which can be approximated pointwise by the members f_α of an equiuniformly continuous family of functions is uniformly continuous; in fact, the closure of $\{f_\alpha\}$ in the product space $F(X, Y)$ is an equiuniformly continuous family.*

Proof. Points $f(p)$, $f(q)$ are near of order $\mathcal{V}^*$ whenever every pair $f_\alpha(p)$, $f_\alpha(q)$ is near of order $\mathcal{V}$.

An equiuniformly continuous family of functions on a fine space is called an *equicontinuous* family. Note that every proposition on equiuniformly continuous families applies to equicontinuous families.

36. Theorem. *If X and Y are precompact, then every equiuniformly continuous family in $U(X, Y)$ is precompact.*

PROOF. For any uniform covering $\{V_i\}$ of Y, there is a uniform covering $\{U_j\}$ of X refining $f^{-1}(\{V_i\})$ for all f in the equiuniformly continuous family E; and we may suppose each covering is finite. Each f in E takes finitely many sets U_j to subsets of finitely many sets V_i in some manner; choose a finite set of f's which do this in all possible ways. Then every member of E is within $\{V_i\}$ of one of these; so E is precompact.

If just Y is precompact, then for each single f we could use finite coverings of X, but there may be equiuniformly continuous families E for which no finite covering of X suffices. The hypothesis of the theorem does admit one weakening which is sometimes useful for applications. A family $\{f_\alpha\}$ of functions is called *pointwise precompact* (or sometimes *pointwise bounded*) if for each point x the set of all values $f_\alpha(x)$ is precompact.

37. ASCOLI THEOREM. *An equiuniformly continuous, pointwise precompact family of functions on a precompact domain is precompact. On every space X which is not precompact there is an equiuniformly continuous family of functions into I which is not precompact*

PROOF. Let E be equiuniformly continuous and pointwise precompact in $U(X, Y)$. Let $\{V_\alpha\} = \mathscr{V}$ be a uniform covering of Y, and $\{U_i\}$ a finite uniform covering of X refining all $f^{-1}(\mathscr{V})$, $f \in E$. Select $x_i \in U_i$ for each i. The set $\{f(x_i) : f \in E\}$, being precompact, is covered by a finite set $\{V_\alpha : \alpha \in J_i\}$. Each f in E maps U_i into some $\mathrm{St}(V_\alpha, \mathscr{V})$, with $\alpha \in J_i$. Again select a finite set of f's which do this in all possible ways; every member of E is within $\mathscr{V}^*$ of one of these.

If X is not precompact, there exist an infinite set $\{x_n\}$ in X and a uniformly continuous mapping h of X into a metric space such that $d(h(x_n), h(x_m)) \geqq 1$ for $m \neq n$. Then defining $f_n(x) = \min(1, d(h(x), h(x_n)))$, we have an equiuniformly continuous family of functions in $U(X, I)$ all separated by distance 1.

There is a further refinement on the Ascoli theorem for continuous functions, i.e., for fine spaces; the equicontinuity can be taken pointwise. A family of functions f_α defined on a topological space X with values in a uniform space νY is said to be *equicontinuous at* a point $x \in X$ if for every covering $\mathscr{V} \in \nu$ there is a neighborhood N of x such that every image $f_\alpha(N)$ lies in a single element of $\mathscr{V}$.

38. *A family of functions on a fine space μX to a uniform space νY that is equicontinuous at every point of X is equicontinuous.*

PROOF. Let the family of functions be $\mathscr{F} = \{f_\alpha : \alpha \in A\}$; consider the coverings in ν, $\mathscr{V}^m = \{V^m_\beta : \beta \in B_m\}$. Let F_m be the set of all functions $\phi : A \to B_m$. For each $\phi \in F_m$ let $W_\phi = \bigcap [f_\alpha^{-1}(V^m_{\phi(\alpha)}) : \alpha \in A]$; let U_ϕ be the interior of W_ϕ.

For each point x of X, equicontinuity of $\mathscr{F}$ at x implies the existence of $\phi \in F_m$ such that W_ϕ contains a neighborhood of x; hence $\{U_\phi\} = \mathscr{F}^{-1}(\mathscr{V}^m)$ is an open covering. To see that it is a normal covering, it suffices to check that $\mathscr{V}^2 <^* \mathscr{V}^1$ implies $\mathscr{F}^{-1}(\mathscr{V}^2) <^* \mathscr{F}^{-1}(\mathscr{V}^1)$. For any member U_ψ of $\mathscr{F}^{-1}(\mathscr{V}^2)$, there is $\phi \in F_1$ such that $\mathrm{St}(V^2_{\psi(\alpha)}, \mathscr{V}^2) \subset V^1_{\phi(\alpha)}$ for all α. Then $\mathrm{St}(U_\psi, \mathscr{F}^{-1}(\mathscr{V}^2)) \subset W_\phi$; since the star is an open set, it is contained in U_ϕ.

Exercises.

1. Y-REFLECTION. We noted after Proposition 3 the unsolved problem whether the reflection mapping $r: X \to \mathrm{Ref}(X, Y)$ always has the property that $U(r, 1): U(\mathrm{Ref}(X, Y), Y) \to U(X, Y)$ is an isomorphism. An easier exercise is:

(a) Show that $\mathrm{Ref}(X, Y)$ is a reflection of X.

(b) If it is true that $U(r, 1)$ is always an isomorphism then every one-to-one onto mapping of the form $U(f, 1)$ is an isomorphism.

*2. CARDINAL REFLECTION.

(a) Prove that in the category of all spaces having no uniformly discrete subspace of power m (an infinite cardinal) every space X has a reflection X_m.

(b) Using an indexing device like those in the proof of Theorem 28, show that every countable uniform covering has a countable uniform star-refinement. Hence every such covering is realized by the reflection mapping $r: X \to X_{\aleph_1}$.

It is an open problem whether (b) remains valid for higher cardinals.

3. LIFTING.

(a) Prove all affirmative statements about $U(1, g)$ in Proposition 6. (The affirmative part of the second paragraph can be strengthened if desired.)

(b) Check that if $g: Y' \to Y$ has no right inverse then $U(1, g): U(Y, Y') \to U(Y, Y)$ is not onto.

Suppose X is not uniformly discrete, and even has no uniform covering consisting of sets of 1 or 2 points. Let Y' be a sum of two-point sets $\{x_\alpha, y_\alpha\}$, one for each pair of distinct points x, y of X. For each uniform covering $\mathscr{U}$ of X, define $\mathscr{V}(\mathscr{U})$ on Y' to consist of those pairs $\{x_\alpha, y_\alpha\}$ for which x and y are near of order $\mathscr{U}$, and all the remaining single points of Y'.

(c) The coverings $\mathscr{V}(\mathscr{U})$ form a basis for a uniformity on Y'.

(d) Map Y' to X by q, defined $q(x_\alpha) = x$, $q(y_\alpha) = y$. Then q is a quotient mapping.

(e) For $1: X \to X$, $U(1, q)$ is not onto.

(f) If X is not uniformly discrete but has a uniform covering consisting of sets of at most 2 points, find a quotient space Y of X which has no such covering and modify the preceding to show that $U(1, q)$ is not onto for some quotient mapping $q: Y' \to Y$.

(g) If $1: X \to X$ is not a limit of mappings into uniformly discrete subspaces, then X has no uniform covering consisting of sets of at most 2 points.

Conclude the proof of Proposition 6.

4. Prove Proposition 7. (Use $F(X', Y)$ and Theorem 31.)

5. (a) For Proposition 11, let R be a real line isometrically embedded in a metric space M. For each $x \in M$, let $r(x)$ be $\sup\{y - d(x, y) : y \in R\}$.

(b) Formulate and prove the extension theorem corresponding to this retraction theorem.

6. Weak uniformity. In the usual two-type set theory having sets and proper classes, the precautions taken in defining semi-uniform product are not necessary. For any set X and any class of functions on X into various uniform spaces, there is a coarsest preuniformity on X admitting all these functions.

7. Separate continuity. On the product set $X \times Y$ of two uniform spaces, there is a uniformity such that $f : X \times Y \to Z$ is uniformly continuous if and only if each $f(x, \ \ )$ and each $f(\ \ , y)$ is uniformly continuous. However, it need not even be precompact when X and Y are compact.

This uniformity makes $X \times Y$ into a space $X \# Y$ such that mappings $X \# Y \to Z$ correspond to mappings $X \to U(Y, Z)$ with the pointwise uniformity. This seems to be the only way to put a uniformity on the function space so as to get a commutative "tensor product" $X \# Y$.

8. The functor *.

(a) Mappings $f : A \to A'$, $g : B \to B'$, induce a mapping $f^*g : A^*B \to A'^*B'$. * is a covariant functor of two variables. If A and B are nonempty, f^*g is an isomorphism if and only if f and g are.

(b) If f and g are quotient mappings, so is f^*g. The essential point is that when $f : A \to A'$ is a quotient mapping and $\{h_\alpha\}$ a family of mappings on A' to Y, $\{h_\alpha f\}$ is equiuniformly continuous if and only if $\{h_\alpha\}$ is. (Use function spaces.)

(c) Quotient mappings $f : A \to A'$, $g : B \to B'$, induce a quotient mapping $f \times g : A \times B \to A' \times B'$.

9. Rectangular compactness.

(a) In a function space $U(X, Y)$, a *closed rectangle* H is determined by a function h assigning to each point of X a closed subset of Y, and consists of all $f \in U(X, Y)$ such that $f(x) \in h(x)$ for each x. A closed rectangle is a closed set.

(b) If Y is compact, then in $F(X, Y)$ every family of closed rectangles having the finite intersection property has a common point.

(c) Every nonempty open set in $U(X, Y)$ (for any Y) contains a closed rectangle with nonempty interior.

(d) If X is uniformly discrete and Y is compact, then in $U(X, Y)$ a countable intersection of dense open sets is dense.

However:

(e) There exists a compact space Y having no isolated points and containing no infinite closed metrizable subspace. For this, let N be a countably infinite discrete space, βN its Stone-Čech compactification, $Y=\beta N-N$. Show that Y has no isolated points.

(f) No sequence $\{y_n\}$ in Y is convergent. For the union of N and $\{y_n\}$ is a countable completely regular space, and therefore normal. Thus any bounded continuous real-valued function on $\{y_n\}$ can be extended over N, and then over βN.

(g) A continuous mapping of the Cantor set C into Y takes only finitely many values.

(h) $U(C, Y)$ is a countable union of nowhere dense sets.

10. Graph space. The functions in $U(X, Y)$ are determined by their graphs, which are closed sets in $X \times Y$ forming a subspace of $H(X \times Y)$. This space, the *graph space* of (X, Y), would be a poor substitute for the function space; it need not be closed in $H(X \times Y)$, e.g. for X a Cantor set and Y a two-point space. However, it is homeomorphic with $U(X, Y)$.

Research Problem A.

Characterize U. It would clarify the properties of function spaces considerably if a simple and convincing nonconstructive definition of $U(X, Y)$ could be found which would determine uniquely the uniform function space as defined here. Such a definition should resemble the nonconstructive definitions of sums, products, and more particularly reflections and quotients. There seems to be no reason to expect it to be any simpler than the triune definition of a quotient mapping. Since U is a fairly complicated functor, the definition might resemble typographically the Eilenberg-Steenrod axioms for homology theory; and its simplicity and convincingness might not be generally granted for some years after its publication.

It is unnecessary to belabor the advantages of an axiomatic treatment here.

In fact the existing theory of U is not entirely convincing; it is conceivable that a different uniformity for function spaces can be found which is more worthy of the title. The importance of the present U is sufficiently clear. Even the derivative construction * seems firmly established, by its contribution to the extension problem in III.25 and III.29. What is not proven (though probably true) is that the analogy of $U(X, Y)$ with $\mathrm{Hom}(A, B)$, of X^*Y with $A \otimes B$, is sound.

The problem has no literature. The obvious method of attack is by a detailed study of properties which the space $U(X, Y)$, or the functor U, has and those which it might reasonably have but doesn't. Among the latter, recall that the natural function on X to $U(U(X, Y), Y)$ is not a mapping.

Research Problem B_1.

Describe X^*Y. As we already noted, the problem whether Theorem III.28 and its corollaries hold without restriction on the spaces is open.

It is even an open problem whether every uniform space has a basis of σ-disjoint coverings. If this could be proved it would show that III.28 is already without restrictions. It seems more likely that not every uniform space has such a basis. However, that problem is related to other important problems discussed later (B_2, B_3).

To generalize III.28 is a straightforward infinite combinatorial problem that, like many such problems, looks terribly difficult to carry out or to prove impossible. As long as it is not done, even if it is proved impossible, the corollaries (which are more interesting) might be treated separately. An obvious approach to simultaneous extension is along the lines of III.30. The theory of Banach spaces might be applied to the problem in several ways. The literature on simultaneous extension is mainly concerned with linear operators extending bounded real-valued functions; see Day [**D**], Isbell-Semadeni [1], McShane [1]. Similar remarks apply to the extension problem, but it seems to be in better shape.

Notes. There is more on injective spaces in Isbell [2] and [5]. In the latter, it is proved that if the power of the continuum is a number $\aleph_n$ (i.e., less than $\aleph_\omega$), then the real line is not even a neighborhood retract of an injective space in any uniformity not equivalent to the usual one. In any case it cannot be injective; a locally compact injective uniform space is compact.

In the proof of III.30, the proposition (*) is a little stronger than is needed. Not every quotient mapping has the property that the inverse image of each nonuniform covering is nonuniform. Every mapping having a right inverse does have this property.

There are many versions of the Ascoli theorem. Note that Kelley [**K**] gives several.

Exercise 5 is from McShane [1]. There are several interesting notions of injectivity for metric spaces, besides the uniform notion (Isbell [5]) and the strict notion $d(f(x), f(y)) \leqq d(x, y)$ (Aronszajn-Panitchpakdi [1]). Each of these three papers considers some variants.

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CHAPTER IV

MAPPINGS INTO POLYHEDRA

Polyhedra occupy a central position in the theory of uniform spaces for two reasons: the simplicity of their structure, and the ease of mapping other spaces into them. This chapter develops the second notion. The simplicity of the structure of infinite polyhedra, indeed, recedes as one looks more closely, and turns out to enfold a very large part of the structure of the whole category of uniform spaces. The point remains that general spaces are more easily approached by way of polyhedra.

Uniform complexes. An (abstract) *simplicial complex* K consists of a set V of *vertices* and a family of nonempty finite subsets of V called *simplexes*, such that every single vertex is a simplex and every nonempty subset of a simplex is a simplex. Vertices are said to have dimension 0, and generally a simplex having n vertices is said to have *dimension* $n-1$. The subcomplex $K^{(p)}$ consisting of all the vertices of K and all simplexes of dimension p or less is called the *p-skeleton*.

A simplex s contained in a simplex t is called a *face* of t. The *closure* of a simplex s is the subcomplex consisting of all its faces (including s itself). The *star* of s is the collection of all simplexes which have s as a face.

The set of simplexes of a simplicial complex K may be regarded as a topological space (satisfying the T_0 separation axiom, but no more) in which the closed sets are just the subcomplexes. This is at least a convenient device for organizing definitions and relations. For example, the star of a simplex is the smallest open set containing it. There is another example in the description of simplicial mappings that follows here. This point of view could be used also in connection with products of simplicial complexes, which we shall need later in the chapter.

A *simplicial mapping* from a complex K to a complex L is a function f from the simplexes of K to simplexes of L which takes vertices to vertices and, for each simplex s of K, takes s to exactly that simplex $f(s)$ of L whose vertices are the images $f(v)$ of vertices v of s. Thus f is determined by the *vertex mapping* $f_0=f\,|\,K^{(0)}$. A vertex mapping is just a function from vertices to vertices which takes the set of vertices of a simplex to the set of vertices of a simplex.

In the T_0-topological interpretation, a simplicial mapping is just a closed continuous mapping.

We want a more geometrical interpretation of the simplicial complexes, as polyhedra. Only a single type of polyhedra will be used; but others are possible, and for further progress in the theory others seem to be necessary. Therefore we shall spend some time on preliminary descriptions.

NOTATION: $\bar{s}^n$ is a complex consisting of an n-dimensional simplex with its faces; $\dot{s}^n$ is the complex $\bar{s}^n - s^n$.

The *category of abstract simplicial complexes* $\mathscr{L}$ is the category whose objects are simplicial complexes and mappings, simplicial mappings. A *topological realization* of $\mathscr{L}$ is a functor r embedding $\mathscr{L}$ isomorphically in the category of topological spaces, so that

(1) (NORMALIZATION). *Each complex $\bar{s}^n$ is realized by a closed n-cell, and $r(\dot{s}^n)$ is its boundary.*

(2) (PRESERVATION OF SUBCOMPLEXES). *A mapping embedding a subcomplex is realized by a mapping embedding a subspace.*

(3) (DOMINATION). *For each complex K, every point of the realization $r(K)$ belongs to $r(\bar{s})$ for some simplex s of K.* A *uniform realization* is a functor embedding $\mathscr{L}$ isomorphically in the category of uniform spaces, satisfying the same three conditions. Relative to a realization r, the spaces isomorphic with spaces $r(K)$ are *polyhedra.*

These conditions (1)-(3) are probably not enough for a full axiomatic development of polyhedra. They do suffice for proving:

1. *Embeddings of subcomplexes are realized by embeddings of closed subspaces.*

2. *For each complex K, for each point p of $r(K)$, there is a unique simplex s of K such that for a subcomplex K' of K, p is in $r(K')$ if and only if s is in K'.*

3. *The space realizing a finite complex is compact and topologically unique.*

For the proofs, first note that $r(K^{(0)})$ is discrete. Every two of its points form a discrete space, in view of (1) and (2); the rest follows since every function on $K^{(0)}$ to itself is a simplicial mapping.

PROOF OF PROPOSITION 1. Let the subcomplex $K' \subset K$ be realized by a subspace $L' \subset L$. Let p be a point of $L - L'$. Consider two cases: (I) there is not, or (II) there is a simplex s^n of K of dimension $n > 0$ such that $p \in r(\bar{s}^n)$. In case (I), p is in $r(v)$ for some isolated vertex v of K (in view of (3)). Then p is separated from L' by a continuous mapping into $r(K^{(0)})$; so p is not a limit point of L'. In case (II), there is a vertex v of s^n such that $r(v) \neq p$. Let

f: $K \to K$ be the simplicial mapping which leaves the vertices of s^n fixed and takes all other vertices to v. Then $r(f)$ is continuous and coincides with the identity on $r(\bar{s}^n)$. It maps all of L' (using (3) here) into subsets $r(t)$ of $r(\bar{s}^n)$, where t is a face of s^n belonging to K'. This is a finite family of compact sets not containing p. Thus $r(f)$ separates L' from p, and L' is closed.

Proof of Proposition 2. Let s and t be any two simplexes of K having a largest common face u. Let v be a vertex of u, and let f be the simplicial mapping of K to K which leaves s fixed and takes all other vertices to v. Then consider any point p of $r(\bar{s}) \cap r(\bar{t})$. Since p is in $r(\bar{s})$, $r(f)(p) = p$; since p is in $r(\bar{t})$, $r(f)(p) \in r(\bar{u})$. Hence p is in $r(\bar{u})$, i.e., $r(\bar{s}) \cap r(\bar{t}) = r(\bar{u})$.

Suppose s and t have no common face. If both are zero-dimensional we already know $r(\bar{s})$ and $r(\bar{t})$ are disjoint; if one has two or more vertices, an obvious modification of the preceding paragraph shows $r(\bar{s})$ and $r(\bar{t})$ are disjoint.

Now every point p of $r(K)$ is in $r(\bar{z})$ for at least one simplex z of K. By the preceding, among the faces s of z having p in $r(\bar{s})$ there is a smallest s. For any simplex t, $p \in r(\bar{t})$ implies s is a face of t. With (3), the proof is complete.

Proof of Proposition 3. A finite union of n-cells is of course compact Moreover, if r_1 and r_2 are topological realizations of $\mathscr{L}$, a homeomorphism h: $r_1(K) \to r_2(K)$ (K finite) may be defined as follows. $r_1(K^{(0)})$ and $r_2(K^{(0)})$ are homeomorphic by the correspondence $h(r_1(v)) = r_2(v)$. Having defined h: $r_1(K^{(i)}) \to r_2(K^{(i)})$, consider each $(i+1)$-simplex s of K. We have a homeomorphism h from the sphere $r_1(\dot{s})$ to $r_2(\dot{s})$; then it may be extended radially to a homeomorphism of $r_1(\bar{s})$ upon $r_2(\bar{s})$. In view of Proposition 2, this gives a homeomorphism of the $(i+1)$-skeletons, and the recursion completes the proof.

The simplex s associated with a point p of $r(K)$ by Proposition 2 is called the *carrier* of p. Further, an isomorphism $f: r(K) \to X$ is called a *triangulation* of X. Relative to a triangulation f, the carrier of a point $f^{-1}(x)$ is also called the carrier of x. The set of all points of X having a given carrier s is an *open simplex* of f, and it may be notationally identified with s when this is convenient. Its closure $f(r(\bar{s}))$ is a *closed simplex* of the triangulation.

No generally satisfactory uniform realization of all simplicial complexes is known. In fact, the "realization" in most common use takes some simplicial mappings to nonuniformly continuous functions. Since all our applications are in the finite-dimensional case, where this difficulty does not occur, we shall not try to correct it here.

For any simplicial complex K, the *uniform complex* K_u is a metric space whose points are those non-negative real-valued functions p on the vertices of K such that, for some simplex s of K, $p(v) = 0$ for all vertices v not in s and $\sum_{v \in s} p(v) = 1$. Distance in K_u is defined $d(p, q) = \max_v |p(v) - q(v)|$. The vertices of K then correspond to their characteristic functions, and each simplex consists of all weighted averages of its vertices. A simplicial mapping f: $K \to L$ is realized by f_u: $K_u \to L_u$, coinciding with f on vertices and defined in general $f(\sum \alpha_v v) = \sum \alpha_v f(v)$. Evidently

4. *Uniform complexes constitute a uniform realization of finite-dimensional simplicial complexes.*

We shall usually suppress the subscript u on a uniform complex. Also we may refer to simplexes of K ($=K_u$); this is relative to the triangulation $1: K_u \to K_u$ unless specified otherwise. *Dimension* of a simplex is 1 less than the number of vertices; the uniform significance of the concept appears in the next chapter. Further, the dimension of a complex (abstract or uniform) is the least upper bound in $\{0, 1, \cdots, \infty\}$ of the dimensions of its simplexes.

5. *A uniform complex is complete if and only if it is finite-dimensional.*

PROOF. The coordinates of the centroid of an n-dimensional simplex are $1/(n+1)$, $n+1$ times, and 0; if there are simplexes of dimensions $n_i \to \infty$, then their centroids form a Cauchy sequence which is not convergent. On the other hand, suppose X is an n-dimensional uniform complex and $\{p_i\}$ a Cauchy sequence in X. For each vertex v, the values $p_i(v)$ form a Cauchy sequence of non-negative real numbers with limit $p(v)$. Let s be the set of all v for which $p(v) > 0$. There cannot be more than $n+1$ elements in s; for if p had $n+2$ positive values, all $\geqq 2\epsilon$, then those p_i which are within ϵ of p would also have at least $n+2$ positive values. By the same argument, s is a simplex. Obviously $\sum_{v\in s} p(v) \leqq 1$. Suppose the sum is $1-\delta$. Then for $\gamma < \delta/(n+1)$, and i such that for $j > i$, $d(p_i, p_j) < \gamma$, we have $\sum p_i(v) < 1$, a contradiction. Thus p is a point of X, and $p_i \to p$.

Another characteristic property of finite-dimensional uniform complexes is that the stars of the vertices form a uniform covering. (The best Lebesgue number, in an n-dimensional complex, is $1/(n+1)$.) But in this case we have more, as follows.

A triangulation $g: L_u \to X$ is a *subdivision* of a triangulation $f: K_u \to X$ in case every closed simplex of g is a subset of a closed simplex of f. Equivalently, every open simplex of g is a subset of an open simplex of f.

There is an abstract notion of subdivision, too. The *first barycentric subdivision* $K^{[1]}$ of an abstract complex K is the abstract complex whose vertices are the simplexes of K and whose simplexes are finite sets of simplexes of K totally ordered by inclusion (the face relation). Successive barycentric subdivisions are defined $K^{[n+1]} = K^{[n][1]}$.

A realized subdivision g: $K_u^{[1]} \to K_u$ need not exist (as a uniform isomorphism). The idea behind the definition of $K^{[1]}$ is that a vertex s of $K^{[1]}$ corresponds to the centroid of s in K_u. Specifically, if s has vertices $v_0, \cdots, v_n$ in K, and we identify vertices with their characteristic functions, then g: $K_u^{[1]} \to K_u$ is defined (a) $g(s) = \sum [1/(n+1) v_i$: $i = 0, \cdots, n]$; (b) for $p = \sum \alpha_j s_j$ in $K_u^{[1]}$, $g(p) = \sum \alpha_j g(s_j)$. This is a function into K_u, because the s_j are totally ordered, and all $g(s_j)$ lie in the closure of the largest s_j in K_u. It is uniformly continuous because, as one may compute without difficulty, it is distance-decreasing. Moreover, g is one-to-one and onto. To see this, consider any p in K_u, with carrier s. If p is the centroid of s, it is clear; p is the image of the vertex s of $K_u^{[1]}$ and of no other point. Otherwise consider the ray (in the linear space of all functions on the vertices of K) emanating from the centroid of s and passing through p. Clearly this ray meets $\dot{s}$ in just one point p_0. Since p_0 has a carrier of lower dimension, we may assume inductively that $p_0 = g(x_0)$ for just one x_0 in $K_u^{[1]}$. Then p is the image of a suitable average of x_0 and s, but not of any other point.

Suppose K is finite-dimensional. Note that $K^{[1]}$ has the same dimension; the longest totally ordered sets of simplexes are those which include a simplex of highest dimension and a face of it of each dimension through 0. Further, in this case g^{-1} is also uniformly continuous. Indeed, if n is the dimension of K, then the centroids of any two simplexes of K_u are distant at least $1/(n+1)$ from each other, and one may compute that g^{-1} increases no distance by more than a factor of $n+1$.

Even if K is infinite-dimensional, the mapping g is called the first barycentric subdivision of K_u. Similarly for higher order subdivisions. We again identify, speaking of sets $g(\sigma)$ in K_u (σ a simplex of $K_u^{[1]}$) as simplexes of the first barycentric subdivision, and so on. Then:

6. THEOREM. *In a finite-dimensional uniform complex X, let $\mathcal{U}_n$ denote the covering consisting of the stars of vertices of the nth barycentric subdivision. Then the coverings $\mathcal{U}_n$ form a basis for the uniformity.*

PROOF. We know already that the stars of the vertices in a complex $K_u^{[n]}$ form a uniform covering; this remains true after an isomorphic mapping to X. Then we need only check that the diameters of simplexes in successive barycentric subdivisions of X approach 0, since the star of a vertex has at most twice the diameter of a simplex. These simplexes lie in simplexes of X, of dimension at most p; such a simplex is uniformly equivalent with a simplex in Euclidean p-space, by a mapping which is linear (so preserves centroids) and independent of the choice of simplex in X. Then we need only check that barycentric subdivision of a simplex of diameter M in E^p cuts it into a number of simplexes of diameter at most $pM/(p+1)$. Since the diameter is the maximum distance between two vertices, this is clear, and the proof is complete.

7. *The real line is a uniform polyhedron.*

8. *The product of two finite-dimensional uniform polyhedra is a uniform polyhedron.*

PROOF. Let K and L be abstract complexes corresponding to the polyhedra X and Y. The uniform space $X \times Y$ is of course metric; the product uniformity is induced, for example, by the distance $d((x, y), (x', y')) = \max(d(x, x'), d(y, y'))$. We define an abstract complex M: the vertices of M are all the ordered pairs (s, t) consisting of a simplex of K and a simplex of L; a partial ordering of the vertices is defined $(s, t) > (s', t')$ if $s \supset s'$ and $t \supset t'$; a simplex of M is a totally ordered set of vertices. Let $Z = M_u$. We define $i: Z \to X \times Y$ as follows. For each vertex (s, t) of Z, $i(s, t) = (p, q)$, where p is the centroid of s and q is the centroid of t. Then any point z of Z can be written uniquely as a weighted average of vertices $\sum \alpha_k v_k$; $i(z)$ is $\sum \alpha_k i(v_k)$. This is a mapping into $X \times Y$ because the vertices v_k are totally ordered. Moreover, examination of the definitions shows that i (like g: $K_u^{[1]} \to K_u$) is one-to-one, onto, and distance-decreasing. Finally, if n is the larger of the dimensions of K and L, any two centroids $i(v)$, $i(v')$ in $X \times Y$ are distant at least $1/(n+1)$; and i^{-1} increases distances at most by a factor of $n+1$.

Canonical mappings. The *nerve* of a covering $\mathscr{V}$ of a set S is an abstract simplicial complex N defined as follows. The vertices of N are the nonempty elements of $\mathscr{V}$. A finite set of vertices, $V_1, \cdots, V_n$, is a simplex if $V_1 \cap \cdots \cap V_n$ is a nonempty subset of S.

9. *The nerve of the covering of a simplicial complex K which consists of the stars of the vertices is isomorphic with K.*

Indeed, the elements of the covering are already indexed by the vertices of K, and the stars of a set of vertices have nonempty intersection if and only if there is a simplex having all those among its vertices—which implies there is a simplex having exactly those vertices. Note that the same thing is true for the covering of a topological realization which consists of the stars of the vertices.

Henceforward the term *nerve* will usually refer to the uniform complex determined by an abstract nerve.

A *canonical mapping* of a (uniform) space X into the nerve of a covering $\mathscr{V}$ of X is a (uniformly continuous) mapping such that the inverse image of the star of each vertex V is contained in the set V. The coordinates f_V of a canonical mapping f form a family of non-negative real-valued functions with sum 1, which is called a *partition of unity*. The condition on stars means precisely that each function f_V vanishes outside the set V; this condition is commonly expressed by saying that the family $\{f_V: V \in \mathscr{V}\}$ is *subordinated* to $\mathscr{V}$. Uniform continuity of the mapping f is equivalent to equiuniform continuity of its coordinates; and finally, for the f_V to give a mapping into the nerve they must be a *point-finite* family, i.e., at each point all but finitely many vanish.

10. *A family of functions f_V on a uniform space, indexed by the elements of a covering $\mathcal{V}$, forms the family of coordinates of a canonical mapping into the nerve of $\mathcal{V}$ if and only if it is a point-finite equiuniformly continuous partition of unity subordinated to $\mathcal{V}$.*

In case $\mathcal{V}$ is finite-dimensional (i.e., has a finite-dimensional nerve) a family of functions subordinated to $\mathcal{V}$ is automatically point-finite. In this case, further, a canonical mapping yields a realization of $\mathcal{V}$, since the covering of the nerve by stars of vertices is uniform. For the infinite-dimensional case there are enough difficulties to make the canonical mappings, as defined here, useless; however, canonical mappings are useful for infinite-dimensional spaces in topology, and there is the possibility of a modification. We note that two canonical mappings of X into the nerve of $\mathcal{V}$ are homotopic; their values at each point of X can be joined by a straight line segment of length at most 1.

11. THEOREM. *To every uniform covering of a uniform space one can subordinate an equiuniformly continuous partition of unity.*

PROOF. Let $\{U_\alpha\}$ be a uniform covering indexed by a well-ordered index set. Let d be a uniformly continuous pseudometric, as in I.14, such that every set of d-diameter 2 or less is contained in some U_α. By transfinite induction we define functions g_α: $g_\alpha(x)$ is the minimum of 1 and $\sup\{d(x, X-U_\beta) : \beta<\alpha\}$. For the first index α_0, we interpret the supremum over the empty set to be 0. Then let $f_\alpha = g_{\alpha+1} - g_\alpha$.

For any two points x, y, $|g_\alpha(x)-g_\alpha(y)| \leqq d(x, y)$, $|f_\alpha(x)-f_\alpha(y)| \leqq 2d(x, y)$; hence the functions f_α are equiuniformly continuous. For each x, the real numbers $g_\alpha(x)$ increase monotonically to 1; and for a limit index α, $g_\alpha = \sup\{g_\beta : \beta<\alpha\}$. Hence the functions f_α form a partition of unity, which is obviously subordinated to $\{U_\alpha\}$.

12. COROLLARY. *Every finite-dimensional uniform covering can be realized by a canonical mapping into its nerve.*

Having mapped a space into a nerve, one does not usually stop there. Among the mappings of uniform complexes we have the *simplicial mappings* —more correctly, the mappings realizing simplicial mappings. Among these note especially the *connecting mappings* between nerves. Here we must have two coverings $\mathcal{U}$ and $\mathcal{V}$ of the same space, with $\mathcal{V} < \mathcal{U}$. Each $V_\beta \in \mathcal{V}$ then is contained in at least one U_α, $\alpha = f(\beta)$, in $\mathcal{U}$. A function f selecting such an α for each β defines a vertex mapping from the nerve N of $\mathcal{V}$ to the nerve M of $\mathcal{U}$; and from the definition of nerve it is clear that f takes simplexes to simplexes. The realization of f is a connecting mapping. We remark that the notion depends not only on the triangulations of M and N but on the whole situation, with $\mathscr{V}<\mathscr{U}$ on some space X.

13. *If $f:X\to N$ is a canonical mapping and $g:N\to M$ is a connecting mapping, then gf is canonical.*

PROOF. Let N be the nerve of $\mathscr{V}$, M of $\mathscr{U}$. For each U in $\mathscr{U}$, each point of $X-U$ is mapped by f into the subcomplex complementary to the union of the stars of the vertices V_β of N satisfying $V_\beta\subset U$; so gf does not map it into the star of U.

A covering $\mathscr{V}$ refining $\mathscr{U}$ is called a *shrinking* of $\mathscr{U}$ if some connecting mapping is an embedding; i.e., there is a one-to-one correspondence between all the $V_\beta\in\mathscr{V}$ and some of the $U_\beta\in\mathscr{U}$ such that $V_\beta\subset U_\beta$. There is a roughly dual notion which we shall want more than once; it has no name, so we may as well call it a *coshrinking*. It is not clear how the definition should read; let us select the simplest version. $\mathscr{V}$ is a coshrinking of $\mathscr{U}$ if $\mathscr{V}<\mathscr{U}$ and some connecting mapping is onto.

14. *Every refinement is a coshrinking of a shrinking.*

PROOF. Given $\mathscr{V}=\{V_\beta\}<\mathscr{U}=\{U_\alpha\}$, select a function f taking each index β to $\alpha=f(\beta)$ so that $U_\alpha\supset V_\beta$. If $f^{-1}(\alpha)$ is not empty, define W_α as $\bigcup[V_\beta: f(\beta)=\alpha]$. Then $\mathscr{W}=\{W_\alpha\}$ is obviously a shrinking of $\mathscr{U}$, and $\mathscr{V}$ is a coshrinking of $\mathscr{W}$.

A larger class of mappings of polyhedra, still determined by their effect on vertices, are the *semilinear mappings* $f:X\to Y$. Here X must be a polyhedron with a distinguished triangulation; Y may be a triangulated polyhedron or a linear topological space, as long as the notion of a convex combination or weighted average is meaningful in Y. Then f is a semilinear mapping if f is well-defined and its restriction to each closed simplex of X is linear; i.e., on any combination of vertices $p=\sum\alpha_i v_i$, $f(p)$ is $\sum\alpha_i f(v_i)$.

It is convenient not to put uniform continuity into the definition of a semilinear mapping. Note that on a finite (i.e., compact) polyhedron every semilinear mapping is uniformly continuous.

15. *Every semilinear mapping of an n-dimensional finite uniform complex into the Euclidean space E^{2n+1} can be approximated uniformly by semilinear embeddings.*

PROOF. All we need, because of compactness, is a one-to-one semilinear mapping. Given semilinear f, taking the vertices $v_1,\cdots,v_k$ to points $p_1,\cdots,p_k$, in order to approximate within ϵ, we must find points $q_1,\cdots,q_k$ with $d(p_i,q_i)<\epsilon$ such that no $2n+2$ points q_i lie in a $2n$-dimensional linear set in

E^{2n+1}. This can be done recursively, starting with $q_1 = p_1$, selecting each q_i near p_i but not in any of the proper linear subsets determined by the preceding q_j. Then the semilinear mapping taking each v_i to q_i is the required approximation.

We are not yet prepared for dimension theory, but one of the basic results is so close that we may as well reach out and take it. The *uniform dimension* δdX of a space X is said to be $\leqq n$ provided every finite uniform covering of X has a uniform refinement of dimension at most n. The precise number δdX is the smallest such n, or -1 for the special case that X is empty; and we write $\delta dX = \infty$ if there is no such n.

16. THEOREM. *Every precompact n-dimensional metric space can be embedded in E^{2n+1}; and every mapping into E^{2n+1} can be approximated by embeddings.*

PROOF. We use III.34. Given X, precompact n-dimensional metric, and given a uniform covering $\mathscr{U}$ of X, we must show that any mapping $f: X \to E^{2n+1}$ can be approximated by mappings realizing $\mathscr{U}$. We may suppose $\mathscr{U}$ is finite; and if $\mathscr{V}$ is a uniform refinement of dimension at most n, then Proposition 14 gives us a shrinking $\mathscr{W}$ of $\mathscr{U}$ which is finite and at most n-dimensional. To realize $\mathscr{U}$ it suffices to follow a canonical mapping into the nerve of $\mathscr{W}$ with an embedding. Now to approximate $f: X \to E^{2n+1}$ to within ϵ, we make a preliminary modification, replacing $\mathscr{U}$ by a finer covering $\{U_\alpha\}$ such that each $f(U_\alpha)$ has diameter at most $\epsilon/2$. Then the shrinking $\{W_\alpha\}$ has the same property. We use a semilinear mapping h from the nerve of $\{W_\alpha\}$ into E^{2n+1} which takes each vertex W_α to a point of the set $f(W_\alpha)$. Composing with a canonical mapping g, each point $hg(x)$ is an average of points within $\epsilon/2$ of $f(x)$; so hg is within $\epsilon/2$ of f. Then approximating h within $\epsilon/2$ by an embedding, we have it.

For infinite polyhedra matters are more complicated (and the problem of embedding in Euclidean spaces is not solved). The first results needed for the infinite case are these two.

17. *A semilinear mapping of a finite-dimensional uniform complex into a Euclidean space is uniformly continuous if and only if the images of simplexes have bounded diameters.*

18. *For a uniformly continuous semilinear mapping of a uniform complex into a Euclidean space to be an embedding it is necessary and sufficient that for some $\epsilon > 0$, every two disjoint closed simplexes have images distant at least ϵ from each other.*

PROOF OF PROPOSITION 17. Necessity is obvious. (The proof is the same as in III.10; here not every pair of points lies in an interval, but every pair in a simplex is in an interval.) The converse follows from the fact that in an n-dimensional complex, two points at distance less than $1/(n+1)$ lie together in the star of some vertex.

PROOF OF PROPOSITION 18. Necessity is obvious, since disjoint closed simplexes are distant by 1. For the converse it is enough to show that for each vertex v, if $0 \leqq s < t \leqq 1$, then any point x in the given complex whose coordinate x_v exceeds t has an image distant at least $\epsilon(t-s)$ from the image of any y such that $y_v \leqq s$. First suppose $s=0$. Let A be the closure of the carrier of x, B the closure of the carrier of y, and $F = A \cap B$. (If F were empty, the images would be distant by ϵ, as required.) Let B_0 be the face of B opposite F; then the images of A and B_0 are at least ϵ apart, and similarly for the face A_0 of A opposite F. Consider the line segment axb through x which joins a point a of A_0 to a point b of F. (It exists because x is a convex combination of the vertices of A.) On axb the vth coordinate is a positive linear function vanishing at b; the distance of the image from the image of B is piecewise linear, vanishing at b, and increasing to a value $\lambda \geqq \epsilon$ either at a or at the first point m where the nearest point of the image of B becomes a point of the image of B_0. Between b and m (between b and a, if there is no such point as m) the latter function is linear, and after m it remains at least ϵ. Hence the distance of the image of x from the image of B is at least $t\epsilon$. Finally for $s>0$, if $y_v \leqq s$, then y is a convex combination $rv+(1-r)y'$, where $r \leqq s$ and $y'_v = 0$. Solving $x = rv+(1-r)x'$ for x', we have similar triangles vxy, $vx'y'$, and reduce to the previous case.

