

Dimension Theory

RYSZARD ENGELKING

Warsaw University

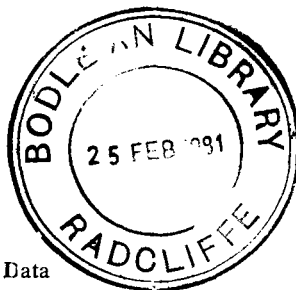


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PREFACE

Dimension theory is a branch of topology devoted to the definition and study of the notion of dimension in certain classes of topological spaces. It originated in the early twenties and rapidly developed during the next fifteen years. The investigations of that period were concentrated almost exclusively on separable metric spaces; they are brilliantly recapitulated in Hurewicz and Wallman's book *Dimension Theory*, published in 1941. After the initial impetus, dimension theory was at a standstill for ten years or more. A fresh start was made at the beginning of the fifties, when it was discovered that many results obtained for separable metric spaces can be extended to larger classes of spaces, provided that the dimension is properly defined. The last reservation necessitates an explanation. It is possible to define the dimension of a topological space X in three different ways, the small inductive dimension $\text{ind}X$, the large inductive dimension $\text{Ind}X$, and the covering dimension $\text{dim}X$. The three dimension functions coincide in the class of separable metric spaces, i.e., $\text{ind}X = \text{Ind}X = \text{dim}X$ for every separable metric space X . In larger classes of spaces the dimensions ind , Ind , and dim diverge. At first, the small inductive dimension ind was chiefly used; this notion has a great intuitive appeal and leads quickly and economically to an elegant theory. The dimension functions Ind and dim played an auxiliary role and often were not even explicitly defined. To attain the next stage of development of dimension theory, namely its extension to larger classes of spaces, first and foremost to the class of metrizable spaces, it was necessary to realize that in fact there are three theories of dimension and to decide which is the proper one. The adoption of such a point of view immediately led to the understanding that the dimension ind is practically of no importance outside the class of separable metric spaces and that the dimension dim prevails over the dimension Ind . The greatest achievement in dimension theory during the fifties was the discovery that $\text{Ind}X = \text{dim}X$ for every metric space X and the creation of a satisfactory dimension theory for metrizable spaces. Since that time many important results on dimension of topological spaces

have been obtained; they primarily bear upon the covering dimension $\dim$. Included among them are theorems of an entirely new type, such as the factorization theorems, with no counterpart in the classical theory, and a few quite complicated examples, which finally demarcated the range of applicability of various dimension functions.

The above outline of the history of dimension theory helps to explain the choice and arrangement of the material in the present book. In Chapter 1, which in itself constitutes more than half of the book, the classical dimension theory of separable metric spaces is developed. The purpose of the chapter is twofold: to present a self-contained exposition of the most important section of dimension theory and to provide the necessary geometric background for the rather abstract considerations of subsequent chapters. Chapters 2 and 3 are devoted to the large inductive dimension and the covering dimension, respectively. They contain the most significant results in dimension theory of general topological spaces and exhaustive information on further results. Chapter 4, the last in the book, develops the dimension theory of metrizable spaces. The interdependence of Chapters 2–4 is rather loose. After having read Chapter 1, the reader should be able to continue the reading according to his own interests or needs; in particular, he can read small fragments of Sections 3.1 and 3.2 and pass to Chapter 4 (cf. the introduction to that chapter).

Chapter 1 is quite elementary; the reader is assumed to be familiar only with the very fundamental notions of topology of separable metric spaces. The subsequent chapters are more difficult and demand from the reader some acquaintance with the notions and methods of general topology.

Each section ends with historical and bibliographic notes. Those are followed by problems which aim both at testing the reader's comprehension of the material and at providing additional information; the problems usually contain detailed hints, which, in fact, are outlines of proofs.

The mark $\square$ indicates the end of a proof or of an example. If it appears immediately after the statement of a theorem, a proposition or a corollary, it means that the statement is obviously valid.

Numbers in square brackets refer to the bibliography at the end of the book. The papers of each author are numbered separately, the number being the year of publication. In referring to my *General Topology* (Engelking [1977]), which is quite often cited in the second half of the present book, the symbol [GT] is used.

