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The coshape invariant and continuous extensions of functors

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ABSTRACT

The coshape invariant and continuous extensions of group-valued covariant and contravariant functors, defined on the category of pairs of spaces with the homotopy type of a pair of finite CW-complexes, are constructed.

0. Introduction

The problem of extension of functors from the subcategory of spaces with the homotopy type of "good" spaces to the category of topological spaces is one of the important problems of algebraic topology [5,6,8,9,13,14,17]. The achievements in the solution of this problem have interesting applications in different branches of modern topology and algebra.

The coshape theory [2,7,11,15] is closely connected with the problem of extension of functors from the category of spaces with the homotopy type of polyhedra to the category of topological spaces. In particular, the spectral (co)homotopy groups [11] and the spectral singular (co)homology groups [6] of spaces are invariant functors of coshape theory. Besides, the (co)homotopy and (co)homology, inj-groups and pro-groups of spaces [3,11,16] also induce important coshape invariant functors because they contain much more information about direct and inverse systems than their limits, even if these limits exist.

The main aim of the present paper is to study the extension problem of functors. To achieve this aim, a coshape theory of pairs of topological spaces is developed. Obtained results lead to a construction of coshape invariant and continuous extensions of group-valued covariant and contravariant functors from the homotopy category of pairs of spaces with the homotopy type of a pair of finite CW-complexes to the homotopy category of pairs of topological spaces (cf. [1,3,7,8,10,11, 14,15,18]).

1. Preliminaries

We use the notation of [2,11,12,14,16]. A space and a map considered here mean a topological space and a continuous map, respectively.

Let $\mathbf{Top}^2$ ($\mathbf{Top}^2_*$) be the category of pairs (pointed pairs) of spaces. By $\mathbf{CW}^2_f$ ($\mathbf{CW}^2_{f^*}$) we denote the full subcategory of $\mathbf{Top}^2$ ($\mathbf{Top}^2_*$) consisting of pairs (pointed pairs) of finite CW-complexes. We write $\mathbf{HTop}^2$ ($\mathbf{HTop}^2_*$) for the homotopy (pointed

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homotopy) category of the category $\mathbf{Top}^2$ ($\mathbf{Top}^2_*$). Let $\mathbf{HCW}^2_{\boldsymbol{f}}$ ($\mathbf{HCW}^2_{\boldsymbol{f}*}$) denote the full subcategory of $\mathbf{HTop}^2$ ($\mathbf{HTop}^2_*$) whose objects are pairs (pointed pairs) of spaces homotopy equivalent to a pair (pointed pair) of finite CW-complexes.

Write $\mathbf{Ssc}^2$ for the category of pairs of semisimplicial complexes (ssc) and semisimplicial maps (ssm) and **Gr** for the category of groups and homomorphisms.

First, we recall some results from the theory of semisimplicial complexes [12].

Let $S_n(X)$ be the collection of all continuous maps $\sigma : \Delta^n \longrightarrow X$ of the standard n-simplex Δ^n into a topological space X. Let $S(X) = \{S_n(X) \mid n = 0, 1, 2, \ldots\}$ and let $d_i^* : \Delta^n \longrightarrow \Delta^{n+1}$ and $s_j^* : \Delta^n \longrightarrow \Delta^{n-1}$ be the maps given by formulas:

$$d_i^*(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n), \quad (t_0, \ldots, t_n) \in \Delta^n,$$
$$s_j^*(t_0, \ldots, t_n) = (t_0, \ldots, t_{j-1}, t_j + t_{j+1}, t_{j+2}, \ldots, t_n), \quad (t_0, \ldots, t_n) \in \Delta^n.$$

Let $d_i : S_n(X) \longrightarrow S_{n-1}(X)$ and $s_j : S_n(X) \longrightarrow S_{n+1}(X)$ be the maps sending $x_n \in S_n(X)$ into $x_n d_i^*$ and $x_n s_j^*$, respectively. It is clear that $S(X)$ is ssc. If $f : X \longrightarrow Y$ is a continuous map, then it induces an ssm $S(f) : S(X) \longrightarrow S(Y)$. By definition, $S(f)(\sigma) = f \cdot \sigma$, $\sigma : \Delta^n \longrightarrow X$. If $f : (X, X_0) \longrightarrow (Y, Y_0)$ is a continuous map of pairs of topological spaces, then $S(f)$ is the ssm $S(f) : (S(X), S(X_0)) \longrightarrow (S(Y), S(Y_0))$ of pairs of ssc's.

Now, we associate to a given ssc X and ssm $f : X \longrightarrow Y$ their geometric realizations, a CW-complex $|X|$ and a continuous map $|f| : |X| \longrightarrow |Y|$, respectively.

Let $M(X)$ be the topological disjoint union of all copies (Δ^n, x_n), $x_n \in X_n$, i.e. $M(X) = \bigsqcup_{n=1}^{\infty} (\Delta^n \times X_n)$. Let E be an equivalence relation on $M(X)$ given by the following conditions

$$\big(d_i^* t, x_n\big) E (t, d_i x_n), \quad t \in \Delta^{n-1},$$
$$\big(s_j^* t, x_n\big) E (t, s_j x_n), \quad t \in \Delta^{n+1}.$$

We say that the pairs (t, x) and (u, y) of $M(X)$ are E equivalent, $(t, x)\ E(u, y)$, if there exists a finite chain of such type equivalences beginning at (t, x) and ending at (u, y). Let $|X| = M(X)/E$ and $\eta : M(X) \longrightarrow |X|$ be the quotient map given by the formula:

$$\eta\big((t, x)\big) = \big[(t, x)\big], \quad (t, x) \in M(X).$$

Each ssm $f : X \longrightarrow Y$ induces a map $M(f) : M(X) \longrightarrow M(Y)$. By definition,

$$M(f)(t, x_n) = \big(t, f(x_n)\big), \quad x_n \in X_n,\ t \in \Delta^n.$$

There exists a continuous map $|f| : |X| \longrightarrow |Y|$ defined by

$$|f|\big(\big[(t, x_n)\big]\big) = \big[\big(t, f(x_n)\big)\big], \quad x_n \in X_n,\ t \in \Delta^n.$$

Note that the semisimplicial subcomplexes of an ssc X are in a one-to-one correspondence with the subcomplexes of the CW-complex $|X|$ (see [12, Lemma III.4.10]).

Let $\mathrm{S} : \mathbf{Top}^2 \longrightarrow \mathbf{Ssc}^2$ and $\mathrm{R} : \mathbf{Ssc}^2 \longrightarrow \mathbf{Top}^2$ be the singular functor and the geometric realization functor given by formulas:

$$\mathrm{S}\big((X, X_0)\big) = \big(S(X), S(X_0)\big), \quad (X, X_0) \in \mathbf{Top}^2,$$
$$\mathrm{S}(f) : \big(S(X), S(X_0)\big) \longrightarrow \big(S(Y), S(Y_0)\big), \quad \big(f : (X, X_0) \longrightarrow (Y, Y_0)\big) \in \mathbf{Top}^2,$$
$$\mathrm{R}\big((X, X_0)\big) = \big(|X|, |X_0|\big), \quad (X, X_0) \in \mathbf{Ssc}^2,$$
$$\mathrm{R}(f) = |f| : \big(|X|, |X_0|\big) \longrightarrow \big(|Y|, |Y_0|\big), \quad \big(f : (X, X_0) \longrightarrow (Y, Y_0)\big) \in \mathbf{Ssc}^2.$$