Extensions and modifications. It is convenient to give a narrow definition of extension of coverings, so as to avoid repeating all the conditions satisfied by the extensions we shall be interested in. Relative to an embedding $X \subset Y$, an *extension over* Y of a uniform covering $\{V_\beta\}$ of X is a uniform covering $\{U_\beta\}$ of Y, where the notation indicates a one-to-one correspondence between all V_β and some of the U_β, such that $V_\beta = U_\beta \cap X$ for each V_β. From the definition of an embedding, every uniform covering of a subspace has an extension. An extension $\{U_\beta\}$ of $\{V_\beta\}$ is *isomorphic* if the indexed nerves are isomorphic, i.e., no finite family of sets U_{β_i} has a common point unless they have a common point in the subspace X. (Applied to one-member families, this means every U_β meets X.)

For uniform coverings of a single space, $\{V_\alpha\}$ is a *strict shrinking* of $\{U_\alpha\}$ if there is a uniform covering $\mathscr{W}$ such that for all α, $\mathrm{St}(V_\alpha, \mathscr{W}) \subset U_\alpha$.

19. *Every uniform covering has a uniform strict shrinking.*

PROOF. In view of I.14, it suffices to consider a uniform covering of a metric space, having a Lebesgue number ϵ. The family of all $V_\alpha = \{x: d(x, X-U_\alpha) > \epsilon/3\}$ is a strict shrinking.

20. THEOREM. *For any subspace X of a uniform space Y, every uniform covering $\{V_\beta\}$ of X has an isomorphic extension $\{U_\beta\}$ over a uniform neighborhood of X in Y. One can secure that $U_\beta \subset \mathrm{St}(V_\beta, \mathscr{Y})$ for any given uniform covering $\mathscr{Y}$ of Y.*

PROOF. Let $\{W_\beta\}$ be a strict shrinking of $\{V_\beta\}$. Choose $\mathscr{Y}$ so fine that $\mathrm{St}(W_\beta, \mathscr{Y}) \cap X \subset V_\beta$ for all β. Then apply I.14 to $\mathscr{Y}$. We get a uniformly continuous pseudometric d on Y such that for each β, $d(W_\beta, X - V_\beta) \geqq 1$. Let U_β consist of V_β together with all points of Y within d-distance $1/4$ of X and $1/3$ of W_β. Then on the $1/4$ neighborhood of X, $\{U_\beta\}$ is a uniform covering. Each $U_\beta \cap X$ is V_β, so that we have an extension. Finally, if some point p is common to a family of sets U_β, $\beta \in I$, then for p in X, p is in each V_β. For p not in X, p is within $1/3$ of every W_β $(\beta \in I)$; then a point q of some W_β within $1/3$ of p is within $2/3$ of each of the other W_β's and hence in all the V_β $(\beta \in I)$.

COROLLARIES.

21. *Every uniform covering of a uniform space has an isomorphic extension over the completion.*

22. *If X is a subspace of Y and $\mathscr{U}$ a uniform covering of X, then there is an extension of $\mathscr{U}$ to a uniform covering of Y which consists of an isomorphic extension plus one more set, disjoint from X.*

Another basic result of dimension theory follows now.

23. *If X is a subspace of Y, then $\delta dX \leqq \delta dY$. If X is a dense subspace of Y, then $\delta dX = \delta dY$.*

PROOF. To get n-dimensional refinements of finite uniform coverings of X, it suffices to have n-dimensional refinements of their extensions over Y. Conversely, if X is dense and $\delta dX \leqq n$, we treat any finite uniform covering $\{U_i\}$ of Y as follows. Let $\{V_i\}$ be a strict shrinking, and $\{W_i\}$ an n-dimensional uniform covering of X shrinking $\{V_i \cap X\}$(constructed as in IV.16). Let $\{Z_i\}$ be an isomorphic extension of $\{W_i\}$ over Y, so chosen that we have $Z_i \subset \mathrm{St}(W_i, \mathscr{Y}) \subset U_i$ for a suitable $\mathscr{Y}$.

Next we consider ways of picking a covering apart. A priori the most natural way is to "color $\mathscr{U}$ in p colors", i.e., to write $\mathscr{U}$ as a union of subcollections $\mathscr{U}_1, \cdots, \mathscr{U}_p$, so that within each subcollection $\mathscr{U}_i$, no two members have a common point. What we shall use is a pair of related partitioning notions, one stronger, one weaker.

A pair of vertices of a simplicial complex are *neighbors* if they lie in a simplex. The complex is *star-bounded,* of *density* $k < \infty$, if no vertex has more than k distinct neighbors and some vertex has k neighbors. This implies of course that the complex is at most k-dimensional. Also, it is stronger than the notion *star-finite* mentioned in Exercise I.7 (each vertex has only finitely many neighbors). Note that whether a complex K is star-bounded (or star-finite) is determined by the 1-skeleton $K^{(1)}$. Also, as with dimension, a covering is called star-bounded (star-finite) if its nerve is. This applies to all combinatorial properties of complexes.

24. *A covering $\mathscr{U}$ is star-bounded if and only if it is a finite union of subcollections $\mathscr{U}_i$ such that no member of $\mathscr{U}$ meets two members of any $\mathscr{U}_i$.*

PROOF. If $\mathscr{U}$ is a union of k such subcollections, then each $U \in \mathscr{U}$, being itself in one of the subcollections, has at most $k-1$ neighbors. Conversely, if $\mathscr{U}$ has density l, let $\mathscr{U}_1$ be a maximal subset of $\mathscr{U}$ no two of whose members have a common neighbor. Let $\mathscr{R}_1 = \mathscr{U} - \mathscr{U}_1$, and recursively let $\mathscr{U}_{i+1}$ be a maximal subset of $\mathscr{R}_i$ no two of whose members have a common neighbor in $\mathscr{U}$. With $\mathscr{R}_{i+1} = \mathscr{R}_i - \mathscr{U}_{i+1}$, $\mathscr{R}_{l2+1}$ must be empty. For when $U \in \mathscr{R}_{l2}$, its l neighbors must have neighbors in all preceding $\mathscr{U}_i$, or be themselves in $\mathscr{U}_i$. Each neighbor of U has at most $l-1$ other neighbors. So U cannot be left out of the maximal subset $\mathscr{U}_{l2+1}$.

Here the number of subcollections does not determine the density (as one can easily check). In the next partitioning, the number of subcollections does correspond exactly to the dimension; but the result is not combinatorial, for we need a uniform refinement.

25. *A uniform covering has an n-dimensional uniform refinement if and only if it has a uniform refinement which is a union of $n+1$ uniformly discrete subcollections.*

PROOF. "If" is trivial. For the converse, let f be a canonical mapping into the nerve N of the n-dimensional uniform covering $\mathscr{U}$. The stars of vertices in the subdivision $N^{[1]}$ form a uniform covering $\mathscr{V}$ which is a union of $n+1$ collections each of which is a disjoint collection, namely the collections $\mathscr{V}_i$ of stars of vertices which are centroids of i-dimensional simplexes for $i = 0, \cdots, n$. If $\mathscr{W}$ is a uniform strict shrinking of $\mathscr{V}$, then $\mathscr{W}$ is a union of $n+1$ uniformly discrete collections; and the same is true for $f^{-1}(\mathscr{W})$, which refines $\mathscr{U}$.

A uniform complex is called *Euclidean* (and with it the corresponding abstract complex and any covering having it for nerve) if it is embeddable in some Euclidean space. The problem of Euclidean complexes can be quickly stated. Clearly a Euclidean complex must be countable and star-bounded. One can show (we leave it for an exercise) that there is a higher-order restriction on the number of neighbors; it must grow no faster, as we take neighbors, neighbors of neighbors, and so on, than some polynomial. The problem is whether these conditions characterize Euclidean complexes. It has hardly been touched. Corollary 27 throws a dim light on it.

26. THEOREM. *Suppose N is a p-dimensional uniform complex and $n \geqq 2p+1$. Among the semilinear mappings of N into E^n, every mapping that is within finite distance of an embedding is a uniform limit of embeddings. These mappings are characterized by the conditions*:

(1) *the images of simplexes of N have bounded diameter*;

(2) *there exist $\delta > 0$ and an integer m such that no $m+1$ vertices of N are mapped into a sphere of radius δ.*

PROOF. We have noted (IV.17) that (1) is the condition for uniform continuity; evidently if f satisfies (1), so does any function within finite distance of f. Similarly (2) is satisfied by embeddings and by all functions within finite distance of them; by embeddings, since a sphere of fixed radius can hold only a limited number of points at distances bounded from zero, and by functions within finite distance d, since a sphere of radius $d+\delta$ can be covered by a limited number of spheres of radius δ. It remains to approximate a semilinear mapping f satisfying (1) and (2), to within any assigned $\epsilon > 0$, by a semilinear embedding g. The conditions g must satisfy are (1) —which is automatic—and, from IV.18: for some fixed $\gamma > 0$, g maps any two disjoint closed simplexes to sets distant by at least γ.

Observe that E^n can be covered with a family of sets which are all isometric to a convex bounded open set U_0, sav a cube, so that the covering is a finite union of uniformly discrete subcollections, and (U_0 not being preassigned) so that it has any preassigned Lebesgue number; call such a covering a *standard* covering of E^n. Let θ be an upper bound for the diameters of images of simplexes under f, and let $\mathscr{U}$ be a standard covering with Lebesgue number $2\theta+5\epsilon$.

Then for any two simplexes of N, either f maps them into disjoint sets at least 3ϵ apart or f maps their union into a set whose ϵ-neighborhood is contained in one element of $\mathscr{U}$. In the first case every mapping within ϵ of f keeps these simplexes ϵ apart, in the second case every mapping within ϵ of f maps them into a single element $\mathscr{U}$. Now for each element U of $\mathscr{U}$, let $\alpha(U)$ be the sub-complex of N consisting of all simplexes whose vertices f maps into U. The complexes $\alpha(U)$ have bounded numbers of vertices (as we noted earlier, this follows from (2)) and hence may all be embedded isometrically in some finite p-dimensional complex T. Let U_0 be an isometric copy of all the elements of the standard covering $\mathscr{U}$, and let M be the totally bounded metric space of all semilinear mappings of T into U_0. Next recall that $\mathscr{U}$ is a union of uniformly discrete subfamilies $\mathscr{U}_1, \cdots, \mathscr{U}_s$. We shall define a semilinear mapping $f_1 \colon N \to E^n$ by defining its effect on the vertices v. If $f(v)$ is not in an element of $\mathscr{U}_1$, then $f_1(v) = f(v)$. For the others, choose a finite set of embeddings ϕ_i in M such that every point of M is within $\epsilon/2$ of some ϕ_i (by IV.15, this is possible), and on each $\alpha(U)$, U in $\mathscr{U}_1$, modify f by less than $\epsilon/2$ to make it agree with some ϕ_i. Then there is a positive number ϵ_1 such that any two disjoint simplexes of T are mapped by each ϕ_i into sets at distance at least ϵ_1 from each other; and for any two disjoint simplexes contained in some $\alpha(U)$, U in $\mathscr{U}_1$, any mapping within $\epsilon_1/3$ of f_1 will still take them to disjoint sets at least $\epsilon_1/3$ apart. Thus we can go on to construct f_2 near f_1, mapping the vertices sent into $\mathscr{U}_2$ by f_1 to new images so that disjoint closed simplexes spanned by these vertices will have images at least ϵ_2 apart, for some $\epsilon_2>0$; on the other vertices f_2 coincides with f_1. The recursive construction concludes with a mapping f_s satisfying all conditions; f_s is a semilinear embedding within ϵ of f.

27. Corollary. *A uniform complex having a Euclidean* 1-*skeleton is Euclidean.*

Proof. First, any embedding of the 1-skeleton gives a semilinear mapping coinciding with it on the vertices and satisfying (1) and (2) of the theorem. Then, since the 1-skeleton is Euclidean, it is star-bounded, and the complex is finite-dimensional.

We showed in II.30 and in Exercise III.2 that finite and countable uniform coverings generate uniformities. There are corresponding results for most of the usual combinatorial properties of coverings. We add that a covering is called *point-finite* if each point is in only finitely many of the elements.

28. *Let* $\mathfrak{P}$ *denote the property of point-finiteness, star-finiteness, star-boundedness, or finite dimensionality. Then if* $\mathscr{U}$ *is a covering of a set having property* $\mathfrak{P}$ *and* $\mathscr{V}$ *is a star-refinement of* $\mathscr{U}$, *there is a covering* $\mathscr{W}$ *having property* $\mathfrak{P}$ *which satisfies* $\mathscr{V} < \mathscr{W} <^* \mathscr{U}$.

29. *Every Euclidean uniform covering has a Euclidean uniform star-refinement.*

Proof of Proposition 28. We modify the proof of IV.14. For each V in $\mathscr{V}$, the set $i(V)$ of all members of $\mathscr{U}$ that contain V is nonempty and finite, and the set $j(V)$ of all members of $\mathscr{U}$ that contain $\mathrm{St}(V, \mathscr{V})$ is a nonempty subset of $i(V)$. For each finite subset i of U and each nonempty subset j or i, let W_{ij} be the union of all V such that $i(V)=i$ and $j(V)=j$. Then the covering W consisting of all W_{ij} is as required; verification is left as an exercise

Proof of Proposition 29. The covering is refined by $f^{-1}(\mathscr{U})$ for some mapping f into an E^n and some uniform covering $\mathscr{U}$ of E^n. Then take $f^{-1}(\mathscr{V})$, where $\mathscr{V} <^* \mathscr{U}$.

Inverse limits. We come to the transition from various mappings of a space into polyhedra to systems of mappings into polyhedra, bound together more tightly than the canonical mappings are bound by connecting mappings. There is a reasonable analogy with the transition, in elementary calculus, from approximation by differentials to development in series. In terms of this analogy, we shall not get to Taylor series in this chapter. The main result (Theorem 37) corresponds to the closure of the analytic functions under summation of uniformly convergent series.

An (abstract) *inverse mapping system* is a category $\mathscr{C}$ such that

(1) $\mathscr{C}$ has only a set S of objects;

(2) each $\mathrm{Map}(X, Y)$ in $\mathscr{C}$ has at most one member;

(3) S is directed by the relation $<$ defined $X<Y$ if $\mathrm{Map}(X, Y)$ is non-empty.

An *inverse mapping system of uniform spaces* is an inverse mapping system composed of uniform spaces and uniformly continuous mappings.

The mappings in an inverse mapping system are called *bonding mappings.* It is convenient to write the objects with indices, X_α, and to write $f_{\alpha\beta}$ for a bonding mapping from X_α to X_β. Then $\{X_\alpha;\ f_{\alpha\beta}\}$ is a notation for the whole system.

Relative to an inverse mapping system $\{X_\alpha;\ f_{\alpha\beta}\}$, a *left compatible family* is a family of mappings $\{g_\alpha\colon W\to X_\alpha\}$ satisfying $f_{\alpha\beta}g_\alpha=g_\beta$ for every bonding mapping $f_{\alpha\beta}$. An *inverse limit* of the system is a left compatible family $\{g_\alpha\colon W\to X_\alpha\}$ such that for every left compatible family $\{h_\alpha\colon V\to X_\alpha\}$ there is a unique mapping $k\colon V\to W$ satisfying $g_\alpha k=h_\alpha$ for all α.

30. *The inverse limit of an inverse mapping system is unique up to isomorphism.*

Proof. If $\{g_\alpha\colon W\to X_\alpha\}$ and $\{h_\alpha\colon V\to X_\alpha\}$ are both inverse limits, then there are $k\colon V\to W$ and $j\colon W\to V$ satisfying $g_\alpha k=h_\alpha$, $h_\alpha j=g_\alpha$. Hence $g_\alpha kj=g_\alpha$; by uniqueness, kj is the identity 1_W. Similarly $jk=1_V$.

We shall speak simply of "the" inverse limit. It consists of the *limit space* W and the *canonical* or *coordinate projections* g_α; however, as with products, we may use the term "inverse limit" for the limit space alone.

31. *Every inverse mapping system of uniform spaces $\{X_\alpha;\ f_{\alpha\beta}\}$ has an inverse limit. The limit space is a closed subspace of the product of the spaces X_α. Every uniform covering of it is realized by some coordinate projection.*

Proof. Let P denote the product of all X_α, with coordinate projections $p_\alpha\colon P\to X_\alpha$. A point x of P is called a *thread* if for every bonding map $f_{\alpha\beta}$, $f_{\alpha\beta}p_\alpha(x)=p_\beta(x)$. Let L denote the set of all threads—which is clearly a closed

subspace of P. Define g_α: $L \to X_\alpha$ as $p_\alpha | L$. From the definition of L, $\{g_\alpha\}$ is left compatible. Moreover, any family $\{h_\alpha$: $V \to X_\alpha\}$ is $p_\alpha k$ for a unique k: $V \to P$, and to say that $\{h_\alpha\}$ is left compatible is just to say $k(V) \subset L$. Thus L, with canonical projections g_α, is an inverse limit.

Every uniform covering $\mathscr{U}$ of L can be extended to a uniform covering $\mathscr{V}$ of P, which (by Exercise II.2) is refined by a finite intersection of coverings $p_{\alpha_i}^{-1}(\mathscr{W}_i)$, $\mathscr{W}_i$ a uniform covering of X_{α_i}. Since $\{X_\alpha; f_{\alpha\beta}\}$ is an inverse mapping system, there is λ such that $f_{\lambda\alpha_i}$ exists for all i. The inverse images of the $\mathscr{W}_i$ under these mappings are uniform and have a common uniform refinement $\mathscr{W}$. Then $\mathscr{U}$ is realized by g_λ.

Among the corollaries of Proposition 31 we note

32. *If L is an inverse limit of spaces X_α with $\delta d\, X_\alpha \leqq n$, then $\delta d\, L \leqq n$.*

This would follow immediately if we already had

33. *Every finite uniform covering $\mathscr{U}$ realized by a mapping $g: X \to Y$ is refined by $g^{-1}(\mathscr{V})$ for some finite uniform covering $\mathscr{V}$ of Y.*

To prove this: we have $\mathscr{U}$ refined by $g^{-1}(\mathscr{W})$ for some uniform covering $\mathscr{W} = \{W_\alpha\}$ of Y. $g^{-1}(\mathscr{W})$ is a coshrinking of a shrinking $\mathscr{G}$ of $\mathscr{U}$. Then group the sets W_α into equivalence classes by means of a connecting mapping h from the nerve of $g^{-1}(\mathscr{W})$ onto the nerve of $\mathscr{G}$: W_α is equivalent to W_β provided either $h(g^{-1}(W_\alpha)) = h(g^{-1}(W_\beta))$ or both $g^{-1}(W_\alpha)$ and $g^{-1}(W_\beta)$ are empty. Since $\mathscr{G}$ is a shrinking of $\mathscr{U}$, there are only finitely many equivalence classes. Then their unions form the required covering $\mathscr{V}$.

Inverse mapping systems admit an extensive theoretical apparatus. They can be mapped one to another; they can converge; and it is important in some applications that they can be asymptotic. (Two inverse mapping systems on the same spaces are asymptotic if their bonding mappings are in a suitable sense finally arbitrarily close together.) Here we want two notions from the mapping theory. A *subsystem* $\{Y_\alpha; e_{\alpha\beta}\}$ of $\{X_\alpha; f_{\alpha\beta}\}$ is an inverse mapping system with each $Y_\alpha \subset X_\alpha$ and $e_{\alpha\beta} = f_{\alpha\beta} | Y_\alpha$. The subspaces Y_α of course must satisfy $f_{\alpha\beta}(Y_\alpha) \subset Y_\beta$. A *cofinal part* of an inverse mapping system $\{X_\alpha; f_{\alpha\beta}\}$, with the indices α forming a directed set A, is the mapping system consisting of those X_α whose indices lie in some cofinal subset B of A, with whatever bonding mappings connect them.

34. *Every closed subspace of the limit space of an inverse mapping system $\{X_\alpha; f_{\alpha\beta}\}$ is the limit space of a subsystem of closed subspaces of the X_α.*

Proof. Consider the limit L as the set of all threads in the product space. For any closed subspace F of L, define $Y_\alpha = g_\alpha(F)^-$ for each α. Evidently the Y_α form a subsystem; and every thread in F is a thread of $\{Y_\alpha; f_{\alpha\beta} | Y_\alpha\}$.

For any point x of $L-F$, there is a uniform covering of L with respect to which x and F have disjoint stars. This covering is realized by some g_α; so $g_\alpha(x)$ is not in Y_α.

35. *A cofinal part of an inverse mapping system has the same limit space.*

Proof. We remark that this is true in any category; but it can be proved more quickly with threads. If $\{X_\lambda; f_{\lambda\mu}\}$ is a cofinal part of $\{X_\alpha; f_{\alpha\beta}\}$, then any thread $x=\{x_\alpha\}$ determines by restriction $x'=\{x_\lambda\}$. On the other hand, for every α there is λ such that $f_{\lambda\alpha}$ exists; and x' determines x by the rule $x_\alpha = f_{\lambda\alpha}(x_\lambda)$, which is independent of the choice of λ. The mapping $x \rightarrow x'$ is uniformly continuous because its coordinate functions g_λ are uniformly continuous. Each coordinate h_α of the mapping $x' \rightarrow x$ is also uniformly continuous, for it is a coordinate projection followed by a mapping $f_{\lambda\alpha}$. Hence this correspondence between the limit spaces is an isomorphism.

In view of Proposition 35, an inverse mapping system with only countably many spaces in it can be replaced by a sequence, since every countable directed set has a cofinal subsequence. (The proof of this is a straightforward recursion, choosing indices n_{i+1} which precede both the n_i just chosen and $i+1$.)

36. *Let $\{X_m; f_{mn}\}$ be an inverse mapping system of countably many spaces, with all bonding mappings onto. Then all canonical projections are onto.*

Proof. It suffices to consider a cofinal sequence, beginning with an arbitrary X_m, relabeled as X_1. Let x_1 be any point of X_1. Recursively, having defined x_n in X_n, choose x_{n+1} in X_{n+1} mapping by $f_{n+1,n}$ to x_n; this is possible since the bonding mapping is onto. Completing the recursion, we have a thread x with $g_1(x)=x_1$.

The final theorem refers to sequences of metric spaces. We note, as another corollary of Proposition 31, that the limit space of a sequence of metric spaces is metric. The statement of the theorem is complicated at best; we save some complication by imposing particular distance functions on the spaces, though of course it does not matter which distance functions are chosen. The point of this result is a sharpening of the embedding method of III.34. There we required that *every* mapping into the proposed range Y can be approximated by mappings satisfying certain conditions; the following result can be applied, sometimes, when we know only that certain classes of mappings can be suitably approximated. The main application involves the mappings which are approximated by embeddings in Theorem 26; but it will require further preparation.

37. THEOREM. *If the metric space (X, d) is the limit space of an inverse mapping sequence of metric spaces $\{(X_m, d); f_{mn}\}$, then there exists a sequence of positive numbers ϵ_m such that the following is true. Suppose all X_m are embedded in a complete metric space (E, d) by embeddings e_m. Let $\delta_m = \inf[d(e_m(x), e_m(y)): d(x, y) > \epsilon_m]$. If $\delta_m \to 0$ and whenever $m > n$, $3d(e_m, e_n f_{mn}) < \delta_n$, then the mappings $e_m g_m$ converge uniformly to an embedding of X in E.*

PROOF. For each natural number i there exist (by IV.31) a natural number n_i and a positive number θ_i such that any two points of X at distance more than $1/i$ from each other have n_ith coordinates distant by at least θ_i. Since all f_{mn} are uniformly continuous, we can select the indices n_i to be all different. For each m, if $m = n_i$ for some i, define ϵ_m as min $(\theta_i, 1/m)$; otherwise let $\epsilon_m = 1/m$.

Suppose the embeddings e_m are as described. Then the functions $e_m g_m$: $X \to E$ form a Cauchy sequence; for when $m > n$, $e_n g_n = e_n f_{mn} g_m$ is within $\delta_n/3$ of $e_m g_m$. Thus they converge to a uniformly continuous limit function $e: X \to E$. For each i, two points x, y of X at distance more than $1/i$ from each other have images under some $e_n g_n$ $(n = n_i)$ at least δ_n apart; hence for all $m > n$, $d(e_m g_m(x), e_m g_m(y)) > \delta_n/3$, and $d(e(x), e(y)) \geqq \delta_n/3$. Thus e is an embedding.

38. COROLLARY. *Any sufficiently rapidly convergent sequence of embeddings of a metric space in a complete metric space converges to an embedding.*

A precise $\epsilon - \delta$ statement comes from the theorem on putting $X_m = X, f_{mn} = 1$. This corollary will be applied in one place (VI.25); and the numbers will be omitted.

Exercises.

1. If g is a subdivision of a triangulation f of a space, every simplex of f (closed or open) is a finite union of simplexes of g.

2. OBSTRUCTION. Given a uniform covering $\mathcal{U}$ of $X \subset Y$, consider the closed subspaces S_α of Y, containing X, over which $\mathcal{U}$ can be isomorphically extended. Show that in the hyperspace $H(Y)$ they have Y as a limit point.

One might develop a set-theoretic approach to obstruction theory, beginning here, by considering the system of sets $Y - S_\alpha$. It is not clear whether this would throw any light on algebraic obstruction theory.

3. COLORING. A covering even of dimension 1 need not be colorable in p colors for any finite p. Colorability of the nerve K is equivalent to colorability of the 1-skeleton $K^{(1)}$. Thus if K can be colored in p colors then $K^{(1)}$ is not the 1-skeleton of any p-dimensional complex.

Dirac has conjectured [1] that if $K^{(1)}$ admits no monotone mapping (i.e., a mapping under which inverse images of points are connected) upon the 1-skeleton of a p-dimensional complex, then K is colorable in p colors. This would imply the four-color theorem.

4. **Polynomial growth.** A uniform complex K can be embedded in E^n only if there is an nth degree polynomial P such that any vertex v of K has at most $P(m)$ vertices joined to it by connected paths of m or fewer edges, for each natural number m. Hence a Euclidean complex is star-bounded.

5. **Continuous properties.** A property of uniform spaces is called *left continuous* if every inverse limit of spaces having the property has the property. Generalize Proposition 32 along the following lines. The definition of $\delta dX \leqq n$ has the form: Every uniform covering having property $\mathfrak{P}$ has a uniform refinement having property $\mathfrak{Q}$.

In the generalization, apparently $\mathfrak{Q}$ should be a hereditary property of nerves, so that $\mathfrak{Q}(\mathscr{V})$ implies $\mathfrak{Q}(g^{-1}(\mathscr{V}))$. $\mathfrak{P}$ should satisfy Proposition 33; but note that one can do the proof more carefully, using the subspace $g(X)$ and using e.g. Proposition 22.

This problem is open-ended. The version with $\mathfrak{P}$ vacuous and $\mathfrak{Q}$ a hereditary property of nerves is noted in Isbell [**3**]. Proposition 32 is proved in another way in Isbell [**5**]; the powers of the method used there and the method indicated here have not been compared in the literature.

6. **Limits of polyhedra.**

(a) Suppose X is an inverse limit of uniform complexes and simplicial mappings $\{K_\alpha; f_{\alpha\beta}\}$. Then each projection $g_\alpha(X)$ is a subcomplex X_α of K_α. (Hint: If $f_{\alpha\beta}$ takes a simplex s onto an n-simplex t^n, then there is an n-dimensional face s^n of s mapped isomorphically on t^n. Curiously, this proof seems to require the axiom of choice.)

(b) Every component of X containing more than one point contains an interval.

Thus not every compact subset of E^2 is an inverse limit of polyhedra under simplicial mappings. On the other hand:

7. **More limits of polyhedra.** Two inverse mapping systems $\{X_\alpha; f_{\alpha\beta}\}$, $\{X_\alpha; \phi_{\alpha\beta}\}$, on the same directed family of spaces, with limits L, Λ, and canonical projections g_α, γ_α, are *asymptotic* provided for each β the net of mappings $\phi_{\alpha\beta}\, g_\alpha$ converges to a limit $i_\beta: L \rightarrow X_\beta$, and the mappings i_β are the coordinates of an isomorphism $i: L \rightarrow \Lambda$.

(a) Given an inverse mapping sequence of complete metric spaces $\{X_m; f_{mn}\}$; given classes $\Phi_{n+1,n} \subset U(X_{n+1}, X_n)$ with each $f_{n+1,n}$ a limit point of $\Phi_{n+1,n}$; there exists $\{X_m; \phi_{mn}\}$ asymptotic to $\{X_m; f_{mn}\}$, with $\phi_{n+1,n} \in \Phi_{n+1,n}$ for each n.

(b) Every uniformly continuous mapping of a finite-dimensional uniform complex K into a uniform complex or linear topological space can be approximated uniformly by mappings which are semilinear on various barycentric subdivisions $K^{[n]}$. Thus if a space is an inverse limit of a sequence of finite-dimensional polyhedra, under some mappings, then it is an inverse limit of the same polyhedra under semilinear mappings.

(c) One should prove also that asymptoticity is an equivalence relation.

*8. FURTHER LIMITS OF POLYHEDRA. Equivalent conditions:

(1) X is an inverse limit of finite-dimensional polyhedra.

(2) X is complete and has a basis of finite-dimensional uniform coverings.

(3) X is a closed subspace of a product of finite-dimensional polyhedra.

(4) X is an inverse limit of finite-dimensional polyhedra under semilinear. mappings.

In fact

(a) the implications (4) $\Rightarrow$(1) $\Rightarrow$(2) $\Rightarrow$(3) are trivial after the results in the text.

To conclude, prove:

(b) A product space $\Pi [X_\alpha: \alpha \in A]$ is the inverse limit of the partial products $\Pi [X_\alpha: \alpha \in F]$ over finite subsets F of A, bonded by the natural projections.

(c) Finish. (Note Theorem 6. Note the semilinearity in Proposition 8. Use the method of Proposition 34.)

9. EMPTY LIMITS. An inverse mapping system of nonempty spaces and onto mappings can have an empty limit space. Consider all the countable ordinals α. The set of predecessors of α can be embedded, preserving order, in the set Q of rational numbers. Moreover, every embedding f which goes into a subset of Q that is bounded above can be extended to an embedding g of all the predecessors of β, for any larger countable ordinal β. Let X_α be a discrete space whose points are all these bounded embedding functions f. For $\beta \geqq \alpha$, define the bonding mapping $f_{\beta\alpha}$ by restriction; $f_{\beta\alpha}(f)$ is f restricted to the set of predecessors of α.

Since Q contains no isomorphic copy of the set of all countable ordinals, this defines an onto inverse mapping system with empty limit space.

This example is simplified from Higman-Stone [1]; they used only certain countable subsets of the sets X_α. There is no such example using only finite sets, as the next exercise shows.

*10. COMPACT LIMITS.

(a) An inverse mapping system of nonempty compact spaces has a nonempty limit space, even if the bonding mappings are not onto.

(b) Moreover, if $\{X_\alpha; f_{\alpha\beta}\}$ is an inverse system of compact spaces with limit L, every point $x \in X_\beta$ which is in every image $f_{\alpha\beta}(X_\alpha)$ is in $g_\beta(L)$. No such result holds for inverse mapping sequences of arbitrary spaces.

11. EMBEDDING IN E^{2n}.

(a) Consider Euclidean space E^{2n} as a product $A \times B$ of two copies of E^n. Call a subset S of $A \times B$ *flat* if there is a homeomorphism $h: A \times B \rightarrow A \times B$ such that $h(S) \subset A \times \{0\}$. Then the graph of a continuous function f from a closed subset of A into B is a flat set.

(b) If S is a compact space embeddable in E^n, then any continuous mapping of S into a flat subset of E^{2n} can be uniformly approximated by embeddings into flat subsets.

(c) The inverse limit of a sequence of compact subspaces of E^n can be embedded in E^{2n}. (Use IV.37.)

The fact that not every n-dimensional compact metric space can be embedded in E^{2n} was established by Flores [1]. In fact, for each n, the n-skeleton of a closed simplex of dimension $2n+2$ cannot be embedded in E^{2n}; but the proof is beyond the scope of this book.

Research Problem B_2.

INFINITE-DIMENSIONAL POLYHEDRA. There is a large problem here, namely the systematic investigation of topological and uniform realizations of abstract simplicial complexes. One important paper in the literature (Dowker [**2**]) has examined this problem, not from a categorical viewpoint. Dowker's work tends to confirm, what many successful applications suggest, that for topology J. H. C. Whitehead's realization by CW-complexes has strong claims to preference. Its definition is as simple as could be: it satisfies our axioms (1), (2), (3), hence also Proposition IV.2—this determines the point set $r(K)$ for every K—and the topology is the finest compatible with axiom (1). But Dowker's work highlights the point that the suitability of CW-complexes for homology and homotopy is not conclusive; many realizations are topologically distinct but homotopy equivalent.

By now substantial experience in uniform spaces supports the pretensions of uniform complexes, in the finite-dimensional case only. (In any case they are homotopy equivalent (topologically) with CW-complexes; Dowker [**2**].) In general they are not satisfactory, e.g. because they lack subdivisions. One can save the subdivisions, or any sufficiently narrow requirement, by tailoring a definition to fit. (Kuzminov and Švedov [**1**] define a realization for which IV.6 is always valid; but all their applications are in the finite-dimensional case.) The real problem holding up progress is, what applications can be made of infinite-dimensional polyhedra in the general theory of uniform spaces? It would probably be beside the point to carry out a formal investigation of realizations with no specific applications in mind.

We have not fully stated the problem yet. Cf. Research Problem B_3, where B_1 and B_2 will be brought together.

Notes. The theory of simplicial complexes branches off in at least two main directions: to cell complexes, of which the first main examples are the finite products of simplicial complexes, and to semi-simplicial complexes, of which the first main example is the singular complex of a topological space. For more on complexes, relevant to the present subject, see Lefschetz [**L**].

The theory of inverse limits branches off in many directions, but the literature is not so extensive. Eilenberg-Steenrod [**ES**] give a fairly selfcontained chapter on them; Freudenthal's basic results [1] on compact metric spaces and finite polyhedra have been partially generalized (Isbell [4]); and there is more in later chapters here.

The result of Exercise 6(b) was observed by James Case in 1957. The results of Exercise 7 are implicit in Freudenthal [1]; 7(a) is explicit in Brown [1].

The analogy between inverse limits and series expansion of functions is suggested in Freudenthal [1]. It does not seem profitable to press it very far; in particular, the special representations we shall use (V.33) are highly non-unique and so not analogous to Taylor series. But three remarks demand to be made here. (1) Freudenthal developed devices for normalizing certain inverse mapping systems, which do have applications (Freudenthal [1]); and there is a qualified uniqueness theorem for normalized representations of metric spaces. (2) However, Freudenthal's existence theorem for normalized representations in the compact case does not generalize; the normalization steps make sense, but the process cannot be completed (Isbell [4]). (3) With a different approach, Pasynkov [2] has given a unique inverse mapping system whose limit is any prescribed compact space. The cardinal number of the system is high.

In connection with the Research Problem we should note two theorems in the literature that are significant for uniform spaces (though basically topological) and originally proved by means of infinite-dimensional polyhedra: Dugundji's Extension Theorem (Dugundji [1]), which (with Dowker [3]) determines the fine spaces all of whose closed subspaces are fine, and a theorem of Dowker [1] which may be restated as asserting the coincidence of two dimension functions for fine spaces (namely δd defined above and Δd defined in the next chapter). Both theorems have simpler proofs not involving realizations of complexes; see Arens [1], and VIII.3, VIII.4 below.

Zeeman has questioned the value of any topology for infinite complexes, and at the same time formalized a theory of finite, not necessarily closed, triangulations (in [F, pp. 57-70]). The nonclosed triangulations will not give a complete uniformity; so infinite triangulations of uniform spaces are indispensable. This scarcely diminishes the relevance of Zeeman's work to uniform polyhedra; it merely shows that the problems are not the same.

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CHAPTER V

DIMENSION (1)

This chapter is a systematic treatment of the dimension theory of general uniform spaces.

Covering dimension. The *large dimension* ΔdX of a space X is said to be $\leqq n$ provided every uniform covering of X has a uniform refinement of dimension at most n. As with uniform dimension, the number ΔdX is the smallest such n, -1 for empty X, ∞ when there is no such n. Note

1. $\delta dX = \Delta dpX$.

PROOF. The condition for $\delta dX \leqq n$ is that every finite uniform covering $\mathscr{U}$ has a uniform refinement $\mathscr{V}$ of dimension at most n. For $\Delta dpX \leqq n$ we must be able to take $\mathscr{V}$ finite. But if $\mathscr{V}$ is a coshrinking of a shrinking $\mathscr{W}$, then $\mathscr{W}$ is finite and will do.

Obviously Δd is a uniform invariant. In view of II.35, δd is a δ-invariant.

2. *If X is a subspace of Y then $\delta dX \leqq \delta dY$ and $\Delta dX \leqq \Delta dY$. If X is dense in Y there is equality in both cases. Hence the Samuel compactification of X has the same uniform dimension as X.*

The proof of IV.23, with the remark $\delta dX = \delta dpX$, proves this. For the next two theorems we want a patching lemma.

3. LEMMA. *If $\{U_i\}$ is a finite uniform covering and $\mathscr{V}$ is a covering whose trace on each set U_i is uniform, then $\mathscr{V}$ is uniform.*

PROOF. Since the trace of $\mathscr{V}$ on U_i is uniform, it is the trace on U_i of some uniform covering $\mathscr{V}^i$ of the whole space. The finite intersection of all $\mathscr{V}^i$ and $\{U_i\}$ is a refinement of $\mathscr{V}$.

A related lemma, to be used shortly:

4. LEMMA. *If $\{A_i\}$ is a finite covering and for each i, N_i is a uniform neighborhood of A_i, then $\{N_i\}$ is a uniform covering.*

PROOF. If $N_i \supset \mathrm{St}(A_i, \mathscr{U}^i)$, then the intersection of all $\mathscr{U}^i$ refines $\{N_i\}$.

5. THEOREM. *If ΔdX is finite then $\delta dX = \Delta dX$. In any case $\delta dX \leqq \Delta dX$. More generally, if every finite uniform covering has an n-dimensional uniform refinement then so does every finite-dimensional uniform covering.*

PROOF. Note first that $\delta dX \leqq \Delta dX$, immediately from the definitions. Now suppose $\delta dX = n$, and consider any finite-dimensional uniform covering of X. By IV.25, we may suppose the covering has the form $\{U_{i\alpha}\}$, where i takes only finitely many values and for each fixed i, $\{U_{i\alpha}\}$ is a uniformly discrete collection. If H_i is the union of all $U_{i\alpha}$, then $\{H_i\}$ is a finite uniform covering. The hypothesis implies that $\{H_i\}$ has a shrinking $\{K_i\}$ which is at most n-dimensional. Then let $V_{i\alpha} = U_{i\alpha} \cap K_i$. The trace of $\{V_{i\alpha}\}$ on K_i is the trace on $K_i \subset H_i$ of $\{U_{i\alpha}\}$, which is uniform there; so by Lemma 3, all $V_{i\alpha}$ form a uniform covering of X. Since no two $V_{i\alpha}$ with the same first index meet, the dimension does not exceed the dimension of $\{K_i\}$.