In 1971–1973 I gave a two-year course of lectures on dimension theory at the Warsaw University; this book is based on the notes from those

lectures. When preparing the present text, I availed myself of the comments of my students and colleagues. Thanks are due to K. Alster, J. Chaber, J. Kaniewski, P. Minc, R. Pol, T. Przymusiński, J. Przytycki and K. Wojtkowska. I am particularly obliged to Mrs. E. Pol, the first reader of this book, for her helpful cooperation, and to J. Krasinkiewicz for his careful reading of Chapter 1.

Ryszard Engelking

Warsaw, February 1977

CHAPTER 1

DIMENSION THEORY OF SEPARABLE METRIC SPACES

In the present chapter the classical dimension theory of separable metric spaces is developed. Practically all the results of this chapter were obtained in the years 1920–1940. They constitute a canon on which, in subsequent years, dimension theory for larger classes of spaces was modelled. Similarly, in Chapters 2–4 we shall follow the pattern of Chapter 1 and constantly refer to the classical theory. This arrangement influences our exposition: the classical material is discussed here in relation to modern currents in the theory; in particular, the dimension functions Ind and dim are introduced at an early stage and are discussed simultaneously with the dimension function ind .

To avoid repetitions in subsequent chapters, a few definitions and theorems are stated in a more general setting, not for separable metric but for topological, Hausdorff, regular or normal spaces; this is done only where the generalization does not influence the proof. If the reader is not acquainted with the notions of general topology, he should read “metric space” instead of “topological space”, “Hausdorff space”, “regular space”, and “normal space”. Reading the chapter for the first time, one can omit Sections 1.4 and 1.12–1.14, which deal with rather special topics; similarly, the final parts of Sections 1.6, 1.8 and 1.9 can be skipped.

Let us describe briefly the contents of this chapter.

Section 1.1 opens with the definition of the small inductive dimension ind ; in the sequel some simple consequences and reformulations of the definition are discussed. Sections 1.2 and 1.3 are devoted to a study of zero-dimensional spaces. We prove several important theorems, specified in the titles of the sections, which are generalized to spaces of higher dimension in Sections 1.5, 1.7 and 1.11.

In Section 1.4 we compare the properties of zero-dimensional spaces with the properties of different highly disconnected spaces. From this

comparison it follows that the class of zero-dimensional spaces in the sense of the small inductive dimension is the best candidate for the zero level in a classification of separable metric spaces according to their dimension. The results of this section are not used in the sequel of the book.

Section 1.5 contains the first group of basic theorems on n -dimensional spaces. As will become clear further on, the theorems in this group depend on the dimension ind , whereas the theorems that follow them depend on the dimension dim . Besides the generalizations of five theorems proved in Sections 1.2 and 1.3 for zero-dimensional spaces, Section 1.5 contains the decomposition and addition theorems.

In Section 1.6 the large inductive dimension Ind and the covering dimension dim are introduced; they both coincide with the small inductive dimension ind in the class of separable metric spaces. In larger classes of spaces the dimensions ind , Ind and dim diverge. This subject will be discussed thoroughly in the following chapters. In particular, it will become evident that the dimension ind , though excellent in the class of separable metric spaces, loses its importance outside this class.

Section 1.7 opens with the compactification theorem. The location of this theorem at such an early stage in the exposition of dimension theory is a novelty which, it seems, permits a clearer arrangement of the material. From the compactification theorem the coincidence of ind , Ind , and dim for separable metric spaces is deduced.

In Section 1.8 we discuss the dimensional properties of Euclidean spaces. We begin with the fundamental theorem of dimension theory, which states that $\text{ind} R^n = \text{Ind} R^n = \text{dim} R^n = n$; then we characterize n -dimensional subsets of R^n as sets with a non-empty interior, and we show that no closed subset of dimension $\leq n-2$ separates R^n . This last result is strengthened in Mazurkiewicz's theorem, which is established in the final part of the section with the assistance of Lebesgue's covering theorem.

Section 1.9 opens with the characterization of dimension in terms of extending mappings to spheres from a closed subspace over the whole space. From this characterization the Cantor-manifold theorem is deduced. In the final part of the section we give some information on the cohomological dimension.