For each pair $(X, X_0) \in \mathbf{Top}^2$ define a map

$$j_{(X, X_0)} : \big(\big|S(X)\big|, \big|S(X_0)\big|\big) \longrightarrow (X, X_0).$$

By definition,

$$j_{(X, X_0)}\big(\big[(t, \sigma)\big]\big) = \sigma(t), \quad t \in \Delta^2,\ \sigma : \Delta^n \longrightarrow X.$$

Let $f : (X, X_0) \longrightarrow (Y, Y_0)$ be a continuous map of pairs of spaces. The following diagram is commutative:

$$\begin{array}{ccc}
(|S(X)|, |S(X_0)|) & \xrightarrow{\ |S(f)|\ } & (|S(Y)|, |S(Y_0)|) \\
\Big\downarrow{\scriptstyle j_{(X, X_0)}} & & \Big\downarrow{\scriptstyle j_{(Y, Y_0)}} \\
(X, X_0) & \xrightarrow{\quad f \quad} & (Y, Y_0).
\end{array}$$

Consequently, $j = \{j_{(X,X_0)} \mid (X, X_0) \in \mathbf{Top}^2\}$ is a natural transformation of the composition $\mathrm{R} \cdot \mathrm{S}$ of the singular and geometric realization functors to the identity functor $1_{\mathbf{Top}^2} : \mathbf{Top}^2 \longrightarrow \mathbf{Top}^2$.

Now, we recall some notions and facts on the inj-category whose detailed description was given in [2].

Let $\mathcal{T}$ be an arbitrary category. A *direct system* in $\mathcal{T}$ is a covariant functor $\boldsymbol{X}$ from the category determined by a directed set $(A, \leqslant)$ to the category $\mathcal{T}$, i.e. a direct system $\boldsymbol{X}$ in $\mathcal{T}$ is a family $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A)$, where X_α, $\alpha \in A$ is an object of $\mathcal{T}$ and $p_{\alpha\alpha'} : X_\alpha \longrightarrow X_{\alpha'}$, $\alpha \leqslant \alpha'$ is a bonding morphism with properties $p_{\alpha\alpha} = 1_{X_\alpha} : X_\alpha \longrightarrow X_\alpha$, $\alpha \in A$ and $p_{\alpha'\alpha''} \cdot p_{\alpha\alpha'} = p_{\alpha\alpha''}$, $\alpha \leqslant \alpha' \leqslant \alpha''$. For every object $X \in \mathcal{T}$ by (X) we denote the direct system indexed by a singleton and having only one term X.

A morphism $(f_\alpha, \varphi) : \boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \longrightarrow \boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ of $\mathbf{dir}\text{-}\mathcal{T}$, called a mapping of direct systems, consists of a function $\varphi : A \longrightarrow B$ and a collection of morphisms $f_\alpha : X_\alpha \longrightarrow Y_{\varphi(\alpha)}$, $\alpha \in A$, such that for each pair $\alpha \leqslant \alpha'$ there is an index $\beta \geqslant \varphi(\alpha), \varphi(\alpha')$ with $q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\varphi(\alpha')\beta} \cdot f_{\alpha'} \cdot p_{\alpha\alpha'}$.

The composition (h_α, ζ) of morphisms $(f_\alpha, \varphi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $(g_\beta, \psi) : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ is defined in the usual manner. The mapping of direct systems $(h_\alpha, \zeta) : \boldsymbol{X} \longrightarrow \boldsymbol{Z}$ consists of the function $\zeta = \psi \cdot \varphi$ and the collection of morphisms $h_\alpha = g_{\varphi(\alpha)} \cdot f_\alpha : X_\alpha \longrightarrow Z_{h(\alpha)}$. The family $(1_{X_\alpha}, 1_A)$ is the identity mapping of the direct system $\boldsymbol{X}$.

The direct systems of the category $\mathcal{T}$ and their morphisms form a category $\mathbf{dir}\text{-}\mathcal{T}$.

Two mappings of direct systems $(f_\alpha, \varphi), (g_\alpha, \psi) : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ are said to be equivalent, $(f_\alpha, \varphi) \sim (g_\alpha, \psi)$, if for each index $\alpha \in A$ there is an index $\beta \geqslant \varphi(\alpha), \psi(\alpha)$ such that $q_{\varphi(\alpha)\beta} \cdot f_\alpha = q_{\psi(\alpha)\beta} \cdot g_\alpha$.

The relation $\sim$ is an equivalence on the set of morphisms of $\boldsymbol{X}$ to $\boldsymbol{Y}$.

Let $\boldsymbol{f} = [(f_\alpha, \varphi)] : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{g} = [(g_\beta, \psi)] : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ be the equivalence classes of morphisms in $\mathbf{dir}\text{-}\mathcal{T}$. The composition $\boldsymbol{g} \cdot \boldsymbol{f}$ is well defined:

$$\boldsymbol{g} \cdot \boldsymbol{f} = \big[(g_\beta, \psi)\big] \cdot \big[(f_\alpha, \varphi)\big] = \big[(g_\beta, \psi)\big] \cdot \big[(f_\alpha, \varphi)\big].$$

Note that $\mathbf{1}_{\boldsymbol{Y}} \cdot \boldsymbol{f} = \boldsymbol{f} = \boldsymbol{f} \cdot \mathbf{1}_{\boldsymbol{X}}$ and $\boldsymbol{h} \cdot (\boldsymbol{g} \cdot \boldsymbol{f}) = (\boldsymbol{h} \cdot \boldsymbol{g}) \cdot \boldsymbol{f}$ for each equivalence classes $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$, $\boldsymbol{g} : \boldsymbol{Y} \longrightarrow \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Z} \longrightarrow \boldsymbol{W}$.

Consequently, there is a quotient category

$$\mathbf{inj}\text{-}\mathcal{T} = \mathbf{dir}\text{-}\mathcal{T}/_\sim$$

whose objects are objects of $\mathbf{dir}\text{-}\mathcal{T}$ and whose morphisms are equivalence classes $\boldsymbol{f} = [(f_\alpha, \varphi)]$ of morphisms (f_α, φ) from $\mathbf{dir}\text{-}\mathcal{T}$. The category $\mathbf{inj}\text{-}\mathcal{T}$ is dual to the pro-category $\mathbf{pro}\text{-}\mathcal{T}$ [14].

Let $\mathcal{P}$ be a full subcategory of the category $\mathcal{T}$. Let X be an object of the category $\mathcal{T}$. A *$\mathcal{T}$-coexpansion* of X is a morphism $\boldsymbol{p} : \boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \longrightarrow (X)$ in $\mathbf{inj}\text{-}\mathcal{T}$ of a direct system $\boldsymbol{X}$ in the category $\mathcal{T}$ to a direct system (X) with the condition:

For each direct system $\boldsymbol{Y} = (Y_\beta, q_{\beta\beta'}, B)$ in the subcategory $\mathcal{P}$ and each morphism $\boldsymbol{g} : \boldsymbol{Y} \longrightarrow (X)$ in $\mathbf{inj}\text{-}\mathcal{T}$ there exists a unique morphism $\boldsymbol{f} : \boldsymbol{Y} \longrightarrow \boldsymbol{X}$ in $\mathbf{inj}\text{-}\mathcal{T}$ such that $\boldsymbol{p} \cdot \boldsymbol{f} = \boldsymbol{g}$.