There is no difficulty in constructing an example X with $\Delta dX = \infty$, $\delta dX = 0$, except that at this point we are not equipped to prove $\Delta dX = \infty$. Let us assume that we have an infinite-dimensional uniform space Y, with a uniform covering $\{U_\alpha\}$ that has no finite-dimensional uniform refinement. We define X to consist of the points of Y, and a covering $\mathscr{V}$ to be uniform if Y can be expressed as a union of finitely many sets $A_1, \cdots, A_n$ so that the trace of $\mathscr{V}$ on each A_i is uniform. Then X is a uniform space and $\delta dX = 0$. Let $\{V_\alpha\}$ be a uniform covering of Y which is a strict shrinking of $\{U_\alpha\}$. Then $\{V_\alpha\}$ is a uniform covering of X. Suppose it has a finite-dimensional refinement which is uniform on X, thus uniform in Y on each member of some finite covering $\{A_i\}$. Apply the extension theorem IV.20; we get a finite number of coverings of neighborhoods N_i, each finite-dimensional and finer than $\{U_\alpha\}$. By Lemmas 3, 4, the union makes a uniform covering of Y, a finite-dimensional refinement of $\{U_\alpha\}$. This is a contradiction. Therefore $\Delta dX = \infty$.

6. THEOREM. *If A is a subspace of X, $\Delta dA \leqq n$, and $\Delta dB \leqq n$ for every subspace B of X which is far from A, then $\Delta dX \leqq n$.*

PROOF. Given a uniform covering $\mathscr{U}$ of X, take a strict shrinking, restrict to A, take an n-dimensional uniform refinement, and extend isomorphically over a uniform neighborhood N of A to a covering $\{U_\alpha\}$ which is still finer than $\mathscr{U}$. Now let M be a uniform neighborhood of A so small that N is a uniform neighborhood of M; let L be a uniform neighborhood of A so small that M is a uniform neighborhood of L. Then $B = X - L$ is far from A, so that $\Delta dB \leqq n$. Cover B with the sets $U_\alpha - L$ and the trace of $\mathscr{U}$ on $X - M$. This covering is uniform on $N - L$ and on $X - M$; by Lemma 3, it is uniform on B. Let $\{V_\alpha\}$ be an n-dimensional uniform shrinking of it. Finally, for each U_α, define W_α as $(U_\alpha \cap M) \cup (V_\alpha - M)$; for each V_α which does not correspond to a U_α, let $W_\alpha = V_\alpha$. The covering $\{W_\alpha\}$ of X is uniform on B, since it is coarser than $\{V_\alpha\}$, and it is uniform on M. By construction it is finer than $\mathscr{U}$. It is at most n-dimensional on M and also on $X-M$; so the proof is complete.

7. Corollary. *Theorem* 6 *applies also to* δd.

For precompact reflection preserves all the relations involved.

8. Corollary. *If* X *is a finite union of subspaces* A_i *then* $\Delta dX = \max \Delta dA_i$ *and* $\delta dX = \max \delta dA_i$.

Let us mention another special case of the theorem. $d\infty X$ is defined as $\delta d(\beta(X)-X)$ (here β is Samuel compactification). rdX is the supremum of δdB over all compact subspaces B of X; note that rd is a topological invariant. Since the subspaces of $\beta(X)$ far from $\beta(X)-X$ are just the subspaces of X having compact closure; and since β preserves δd; we have

9. Corollary. $\delta dX = \max(rdX, d\infty X)$.

For the next theorem we want Sperner's celebrated lemma on subdivisions. It is actually valid for arbitrary subdivisions. We need it only for barycentric subdivisions. Let us introduce an intermediate notion; then the generalization can easily be covered in the exercises at the end of the chapter.

We call a subdivision $g: L_u \to K_u$ *semilinear* if it is a semilinear mapping. From elementary geometry, a semilinear subdivision cannot take an n-simplex into an $(n-1)$-simplex; and when it takes an $(n-1)$-simplex t into an open n-simplex s, t must be a face of exactly two n-simplexes of L_u.

10. Sperner lemma. *Let* K *be a polyhedron realizing a closed* n*-simplex with vertices* $v_0, \cdots, v_n$, *and let* $g: L \to K$ *be a semilinear subdivision. For each vertex* w_i *of* L *let* $h(w_i)$ *be a vertex of* K *whose star includes* $g(w_i)$. *Then there is a simplex of* L *whose vertices are mapped by* h *onto the set of vertices of* K.

Proof. Call a simplex of L "good" if it satisfies the concluding condition. We wish to prove the number of good simplexes is not 0; it will be easier to prove that it is odd.

For $n=0$, the number of good simplexes is 1, which is odd. Now assume this has been proved for $n-1$. We call an $(n-1)$-simplex of L a *marked face* if h maps its vertices onto the set $\{v_1, \cdots, v_n\}$. Evidently, among the faces of an n-simplex of L, at most two are marked. Moreover, an n-simplex is good if and only if it has exactly one marked face. Then if $m(t)$ denotes the number of marked faces of t, the sum of $m(t)$ over all n-simplexes t of L has the same parity as the number of good simplexes. This number may be obtained also by adding, over the marked faces u, the number $k(u)$ of n-simplexes of L having u as a face. This number is 2 if $g(u)$ is in the interior of K and 1 otherwise. We need only attend to the 1's, of which (by the inductive hypothesis) an odd number occur in the face of K whose vertices are $v_1, \cdots, v_n$. But no simplex of K other than this face and the interior can possibly contain the image of a marked face of L, for every other simplex of K is disjoint from the star of one of the vertices $v_1, \cdots, v_n$. This completes the proof.

11. *If a closed n-simplex with vertices $v_0, \cdots, v_n$ is covered by $n+1$ closed sets $A_0, \cdots, A_n$ in such a way that the face spanned by any set of vertices $\{v_i: i \in I\}$ is covered by the corresponding sets A_i, $i \in I$, then all A_i have a common point.*

PROOF. For every positive number ϵ, the simplex has subdivisions $g: L \to K$ all of whose simplexes have diameter $< \epsilon$ in K. Then if every subdivision has a simplex whose closure meets all A_i, there must indeed be a common point. To complete the proof, define a function h on the vertices w of any subdivision L to $\{v_0, \cdots, v_n\}$, so that each $h(w)$ is a vertex v_j of the carrier of $g(w)$ such that $g(w) \in A_j$. Such a choice is possible, since these A_j cover the carrier of $g(w)$. By Sperner's lemma, there is a good simplex; and it must meet all A_i.

12. THEOREM. *For an n-dimensional simplicial complex K, $\Delta dK_u = \delta dK_u = n$.*

PROOF. From IV.6 and IV.9, $\Delta dK_u \leqq n$. In view of V.2 and V.5, it remains only to prove $\delta dK_u \geqq n$ in case K is a closed n-simplex. Let $\mathscr{U}$ be the open covering consisting of the stars of the vertices $v_0, \cdots, v_n$. The specific result (to be cited in Exercise 2):

(*) Any open covering $\mathscr{V}$ finer than $\mathscr{U}$ has dimension at least n.

Indeed, $\mathscr{V}$ is a coshrinking of a shrinking $\{W_0, \cdots, W_n\}$. Take a strict shrinking $\{B_i\}$ of $\{W_i\}$, and let $A_i = B_i^-$. The closed covering $\{A_i\}$ is still a shrinking of $\{W_i\}$. The face spanned by a set of vertices must be covered by the corresponding A_i, for it is disjoint from the others (being disjoint from the stars of those vertices). Thus all A_i have a common point; so $\{W_i\}$ and $\mathscr{V}$ are at least n-dimensional.

Note that the example for $\Delta dX > \delta dX$ can now be completed by using a suitable metric space Y.

Extension of mappings. The next theorem we want is the following.

13. THEOREM. *For a uniform space X, $\delta dX \leqq n$ if and only if every uniformly continuous mapping of any subspace A of X into an n-sphere S^n has a uniformly continuous extension over X.*

This reduces easily (using III.9, III.11, V.2) to a theorem about compact spaces. However, the lemmas for Theorem 13 are done for uniform spaces rather than just for compact spaces.

A space X is called an ANRU (*uniform absolute neighborhood retract*) if X can be embedded in an injective space Y so that for some uniform neighborhood N of X in Y, there is a retraction of N onto X.

14. *If X is an ANRU then whenever $A\subset B$, any uniformly continuous mapping of A into X can be extended over a uniform neighborhood of A in B (and conversely). Also, X is complete.*

PROOF. If $X\subset Y$ and $r:N\to X$ retracts a uniform neighborhood (Y being injective), then consider any mapping $f:A\to X$ as going into Y. There is an extension $g:B\to Y$. Then $g^{-1}(N)$ is a uniform neighborhood of A in B, and rg is an extension of f over $g^{-1}(N)$ with values in X. For the converse, put $A=X$, let B be an injective space containing X, and extend the identity $1:X\to X$. Finally, X must be a retract of a uniform neighborhood in its completion Z, i.e., a retract of Z; hence X is closed in Z, and X is complete.

15. *Every finite-dimensional uniform complex is an ANRU.*

PROOF. Consider an n-dimensional complex K isometrically embedded in an injective metric space Y. Let N be the $1/(3n+3)$ neighborhood of K in Y. For each vertex v_α of K, we define a non-negative real-valued function f_α on N which coincides on K with the αth coordinate function:

$$f_\alpha(p)=\max(0,\sup[x_\alpha-2d(p,x):x\in K]).$$

Thus for $f_\alpha(p)$ to be nonzero, p must be closer than $\lambda>0$ to some x with $x_\alpha\geqq 2\lambda$. Suppose this is true for some finite set of indices $\alpha_1,\cdots,\alpha_m$, with positive numbers $\lambda_1,\cdots,\lambda_m$. Let λ_1 be the smallest of the λ's. There is $x\in K$ within λ_1 of p, thus within $\lambda_1+\lambda_j$ of points of K having α_jth coordinates at least $2\lambda_j\geqq\lambda_1+\lambda_j$. Since the inequality $d(p,x)<\lambda_1$ is strict, x has α_jth coordinate nonzero for all j. Thus the set of α for which $f_\alpha(p)>0$ indexes the vertices of a simplex.

On the other hand, each $p\in N$ is within $(3n+3)^{-1}$ of some $x\in K$; since some x_α is at least $(n+1)^{-1}$, some $f_\alpha(p)$ is at least $(3n+3)^{-1}$. Consequently, $g(p)=(\sum f_\alpha(p))^{-1}$ defines a bounded uniformly continuous function; and $g_\alpha(p)=f_\alpha(p)g(p)$ defines an equiuniformly continuous partition of unity and a retraction of N upon K.

A *homotopy* is a uniformly continuous mapping $h:I\times X\to Y$, where X and Y are arbitrary and I is the closed interval $[0, 1]$. It is called a homotopy *between* the mappings f and g defined $f(x)=h(0, x)$, $g(x)=h(1, x)$; and these mappings are called *homotopic*. Not that since $I\times X=I^*X$ (III.24), a homotopy is essentially the same thing as a path in the function space $U(X, Y)$.

Then it is clear that the relation of being homotopic is an equivalence relation. The equivalence classes are called *homotopy classes*.

16. *If Y is an ANRU, then the homotopy classes in any function space $U(X, Y)$ form a uniform covering.*

PROOF. Embed Y in a product Z of unit balls of ell-infinity spaces. We note some facts about Z. (The point is that Z is a bounded body in a locally convex linear topological space.) First, any two mappings $f: X \to Z$, $g: X \to Z$ are homotopic; the formula $h(t, x) = tg(x) + (1-t)f(x)$ defines a homotopy. Second, Z has arbitrarily fine uniform coverings consisting of convex sets. Now since Y is an ANRU it is a retract of a uniform neighborhood, which contains $\mathrm{St}(Y, \mathscr{U})$ for some uniform covering $\mathscr{U}$ by convex sets. Then any two mappings $f: X \to Y$, $g: X \to Y$, within $\mathscr{U}$ of each other, are homotopic, by retracting the homotopy $tg + (1-t)f$ into Y.

17. HOMOTOPY EXTENSION LEMMA. *Let A be a subspace of X and f and g two uniformly continuous mappings of A into an ANRU, Y. Suppose that f and g are homotopic and that f has an extension f': $X \to Y$. Then g has an extension g': $X \to Y$ which is homotopic to f'.*

PROOF. Select a homotopy $h: I \times A \to Y$ with $h(0, a) = f(a)$, $h(1, a) = g(a)$. Consider the ell-shaped subspace L of $I \times X$ consisting of $I \times A$ and $\{0\} \times X$. Define $k: L \to Y$ to coincide with h on $I \times A$ and with f' on $\{0\} \times X$.

The proof that k is uniformly continuous, while simple, depends on the shape of L. For any uniform covering $\mathscr{V}$ of Y there exist a uniform star-refinement $\mathscr{W}$ of $\mathscr{V}$ and two uniform coverings, $\mathscr{U}_0$ of X, $\mathscr{U}'$ of $I \times A$, such that $f'^{-1}(\mathscr{W})$ is coarser than $\mathscr{U}_0$ and $h^{-1}(\mathscr{W})$ is coarser than $\mathscr{U}'$. Since $I \times A$ is a product space we may assume $\mathscr{U}'$ is a product covering $\mathscr{U}_1 \times \mathscr{U}_2$. We may assume further that $\mathscr{U}_0$ is an extension of $\mathscr{U}_2$. Then $\mathscr{U}_1 \times \mathscr{U}_0$, restricted to L, is finer than $k^{-1}(\mathscr{V})$.

Since Y is an ANRU, there is a uniformly continuous extension j of k over a uniform neighborhood N of L. Moreover, we may assume that N consists of a product neighborhood of $I \times A$ and the set $[0, \epsilon) \times X$, for some $\epsilon > 0$. There is another uniformly continuous function $e: I \times X \to I$ that takes the constant value 1 on L and 0 outside N. Then define h' on $I \times X$ to Y by $h'(t, x) = j(te(x, t), x)$. Because of the shape of N and the size of e, h' is well-defined. Uniform continuity of h' is clear if we note that the functions being multiplied are bounded. By definition h' is an extension of k, and $g'(x) = h'(1, x)$ gives the desired extension of g.

18. *For any ANRU, Y, a space X has the property that every uniformly continuous mapping of a subspace of X into Y has a uniformly continuous extension over X if and only if the completion of X has this property. If Y is compact,*

then X has that property if and only if its Samuel compactification does.

Proof. The second assertion reduces to the first by precompact reflection. Further, if the completion of X, or any other space containing X, has the property that every mapping of any of its subspaces into Y can be extended, then so does X. Conversely, suppose Y is an ANRU. Let Z be the completion of X and $f: A \to Y$ be a mapping defined on a subspace A of Z. Let f_1 be an extension of f over a uniform neighborhood of A, call it N. If the restriction $f_1 | N \cap X$ has an extension f_2 over X, then f_2 can be further extended over the completion Z. This mapping $f_3: Z \to Y$ coincides with $f = f_1 | A$, since it coincides with f_1 on a dense subset of a neighborhood of A.

Proof of Theorem 13. First suppose $\delta d X \leqq n$, and let $f: A \to S^n$ be a uniformly continuous mapping defined on a subspace A of X. To show that f can be extended, it suffices by Propositions 16 and 17 to show that one can approximate f uniformly by mappings extensible over X.

Consider the following scheme for mapping X into S^n. $\mathscr{U}$ is some n-dimensional finite uniform covering of X, and h a canonical mapping from X into the nerve N of $\mathscr{U}$. Further, S^n is considered as the set of all points at distance 1 from the origin in E^{n+1}; and $j: N \to E^{n+1}$ is a semilinear mapping satisfying the following condition. For each vertex v_i of N which corresponds to a set $U_i \in \mathscr{U}$ that meets A, $j(v)$ is a point of S^n which is $f(a)$ for some $a \in U_i$. Finally, note that $j(N)$ must be nowhere dense in E^{n+1}; select a point c near the origin, not in $j(N)$, and let $k: j(N) \to S^n$ be central projection from c. Then kjh is certainly a mapping from X to S^n. For each point p of A, $jh(p)$ is a weighted average of certain points $f(a_i)$, each a_i lying in $\mathrm{St}(p, \mathscr{U})$. Then if $\mathscr{U}$ is fine enough and c close enough to 0, $kjh(p)$ must be arbitrarily near to $f(p)$; there is no need to compute all the details.

Now suppose every uniformly continuous mapping of a subspace A of X into S^n can be extended over X. We prove first that the same is true for mappings $f: A \to S^{n+1}$. Such a mapping can be described by means of its latitude f_0: $A \to I$ and its "longitude" f_1: $A \to S^n$; the longitude is undefined at the poles, however. Then let P and Q be small polar caps on S^{n+1}, and let $A_1 = A - f^{-1}(P) - f^{-1}(Q)$. Let g_0 be a uniformly continuous extension of f_0 over X. Let g_1 be a uniformly continuous mapping from X to S^n extending the well-defined uniformly continuous mapping $f_1: A_1 \to S^n$. Let $g: X \to S^{n+1}$ take each point x to the point at latitude $g_0(x)$ and longitude $g_1(x)$. For x in A, $g(x)$ has the same latitude as $f(x)$ and unless the latitude is high, the longitude is also the same. Hence $g | A$ is as near as we please to f, and f is extensible over X.

It follows that mappings of subspaces of X into S^n or any higher-dimensional sphere can be extended. Let us now assume (as we may) that X is compact. Then let $\mathscr{U}$ be any finite open covering of X, and $h: X \to N$ a canonical mapping into the nerve of $\mathscr{U}$. Suppose B is a simplex of N of maximum dimension, and this dimension m exceeds n. Let S denote the boundary of B, $A=h^{-1}(S)$, $Y=h^{-1}(B)$. The mapping h: $A \to S$ can be extended over all of X; in particular, it has a continuous extension h_1 over Y. Let h': $X \to N$ coincide with h on $X-Y$ and with h_1 on Y. Then h' is continuous, and, moreover, h' is still canonical. Continuing this process we obtain a canonical mapping h^* from X into the n-skeleton of N. But the inverse image under h^* of the covering of N consisting of the stars of the vertices is then an n-dimensional open covering finer than $\mathscr{U}$.

Here is a restatement of the theorem that will be useful later. Note that we could always extend mappings of subspaces into S^n if we allowed the extension to take values in a solid ball I^{n+1} having S^n as boundary. A mapping $f: X \to I^{n+1}$ is called *essential* if the restriction $f \mid f^{-1}(S^n)$ cannot be extended over X with values in S^n.

19. *The uniform dimension of a uniform space X is the largest n such that X has an essential mapping into I^n.*

Separation. In a space X, a set U is said to *δ-separate* two sets A, B if $X-U$ is the union of two sets A', B', far from each other, with $A \subset A'$ and $B \subset B'$. A set W is said to *free* A and B if W is far from $A \cup B$ and every uniform neighborhood of W which is disjoint from $A \cup B$ δ-separates A and B.

20. *A set W frees A and B if and only if the closure of W frees the closures of A and B.*

Since the far sets and the uniform neighborhoods are the same for the given sets as for their closures, this is obvious.

Inductive dimension $\delta\mathrm{Ind}\ X$ of a uniform space X is defined as follows. As usual, $\delta\mathrm{Ind}\ X=-1$ means that X is empty. Recursively, $\delta\mathrm{Ind} X \leqq n$ if every two far sets in X are freed by some subspace W such that $\delta\mathrm{Ind}\ W \leqq n-1$. Then $\delta\mathrm{Ind}\ X$ is the least n such that $\delta\mathrm{Ind} X \leqq n$, ∞ if there is no such n.

This definition parallels the definition of topological "big inductive dimension" Ind. The notion of freeing could not reasonably be replaced by the notion of δ-separating; for example, a set δ-separating the real line must contain an interval.

Let us note before getting into the theorems that there will be no counter-examples at all. It is an open question whether $\delta\mathrm{Ind}\ X=\delta d X$ for all X.

21. *If X is a dense subspace of Y, then* $\delta\mathrm{Ind}\ X \geqq \delta\mathrm{Ind}\ Y$.

Proof. Suppose $\delta\mathrm{Ind}\ X \leqq n$; let A and B be far sets in Y. Let C, D be uniform neighborhoods of A, B, which are still far from each other. Let $E=C \cap X$, $F=D \cap X$. Let W be a subset of X freeing E and F, with $\delta\mathrm{Ind} W \leqq$

$n-1$. Then W frees A and B in Y. To check this it suffices to consider any closed uniform neighborhood V of W that is disjoint from $C \cup D$. Since $V \cap X$ is a uniform neighborhood of W in X, $X-V$ is a sum of far sets H, K, containing E and F, respectively. Since $Y-V$ is open, its intersection with X is dense in it; therefore $Y-V$ is the union of the relative closures of H and K, which are far sets containing the relative closures of E and F, which in turn contain A and B, respectively.

22. **Remark.** *The function* $\delta\mathrm{Ind}$ *would not be changed if we changed the definition to refer only to closed sets.*

This follows from 20 and 21. We remark further that $\delta\mathrm{Ind}$ is a δ-invariant; so Samuel compactification cannot increase it.

23. **Theorem.** *For every uniform space* X, $\delta\mathrm{Ind}\, X \geqq \delta dX$.

Proof. It suffices to prove this for compact spaces. That is, if Y is compact and $\delta\mathrm{Ind}\, Y \leqq n$ then $\delta dY \leqq n$; and we may suppose this has already been done for $n-1$.

Let $\{U_i\}$ be any finite open covering of Y and $\{V_i\}$ a strict shrinking of it. For each i, let W_i be an $(n-1)$-dimensional closed set freeing V_i from $Y-U_i$; the meaning is $\delta\mathrm{Ind} W_i \leqq n-1$, but by the inductive hypothesis this implies $\delta dW_i \leqq n-1$ too. Then the union W of the W_i has dimension $\delta dW \leqq n-1$ (V.8). Using the covering extension theorem IV.20, we can get an $(n-1)$-dimensional open covering $\{P_j\}$ of a neighborhood N of W which is finer than $\{U_i\}$. Let M be a neighborhood of W such that N is a uniform neighborhood of M. Since M is a neighborhood of every W_i, $Y-M$ is a sum of open-closed subsets H_i containing $V_i - M$ and contained in U_i. Let $Q_1 = H_1$, and define Q_i recursively as $H_i - \bigcup [H_j : j < i]$. Then $Y-M$ is the union of the uniformly discrete collection $\{Q_i\}$, which with $\{P_j\}$ forms an open covering of dimension at most n refining $\{U_i\}$, as required.

There is a characterization of δd in terms of separation, derived from the notion of separating opposite faces of a cube. It will be convenient to represent the ball I^{n+1} as the product of intervals $I = [0,1]$ in E^{n+1}. Moreover, we shall triangulate its boundary S^n as a generalized octahedron, with $2n+2$ vertices classified into $n+1$ antipodal pairs; a set of vertices spans a simplex if and only if it contains no antipodal pair. It is convenient to note also (using a round sphere, and the unique shortest paths joining nonantipodal pairs of points) that two mappings of any space into S^n are homotopic unless points of the domain can be found whose two images are arbitrarily nearly antipodal.

A family of sets in a uniform space is called *vanishing* if their complements form a uniform covering.

24. THEOREM. *The following properties of a uniform space X are equivalent:*

(a) $\delta dX \leqq n$.

(b) *For any $n+1$ pairs (A_i, B_i) of sets, each pair far from each other, there is a vanishing family of $n+1$ sets W_i such that each W_i frees A_i and B_i.*

(c) *For any $n+1$ pairs (A_i, B_i) of sets, each pair far from each other, there is a vanishing family of $n+1$ sets U_i such that each U_i δ-separates A_i and B_i.*

PROOF. First, though (b) and (c) seem worth stating separately, their equivalence is trivial. A δ-separating set can be shrunk to a freeing set, and a vanishing family of sets has a vanishing family of uniform neighborhoods.

Suppose (a); so X has no essential mapping into I^{n+1}. Let (A_i, B_i) be $n+1$ pairs of far sets. Take mappings f_i: $X \to I$ with $f_i = 0$ on A_i, $f_i = 1$ on B_i. These are the coordinates of a mapping $F: X \to I^{n+1}$. Since F cannot be essential, there is $G: X \to S^n$ such that $G(x) = F(x)$ whenever $F(x) \in S^n$— in particular, when x is in any A_i or B_i. Then let U_i be the set of all x such that $G(x)$ has ith coordinate in $[1/3, 2/3]$; the U_i form a vanishing family δ-separating A_i and B_i.

Conversely, suppose $\delta dX > n$; so there is an essential mapping $F: X \to I^{n+1}$. Define A_i as the set where $\pi_i F = 0$, B_i by $\pi_i F = 1$. We shall get a contradiction from the assumption that the pairs (A_i, B_i) are δ-separated by a vanishing family $\{U_i\}$. The uniform covering $\{X - U_i\}$ would then be refined by a uniform covering $\{P_1, \cdots, P_{n+1}, Q_1, \cdots, Q_{n+1}\}$, with $A_i \subset P_i$, $B_i \subset Q_i$, and P_i far from Q_i. The nerve of this covering is isomorphic to a subcomplex of S^n. Speccifically, we embed it in the regular generalized octahedron, with P_i corresponding to the center of the face $\pi_i = 0$, Q_i to the center of $\pi_i = 1$; and we use central projection from the center of I^{n+1} to push the nerve into S^n. Now consider a canonical map G' from X to this nerve in S^n. If $F(x) \in S^n$ then $F(x)$ is in some face, $\pi_i = 0$ or $\pi_i = 1$; $G'(x)$ is not in the star of the center of the opposite face. Therefore $G'' = G' | F^{-1}(S^n)$ is homotopic to $F | F^{-1}(S^n)$. By the homotopy extension lemma, $F | F^{-1}(S^n)$ has an extension G: $X \to S^n$; that is, F is inessential, a contradiction.

It is natural to conjecture that the possibly stronger condition $\delta \mathrm{Ind} X \leqq n$ amounts to (c) of Theorem 24 with a change in the order of quantifiers. This is an open question, apparently independent of the problem whether $\delta\mathrm{Ind}$ and δd really differ. There is a partial result.

25. LEMMA. *Suppose E is a subspace of X, $\delta \mathrm{Ind}\ E = 0$, and A and B are far sets in X. Then A and B are δ-separated by some set far from E.*

PROOF. Let C and D be far uniform neighborhoods of A and B. Decompose E into far sets $F \supset C \cap E$, $G \supset D \cap E$. Now F is disjoint from D, hence far from B; similarly G is far from A. Then $A \cup F$ is far from $B \cup G$. Let U and V be far uniform neighborhoods of these sets; then $X - U - V$ is the set we require.

26. **Theorem.** $\delta \mathrm{Ind} X = 0$ *if and only if* $\delta d X = 0$. $\delta \mathrm{Ind} X \leqq 1$ *if and only if any two far sets* A_1, B_1 *can be freed by a set* W_1 *such that any two far sets* A_2, B_2 *can be freed by a set* W_2 *far from* W_1.

Proof. Since $\delta \mathrm{Ind} X < 0$ if and only if $\delta d X < 0$, the first assertion is a special case of Theorem 24. For the second, if $\delta \mathrm{Ind} X \leqq 1$ we take W_1 zero-dimensional and apply the lemma; on the other hand, if $\delta \mathrm{Ind} X > 1$ we can choose A_1, B_1 so that W_1 cannot be zero-dimensional, and then we can choose A_2, B_2 contained in W_1 and not freed in W_1 by the empty set.

Theorem 26 may extend to higher dimensions; Lemma 25 does not. (Exercise V.7)

Metric spaces.

27. **Lemma.** *Let G and H be subspaces of a metric space. Then G contains a set J such that*

(1) *every subset of J which is far from H is uniformly discrete, and*

(2) *every subset of G which is far from J is far from H.*

Proof. To construct J, let U_n denote the intersection of G with the $1/n$ neighborhood of H; let J_n be a maximal set of points of U_n distant at least $1/n$ from each other; let $J = \bigcup J_n$.

28. **Theorem.** *Let M be a metric space, H a nonempty subset of M, and J a subset of M such that every subset of J which is far from H is zero-dimensional. Then* $\delta \mathrm{Ind} J \leqq \delta \mathrm{Ind} H$.

Proof. If $\delta \mathrm{Ind}\, H = 0$, the theorem holds whether M is metric or not. Here $\delta d(H \cup J) = 0$ by Theorem 6, $\delta d J \leqq 0$ by IV.23, and this means $\delta \mathrm{Ind} J \leqq 0$.

Suppose the theorem is established for $\delta \mathrm{Ind} H \leqq n-1$, and consider the case $\delta \mathrm{Ind}\, H = n$. For any far subsets A and B of J, let C and D be far uniform neighborhoods of them, and let E and F be far uniform neighborhoods of C and D, respectively. Then $E \cap H$ and $F \cap H$ are freed in H by some subset V with $\delta \mathrm{Ind} V \leqq n-1$. Applying the lemma to J and V, we get a subset K of J satisfying (1) and (2). Then $W = K - C - D$ also satisfies these conditions, the first a fortiori, the second because a set far from $K - C - D$ is the union of a set far from K and a set contained in any preassigned uniform neighborhood of $C \cup D$. By construction W is far from A and B; by the inductive hypothesis $\delta \mathrm{Ind}\, W \leqq n-1$. It remains to show that for any uniform neighborhood U of W disjoint from A and B, $J - U$ decomposes into two far sets respectively containing A and B. Here $J-U$ is far from V; i.e., V has a uniform neighborhood T disjoint from $J-U$. Choosing T small, it will δ-separate $E\cap H$ and $F\cap H$ in H; so $H-T=P\cup Q$ with P far from Q, $E\cap H\subset P$, $F\cap H\subset Q$. Let R and S be far uniform neighborhoods of $P\cup A$ and $Q\cup B$. Then $R\cup S\cup T$ is a uniform neighborhood of H. Let I be a uniform neighborhood of H far from $J-R-S-T$, and split $J-I$ into far sets Y, Z, containing $(J\cap R)-I$ and $(J\cap S)-I$, respectively. We shall be finished when we verify that $J-U$ decomposes into its intersections with $R\cup(Y-S)$ and with $S\cup(Z-R)$, which are far sets containing A and B, respectively. Indeed, R and S already contain A and B. Those points of $J-U$ which are not in $R\cup S$, being also not in T, are in $J-I$; hence they are in Y or in Z. Thus $J-U\subset R\cup S\cup(Y-S)\cup(Z-R)$. R is far from S, and Y is far from Z. $(J-U)\cap R$ is far from $J\cap(Z-R)$; for $(J-U)\cap R\cap I\subset I-S-T$ (far from $J-R$) and $(J\cap R)-I\subset Y$ (far from Z). Likewise $(J-U)\cap S$ is far from $J\cap(Y-S)$. This completes the proof.

29. Corollary. *For any subspace J of a metric space M, $\delta\mathrm{Ind}J\leqq\delta\mathrm{Ind}M$. If J is dense in M then $\delta\mathrm{Ind}J=\delta\mathrm{Ind}M$.*

Proof. Put $H=M$ in the theorem.

30. Theorem. *For any metric space M, $\delta\mathrm{Ind}M\leqq\Delta dM$.*

Proof. Since completion preserves both dimension numbers, we may assume that M is complete; recall that M is then supercomplete (II.48). Let $\Delta dM=n$. We assume inductively that the theorem is established for spaces of smaller dimension; then it will suffice to show that any two far sets A, B, can be freed by a set H such that $\Delta dH\leqq n-1$. (In view of V.26 we have $n-1\geqq 0$.)

H will be constructed as the limit of a stable filter with basis $\{S_0, S_1,\cdots\}$. For convenience, fix a metric d. Let C and D be far uniform neighborhoods of A and B, and let $S_0=M-C-D$. Recursively, suppose S_{j-1} is a subset of S_0 that δ-separates A and B, its complement being a union of far sets C_{j-1}, D_{j-1} containing A and B, respectively. Let $\mathscr{U}^j$ be a uniform covering of mesh at most 2^{-j} (*mesh*: least upper bound of the diameters of the elements), which is further so fine that each of its elements is either far from C_{j-1} or far from D_{j-1}, and which is the union of uniformly discrete collections $\mathscr{U}^j_i$, $0\leqq i\leqq n$. Let $\mathscr{V}^j=\{V_\alpha\}$ be a uniform strict shrinking of $\mathscr{U}^j=\{U_\alpha\}$. (Let t be a positive number such that each U_α contains the t-neighborhood of V_α.) Thus $\mathscr{V}^j$ is a union of uniformly discrete collections $\mathscr{V}^j_i$ corresponding to the $\mathscr{U}^j_i$. Let E_j be the union of all those elements V_α of $\mathscr{V}^j_0$ such that U_α contains a point of S_{j-1} which belongs to no element of $\mathscr{V}^j_0$; let $S_j=S_{j-1}-E_j$.

Since S_j contains all of S_{j-1} except for a uniformly discrete collection of sets none of which reaches from near C_{j-1} to near D_{j-1}, S_j δ-separates A and B. Moreover, S_j has an $(n-1)$-dimensional uniform covering of mesh at most 2^{-j}. To see this, note that S_j is a union of two far sets; those members of $\mathcal{V}_0^j$ which meet S_j are distant by at least t from the rest of S_j. Now on one of these sets the trace of $\mathcal{U}_0^j$ is a 0-dimensional uniform covering; on the other set the trace of the rest of $\mathcal{U}^j$ is an $(n-1)$-dimensional uniform covering. Finally, by construction, $\mathrm{St}(S_j, \mathcal{U}^j) \supset S_{j-1}$. Therefore the sequence $\{S_j\}$ is a basis of a stable filter, and the limit is the required set H.

31. COROLLARY. *If M is a metric space and $\Delta dM < \infty$, then $\Delta dM = \delta dM = \delta \mathrm{Ind} M$.*

It is an open question whether Corollary 31 holds for all metric spaces. At least the 0-dimensional case is not open.

32. *For a metric space M, if $\delta dM = 0$ then $\Delta dM = 0$. (Thus if one of δdM, $\delta \mathrm{Ind} M$, ΔdM is 0, all are 0.)*

PROOF. Fix a metric. For each positive number δ, define an equivalence relation; two points are equivalent if they cannot be separated by a decomposition of M into two sets distant at least δ from each other. This defines a zero-dimensional covering by equivalence classes, $\mathcal{U}(\delta)$, which certainly has Lebesgue number δ. We shall show that if $\delta dM = 0$ and $\epsilon > 0$ then some $\mathcal{U}(\delta)$ has mesh at most ϵ. Suppose the contrary. Then there is a sequence of pairs (x_n, y_n) of points distant by ϵ, such that no decomposition of M into two far sets separates infinitely many of these pairs. If some infinite set of x's or y's had diameter $< \epsilon/2$, this would mean that the finite uniform covering consisting of the complement of this set S and the $\epsilon/2$ neighborhood of S had no zero-dimensional uniform refinement. But if this is not the case, we can construct an infinite sequence of indices n for which the x_n and y_n form a uniformly discrete set, and again a contradiction. This completes the proof.

For a metric space M, finally, if M is an inverse limit of finite-dimensional polyhedra at all (cf. Exercise IV.8), then M is an inverse limit of the nerves of its uniform coverings. Given any normal sequence of uniform coverings $\mathcal{U}^n$, each of which is point-finite, we define an inverse mapping system on the nerves which is called the *barycentric system* on the sequence $\{\mathcal{U}^n\}$. Let K_n be the nerve of $\mathcal{U}^n$. Observe that for each vertex v_α of K_{n+1}, the corresponding set U_α^{n+1} in $\mathcal{U}^{n+1}$ is contained in at least one element of $\mathcal{U}^n$, but in only finitely many. Moreover, since U_α^{n+1} is nonempty, the corresponding vertices in K_n span a simplex. Now the bonding mappings $f_{m,n}$ will be semilinear. For each vertex v_α of K_{n+1}, $f_{n+1,n}(v_\alpha)$ is the centroid of the simplex whose vertices correspond to elements of $\mathcal{U}^n$ containing U_α^{n+1}.

This clearly takes simplexes into simplexes; so we have a well-defined semilinear mapping. Finally then, $f_{n+k,n}$ is $f_{n+1,n}\cdots f_{n+k,n+k-1}$.

33. **Lemma.** *If X is a complete finite-dimensional metric space and $\{\mathscr{U}^n\}$ a normal sequence of k-dimensional uniform coverings forming a basis for X, then X is uniformly equivalent to the limit space of the barycentric system on $\{\mathscr{U}^n\}$.*

Proof. For each x in X, let $s^n(x)=s^n$ be the closed simplex of K_n corresponding to the set of all elements of $\mathscr{U}^n$ which contain x. Clearly $f_{n+1,n}(s^{n+1})\subset s^n$, so that $\{s^n; f_{m,n}\}$ forms an inverse system. It is an inverse system of compact spaces; so its limit space is a nonempty subspace of the limit space L of the barycentric system (Exercise IV.10). On the other hand, each s^n is a metric space of diameter at most 1. Let us check that $f_{n+1,n}$ restricted to s^{n+1} decreases every distance by at least the ratio $k/(k+1)$. It suffices to check the vertices v_β of s^{n+1}; and there is at least one U_α in $\mathscr{U}^{n+1}$ such that each $f_{n+1,n}(v_\beta)$ has αth coordinate at least $1/(k+1)$, so we have it. Then the inverse limit of $\{s^n(x); f_{m,n}\}$ is a single point $\lambda(x)$ in L.

Now $\lambda: X\to L$ is (1) uniformly continuous. For this it suffices to show that the coordinates $\lambda_n: X\to K_n$ are uniformly continuous. Given n, given $\epsilon>0$, choose an integer r such that $(k/k+1)^r<\epsilon$, and observe that each element U_α of $\mathscr{U}^{n+r}$ is mapped by λ_{n+r} into the closed star of the vertex v_α of K_{n+r}; thus $\lambda_n(U_\alpha)=f_{n+r,n}\lambda_{n+r}(U_\alpha)$ has diameter $<\epsilon$, and λ_n is uniformly continuous. To see that (2) λ is onto, consider any thread $p=(p_n)$ in L. If t^n is the carrier of p_n, the sets $\lambda_n^{-1}(t^n)$ form a Cauchy filter base in X, with limit x. Then $\lambda_n(x)$ is in the closure of t^n, and $\lambda(x)=p$. (3) λ is one-to-one because $\{\mathscr{U}^n\}$ is a basis; moreover, each $\mathscr{U}^n$ is realized by λ_{n+1}, and λ is a uniform equivalence.

34. **Theorem.** *Let X be a complete metric space and $\{\mathscr{U}^n\}$ a basis of uniform coverings for X, each of which is finite-dimensional. Then there exists an inverse mapping system whose spaces are some of the nerves of the coverings $\mathscr{U}^n$ and whose limit space is X.*

The construction of the lemma proves this, if we make the following observation. The only use of the fixed dimension k was in showing that $f_{n+1,n}|s^{n+1}$ decreases distances at least by $k/(k+1)$. This depends on the dimension of the range. To retain a fixed factor $K<1$ with a range K_n of arbitrary dimension p, realize the given covering $\mathscr{U}^n$ by a canonical mapping and introduce a sequence of auxiliary p-dimensional coverings $\mathscr{A}_l<^*\mathscr{A}_{l-1}<^*\cdots<^*\mathscr{U}^n$, where l is so large that $(p/p+1)^l<K$. Since we have a basis, there is $\mathscr{U}^{n+1}$ finer than $\mathscr{A}_l$. The barycentric system on the sequence of coverings so constructed will then have X for limit space; and

there is a cofinal part whose spaces are some of the nerves of the given coverings $\mathscr{U}^n$.

Exercises.

1. CLOSED COVERINGS AND THE CANTOR SET.

(a) A compact space X has $\delta dX \leqq n$ if and only if every finite open covering is refined by an n-dimensional finite covering consisting of closed sets.

(b) If X is compact, $\delta dX \leqq r$, and $f: X \to Y$ is an onto mapping taking at most k points of X to any one point of Y, then $\delta dY \leqq kr + k - 1$.

(c) If X is complete metric and $\Delta dX \leqq n$ then there is a complete metric space Z with $\Delta dZ \leqq 0$ and a mapping of Z onto X taking at most $n+1$ points to any one point. (Use an inverse limit of 0-skeletons of nerves.)