In Section 1.10 we characterize n -dimensional spaces in terms of mappings with arbitrarily small fibers to polyhedra of geometric dimension $\leq n$ and develop the technics of nerves and κ -mappings which are crucial for the considerations of this and the following section.

In Section 1.11 we prove that every n -dimensional space can be embedded in R^{2n+1} and we describe two subspaces of R^{2n+1} which contain topologically all n -dimensional spaces; the second of those is a compact space.

The last three sections are of a more special character. Section 1.12 is devoted to a study of the relations between the dimensions of the domain and the range of a continuous mapping. In Section 1.13 we characterize compact spaces of dimension $\leq n$ as spaces homeomorphic to the limits of inverse sequences of polyhedra of geometric dimension $\leq n$, and in Section 1.14 we briefly discuss the prospects for an axiomatization of dimension theory.

1.1. Definition of the small inductive dimension

1.1.1. Definition. To every regular space X one assigns the *small inductive dimension* of X , denoted by $\text{ind} X$, which is an integer larger than or equal to -1 or the "infinite number" ∞ ; the definition of the dimension function ind consists in the following conditions:

(MU1) $\text{ind} X = -1$ if and only if $X = \emptyset$;

(MU2) $\text{ind} X \leq n$, where $n = 0, 1, \dots$, if for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that

$$x \in U \subset V \quad \text{and} \quad \text{ind Fr } U \leq n-1;$$

(MU3) $\text{ind} X = n$ if $\text{ind} X \leq n$ and $\text{ind} X > n-1$, i.e., the inequality $\text{ind} X \leq n-1$ does not hold;

(MU4) $\text{ind} X = \infty$ if $\text{ind} X > n$ for $n = -1, 0, 1, \dots$

The small inductive dimension ind is also called the *Menger-Urysohn dimension*.

Applying induction with respect to $\text{ind} X$, one can easily verify that whenever regular spaces X and Y are homeomorphic, then $\text{ind} X = \text{ind} Y$, i.e., the small inductive dimension is a topological invariant.

In order to simplify the statements of certain results proved in the sequel, we shall assume that the formulas $n \leq \infty$ and $n + \infty = \infty + n = \infty + \infty = \infty$ hold for every integer n .

Since every subspace M of a regular space X is itself regular, if the

dimension ind is defined for a space X it is also defined for every subspace M of the space X .

1.1.2. The subspace theorem. *For every subspace M of a regular space X we have $\text{ind } M \leq \text{ind } X$.*

Proof. The theorem is obvious if $\text{ind } X = \infty$, so that one can suppose that $\text{ind } X < \infty$. We shall apply induction with respect to $\text{ind } X$. Clearly, the inequality holds if $\text{ind } X = -1$.

Assume that the theorem is proved for all regular spaces whose dimension does not exceed $n-1 \geq -1$. Consider a regular space X with $\text{ind } X = n$, a subspace M of the space X , a point $x \in M$ and a neighbourhood V of the point x in M . By the definition of the subspace topology, there exists an open subset V_1 of the space X satisfying the equality $V = M \cap V_1$. Since $\text{ind } X \leq n$, there exists an open set $U_1 \subset X$ such that

$$x \in U_1 \subset V_1 \quad \text{and} \quad \text{ind Fr } U_1 \leq n-1.$$

The intersection $U = M \cap U_1$ is open in M and satisfies $x \in U \subset V$. The boundary $\text{Fr}_M U$ of the set U in the space M is equal to $M \cap \overline{M \cap U_1} \cap \overline{M \setminus U_1}$, where the bar denotes the closure operation in the space X ; thus the boundary $\text{Fr}_M U$ is a subspace of the space $\text{Fr } U_1$. Hence, by the inductive assumption, $\text{ind Fr}_M U \leq n-1$, which—together with (MU2)—yields the inequality $\text{ind } M \leq n = \text{ind } X$. $\square$

Sometimes it is more convenient to apply condition (MU2) in a slightly different form, involving the notion of a partition.

1.1.3. Definition. Let X be a topological space and A, B a pair of disjoint subsets of the space X ; we say that a set $L \subset X$ is a *partition between A and B* if there exist open sets $U, W \subset X$ satisfying the conditions

$$(1) \quad A \subset U, \quad B \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad X \setminus L = U \cup W.$$

Clearly, the partition L is a closed subset of X .