If $\boldsymbol{X}$ and $\boldsymbol{f}$ are an object and a morphism of $\mathbf{inj}\text{-}\mathcal{P}$, then we say that $\boldsymbol{p}$ is a *$\mathcal{P}$-coexpansion* of X. In this case we also say that $\boldsymbol{X}$ is coassociated with X.

Note that if $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ and $\boldsymbol{p}' : \boldsymbol{X}' \longrightarrow (X)$ are two $\mathcal{P}$-coexpansions of an object $X \in \mathcal{T}$, then there is an isomorphism $\boldsymbol{i} : \boldsymbol{X} \longrightarrow \boldsymbol{X}'$ of the category $\mathbf{inj}\text{-}\mathcal{P}$.

The following theorem gives necessary and sufficient conditions for $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ to be a $\mathcal{T}$-coexpansion ($\mathcal{P}$-coexpansion).

Theorem 1. *Let $\boldsymbol{X} = (X_\alpha, p_{\alpha\alpha'}, A) \in \mathbf{inj}\text{-}\mathcal{T}$ ($\mathbf{inj}\text{-}\mathcal{P}$). A morphism $\boldsymbol{p} = [(p_\alpha)] : \boldsymbol{X} \longrightarrow (X)$ is a $\mathcal{T}$-coexpansion ($\mathcal{P}$-coexpansion) if and only if the morphisms $p_\alpha : X_\alpha \longrightarrow X$, $a \in A$, satisfy the following conditions*:

CAE1) *For any morphism $h : P \longrightarrow X$ in $\mathcal{T}$, $P \in \mathcal{P}$, there exist an index $a \in A$ and a morphism $f : P \longrightarrow X_\alpha$ in $\mathcal{T}$ (in $\mathcal{P}$) for which $h = p_\alpha \cdot f$.*

CAE2) *If for morphisms $f, f' : P \longrightarrow X_\alpha$ the equality $p_\alpha \cdot f = p_\alpha \cdot f'$ holds, then there exists an index $\alpha' \geqslant \alpha$ such that $p_{\alpha\alpha'} \cdot f = p_{\alpha\alpha'} \cdot f'$.*

For a proof we refer the reader to [4].

A subcategory $\mathcal{P} \subset \mathcal{T}$ is called a *codense subcategory* of the category $\mathcal{T}$ provided each object $X \in \mathcal{T}$ admits a $\mathcal{P}$-coexpansion.

Now, we define the *coshape category* for an arbitrary category $\mathcal{T}$ and its full codense subcategory $\mathcal{P}$. Let $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$, $\boldsymbol{p}' : \boldsymbol{X}' \longrightarrow (X)$ and $\boldsymbol{q} : \boldsymbol{Y} \longrightarrow Y$, $\boldsymbol{q}' : \boldsymbol{Y}' \longrightarrow (Y)$ be $\mathcal{P}$-coexpansions of X and Y, respectively. Then there are isomorphisms $\boldsymbol{i} : \boldsymbol{X} \longrightarrow \boldsymbol{X}'$ and $\boldsymbol{j} : \boldsymbol{Y} \longrightarrow \boldsymbol{Y}'$. We say that morphisms $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{f}' : \boldsymbol{X}' \longrightarrow \boldsymbol{Y}'$ are equivalent if $\boldsymbol{f}' \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{f}$. The equivalence class of $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ we denote by F and call a coshape morphism from X to Y. The composition $G \cdot F : X \longrightarrow Z$ of two coshape morphisms $F : X \longrightarrow Y$ and $G : Y \longrightarrow Z$ we can define as the equivalence class of the morphism $\boldsymbol{g} \cdot \boldsymbol{j} \cdot \boldsymbol{f}$, where $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y}' \longrightarrow \boldsymbol{Z}$ are representatives of F and G, respectively. Let I_X be the equivalence class of the identity morphism $1_{\boldsymbol{X}} : \boldsymbol{X} \longrightarrow \boldsymbol{X}$. It is clear that $I_Y \cdot F = F \cdot I_X = F$ and $H \cdot (G \cdot F) = (H \cdot G) \cdot F$ for each coshape morphisms $F : X \longrightarrow Y$, $G : Y \longrightarrow Z$ and $H : Z \longrightarrow W$. We have obtained the abstract coshape category $\mathbf{CSH}_{(\mathcal{T},\mathcal{P})}$, whose objects are all objects of the category $\mathcal{T}$ and whose morphisms are all coshape morphisms.

For each morphism $f : X \longrightarrow Y$ of the category $\mathcal{T}$ and for any $\mathcal{P}$-coexpansions $\boldsymbol{p} : \boldsymbol{X} \longrightarrow (X)$ and $\boldsymbol{q} : \boldsymbol{Y} \longrightarrow (Y)$ there exists a unique morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ in **inj**-$\mathcal{P}$ such that $f \cdot \boldsymbol{p} = \boldsymbol{q} \cdot \boldsymbol{f}$. Let $\mathrm{CS}(f)$ denote the equivalence class of the morphism $\boldsymbol{f}$. If we put $\mathrm{CS}(X) = X$ for each object $X \in \mathcal{T}$, then we obtain a functor $\mathrm{CS} : \mathcal{T} \longrightarrow \mathbf{CSH}_{(\mathcal{T},\mathcal{P})}$ called the coshape functor. For any morphism $\boldsymbol{f} : \boldsymbol{X} \longrightarrow \boldsymbol{Y}$ in **inj**-$\mathcal{P}$ there exists a unique coshape morphism $F : X \longrightarrow Y$ such that $\boldsymbol{q} \cdot \boldsymbol{f} = F \cdot \boldsymbol{p}$. If the objects X and Y are isomorphic in the coshape category $\mathbf{CSH}_{(\mathcal{T},\mathcal{P})}$, then we say that they have the same coshape and write $\mathrm{csh}(X) = \mathrm{csh}(Y)$.

2. The topological coshape category of pairs of spaces

This section contains results which play an essential role in the construction of coshape theory and in the whole paper. We have the following proposition (cf. [12, Proposition III.4.12]).

Proposition 2. *Let (K, K_0) be a pair of ssc's. For each map $g : (|K|, |K_0|) \longrightarrow (X, X_0)$ of $(|K|, |K_0|)$ to the pair $(X, X_0) \in \mathbf{Top}^2$ there exists an ssm $\bar{g} : (K, K_0) \longrightarrow (S(X), S(X_0))$ such that $g = j_{(X,X_0)} \cdot |\bar{g}|$.*

Proof. Indeed, let $\bar{g} : K \longrightarrow S(X)$ be an ssm defined in [12]. Let $|\sigma|$ be any n-cell of $|K|$ and $\varphi_n : \Delta^n \longrightarrow |\sigma|$ its characteristic map, the restriction of η to (Δ^n, σ) in $M(K)$. By the definition of $\bar{g}$,

$$\bar{g}(t) = \eta\big(\varphi_\sigma^{-1}(t), g\varphi_n\big) \in \big|S(X)\big|, \quad t \in |\sigma|.$$

It is easy to see that $\bar{g} : K \longrightarrow S(X)$ is a well-defined ssm, which induces the ssm of pairs $\bar{g} : (K, K_0) \longrightarrow (S(X), S(X_0))$ and satisfies the condition $g = j_{(X,X_0)} \cdot |\bar{g}|$. $\square$

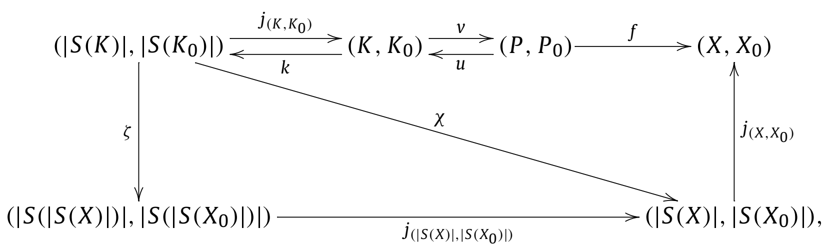
Now, we construct the coshape category $\mathbf{CSH}^2 = \mathbf{CSH}_{(\mathcal{T},\mathcal{P})}$, where $\mathcal{T} = \mathbf{HTop}^2$ and $\mathcal{P} = \mathbf{HCW}^2_{\boldsymbol{f}}$. To achieve this aim we establish the following main theorem.