(d) A compact metric space X without isolated points has dimension at most n if and only if there is a mapping of the Cantor set onto X taking at most $n+1$ points to any one point.

The generalization of this odd characterization poses an unsolved problem. Cf. Aleksandrov and Ponomarev [2].

2. ESSENTIALITY OF CUBES.

(a) The identity mapping 1: $I^n \to I^n$ is essential (Use (*) from the proof of Theorem 12.)

(b) If U is an open set in E^n with compact closure V and $f: V \to V$ is a mapping agreeing with the identity on $V - U$ then $f(V) = V$. (Hint: $E^n = I^n - S^{n-1}$.)

3. FIXED POINT THEOREM. A continuous mapping of I^n into itself must leave at least one point fixed.

4. SUBDIVISIONS. To establish Sperner's lemma for arbitrary subdivisions of a closed n-simplex $\bar{s}^n$. it remains to show that an $(n-1)$-simplex t^{n-1} lying in $\dot{s}^n$ is a face of just one n-simplex and an $(n-1)$-simplex in the interior is a face of two.

(a) This will follow if we show that the star of t^{n-1} is not homeomorphic with any open subset of E^n unless t^{n-1} is a face of just two n-simplexes.

(b) If t^{n-1} is a face of just one n-simplex then its star contains open sets with compact closure which do not satisfy the condition of Exercise 2(b).

(c) Suppose t^{n-1} is a face of at least three n-simplexes s_1, s_2, s_3, and its star is embedded in E^n as an open set G. Let H be the image of $t^{n-1} \cup s_1 \cup s_2$; note that H is also homeomorphic with E^n. Note that there is a retraction $r: G \to H$ such that $r(G - H)$ is nowhere dense in H. Let o be the image of the centroid of t^{n-1}, and N a 3ϵ-neighborhood of o in E^n which is contained in G. Choose p in $G - H$ within ϵ of o. Define $g: G - \{p\} \to G$ as follows: outside the 2ϵ-neighborhood of p, g is the identity; the deleted 2ϵ-neighborhood is retracted radially upon its outer boundary. Define $f: H \to H$ as $rg|H$. Show that on some open set in H with compact closure, f contradicts Exercise 2(b).

Of course there is a more general theorem, that an open set of E^n is not homeomorphic with a nonopen set of E^n. The proof cannot be given without wandering rather too far from our subject. See e.g. Hurewicz and Wallman [**HW**].

5. SUBSPACES OF FULL DIMENSION.

(a) A bounded subset S of E^n has $\delta dS < n$ if and only if S is nowhere dense in E^n.

(b) A subspace S of E^n has $\Delta dS = n$ if and only if for some positive number r, every uniform neighborhood of S contains a ball of radius r.

*6. APPROXIMATION.

(a) $\delta dX \leqq n$ if and only if every uniformly continuous mapping X into any Euclidean space can be approximated by mappings into subspaces of dimension at most n.

(b) $\Delta dX \leqq n$ if and only if every uniformly continuous mapping of X into a Banach space can be approximated by mappings into subspaces S with $\Delta dS \leqq n$.

7. RELATIVE DIMENSION AND SUCCESSIVE SEPARATION. Theorem 26 extends to higher dimensions for metric spaces. Define the *relative inductive dimension* of a family of subsets $C_1, \cdots, C_k$ of X as follows. $\delta \mathrm{Ind}_X(C_1, \cdots, C_k) = -1$ iff $\{C_1, \cdots, C_k\}$ is a vanishing family. $\delta \mathrm{Ind}_X(C_1, \cdots, C_k) \leqq n+1$ iff for any far sets A, B in X there is C_{k+1} freeing A and B such that $\delta \mathrm{Ind}_X(C_1, \cdots, C_{k+1}) \leqq n$. This determines the number $\delta \mathrm{Ind}_X(C_1, \cdots, C_k)$. $\delta \mathrm{Ind}^* X$ is defined as $\delta \mathrm{Ind}_X(X)$.

(a) $\delta \mathrm{Ind}_X(C_1, \cdots, C_k) \geqq \delta \mathrm{Ind}(C_1 \cap \cdots \cap C_k)$. Hence $\delta \mathrm{Ind}^* \geqq \delta \mathrm{Ind}$. By Theorem 26, $\delta \mathrm{Ind}^* X = \delta \mathrm{Ind}\, X$ if $\delta \mathrm{Ind}\, X \leqq 1$.

(b) Define a family of subsets $C_1, \cdots, C_k$ to be *divergent* if its trace on any set far from $C_1 \cap \cdots \cap C_k$ is a vanishing family. (Two very smooth curves in E^n are divergent if and only if they are not asymptotic. However, a subfamily of a divergent family need not be divergent. We have:) If $\{C_1, \cdots, C_{k-1}\}$ is divergent then $\{C_1, \cdots, C_k\}$ is divergent if and only if C_k is divergent from $C_1 \cap \cdots \cap C_{k-1}$.

(c) Key lemma for metric spaces: If $\delta \mathrm{Ind}\, C \leqq n+1$ and A is far from B then there exists D divergent from C, freeing A and B, with $\delta \mathrm{Ind}(C \cap D) \leqq n$.

(d) If $\{C_1, \cdots, C_k\}$ is a divergent family in a metric space X then $\delta \mathrm{Ind}_X(C_1, \cdots, C_k) = \delta \mathrm{Ind}(C_1 \cap \cdots \cap C_k)$. In particular, $\delta \mathrm{Ind}^* X = \delta \mathrm{Ind}\, X$; and $\delta \mathrm{Ind}_X(C) = \delta \mathrm{Ind}\, C$.

The last conclusion in (d) extends Lemma 25. However, there is a nonmetric space X containing an interval I such that $\delta \mathrm{Ind}_X(I) = \infty$. Begin with the *long line* L; the product of the set W of all countable ordinals and an interval $[0, 1)$, lexicographically ordered.

(e) L is connected in the order topology.

(f) Let L^* be the one-point compactification of L. For each nowhere dense set S in I, form a uniform space T_S as follows; from $L^* \times I$ (which is compact) remove $L \times S$. Identify I with the undamaged copy $\{\infty\} \times I$. Then if H frees 0 from 1 in T_S, H has a limit point in $I-S$.

(g) Sewing together all T_S along the edges I, with any uniformity such that all T_S are subspaces, we get the desired space X.

8. Product spaces.

(a) $\Delta d(X \times Y) \leqq \Delta dX + \Delta dY$. (Hint: Note IV.8.)

(b) The inequality (a) does not hold for δd or δInd. In fact, if $\Delta dX = \infty$, then $\delta d(X \times X) = \infty$. (Hence δInd also; V.23.)

A space X is called *dimensionally deficient* with respect to a dimension function d if for some space Y, $d(X \times Y) < dX + dY$. Dimensional deficiency occurs already in 2-dimensional compact metric spaces. [HW]. Its study involves homology and is outside the scope of this book.

9. Embedding in Euclidean spaces. Lemma 33 combined with Theorems IV.26, IV.37 will embed a space X in a Euclidean space if such an embedding is possible. To see this, define a *star-bounded collection* of coverings as a collection, the union of any two of whose members is star-bounded.

(a) X can be embedded in some Euclidean space if and only if X has a basis which is a countable star-bounded collection of Euclidean coverings all of whose nerves are embeddable in E^n for some fixed n.

Restatement of the conditions will not get around the fundamental lack of a characterization of Euclidean coverings; but restatements are of some interest. Immediately from IV.26, we have a restatement in terms of dimension and 1-skeletons.

(b) The conditions can be imposed on different bases. That is, if X has four bases of uniform coverings, one of which is countable, another star-bounded, another k-dimensional, and another having 1-skeletons of nerves embeddable in E^n, then X has a basis satisfying all these conditions.

(c) No three of the conditions suffice. Moreover, if either of the first two conditions is omitted, there is a counter-example that is homeomorphic with E^1 and has a basis of coverings whose nerves are isomorphic with E^1.

(d) The star-bounded basis can be restated in metric terms: for some fixed positive ϵ, for every $\delta > 0$, there is an integer n such that every set of diameter ϵ can be covered by n sets of diameter δ.

(e) A metric space having a star-bounded basis is uniformly equivalent to a metric space in which for every positive M and δ there is n such that every set of diameter M can be covered by n sets of diameter δ.

(f) The condition that X has a basis of coverings with the 1-skeletons of their nerves embeddable in E^n may be replaced by the condition that some mapping $f: X \to E^n$ realizes some covering belonging to a star-bounded basis. ($n \geqq 3$.)

*10. Real extension spaces. X is called an RE *space* if every mapping of a subspace of X into the real line can be extended over X. All known RE spaces are (dense) subspaces of inverse limits of fine spaces.

(a) Every fine space is RE. (Put a bounded metric on the line and use pseudometric extension, III.15. Then use coreflection into fine spaces and Tietze's extension theorem for metrizable spaces.)

(b) Every inverse limit of RE spaces is RE. (Homotopy extension; one need only approximate within finite distance for a homotopy.)

A subspace of an RE space is obviously RE.

(c) The completion of an RE space is RE.

(d) If X is an RE metric space then X has a basis of uniform coverings each consisting of a uniformly discrete collection $\mathscr{D}$ and a finite collection whose union is disjoint from the union of $\mathscr{D}$. (Otherwise an argument like V.32 gives a uniformly discrete sequence of pairs (x_n, y_n) each of which is joined by a finite chain of sets of diameter $1/n$.)

(e) An RE metric space is a dense subspace of an inverse limit of fine spaces. (V.34.)

Seventeen characterizations of the fine metric spaces have been collected by Atsuji [1] and Rainwater [1]. (Levšenko [1] gives others.) Here is one.

(f) A metric space X is fine if and only if the set of all nonisolated points of X is compact and every set far from it is uniformly discrete.

11. SUBCONNECTING MAPPINGS. In the proof of Theorem 34 the introduction of irrelevant canonical mappings is a flaw. The job can be done combinatorially as follows. A semilinear mapping $f: L \to K$ is called *normal* if it is simplicial from L to $K^{[n]}$ for some n. It is natural to extend the notion of connecting mapping to a *subconnecting mapping* $f: L \to K$, where L and K are nerves of coverings $\mathscr{V} < \mathscr{U}$; this is a simplicial mapping from L to $K^{[1]}$ taking each vertex V of L to the centroid of a simplex of K all of whose vertices contain V, or a composition of such mappings.

(a) If $\mathscr{U}$ is a k-dimensional covering and $\mathscr{V}$ is a covering so fine that $\mathrm{St}(\mathrm{St}(\cdots(\mathrm{St}(V, \mathscr{V}), \cdots), \mathscr{V}), \mathscr{V})$ is contained in an element of $\mathscr{U}$ for each V in $\mathscr{V}$ (where St appears $k+1$ times), then there is a subconnecting mapping from the nerve of $\mathscr{V}$ to the nerve of $\mathscr{U}$. (Define $n(U, V)$ as the number of times St can be applied to V without getting out of U. Since V is in at most $k+1$ sets $U, n(U, V)$ must omit a value in $\{0, \cdots, k+1\}$. Use the values greater than all omitted values.)

(b) Verify that an inverse mapping system of nerves of a basis bonded by subconnecting mappings has the correct limit space. (This is true for nonmetric spaces, but no such inverse mapping system need exist; cf. Mardešić [2], Pasynkov [1].)

Research Problem C.

DIMENSION OF METRIC SPACES. Proofs of three conjectures would settle all the unsolved problems apparent in this chapter except for Exercises 1 and 9. These are $\delta\mathrm{Ind} = \delta d$, $\delta\mathrm{Ind}^* = \delta\mathrm{Ind}$ (see Exercise 7), and $\Delta d = \delta d$ in metric spaces. A proof of the last would dispose of all three for metric spaces. Moreover, the methods available for metric spaces are so strong that the question should not remain open long. It has no literature.

If some of these conjectures are false, more complicated questions arise. There is no good reason apparent to the author for expecting either of the first two to be true. (Reasons for expecting the third to be true might begin with II.38.) In any case we should note the problem of a sum theorem for δInd (unsettled even for metric spaces) and the question whether δInd is monotonic on subspaces.

The sum theorems known for dimension functions defined inductively have all been proved indirectly, by means of noninductive characterizations, with some very important exceptions sharing the following peculiarity: the countable sum theorem in dimension $n-1$ is needed to prove the finite sum theorem in dimension n. (There is one such proof in this book, VIII.12, from Cech [1].) A proof like this is quite unusable for general uniform spaces, because there is no remnant of a countable sum theorem. So a direct proof of a sum theorem for δInd (or δInd*: the same thing in metric spaces, but perhaps easier to get at) would have to give us a new geometrical idea.

Notes. The only substantial omission in this chapter is homological dimension. Kuzminov and Švedov [1] give some results for uniform spaces. To read their paper one needs background which can be obtained from [HW]. That book remains the basic book on topological dimension theory; it is the source of Exercises 1-4 above. Exercise 5 is from Smirnov [5]. Exercise 9 is from Isbell [3] and Doičinov [1]. Exercise 10(b)-10(e) is from Corson and Isbell [1].

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CHAPTER VI

COMPACTIFICATIONS

This chapter is a somewhat specialized treatment of compactifications. We introduce several topological definitions of dimension and aim to relate them to uniform notions by examining their behavior under compactification. But compactification, not uniformity, is the main idea.

Dimension-preserving compactifications. In this chapter a space X is, unless specified otherwise, a uniformizable topological space. Juxtaposition of β with the name of a topological space denotes Stone-Čech compactification. Recall that β is a reflection functor and defines mappings $\beta f:\beta X \to \beta Y$. For a uniform space μX, $\beta(\mu X)$ denotes Samuel compactification.

We consider the question: has X a compactification of the same (or lower) dimension? Four dimension functions will be defined. The notations (more or less standard) are $\dim X$, $\operatorname{mindim} X$, $\operatorname{ind} X$, $\operatorname{Ind} X$; but the greater part of the literature on dim and Ind is restricted to normal spaces and uses simpler definitions, so our definitions may be surprising to the reader.

The *covering dimension* $\dim X$ is $\delta d\mu X$, where μ is the fine uniformity. (In normal spaces, in view of Exercise II.13, the definition can be phrased in terms of finite open coverings. The definition so phrased is due to Čech [**2**]. It was Smirnov [**4**] who pointed out that it had led to no results for non-normal spaces in a quarter of a century, and gave the present definition.) The *minimum dimension* $\min\dim X$ is the minimum of $\delta d\nu X$ over all uniform spaces νX homeomorphic with X. The *little inductive dimension* $\operatorname{ind} X$ is (like all dimension numbers) -1, in case X is empty; $\operatorname{ind} X \leqq n+1$ provided X has a basis of open sets U_α whose boundaries B_α have $\operatorname{ind} B_\alpha \leqq n$.

Next recall that two sets A, B in a space X are said to be *separated* if they have disjoint neighborhoods M, N. (M and N will be themselves separated if we took the trouble to choose open neighborhoods.) A set C *separates* A and B if $X - C = M \cup N$, where M and N are separated sets containing A and B (not necessarily neighborhoods). Similarly, a continuous real-valued function f on X *separates* A and B if $f(A)$ and $f(B)$ have disjoint closures in the real line. But separation by a function is stronger than separation by a set;

it is possible if and only if A and B are far in the fine uniformity. Such sets are said to be *completely separated.*

For the dimension function Ind, *big inductive dimension,* we begin with -1 for the empty space. $\operatorname{Ind} X \leqq n+1$ provided every two completely separated sets are separated by a closed set C with $\operatorname{Ind} C \leqq n$. (Very little has been done with this dimension function for non-normal spaces. In normal spaces, since completely separated sets are just sets with disjoint closures, one says "every two disjoint closed sets" instead of "every two completely separated sets". One is tempted to say "every two separated sets". But such a definition would disagree with all current definitions even in compact metric spaces, where all current definitions agree.)

Two trivial relations: $\dim X \geqq \min\dim X$ (immediately); $\operatorname{Ind} X \geqq \operatorname{ind} X$ (by induction).

Back to compactifications; by V.2, $\dim X$ is the same as $\delta d\beta X$. We may restate this as follows.

1. $\dim \beta X = \dim X$.

But there may exist other compactifications of lower covering dimension. We defer the example, for it can be used to establish several other points. (See VI.13.)

For the function min dim, certain desirable properties follow at once from the rather artificial definition.

2. min dim *coincides with* dim *for compact spaces. For any space* X, $\min\dim X$ *is the minimum of* $\dim Y$ *over all compactifications* Y *of* X*—or over all spaces* Y *containing* X. *Consequently, if* $W \subset X$ *then* $\min\dim W \leqq \min\dim X$.

PROOF. Compact spaces have only the fine uniformity. If $\min\dim X$ is $\delta d\nu X$, then it is $\dim\beta(\nu X)$. On the other hand, if $X \subset Y$ and $\dim Y = n$, then $\dim \beta Y = n$ and βY induces on X a uniformity with dimension at most n. The last assertion follows from this.

The difficulties with min dim show up before one goes much further. The main difficulty is that its value for a given space is commonly hard to determine. For the important case of dimension zero, this difficulty vanishes.

3. $\min\dim X = 0$ *if and only if* $\operatorname{ind} X = 0$, *and this is if and only if* X *has a basis of open sets that are at the same time closed sets.*

PROOF. Since the sets with empty boundaries are the open-closed sets, $\operatorname{ind} X = 0$ if and only if X has an open-closed basis. When $\delta d\mu X = 0$, every finite uniform covering is refined by a covering by disjoint open-closed sets;

so $\operatorname{ind} X = 0$. On the other hand, the finite coverings of a space by disjoint open-closed sets always form a basis for a preuniformity; $\operatorname{ind} X = 0$ says that this is a uniformity.

The preuniformity just described enables us to show that *the subcategory of spaces X such that* $\operatorname{ind} X = 0$ *is reflective*. The reflection QZ of a space Z is constructed as follows. The *quasi-component* of a point $z \in Z$ is the intersection of all open-closed sets containing z. Evidently the quasi-components of two points are either identical or disjoint; so we have a decomposition of Z into closed sets. However, the quotient space need not be regular. The space QZ has for points the quasi-components of Z; a set M is a neighborhood of a point K provided the union of M in Z contains an open-closed neighborhood of the set K. QZ is called the *quasi-component space* of Z. QZ has an open-closed basis; so it is completely regular, and $\operatorname{ind} QZ = 0$. One verifies easily that the natural mapping $Z \to QZ$ is a reflection mapping.

The quasi-component space will be used in VI.29 and later; for the present it suggests the qualitative difference between dimension zero and higher dimensions. We remark that if Z is compact, not only is the reflection $Z \to QZ$ a quotient mapping, but also the quasi-components are connected, i.e., they are components. This remark will not be used, but it will be recalled later.

From Propositions 2 and 3, for compact spaces Y, $\operatorname{ind} Y = 0$ if and only if $\dim Y = 0$. Thus if $\operatorname{ind} X = 0$ then X has a compactification Y with $\operatorname{ind} Y = 0$. This does not generalize to higher dimensions. The example showing this (VI.16) depends on the other promised example (VI.13). We have trivially that ind cannot be lowered by compactification.

4. *If* $W \subset X$ *then* $\operatorname{ind} W \leqq \operatorname{ind} X$.

PROOF. Suppose $\operatorname{ind} X \leqq n$; we wish to show $\operatorname{ind} W \leqq n$, and may suppose this has been done for $n-1$. Then if $\{U_\alpha\}$ is a basis of open sets of X with at most $(n-1)$-dimensional boundaries, $\{U_\alpha \cap W\}$ is a basis for W with the same property.

Finally, for Ind, the facts are very nearly the same as for dim. The example of compactification lowering dim will lower Ind also; and the compactification preserving dim preserves Ind also, at least for normal spaces. Three lemmas are needed; the first is the result from general topology that if A is a closed subspace of a normal space X then completely separated sets in A are completely separated in X. (This follows easily from Tietze's extension theorem.)

In general, a subspace W of X is called *normally embedded* if completely separated sets in W are completely separated in X. Equivalently, the reflection βe of the embedding $e: W \to X$ is an embedding.

5. LEMMA. *If W is a normally embedded normal subspace of X, then* $\operatorname{Ind} W \leqq \operatorname{Ind} X$.

PROOF. Suppose $\operatorname{Ind} X \leqq n$; we wish to show $\operatorname{Ind} W \leqq n$, and may suppose this has been done for $n-1$. Let A, B be completely separated sets in W, so also in X. Then some closed set C in X with $\operatorname{Ind} C \leqq n-1$ separates A and B. Therefore $C \cap W$ separates A and B in W. $C \cap W$ is closed in the normal space W, so normally embedded and normal. Then $C \cap W$ is normally embedded even in X, and in particular in C. Hence the inductive hypothesis applies; $\operatorname{Ind}(C \cap W) \leqq n-1$.

6. REMARK. *If W is a normally embedded subspace of X, then* $\dim W \leqq \dim X$.

For $\dim \beta W \leqq \dim \beta X$ by IV.23.

7. LEMMA. *If a closed set C separates A and B in X, then in βX the closure of C separates A and B.*

PROOF. $X-C$ must be the union of two separated (hence disjoint open) sets $U \supset A$, $V \supset B$. Every point of βX not in the closure of C must be a limit point of U or of V; we have the desired separation if we show that such a point p cannot be a limit of both U and V. Suppose it is. Let ϕ be a continuous function on βX to I taking the value 0 at p and equal to 1 on C. Define a function f on X by $f(x)=\phi(x)$ except in case $x \in V$ and $\phi(x)<1/2$, in which case $f(x)=1/2$. Then f is locally continuous on X, hence continuous, but cannot be extended over βX. The contradiction completes the proof.

8. THEOREM. *If X is normal,* $\operatorname{Ind} \beta X = \operatorname{Ind} X$.

PROOF. $\operatorname{Ind} X \leqq \operatorname{Ind} \beta X$ by Lemma 5. For the converse, if $\operatorname{Ind} X \leqq n$, consider any completely separated sets A, B in βX. Let E, F be completely separated neighborhoods of them. Then $E \cap X$, $F \cap X$ are separated in X by a closed set C with $\operatorname{Ind} C \leqq n-1$. Since C is closed in X, its closure in βX is βC We may apply induction to conclude $\operatorname{Ind} \beta C \leqq n-1$, and by Lemma 7, βC separates $E \cap X$ and $F \cap X$. Since $\beta X - \beta C$ is open, it is a union of disjoint open sets $U \supset E \cap X$, $V \supset F \cap X$. Then every point of A, being a limit point of $E \cap X$, is a nonmember of V. It is also not in βC, since E is a neighborhood of A disjoint from C. Thus $A \subset U$; similarly $B \subset V$, and the proof is complete.

The effect of β on Ind for non-normal spaces has never been studied. It seems likely that it changes dimension in an arbitrary fashion, except that it must preserve the distinction between zero and positive dimension.

9. $\operatorname{Ind} X = 0$ *if and only if* $\dim X = 0$.

Proof. Let μX be fine. $\operatorname{Ind} X = 0$ means any two far sets lie in complementary open-closed sets, i.e., every binary normal covering has a normal refinement of dimension 0. In view of II.36, this is equivalent to $0 = \delta d_\mu X = \dim X$.

The assertion just before Proposition 9 is now justified by Proposition 1.

Before resuming compactification we consider a more modest addition to a space.

10. *If p is a point of a space Y containing more points, then* $\dim(Y - \{p\}) \geqq \dim Y$ *and* $\operatorname{Ind}(Y - \{p\}) \geqq \operatorname{Ind} Y$.

Proof. Let $X = Y - \{p\}$. If $\operatorname{Ind} X \leqq n$, and A and B are completely separated sets in Y, let C be a closed subset of X separating $A \cap X$ from $B \cap X$ with $\operatorname{Ind} C \leqq n-1$. If $p \in A \cup B$, then C separates A and B in Y. Otherwise $C \cup \{p\}$ separates them, and by an induction we have $\operatorname{Ind}(C \cup \{p\}) \leqq \operatorname{Ind} C$.

For dim, we write μY and νX for the fine spaces, μX for the subspace X of μY. We wish to show that if $\delta d \nu X \leqq n$ then $\delta d_\mu Y \leqq n$. By V.7, it will suffice to show $\delta d_\mu H \leqq n$ for every subspace H of Y that is far from p. H is a subset of X, and we know $\delta d \nu H \leqq n$. To conclude we shall show that $\nu H = \mu H$. by showing that every continuous mapping of H into an ell-infinity space L which can be extended over X can be extended over Y. Given $f: X \to L$, and taking a continuous real-valued function g on Y that is 1 on H and 0 on a neighborhood of p, we define $h: Y \to L$ by $h(x) = g(x) f(x)$, $h(p) = 0$. Then $h \mid H = f \mid H$, and the proof is complete.

We shall see that equality need not hold in Proposition 10. Moreover, this inequality does not hold for ind or min dim, though the reverse inequality holds trivially. Also trivially, $\operatorname{ind}(Y - \{p\}) \geqq \operatorname{ind} Y - 1$. Whether this holds for min dim is unknown; see the end of Exercise 10, where a possible counterexample is described.

11. $\operatorname{Ind} X \leqq 0$ *if and only if every space Y consisting of X and one more point has* $\operatorname{ind} Y = 0$.

Proof. If $\operatorname{Ind} X \leqq 0$, we have $0 = \operatorname{Ind} Y \geqq \operatorname{ind} Y = 0$. If $\operatorname{Ind} X > 0$, let A and B be completely separated sets which are not separated by the empty set. In βX, consider the points p of A^-. Adjoining one of them to X must give a space in which p (though far from B) is not separated from B by the empty set; otherwise, in view of the compactness of A^-, we should have a finite number of open-closed sets of X whose union contains A and is far from B.

Proposition 10 involves a helpful property of subspaces of fine spaces. The concluding portion of the proof does not depend on the fact that $Y - X$ is a single point; it proves the following. (An excision property.)

12. *If A is a subspace of B and B is a subspace of C, with $C - B$ completely*

separated from A, then the fine uniformities on B and on C induce the same uniformity on A.

Examples.

13. EXAMPLE. *There is a normal space X with* $\operatorname{Ind} X = \dim X = 1$, $\operatorname{ind} X = \min\dim X = 0$, *such that by adjoining a single point one can get a space Y with all four dimension numbers 0 or a space Z with all dimension numbers 1. The compactifications βY and βZ have the same dimension numbers as Y and Z.*

PROOF. Let W be the space of all countable ordinals, in the order topology. In the interval $I = [0, 1]$, consider the congruence classes modulo the rationals. There are c such classes, each dense in I. and $c \geqq \aleph_1$. Take some $\aleph_1$ of these classes Q_α indexed by the elements of W. Then X is the subspace of the product space $W \times I$ consisting of those points (α, p) for which p does not belong to any Q_β, $\beta \geqq \alpha$.

Note first that every initial segment S_α of W (the set of all $\beta \leqq \alpha$) is a separable metrizable space. Hence $S_\alpha \times I$ is separable metrizable. It is open and closed in $W \times I$. Its intersection with X is a subspace of $S_\alpha \times (I - Q_\alpha)$, so has dimension zero in the sense of all four dimension functions. Therefore $\operatorname{ind} X = 0$ (and $\min\dim X = 0$ by VI.3).

Next note

(*) In W, two unbounded closed sets have unbounded intersection.

PROOF. If H and K are unbounded closed sets, then for any $\alpha_0 \in W$ there is α_1 in H after α_0; there is β_1 in K after α_1; and we can get a sequence $\cdots < \alpha_n < \beta_n$, which must converge to a point of $H \cap K$.

Consequently, a finite open covering of W must have an element whose complement is bounded. Moreover, the space W is countably compact (every infinite set has a limit point); so every countable open covering is refined by a finite open covering.

(**) If $\mathcal{U}$ is a finite open covering of X and p is a point of I, then for some $\alpha \in W$ and some neighborhood N of p in I, one element of $\mathcal{U}$ contains $X \cap [(W - S_\alpha) \times N]$.

PROOF. First note that $W \times \{p\}$ is in X except for an initial segment. Thus one open set $U \in \mathcal{U}$ contains $(W - S_\beta) \times \{p\}$ for some β. For each $\alpha > \beta$, U must contain $[\{\alpha\} \times N] \cap X$ for some neighborhood N of p; let ϵ_α be the radius of the largest spherical neighborhood N satisfying this condition. If the numbers ϵ_α had 0 as a limit point, we should have a sequence $\{\alpha_n\}$ in $W - S_\beta$, having a limit point $\alpha_0 > \beta$, with the contradictory conclusion that (α_0, p) is not interior to U. Therefore the ϵ_α are bounded away from 0, and (**) follows.

Now since I is compact, it follows that every finite open covering of X has a refinement of the following form. For some covering $\{N_1, \cdots, N_k\}$ of I

(which we may suppose 1-dimensional and consisting of intervals), for some $\alpha \in W$, we have the trace on X of $\{(W-S_\alpha)\times N_i\}$; and on $X\cap[S_\alpha\times I]$, we have a finite covering by disjoint open-closed sets.

Since these coverings are normal, X is a normal space. We have $\dim X \leqq 1$; and since the boundary of $(W-S_\alpha)\times N_i$ is contained in the topological sum of two copies of W, Ind $X \leqq 1$. However, $\dim X$ is not 0. If it were, the open covering consisting of the complements of the two sets $W\times\{0\}, W\times\{1\}$, would have a finite zero-dimensional refinement, and thus a refinement of the form described above in which $\{N_1,\cdots,N_k\}$ would not contain a chain joining 0 to 1, which is absurd. By VI.9, Ind$X$ is not 0, and we have proved all the assertions about X.

To construct Y we add a point ω_1, and define the sets far from ω_1 to be the sets whose projection in W is bounded. Normality of Y follows from normality of Y minus a point (with regularity of Y, which is evident). Moreover, given two disjoint closed sets in Y, one of them is bounded; hence $\operatorname{Ind} Y=0$. With VI.9 and VI.3, $0=\dim Y=\operatorname{mindim} Y=\operatorname{ind} Y$. The same goes for βY.

To construct Z we add a point $(\omega_1, 0)$; a basis of neighborhoods for it is $(\omega_1,0)\cup[(W-S_\alpha)\times N]$, for all α in W and all neighborhoods N of 0 in I. Z, like Y, is normal. We have ind $Z\neq 0$, since no open-closed set contains $(\omega_1,0)$ and is contained in $Y-[W\times\{1\}]$. Reversing the argument of the preceding paragraph, all dimension numbers of Z are nonzero. But $\operatorname{Ind} Z$ and $\dim Z$ are 1, just as for X; hence so are the smaller numbers. Then $\operatorname{Ind}\beta X$ and $\dim\beta Z$ are 1. Finally, $\operatorname{min dim}\beta Z=\dim\beta Z$ and $\operatorname{ind} Z\leqq\operatorname{ind}\beta Z\leqq\operatorname{Ind}\beta Z$.

This example (X) has a name; it is *Dowker's Example* M. A number of observations have been published on the properties and construction of Example M (see Notes, p. 121). Here we note the corollary:

14. *There is a compact space having an open subspace of higher covering dimension. That is, there is a locally compact space A with* $\dim A > \operatorname{min dim} A$.

Proof. Let X be the space of Example M, and let p: $X\to W$ be the first coordinate projection. Form $\beta p:\beta X\to\beta W$, and let A be $(\beta p)^{-1}(W)$. Since $X\subset A\subset\beta X, \beta A=\beta X$ and $\dim A=\dim X=1$. But every point of A is in the closure in βX of an open-closed subset of X of covering dimension 0. Thus ind $A=0$, and mindim $A=0$.

The space A is normal; this is left as an exercise.

15. *For every locally compact space A,* mindim A *is the supremum* $rd\,A$ *of the covering dimensions of compact subspaces.*

Proof. The one-point compactification B of A has dim $B=rd\,A$ by V.6;

every compactification of A has dimension at least rdA by IV.23.

16. EXAMPLE. *There is a normal space X with* $\operatorname{ind} X = 1$ *such that for every compactification Y of X,* $\operatorname{ind} Y \geqq 2$.

The example has also Ind $X = 2$, dim $X = \operatorname{mindim} X = 1$. This will give us

17. COROLLARY. *The four functions* dim, mindim, ind, Ind *are all different for normal spaces.*

PROOF. After Example M it remains only to distinguish Ind from dim—X does this—and ind from mindim. βX is a compactification of X, so $\operatorname{ind} \beta X \geqq 2 > \dim X = \operatorname{min\,dim} \beta X$.

The idea for Example 16 is, first, that when $\operatorname{Ind} X \geqq \operatorname{ind} X$ we may be able to assure that for any compactification Y, $\operatorname{ind} Y \geqq \operatorname{Ind} X$. Certainly this cannot be done with $\operatorname{ind} X = 0$, because of VI.2, VI.3, and VI.9. The trouble is that two completely separated sets A_1, A_2 in X which are hard to separate may not be completely separated in Y. So we may think of constructing a sequence of such sets A_n converging to a point of X. If the difficulty of separating the A_n from each other is localized near single points p_n of βA_n (as VI.11 suggests), and if the p_n converge to a point of X, then we shall have it; for in a compactification the p_n cannot all be identified to a single point. It is the limit of $\{p_n\}$ that enforces ind $X > 0$.

The details are based on Example M. For the localization in βA_n, we have

18. *The only compactification of the space W of countable ordinals is the one-point compactification.*

PROOF. If W had a compactification adding more than one point, W would contain disjoint closed noncompact sets; but VI.13(*) shows that it does not.

Now in Example M the design is right but the dimension is wrong. To raise it we use the long line L (Exercise V.7); the lexicographically ordered product $W \times [0, 1)$. We regard W as embedded in L, α corresponding to $(\alpha, 0)$; note that this is a homeomorphic embedding. As before, index $\aleph_1$ disjoint dense sets Q_α in I. Define the subspace S of the topological product $L \times I$ to consist of $W \times I$ and those points $((\alpha, x), p)$ for which $x \neq 0$, $p \in (0, 1)$, and p does not belong to any $Q_\beta, \beta \geqq \alpha$.

Let L_α denote the set of all $(\beta, x) \leqq (\alpha, 0)$ in L. Let $S_\alpha = S \cap [L_\alpha \times I]$. Then L_α is homeomorphic with I. Also (i) dim $S_\alpha \leqq 1$ and (ii) ind $S_\alpha \leqq 1$. For (i), it is easiest to apply V.7 and VI.12; the compact set $[W \times I] \cap S_\alpha$ has covering dimension 1, and the rest of the space lies in $L_\alpha \times (I - Q_\alpha)$. For (ii), every point has a basis of rectangular neighborhoods whose top and bottom are subspaces of W and whose ends are subspaces of $I - Q_\alpha$ for some α. It follows that ind $S \leqq 1$. On the other hand, S contains intervals and is not zero-dimensional in any sense.

(**) If $\mathscr{U}$ is a finite open covering of S and p is a point of I, then for some $\alpha \in W$ and some neighborhood N of p in I, one element of U contains $S \cap [(L - L_\alpha) \times N]$.

PROOF. As in VI.13, one open set $U \in \mathscr{U}$ contains all of $W \times \{p\}$ except an initial segment. We shall want the specific conclusion (***) that this implies (**) for the set U. For each $\alpha \in W$, define $\epsilon_\alpha \geqq 0$ as the infimum of those t such that $S - U$ contains a point $((\alpha, x), p \pm t)$. After the excepted initial segment of $W \times \{p\}$, ϵ_α cannot take the value 0 infinitely often nor can it take a sequence of values converging to 0; either way, this would give us a sequence of points of $S - U$ converging to a point of U. This proves (***) and (**).

Just as in VI.13, it follows that S is normal and $\dim S \leqq 1$.

For sets H in S, let us abbreviate the expression "$H - S_\alpha$, for some α" to almost all of H". Next suppose A contains almost all of $W \times \{0\}$, and B almost all of $W \times \{1\}$. We shall show that a closed set C separating A and B must contain an interval. Certainly it does unless it is nowhere dense; so we may assume $S - C = U \cup V$, where U and V are disjoint open sets with dense union, $U \supset A$, $V \supset B$, $U^- \cap V^- = C$. Let U_0 (resp. V_0) be the set of all $p \in I$ such that U (resp. V) contains almost all of $W \times \{p\}$. By (***), $p \in U_0$ implies U contains almost all of $L \times \{p\}$, and similarly for V. Consequently if U_0 and V_0 have a common limit point $q \in I$, C contains almost all of $L \times \{q\}$(since q would be the limit of countable subsets of U_0 and V_0). In the remaining case there is an interval $J \subset I - U_0 - V_0$; for each t in J, neither $U^- \cap [W \times \{t\}]$ nor $V^- \cap [W \times \{t\}]$ is bounded. Then, since U is open, for every $\alpha \in W$ there exist $(\beta, x) > \alpha$ in L and two rational numbers $m < n$ in J such that U contains $S \cap [\{(\beta, x)\} \times [m, n]]$. Since there are only countably many rationals, this remains true for some fixed m and n; the set U^0 of all (β, x) such that U contains the dense part of $\{(\beta, x)\} \times [m, n]$ that lies in S is unbounded in L. But the same argument applies to V and $J' = [m, n] \subset J$ giving an unbounded set V^0 and an interval $J'' \subset J'$. Taking $l_1 < l_2 < \cdots$ alternately from U^0 and V^0, we get a common limit point λ in W. Then C contains the interval $\{\lambda\} \times J''$.

To construct the final space X, consider $L \times [0, \infty)$. Define S_n as the set of all points $(l, n+p)$, where $(l, p) \in S$. X consists of the union of all S_n, $n = 0, 1, \cdots$, plus a point ∞; a set is far from ∞ if and only if it is contained in a finite union of S_n's.

Given two disjoint closed sets in X, one of them is contained in a finite union of S_n's; so X is normal. Since ∞ has a basis of neighborhoods with boundaries homeomorphic to W, $\operatorname{ind} X = 1$. It is clear that a finite union of S_n's has covering dimension 1; so by V.7, $\dim X = 1$. $\operatorname{Ind} X$ is at least 2 because of what we just proved about S; and since it is not important for us that $\operatorname{Ind} X$ is just 2, we leave it as a (tedious) exercise.

Let A_n be the set $W \times \{n\}$ in X. Let Y be any compactification of X.

By VI.18, A_n^- is the union of A_n with a point p_n. Since every neighborhood of ∞ contains almost all A_n, the sequence $\{p_n\}$ converges to ∞. Since $p_n \neq \infty$ for all n, $p_n \neq p_{n+1}$ for some n. Given any basis of neighborhoods of p_n in Y with boundaries B_α, their intersections with S_n form a family of relatively open sets with boundaries $C_\alpha \subset B_\alpha$; and some C_α separate almost all of A_n from almost all of A_{n+1} in S_n. Hence ind $C_\alpha > 0$ for some α, and ind $Y \geqq 2$.

Metric case.

19. THEOREM. *Any basis for a uniformity on a separable metrizable space contains a basis for a metric uniformity.*

PROOF. Let $\mathscr{B}$ be a basis for a uniformity μ on X and let $\{U_i\}$ be a countable basis of open sets. Enumerate in some order all those ordered pairs $p_n = (U_i, U_j)$ such that U_j is a uniform neighborhood of U_i. Select coverings $\mathscr{V}_n$ from $\mathscr{B}$ recursively so that $\mathscr{V}_{n+1}$ is a star-refinement of $\mathscr{V}_n$ and so that in $p_n = (U_i, U_j)$, U_j contains $\mathrm{St}(U_i, \mathscr{V}_n)$. Then the coverings $\mathscr{V}_n$ form a basis for a metric uniformity inducing the given topology.

20. COROLLARY. *If X is separable metrizable and* min dim$X = n$ *then X can be embedded in a compact metric space Y with* dim$Y = n$. *Hence X can be embedded in E^{2n+1}.*

21. COROLLARY. *A topological space X is homeomorphic with a closed subset of a Euclidean space if and only if X is separable metrizable and locally compact and* dim X *is finite*; *in case* mindim $X = n$, *X is homeomorphic with a closed subset of E^{2n+1}.*

PROOF. Necessity of the conditions is clear if we recall (VI.6) that for normally embedded subspaces dim is monotonic. Supposing X satisfies these conditions, the one-point compactification Y of X is compact, has a countable basis, and has dim $Y = n$ (V.6). Then Y can be embedded in E^{2n+1}. Now observe that the one-point compactification of E^{2n+1} is the sphere S^{2n+1}; in particular, it is homogeneous. Hence if we add to E^{2n+1} a point at infinity and remove the point corresponding to $Y - X$, we have another copy of E^{2n+1} containing X as a closed set.