The notion of a partition is related to the notion of a separator. Let us recall that a set $T \subset X$ is a *separator between A and B* , or *T separates the space X between A and B* , if there exist two sets U_0 and W_0 open in the subspace $X \setminus T$ and such that $A \subset U_0$, $B \subset W_0$, $U_0 \cap W_0 = \emptyset$ and $X \setminus C = U_0 \cup W_0$. Obviously, a set $L \subset X$ is a partition between A and B if and only if L is a closed subset of X and L is a separator between A and B .

Separators are not to be confused with cuts, a related notion we will refer to in the notes below and in Section 1.8. Let us recall that a set $T \subset X$ is a *cut between A and B* , or *T cuts the space X between A and B* , if the sets A , B and T are pairwise disjoint and every continuum, i.e., a compact connected space $C \subset X$, intersecting both A and B intersects the set T . Clearly, every separator between A and B is a cut between A and B , but the two notions are not equivalent (see Problems 1.1.D and 1.8.F).

1.1.4. Proposition. *A regular space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if for every point $x \in X$ and each closed set $B \subset X$ such that $x \notin B$ there exists a partition L between x and B such that $\text{ind} L \leq n-1$.*

Proof. Let X be a regular space satisfying $\text{ind} X \leq n \geq 0$; consider a point $x \in X$ and a closed set $B \subset X$ such that $x \notin B$. There exist a neighbourhood $V \subset X$ of the point x such that $\bar{V} \subset X \setminus B$ and an open set $U \subset X$ such that $x \in U \subset V$ and $\text{ind} \text{Fr} U \leq n-1$. One easily sees that the set $L = \text{Fr} U$ is a partition between x and B ; the sets U and $W = X \setminus \bar{U}$ satisfy conditions (1).

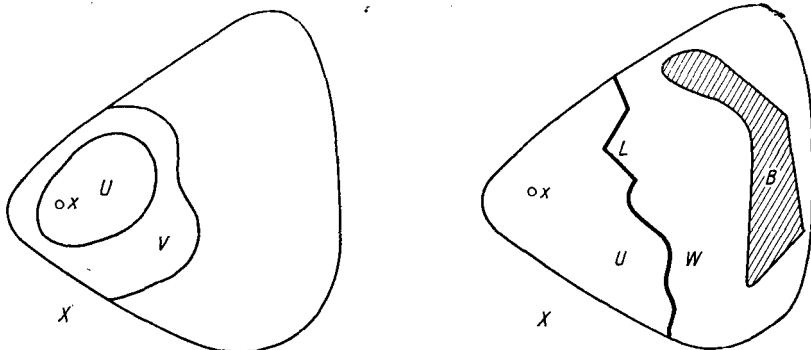


Fig. 1

Now, assume that a regular space X satisfies the condition of the theorem; consider a point $x \in X$ and a neighbourhood $V \subset X$ of the point x . Let L be a partition between x and $B = X \setminus V$ such that $\text{ind} L \leq n-1$ and let $U, W \subset X$ be open subsets of X satisfying conditions (1). We have

$$x \in U \subset X \setminus W \subset X \setminus B = V$$

and

$$\text{Fr} U \subset (X \setminus U) \cap (X \setminus W) = X \setminus (U \cup W) = L,$$

so that $\text{ind} \text{Fr} U \leq n-1$ by virtue of 1.1.2. Hence $\text{ind} X \leq n$. $\square$

Obviously, a regular space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if X has a base $\mathcal{B}$ such that $\text{ind Fr } U \leq n-1$ for every $U \in \mathcal{B}$. In the realm of separable metric spaces this observation can be made more precise.