Theorem 3. *The homotopy category $\mathbf{HCW}^2_{\boldsymbol{f}}$ is a codense subcategory of the homotopy category $\mathbf{HTop}^2$.*

The proof of this theorem is based on Theorem 1 and on the following two lemmas.

Lemma 4. *Let $f : (P, P_0) \longrightarrow (X, X_0)$ be a map of the pair $(P, P_0) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ to the pair $(X, X_0) \in \mathbf{Top}^2$. Then it factorizes through a pair of finite CW-simplicial complexes of a small subcategory of the category $\mathbf{HCW}^2_{\boldsymbol{f}}$.*

Proof. By assumption there is a pair (K, K_0) of a finite CW-complex K and its subcomplex K_0 and maps $u : (P, P_0) \longrightarrow (K, K_0)$ and $v : (K, K_0) \longrightarrow (P, P_0)$ such that $v \cdot u \simeq 1_{(P,P_0)}$ and $u \cdot v \simeq 1_{(K,K_0)}$. Consider the following diagram:



where $j_{(K,K_0)}$, k, ζ and χ are maps such that

$$j_{(K,K_0)} \cdot k = 1_{(K,K_0)}, \qquad j_{(X,X_0)} \cdot \chi = f \cdot v \cdot j_{(K,K_0)}, \qquad j_{(|S(X)|,|S(X_0)|)} \cdot \zeta = \chi.$$

The existence of these maps follows from Proposition 2. Let $h = \zeta \cdot k \cdot u : (P, P_0) \longrightarrow (|S(|S(X)|)|, |S(|S(X_0)|)|)$ and $j = j_{(X,X_0)} \cdot j_{(|S(X)|,|S(X_0)|)} : (|S(|S(X)|)|, |S(|S(X_0)|)|) \longrightarrow (X, X_0)$. Note that

$$\begin{aligned} j \cdot h &= j_{(X,X_0)} \cdot j_{(|S(X)|,|S(X_0)|)} \cdot \zeta \cdot k \cdot u = j_{(X,X_0)} \cdot \chi \cdot k \cdot u \\ &= f \cdot v \cdot j_{(K,K_0)} \cdot k \cdot u = f \cdot v \cdot 1_{(K,K_0)} \cdot u = f \cdot v \cdot u \simeq f \cdot 1_{(P,P_0)} = f. \end{aligned}$$

Thus, $f \simeq j \cdot h$. It is clear that the pair $(|S(|S(X)|)|, |S(|S(X_0)|)|)$ is a pair of CW-simplicial complexes (see [12, Lemma III.4.10 and Corollary IV.3.6]).

Let $\mathscr{P}' = \{(X_\alpha, X_{0\alpha}) \mid \alpha \in A\}$ be the set of all pairs of finite CW-simplicial subcomplexes of the pair $(|S(|S(X)|)|, |S(|S(X_0)|)|)$.

We have the following inclusion:

$$h\big((P,P_0)\big) = (\zeta\cdot k\cdot u)\big((P,P_0)\big) \subset \zeta k\big(u(P,P_0)\big) \subset \zeta k\big((K,K_0)\big) = \big(\zeta k(K), \zeta k(K_0)\big).$$

The compact pair $(\zeta k(K), \zeta k(K_0))$, and hence the pair $h((P,P_0))$, is contained in some pair $(X_\alpha, X_{0\alpha}) \in \mathscr{P}'$.

Let $j_\alpha = j_{|(X_\alpha, X_{0\alpha})} : (X_\alpha, X_{0\alpha}) \longrightarrow (X, X_0)$ and let $h_\alpha = h^{|(X_\alpha, X_{0\alpha})} : (P,P_0) \longrightarrow (X_\alpha, X_{0\alpha})$. Clearly, $f \simeq j_\alpha \cdot h_\alpha$. This is the desired factorization. □

Lemma 5. *Let $(X,X_0) \in \mathbf{HTop}^2$, $(P,P_0), (P',P'_0) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ and let $f' : (P',P'_0) \longrightarrow (X,X_0)$, $h_1, h_2 : (P,P_0) \longrightarrow (P',P'_0)$ be maps such that $f'\cdot h_1 \simeq f'\cdot h_2$. Then there exist a pair $(P'',P''_0) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ and maps $f'' : (P'',P''_0) \longrightarrow (X,X_0)$ and $h : (P',P'_0) \longrightarrow (P'',P''_0)$ such that $f''\cdot h = f'$ and $h\cdot h_1 \simeq h\cdot h_2$.*

Proof. Let $H : (P,P_0)\times I \longrightarrow (X,X_0)$ be a homotopy between $f'\cdot h_1$ and $f'\cdot h_2$. Let $f'_0 = f'_{|P'_0} : P'_0 \longrightarrow X_0$, $h_{01} = h_{1|P_0} : P_0 \longrightarrow P'_0$, $h_{02} = h_{2|P_0} : P_0 \longrightarrow P'_0$. Note that $H_{|P_0\times I} = H_0 : f'_0 \cdot h_{01} \simeq f'_0 \cdot h'_{02}$. Consider the pair $(S,S_0) = (P\times I \cup \mathrm{Cyl}(g), P_0 \times I \cup \mathrm{Cyl}(g_0))$, where $\mathrm{Cyl}(g)$ and $\mathrm{Cyl}(g_0)$ are the mapping cylinders of maps $g = h_1 \oplus h_2 : P^1 \oplus P^2 \longrightarrow P'$, $P^1 = P$, $P^2 = P$ and $g_0 = h_{01} \oplus h_{02} : P^1_0 \oplus P^2_0 \longrightarrow P'_0$, $P^1_0 = P_0$, $P^2_0 = P_0$, respectively.

Consider the following relation on S:

$$\begin{aligned}
&(p,1) \sim \big[(p,0)\big], \quad (p,1) \in P\times I,\ \big[(p,0)\big] \in \mathrm{Cyl}(g),\ p \in P^1;\\
&(p,0) \sim \big[(p,0)\big], \quad (p,0) \in P\times I,\ \big[(p,0)\big] \in \mathrm{Cyl}(g),\ p \in P^2;\\
&(p,1) \sim \big[(p,0)\big], \quad (p,1) \in P_0\times I,\ \big[(p,0)\big] \in \mathrm{Cyl}(g_0),\ p \in P^1_0;\\
&(p,0) \sim \big[(p,0)\big], \quad (p,0) \in P_0\times I,\ \big[(p,0)\big] \in \mathrm{Cyl}(g_0),\ p \in P^2_0.
\end{aligned}$$

Let $P'' = S/\sim$ and $P''_0 = S_0/\sim$ and let $q : S \longrightarrow P''$ be the quotient map.