This gives us the interesting corollary that for these spaces, $rdX \leqq n$ implies dim$X \leqq 2n+1$. Perhaps we should pause to prove the theorem mindim$=$dim. Recall that a *σ-compact* space (also called "denumerable at infinity") is a space which is a countable union of compact sets. It is then an expanding union of compact sets $K_1 \subset K_1 \cup K_2 \subset \cdots$; if it is locally compact too, then every compact set has a compact neighborhood and the space is a union of compact sets K_n with each K_{n+1} a neighborhood of K_n. This makes it obvious that the space is normal.

22. LEMMA. *For a locally compact σ-compact space X,* $\dim X = rdX = \operatorname{mindim} X$.

PROOF. It suffices to prove that the uniform dimension of X in the fine uniformity is at most n, assuming this is true for all compact subspaces. Let $X = \bigcup K_m$, with each K_{m+1} a compact neighborhood of K_m. Let A be a subspace of X and $f: A \to S^n$ a uniformly continuous function. Then f has a continuous extension f_1 over the closure A_1 of A. Since $\dim K_2 \leqq n$, $f_1 | A_1 \cap K_2$ has a continuous extension f^2 over K_2. Recursively, having f^m: $K_m \to S^n$, agreeing with f_l on $A_1 \cap K_m$, define f_m: $A_1 \cup K_m \to S^n$ to agree with f_1 on A_1 and with f^m on K_m. Since A_1 and K_m are closed, f_m is continuous. Then $f_m | K_m \cup (A_1 \cap K_{m+1})$ has a continuous extension f^{m+1} over K_{m+1}, and the recursion runs. It defines a function $g: X \to S^n$ which is locally continuous, hence continuous, and extends f.

REMARKS. (1) Of course Lemma 22 can be proved directly by cutting and fitting a finite open covering. (2) It is true without the assumption of local compactness. (VIII.7) (3) We should mention that rdX need not approach the genuine dimension numbers unless some compactness condition is imposed. (Exercise 6.)

A space has the *Lindelöf property* (or is a *Lindelöf space*) if every open covering has a countable subcovering.

23. THEOREM. *If X has the Lindelöf property then* $\operatorname{mindim} X = \dim X$.

PROOF. Assume $\operatorname{mindim} X = n$; so X is contained in a compact space Y with $\dim Y = n$. We shall show that every finite open covering $\{U_i\}$ of X has an at most n-dimensional normal refinement; this will prove the theorem besides proving X is a normal space. Each U_i is $V_i \cap X$ for some open set V_i of Y. The complement of the union of the V_i's is a closed set K. Each point x of X has a neighborhood N_x in Y completely separated from K; moreover, using a continuous function f_x that is 1 at x and 0 on K, we may suppose N_x is open and σ-compact. Since X has the Lindelöf property it is contained in a countable union Z of the sets N_x. Then Z is locally compact and σ-compact. Since every compact subset of Z, being a closed subset of Y, has covering dimension at most n, so does Z (Lemma 22). Since Z is normal and contained in $Y - K$, the sets $V_i \cap Z$ cover it and there is an n-dimensional refinement $\{W_j\}$. Since $\{W_j\}$ is a normal covering of Z, $\{W_j \cap X\}$ is a normal refinement of $\{U_i\}$ of dimension at most n.

No substantial extension of Theorem 23 beyond Lindelöf spaces is known. (See, however, Exercise 5.) It does not extend to arbitrary metrizable spaces, by a recent example of Roy [1].

The coincidence theorems tying in Ind and ind for separable metrizable spaces, where all four dimension functions coincide, involve some other ideas; see VIII.15 and VIII.16.

For an n-dimensional separable metrizable space X, we know (Exercise V.6) that any continuous mapping into a Euclidean space can be approximated by mappings into n-dimensional subspaces; and we may suspect (from IV.16) that any mapping into E^{2n+1} can be approximated by embeddings. The next theorem combines these approximations.

24. Lemma. *Suppose given a metrizable space X, a metric uniformity μ on X, and a basis $\mathscr{B}$ for a uniformity finer than μ. Then $\mathscr{B}$ contains a basis for a metric uniformity finer than μ.*

Proof. If $\{\mathscr{U}_n\}$ is a basis for μ, we need only select coverings $\mathscr{V}_n$ from B such that $\mathscr{V}_{n+1} <^* \mathscr{V}_n < \mathscr{U}_n$.

25. Theorem. *If X is separable metrizable,* $\dim X = n$, *and* $m \geqq 2n+1$, *then every continuous mapping of X into E^m can be approximated uniformly by embeddings into n-dimensional closed sets.*

Proof. Given the continuous mapping $f: X \to E^m$, we shall first construct a locally compact σ-compact metric space Z containing X, over which f can be continuously extended so that the inverse image of any compact subset of E^m is compact n-dimensional. Select a basis $\mathscr{B}$ for the Čech uniformity of X (the finest precompact uniformity) consisting of n-dimensional coverings; select a precompact metric uniformity ν on X. Let X_r denote the subspace $\{x \in X: d(f(x), 0) \leqq r\}$. Let e_r denote the one-to-one function from X_r to νX defined $e_r(x) = x$; let f_r denote $f \mid X_r$, considered as a mapping into a uniform space (e.g. into the compact r-neighborhood of 0, with its unique uniformity). The weak preuniformity induced by e_r and f_r is a compatible uniformity, since both functions are continuous and e_r is a topological embedding. It is a precompact metric uniformity μ_r; so Lemma 24 applies. In particular, X_2 has an n-dimensional metric compactification C_2 over which f extends continuously. Let D_2 be the subspace of C_2 consisting of X_2 and the closure of X_1. A topological space Y_2 is defined as follows: it is the union of D_2 and X_3 (which have intersection X_2), with a set defined to be closed if its intersections with D_2 and with X_3 are closed. Evidently Y_2 is regular and has a countable basis of open sets; i.e., Y_2 is separable metrizable, and then $\dim Y_2 \leqq n$ in view of VI.12. Since f_2 extends continuously over C_2, f_3 extends continuously over Y_2. Using any precompact metric uniformity ν on Y_2, we get as before an n-dimensional metric compactification C_3 over which f_3 extends. To continue the recursion, define D_{k+1}, Y_{k+1}, C_{k+2}, in the same manner as D_2, Y_2, C_3. It gives us the required space $Z = \bigcup C_k$: each C_{k+1} is a neighborhood of C_k, and dim $Z=n$ (by Theorem 23). Let f denote now the (unique) extended mapping of Z into E^m.

To conclude, we need only approximate f by embeddings of Z. The image of Z under any mapping within finite distance of f is closed, since the inverse images of compact sets are compact. We use the sets $Z_r=\{z\in Z: d(f(z), 0)\leqq r\}$. A mapping within finite distance of f is an embedding if its restriction to each Z_r is an embedding. Moreover, we can get the required mapping g as a limit of mappings g_r if on Z_r the sequence $\{g_r, g_{r+1}, \cdots\}$ is a very rapidly convergent sequence of embeddings (IV.38). For each r (recursively) we get a suitable embedding e_r: $Z_{r+1}\to E^m$ from IV.16. We define g_r to coincide with e_r and Z_r and with f outside Z_{r+1}; if $d(f(z),0)=r+t$ for some t in $(0,1]$, we define $g_r(z)=tf(z)+(1-t)e_r(z)$.

Freudenthal compactification. This concluding portion of the chapter presents a specialized theory that has grown out of a question which we may state roughly, so: Is there a natural way to compactify an open manifold? Had the development of the theory stayed very close to manifold theory, it would not appear here. Perhaps the best way to introduce what does appear here is with the following question. *What spaces X have compactifications Y such that the remainder $Y-X$ is zero-dimensional?* Obviously the open manifolds, and all other locally compact spaces, are included. In effect we expect to get naturality by working for generality.

26. For any compactification Y of X, with the embedding e: $X\to Y$, we have βe: $\beta X\to Y$. Y is determined as a quotient space of βX by an upper semi-continuous decomposition, in which the single points of X are elements. (For if $\beta e(x)=\beta e(z)$, $x\in X$, $z\neq x$, there would be a closed set H of X far from x but not from z; then $\beta e(H)=e(H)$ would have x as a limit point, so that e is no embedding.) Hence the decomposition is determined by its restriction to $X^*=\beta X-X$. It is an upper semi-continuous decomposition of X^* into compact sets, but not an arbitrary one. However, the restrictions imposed on it depend only on X^{*-} in βX. (For a decomposition is upper semi-continuous if it is upper semi-continuous on some closed set containing all its nondegenerate elements.) We write $X^{*-}=K(X), K(X)\cap X=R(X)$. $R(X)$ is the set of all points of X at which X is not locally compact.

27. If S is a subset of a space X, a *neighborhood* of S is a set whose interior contains S. A *basis* at S is a cofinal subset of the family of all neighborhoods of S ordered by inclusion. Suppose X is a dense subspace of Y. Then X-neighborhoods U and Y-neighborhoods V correspond (roughly) by $U\to U^-$, $V\to V\cap X$. In case S is compact, *this correspondence takes bases to bases.*

Proof. The point is that a compact set has a basis of closed neighborhoods. If V is a closed neighborhood of S in Y, $V \cap X \supset U$ implies $V \supset U^-$. If U is a closed X-neighborhood, $U^- \supset V$ implies $U \supset V \cap X$.

When $S \subset X \subset Y$, X not dense in Y, the correspondence between Y-neighborhoods and X-neighborhoods is many-one, and worst, boundaries cannot be controlled. The following lemma is the reason for a critical turn in the treatment of our topic question, which we shall take after Proposition 33.

28. **Lemma.** *Suppose X is a closed subspace of a normal space Y and U is an open subset of X whose X-boundary is contained in $B \cap X$ for some closed G_δ set B of Y. Then there exist open sets V of Y such that $V \cap X = U$ and the boundary of V meets X in $B \cap X$; in fact, every Y-neighborhood of U^- contains such a set V.*

Proof. B is $f^{-1}(0)$ for some continuous non-negative real-valued function f on Y. (If B is the intersection of open sets W_n, let $f = \sum 2^{-n} f_n$. where f_n is 0 on B and 1 outside W_n.) Define $g: X \to E^1$ by $g(x) = -f(x)$ for $x \in U, g(x) = f(x)$ otherwise. Then g is continuous and has a continuous extension h over Y. Moreover, if N is a neighborhood of U^-, we can make sure h is positive outside N by adding a suitable function vanishing on U^-. Then the set where h is negative is the required set V.

In connection with G_δ sets we shall want two more results from general topology. (1) A completely regular space has a basis of open F_σ sets; in fact, any two completely separated sets have disjoint open F_σ neighborhoods. (Moreover, we have the proof in the proof of Theorem 23.) (2) In a compact space, a closed G_δ has a countable neighborhood base.

Proof. It is the intersection of a shrinking sequence $\{U_n\}$ of open sets with $U_{n+1}^- \subset U_n$; any closed set disjoint from it is disjoint from some U_n^-, so the U_n's form a base.

Returning to compactifications $e: X \to Y$ and zero-dimensional remainders $Y - X$: Since Y provides a uniformity, there are seven dimension functions available. However, for dimension zero they reduce to three, δd, dim, and ind (by V.1, V.26, VI.3, and VI.9). The problem on $\delta d(Y - X) = 0$ is the only one of the three that is completely solved.

29. *X has a compactification Y in which $\delta d(Y - X) = 0$ if and only if X has a basis of open sets whose boundaries have compact neighborhoods.*

Proof. $\delta d(Y - X) = 0$ if and only if the compact space $K(X)$ is zero-dimensional. If it is, the mapping $\beta e | K(X)$ factors across the zero-dimensional reflection q: $K(X) \to QK(X)$. In any case q determines a decomposition of βX (cf. VI.26), with quotient space Y_0. The question is whether Y_0

is a compactification of X, i.e., whether every single point of $R(X)$ is a component of $K(X)$. This means points of $R(X)$ have neighborhood bases in $K(X)$ consisting of open-closed sets; equivalently (Lemma 28), they have neighborhood bases in βX whose boundaries are contained in $X-R(X)$. VI.27 translates the question into X. Now the closed sets of X that have compact neighborhoods are just the compact subsets of $X-R(X)$. Then the proof is complete when we note that points of $X-R(X)$ always have neighborhoods of the required sort.

The topological dimensions dim, ind are harder to manage. The reflection device does not quite work for $\mathrm{ind}(Y-X)$, and for $\dim(Y-X)$ there is no such device to try. We can conclude that if $\mathrm{ind}(Y-X)=0$ for some compactification Y of X, then the restriction $\beta e|X^*=\beta^*e$: $X^*\to Y-X$ factors across q: $X^*\to QX^*$. It follows easily that the points of X and the quasi-components of X^* constitute an upper semi-continuous decomposition of βX. It is not known whether this necessary condition is sufficient for the existence of such a compactification of X.

We reformulate the problem to make it easier. What if not only $\mathrm{ind}(Y-X)=0$, but every point of $Y-X$ has a basis of neighborhoods in Y whose boundaries are contained in X? (Cf. Exercise 7.)

A space is called *rim-compact* (also *locally peripherally compact* and *semi-compact*) if it has a basis of open sets with compact boundaries.

30. THEOREM. *Every rim-compact space X has a compactification that has a basis of open sets whose boundaries are contained in X.*

PROOF. We define a proximity relation δ on X: $A\delta B$ unless the closures of A and B are separated by some compact set. For the axioms: (i) If A is not near B or C, there are compact sets H, K separating A^- from B^- and C^-, respectively. Then $X-H$ contains a relatively open-closed set U containing A^- and disjoint from B^-, and $X-K$ contains such a set $V\supset A^-$, disjoint from C^-. $U\cap V$ is an open set containing A^- whose boundary is a compact set separating A^- from $(B\cup C)^-$. Axioms (ii), (iii) are obvious. (iv) If A is not near B, then some compact set K separates X into open sets $U\supset A^-$, $V\supset B^-$. X is the union of $U\cup K$ and $K\cup V$. A is not near $K\cup V$, because K has a finite covering by open sets W_i with compact boundaries, and the union of K, V, and all W_i has a compact boundary L separating $K\cup V$ from A^-. Similarly B is not near $U\cup K$. Thus δ is a proximity; and rim-compactness of X says precisely that a point is near a set S if and only if it is in the closure of S. So δ determines a precompact uniformity μ and a completion $\beta(\mu X)$; call it Y. Now in Y, two sets are far if and only if they have neighborhoods whose intersections with X are far. In particular, if a compact set K in X separates X into open sets U, V, then each point p of $U^- - K$ in Y is far from V; so K separates Y into $U^- - K$, $V^- - K$. It follows that every point of Y has a basis of neighborhoods whose boundaries lie in X, as was to be shown.

31. COROLLARY. *Every rim-compact space X has a compactification Y with* $\operatorname{ind}(Y-X)=0$.

The compactification constructed in Theorem 30 is called the *Freudenthal compactification FX* of the rim-compact space X. Generally, a subset Z of a space Y is said to be *zero-dimensionally embedded* if Y has a basis of open sets whose boundaries are disjoint from Z. We note

32. *A compactification Y of X has zero-dimensionally embedded remainder if and only if the uniformity induced on X as a subspace of Y has a basis of finite open coverings by sets with compact boundaries. Y is the Freudenthal compactification if every such covering of X is uniform (and not otherwise).*

PROOF. $Y-X$ is zero-dimensionally embedded if and only if Y has a basis of finite open coverings by sets whose compact boundaries are contained in X, and this is clearly equivalent to the condition asserted. In proving Theorem 30(iv) we showed that two closed sets in X separated by a compact set are far if every finite covering by open sets with compact boundaries is uniform. The converse is clear since every open covering of FX is uniform.

33. $\dim Z=0$ *if and only if Z is zero-dimensionally embedded in every compactification.*

PROOF. If U is an open set of βZ whose boundary is disjoint from Z, then $U \cap Z$ is closed. If βZ has a basis of open sets of this sort, then it also has a basis of finite open coverings consisting of these sets. Hence every finite normal covering of Z is refined by a finite covering consisting of open-closed sets. But every such covering has an open refinement of dimension zero.

Suppose $\dim Z=0$, and let Y be a compactification of Z. Points of Z of course have bases of neighborhoods that are closures of open-closed sets of Z, so have boundaries disjoint from Z. But for $p \in Y-Z$, $\dim(Z \cup \{p\})$ is still 0 (VI.10). Thus every point of Y has the required basis of neighborhoods.

REMARKS. (1) The condition for Z to be zero-dimensionally embedded in some compactification is $\operatorname{ind} Z=0$. (2) Proposition 33 does not extend to arbitrary spaces containing Z, because Z-neighborhoods of points of Z do not extend suitably; we must fall back on Lemma 28. The counterexample requires preparation, though. (VII.25) It will give us a space X that is not rim-compact but has a compactification Y with the remainder $Y-X$ zero-dimensional in the strongest topological sense: $\dim(Y-X)=0$.

The G_δ sets of VI.28 need not be preserved by compactification; but by

VI.27, if S is compact and has a countable basis of neighborhoods in X, then S is a closed G_δ in any compactification of X. A set having a countable neighborhood base is said to have *countable character*.

34. LEMMA. *Suppose every compact subset of $R(X)$ is contained in a compact set of countable character in X, and X has a compactification Y with* $\dim(Y-X)=0$. *Then $Y-X$ is zero-dimensionally embedded in Y;* so *X is rim-compact.*

PROOF. Let $K=(Y-X)^-$. Let p be a point of K, U a neighborhood of p in Y. $U^- \cap K$ is a K-neighborhood of p; it contains (Proposition 33) an open K-neighborhood U_0 whose boundary is a compact subset B_0 of $R(X)$. By hypothesis B_0 is contained in a compact set of countable character $B_1 \subset X$. Also (since Y is completely regular) there is an open F_σ set N containing p and disjoint from $K-U_0$. Then $B=B_1-N$ is a compact G_δ in Y whose intersection with K contains the boundary of U_0-B. By Lemma 28, every neighborhood of U^- contains a neighborhood V whose boundary meets K in $B \cap K$; in particular, the compact boundary of V lies in X. Since the points of $X-R(X)$ have compact neighborhoods, the proof is complete.

The largest familiar class of spaces satisfying the first condition of Lemma 34 is the class of metrizable spaces, in which every compact set has countable character.

PROOF. For a closed set H disjoint from a compact set S, the minimum on S of $d(x, H)$ exceeds $1/n$ for some n.

It is advantageous to introduce an intermediate class: those spaces X in which every compact set is contained in a compact set of countable character. These spaces are said to be *Lindelöf at infinity*, for the following reason.

35. *The following conditions on a space X are equivalent:*

(a) *X is Lindelöf at infinity.*

(b) *$X=Y-Z$ for some compact space Y and some subspace Z having the Lindelöf property.*

(c) *For every compactification Y of X, $Y-X$ has the Lindelöf property.*

PROOF. It suffices to prove the equivalence of (a) and (c) for each compactification Y of X. Now (a) means that closed sets contained in X lie in closed G_δ sets contained in X; that is, open sets containing $Y-X$ contain open F_σ sets containing $Y-X$. Clearly this implies $Y-X$ is Lindelöf. Conversely, if $Y-X$ is Lindelöf, consider any open set U containing it. Each point α of $Y-X$ has a $Y-X$ neighborhood N_α whose Y-closure is contained in U; moreover, we can find an open F_σ set containing α and contained in N_α^-. Some countable union of these contains $Y-X$, and it is an open F_σ set contained in U.

36. **Theorem.** *If X is Lindelöf at infinity, the following conditions are equivalent.*

(a) *X is rim-compact.*

(b) *X has a compactification Y with* $\dim(Y-X)=0$.

(c) *X has a compactification Y with* $\operatorname{ind}(Y-X)=0$.

Proof. Since $\operatorname{ind}(Y-X)=0$ means $\operatorname{mindim}(Y-X)=0$, Theorem 23 and Proposition 35 show that (b) and (c) are equivalent. Lemma 34 gives (b) $\Longrightarrow$ (a), and Corollary 31 gives (a) $\Longrightarrow$ (c).

37. **Corollary.** *A separable metrizable space X has a metric compactification Y with zero-dimensional remainder if and only if X is rim-compact.*

Proof. Necessity is immediate. Conversely, when X is rim-compact, a compactification Y has zero-dimensional remainder if and only if (VI.32, VI.34) X has a basis of uniform coverings (as a subspace of Y) that are finite coverings by open sets with compact boundaries. Since one such compactification exists, there exists one such basis $\mathscr{B}$ for a uniformity. By Theorem 19, $\mathscr{B}$ contains a basis for a metric uniformity. It is a precompact uniformity; so the completion is the desired metric compactification of X.

So far, the problem of zero-dimensional remainders is distinguished mostly by difficulties. And we should not leave the general problem without mentioning that it is still unknown whether every rim-compact space X has a compactification Y with $\dim(Y-X)=0$. However, with this exception, the Freudenthal compactification is a successful one.

Recall the property of Stone-Čech compactification from VI.7: if a closed set C separates A and B in X, then C^- separates A and B in βX. In general, a compactification Y of X for which this holds is called *perfect*.

38. *The following conditions on a compactification $e: X \to Y$ of a space X are equivalent.*

(a) *Y is a perfect compactification.*

(b) *If $p \in Y-X$, then for any neighborhood U of p, $U \cap X$ is not the union of two disjoint relatively open sets both having p as a limit point.*

(c) *Under $\beta e: \beta X \to Y$, the inverse images of points are connected sets.*

Proof. (a) $\Longrightarrow$ (b). Suppose (b) violated by $U \cap X = V \cup W$. The disjoint sets V, W are separated by $C = X - V - W$; but in Y, C^- does not contain p and cannot separate V from W. (b) $\Longrightarrow$ (c). If $(\beta e)^{-1}(p)$ were disconnected, it would be a union of disjoint closed sets H, K, having disjoint neighborhoods V, W. $\beta e(V \cup W)$ would be a neighborhood of p (since βe is a closed mapping) whose intersection with X splits into $V\cap X$, $W\cap X$, contrary to (b). Finally, (c) $\Longrightarrow$ (a). Suppose C separates A from B in X; by VI.7, the closure C^β in βX separates βX into open sets $U\supset A$, $V\supset B$. If C^- is the closure in Y, then $C^* = (\beta e)^{-1}(C^-)$ contains C^β, so separates $U - C^*$ from $V - C^*$. The sets $\beta e(U - C^*)$, $\beta e(V - C^*)$ are disjoint since the inverse image of a point is connected. Hence both $U - C^*$ and $V - C^*$ are complete inverse sets. Since βe is a quotient mapping, $\beta e(U - C^*)$ and $\beta e(V - C^*)$ are open; so C^- separates A from B.

39. *The Freudenthal compactification of a rim-compact space is perfect.*

PROOF. We use (c) of Proposition 38. Consider $\beta e : \beta X \to FX$. Suppose $(\beta e)^{-1}(p)$ is disconnected for some p in FX. Then $(\beta e)^{-1}(p)$ is the union of two closed sets H, K with disjoint neighborhoods V, W. For any sufficiently small neighborhood U of p, $(\beta e)^{-1}(U) \subset V \cup W$. This gives us a separation $(\beta e)^{-1}(U) = V_1 \cup W_1$. By the proof of Theorem 30 we may suppose the boundary of U is a compact subset B of X. Every boundary point of V_1 or of W_1 is in $(\beta e)^{-1}(B) = B$; so B separates V_1 from W_1. Taking smaller closed neighborhoods V_2, W_2 of H and K, we have $V_2 \cap X$ and $W_2 \cap X$ closed and separated by a compact set; therefore $V_2 \cap X$ and $W_2 \cap X$ are far in FX. The contradiction completes the proof.

It follows that $\beta^* e : X^* \to FX - X$ is just the reflection mapping $q: X^* \to QX^*$; for the inverse images of points are connected, and the image satisfies $\operatorname{ind}(FX - X) = 0$. In fact, this shows that the space X^* (the Stone-Čech remainder of a rim-compact space) has the property that quasi-components are components.

The Freudenthal compactification of a metrizable space of course need not be metric; in fact, if the Stone-Čech compactification is zero-dimensional, that is the only perfect compactification. The Freudenthal compactification is a quotient space of every perfect compactification (by the preceding paragraph); it follows that FX is metric if any perfect compactification is. This depends on a well-known result from general topology which we need to amplify.

40. LEMMA. *A continuous image of a separable metrizable space has the Lindelöf property; and if it is compact Hausdorff, it is metric.*

PROOF. If $f: X \to Y$ is continuous onto and X has the Lindelöf property, then for any open covering $\mathscr{V}$ of Y, $f^{-1}(\mathscr{V})$ has a countable subcovering; hence so does $\mathscr{V}$. Then $Y \times Y$ also is an image of $X \times X$, by $f^2(x_1, x_2) = (f(x_1), f(x_2))$; moreover, $Y \times Y$ minus the diagonal is the image of its inverse image in $X \times X$. If Y is compact (Hausdorff), this implies the diagonal in $Y \times Y$ has countable character (since the complement has an open covering by F_σ sets of $Y\times Y$). Therefore the unique uniformity of Y has a countable basis of entourages (in view of Exercise II.14) and is metric.

41. LEMMA. *In a separable metrizable space, every basis of open sets contains a countable basis.*

This is the topological counterpart to Theorem 19 and is proved in the same way.

42. THEOREM. *The Freudenthal compactification FX of a rim-compact space X is metric if and only if X is separable metrizable and the quasi-component space QX is compact.*

PROOF. If any compactification of X is metric, X is separable metrizable. Suppose FX is metric; then so is its quotient space QFX. Reflect $e: X\to FX$ to $f: QX\to QFX$, an embedding. If $f(QX)\neq QFX$, then there is a sequence of distinct quasi-components K_n of X converging to a point p of $FX-X$. The union H of the odd-numbered K_n and the union J of the even-numbered ones are closed sets. By Lemma 40, QX has the Lindelöf property; so $\operatorname{Ind} QX=0$. Hence H and J are separated in X by the empty set. Since FX is a perfect compactification, H and J must be separated by the empty set in FX, a contradiction.

For the converse, we want

(*) If QX is compact and U is an open subset of X with compact boundary B, then $Q(X-U)$ is compact.

PROOF. A quasi-component of $X-U$ that does not meet B is a quasi-component of X. Hence any ultrafilter of quasi-components of $X-U$ either is finally an ultrafilter of quasi-components of X or has a cluster point in B. In either case it is convergent.

Then suppose X is rim-compact separable metrizable and QX is compact. Let $\mathscr{B}$ be a countable basis of open sets of X with compact boundaries (Lemma 41). We wish to exhibit a countable basis for the uniformity μ induced on X by embedding in FX, which is described in VI.32. We describe a countable family of coverings as follows. Let C, D be finite unions of elements of $\mathscr{B}$ with $C^-\subset D=U_1\cup\cdots\cup U_n$. The zero-dimensional finite open coverings of $X-C$ have a countable basis $\mathscr{C}$ by (*) and Lemma 40; let $\{V_1,\cdots,V_m\}$ be a member of $\mathscr{C}$. Then $\{U_1,\cdots,U_n,V_1,\cdots,V_m\}\in\mu$.

Let $\{W_1,\cdots,W_s\}$ be any finite open covering of X by sets with compact boundaries. The union of the boundaries B is compact; on some neighborhood of it we can refine the given covering by a finite family $\{U_1,\cdots,U_n\}\subset\mathscr{B}$. Another finite subset of $\mathscr{B}$ has union $C\supset B$ with C^- covered by $U_1,\cdots,U_n$; and $\{W_i\}$ is refined by one of the coverings described above. This completes the proof.

Exercises.

1. DIMENSION 1. For compact spaces, there is just one covering dimension, i.e., mindim coincides with dim. It is an unsolved problem whether the inductive dimensions ind and Ind coincide. One has trivially $\mathrm{Ind} \geqq \mathrm{ind}$. One can show

(a) If X is compact and $\mathrm{ind}\, X=1$ then $\mathrm{Ind}\, X=1$, using finite coverings of closed sets by open sets with zero-dimensional boundaries, and using the sum theorem V.8. The sum theorem for *covering* dimension is applicable because for compact spaces, covering dimension zero means inductive dimension zero.

Several somewhat different examples show that there is no sum theorem for inductive dimension in compact spaces. In order to use this to make an example for $\mathrm{Ind} > \mathrm{ind}$, one would have to carry out a very intricate condensation-of-singularities procedure. Perhaps it is not possible.

One way to construct $X = X_1 \cup X_2$ with $\mathrm{Ind} X_i = 1$, $\mathrm{ind} X = 2$, is as follows. Let C be the ternary Cantor set of all reals having representations $\sum a_n 3^{-n}$ $(n=1, 2, \cdots)$ with each $a_n = 0$ or 2. Let f: $C \to I$ take $\sum a_n 3^{-n}$ to $\sum a_n 2^{-n-1}$; f is onto, and it is one-to-one except for a countable dense set $S \subset I$, where f is two-to-one. Form X_1 from $L \times C$ (L the long line) by compactifying, adjoining $\{\omega_1\} \times C$, and decomposing $\{\omega_1\} \times C$ according to f so that it becomes an interval. Let X_2 be a homeomorphic copy of X_1.

(b) For any two countable dense sets S, T in I, neither containing 0 or 1, there is a homeomorphism $g: I \to I$ such that $g(S) = T$.

(c) Use (b) to identify the edges I in X_1 and in X_2 in such a way that the exceptional sets S do not match at any point. The result is a compact space $X = X_1 \cup X_2$ having the required properties.

2. TOPOLOGICAL PRODUCT.

(a) For any spaces X, Y, $\mathrm{min\,dim}\, (X \times Y) \leqq \mathrm{min\,dim} X + \mathrm{min\,dim} Y$. (Cf. Exercise V.8)

(b) If a compact space X is the union of two closed sets A, B, of inductive dimension 1, with zero-dimensional intersection, then $\mathrm{Ind} X = 1$.

(c) For compact X, Y, of inductive dimension 1, $\mathrm{Ind}\, (X \times Y) \leqq 2$.

No example is known in which any of these dimension functions takes a larger value for a product than the sum of the values for the factors.

3. Reflection q: $X \to QX$ factors across a topological quotient space that is always Hausdorff but need not be regular.

4. The space A of Proposition 14 is normal. For this, let $q: X \to I$ be the second coordinate projection. Form $\beta q: \beta X \to \beta I$. Prove: If H and K are disjoint closed sets in A then for some α, $H - [S_\alpha \times I]$ and $K - [S_\alpha \times I]$ have completely separated images under βq.

5. A space is said to have the *star-finite property* if every open covering has an open star-finite refinement. Since a star-finite collection is a disjoint union

of countable collections, one can prove that mindim and dim coincide for these spaces in the same manner as for spaces with the Lindelöf property.

6. Rim-compactness is not enough to make rd a reasonable dimension function; for there is a connected rim-compact plane set X with $rd\,X=0$. The *Sierpiński curve* is the subset of $I\times I$ given by $[I\times C]\cup[C\times I]$, C the Cantor set.

(a) The Sierpiński curve is not separated by any countable set.

(b) An uncountable compact metric space has the power of the continuum. (No more, of course; but it contains a closed dense-in-itself set, and that contains a homeomorphic copy of C.)

(c) Since the Sierpiński curve contains the same number of (1) points, (2) compact connected sets, and (3) closed separating sets; and since all the sets (2) and (3) are the same size; therefore one can remove from it a set S which meets every set (2) and contains no set (3), avoiding enough separating sets to make certain that $\operatorname{ind} S=0$. Then the complement of S is the required example X.

7. We have not answered the question preceding Theorem 30: what if every point of $Y-X$ has a basis of neighborhoods in Y whose boundaries are contained in X? This does not imply that $Y-X$ is zero-dimensionally embedded in Y; and though Y is compact, X need not be rim-compact. For this, use the compactification βZ of Example M (Example 13). Let Y be the product of βZ with the space of all ordinals $\leqq\omega_2$; let $Y-X$ be the product of Example M by the point ω_2. This construction is more complicated than necessary, but the complication makes Y the Stone-Čech compactification of its subspace X.

8. A rim-compact space need not have a compactification which at the same time preserves dimension and introduces a zero-dimensional remainder. Let X be $I\times I$ minus the set of all points both of whose coordinates are irrational. Clearly that is a zero-dimensional set; so X is rim-compact.

(a) $\operatorname{mindim} X=1$.

(b) Two subsets of X are far in $I\times I$ if and only if their closures in X are separated by a compact set; so $I\times I$ is the Freudenthal compactification.

In view of Propositions 32 and 34, every compactification of X with zero-dimensional remainder is a quotient space of $I\times I$. But consider the closed subset A of X, $A=[I\times\{0,1\}]\cup[\{0,1\}\times I]$. The obvious mapping $f\colon A\to S^1$ cannot be continuously extended over $I\times I$. Therefore

(c) it cannot be continuously extended over any compactification of X with zero-dimensional remainder. Thus every such compactification has covering dimension at least 2.

9. Completely metrizable spaces. A metrizable space X is complete in some metric (i) if and only if it is a G_δ subset of (ii) some compactification, or (iii) some metric completion, or (iv) each space Y in which X is embedded

as a dense subspace. Trivial implications: (iv) $\Longrightarrow$ (ii), (iv) $\Longrightarrow$ (iii).

(a) (i) $\Longrightarrow$ (iv). Consider a uniform basis of open coverings of X, and the open subspaces of Y over which they can be extended.

(b) (iii) $\Longrightarrow$ (i). First, an open subset $M-A$ of a complete metric space (M, d) is complete in the metric $e(x, y)=d(x, y)\int_0^1 f(t)^{-1}dt$, where $f(t)=td(x, A)+(1-t)d(y, A)$. Then, an intersection of subspaces U_i of a space is an inverse limit of the U_i, thus a closed subspace of their product.

(c) (ii) $\Longrightarrow$ (i). This seems to require a recursive construction of a suitable normal sequence $\{\mathscr{V}^n\}$ related to a given metric basis $\mathscr{U}^n$ and a sequence of open sets $U_n \supset X$.

10. Properties at infinity. For any topological property $\mathfrak{P}$, a space X is said to have $\mathfrak{P}$ *at infinity* if X^* has the property $\mathfrak{P}$. This is of interest primarily when the remainders of X in various compactifications must either all have $\mathfrak{P}$ or all lack $\mathfrak{P}$—as with the Lindelöf property. Some further properties for which this is true are (b) compactness, (c) local compactness, (d) σ-compactness, (e) paracompactness, (f) the property of being Lindelöf at infinity.

(b) "Compact at infinity" means locally compact.

(c) X is locally compact at infinity if and only if $R(X)$ is compact.

(f) X is Lindelöf at infinity, at infinity if and only $R(X)$ has the Lindelöf property.

(g) Generalize (c) and (f).

(a) Consider VI.26. A *proper mapping* is a closed continuous mapping $f: A \to B$ such that each inverse set $f^{-1}(b)$, $b \in B$, is compact. A proper mapping onto is called a *fitting mapping*. A topological property $\mathfrak{P}$ is called a *fitting property* if when $f: A \to B$ is fitting, both or neither of A and B must have $\mathfrak{P}$. Show that if $\mathfrak{P}$ is a fitting property and X has a compactification in which the remainder has $\mathfrak{P}$, then the remainder in any compactification of X has $\mathfrak{P}$.

(d) Compactness, hence also σ-compactness, is a fitting property. A space that is σ-compact at infinity is called an *absolute* G_δ. (The objectionable term *topologically complete* is also in use, because of Exercise 9.)

(e) Paracompactness is a fitting property.

For properties involving metrics and dimension, it is not so easy. It is clear that none of those properties is fitting, perhaps excepting rim-compactness. For that, consider the coordinate projection $Q \times I \to Q$, where Q is the space of rational numbers.

Nevertheless the relations between fitting mappings and dimensional properties call for much more study. It is an open question whether every locally compact space admits a proper mapping into a 1-dimensional space (in the sense, say, of mindim), or even whether every uniformizable space admits such a mapping. The product Z of a long line with $[0, 1)$ may be a counterexample. If it is, one can construct a space Y having a point p such that mindim $(Y-\{p\})=\text{mindim}\,Y-2$. One need only build along the lines of Example M so that $\text{ind}(Y-\{p\})=0$ but Y^* contains Z and the closure in βY is $Z\cup\{p\}$. (Z, like I, is the union of an expanding family of $\aleph_1$ dense zero-dimensional subspaces. One would need the lemma that a continuous real-valued function on $W\times W$ is constant on a rectangle $\{(\alpha,\beta)\colon \alpha\geqq\alpha_0, \beta\geqq\beta_0\}$; VII.34 implies this.)

11. NATURAL COMPACTIFICATION.

(a) A continuous mapping $f\colon X\to Y$ is proper if and only if $\beta f(X^*)\subset Y^*$.

(b) If X and Y are rim-compact, every proper mapping $f\colon X\to Y$ has a continuous extension $Ff\colon FX\to FY$.

From Proposition 38(c), a space X can have at most one perfect compactification Y with $\text{ind}(Y-X)=0$; for $Y-X$ must be QX^*. Moreover,

(c) A space X can have at most one perfect compactification Y with $rd(Y-X)=0$.

A space Z is called *punctiform* if $rd\ Z=0$. A subset Z of a space Y is called *nowhere cutting* if $Y-Z$ is dense and no neighborhood of any point p of Z is a union of two disjoint relatively open sets of $Y-Z$ both having p as a limit point. (Cf. Proposition 38(b).)

(d) Suppose A and B are compact spaces having nowhere cutting punctiform subspaces C, D, respectively. Then any proper mapping $f\colon A-C\to B-D$ has a continuous extension $f^-\colon A\to B$. If $A-C$ and $B-D$ are homeomorphic, so are A and B.

The Freudenthal compactification is not indisputably the most natural compactification for open manifolds; for example, $S^1\times E^2$ has the perfect compactification $S^1\times S^2$, but the Freudenthal compactification is not a manifold. It is not likely that any general construction can give the compactification $S^1\times S^2$ in this case, since there exist nonhomeomorphic compact spaces A, B whose products with E^2 (even with E^1) are homeomorphic. (Bing [1]) From this point of view it seems natural to confine attention to perfect compactifications. One could regard FX as a mere preliminary construction and go on to study boundaries of neighborhoods of the points of $FX-X$ with a view to removing singularities.

12. COMPACT QUASI-COMPONENT SPACE. The first part of the proof of Theorem 42 proves more than is stated, in two ways. If X has a perfect metric compactification, then QX is compact and the reflection mapping $q\colon X\to QX$ is a quotient mapping.

13. A METRIZATION THEOREM.

(a) A compact space X is metric if it is a countable union of separable metrizable subspaces. (Use Lemma 40.)

(b) This generalizes from compact spaces to absolute G_δ's. For the diagonal in $X\times X$ is still a G_δ. Embed X in a compactification K; $X\times X$ becomes a G_δ in $K\times K$. Construct a preuniformity with countable basis on K that induces a compatible uniformity on X.

Research Problem D.

n-COMPACTIFICATION AND DEFECT. The *defect* of a space X relative to a dimension function d is the least n such that X has a compactification Y such that $d(Y-X)=n$. We have been studying the case of defect zero. Very little is known about higher defect. The concept has been introduced, and the problem stressed, by de Groot. The present remarks are based on Professor de Groot's 1960 lectures on the problem.