1.1.5. Lemma. *If a topological space X has a countable base, then every base $\mathcal{B}$ for the space X contains a countable family $\mathcal{B}_0$ which is a base for X .*

Proof. Let $\mathcal{D} = \{V_i\}_{i=1}^{\infty}$ be a countable base for the space X . For $i = 1, 2, \dots$ define

$$\mathcal{B}_i = \{U \in \mathcal{B} : U \subset V_i\};$$

as $\mathcal{B}$ is a base for X , we have $\bigcup \mathcal{B}_i = V_i$. The subspace V_i of the space X also has a countable base, so that the open cover $\mathcal{B}_i$ of V_i contains a countable subcover $\mathcal{B}_{0,i}$. The family $\mathcal{B}_0 = \bigcup_{i=1}^{\infty} \mathcal{B}_{0,i} \subset \mathcal{B}$ is countable and is a base for X ; indeed, every non-empty open subset of X can be represented as the union of a subfamily of $\mathcal{D}$, and thus can also be represented as the union of a subfamily of $\mathcal{B}_0$. $\square$

1.1.6. Theorem. *A separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if X has a countable base $\mathcal{B}$ such that $\text{ind Fr } U \leq n-1$ for every $U \in \mathcal{B}$. $\square$*

Historical and bibliographic notes

The dimension of simple geometric objects is one of the most intuitive mathematical notions. There is no doubt that a segment, a square and a cube have dimension 1, 2 and 3, respectively. The necessity of a precise definition of dimension became obvious only when it was established that a segment has exactly as many points as a square (Cantor 1878), and that a square has a continuous parametric representation on a segment, i.e., that there exist continuous functions $x(t)$ and $y(t)$ such that points of the form $(x(t), y(t))$ fill out a square when t runs through a segment (Peano 1890). First and foremost the question arose whether there exists a parametric representation of a square on a segment which is at the same time one-to-one and continuous, i.e., whether a segment and a square are homeomorphic, and—more generally—whether the n -cube I^n and the m -cube I^m are homeomorphic if $n \neq m$; clearly, a negative answer was expected. Between 1890 and 1910 a few faulty proofs of the fact that I^n

and I^m are not homeomorphic if $n \neq m$ were produced and it was established that I , I^2 and I^3 are all topologically different.

The theorem that I^n and I^m are not homeomorphic if $n \neq m$ was proved by Brouwer in [1911]. The idea suggests itself that to prove this theorem one should define a function d assigning to every space a natural number, expressing the dimension of that space, such that to every pair of homeomorphic spaces the same natural number is assigned and that $d(I^n) = n$. It was none too easy, however, to discover such functions; the search for them gave rise to dimension theory. In Brouwer's paper [1911] no function d is explicitly defined, yet an analysis of the proof shows that to differentiate I^n and I^m for $n \neq m$ the author applies the fact that for a sufficiently small positive number ε it is impossible to transform the n -cube $I^n \subset R^n$ into a polyhedron $K \subset R^n$ of geometric dimension less than n by a continuous mapping $f: I^n \rightarrow K$ such that $\varrho(x, f(x)) < \varepsilon$ for every $x \in I^n$. As we shall show in Section 1.10, this property characterizes compact subspaces of R^n which have dimension equal to n . Another topological property of the n -cube I^n was discovered by Lebesgue in [1911], viz. the fact that I^n can be covered, for every $\varepsilon > 0$, by a finite family of closed sets with diameters less than ε such that all intersections of $n+2$ members of the family are empty, and cannot be covered by a finite family of closed sets with diameters less than 1 such that all intersections of $n+1$ members of the family are empty. Obviously, Lebesgue's observation implies that I^n and I^m are not homeomorphic if $n \neq m$. Though the proof outlined by Lebesgue contains a gap (filled by Brouwer in [1913] and by Lebesgue in [1921]), nevertheless the discovery of the new invariant was an important achievement which eventually led to the definition of the covering dimension. Lebesgue's paper [1911] contains one more important discovery. The author formulated the theorem (the proof was given in his paper [1921]) that for every continuous parametric representation $f(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of the n -cube I^n on the closed unit interval I , some fibres of f , i.e., inverse images of one-point sets, have cardinality at least $n+1$, and that I^n has a continuous parametric representation on I with fibres of cardinality at most $n+1$.

A decisive step towards the definition of dimension was made by Poincaré in [1912], where he observed that the dimension is related to the notion of separation and could be defined inductively. Poincaré called attention to the simple fact that solids can be separated by surfaces, surfaces by lines, and lines by points. It was due to the character of the journal for which Poincaré was writing and also to his death in the same year

1912 that Poincaré's important ideas were not presented as a precise definition of dimension.