It is clear that q maps the pair (S,S_0) onto the pair (P'',P''_0). Now define maps $h : P' \longrightarrow P''$ and $f'' : P'' \longrightarrow X$. By definition,

$$h\big(p'\big) = \big[p'\big], \quad p' \in P';$$

$$f''(z) = \begin{cases} H(p,t),\ z = q([(p,t)]),\ p \in P,\ 0 \leqslant t \leqslant 1,\\ f'h_1(p),\ z = q([(p,t)]),\ p \in P^1,\ 0 \leqslant t \leqslant 1,\\ f'h_2(p),\ z = q([(p,t)]),\ p \in P^2,\ 0 \leqslant t \leqslant 1,\\ f'(p'),\ z = q([(p',t)]),\ p' \in P'. \end{cases}$$

It is clear that $h(P'_0) \subseteq P''_0$ and $f''(P''_0) \subseteq X_0$, i.e. h and f'' are maps of pairs. The pair (P'',P''_0) and maps $f'' : (P'',P''_0) \longrightarrow (X,X_0)$ and $h : (P',P'_0) \longrightarrow (P'',P''_0)$ satisfy the conditions of the lemma. □

Let $\mathbf{HTop}^2_*$ be the pointed homotopy category of pointed pairs and let $\mathbf{HCW}^2_{\boldsymbol{f}*}$ be the pointed homotopy category of pairs with the homotopy type of pointed pair of finite CW-complexes. Similarly, we can prove the pointed versions of Lemma 4 and Lemma 5. Consequently, we have the following theorem.

Theorem 6. *The pointed homotopy category $\mathbf{HCW}^2_{\boldsymbol{f}*}$ is the codense subcategory of the pointed homotopy category $\mathbf{HTop}^2_*$.*

The pointed coshape category $\mathbf{CSH}^2_*$ of pairs of spaces is the abstract coshape category $\mathbf{CSH}_{(\mathcal{T},\mathcal{P})}$, where $\mathcal{T} = \mathbf{HTop}^2_*$ and $\mathcal{P} = \mathbf{HCW}^2_{\boldsymbol{f}*}$.

By $\mathrm{csh}(X,X_0)$ ($\mathrm{csh}(X,X_0,*)$) we denote the coshape (the pointed coshape) of the pair (X,X_0) (the pointed pair $(X,X_0,*)$).

Remark 1. Applying Lemma 4 we can conclude that for each pair $(X,X_0) \in \mathbf{HTop}^2$ ($(X,X_0) \in \mathbf{HTop}^2_*$) there exists a coassociated with (X,X_0) ($(X,X_0,*)$) direct system consisting of pairs (pointed pairs) of finite CW-simplicial complexes.

3. On extensions of functors

The purpose of this section is to construct the coshape invariant and continuous extensions of covariant (contravariant) functors from the category $\mathbf{HCW}^2_{\boldsymbol{f}}$ ($\mathbf{HCW}^2_{\boldsymbol{f}*}$) to the category $\mathbf{HTop}^2$ ($\mathbf{HTop}^2_*$).

Let $\mathrm{T} : \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ be a covariant (contravariant) functor of the category $\mathbf{HCW}^2_{\boldsymbol{f}}$ to the category $\mathbf{Gr}$. Let $(\boldsymbol{X}, \boldsymbol{X}_0) = ((X_\alpha, X_{0\alpha}), p_{\alpha\alpha'}, A)$ be a direct system in $\mathbf{HCW}^2_{\boldsymbol{f}}$. The covariant (contravariant) functor T forms a direct (inverse) system

$\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)=(\mathrm{T}(X_\alpha,X_{0\alpha}),\mathrm{T}(p_{\alpha\alpha'}),A)$ in the category **Gr**. Let $(f_\alpha,\varphi):(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(\boldsymbol{Y},\boldsymbol{Y}_0)$ be a morphism of the category **dir-HCW**$^2_{\boldsymbol{f}}$. Then we have the morphism $(\mathrm{T}(f_\alpha),\varphi):\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)$ $((\mathrm{T}(f_\alpha),\varphi):\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)\longrightarrow\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0))$ of the category **dir-Gr** (**inv-Gr**). It is clear that if $(f_\alpha,\varphi)\sim(f'_\alpha,\varphi')$, then $(\mathrm{T}(f_\alpha),\varphi)\sim(\mathrm{T}(f'_\alpha),\varphi')$ in the category **dir-Gr** (**inv-Gr**). Consequently, a morphism $\boldsymbol{f}=[(f_\alpha,\varphi)]:(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(\boldsymbol{Y},\boldsymbol{Y}_0)$ of the category **inj-HCW**$^2_{\boldsymbol{f}}$ induces the morphism $\mathrm{T}(\boldsymbol{f})=[(\mathrm{T}(f_\alpha),\varphi)]:\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)$ $(\mathrm{T}(\boldsymbol{f})=[(\mathrm{T}(f_\alpha),\varphi)]:\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)\longrightarrow\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0))$ of the category **inj-Gr** (**pro-Gr**). Thus, we have defined a covariant (contravariant) functor, which for simplicity we again denote by

$$\mathrm{T}(-,-):\textbf{inj-HCW}^2_{\boldsymbol{f}}\longrightarrow\textbf{inj-Gr}$$
$$\big(\mathrm{T}(-,-):\textbf{inj-HCW}^2_{\boldsymbol{f}}\longrightarrow\textbf{pro-Gr}\big).$$

Let $(X,X_0)\in\mathbf{HTop}^2$ and let $\boldsymbol{p}=[(p_\alpha)]:(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(X,X_0)$ be an **HCW**$^2_{\boldsymbol{f}}$-coexpansion of (X,X_0). Note that for each other **HCW**$^2_{\boldsymbol{f}}$-coexpansion $\boldsymbol{p}'=[(p'_{\alpha'})]:(\boldsymbol{X},\boldsymbol{X}_0)'\longrightarrow(X,X_0)$, the isomorphism $\boldsymbol{i}:(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(\boldsymbol{X},\boldsymbol{X}_0)'$ induces the isomorphism $\mathrm{T}(\boldsymbol{i}):\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)'$ $(\mathrm{T}(\boldsymbol{i}):\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)'\longrightarrow\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0))$. The equivalence class of $\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)$ is denoted by inj-$T(X,X_0)$ (pro-$T(X,X_0)$).