First, there is little point to going beyond separable metrizable spaces (where d is unique) until more is known in that context. It may be desirable to insist that the compactifications be metric too. In *that* context, one can prove the necessity of a condition for *defect* $X \leqq n$ which may be necessary and sufficient for arbitrary compactifications of separable metrizable spaces. The key lemma is that if Y is compact metric and Z is an n-dimensional subset, then any two far sets are separated by a closed set C such that $C \cap Z$ has dimension at most $n-1$. Then one defines cmp X inductively: cmp $X = -1$ if and only if X is compact; cmp $X \leqq n+1$ if X has a basis of open sets U_i whose boundaries B_i satisfy cmp $B_i \leqq n$.

The conjecture is cmp $X =$ *defect* X. The theorem is cmp $X \leqq$ *defect* X. Moreover, one can introduce "Cmp", defined like cmp except that the U_i must contain a basis of neighborhoods at every closed subset of X (starting not at -1 but with Cmp $X=0$ for rim-compact X); then cmp $X \leqq$ Cmp $X \leqq$ *defect* X. This splits the problem.

Another splitting is given in Sklyarenko [4]. If we define Skl $X \leqq n$ to mean X has a basis of open sets, any $n+1$ of whose boundaries have compact intersection, we get cmp $X \leqq$ Skl $X \leqq$ *defect* X.

What is lacking seems to be a novel idea or cluster of ideas, perhaps comparable with Sperner's Lemma or the idea of degree of a mapping. Spaces are known having arbitrarily high defect; the defect of a closed n-simplex minus an open $(n-1)$-dimensional face is $n-1$. (The proof is like Exercise 6, with an appeal to the fact that S^{n-2} is an absolute neighborhood retract.) There are spaces X known to have cmp $X=n$ for each n; but for the examples just described, cmp is not known to exceed 2.

The problem has no literature. The possibility exists that the conjecture cmp $X =$ *defect* X might go long unsettled (like the Hauptvermutung) while a substantial theory of defect grows up without it; but that is the fate of few conjectures.

Notes. Example M, and a number of other counterexamples in dimension theory, are inDowker [5]. More such examples, partly based on Example M, are in Smirnov [6; 7]. The latter paper gives the first known example proving VI.16; however, a more complicated auxiliary space than Example M is used, and the proof seems to require assuming some weak analog of the continuum hypothesis. (There is an argument by cases which is not exhaustive if $2^{\aleph_0}$ is the sum of a smaller number of smaller cardinals.)

It is an unsolved problem whether separability is a redundant assumption in Theorem 42. Even if X is connected, metrizable, and rim-compact, must X be separable? For some 25 years it was unsolved even if "rim-compact" is weakened to "rim-separable"; but Treybig [1] gave a counterexample to that.

In Exercise 1, (a) is a theorem of Vedenisov [1]; the example is Lokuciev-skiĭ's [1]. The first such example was given by Lunc [1]. Mardešić [1] gives an example having a basis of coverings whose nerves are intervals; all three examples have dim $X=1$, ind $X=2$, and are therefore not inverse limits of their nerves. (Mardešić [2], Pasynkov [2].)

On Exercise 2, see Morita [4]. In particular, the conclusion (c) can be strengthened to equality. Exercise 5, and the theorem it generalizes (23), are due to Morita [4]. Morita proved earlier (Morita [1]) that the Lindelöf property implies the star-finite property (hence it implies paracompactness). Spaces with the star-finite property are sometimes called *strongly paracompact*.

The Sierpiński curve (Exercise 6) contains a topological image of every 1-dimensional compact plane set (Sierpiński [1]); compare Menger [1] (or Hurewicz and Wallman [HW]) for n-dimensional compact metric spaces in which every n-dimensional compact metric space can be embedded.

Exercise 7 is from Smirnov [6]. Exercise 8, Zippin [1], Nishiura [1], Sklyarenko [3]. Exercise 9, Hausdorff [1] and Čech [3]. Exercise 10(d) is also from Čech [3]. Much of Exercise 10 was independently discovered several times, e.g. by Vainštein [1], Hanai [1], Henriksen-Isbell [2]. Cf. Aleksandrov [1].

Exercise 11(c) and (d), again Sklyarenko [3]. Cf. Michael [2]. Exercise 13(a), Smirnov [3]; 13(b), Arhangelskii [1]. Lemma 40 (the latter part) and the application to Exercise 13 are unpublished results of Corson.

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CHAPTER VII

LOCALLY FINE SPACES

The theory of fine uniform spaces may be described fairly accurately as an unexplored wilderness in which a few patches have been cleared; so we have here a rather loosely connected chapter. The most basic result is Stone's theorem that every metrizable space is paracompact. We shall strike a keynote by introducing some larger classes of spaces and generalizing Stone's theorem.

The functor λ. To begin with, we have characterizations of fine spaces in I.20 and I.21 (normal coverings; continuous functions). No better characterization (in fact, essentially no other characterization) is known. The fine spaces form a coreflective subcategory; and we know something about the coreflection, e.g. on Lindelöf spaces it cannot increase uniform dimension. Note that coreflection upon fine spaces takes sums to sums, and a sum of fine spaces is fine. Although the functor does not preserve quotients (Exercise II.5), a quotient of a fine space is fine.

A subspace of a fine space is called *subfine.*

1. THEOREM. *Every uniform space μX has a subfine coreflection $l\mu X$. If μX is embedded in any injective space νY, the uniformity induced on X by the fine uniformity on Y is $l\mu$.*

PROOF. Write $l\mu$ for the uniformity induced on X by the fine uniformity $l\nu$ on Y. Consider any mapping $f: \sigma A \to \mu X$, where σA is a subspace of a fine space σB. It has an extension $g: \sigma B \to \nu Y$. Since σB is fine, the continuous function g is uniformly continuous on σB into $l\nu Y$; so the restriction to A is uniformly continuous, and that is $f: \sigma A \to l\mu X$.

2. COROLLARY. *The subfine coreflection of an injective space, or of a complete metric space, is fine.*

PROOF. The injective case is immediate from the theorem. If μX is complete metric, it is a closed subspace of a metric injective space νY; since every open covering of X extends over Y, the fine uniformity on Y induces the fine uniformity on X.

The existence of many subfine uniformities is easy to establish. For example, every precompact uniformity is subfine. Every non-normal space has closed subspaces that are subfine but not fine, and many examples are complete. We can give more explicit examples after Corollary 24.

The fact that every uniform covering can be realized in a metric space implies that every normal covering, and in fact every uniform covering of a subfine space, has whatever combinatorially simple refinements the open coverings of metrizable spaces have. The main theorem in this connection is not combinatorial as usually stated: every open covering of a metrizable space has a locally finite open refinement. Recall another theorem: Every open covering of a metrizable space has a σ-disjoint open refinement.

A collection $\mathscr{H}$ of sets in a uniform space is called *uniformly locally finite* if there is a uniform covering $\mathscr{U}$ each of whose elements meets only finitely many elements of $\mathscr{H}$. It is *σ-uniformly discrete* if it is a countable union of uniformly discrete collections.

3. Lemma. *Every point-finite uniform covering has a uniformly locally finite uniform shrinking. Every σ-disjoint uniform covering has a σ-uniformly discrete uniform shrinking.*

Proof. Any uniform strict shrinking (IV.19) will do.

4. Stone theorem. *Every subfine uniform space has a basis of uniformly locally finite uniform coverings and a basis of σ-uniformly discrete uniform coverings.*

Proof. This follows immediately from the lemma and the preceding remarks.

We observe that the information in Theorem 4, though welcome, does not focus very sharply; as a matter of fact, no example is known to refute the seemingly ridiculous conjecture that Theorem 4 generalizes to arbitrary uniform spaces.

There is a much stronger property of subfine spaces, which may characterize them. A covering $\mathscr{V}$ of a uniform space is called *uniformly locally uniform* if there is a uniform covering $\{U_\alpha\}$ such that the trace of $\mathscr{V}$ on each subspace U_α is uniform. A space μX is called *locally fine* if every uniformly locally uniform covering is uniform.

We show in VII.6 that every subfine space is locally fine. Some comment must be made on the term. First, if it can be shown that every locally fine space is subfine, the term "locally fine" will no longer be needed. In any case it is intended to suggest two main ideas about the spaces; they are like fine spaces, and local arguments (more precisely, uniform local arguments) can be used in them. Briefly, the same local arguments can be used in locally fine spaces as in fine spaces. The property of being locally fine is not a local property, but that should not cause confusion because no interesting uniform properties are local. Should our terminology be codified by an academy, "localizationally fine" would be more precise.

A typical uniformly locally uniform covering has the form $\{U_\alpha \cap V_\beta^\alpha\}$, where $\{U_\alpha\}$ is a uniform covering, and for each α we may suppose that $\{V_\beta^\alpha\}$ is either a uniform covering of the subspace U_α or a uniform covering of the whole space. The obvious method for constructing locally fine uniformities is transfinite iteration of this construction. We can prove that the method works, using Stone's theorem; but it is an unsolved problem whether the intermediate constructs are uniformities. Therefore, for the nonce only, we introduce generalized uniformities.

A *quasi-uniformity* μ on a set X is a family of coverings of X forming a filter in the refinement ordering. X with μ is a *quasi-uniform space* μX. A *subspace* is a subset A of X with the traces on A of coverings in μ. Coverings in μ are *uniform*; a covering $\mathscr{V}$ is *uniformly locally uniform* if for some uniform covering $\{U_\alpha\}$, the trace of $\mathscr{V}$ on each U_α is uniform.

The family of all uniformly locally uniform coverings of a quasi-uniform space μX is a quasi-uniformity $\mu^{(1)}$, the *derivative* of μ. For any ordinal $\alpha+1$, $\mu^{(\alpha+1)}$ is defined as $\mu^{(\alpha)(1)}$; for a limit ordinal $\beta, \mu^{(\beta)}$ is the union of all preceding $\mu^{(\alpha)}$ The successive derivatives form an expanding family of quasi-uniformities. Since X has a limited number of coverings, there is a last derivative $\mu^{(\alpha)} = \mu^{(\alpha+1)}$; it is called $\lambda\mu$, and its members are λ-*uniform* coverings.

By an easy induction we see that every λ-uniform covering of a uniform space has an open λ-uniform refinement.

Call a uniformity or preuniformity *point-finite* if it has a basis of point-finite coverings. (Equivalently, uniformly locally finite coverings; but the notion of point-finiteness is more convenient when we have several uniformities.) In view of I.10, every quasi-uniformity μ contains a finest point-finite preuniformity $pf\mu$. If μ contains a uniformity ν, then $pf\mu$ contains $p\nu$; so $pf\mu$ is a uniformity.

5. *If μ is a point-finite uniformity then so is $\mu^{(1)}$.*

PROOF. Let $\{U_\alpha \cap V_\beta^\alpha\}$ be a covering in $\mu^{(1)}$, with $\{U_\alpha\}$ and each $\mathscr{V}^\alpha = \{V_\beta^\alpha\}$ point-finite coverings in μ. Let $\{Y_\gamma\} = \mathscr{Y}$ be a point-finite star-refinement of $\{U_\alpha\}$, and $\mathscr{W}^\alpha <^* \mathscr{V}^\alpha$, all in μ. Each Y_γ is contained in only finitely many U_α; let $\mathscr{X}^\gamma$ be the intersection of the corresponding $\mathscr{W}^\alpha$, and let $\mathscr{Z}^\gamma = \{Z_\delta^\gamma\}$ be a point-finite uniform refinement of $\mathscr{X}^\gamma$. Then $\{Y_\gamma \cap Z_\delta^\gamma\}$ is a point-finite covering in $\mu^{(1)}$. It remains to check that this covering, Ω, is a star-refinement of $\{U_\alpha \cap V^\alpha_\beta\}$. For any γ and δ, choose U_α containing $\mathrm{St}(Y_\gamma, \mathscr{Y})$ and V^α_β containing $\mathrm{St}(Z^\gamma_\beta, \mathscr{W}^\alpha)$. If $Y_\sigma \cap Z^\sigma_\tau$ meets $Y_\gamma \cap Z^\gamma_\delta$, then $Y_\sigma \subset \mathrm{St}(Y_\gamma, \mathscr{Y}) \subset U_\alpha$; so $\mathscr{Z}^\sigma < \mathscr{X}^\sigma < \mathscr{W}^\alpha$. Hence $Y_\sigma \cap Z^\sigma_\tau \subset U_\alpha \cap V^\alpha_\beta$, as was to be shown.

6. *If μX is a subspace of νY, $\mu^{(1)}X$ is a subspace of $\nu^{(1)}Y$. Every subfine space is locally fine.*

PROOF. The first assertion is immediate from the definitions. The second follows by VII.4 and VII.5.

7. LEMMA. *Let P be a partially ordered set in which every descending chain is finite. Then there exists an ordinal-valued function f on P such that for each p in P, $f(p)$ is the least ordinal exceeding $f(q)$ for all successors q of p.*

PROOF. Define $f(p)=0$ if p is minimal in P; inductively, for each α, let $f(p)=\alpha$ if p is minimal among the elements of P for which f has not yet been defined. When the cardinal of α passes the cardinal of P, the process must have concluded. Any p for which f was not defined would have an infinite descending chain of successors at which f was not defined; so f was defined on all of P.

We remark that the function f is unique and exists only when there are no infinite descending chains. It is the *height function* of P; $f(p)$ is the *height* of p.

8. LEMMA. *Let $\{U_p\}$ be a family of subsets of a quasi-uniform space μX, indexed by a partially ordered set P. Let S be a subset of P such that every descending chain in $P-S$ is finite. Suppose* (a) *the set of all U_p is a uniform covering, and* (b) *for every p in P, the trace on U_p of the family of all U_q, q a successor of p, is uniform. Then $\mathscr{U}_S=\{U_p : p\in S\}$ is a λ-uniform covering of X.*

PROOF. We use the height function f on $P-S$. If $f(p)=0$, then $\mathscr{U}_S$ is uniformly locally uniform on U_p. If there were some U_p on which $\mathscr{U}_S$ is not the trace of a λ-uniform covering, there would be one of least height; but then the successors U_q form a uniform covering, and $\mathscr{U}_S$ coincides on each of them with a λ-uniform covering. Thus $\mathscr{U}_S$ is λ-uniform on every U_p, so also on X.

9. *For any complete metric space μX, $\lambda\mu$ is the fine uniformity on X.*

PROOF. First, every covering $\{U_\alpha\}$ in μ has the property that the interiors of its elements cover X. By a trivial transfinite induction, the same is true for $\lambda\mu$; and $\lambda\mu$ is contained in the fine uniformity.

For the converse, let $\{V_\beta\}$ be any open covering of X. For each n, let

$\mathscr{U}^n = \{U_{n,\alpha}\}$ be a uniform covering of μX consisting of sets of diameter at most 2^{-n}. Partially order the set P of all indices (n,α) as follows: $(n+1,\beta) < (n,\alpha)$ if $U_{n+1,\beta}$ meets $U_{n,\alpha}$; $(n+k,\gamma) < (n,\alpha)$ if there is a chain $(n+k,\gamma) < (n+k-1,\delta) < \cdots < (n,\alpha)$. Let S be the set of all (n,α) such that $U_{n,\alpha}$ is contained in some V_β. Since μX is complete and each $U_{n,\alpha}$ has diameter at most 2^{-n}, each infinite descending chain in P indexes a sequence of sets converging to a point x. Some ϵ-neighborhood of x is in some V_β; so only a finite number of the indices in the chain can be in $P-S$. The conditions (a) and (b) of Lemma 8 are satisfied. Thus S indexes a λ-uniform covering, and since $\{V_\beta\}$ is coarser, the proof is complete.

REMARK. If Proposition 9 is true for injective spaces then every locally fine space is subfine, and $\lambda = l$.

10. THEOREM. *For any uniformity μ, $\lambda\mu$ is a point-finite uniformity. It is the coarsest locally fine uniformity finer than μ. Every mapping $f: \mu X \to \nu Y$ is uniformly continuous on $\lambda\mu X$ to $\lambda\nu Y$; so λ defines a coreflection upon the subcategory of locally fine spaces. The coreflection takes embeddings to embeddings.*

PROOF. We wish to show that $\lambda\mu$ is the uniformity $pf\lambda\mu$, i.e., each covering in any $\mu^{(\alpha)}$ is in $pf\lambda\mu$. For a covering $\mathscr{U}$ in μ itself, we may realize $\mathscr{U}$ by a mapping into a complete metric space, $f: \mu X \to \nu Y$. By Proposition 9, there is a covering $\mathscr{V}$ in $pf\lambda\nu$ such that $f^{-1}(\mathscr{V}) < \mathscr{U}$; and clearly $f^{-1}(\mathscr{V}) \in pf\lambda\mu$. Then μ is contained in $\rho = pf\lambda\mu$. From $\mu \subset \rho \subset \lambda\mu$ we have $\mu^{(1)} \subset \rho^{(1)} \subset \lambda\mu$. Since $\rho^{(1)}$ is a point-finite uniformity (VII.5), it is $\rho \supset \mu^{(1)}$. By induction, $\mu^{(\alpha)} \subset \rho$ for all α, and $\lambda\mu = \rho$.

Any locally fine uniformity containing μ contains $\mu^{(1)}$, and by induction contains $\lambda\mu$; so $\lambda\mu$ is the coarsest. If $f: \mu X \to \nu Y$ is a mapping, then the inverse image of a uniformly locally uniform covering is uniformly locally uniform; so by induction, $f: \lambda\mu X \to \lambda\nu Y$ is a mapping. This establishes the coreflection. Induction applied to VII.6 shows that embeddings are preserved.

11. COROLLARY. *Every locally fine uniform space has a basis of uniformly locally finite coverings and a basis of σ-uniformly discrete uniform coverings.*

Shirota's theorem.

12. *The functors l and λ commute with completion. In particular, the completion of a locally fine space is locally fine.*

PROOF. Since both coreflections preserve embeddings and topology, the embedding of X in its completion Y embeds $lX(\lambda X)$ as a dense subspace of lY (resp. λY). Since Y is complete in the given uniformity, it is complete in any finer uniformity inducing the same topology, which these do.

13. *The completion of a locally fine space X is an inverse limit of fine metrizable spaces; so every uniformly continuous real-valued function on a subspace of X can be extended over X.*

Proof. We appeal to Exercises IV.8, V.10. Any complete uniform space Y is an inverse limit of complete metric spaces Z_α, since it is a closed subspace of a product of them. Applying the functor λ, we get a left compatible family of mappings of λY into the complete, fine metrizable spaces λZ_α. Y is still the set of points of the limit space. Assuming Y was already locally fine, every uniform covering is realized by projection into one of the Z_α; so Y is the limit space. The rest is immediate from Exercise V.10.

The next two theorems will establish a connection between completion and real-valued functions for locally fine spaces. They say almost that two functors commute on the image of λ: completion and reflection upon subspaces of products of real lines. But there is a hedging clause arising from an unsolved problem in set theory.

The *Problem of Ulam* is this. Does there exist a countably additive measure μ defined on all subsets of some set S, taking no value other than 0 and 1, vanishing for all one-point sets, but not vanishing identically? The problem is unsolved even if the restriction to two values is abandoned; on the other hand, even if that restriction is abandoned, it is known that (if set theory is consistent) there exist models for set theory in which there are no such measures. We shall not go into that (see Notes, p. 144); but without constructing special models we can show that the (restricted) Ulam Problem has a negative solution unless the cardinal number of the set S of the counterexample far exceeds all the well-known cardinals.

Call a cardinal m *nonmeasurable* if when S has power m, every countably additive two-valued measure defined on all subsets of S and vanishing on singletons vanishes. Obviously $\aleph_0$ is nonmeasurable. Obviously if $m < n$ and n is nonmeasurable then m is nonmeasurable.

14. Ulam theorem. *A product of nonmeasurably many nonmeasurable cardinals is nonmeasurable.*

Proof. Let S be a product set of factors F_α ($\alpha \in A$), where A and each F_α have nonmeasurable powers. Let μ be a countably additive two-valued measure on subsets of S, vanishing on singletons. Suppose $\mu(S) = 1$. Consider the projections $p_\alpha : S \to F_\alpha$. We define a measure μ_α for all subsets G of F_α: $\mu_\alpha(G) = \mu(p_\alpha^{-1}(G))$. Then μ_α is a countably additive two-valued measure that does not vanish; so $\mu_\alpha(\{x_\alpha\}) = 1$ for some point $x_\alpha \in F_\alpha$. This gives us a set $T_\alpha = p_\alpha^{-1}(x_\alpha)$, with $\mu(T_\alpha) = 1$. The intersection of the sets T_α ($\alpha \in A$) is a single point x; $\mu(\{x\}) = 0$. $S - \{x\}$ is then a union of nonmeasurably many sets R_α of measure zero. We can write it as a disjoint union by means of a well-ordering of A; $R'_\alpha = R_\alpha - \bigcup[R_\beta: \beta < \alpha]$ Define a measure ν on A: $\nu(I) = \mu(\bigcup[R'_\alpha: \alpha \in I])$. Again ν is countably additive two-valued, and it vanishes on points by construction. We conclude $\nu(A) = 0 = \mu(S - \{x\}) = \mu(S)$, a contradiction.

This theorem and a little cardinal arithmetic show that it is not possible to construct "arithmetically" a measurable cardinal number. The model-theoretic results we noted above make this statement of impossibility precise.

We call a uniform space *nonmeasurable* if it has a basis of uniform coverings of nonmeasurable power. It is the same (by II.33) to say that there is no uniformly discrete subspace of measurable power.

15. *If μX is locally fine and nonmeasurable, then a filter in X which meets every countable uniform covering is Cauchy in μX.*

Proof. Let $\mathscr{F}$ be a filter meeting every countable uniform covering. It will suffice (by Corollary 11) to show $\mathscr{F}$ meets every σ-uniformly discrete nonmeasurable uniform covering $\{U_{n\alpha}\}$, where each $\{U_{n\alpha}\}_{n=n_0}$ is uniformly discrete. The unions W_n of the sets $U_{n\alpha}$ form a countable uniform covering; so at least one W_n belongs to $\mathscr{F}$. On W_n, $\{U_{n\alpha}: \alpha \in A\}$ is a uniform covering. For any countable partition $\{A_i\}$ of A, $\mathscr{F}$ contains exactly one set $\bigcup[U_{n\alpha}: \alpha \in A_i]$. Then define a measure ν on A: $\nu(I) = 1$ if $\bigcup[U_{n\alpha}: \alpha \in I] \in \mathscr{F}$, $\nu(I) = 0$ otherwise. This is countably additive two-valued, and it does not vanish. Hence ν takes the value 1 on a singleton, and $\mathscr{F}$ meets $\{U_{n\alpha}\}$.

Let us name the reflection upon spaces having a basis of countable uniform coverings: e. (The existence of such a reflection was noted in Exercise III.2.) By Exercise III.2, every countable uniform covering of μX is a uniform covering of $e\mu X$. We define $c\mu X$ as the set X with the weak uniformity induced by all real-valued functions uniformly continuous on μX. We remark that this is a reflection too (also $ce = ec = c$), but we shall not use the fact. We shall use the obvious relation $pe = p$, or more specifically its consequence that $\beta(\mu X)$ is the Samuel compactification of the completion of $e\mu X$.

16. *If μX is locally fine, so is $e\mu X$.*

Proof. A uniformly locally uniform covering of $e\mu X$ has a refinement of the form $\{U_i \cap V^i_j\}$ that is countable and (μX being locally fine) uniform.

Obviously the functor e takes embeddings to embeddings. On the other hand, c does not preserve either local fineness or embeddings. We want a conservative property of c that could be stated in either of two ways; since both are easy to prove, let us have both.

17. *If μX is a subspace of μY, and either X is dense in Y or μY is locally fine, then $c\mu X$ is a subspace of $c\mu Y$.*

PROOF. In either case every real-valued uniformly continuous function on μX extends over μY; so the weak uniformity induced on X by all uniformly continuous real-valued functions on μY is the same as that induced by the real functions on μX.

18. SHIROTA THEOREM. *If μX is locally fine, then every Cauchy filter in $c\mu X$ is Cauchy in $e\mu X$; thus it is Cauchy, if μX is nonmeasurable. Then if μY is the completion of μX, $c\mu Y$ is the completion of $c\mu X$.*

PROOF. Let $\mathcal{F}$ be a Cauchy filter in $c\mu X$ and $\mathcal{U} = \{U_i\}$ a countable uniform covering of μX. $\mathcal{U}$ is realized by a mapping into a complete separable metric space, $f: \mu X \to \nu Z$. For some locally finite countable open covering $\{V_j\}$ of Z, $j = 1, 2, \cdots, f^{-1}(\{V_j\})$ refines $\mathcal{U}$. Subordinate a partition of unity $\{g_j\}$ to $\{V_j\}$, and define $h: Z \to E^1$ as $\sum jg_j$. h is locally a finite sum; so it is well-defined and continuous. By Theorem 9, hf is uniformly continuous. Therefore $\mathcal{F}$ has a member F on which hf is bounded. This means $f(F)$ is contained in a finite union of V_j's; for outside $V_1 \cup \cdots \cup V_n$, h exceeds n. Consequently F is contained in a finite union of U_i's. So $\mathcal{F}$ is semi-Cauchy in $e\mu X$ and converges to a compact set in the completion.

Now consider $p\mu X$. There $\mathcal{F}$ is Cauchy and converges to a single point y of $\beta(\mu X)$. The completion of $e\mu X$ is topologically a subspace of $\beta(\mu X)$; therefore the compact set to which $\mathcal{F}$ converges is just y, and $\mathcal{F}$ is Cauchy in $e\mu X$.

If μX is nonmeasurable, $\mathcal{F}$ is Cauchy by Proposition 15. Passing to the completion μY, it is complete, locally fine, and nonmeasurable; so $c\mu Y$ is complete. By Proposition 17, $c\mu X$ is a dense subspace, and the theorem is proved.

REMARK. The fact that this theorem holds for locally fine spaces as well as fine spaces is not at all deep; all properties mentioned in the conclusion are continuous properties inherited by arbitrary subspaces.

Products of separable spaces. The fact that products of separable topological spaces are in some ways "almost separable" should be set down as a discovery of E. Szpilrajn (= E. Marczewski). Several results of this type are described in the Notes, p. 145. The uniform applications that have been discovered so far can all (all three of them) be derived from the following new result of A. Gleason.

On a product space X of factors $X_\alpha (\alpha \in A)$, a *countable projection* is a mapping $p: X \to Y$ upon the partial product Y of the spaces X_α $(\alpha \in B)$, for some countable subset B of A; $p(x)$ has the same αth coordinate as x, for each α in B.

19. GLEASON THEOREM. *Every continuous function $f: X \to M$ from a product X of separable topological spaces to a Hausdorff space M in which every*

point is a G_δ set has the form $g\pi$, where $\pi : X \to Y$ is a countable projection and $g : Y \to M$ is a continuous function.

PROOF. Let X be the product of $\{X_\alpha : \alpha \in A\}$. We may suppose no X_α is empty. Select a point 0_α, or 0, in each X_α; let p^0 be the point of X having all coordinates 0. Let $S^0 = f^{-1}f(p^0)$. This is a countable intersection of open sets, and it contains a countable intersection of basic open sets of the product containing p^0; so there is a countable subset B_0 of A such that each point x having the same coordinates in B_0 as p^0 has the same image under f. Generally, having a countable subset B_n of A, select a countable dense set $\{q_i^{n+1}\}$ in the partial product of $\{X_\alpha : \alpha \in B_n\}$. Let $p_i^{n+1} \in X$ have the same αth coordinates for $\alpha \in B_n$ as q_i^{n+1}, with all other coordinates 0. Let $S_i^{n+1} = f^{-1} f(p_i^{n+1})$; again there is a countable set of indices B_i^{n+1} such that each point having the same coordinates in B_i^{n+1} as p_i^{n+1} has the same image under f. Let $B_{n+1} = \bigcup_i B_i^{n+1}$.

Complete the induction and form $B = \bigcup_n B_n$. For any x in X, let x' have the same coordinates in B as x, 0 otherwise. Evidently x' is a limit of points p_i^n. If r_i^n has the same coordinates in B as p_i^n and the same coordinates in $A - B$ as x, then $\{r_i^n\}$ converges to x whenever $\{p_i^n\}$ converges to x'. But $f(r_i^n) = f(p_i^n)$. Since limits are unique in a Hausdorff space, $f(x) = f(x')$. Thus $f = g\pi$, where π is the projection upon the partial product of $\{X_\alpha : \alpha \in B\}$ and g is a function. Since π is a quotient mapping and f is continuous, g is continuous.

20. COROLLARY. *Every metrizable continuous image of a product of separable topological spaces is separable.*

21. COROLLARY. *If μX is a product of complete separable metric spaces, then $\lambda\mu X$ is fine.*

PROOF. Every normal covering of X is realized by a continuous function $g\pi$, where $\pi : X \to Y$ is a countable projection. Y is complete metric in the product uniformity ν, and π is uniformly continuous from μX to νY. So $\pi : \lambda\mu X \to \lambda\nu Y$ is uniformly continuous. $\lambda\nu Y$ is fine (by VII.9); so g is uniformly continuous on $\lambda\nu Y$. Therefore every normal covering of X is λ-uniform.

22. COROLLARY. *A locally fine space containing no uncountable uniformly discrete subspace is subfine.*

PROOF. Such a space μX can be embedded in a product ρR of complete separable metric spaces, by II.33 and II.13-II.14. Then $\lambda\mu X = \mu X$ is embedded in the fine space $\lambda\rho R$.

In view of VII.16, for any locally fine space μX, $e\mu X$ can be embedded in a fine space Y. The real-valued uniformly continuous functions on μX may be identified as the functions continuously extensible over Y.

For the last application (an example), we want a modified form of Gleason's theorem. Though it is a corollary in spirit, it contains the theorem; so we label it:

23. Gleason theorem. *Every continuous function $f: U \to M$ from an open subset U of a product X of separable topological spaces to a Hausdorff space M in which every point is a G_δ set has the form $g\pi | U$, where $\pi: X \to Y$ is a countable projection and $g: \pi(U) \to M$ is a continuous function.*

Proof. Conduct the same argument with the following modifications. p^0 should be a point of U. Having a countable index set $B_n \subset A$, let Y_n be the partial product of $\{X_\alpha: \alpha \in B_n\}$, U_n the projection of U in Y_n. As coordinate projections are open mappings, U_n is an open set. Hence it has a countable dense subset $\{q_i^{n+1}\}$. For each i, there is u_i^{n+1} in U projecting upon q_i^{n+1}. Since U is open in the product topology, there are finite sets of indices $C(n+1, i)$ such that every point of X agreeing with u_i^{n+1} in these indices is in U. Let p_i^{n+1} have the same αth coordinate as u_i^{n+1} for α in $B_n \cup C(n+1, i)$, all other coordinates 0. Construct B_i^{n+1} as before and let B_{n+1} be the union of these sets and the $C(n+1, i)$. Again let $B = \bigcup_n B_n$.

Consider any x, y in U having the same coordinates in B. Let x' be the point of X having these coordinates for indices in B, all other coordinates 0. Still x' is a limit of points p_i^n. Let r_i^n and s_i^n have the same coordinates in B as p_i^n; let the other coordinates of r_i^n be the same as for x, the other coordinates of s_i^n the same as for y. Then $f(r_i^n) = f(p_i^n) = f(s_i^n)$ for each n and i, and if we select a net $\{p_i^n\}$ converging to x' we get nets $\{r_i^n\} \to x$, $\{s_i^n\} \to y$. Hence $f(x) = f(y)$. The conclusion follows as before since $\pi | U$ is an open mapping.

24. Corollary. *Any two disjoint open sets in a product of separable metric spaces are mapped into disjoint (open) sets by some countable projection.*

25. Example. *There exist a uniformizable space Z with* $\dim Z = 0$ *and a compact space Y containing Z that has no basis of open sets whose boundaries are disjoint from Z. $Y - Z$ is not rim-compact.*

Proof. Let Ω consist of an interval $I = [0, 1]$ and for each point t of I a copy W_t of the space of all countable ordinals, converging to t. Each W_t is an open set; a subset U of Ω is a neighborhood of $t \in I$ if U contains an I-neighborhood N of t and, for each $s \in N, U$ contains almost all of W_s. Let $Z \subset \Omega$ consist of $0 \in I$ and all isolated points. Since Z has only one nonisolated point, $\dim Z = 0$. (Incidentally, this makes Z paracompact; the example cannot have the Lindelöf property, by VI.36.) Since every continuous real-valued function on W_t is finally constant, every continuous real-valued function on Ω that is 0 at 0 and 1 at 1 takes the value 1/2 on Z. Then let C be the set of all bounded real-valued continuous functions on Ω, and embed Ω in a product Y of closed intervals indexed by C, each $p \in \Omega$ going to the point whose cth coordinate ($c \in C$) is $c(p)$. In Y, the points 0 and 1 of Ω are not separated by any set disjoint from Z; for if M and N are disjoint open sets of Y, Corollary 24 gives a mapping of Y to a metric space taking M and N into disjoint open sets, and thus a continuous function whose values are $<1/2$ on M and $>1/2$ on N. Consequently the point 1 of $Y-Z$ has no basis of neighborhoods whose boundaries are compact subsets of $Y-Z$.

Glicksberg's theorem. When is a product of fine spaces fine? The complete answer is not known, but there is an answer in terms of reasonable topological properties of the factor spaces *and* the product. Perhaps the term "reasonable" should be qualified; except for the compact and discrete cases, products of the most familiar spaces are never fine.

First we consider the problem for locally fine spaces. If a product space is going to be fine, it must be locally fine, and the conditions for this are already so restrictive that only some obvious supplementary conditions need be added to make the product fine. (What has to be added to *prove* the product fine is less obvious.)

A uniform space is said to *admit* a cardinal number m if every m uniform coverings have a uniform common refinement. The finite cardinals are of course admitted by all spaces.

26. *If μX admits an infinite cardinal, $\Delta d\mu X=0$.*

PROOF. A uniform covering $\mathscr{U}$ has a normal sequence of refinements, $\cdots <^* \mathscr{U}_1 <^* \mathscr{U}$; and a uniform common refinement $\mathscr{V}$ for the sequence. Every finite chain of intersecting elements of $\mathscr{V}$ lies in one element of $\mathscr{U}_1$. Then it follows that the finest partition $\mathscr{P}(\mathscr{V})$ coarser than $\mathscr{V}$ is finer than $\mathscr{U}$. (Cf. Exercise I.3, if necessary.)

27. *A uniform space that admits a cardinal m and is not uniformly discrete has a uniformly discrete subspace of power m.*

PROOF. This is obvious for finite m. If μX admits $\aleph_\alpha$ and is not uniformly discrete, we can find a descending chain of ω_α uniform partitions $\mathscr{U}_\lambda$, all distinct. Their intersection is another uniform partition $\mathscr{V}$. Suppose $\mathscr{V}$ has fewer than $\aleph_\alpha$ elements V_β, say n. Each V_β lies in a unique element U_β^λ of each $\mathscr{U}_\lambda$; moreover, U_β^λ is a union of elements of $\mathscr{V}$. For fixed β, there are at most n values λ such that $U_\beta^\lambda \neq U_\beta^{\lambda+1}$. Then there are only $n^2 < \aleph_\alpha$ indices λ such that $\mathscr{U}_\lambda \neq \mathscr{U}_{\lambda+1}$, a contradiction. This shows there is a uniformly discrete subspace of power $\aleph_\alpha$.

28. *The product of two locally fine spaces μX, νY is locally fine if and only if each factor admits the power of every uniformly discrete subspace of the other factor. Unless one factor is uniformly discrete, there is a unique cardinal m such that each factor has uniformly discrete subspaces of all powers less than m but not of power m, and admits all cardinals less than m but not m.*

PROOF. If μX fails to admit the power of a uniformly discrete subspace of νY, say D of power m, consider a uniform covering of νY consisting of $Y-D$ and disjoint neighborhoods N_δ of the points of D. Let $\{\mathscr{U}^\delta\}$ be a family of coverings in μ having no uniform common refinement; let $\mathscr{U}^\delta = \{U^\delta_\gamma\}$. Cover $\mu X \times \nu Y$ by $X \times (Y-D)$ and the sets $U^\delta_\gamma \times N_\delta$. This covering is uniformly locally uniform, but if it were uniform it would be refined by a product covering $\{U_\alpha \times V_\beta\}$, and $\{U_\alpha\}$ would be a common refinement of all $\mathscr{U}^\delta$.

This shows that the proposed condition is necessary. If neither factor is uniformly discrete, let m be the smallest cardinal μX does not admit. By Proposition 27, μX has uniformly discrete subspaces of all smaller powers; so νY also admits all smaller cardinals, and νY has uniformly discrete subspaces of all those powers. Since μX does not admit m, νY has no uniformly discrete subspace as big as that and does not admit m; so μX also has no uniformly discrete subspaces as big as that.

Suppose the condition satisfied. If one factor space is uniformly discrete, the product is clearly locally fine. If both factors are precompact, so is the product. In the remaining case, each space admits an infinite cardinal; so each space has a basis of uniform partitions, and their powers are cardinals admitted by both factors. For any uniformly locally uniform covering $\{(U_\alpha \times V_\beta) \cap (Y^{\alpha\beta}_\gamma \times Z^{\alpha\beta}_\delta)\}$, the total number of coverings involved is so small that we can intersect them all; and the covering is uniform on the product space.

Two remarks ought to be added. (1) Whether a space has a basis of uniform partitions or not, the cardinal numbers of its uniformly discrete subspaces are the cardinals of the uniform coverings in a basis, by II.33. The smallest cardinal *greater* than all these is called the *covering character* of the space. (2) The smallest cardinal m not admitted by a space is necessarily a regular cardinal, i.e., it is not the sum of fewer than m smaller cardinals. (Proof is obvious.)

29. *An infinite product of uniform spaces each containing more than one point is locally fine if and only if all factors are precompact.*

PROOF. If all factors are precompact, so is the product. If one factor is not precompact, then the product space contains the product of an infinite discrete space and a Cantor set, which is not locally fine.

Among the fine spaces we have the same cases to consider. A uniformizable space is called *pseudodiscrete* (usually abbreviated: a *P-space*) if every G_δ set is open; we shall call it *m-discrete* if every intersection of at most m open sets is open. A space is *m-compact* if every open covering has a subcovering of power less than m; we shall call it *pseudo-m-compact* if every normal covering has such a subcovering, i.e., the covering character in the fine uniformity is at most m. The term *pseudocompact* is standard for precompact fine spaces. So we have pseudocompact = pseudo-$\aleph_0$-compact, pseudodiscrete = $\aleph_0$-discrete.

30. *A fine space admits an infinite cardinal m if and only if it is m-discrete. It has a uniformly discrete subspace of power m if and only if it is not pseudo-m-compact. It admits the powers of all its uniformly discrete subspaces if and only if for every cardinal m it is either m-discrete or pseudo-m-compact.*

PROOF. Suppose the fine space X admits m, and $\{U_\alpha\}$ is a family of at most m open sets. For any point x of $\bigcap U_\alpha$, consider the binary coverings $\{X-x, U_\alpha\}$. There is a normal covering $\mathscr{W}$ refining all of them; then $\mathrm{St}(x, \mathscr{W})$ is a neighborhood of x contained in $\bigcap U_\alpha$. Thus X is m-discrete. Now suppose X is m-discrete, m infinite. Then a normal covering $\mathscr{U}$ and a whole normal sequence refining it are refined by their intersection, an open partition. It remains to check that the intersection of m or fewer open partitions is an open partition, which is immediate. Thus X admits m.

Then the other two assertions are obvious.

Here are some further facts about these concepts; some for later use, some not.

31. *A finite product of m-discrete spaces is m-discrete. An m-discrete pseudo-m-compact space is discrete. An $\aleph_\alpha$-discrete $\aleph_{\alpha+2}$-compact space is paracompact. Pseudocompact spaces are characterized by the property that every real-valued continuous function is bounded, or by the property that every locally finite family of nonempty open sets is finite. Pseudodiscrete spaces are characterized by the property that every continuous real-valued function is locally constant.*

PROOF. The first assertion is obvious. The second is a special case of VII.27. Next suppose X is $\aleph_\alpha$-discrete (so has an open-closed basis) and $\aleph_{\alpha+2}$-compact. Then any open covering is refined by a covering consisting of at most $\aleph_{\alpha+1}$ open-closed sets V_β, which can be well-ordered so that each one has at most $\aleph_\alpha$ predecessors. Then define V_β'' as V_β minus the union of the predecessors, and we have an open partition. Thus X is fully normal, which means X is paracompact.