The first definition of a dimension function was given by Brouwer in [1913], where he defined a topological invariant of compact metric spaces, called *Dimensionsgrad*, and proved that the *Dimensionsgrad* of the n -cube I^n is equal to n . In conformity with Poincaré's suggestion, the definition is inductive and refers to the notion of a cut: Brouwer defined the spaces with *Dimensionsgrad* 0 as spaces which do not contain any continuum of cardinality larger than one (i.e., as punctiform spaces; cf. Section 1.4), and stated that a space X has *Dimensionsgrad* less than or equal to $n \geq 1$ if for every pair A, B of disjoint closed subsets of X there exists a closed set $L \subset X$ which cuts X between A and B and has *Dimensionsgrad* less than or equal to $n-1$. Brouwer's notion of dimension is not equivalent to what we now understand by the dimension of a compact metric space; however, the two notions coincide in the realm of locally connected compact metric spaces (the proof is based on the fact that in this class of spaces the notions of a separator and a cut are equivalent for closed subsets; cf. Kuratowski [1968], p. 258). Brouwer did not study the new invariant closely: he only used it to give another proof that I^n and I^m are not homeomorphic if $n \neq m$.

Referring to the second part of Lebesgue's paper [1911], Mazurkiewicz proved in [1915] that for every continuous parametric representation of the square I^2 on the interval I , some fibres of f have cardinality at least 3, and showed that every continuum $C \subset R^2$ whose interior in R^2 is empty can be represented as a continuous image of the Cantor set under a mapping with fibres of cardinality at most 2. These results led him to define the dimension of a compact metric space X as the smallest integer n with the property that the space X can be represented as a continuous image of a closed subspace of the Cantor set under a mapping f such that $|f^{-1}(x)| \leq n+1$ for every $x \in X$. As was proved later (cf. Problem 1.7.D), this definition is equivalent to the definition of the small inductive dimension, but Mazurkiewicz's paper had no influence on the development of dimension theory.

The definition of the small inductive dimension ind was formulated by Urysohn in [1922] and by Menger in [1923], both papers contain also Theorem 1.1.2. Menger and Urysohn, working independently, built the framework of the dimension theory of compact metric spaces, but Urysohn was ahead of Menger by a few months and was able to establish a larger number of basic properties of dimension. Urysohn's results are presented

in a two-part paper, [1925] and [1926], published after the author's death in 1924, whereas Menger's results are contained in his papers [1923] and [1924] and in his book [1928]. A generalization of dimension theory to separable metric spaces is due to Tumarkin ([1925] and [1926]) and Hurewicz ([1927] and [1927b]). In [1927] Hurewicz, in a particularly successful way, made use of the inductive character of dimension and greatly simplified the proofs of some important theorems, e.g., the sum theorem and the decomposition theorem. Moreover, owing to his discovery of the compactification theorem, Hurewicz reduced, in a sense, the dimension theory of separable metric spaces to the dimension theory of compact metric spaces.

When the work of Menger and Urysohn drew the attention of mathematicians to the notion of dimension, Brouwer (in [1923], [1924], [1924a] and [1924b]) ascertained that the definition of his *Dimensionsgrad* was marred by a clerical error and that it should read exactly as the definition of the large inductive dimension (see Section 1.6) and thus should lead to the same notion of dimension for compact metric spaces; he also commented that even the original faulty definition of *Dimensionsgrad* could serve as a basis for an equally good, although different, dimension theory. Brouwer's arguments do not seem quite convincing. After the publication of Menger's book [1928] a heated discussion arose between Brouwer ([1928]) and Menger ([1929a], [1930], [1933]) concerning priority in defining the notion of dimension; a good account of this discussion is contained in Freudenthal's notes in the second volume of Brouwer's *Collected Papers* (Brouwer [1976]). The history of the first years of dimension theory and, in particular, an evaluation of the contributions of Menger and Urysohn can be found in Alexandroff [1951].

Problems

1.1.A. Observe that a metric space X satisfies the inequality $\text{ind} X \leq n \leq 0$ if and only if for every point $x \in X$ and each positive number ε there exists a neighbourhood $U \subset X$ of the point x such that $\delta(U) < \varepsilon$ and $\text{ind Fr } U \leq n - 1$.