Let $F:(X,X_0)\longrightarrow(Y,Y_0)$ be a coshape morphism and let $\boldsymbol{f}:(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow(\boldsymbol{Y},\boldsymbol{Y}_0)$ be its representative. For another representative $\boldsymbol{f}':(\boldsymbol{X},\boldsymbol{X}_0)'\longrightarrow(\boldsymbol{Y},\boldsymbol{Y}_0)'$ we have $\boldsymbol{f}'\cdot\boldsymbol{i}=\boldsymbol{j}\cdot\boldsymbol{f}$. Consequently,

$$\mathrm{T}\big(\boldsymbol{f}'\big)\cdot\mathrm{T}(\boldsymbol{i})=\mathrm{T}(\boldsymbol{j})\cdot\mathrm{T}(\boldsymbol{f})\quad\big(\mathrm{T}(\boldsymbol{f})\cdot\mathrm{T}(\boldsymbol{j})=\mathrm{T}(\boldsymbol{i})\cdot\mathrm{T}\big(\boldsymbol{f}'\big)\big).$$

The morphisms $\mathrm{T}(\boldsymbol{f}):\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)\longrightarrow\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)$ and $T(\boldsymbol{f}'):\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)'\longrightarrow\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)'$ $((\mathrm{T}(\boldsymbol{f}):\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)\longrightarrow\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0))$ and $\mathrm{T}(\boldsymbol{f}'):\mathrm{T}(\boldsymbol{Y},\boldsymbol{Y}_0)'\longrightarrow\mathrm{T}(\boldsymbol{X},\boldsymbol{X}_0)')$ coincide. Thus, the coshape morphism $F:(X,X_0)\longrightarrow(Y,Y_0)$ induces the morphism

$$\text{inj-}T(F):\text{inj-}T(X,X_0)\longrightarrow\text{inj-}T(Y,Y_0)$$
$$\big(\text{pro-}T(F):\text{pro-}T(Y,Y_0)\longrightarrow\text{pro-}T(X,X_0)\big).$$

Thus, we have defined the covariant (contravariant) functor

$$\text{inj-T}(-,-):\mathbf{CSH}^2\longrightarrow\textbf{inj-Gr}$$
$$\big(\text{pro-T}(-,-):\mathbf{CSH}^2\longrightarrow\textbf{pro-Gr}\big).$$

By definition,

$$(\text{inj-T})\big((X,X_0)\big)=\text{inj-}T(X,X_0),\quad (X,X_0)\in\mathbf{CSH}^2,$$
$$(\text{pro-T})\big((X,X_0)\big)=\text{pro-}T(X,X_0),\quad (X,X_0)\in\mathbf{CSH}^2,$$
$$(\text{inj-T})(F)=\text{inj-}T(F),\quad F\in\mathbf{CSH}^2,$$
$$(\text{pro-T})(F)=\text{pro-}T(F),\quad F\in\mathbf{CSH}^2.$$

Analogously, we can define the covariant (contravariant) functor

$$\text{inj-T}(-,-):\mathbf{CSH}^2_*\longrightarrow\textbf{inj-Gr}$$
$$\big(\text{pro-T}(-,-):\mathbf{CSH}^2_*\longrightarrow\textbf{pro-Gr}\big).$$

The objects of the category **inj-Gr** are called inj-groups [3,16] and the objects of the category **pro-Gr** are called pro-groups [14].

We have obtained the following propositions.

Proposition 7. *Let* $(X,X_0),(Y,Y_0)\in\mathbf{HTop}^2$ *and* $\operatorname{csh}(X,X_0)=\operatorname{csh}(Y,Y_0)$. *Then* inj-$T(X,X_0)=$ inj-$T(Y,Y_0)$ *and* pro-$T(X,X_0)=$ pro-$T(Y,Y_0)$.

Proposition 8. *Let* $(X,X_0,*),(Y,Y_0,*)\in\mathbf{HTop}^2_*$ *and* $\operatorname{csh}(X,X_0,*)=\operatorname{csh}(Y,Y_0,*)$. *Then* inj-$T(X,X_0,*)=$ inj-$T(Y,Y_0,*)$ *and* pro-$T(X,X_0,*)=$ pro-$T(Y,Y_0,*)$.

For each pair (X,X_0) and each coshape morphism $F:(X,X_0)\longrightarrow(Y,Y_0)$ define spectral groups

$$\hat{T}(X,X_0)=\varinjlim\text{inj-}T(X,X_0)$$
$$\big(\overset{\vee}{T}(X,X_0)=\varprojlim\text{pro-}T(X,X_0)\big)$$

and homomorphisms

$$\hat{F} = \varinjlim \text{inj-}T(F) : \hat{T}(X, X_0) \longrightarrow \hat{T}(Y, Y_0)$$
$$\big(\overset{\vee}{F} = \varprojlim \text{pro-}T(F) : \overset{\vee}{T}(Y, Y_0) \longrightarrow \overset{\vee}{T}(X, X_0)\big).$$

Thus, the covariant (contravariant) functor $\mathrm{T} : \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ induces the covariant (contravariant) functor $\overset{\wedge}{\mathrm{T}} : \mathbf{CSH}^2 \longrightarrow \mathbf{Gr}$ ($\overset{\vee}{\mathrm{T}} : \mathbf{CSH}^2 \longrightarrow \mathbf{Gr}$). By definition,

$$\overset{\wedge}{\mathrm{T}}\big((X, X_0)\big) = \hat{T}(X, X_0), \quad (X, X_0) \in \mathbf{CSH}^2$$
$$\big(\overset{\vee}{\mathrm{T}}\big((X, X_0)\big) = \overset{\vee}{T}(X, X_0), \quad (X, X_0) \in \mathbf{CSH}^2\big),$$
$$\overset{\wedge}{\mathrm{T}}(F) = \hat{F}, \quad F \in \mathbf{CSH}^2$$
$$\big(\overset{\vee}{\mathrm{T}}(F) = \overset{\vee}{F}, \quad F \in \mathbf{CSH}^2\big).$$

Analogously, the covariant (contravariant) functor $\mathrm{T} : \mathbf{HCW}^2_{\boldsymbol{f}}* \longrightarrow \mathbf{Gr}$ induces the covariant (contravariant) functor $\overset{\wedge}{\mathrm{T}} : \mathbf{CSH}^2_* \longrightarrow \mathbf{Gr}$ ($\overset{\vee}{\mathrm{T}} : \mathbf{CSH}^2_* \longrightarrow \mathbf{Gr}$).

The composition $\overset{\wedge}{\mathrm{T}} \cdot \mathrm{CS}$ ($\overset{\vee}{\mathrm{T}} \cdot \mathrm{CS}$) of the constructed functor $\overset{\wedge}{\mathrm{T}}$ ($\overset{\vee}{\mathrm{T}}$) with the coshape functor CS is a coshape invariant extension of the functor T. For simplicity we again denote it by $\overset{\wedge}{\mathrm{T}}$ ($\overset{\vee}{\mathrm{T}}$). Hence, we have the following propositions.

Proposition 9. *If* $(X, X_0), (Y, Y_0) \in \mathbf{HTop}^2$ *and* $\mathrm{csh}(X, X_0) = \mathrm{csh}(Y, Y_0)$, *then* $\hat{T}(X, X_0) = \hat{T}(Y, Y_0)$ *and* $\overset{\vee}{T}(X, X_0) = \overset{\vee}{T}(Y, Y_0)$.

Proposition 10. *If* $(X, X_0, *), (Y, Y_0, *) \in \mathbf{HTop}^2_*$ *and* $\mathrm{csh}(X, X_0, *) = \mathrm{csh}(Y, Y_0, *)$, *then* $\hat{T}(X, X_0, *) = \hat{T}(Y, Y_0, *)$ *and* $\overset{\vee}{T}(X, X_0, *) = \overset{\vee}{T}(Y, Y_0, *)$.