On a pseudocompact space there is clearly no unbounded real-valued continuous function. In turn this implies there is no locally finite infinite sequence of nonempty open sets U_n; for there would be functions f_n to $[0,1]$ vanishing outside U_n but taking the value 1 somewhere inside, and $\sum nf_n$ would be continuous. To close the cycle of implications, a space that is not pseudocompact has even a uniformly discrete sequence of open sets.

Finally, a real-valued continuous function is constant on G_δ's, so locally constant when all G_δ's are open. If some G_δ has a noninterior point x, we get a function not locally constant at x by a formula $\sum f_n/n$.

32. *For the product of two fine spaces to be fine, one of the following must hold.* (a) *One factor is discrete of power at most m and the other is m-discrete.* (b) *Both factor spaces are pseudocompact*; *and so is the product.* (c) *For some* $\aleph_\alpha$, *both factors and the product are pseudo-*$\aleph_\alpha$*-compact and m-discrete for all* $m<\aleph_\alpha$.

For an infinite product of fine spaces, each having two or more points, to be fine, the factors and the product must be pseudocompact.

PROOF. Most of this is a specialization of Propositions 28 and 29. A product of fine pseudocompact spaces is certainly precompact; so if it is fine, it is pseudocompact. Pseudo-$\aleph_\alpha$-compactness of the product in case (c) is deduced in the same way, and m-discreteness from Proposition 31.

We show next that the four alternative necessary conditions of VII.32 are sufficient.

33. *A product of two fine spaces is fine if one is m-discrete and the other is discrete of power at most m.*

PROOF. The product is a topological sum of copies of an m-discrete space. Any normal covering has a normal trace on each copy; so it is refined by a product covering.

34. GLICKSBERG THEOREM. *A product of fine spaces is fine if it is pseudocompact.*

PROOF. First we consider a product $X\times Y$ of two spaces. Since the uniformities involved are precompact, it suffices to show that every continuous real-valued function f on $X\times Y$ is uniformly continuous. By III.21, this means the families of functions $f(x,\ \ )$ and $f(\ \ ,y)$ are equicontinuous.

We reduce the problem by means of the Ascoli theorem III.37. Since these families of functions are bounded, we have to show they are precompact. This means there is no infinite uniformly discrete subset; so it suffices to show that any countable set of them is equicontinuous. In view of III.38, we need only show that a sequence of functions $f(\ \ , y_n)$ must be equicontinuous at a point x_0. (The same proof will show the equicontinuity of $f(x,\ \ )$.)

Suppose the contrary. For some $\epsilon > 0$, every product neighborhood $W \times Y$ of $\{x_0\} \times Y$ contains a point (x, y_n) such that $|f(x, y_n) - f(x_0, y_n)| > \epsilon$. By recursion we can get a sequence $\{(x_k, y_{n_k})\}$ of such points, lying in open neighborhoods $U_k \times Y$ such that (a) $\bar{U}_{k+1} \subset U_k$, (b) for some open neighborhood V_k of y_{n_k}, the variation of f on $U_{k+1} \times V_k$ is at most $1/k$. Then in $U_k \times V_k$, let $U'_k \times V'_k$ be an open neighborhood of (x_k, y_{n_k}) in which $|f(x, y) - f(x_0, y)| > \epsilon/2$. The sequence of open sets $U'_k \times V'_k$ cannot be locally finite (VII.31); it must have a cluster point $(\bar{x}, \bar{y})$. By continuity, $|f(\bar{x}, \bar{y}) - f(x_0, \bar{y})| \geqq \epsilon/2$. On the other hand, since $U'_k \subset U_k$, $\bar{x} \in \bigcap \bar{U}_k = \bigcap U_k$. Therefore for $y \in V_{k-1}$, $U_k \times V_{k-1}$ contains both (x_0, y) and $(\bar{x}, y)$; so $|f(\bar{x}, y) - f(x_0, y)| < 1/(k-1)$. Since $\bar{y}$ is a cluster point of $\{V'_k\}$, this is a contradiction; and the finite case is proved.

For the infinite case, we observe first that every finite partial product, being a continuous image of a pseudocompact space, is pseudocompact. Then it suffices to show that given a continuous real-valued function f on a pseudocompact product of factors X_α $(\alpha \in A)$, given $\epsilon > 0$, there is a finite subset F of A such that if points x, y have the same αth coordinates for $\alpha \in F$, $|f(x) - f(y)| < \epsilon$. Suppose this is not the case. We construct points p_n, q_n, x_n, y_n, and finite sets F_n, as follows. $|f(p_1) - f(q_1)| \geqq \epsilon$. In basic (product) neighborhoods of p_n and q_n we find points x_n, y_n such that x_n (resp. y_n) has the same αth coordinates as p_n (resp. q_n) for α in $F_1 \cup \cdots \cup F_{n-1}$, $|f(x_n) - f(p_n)| < \epsilon/3 > |f(y_n) - f(q_n)|$, and x_n and y_n have the same αth coordinates for all α excepting a finite set F_n. Then p_{n+1} and q_{n+1} are to have the same coordinates over the finite set $F_1 \cup \cdots \cup F_n$, with $|f(p_{n+1}) - f(q_{n+1})| \geqq \epsilon$. Thus for x_n, y_n, F_n, we have $|f(x_n) - f(y_n)| > \epsilon/3$, x_n and y_n differ only in the coordinates indexed by F_n, and (since p_n and q_n agree up to F_{n+1}) the finite sets F_n are disjoint.

Each x_n and y_n have neighborhoods U_n, V_n such that $|f(x) - f(y)| > \epsilon/4$ for $x \in U_n$, $y \in V_n$; moreover, we may assume U_n and V_n are basic neighborhoods having identical coordinate projections except for the coordinates (in F_n), where x_n and y_n differ. By pseudocompactness, $\{U_n\}$ has a cluster point $\bar{x}$. The fact is, $\{U_n\}$ and $\{V_n\}$ have the same cluster points. To see this let N be a basic neighborhood of $\bar{x}$, projecting onto X_α except for α in the finite set F. Whatever $F \cap \bigcup F_n$ may be, it is contained in some $F_1 \cup \cdots \cup F_k$. After k, U_n meets N if and only if V_n meets N. This shows that every neighborhood of $\bar{x}$ contains points where the values of f differ by $\epsilon/4$, a contradiction that completes the proof.

35. THEOREM. *A product of fine spaces is fine if, for some $\aleph_\alpha$, it is pseudo-$\aleph_\alpha$-compact and m-discrete for all $m < \aleph_\alpha$.*

Proof. The case $\alpha=0$ is the case of the preceding theorem. For $\alpha>0$, the product space is pseudodiscrete and is therefore not an infinite product. So we need only consider the case $X\times Y$.

The fine uniformity has a basis of partitions, by VII.26. Since these partitions are smaller than $\aleph_\alpha$, it will suffice to prove that every continuous function f on $X\times Y$ to a two-point space T is uniformly continuous: for then every open partition is refined by a product of open partitions.

As before, it suffices to show that any $\aleph_\alpha$ functions $f(\ \ ,y)$ are equicontinuous. This is so for the following reasons. The function space $U(Y, T)$ is uniformly discrete; so if the induced function on X to $U(Y,T)$, $x\to f(x,\ \ )$, is actually continuous, it takes fewer than $\aleph_\alpha$ values. The converse is also true; any $m<\aleph_\alpha$ continuous functions on X are equicontinuous, since X is m-discrete. Thus if the functions $f(\ \ ,y)$ are not equicontinuous on X, it is because there are $\aleph_\alpha$ different functions $f(x_\beta,\ \ )$; in that case these functions differ already on some $\aleph_\alpha$ points y_γ, and $\{f(\ \ ,y_\gamma)\}$ is a nonequicontinuous family.

Now note that if $X\times Y$ admits $\aleph_\alpha$(in any uniformity), it is discrete. Otherwise $\aleph_\alpha$ is the least cardinal not admitted by the fine uniformity; so it is regular. It follows that any family of $\aleph_\alpha$ open-closed sets of $X\times Y$ must have a complete cluster point (a point, each of whose neighborhoods meets $\aleph_\alpha$ of the sets). This is clear if the characteristic functions of the sets do not form an equicontinuous family. If they do, these sets are various unions of elements of a single partition of power $<\aleph_\alpha$, and $\aleph_\alpha$ of them must have a common point.

Then suppose (as before) that $\{f(\ \ ,y_\gamma): \gamma<\omega_\alpha\}$ is not equicontinuous at x_0. We can get a cofinal subsequence $\{z_\gamma\}$ (necessarily of power $\aleph_\alpha$) as follows. For points x_γ in a shrinking sequence of open-closed neighborhoods U_γ of x_0, $f(x_\gamma, z_\gamma)\neq f(x_0, z_\gamma)$; but f is constant on $U_{\gamma+1}\times V_\gamma$, for some neighborhood V_γ of z_γ. Pick open-closed neighborhoods $U'_\gamma\times V'_\gamma$ of (x_γ,z_γ), in $U_\gamma\times V_\gamma$, on which $f(x, y)\neq f(x_0, y)$. Consider a complete cluster point $(\bar{x},\bar{y})$ of these sets. By continuity, $f(\bar{x},\bar{y})\neq f(x_0,\bar{y})$. On the other hand, for $y\in V_\gamma$, $U_{\gamma+1}\times V_\gamma$ contains both (x_0,y) and $(\bar{x},y)$; so continuity gives the contradictory conclusion $f(\bar{x},\bar{y})=f(x_0,\bar{y})$, and the proof is complete.

The original Glicksberg theorem on pseudocompact spaces is actually the union of Theorem 34, several preceding results, and an additional construction. For pseudocompact spaces with pseudocompact product, it is the same thing to say the two uniformities (fine and product) coincide as to say their completions coincide. More precisely, for any spaces X_α, the reflection mappings embedding X_α in βX_α embed the product $\prod X_\alpha$ in $\prod\beta X_\alpha$. We write $\beta\prod X_\alpha=\prod\beta X_\alpha$ when this embedding is a normal embedding.

36. Glicksberg theorem. *For any family of infinite uniformizable spaces* X_α, $\beta\prod X_\alpha=\prod\beta X_\alpha$ *if and only if* $\prod X_\alpha$ *is pseudocompact.*

Sufficiency has already been proved. For necessity, a trivial lemma reduces to the case of finite products, which is then settled by a nontrivial lemma.

37. LEMMA. *If* $\beta\prod X_\alpha = \prod\beta X_\alpha$ $(\alpha\in A)$, *then for any subset* A' *of* A, $\beta\prod X_\alpha = \prod\beta X_\alpha$ $(\alpha\in A')$.

38. *If* X *and* Y *are infinite spaces and* $X\times Y$ *is not pseudocompact, then* $X\times Y$ *has an infinite uniformly discrete family of open product sets* $U_n\times V_n$ *such that the sets* U_n *are disjoint and the sets* V_n *are disjoint.*

PROOF. If one factor, say X, is not pseudocompact, then we can take $\{U_n\}$ uniformly discrete, and the V_n need only be disjoint. The sets V_n can be constructed by a recursion in which the complement of $V_1^-\cup\cdots\cup V_k^-$ is always infinite. In case both factors are pseudocompact, begin with a uniformly discrete family $\{U_n^*\times V_n^*\}$. The projections U_n^* must have an infinite union (since Y is pseudocompact). Then recursively we can select points from infinitely many of them and surround these points x_k with neighborhoods $U_k'\subset U_{n_k}^*$, disjoint from the preceding U_j', so that the remaining projections U_n' contain infinitely many points not in $U_1'^-\cup\cdots\cup U_k'^-$. Then a similar construction of V's establishes the lemma.

PROOF OF THEOREM 36 (CONCLUSION). Suppose $\beta\prod X_\alpha = \prod\beta X_\alpha$, where all X_α are infinite. For any one factor X, let Y denote the product of the other factors. $\beta(X\times Y) = \beta X\times\beta Y$ by Lemma 37. If the product is not pseudocompact, apply Lemma 38. For each n, there is a continuous function f_n on $X\times Y$ to I that vanishes outside $U_n\times V_n$ but takes the value 1 at some point (x_n, y_n). Since $\{U_n\times V_n\}$ is uniformly discrete, $\sum f_n = f$ is continuous. It must have a continuous extension g over $\beta X\times\beta Y$. At any cluster point (x^*, y^*) of $\{(x_n, y_n)\}$ in $\beta X\times\beta Y$, g must have the value 1. But (x^*, y^*) is also a cluster point of $\{(x^*, y_n)\}$; and each (x^*, y_n) is a cluster point of the points (x_m, y_n), at almost all of which f vanishes. This is a contradiction, and the proof is complete.

The whole problem can be generalized, for example, as follows.

39. *Let* X *and* Y *be fine spaces. The semi-uniform product* X^*Y *is fine if and only if the topological product* $X\times Y$ *is normally embedded, by the natural embedding, in* $X\times\beta Y$. *This is true if the uniform product is fine, and in other cases; e.g. it is always true if* X *is discrete or* Y *compact.*

The proof is outlined in Exercise 7. Some conditions for a product of pseudocompact spaces to be pseudocompact appear in Exercises 4, 5. Very little is known about the spaces of the "third case" (VII.32(c)).

Supercomplete spaces. Supercomplete spaces can be characterized in terms of the locally fine coreflection λ. It is not clear why this should be, as λ does not preserve the topology of the hyperspace. (In effect, no change in the given space preserves the topology of the hyperspace, by Exercise II.17.) For the proof we replace convergence of sets with another type of convergence that is invariant under λ.

For any partially ordered set P, a function $f: P \to \mu X$ is said to be *convergent* if for every uniform covering $\{U_\alpha\}$ of μX there exists a family of residual sets S_α in P whose union is cofinal in P such that $f(S_\alpha) \subset U_\alpha$ for each α. Here μX may be any quasi-uniform space; since the only application intended is in case $\mu = \nu^{(\beta)}$ for some uniformity ν and ordinal β, we have a uniform topology on X (induced by ν). For $f: P \to \mu X$, then, a point x of X is a *cluster point* of f if for every neighborhood U of x, $f^{-1}(U)$ contains a nonempty residual set.

40. *If $f: P \to \mu X$ is convergent then $f: P \to \lambda\mu X$ is convergent.*

Proof. Suppose not; consider the least ordinal α for which $f: P \to \mu^{(\alpha)} X$ is not convergent. α must be a nonlimit ordinal $\beta + 1$, as otherwise $\mu^{(\alpha)}$ is just the union of its predecessors. But consider any covering $\{U_\gamma \cap V_\delta^\gamma\}$, $\{U_\gamma\}$ and each $\{V_\delta^\gamma\}$ in $\mu^{(\beta)}$. Any $p \in P$ has a successor q all of whose successors are mapped into one set U_γ. For that γ, q has a successor r all of whose successors are mapped into one V_δ^γ. Thus there is a cofinal union of residual sets each mapped into one set $U_\gamma \cap V_\delta^\gamma$, and we have a contradiction.

41. Theorem. *The following conditions on a uniform space μX are equivalent.*

(a) *μX is supercomplete.*

(b) *X is paracompact and $\lambda\mu X$ is fine.*

(c) *Every convergent function into μX has a cluster point.*

Proof. (a)$\Longrightarrow$(b). Condition (b) means that every open covering is λ-uniform. Suppose this is not the case, $\mathscr{U} \notin \lambda\mu$. We construct a net N of points of the hyperspace $H(\mu X)$, ordered by inclusion; N consists of all those closed sets $S \subset X$ such that on some uniform neighborhood of $X - S$ the trace of $\mathscr{U}$ is λ-uniform. Because of the requirement of a uniform neighborhood, N is directed by inclusion; that is, N is a filter base. To show the filter is stable, we must show that for every $\mathscr{V} \in \mu$ there is $S \in N$ such that every $T \subset S$ in N is within $\mathscr{V}$ of S—or within $\mathscr{V}^*$ of S, since $\mathscr{V}^*$ is arbitrarily fine if $\mathscr{V}$ is. For this, let S be the closure of the union of all $V \in \mathscr{V}$ on which $\mathscr{U}$ is not λ-uniform. Then $S \in N$. For, the union of all remaining V in $\mathscr{V}$ is a uniform neighborhood of $X - S$, containing $W = \mathrm{St}(X - S, \mathscr{V})$; if we introduce a uniform neighborhood Z of $X - S$ that is far from $X - W$, then $\mathscr{U}$ is uniformly locally λ-uniform on Z, proving $S \in N$. Next, for any $T \in N$,

$\mathrm{St}(T, \mathscr{V})$ contains at least a dense subset of S. For if x is in a set $V \in \mathscr{V}$ on which $\mathscr{U}$ is not λ-uniform, that V is not contained in $X - T$ (since $T \in N$), and $x \in \mathrm{St}(T, \mathscr{V})$. So $\mathrm{St}(T, \mathscr{V}^*) \supset S$, and N is stable.

However, N is not hyperconvergent. In fact, it has no cluster points, since $\mathscr{U}$ is locally uniform.

(b) $\Longrightarrow$ (c). By Lemma 40, we need only show that a convergent function $f: P \to \lambda\mu X$ has a cluster point. If it did not, X would be covered by open sets whose inverse images contain no nonempty residual set. Considering (b), this contradicts convergence.

(c) $\Longrightarrow$ (a). Suppose $\mathscr{F}$ is a stable filter in μX that is not hyperconvergent. Then the set K of cluster points of $\mathscr{F}$ has a uniform neighborhood U containing no element of $\mathscr{F}$. In fact, we may suppose some set far from U meets every element of $\mathscr{F}$. Then let P be the partially ordered set, ordered by inclusion, of all subsets of $X - U$ that are uniform neighborhoods of sets meeting every element of $\mathscr{F}$. Let $f: P \to \mu X$ be any choice function. Obviously f has no cluster point; but we shall see that f is convergent. Let $\{U_\alpha\}$ be any uniform covering of μX. For any $S \in P$, S is a uniform neighborhood of a set T meeting every element of $\mathscr{F}$. There is a uniform covering $\mathscr{V}$ such that $\mathrm{St}(T, \mathscr{V}^*) \subset S$ and $\mathscr{V}^{**} < \{U_\alpha\}$. In the stable filter $\mathscr{F}$ let A be a member that is contained in every star $\mathrm{St}(B, \mathscr{V})$, $B \in \mathscr{F}$. Let x be a point of $T \cap A$. Some U_α contains $R = \mathrm{St}(x, \mathscr{V}^*) \subset \mathrm{St}(T, \mathscr{V}^*) \subset S$; moreover, since $x \in A$, every B in $\mathscr{F}$ meets $\mathrm{St}(x, \mathscr{V})$. Thus R is a uniform neighborhood of a set meeting all B in $\mathscr{F}$, and R is a successor of S in P, all of whose successors are mapped into U_α by f.

Exercises.

1. STAR-UNIFORM COVERINGS. The fine spaces satisfy a stronger condition than that every uniformly locally uniform covering is uniform; but the condition has not been successfully developed. Call an open covering $\{V_\beta\} = \mathscr{V}$ *star-uniform* if its trace on each set $\mathrm{St}(V_\beta, \mathscr{V})$ is uniform.

(a) A uniformly locally uniform covering has a star-uniform refinement.

(b) In a fine space, a star-uniform star-finite covering is uniform.

(c) If the uniformizable space X has the star-finite property (defined in Exercise VI.5), then a uniformity on X is fine if and only if every star-uniform covering is uniform. In fact, for any uniformity on X, every open covering has a star-uniform refinement. (Hint: Reduce to the locally compact case.)

It is an open question whether a star-uniform covering must be normal. The definition is suggested by consideration of the reciprocals of uniformly continuous nowhere vanishing real-valued functions; but perhaps the definition should be modified. Unlike the definition of "uniformly locally uniform", it involves the specific requirement that the covering is open.

(d) Let W be the space of countable ordinals. In the coarsest uniformity on $W \times I$, every star-uniform covering is uniform.

2. DERIVATIVE UNIFORMITIES. The quasi-uniformities $\mu^{(\alpha)}$ are probably not of much interest; but it would be desirable to free the theory of locally fine spaces from quasi-uniformities. By the same device as in III.28, one can show that $\mu^{(1)}$ is a uniformity if μ is a uniformity having either a countable basis or a basis of σ-disjoint coverings. In the latter case, $\mu^{(1)}$ has the same property; so all $\mu^{(\alpha)}$ are uniformities.

3. (a) Do all the reflections and coreflections in this book generate a finite or an infinite semigroup of functors under composition?

The author does not know the answer, nor whether the problem is difficult. One problem in this semigroup that has been the subject of several unsuccessful attempts: does e (of VII.16) commute with completion on arbitrary nonmeasurable spaces? This is true if the spaces have bases of σ-disjoint coverings; but it seems likely that not every space has such a base, which would leave this question remaining.

(b) c does not commute with completion.

4. NON-PSEUDOCOMPACT PRODUCT. The space βN (cf. Exercise III.10) has two dense countably compact subspaces A, B whose intersection is N. So $A \times B$ contains N as an open-closed subset (a "diagonal") and is not pseudocompact.

5. PSEUDOCOMPACT PRODUCT.

(a) If K is compact and Y is pseudocompact, then $K \times Y$ is pseudocompact. (If $\{U_\alpha \times V_\alpha\}$ is a locally finite family of nonempty open sets, then $\{V_\alpha\}$ is locally finite.)

Consider the class $\mathfrak{P}$ of all spaces X such that $X \times Y$ is pseudocompact for every pseudocompact space Y. (a) shows that $\mathfrak{P}$ contains the compact spaces. Obviously $\mathfrak{P}$ is contained in the pseudocompact spaces.

(b) A pseudocompact space X belongs to $\mathfrak{P}$ if every point of X has a neighborhood belonging to $\mathfrak{P}$.

(c) A pseudocompact space satisfying the first axiom of countability belongs to $\mathfrak{P}$. (If $\{U_n \times V_n\}$ is a locally finite sequence of nonempty open sets, then $\{U_n\}$ is locally finite.)

6. EQUICONTINUITY AND CHARACTER. The proofs of Theorems 34 and 35 suggest some questions about the following cardinal invariant for uniform spaces μX: the smallest cardinal m such that in every function space $U(\mu X, \nu Y)$, each family that is not equiuniformly continuous has a subfamily of power at most m that is not equiuniformly continuous. This might be called the *equicharacter* of μX.

(a) Dense subspaces of μX have the same equicharacter.

(b) Equicharacter cannot exceed *density character* (smallest cardinal of a dense subspace. Use entourages).

(c) A precompact space μX has equicharacter at most $\aleph_0$. (If every $\aleph_0$ members of $F \subset U(\mu X, \nu Y)$ form an equiuniformly continuous family, prove first that the induced function $\mu X \to U(F, \nu Y)$ has precompact image; then that it is uniformly continuous.)

(d) If μX has covering character m and admits all cardinals $n < m$, then the equicharacter of μX is at most m.

It is an open question whether (d) remains true without any assumption on admitting cardinals.

(e) This is true in metric spaces, but in this case the result is strictly weaker than (b).

The method of (c) applies partially, if μX has covering character m and $F \subset U(\mu X, \nu Y)$ is not equiuniformly continuous; one finds a subfamily F' that is not equiuniformly continuous but is the union of at most m subfamilies that are. This will prove the conjectured inequality assuming any of several side conditions.

7. SEMI-UNIFORM PRODUCTS. For fine spaces X, Y,

(a) X^*Y is fine if and only if for every continuous function f on $X \times Y$ to any uniformizable space Z, $\{f(\ \ , y)\}$ is an equicontinuous family. Then for any epimorphism $Y \to Y^-$ such that every continuous mapping from Y to Z factors over Y^-, every continuous mapping from $X \times Y$ to Z factors over $X \times Y^-$.

(b) If $f: X \times Y \to Z$ is continuous and $\{f(\ \ , y)\}$ is not equicontinuous, there is a continuous function $g: Z \to I$ such that $\{gf(\ \ , y)\}$ is not equicontinuous.

(c) Conclude the proof of Proposition 39.

8. SUPERCOMPLETENESS IN PRODUCTS. For a product of fine spaces X_α, is $\lambda \prod X_\alpha$ fine? This question is of interest in at least two cases: the paracompact case, which is the supercomplete case, and the case of uncountably many complete metric spaces. Complete metric spaces are usually not isomorphic with fine spaces, but that does not matter; for

(a) $\lambda \prod (\lambda X_\alpha) = \lambda \prod X_\alpha$.

If Corollary 21 generalizes to a larger class of products of complete metric spaces (or perhaps injective metric spaces), so does Corollary 22. One would have to use a different method; for

(b) Both Theorem 19 and Corollary 24 fail for a product of uncountably many metric spaces containing two or more points each, unless all factor spaces are separable.

Note also that even Corollary 21 fails for arbitrary fine spaces—by Exercise 4. Let us change the subject.

(c) The product of a supercomplete space and a compact space is supercomplete.

(d) Let $\mu X, \nu Y$ be fine spaces topologically embedded in compact metric spaces A, B. Then the open λ-uniform coverings of $\mu X \times \nu Y$ are those which can be extended to open coverings of $M \times N$, for some G_δ sets $M \supset X$ in A,

$N \supset Y$ in B. (Use Exercise VI.9.)

(e) A fine separable metrizable space X has the property that $X \times Y$ is supercomplete for every fine separable metrizable space Y if and only if X is locally compact.

Research Problem B_3.

WHAT COVERINGS SUFFICE? The primary problem here is whether Stone's theorem generalizes to arbitrary uniform spaces. That is, (1) does every space have a basis of uniformly locally finite coverings? (2) Does every space have a basis of σ-uniformly discrete coverings? One may conjecture that neither is true, that no infinite-dimensional Banach space has a basis of uniformly locally finite coverings, and that no nonseparable Banach space has a basis of σ-uniformly discrete coverings. But what is known falls far short of settling either question. The locally finite coverings constructed by Stone [1] in metrizable spaces, and earlier by Dieudonné [1] in the separable case, are certainly not often uniform with respect to the metric used for the construction. On the other hand, we have no global properties of locally finite coverings to use in proving their nonexistence. In topology, such coverings give continuous mappings into polyhedra; but not here. For question (2), the generalization would imply generalizations of III.25 (injective function spaces) and VII.15 (Cauchy filters) that can perhaps be shown to be false. Note, though, that results like these certainly need not require a whole basis of coverings of the appropriate special form; consider the statement of VII.15, which refers to countable coverings, or consider the use of binary partitions in VII.35. So our heading "What coverings suffice?" might be reinterpreted, even to cover the whole problem of what can be done with special coverings. But that is not a problem one may propose for "solution". The specific problem (1) is. Moreover, its solution will probably be important even if locally finite uniform coverings are not important (cf. Research Problem B_2), because the arguments used to solve it will certainly tell us more about the elementary spaces. This is so because, if every ell-infinity space has a basis of uniformly locally finite uniform coverings, so does every uniform space.

The problem has no literature. However, it may be related to the problem, given a basis $\mathscr{B}$ of open sets in a paracompact space, when does every open covering have a locally finite refinement consisting of elements of $\mathscr{B}$? On that, see Corson [2], and the abstract Corson-McMinn-Michael-Nagata [1].

The question (1) was first raised in Stone [2]. There is a rather indirect connection between this question and the results of that paper.

Notes. For the construction of a relative model for set theory in which there are no measurable cardinals, see Shepherdson [1]. Recent results of Hanf, Tarski, and Scott (see [NST] and Scott [1]) greatly extend Ulam's theorem. Tarski extends it directly. The first cardinal which is not excluded by the theorem of Ulam (let us say, Theorem 0) is excluded by a similar Theorem 1 of Tarski; these theorems continue until all cardinals shown to be nonmeasurable have been used for numbering theorems, and past that point. Scott's result implies that if m is a cardinal in some model for set theory (with the axiom of choice), there is another model containing a cardinal "as large as" m which is not measurable.

It should be noted that Shirota's theorem cannot be disassociated from the Problem of Ulam; in a set theory in which a measurable cardinal m exists, Shirota's theorem fails for a discrete space of power m.

On products of separable spaces, Marczewski showed in [**1**] that they have no uncountable disjoint collection of open subsets and in [**2**] that they have a stronger countable covering property. Šanin [1; 2] investigated related topological properties very extensively; some lying between separability and absence of uncountable disjoint collections of open sets are preserved under arbitrary products. Meanwhile Mazur [**1**] gave the first factorization theorem like Theorem 19, a remarkable result related to Ulam's theorem in which the given mapping is assumed only to be relatively continuous on every convergent sequence in X and turns out to be continuous because of the factorization. The next significant step was Bockstein's [**1**] discovery of the separation theorem VII.24 for products of separable metrizable spaces. Much of VII.19—VII.23, and much more, can be deduced from Bockstein's theorem; this is a discovery of H. H. Corson, see [**1**], [**3**], and Corson-Isbell [**1**].

Note that the (so far) maximal Gleason Theorem 23 contains Marczewski's original result as well as Bockstein's separation theorem. However, the maximal results of Šanin [**2**], Mazur [**1**], Corson [**1**] go off in three different directions. A systematic investigation in the spirit of Šanin [**2**] would now be a much more involved and more interesting piece of work.

Onuchic [1; 2] investigated case (c) of Proposition 32 in considerable detail for a product $X \times X$ with $\alpha = 1$.

Exercises 2 and 3(b) are from Ginsburg-Isbell [**1**], with the question whether e commutes with completion. A stronger conjecture was raised earlier by Henriksen [**B**, p. 130], involving a uniformity (shall we say $h\mu$?) between $e\mu$ and $c\mu$.

Exercise 4 is from Novák [**1**]. Exercise 5 is mainly from Frolík [**1**]. The results (a) and (c) were known earlier. See Bagley-Connell-McKnight [**1**], and see Glicksberg [**2**], where weaker sufficient conditions are given and there are some results on infinite products. Frolík [**1**] also has further results. Exercises 6(c) and 8(b) are due to Corson.

Finally, in connection with quasi-uniformities, some work has been done in which the star-refinement axiom is replaced by some weaker axiom of similar tendency. See in particular Morita [**3**]. Morita's *regular* (quasi-) *uniformities* (associated with regular topologies) have considerable structure and are closed under taking derivatives.

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CHAPTER VIII

DIMENSION (2)

This chapter concerns special classes of spaces in which such theorems hold as $\Delta d = \delta d$ and dim = ind.

Essential coverings. A uniform covering $\mathscr{U}$ is called *essential* if every uniform refinement of $\mathscr{U}$ is a coshrinking.

1. *If $\mathscr{U}$ is an essential uniform covering then for any uniform refinement $\mathscr{V}$ of $\mathscr{U}$, every connecting mapping from the nerve of $\mathscr{V}$ to the nerve of $\mathscr{U}$ is onto.*

PROOF. First, if $\mathscr{U} = \{U_\alpha\}$ is essential, then for each α, $U_\beta \subset U_\alpha$ implies $\beta = \alpha$. Otherwise (for fixed α) consider $\mathscr{U}'$, the set of all U_β such that $U_\beta \not\subset U_\alpha$ or $\beta = \alpha$. $\mathscr{U} < \mathscr{U}'$; so $\mathscr{U}'$ is uniform. $\mathscr{U}' < \mathscr{U}$; but $\mathscr{U}'$ is not a coshrinking.

Then suppose $\{U_\alpha\}$ is essential but $\mathscr{V}$ is a uniform refinement of $\{U_\alpha\}$ with a nononto connecting mapping ϕ. Factoring ϕ, we get a shrinking $\{W_\alpha\}$ of $\{U_\alpha\}$ such that for some set of indices $\{0, \cdots, n\}$, $U_0 \cap \cdots \cap U_n \neq 0 = W_0 \cap \cdots \cap W_n$. Define $Z_\alpha = W_\alpha$ $(\alpha \leqq n)$, $Z_\alpha = U_\alpha (\alpha > n)$. Then $\{W_\alpha\} < \{Z_\alpha\} < \{U_\alpha\}$. A connecting mapping ψ from the nerve of $\{Z_\alpha\}$ to the nerve of $\{U_\alpha\}$ must take Z_α to U_α for $\alpha > n$, since no other U_β contains Z_α. Whatever ψ does on $Z_0, \cdots, Z_n$, it takes no simplex onto the simplex spanned by $U_0, \cdots, U_n$. This contradicts the assumption of essentiality.

Essentiality is generalized and specialized as follows. A space μX is *essential* if it has a basis of essential uniform coverings. A finite subset F of a uniform covering $\mathscr{U}$ of μX is *essential in* $\mathscr{U}$ if it spans a simplex σ in the nerve of $\mathscr{U}$ and, whenever $\mathscr{V}$ is a uniform refinement of $\mathscr{U}$ and ϕ a connecting mapping, ϕ maps some simplex of the nerve of $\mathscr{V}$ onto σ. In view of Proposition 1, $\mathscr{U}$ is essential if and only if every subset spanning a simplex is essential.

2. *$\delta d \mu X$ is the supremum of the dimensions of essential uniform coverings of μX, or equivalently, the dimensions of essential subsets of uniform coverings.*

PROOF. Suppose $\delta d \mu X \geqq n$. Then there is an essential mapping f of μX to a closed n-simplex I^n. Let $\mathscr{V}$ be the covering of I^n consisting of the stars of the vertices. Then $f^{-1}(\mathscr{V})$ is essential. Suppose it were not; then $f^{-1}(\mathscr{V})=\{U_0,\cdots,U_n\}$ would have a uniform shrinking $\{W_0,\cdots,W_n\}$ with $W_0\cap\cdots\cap W_n=0$. Identifying the nerve of $\{W_0,\cdots,W_n\}$ with its image in I^n (a subcomplex of the boundary S^{n-1}), and taking a canonical mapping, we get $g:\mu X\to S^{n-1}$. On $f^{-1}(S^{n-1})$, g need not coincide with f, but it is homotopic, since every member of $\mathscr{V}$ containing $f(x)$ contains $g(x)$. This contradicts the assumed essentiality of f.

Suppose $\mathscr{U}=\{U_\alpha\}$ is a uniform covering of μX in which $\{U_0,\cdots,U_n\}$ is essential. Let $\mathscr{V}=\{V_\alpha\}$ be a uniform strict shrinking of $\mathscr{U}$. Let $V^*=\bigcup[V_\alpha:\alpha>n]$. Then $\{V^*,V_0,\cdots,V_n\}$ is a finite uniform covering. Suppose it had a uniform shrinking $\{W^*,W_0,\cdots,W_n\}$ of dimension $<n$. Define $Y_\alpha=W_\alpha$ for $\alpha\leqq n$, $Y_\alpha=U_\alpha$ for $\alpha>n$. It remains to check that $\{Y_\alpha\}$ is a uniform covering; since $Y_0\cap\cdots\cap Y_n=0$, this will be a contradiction of the essentiality of $\{U_0,\cdots,U_n\}$. Now with respect to some uniformly continuous pseudometric, $\{W^*,W_0,\cdots,W_n\}$ has a Lebesgue number 2ϵ and the ϵ-neighborhood of each V_α is contained in U_α. Then consider any point x of X. It is ϵ inside some of $W_0,\cdots,W_n$, in $\{Y_\alpha\}$, or ϵ inside W^*. In the latter case it belongs to V^*, hence to some V_α $(\alpha>n)$, and it is ϵ inside U_α. Thus $\{Y_\alpha\}$ has Lebesgue number ϵ. The contradiction proves $\delta d\mu X\geqq n$.

3. THEOREM. *For an essential space μX, $\Delta d\mu X=\delta d\mu X$.*

PROOF. If $\delta d\mu X\leqq n$, then every essential uniform covering has dimension at most n by Proposition 2; since these form a basis, $\Delta d\mu X\leqq n$. In view of V.5, this completes the proof.

The principal application of Theorem 3 is given by the following theorem.

4. THEOREM. *Every locally fine space is essential.*

PROOF. Let $\{U_0,\cdots,U_n\}$ be an inessential subset of a uniform covering $\{U_\alpha\}$ of any space μX. Then there is a uniform shrinking $\{V_\alpha\}$ such that $V_0\cap\cdots\cap V_n=0$. Put $W_\alpha=V_\alpha\cup[U_\alpha-(U_0\cap\cdots\cap U_n)]$. Then $W_0\cap\cdots\cap W_n=0$, and $\{W_\alpha\}$ coincides with $\{U_\alpha\}$ except on $U_0\cap\cdots\cap U_n$.

If $\mathscr{U}$ is a uniformly locally finite uniform covering, then the collection of all intersections M_F of finite subsets F of $\mathscr{U}$ is also uniformly locally finite. Suppose the space is locally fine. Write $\mathscr{U}=\{U_\alpha:\alpha\in A\}$. Well-order the set of all finite subsets F_γ of A that index subsets of $\mathscr{U}$ which have non-empty intersection M_γ but are inessential. Let $\mathscr{V}^1=\{V_\alpha^1\}$ be a uniform shrinking of $\mathscr{U}$, coinciding with $\mathscr{U}$ except on M_0, such that $\{V_\alpha^1:\alpha\in F_0\}$ has empty intersection. Recursively, having $\mathscr{V}^\gamma$, let $\mathscr{V}^{\gamma+1}$ be a uniform shrinking of $\mathscr{V}^\gamma$, coinciding with $\mathscr{V}^\gamma$ except on M_γ, such that $\{V_\alpha^{\gamma+1}:$

$\alpha \in F_\gamma\}$ has empty intersection—if possible; it may be that F_γ indexes an essential set in $\mathscr{V}^\gamma$, in which case we put $\mathscr{V}^{\gamma+1} = \mathscr{V}^\gamma$. For a limit ordinal β, $\mathscr{V}^\beta$ is defined by $V^\beta_\alpha = \bigcap [V^\gamma_\alpha : \gamma < \beta]$. Uniformly locally, there are only finitely many different coverings $\mathscr{V}^\gamma$; so $\mathscr{V}^\beta$ coincides with one of them. Thus the recursion runs and gives a uniform shrinking $\{V_\alpha\}$ of $\mathscr{U}$. For any finite subset of $\mathscr{V}$ spanning a simplex σ of the nerve, the natural embedding of the nerve of $\mathscr{V}$ in the nerve of $\mathscr{U}$ or in the nerve of some $\mathscr{V}^\gamma$ takes σ (and no other simplex) to an essential simplex. So $\mathscr{V}$ is essential.

It is an open question whether locally fine coreflection λ can ever raise dimension, either δd or Δd. From VI.23 we have

5. *If μX is a supercomplete space of covering character $\aleph_1$, then $\Delta d\lambda\mu X = \delta d\lambda\mu X \leqq \delta d\mu X$.*

Proof. $\lambda\mu X$ is fine by VII.41; so $\Delta d\lambda\mu X = \dim X$. Since $\lambda\mu X$ has covering character $\aleph_1$ (VII.16), X is pseudo-$\aleph_1$-compact; being paracompact, it has the Lindelöf property. Then VI.23 gives $\dim X \leqq \delta d\mu X$, as required.

6. *If μX has covering character $\aleph_1$, then $\Delta d\lambda\mu X \leqq \Delta d\mu X$.*

Proof. Suppose $\Delta d\mu X \leqq n$. By VII.22, $\lambda\mu X$ is embeddable in a fine space $\lambda\nu\Pi$, where $\nu\Pi$ is a product of complete separable metric spaces and contains μX. Now any uniform covering of $\lambda\mu X$ is realized by a mapping to a metric space J, which we may take to be injective. Thus the mapping extends to $f : \lambda\nu\Pi \rightarrow J$. By VII.19, f factors as $g\pi$, where $\pi : \lambda\nu\Pi \rightarrow \lambda\sigma P$ is a countable projection. Thus π is uniformly continuous on $\nu\Pi$ to σP, and its restriction $\pi | X$ is a uniformly continuous mapping q from μX to a complete metric space. If $\{\mathscr{U}^m\}$ is a basis for σP, $\{q^{-1}(\mathscr{U}^m)\}$ is refined by a normal sequence of (at most) n-dimensional countable uniform coverings of μX; so q factors as sr, where r is a uniformly continuous mapping from μX to a separable metric space ρY with $\Delta d\rho Y < n$. We may suppose ρY is complete, so that $\lambda\rho Y$ is fine. By VI.23, $\dim Y \leqq n$. Then the given mapping $f | X$ is gsr, $\lambda\mu X \xrightarrow{r} \lambda\rho Y \xrightarrow{s} \lambda\sigma P \xrightarrow{g} J$. The coverings realized by gsr are realized by r in an n-dimensional space. This completes the proof.