Theorem 11. *Let* $\boldsymbol{p} = [(p_\alpha)] : (\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow (X, X_0)$ *be an* $\mathbf{HTop}^2$*-coexpansion of a pair* $(X, X_0) \in \mathbf{HTop}^2$ *and let* $\hat{T}(\boldsymbol{p}) : \hat{T}(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \hat{T}(X, X_0)$ ($\overset{\vee}{T}(\boldsymbol{p}) : \overset{\vee}{T}(X, X_0) \longrightarrow \overset{\vee}{T}(\boldsymbol{X}, \boldsymbol{X}_0)$) *be the induced morphism of* **inj-Gr** (**pro-Gr**)*. Then the homomorphism*

$$\hat{p} = \varinjlim \hat{T}(\boldsymbol{p}) : \varinjlim \hat{T}(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \hat{T}(X, X_0)$$
$$\big(\overset{\vee}{p} = \varprojlim \overset{\vee}{T}(\boldsymbol{p}) : \overset{\vee}{T}(X, X_0) \longrightarrow \varprojlim \overset{\vee}{T}(\boldsymbol{X}, \boldsymbol{X}_0)\big)$$

induced by $\hat{T}(\boldsymbol{p})$ ($\overset{\vee}{T}(\boldsymbol{p})$) *is an isomorphism.*

Proof. For simplicity, we denote the object $\hat{T}(X, X_0)$ by $T(X, X_0)$ for each object $(X, X_0) \in \mathcal{T}$, the homomorphism $\hat{T}(f)$ by $\hat{f}$ for each morphism $f : (X, X_0) \longrightarrow (Y, Y_0)$ in $\mathcal{T}$ and the direct system $\hat{T}(\boldsymbol{X}, \boldsymbol{X}_0) = (T(X_\alpha, X_{0\alpha}), \hat{p}_{\alpha\alpha'}, A)$ in **Gr** by $T(\boldsymbol{X})$ for each direct system $(\boldsymbol{X}, \boldsymbol{X}_0)$ in $\mathcal{T}$. Analogously, we denote by $\overset{\wedge}{\boldsymbol{p}} = (\hat{p}_\alpha)$ the morphism $T(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow T(X, X_0)$ given by homomorphisms $\hat{p}_\alpha : T(X_\alpha, X_{0\alpha}) \longrightarrow T(X, X_0)$, $\alpha \in A$. Finally, by $\hat{p}$ we denote the homomorphism $\varinjlim T(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow T(X, X_0)$ for which $\hat{p} \cdot \pi_\alpha = \hat{p}_\alpha$, $\alpha \in A$, where $\pi_\alpha : T(X_\alpha, X_{0\alpha}) \longrightarrow \varinjlim(\boldsymbol{X}, \boldsymbol{X}_0)$ is the injection homomorphism. Besides, also note that for each pair $\alpha \leqslant \alpha'$ the equality $\pi_{\alpha'} \cdot \hat{p}_{\alpha\alpha'} = \pi_\alpha$ holds.

Let $\boldsymbol{q} : (\boldsymbol{Y}, \boldsymbol{Y}_0) = ((Y_\beta, Y_{0\beta}), q_{\beta\beta'}, B) \longrightarrow (X, X_0)$ be a $\mathcal{P}$-coexpansion of (X, X_0). It is clear that $\overset{\wedge}{\boldsymbol{q}} = (\hat{q}_\beta) : T(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow T(X, X_0)$ is a direct limit and there exists a morphism $\boldsymbol{f} : (\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow (\boldsymbol{X}, \boldsymbol{X}_0)$ of the category **inj-**$\mathcal{T}$ such that $\boldsymbol{p} \cdot \boldsymbol{f} = \boldsymbol{q}$. Let (f_β, φ) be some representative of $\boldsymbol{f}$. The homomorphisms $\hat{f}_\beta : T(Y_\beta, Y_{0\beta}) \longrightarrow T(X_{\varphi(\beta)}, X_{0\varphi(\beta)})$, $\beta \in B$ induce a morphism of inj-groups $\overset{\wedge}{\boldsymbol{f}} = (\hat{f}_\beta, \varphi) : T(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow T(\boldsymbol{X}, \boldsymbol{X}_0)$. Note that $\overset{\wedge}{\boldsymbol{f}} = \overset{\wedge}{\boldsymbol{p}} \cdot \overset{\wedge}{\boldsymbol{q}}$ and $\overset{\wedge}{\boldsymbol{f}}$ induces a homomorphism of groups $\hat{f} : T(X, X_0) \longrightarrow \varinjlim T(\boldsymbol{X}, \boldsymbol{X}_0)$ for which $\boldsymbol{\pi} \cdot \overset{\wedge}{\boldsymbol{f}} = \hat{f} \cdot \overset{\wedge}{\boldsymbol{q}}$, where $\boldsymbol{\pi} : T(\boldsymbol{X}, \boldsymbol{X}_0) \longrightarrow \varinjlim T(\boldsymbol{X}, \boldsymbol{X}_0)$ is a morphism induced by (π_α). For each index $\beta \in B$ we have $\pi_{\varphi(\beta)} \cdot \hat{f}_\beta = \hat{f} \cdot \hat{q}_\beta$. Besides,

$$\hat{p} \cdot \hat{f} \cdot \hat{q}_\beta = \hat{p} \cdot \pi_{\varphi(\beta)} \cdot \hat{f}_\beta = \hat{p}_{\varphi(\beta)} \cdot \hat{f}_\beta = \hat{q}_\beta, \quad \beta \in B.$$

Thus, $\hat{p} \cdot \hat{f} \cdot \overset{\wedge}{\boldsymbol{q}} = \overset{\wedge}{\boldsymbol{q}}$. Note that $\overset{\wedge}{\boldsymbol{q}} : T(\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow T(X, X_0)$ is a direct limit of $T(\boldsymbol{Y}, \boldsymbol{Y}_0)$. Consequently, $\hat{p} \cdot \hat{f} = 1_{T(X,X_0)}$.

Now, we prove $\hat{p} \cdot \hat{f} = 1_{\varinjlim T(\boldsymbol{X},\boldsymbol{X}_0)}$. Let $\boldsymbol{r} = (r_\gamma) : (\boldsymbol{Z}, \boldsymbol{Z}_0) = ((Z_\gamma, Z_{0\gamma}), r_{\gamma\gamma'}, C) \longrightarrow (X_\alpha, X_{0\alpha})$ be an $\mathbf{HCW}^2_{\boldsymbol{f}}$-coexpansion of $(X_\alpha, X_{0\alpha})$. Since $\boldsymbol{q} : (\boldsymbol{Y}, \boldsymbol{Y}_0) \longrightarrow (X, X_0)$ is an $\mathbf{HCW}^2_{\boldsymbol{f}}$-coexpansion of (X, X_0) and $(Z_\gamma, Z_{0\gamma}) \in \mathbf{HCW}^2_{\boldsymbol{f}}$ there is an index $\beta \in B$ and a morphism $g : (Z_\gamma, Z_{0\gamma}) \longrightarrow (Y_\beta, Y_{0\beta})$ for which $p_\alpha \cdot r_\gamma = q_\beta \cdot g$ holds. Note that $q_\beta = p_{\varphi(\beta)} \cdot f_\beta$, $\beta \in B$ and there exists an index $\alpha' \geqslant \alpha, \varphi(\beta)$ such that

$$p_{\alpha'} \cdot p_{\alpha\alpha'} \cdot r_\gamma = p_{\alpha'} \cdot p_{\varphi(\beta)\alpha'} \cdot f_\beta \cdot g.$$

By the condition CAE2) there also exists an index $\alpha'' \geqslant \alpha'$ such that

$$p_{\alpha'\alpha''} \cdot p_{\alpha\alpha'} \cdot r_\gamma = p_{\alpha'\alpha''} \cdot p_{\varphi(\beta)\alpha'} \cdot f_\beta \cdot g,$$

i.e. $p_{\alpha\alpha''} \cdot r_\gamma = p_{\varphi(\beta)\alpha''} \cdot f_\beta \cdot g$. Besides,

$$\begin{aligned}\hat{f} \cdot \hat{p}_\alpha \cdot \hat{r}_\gamma &= \hat{f} \cdot \hat{p}_{\alpha''} \cdot \hat{p}_{\alpha\alpha''} \cdot \hat{r}_\gamma = \hat{f} \cdot \hat{p}_{\varphi(\beta)} \cdot \hat{f}_\beta \cdot \hat{g}_\beta = \pi_{\varphi(\beta)} \cdot \hat{f}_\beta \cdot \hat{g} = \pi_{\alpha''} \cdot \hat{p}_{\varphi(\beta)\alpha''} \cdot \hat{f}_\beta \cdot \hat{g} \\ &= \pi_{\alpha''} \cdot \hat{p}_{\alpha\alpha''} \cdot \hat{r}_\gamma = \pi_\alpha \cdot \hat{r}_\gamma.\end{aligned}$$

Since $\hat{\boldsymbol{r}} = (\hat{r}_\gamma) : T(\boldsymbol{Z}, \boldsymbol{Z}_0) \longrightarrow T(X_\alpha, X_{0\alpha})$ is a direct limit, we have $\hat{f} \cdot \hat{p}_\alpha = \pi_\alpha$, $\alpha \in A$. Hence, $\hat{f} \cdot \hat{p} \cdot \pi_\alpha = \pi_\alpha$, $\alpha \in A$, i.e. $\hat{f} \cdot \hat{p} = 1_{\varinjlim T(\boldsymbol{X},\boldsymbol{X}_0)}$.

Analogously, we can prove that $\check{T}(X, X_0)$ and $\varprojlim \check{T}(X, X_0)$ are isomorphic objects of the category **Gr**. □

Similar arguments prove the pointed version of Theorem 11.

Theorem 12. *Let* $\boldsymbol{p} = [(p_\alpha)] : (\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow (X, X_0, *)$ *be an* $\mathbf{HTop}^2_*$*-coexpansion of a pair* $(X, X_0, *) \in \mathbf{HTop}^2_*$ *and let* $\hat{T}(\boldsymbol{p}) : \hat{T}(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow \hat{T}(X, X_0, *)$ $(\check{T}(\boldsymbol{p}) : \check{T}(X, X_0, *) \longrightarrow T(\boldsymbol{X}, \boldsymbol{X}_0, *))$ *be the induced morphism of* **inj-Gr** (**pro-Gr**). *Then the homomorphism*

$$\hat{p} = \varinjlim \hat{T}(\boldsymbol{p}) : \varinjlim \hat{T}(\boldsymbol{X}, \boldsymbol{X}_0, *) \longrightarrow \hat{T}(X, X_0, *)$$

$$\bigl(\check{p} = \varprojlim \check{T}(\boldsymbol{p}) : \check{T}(X, X_0, *) \longrightarrow \varprojlim \check{T}(\boldsymbol{X}, \boldsymbol{X}_0, *)\bigr)$$

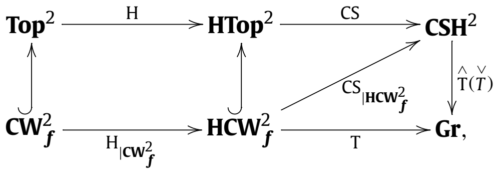
induced by $\hat{T}(\boldsymbol{p})$ $(\check{T}(\boldsymbol{p}))$ *is an isomorphism.*

Let $\mathrm{L} : \mathbf{CW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ be a covariant (contravariant) functor satisfying the homotopy axiom, i.e. if $f \simeq g$, $f, g : (X, X_0) \longrightarrow (Y, Y_0)$, then $\mathrm{L}(f) = \mathrm{L}(g)$. Let $\mathrm{T} : \mathbf{HCW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$ be the covariant (contravariant) functor defined by

$$\mathrm{T}(X, X_0) = \mathrm{L}(X, X_0), \quad (X, X_0) \in \mathbf{HCW}^2_{\boldsymbol{f}},$$

$$\mathrm{T}\bigl([f]\bigr) = \mathrm{L}(f), \quad \bigl([f] : (X, X_0) \longrightarrow (Y, Y_0)\bigr) \in \mathbf{HCW}^2_{\boldsymbol{f}}.$$

Consider the following commutative diagram:



where $\mathrm{H} : \mathbf{Top}^2 \longrightarrow \mathbf{HTop}^2$ is the homotopy functor.

The covariant (contravariant) functor

$$\hat{\mathrm{L}} = \hat{\mathrm{T}} \cdot \mathrm{CS} \cdot \mathrm{H} : \mathbf{Top}^2 \longrightarrow \mathbf{Gr}$$

$$\bigl(\check{\mathrm{L}} = \check{\mathrm{T}} \cdot \mathrm{CS} \cdot \mathrm{H} : \mathbf{Top}^2 \longrightarrow \mathbf{Gr}\bigr)$$

satisfies the homotopy axiom and is a coshape invariant extension of the covariant (contravariant) functor $\mathrm{L} : \mathbf{CW}^2_{\boldsymbol{f}} \longrightarrow \mathbf{Gr}$.

It is clear that such type of extensions exists for a covariant (contravariant) functor $\mathrm{L}:\mathbf{CW}^2_{\boldsymbol{f}*} \longrightarrow \mathbf{Gr}$ which satisfies the relative homotopy axiom, i.e. if $f \simeq g$ rel$\{*\}$, $f, g : (X, X_0, *) \longrightarrow (Y, Y_0, *)$, then $\mathrm{L}(f) = \mathrm{L}(g)$.

Remark 2. Let $K : \mathbf{Top}^2 \longrightarrow \mathbf{Gr}$ be a covariant (contravariant) functor satisfying the homotopy axiom. If each $\mathbf{HTop}^2$-coexpansion $\boldsymbol{p} : (\boldsymbol{X}, \boldsymbol{X}_0) = ((X_\alpha, X_{0\alpha}), p_{\alpha\alpha'} = [\pi_{\alpha\alpha'}], A) \longrightarrow (X, X_0)$ induces a direct (an inverse) limit $K(\boldsymbol{p}) : K(\boldsymbol{X}, \boldsymbol{X}_0) = (K(X_\alpha, X_{0\alpha}), K(\pi_{\alpha\alpha'}), A) \longrightarrow K(X, X_0)$ $(K(\boldsymbol{p}) : K(X, X_0) \longrightarrow K(\boldsymbol{X}, \boldsymbol{X}_0) = (K(X_\alpha, X_{0\alpha}), K(\pi_{\alpha\alpha'}), A))$, then the restriction functor L of the functor K to the subcategory $\mathbf{CW}^2_{\boldsymbol{f}}$ has an extension $\check{\mathrm{L}}$ $(\hat{\mathrm{L}})$ such that K and $\check{\mathrm{L}}$ $(\hat{\mathrm{L}})$ are naturally equivalent.

Remark 3. The technique developed here may be used to construct the coshape invariant and continuous extensions $\hat{\mathrm{T}} : \mathcal{T} \to \mathcal{C}$ $(\check{\mathrm{T}} : \mathcal{T} \to \mathcal{C})$ of covariant (contravariant) functors $\mathrm{T} : \mathcal{P} \to \mathcal{C}$ from a codense subcategory $\mathcal{P}$ of the category $\mathcal{T}$ in a category $\mathcal{C}$ with direct limits and inverse limits.

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