Sum and subset. The basic sum theorem for topological dimension is a theorem in normal spaces. We shall need finite collections of open sets which are strict shrinkings of collections of dimension at most n. Note that, since disjoint closed sets are far, this means just that the closures form an n-dimensional collection (at most n). Let us call such a finite collection of open sets in a normal space *strictly n-dimensional*.

7. **Theorem.** *If the normal space X is the union of a countable family of*

closed subspaces X_i, *and* $\dim X_i \leqq n$ *for each* i, *then* $\dim X \leqq n$.

PROOF. Let $\{U_j: j \leqq m\}$ be a finite open covering of X. To begin a recursion, let $V_j^0 = 0$ for $j \leqq m$. Having a strictly n-dimensional collection of open sets V_j^i covering at least $X_1, \cdots, X_i$, with each $(V_j^i)^- \subset U_j$, insert open sets W_j so that $(V_j^i)^- \subset W_j$, $W_j^- \subset U_j$, and $\{W_j\}$ is strictly n-dimensional. Let $\{Z_k\}$ be a finite open covering of a neighborhood of X_{i+1} with $\{Z_k^-\}$ n-dimensional and finer than $\{U_j\}$, and so that if Z_k^- meets $(V_j^i)^-$ then $Z_k \subset W_j$. To each Z_k we associate an index $j = \lambda(k)$ such that $Z_k^- \subset U_j$ and (if there is such a j) Z_k^- meets $(V_j^i)^-$. Then V_j^{i+1} is the union of V_j^i and all Z_k such that $j = \lambda(k)$.

Evidently $\{V_j^{i+1}: j \leqq m\}$ covers $X_1, \cdots, X_{i+1}$, and the closure of V_j^{i+1} is contained in U_j. Consider any point p in some $(V_j^i)^-$. For any V_t^{i+1} whose closure contains p, W_t^- also contains p. This is clear if p is in $(V_t^i)^-$; otherwise p is in some Z_s^-, $\lambda(s) = t$, and since Z_s^- meets $(V_j^i)^-$ it also meets $(V_t^i)^-$, which proves the assertion. Hence such a p cannot be in more than $n+1$ sets $(V_t^{i+1})^-$; and for the remaining points p, they are in no more different $(V_j^i)^-$ than they are in Z_k^-. Consequently the recursion runs, and the sets $V_j = \bigcup_i V_j^i$ form an n-dimensional open shrinking of $\{U_j\}$.

A *perfectly normal* space is a normal space in which every open set is an F_σ. We have

8. *Every subspace of a perfectly normal space is perfectly normal.*

9. *If A is a subspace of a perfectly normal space X,* $\dim A \leqq \dim X$.

PROOF OF PROPOSITION 8. Obviously every relatively open set in a subspace is a relative F_σ. We have to show that two disjoint relatively closed sets B and C in the subspace A of X lie in disjoint open sets. Consider the open subspace U of X which is the complement of $B^- \cap C^-$. U contains A; and it will suffice to show that $B^- \cap U$ and $C^- \cap U$ lie in disjoint open sets. This does not depend on the fact that U is open; the proposition is

An F_σ subspace of a normal space is normal.

Let U be the union of closed sets F_i; put $B_i = B^- \cap F_i$, $C_i = C^- \cap F_i$. B_1 has an open neighborhood M_1 whose closure does not meet C^-. Having an open set M_k whose closure does not meet $C^- \cup N_{k-1}^-$, let N_k be the union of N_{k-1} with an open neighborhood of C_k whose closure does not meet $B^- \cup M_k^-$. Let M_{k+1} be the union of M_k with an open neighborhood of B_{k+1} whose closure does not meet $C^- \cup N_k^-$. Then M_{k+1} also satisfies this condition, and the recursion runs.

PROOF OF PROPOSITION 9. A finite open covering of A is the trace on A of a finite open covering of an open subspace U. U is an F_σ, and by VI.6 and VIII.7, $\dim U \leqq \dim X$. So $\dim A \leqq \dim X$.

It is an unsolved problem whether Proposition 9 generalizes to all *completely normal* spaces (spaces in which every subspace is normal). From VI.13, it is not enough to assume that the space X and the subspace A are normal.

For any further substantial results, we shall have to assume normality or (usually) much more. Note that (since Δd coincides with the proximity invariant δd for fine spaces) inductive uniform arguments are applicable to closed subspaces of fine normal spaces, since they are normally embedded and thus δ-isomorphic with fine spaces. We apply this remark for $\delta\mathrm{Ind}$ in the next proof.

10. *In a fine uniform space X, if a closed set A separates two far sets B, C, and is far from them, then A frees B and C. If X is normal,* $\mathrm{Ind}\, X \geqq \dim X$.

PROOF. Given that $X-A$ is a topological sum $U \cup V$, with $B \subset U$, $C \subset V$. For any uniform neighborhood N of A disjoint from $B \cup C$, consider $U-N \supset B$ and $V-N \supset C$. There is a continuous real-valued function f that is 0 on A and 1 outside N. Define g to be f on $A \cup U$ and $-f$ on V. Then g is continuous and separates $U-N$ from $V-N$; so A frees B and C.

To finish, it will suffice (in view of V.23) to prove that if $\mathrm{Ind}\, X \leqq n$ then $\delta\mathrm{Ind}\, X \leqq n$; and we may suppose this has been done for $n-1$. Consider two far sets B, C. They have far uniform neighborhoods D, E, which are separated by a closed set A with $\mathrm{Ind}\, A \leqq n-1$. As A is far from B and C, it frees them, and the proof is complete.

11. We want an analog of VI.28. In a completely normal space X, two sets A and B are separated (have disjoint neighborhoods) if neither has a limit point in the other; for they lie in disjoint closed sets in the open normal subspace $X-(A^- \cap B^-)$. In particular, *if U is a relatively open set in a subspace S, U can be extended to an open set V such that $V^- \cap S = U^- \cap S$.*

We want some notation. $\mathrm{Fr}(A)$ will denote the boundary $A^- \cap (X-A)^-$. In case several spaces are involved we write $\mathrm{Fr}(A, X)$ for the boundary of A in X. $\mathrm{Ind}(A, X)$, A a closed subspace of X, will be the least n such that every set far from A can be separated from A by a closed set W with $\mathrm{Ind} W \leqq n-1$. X being normal, $\mathrm{Ind}(A, X) \leqq n$ if and only if every neighborhood U of A in X contains a neighborhood V such that $\mathrm{Ind}\, \mathrm{Fr}(V, X) \leqq n-1$.

12. THEOREM. *If the perfectly normal space X is the union of a countable family of closed subspaces X_i, and* $\mathrm{Ind} X_i \leqq n$ *for each i, then* $\mathrm{Ind} X \leqq n$.

PROOF. This is true for $n = -1$. Suppose it has been established for $n-1$.

(1) Let S be a closed subspace, and A, B^*, two closed subsets of S separated (in S) by an $(n-1)$-dimensional closed set W^*. Suppose further that for each closed subset F of X disjoint from S, $\operatorname{Ind}(F, X) \leqq n$. Let B be any closed subset of X such that $B \cap S = B^*$. Then A and B can be separated by a closed subset W with $\operatorname{Ind} W \leqq n-1$.

To prove this, let $S - W^*$ break into summands $P \supset A$, $Q \supset B^*$. Since W^* is closed, P and Q are open in S. Further P^- and B are disjoint closed sets, and thus can be separated by open sets $U \supset P^-$, $T \supset B$. There is an open set $V \subset U$, $V \cap S = P$, $V^- \cap S = P^-$. Then $\operatorname{Fr}(V, X) \cap S = \operatorname{Fr}(P, S) \subset W^*$.

Let $K = \operatorname{Fr}(V, X)$. Write K as the intersection of a shrinking sequence of open sets E_i. Further, $X - S$, and hence $K - S$, is the union of a sequence of closed sets F_j. Since F_j is disjoint from S, $\operatorname{Ind}(F_j, X) \leqq n$. Select a neighborhood N of K with N^- disjoint from B, and let W_j be a neighborhood of F_j contained in $E_j \cap N - S$, with $\operatorname{Ind} \operatorname{Fr}(W_j) \leqq n-1$.

Let R be the union of V and all the W_j. A boundary point of R which is not in the boundary of any W_j must certainly be in $K = \operatorname{Fr}(V)$, for near every point not in K the collection $\{W_j\}$ is locally finite. But $K - S$ is interior to R. Hence $W = \operatorname{Fr}(R)$ is contained in the union of W^* and the $\operatorname{Fr}(W_j)$, so that $\operatorname{Ind} W \leqq n-1$ by induction; and clearly W separates A and B.

(2) If X is a finite union of closed subspaces with $\operatorname{Ind} \leqq n$, then $\operatorname{Ind} X \leqq n$.

For (2), we need only treat the case $X = S \cup T$. Let A be a closed set and U a neighborhood. By (1), $A \cap S$ and $X - U$ can be separated by an $(n-1)$-dimensional set, so that there is a neighborhood V_1 of $A \cap S$ contained in U, with $\operatorname{Ind} \operatorname{Fr}(V_1) \leqq n-1$. Applying the same argument to $A \cap T$, and the inductive hypothesis again, we have it.

(3) For the infinite union, we may now suppose $X_{i+1} \supset X_i$. Let A be a closed set and Z a neighborhood. To begin, insert an expanding sequence of open sets Z_j, $A \subset Z_j \subset Z_j^- \subset Z_{j+1} \subset Z$. We shall construct three more sequences of open sets, U_j, V_j, T_j, so that the following conditions are satisfied:

$$A \cap X_j \subset U_j \subset Z_j \tag{a_j}$$

$$U_{j-1} \subset U_j \tag{b_j}$$

$$\operatorname{Ind}(X_j \cap \operatorname{Fr}(U_j)) \leqq n-1 \tag{c_j}$$

$$U_j - U_{j-1} \subset V_{j-1} \tag{d_j}$$

$$\operatorname{Fr}(U_j) - X_{j-1} \subset V_{j-1} \tag{e_j}$$

$$\operatorname{Fr}(U_{j-1}) - X_{j-1} \subset V_{j-1} \tag{f_j}$$

$$A \subset V_{j-1} \tag{g_j}$$

$$X_{j-1} - U_{j-1}^- \subset T_{j-1} \tag{h_j}$$

$$T_{j-2} \subset T_{j-1} \tag{i_j}$$

$$T_{j-1} \cap V_{j-1} = 0. \tag{j_j}$$

To begin, let X_0, T_0 and T_{-1} be the empty set, $V_0 = X$. We shall construct U_1 so that (a_1)-(j_1) are satisfied. Then recursively, assuming U_k, V_{k-1}, T_{k-1} and the preceding sets satisfy (a_k)-(j_k), we shall construct V_k and T_k so as to satisfy (f_{k+1})-(j_{k+1}) and then U_{k+1} so as to satisfy (a_{k+1})-(e_{k+1}).

Since $\operatorname{Ind} X_1 \leqq n$, $A < X_1$ has a relative neighborhood in X_1, contained in $Z_1 \cap X_1$, whose X_1-boundary has Ind at most $n-1$. Since X is hereditarily normal, this neighborhood can be extended to an open set U^* of X without picking up any new limit points in X_1. Then (a_1)-(j_1) are satisfied by $U_1 = U^* \cap Z_1$, $U_0 = U_1$.

Having gone as far as U_k, we check that $A \cup [\operatorname{Fr}(U_k) - X_k]$ and $T_{k-1} \cup (X_k - U_k^-)$ are separated. A is separated from T_{k-1} by (g_k) and (j_k); and from $X_k - U_k^-$ by (a_k); $\operatorname{Fr}(U_k) - X_k$ is separated from T_{k-1} by (e_k) (recall $X_{k-1} \subset X_k$) and (j_k), and from $X_k - U_k^-$, as one sees at a glance. Consequently these two sets are contained in disjoint open sets V_k, T_k, i.e., (f_{k+1})-(j_{k+1}) can be satisfied.

Next, the closed set $[A \cup \operatorname{Fr}(U_k)] \cap X_{k+1}$ is the intersection of a shrinking sequence of open sets E_i, and there is a sequence of closed sets F_j whose union is $[A \cup \operatorname{Fr}(U_k)] \cap (X_{k+1} - X_k)$. By (f_{k+1}) and (g_{k+1}), each F_j is contained in V_k; so there are open sets $P_j \supset F_j$, $P_j^- \subset V_k$. Then F_j is contained in $(P_j \cap E_j \cap Z_{k+1}) - X_k$ (in Z_{k+1} because that set contains A and $Z_k^- \supset U_k^-$). Since $\operatorname{Ind} X_{k+1} \leqq n$, there are open sets W_j containing F_j, contained in $(P_j \cap E_j \cap Z_{k+1}) - X_k$, with $\operatorname{Ind}(\operatorname{Fr}(W_j) \cap X_{k+1}) \leqq n-1$. We define U_{k+1} as U_k together with $\bigcup_j W_j$.

Then (a_{k+1}) and (b_{k+1}) are satisfied by construction, and so is (d_{k+1}). As in part (1) of the proof, a boundary point of U_{k+1} which is outside $\bigcap_i E_i$ must be in $\bigcup_j \operatorname{Fr}(W_j)$ or in $\operatorname{Fr}(U_k)$. If it is not in X_k, then in view of (f_{k+1}) it is in V_k. Further, a boundary point of U_{k+1} in $\bigcap_i E_i$ cannot be in $\bigcup_j F_j \subset U_{k+1}$. Then all such points are in X_k, which establishes (e_{k+1}). Further, these points are not in A, by (a_k). Hence $X_{k+1} \cap \operatorname{Fr}(U_{k+1})$ is contained in the union of $X_k \cap \operatorname{Fr}(U_k)$ and all $X_{k+1} \cap \operatorname{Fr}(W_j)$; so (c_{k+1}) follows, and the recursion runs.

Finally we define U as the union of all U_j. By (a_j), U contains A and is contained in Z. We shall check that its boundary is contained in the union of the sets $X_k \cap \operatorname{Fr}(U_k)$; by the inductive hypothesis, this will finish the proof.

For each k, $U - U_k$ is contained in the union of the V_j for $j = k, k+1, \cdots$, by (d_j). T_k is disjoint from that union, by (i_j) and (j_j). Hence $\operatorname{Fr}(U) \cap T_k \subset U_k^-$. Then by (h_{k+1}), $\operatorname{Fr}(U)$ is disjoint from $X_k - U_k^-$, so that $X_k \cap \operatorname{Fr}(U)$ is contained in U_k^-. It must be contained in $\operatorname{Fr}(U_k)$, since $U_k \subset U$; and the proof is complete.

13. Corollary. *If A is a subspace of a perfectly normal space X,* $\operatorname{Ind} A \leqq \operatorname{Ind} X$.

Proof. Disjoint closed sets B, C of A lie in disjoint closed sets of the open F_σ subspace $X - (B^- \cap C^-)$; so the result follows from Theorem 12 by induction.

Coincidence theorems.

14. Lemma. *In a metrizable space X, any basis for the fine uniformity contains a basis for a metric uniformity compatible with the topology.*

Proof. Let $\{\mathscr{U}^n\}$ be a basis for some metric uniformity on X, and $\mathscr{B}$ a basis for the fine uniformity. Recursively construct a normal sequence $\{\mathscr{V}^n\}$ taken from $\mathscr{B}$ so that $\mathscr{V}^n < \mathscr{U}^n$.

15. Theorem. *For a metrizable space X, $\dim X = \operatorname{Ind} X$, and this is also the minimum of $\Delta d_\mu X$ for all compatible metric uniformities μ on X.*

Proof. Choose a metric uniformity μ minimizing $\Delta d_\mu X$. By the lemma, with Theorems 3 and 4, $\Delta d_\mu X \leqq \dim X$. By Proposition 10 $\dim X \leqq \operatorname{Ind} X$. It remains to prove $\operatorname{Ind} X \leqq \Delta d_\mu X$; or, when $\Delta d_\mu X \leqq n$, then $\operatorname{Ind} X \leqq n$.

Take a normal sequence of n-dimensional coverings $\mathscr{U}_i$ forming a basis for μ. Let E and F be disjoint closed sets. Recursively we split each $\mathscr{U}_i$, and the space X, into three parts; at the end, one of the parts of the space will separate E and F and parts of the coverings $\mathscr{U}_i$ will define some $(n-1)$-dimensional metric uniformities which we can use, assuming the present theorem established for $n-1$.

Let $M_0 = N_0 = 0$. Having M_{i-1} and N_{i-1}, let $\mathscr{G}_i$ be the family of all elements of $\mathscr{U}_i$ whose stars (with respect to $\mathscr{U}_i$) do not meet $F \cup N_{i-1}$; $\mathscr{H}_i$ the set of all elements of $\mathscr{U}_i$ whose stars meet $F \cup N_{i-1}$ but do not meet $E \cup M_{i-1}$; $\mathscr{J}_i$ the remaining elements whose stars meet both sets. Let G_i be the union of $\mathscr{G}_i$, H_i the union of $\mathscr{H}_i$, J_i the union of $\mathscr{J}_i$. Let $M_i = X - H_i - J_i$, $N_i = X - G_i - J_i$. Clearly $M_i \subset G_i$, so that M_i and N_i are disjoint.

For any $U \in \mathscr{U}_{i+1}$ at least one V in $\mathscr{U}_i$ contains U^*. We check that

(1) if $V \in \mathscr{G}_i$ then $U^* \cap (F \cup N_i) = 0$,

(2) if $V \in \mathscr{H}_i$ then $U^* \cap (E \cup M_i) = 0$,

(3) if $V \in \mathscr{J}_i$ then $U^* \cap (M_i \cup N_i) = 0$.

For (1), U^* is contained in V, which is disjoint from F; also $V \subset G_i \subset X - N_i$. In (2), V is disjoint from E and contained in $H_i \subset X - M_i$; for (3), both M_i and N_i are disjoint from J_i, which contains V.

From (1) and (3), if U^* meets N_i then V is in $\mathscr{H}_i$; by (2) then, $U^* \cap (E \cup M_i) = 0$. Similarly if U^* meets M_i then $U^* \cap (F \cup N_i) = 0$.

(4) If $U \in \mathscr{J}_{i+1}$ then $U^* \cap (M_i \cup N_i) = 0$, but $U^* \cap E \neq 0$ and $U^* \cap F \neq 0$.

From (4), $\operatorname{St}(J_{i+1}, \mathscr{U}_{i+1})$ is disjoint from $M_i \cup N_i$. By construction G_{i+1} is far from N_i and H_{i+1} is far from M_i. It follows that M_i is interior to M_{i+1} $(= X - H_{i+1} - J_{i+1})$ and N_i is interior to N_{i+1}.

Let M be the union of all M_i, N the union of all N_i. Since the sequences are expanding, M and N are open; since each M_i and N_i are disjoint, M and N are disjoint. Moreover, M contains E and N contains F. For example, for $x \in E$, for sufficiently large i, any U in $\mathscr{U}_i$ which contains x has $U^* \cap F = 0$. Since U^* meets E, U is not in $\mathscr{H}_i$; by (4), U is not in $\mathscr{J}_i$. Thus x is in

$X-H_i-J_i=M_i$. Similarly, each point of F is in some N_i.

Let $W=X-M-N$; we wish to show $\operatorname{Ind} W \leqq n-1$. Consider $W_i=W-J_i$. From (4), every member of $\mathscr{U}_i$ containing a member of $\mathscr{J}_{i+1}$ is a member of $\mathscr{J}_i$; so $J_{i+1} \subset J_i$. Since the members of $\mathscr{J}_i$ are small sets meeting both E and F, the intersection of all J_i is empty and W is the union of the sequence of closed sets W_i. By Theorem 12, then, we need only show $\operatorname{Ind} W_i \leqq n-1$.

Fixing i, let $\mathscr{V}_j$ be the trace of $\mathscr{A}_{i+j}$ on W_i. Since $W_i \subset W \subset X-M_{i+j}-N_{i+j}= (G_{i+j} \cap H_{i+j}) \cup J_{i+j}$, and W_i is disjoint from J_{i+j}, we have $W_i \subset G_{i+j} \cap H_{i+j}$. Hence on the one hand, $\mathscr{V}_j$ covers W_i; on the other hand, each point of W_i is also in at least one element of $\mathscr{G}_{i+j}$, and $\mathscr{V}_j$ is at most $(n-1)$-dimensional.

Let $U \cap W_i$ be a nonempty element of $\mathscr{V}_{j+1}$. Then $U^* \subset V$ for some V in $\mathscr{U}_{i+j}$. Since V meets W_i, V is not contained in J_i and hence V is not an element of $\mathscr{J}_{i+j}$. If V were an element of $\mathscr{G}_{i+j}$ then by (1), $U^* \cap (F \cup N_{i+j})=0$, $U \in \mathscr{G}_{i+j+1}$. It follows that V is in $\mathscr{A}_{i+j}$. A fortiori $V \cap W_i$ contains $\operatorname{St}(U, \mathscr{V}_{j+1})$, and we have a normal sequence. It is clear that its uniform topology on W_i is the original topology, and therefore $\operatorname{Ind} W_i \leqq n-1$, as was to be shown.

16. **Theorem.** *If X has the Lindelöf property and* $\operatorname{ind} X \leqq n$ *then* $\dim X \leqq n$.

Proof. Consider the following lemma.

L_m. *If X has the Lindelöf property and* $\operatorname{ind} X \leqq m$ *then any two disjoint closed sets of X are separated by a closed set C with* $\dim C \leqq m-1$.

Let T_m denote the theorem for $n=m$. We have T_0, from VI.3 and VI.23. We shall prove $\mathrm{T}_m \Longrightarrow \mathrm{L}_{m+1} \Longrightarrow \mathrm{T}_{m+1}$.

Assume T_m, let $\operatorname{ind} X \leqq m+1$, and let A and B be disjoint closed sets of X. Every point x of X has an open neighborhood N_x whose boundary has ind (so also dim) at most m, such that N_x^- does not meet A unless x is in A, and in that case N_x^- does not meet B. A countable number of these neighborhoods cover X. Let $\{U_i\}$ be the list of those which meet A, $\{V_j\}$ the list of the rest. Define $U_k'=U_k-\bigcup[V_j^- : j \leqq k]$. Let U be the union of all U_k'. Then U is open, since each U_k' is open. Each point x of A lies in some U_k; since x lies in no V_j^-, $x \in U_k'$ and $A \subset U$. Similarly every point of B is in some V_j and U^- is disjoint from B. Thus the boundary C of U separates A and B. Now the boundary of each U_k' is contained in a finite union of m-dimensional boundaries of sets N_x; so L_{m+1} follows from Theorem 7, once we check that each point c of C is a boundary point of some U_k'. Suppose not. Then c is a cluster point of $\{U_k'\}$; so it is not in any V_j. Hence $c \in U_i$ for some i, since the U's and V's form a covering. If c is not in the boundary of U_i', then $\bigcup[V_j : j \leqq i]$ is a neighborhood of c, and c cannot be a cluster point of $\{U_k'\}$, a contradiction.

L_{m+1} being established, suppose $\operatorname{ind} X \leqq m+1$, and let $f: A \rightarrow S^{m+1}$ be a uniformly continuous mapping of a subspace A of X into the sphere S^{m+1}. We may suppose A is closed; moreover, we can extend f continuously over a neighborhood U of A. By L_{m+1}. we can replace U by a smaller closed neighborhood V whose boundary C has dim $C \leqq m$. Now the mapping $f|C$ can be approximated arbitrarily closely by a mapping into an m-dimensional subspace (Exercise V.6); so $f|C$ is homotopic to a constant. Therefore $f|C$ can be continuously extended over the closure of $X-V$. This gives us a continuous extension of f over all of X, completing the proof.

17. Corollary. *For a separable metrizable space X,* dim $X=$ mindim $X=$ Ind $X=$ ind X.

Exercises.

1. A separable metrizable space X of dimension at most n is a union of $n+1$ subspaces X_i of dimension at most 0.

2. # 1 characterizes dimension for separable metrizable spaces.

The argument also offers an alternative to Theorem 16 for concluding the proof of coincidence of all four dimension functions.

(a) Suppose ind $X \leqq m$ implies dim $X \leqq m$. Let ind $Y \leqq m+1$. Then using Lemma VI.41 and Theorem VIII.7, $Y=Y_0 \cup X$, where dim $Y_0=0$ and dim $X \leqq m$.

Using Theorem 15, (a) implies ind $X \leqq m$; so induction gives # 1 for $n=m+1$.

(b) Then using VIII.11, prove dim $Y \leqq m+1$, completing the main induction.

(c) # 2 generalizes: If the completely normal space X is a union of subspaces $X_1, \cdots, X_m$, then ind $X \leqq m-1+\sum \operatorname{ind} X_i$. (Use VIII.11.)

3. Every completely uniformizable space is an inverse limit of finite-dimensional polyhedra; and

4. Every completely metrizable space is the limit space of an inverse mapping sequence of finite-dimensional polyhedra; and

5. In # 4, the polyhedra may be taken at most n-dimensional if the given space X has dim $X \leqq n$.

5 follows readily from

(a) In a completely metrizable space, every basis for the fine uniformity contains a basis for a complete metric uniformity.

3 almost follows from Exercise IV.8 and Shirota's theorem—with Euclidean polyhedra; but there is a reservation about measurable cardinals, in case they exist.

(b) Going back to Stone's theorem, complete the proof of # 3.

(c) Prove # 4 for separable X, considering that X is a G_δ in a compact metric space and thus a closed subspace of a countable product of real lines.

(d) Combine the fact of (c) and the method of (b) to finish the proof.

Little more is known about Exercise 3. Exercises 4 and 5 can be brought into a unified treatment by means of

(e) In an essential uniform space, every uniform covering has an essential uniform shrinking. (The proof of VIII.2 shows how to go about it, though the present proof will be longer.)

By VIII.2, a covering of X that is essential in the fine uniformity has dimension at most dim X. By VIII.15, $\aleph_0$ lower-dimensional coverings cannot be enough for a basis of a compatible uniformity. So (e) applied to # 4 gives # 5.

The result (e) is included here for its interest, not because it is necessary. The application is only in the fine case, and the proof of Theorem 4 has already proved (e) in that case. Another interesting result, not unrelated to these questions:

(f) The uniform dimension of an essential complete metric space μX may exceed dim X. (Construct μX as an inverse limit, starting with $X_1 = I$. X_2 is the set of all points (x, y) of the plane, with $y \in I$, such that x is a positive integer and $|y-1/2| \geqq 1/x$. Map X_2 to X_1 by $(x, y) \to y$. Now each component of X_2 is isomorphic with X_1; let its inverse image in X_3 be isomorphic with X_2 and map in the same way, and arrange that the image of X_3 in X_1 omits the values 1/3 and 2/3 as well as 1/2. Continue in the same manner. Then in the limit $\Delta d\mu X = 1$, μX is essential and complete, but dim $X = 0$.)

6. Suppose X is completely uniformizable and mindim $X = n$. Then X is an inverse limit of inverse limits of n-dimensional polyhedra; for X is an inverse limit of completely metrizable spaces with n-dimensional metric uniformities.

7. Covering dimension is lower semi-continuous on inverse mapping sequences of metrizable spaces. To prove this, on any system $\{X_m; f_{mn}\}$ with $\dim X_m \leqq k$, construct k-dimensional metric uniformities making the bonding mappings uniformly continuous. Conclude by applying Theorem 15 to the limit space.

Concerning possible extensions of this result (weakening "sequences" or "metrizable" or replacing dim with another dimension function) little is known. Cf. Pasynkov [**3**], Sitnikov [**1**].

8. A uniformizable space X has $\dim X \leqq n$ if and only if any $n+1$ pairs of completely separated sets (A_i, B_i) can be separated by sets C_i whose complements form a normal covering.

9. Define Ind* in the manner of Exercise V.7 and prove

(a) Ind* $\geqq$ dim for all uniformizable spaces.

(b) Ind* = Ind for completely normal spaces.

It is an open question whether Ind coincides with Ind* or Ind $\geqq$ dim for all spaces.

10. Local dimension. We say locdim $X \leqq n$ if every point of X has a neighborhood U with dim $U \leqq n$.

(b) Prove locdim $X = \dim X$ if X is paracompact, by means of

(a) a suitable trivial extension of Theorem 7.

Similarly, loc Ind $X \leqq n$ if every point has a neighborhood U with Ind $U \leqq n$.

(c) Generalize Theorem 12 trivially, and prove loc Ind $X =$ Ind X if X is paracompact and perfectly normal.

Open questions: Is locdim $=$ dim, or locInd $=$ Ind, for all completely uniformizable spaces?

Notes. Coincidence of Δd and δd for locally fine spaces can be proved by means of canonical mappings; the canonical mappings are easily constructed for uniformly locally finite coverings. The first proof of such a theorem is in Dowker [**1**], for fine spaces. For nonfine spaces one must be more careful about essentiality; cf. Isbell [**4**].

The proof of coincidence of Ind and dim for metrizable spaces given in the text was the third published proof. (Dowker and Hurewicz [**1**].) The earlier proofs of Katětov [**2**] and Morita [**5**] both develop out of special theories devised with this object; each theory seems likely to have further applications, and there have been scattered uses of Katětov's ideas.

Roy [**1**] has constructed an example of a metrizable space for which the dimension numbers differ, ind and mindim being 0, Ind and dim, 1. Its metric uniformities are thus all at least one-dimensional, though apparently not essential. This example seems to mark the boundary between two types of dimension theory. Outside separable metrizable spaces, in spite of Theorem 15, the main feature is the divergence of the several dimension functions. One wants to know what relations restrict their divergence and what happens to other geometric notions when even dimension is ambiguous. The principal geometric notions associated with dimension are homotopy and homology; for the associations, one begins in the separable metrizable case [**HW**]. So the further development of this type of dimension theory depends on the other type, and requires quite another book.

The first proof of dim $\leqq$ ind for Lindelöf spaces (Smirnov [**8**]) differs from both Theorem 16 and Exercises 1 and 2 [**HW**]. Exercises 3-9 are from Isbell [**2**; **4**; **5**], Exercise 10 from Dowker [**4**; **5**]. The results of Exercises 6 and 8 are basically due to Mardešić and Aleksandrov, respectively; see Mardešić [**2**], Sklyarenko [**2**].

APPENDIX

LINE AND PLANE

For the real line E^1, we can give a characterization.

THEOREM A. *A complete metric space which is uniformly locally connected, has a star-bounded basis of uniform coverings, and has a basis of coverings with nerve E^1, is isomorphic with E^1.*

(A star-bounded basis is defined in Exercise V.9.)

No characterization of the plane is known. At least, we can distinguish two problems, for the Euclidean plane and the hyperbolic plane of Bolyai-Lobačevskiĭ.

THEOREM B. *The hyperbolic plane is not embeddable in any Euclidean space.*

We prove first

1. *A complete metric space X that is homeomorphic with E^1 is isomorphic with E^1 if and only if X is uniformly locally connected and has a star-bounded basis.*

PROOF. Clearly E^1 satisfies these conditions. Suppose X satisfies them. Let $\mathscr{V}$ be a covering belonging to some star-bounded basis, and observe that every family of star-bounded uniform coverings finer than $\mathscr{V}$ is a star-bounded family. For, if $\mathscr{T}$ and $\mathscr{U}$ are finer than $\mathscr{V}$, and $\mathscr{W}$ is a common refinement of $\mathscr{T}$ and $\mathscr{U}$ which belongs to a star-bounded family with $\mathscr{V}$, routine counting shows that $\mathscr{T} \cup \mathscr{U}$ is star-bounded.

Let $\mathscr{Y}$ be a uniform covering consisting of connected sets which is a star-refinement of $\mathscr{V}$. Choose some homeomorphism of X with the line and define U_0^1 as the star of the point 0 with respect to $\mathscr{Y}$. Then U_0^1 is an interval. Because of the star-bounded basis, U_0^1 is precompact; because of completeness, it is a finite interval (a, b), with or without endpoints. Define U_1^1 as the union of all elements of $\mathscr{Y}$ which contain b and do not contain 0, U_{-1}^1 as the union of those which contain a and not 0, and so on for all $U_{\pm n}^1$. We get a uniform covering $\mathscr{U}^1$ finer than $\mathscr{V}$, consisting of intervals, and with nerve isomorphic with E^1. For this nerve N_1 we also have a marked point 0, the vertex corresponding to U_0^1; and a simplicial isomorphism $e^1: N_1 \to E^1$ taking the vertices onto the integers.

In the same way we construct a uniform covering $\mathscr{U}^2 <^* \mathscr{U}^1$, whose elements are intervals of diameter (in the metric space X) at most $1/2$, and whose nerve is uniformly isomorphic with E^1. Continuing, we have a star-bounded basis which is a normal sequence, with each nerve a copy of E^1, and the elements of $\mathscr{U}^{n+1}$ contained in any given element of $\mathscr{U}^n$ forming a chain. The limit space L of the barycentric system on $\{\mathscr{U}^n\}$ is isomorphic with X, by V.33. To get an isomorphism of L upon E^1, observe first that every vertex of N_n is the image of at least one vertex of N_{n+1}, since the bonding mappings f_{mn} are onto and take simplexes into closed simplexes. Recursively select a thread of vertices mapping upon 0 in N_1, and call each of these vertices 0 in N_n. Define e_1^2: $N_2 \to N_1$ as follows. $e_1^2(0)=0$. For $m=\pm 1, \pm 2, \cdots$, let v_m be the nearest vertex to 0 in N_2 such that $f_{21}(v_m)=m$, and put $e_1^2(v_m)=m$. On the intervals in N_2 between these vertices, e_1^2 is linear.

The inverse images under e_1^2 of intervals $[m, m+1]$ in N_1 contain bounded numbers of vertices, by star-boundedness. Moreover, each contains at least three vertices, since the mapping f_{21} does not take any simplex to a set of diameter 1.

Define e_n^{n+1} recursively in the same manner, and e_n^m by composition; put $e^m = e^1 e_1^m$. Obviously, the functions $e^m g_m : L \to E^1$ are each uniformly continuous. Moreover, they form a Cauchy sequence, since each e_n^m differs from f_{mn} only by modifications within some intervals whose images under e^n have diameter at most 2^{1-n}. Let e be the limit of $\{e^m g_m\}$. Then e is a uniformly continuous mapping of L onto E^1 realizing all $\mathscr{U}^n$, and the proof is complete.

Now Theorem A could be proved by using a topological characterization of the line, together with the first paragraph of the following argument. Instead we shall indicate a direct proof along the same lines as the proof of Proposition 1.

Proof of Theorem A. Suppose the uniform space X has a basis of coverings with nerve E^1, and A, B, C are three far connected sets each of which meets a closed connected set J; then J contains one of A, B, C. For on the contrary assumption we can find points a, b, c in $A-J$, $B-J$, $C-J$, and a uniform covering $\mathscr{U}$ with nerve E^1 with respect to which J, a, b, and c have disjoint stars and A, B, and C have disjoint stars. Realizing $\mathscr{U}$ by a canonical mapping, we have an absurdity.

Now let $\mathscr{V}$ be a uniform covering belonging to some star-bounded basis. Let $\mathscr{W}^1$ be a finer uniform covering with nerve E^1, and index the elements W_i of $\mathscr{W}^1$ with the integers i so that W_i meets W_j if and only if $|i-j| \leq 1$. Let $\mathscr{Z}^1$ be a finer uniform covering with compact connected sets. (As in Proposition 1, X is uniformly locally compact because of star-boundedness and completeness.) Let H_i be the union of all elements Z of $\mathscr{Z}^1$ such that i is the smallest integer in absolute value such that $Z \subset W_i$. Let $\mathscr{T}^1$ be the covering whose elements are the components of the sets H_i. Evidently $\mathscr{G}^1 < \mathscr{T}^1 < \mathscr{W}^1$.

It is possible for one element K of $\mathscr{T}^1$ to be contained in another element L of $\mathscr{T}^1$; but evidently this implies that L is not again contained in a third element M. Hence we can form the subcovering $\mathscr{U}^1$ consisting of all elements of $\mathscr{T}^1$ which are not contained in other elements of $\mathscr{T}^1$, and $\mathscr{T}^1 < \mathscr{U}^1 < \mathscr{T}^1$. The nerve of $\mathscr{U}^1$ is connected, for there is a finer uniform covering with connected nerve (E^1). No element J of $\mathscr{U}^1$ meets three other elements A, B, C; for any two of A, B, C must be either different components of the same H_i or components of some H_i and H_j with $|i-j|=2$, and (since disjoint compact sets are far) the first paragraph of the proof applies. Now the nerve of $\mathscr{U}^1$ is also infinite, and the only infinite connected complexes in which no vertex is joined to three others are the half-line and line. But $\mathscr{U}^1$ is finer than $\mathscr{W}^1$, and each element of $\mathscr{W}^1$ contains only finitely many elements of $\mathscr{U}^1$. Since the nerve of $\mathscr{W}^1$ is a line, the nerve of $\mathscr{U}^1$ is not a half-line.

Continuing in the same manner we can construct a basis $\{\mathscr{U}^n\}$ to which the argument of Proposition 1 will apply, completing the proof.

2. COROLLARY. *A closed subspace of a Euclidean space that is homeomorphic with E^1 is isomorphic with E^1 if it is uniformly locally connected.*

3. COROLLARY. *A metric space homeomorphic with E^1 is isomorphic with E^1 if it has a transitive group of isometries.*

The first corollary is trivial; the second is easy.

REMARKS. (1) In Corollary 2, if the curve is rectifiable, yet arc length need not be a uniformly continuous function. (2) The 2-dimensional Finsler spaces having transitive groups of isometries are known (Busemann [1]). Up to uniform isomorphism, the only noncompact ones are the Euclidean and hyperbolic planes and $E^1 \times S^1$. But if we do not assume a Finsler space, nothing is known in this context.

For Theorem B, we do not need much background information on the hyperbolic spaces H^n. (One may consult, for example, [**BK**].) Recall that H^n is a metric space homeomorphic with E^n, having an elementary geometric structure (line, plane,$\cdots$; length, angle), having *free mobility* (any isometry of a subspace of H^n into H^n can be extended to an isometry of H^n onto itself), and with the angles of every equilateral triangle less than $\pi/3$, varying from $\pi/3$ to 0 as the side goes from 0 to ∞.

Then we can triangulate H^2 at once. Since there exists an equilateral triangle with each angle $2\pi/7$, the space can be covered with such triangles with exactly seven of them meeting at each vertex. The triangles are isometric with each other and (being compact) uniformly equivalent with ordinary triangles; hence H^2 is uniformly equivalent with the corresponding uniform complex.

In this triangulation, the number of vertices which can be joined to a given vertex by connected chains of n or fewer edges is seven times the sum of the first $2n$ Fibonacci numbers (plus one if we count the given vertex). Recall that the ratios of consecutive Fibonacci numbers approach a limit $>$ $3/2$, so that these numbers exceed $(3/2)^n$.

PROOF OF THEOREM B. Let $\mathscr{U}$ be a uniform covering whose elements have diameter less than the side s of a triangle in the triangulation described above. $\mathscr{U}$ has some Lebesgue number s/m, and the mnth iterated star of an element of $\mathscr{U}$ must contain $(3/2)^n$ different elements of $\mathscr{U}$. No polynomial in mn exceeds$(3/2)^n$, and H^2 does not have a basis of uniform coverings of polynomial growth.

This proof is due to Edward Nelson. A somewhat more sophisticated proof was published by Efremovič [2].

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