1.1.B. To every regular space X and every point $x \in X$ one assigns the *dimension of X at the point x* , denoted by $\text{ind}_x X$, which is an integer larger than or equal to 0 or the infinite number ∞ ; the definition consists

in the following conditions: (1) $\text{ind}_x X \leq n$ if for each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\text{ind Fr } U \leq n-1$; (2) $\text{ind}_x X = n$ if $\text{ind}_x X \leq n$ and $\text{ind}_x X > n-1$; (3) $\text{ind}_x X = \infty$ if $\text{ind}_x X > n$ for $n = 0, 1, \dots$

(a) Note that $\text{ind } X \leq n$ if and only if $\text{ind}_x X \leq n$ for every $x \in X$.

(b) Formulate and prove the counterparts of 1.1.2, 1.1.4 and 1.1.6 for the dimension at a point.

1.1.C. Show that whenever regular spaces X and Y are homeomorphic, then $\text{ind } X = \text{ind } Y$.

1.1.D. Give an example of a subspace X of the plane and of a closed set $C \subset X$ with the property that for a pair A, B of disjoint closed subsets of X the set C is a cut between A and B but is not a separator between A and B .

1.2. The separation and enlargement theorems for dimension 0

A regular space X satisfying the equality $\text{ind } X = 0$ will be called a *zero-dimensional space*.

To begin with, we shall specialize the contents of the previous section to the case of zero-dimensional spaces.

1.2.1. Proposition. *A regular space X is zero-dimensional if and only if X is non-empty and for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open-and-closed set $U \subset X$ such that $x \in U \subset V$. $\square$*

1.2.2. Proposition. *Every non-empty subspace of a zero-dimensional space is zero-dimensional. $\square$*

1.2.3. Proposition. *A regular space X is zero-dimensional if and only if X is non-empty and for every point $x \in X$ and each closed set $B \subset X$ such that $x \notin B$ the empty set is a partition between x and B . $\square$*

1.2.4. Proposition. *A separable metric space X is zero-dimensional if and only if X is non-empty and has a countable base consisting of open-and-closed sets. $\square$*

We shall now discuss a few examples.

1.2.5. Examples. The space of irrational numbers $P \subset R$ is zero-dimensional because it has a countable base consisting of open-and-closed sets, viz., the sets of the form $P \cap (a, b)$, where a and b are rational numbers.

Similarly, the space of rational numbers $Q \subset R$ is zero-dimensional. More generally, if a metric space X satisfies the condition $0 < |X| < \mathfrak{c}$, then $\text{ind} X = 0$. Indeed, for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exist a positive number r such that $B(x, r) \subset V$ and a positive number $t < r$ such that $\varrho(x, y) \neq t$ for every $y \in X$, where ϱ is the metric on the space X ; the set $U = B(x, t)$ satisfies the condition $x \in U \subset V$ and is open-and-closed, because $\text{Fr } U \subset \{y \in X: \varrho(x, y) = t\} = \emptyset$.

A non-empty subspace X of the real line R is zero-dimensional if and only if it does not contain any interval. Since intervals are connected and no connected space containing at least two points is zero-dimensional, the condition is necessary by virtue of Proposition 1.2.2. The condition is also sufficient; indeed, the sets of the form $X \cap (a, b)$, where $a, b \in R$ and $a < x < b$, constitute a base for the space X at the point x and for each $V = X \cap (a, b)$ one can find an open-and-closed set $U \subset V$ such that $x \in U \subset V$, it suffices to define $U = X \cap (c, d)$, where $c \in (a, x) \setminus X$ and $d \in (x, b) \setminus X$.

In particular, the subspace C of the real line consisting of all real numbers in the closed unit interval I that have a tryadic expansion in which the digit 1 does not occur, i.e., the set of all numbers of the form $x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i}$, where x_i is equal to 0 or 1 for $i = 1, 2, \dots$, is zero-dimensional. Indeed, the set C does not contain any interval because $C = \bigcap_{i=1}^{\infty} F_i$, where F_i is the subset of I consisting of all numbers having a tryadic expansion in which 1 does not occur as the j -th digit for $j \leq i$, and F_i contains no interval of length larger than $1/3^i$. One easily sees that the set F_1 is obtained from I by removing the "middle" interval $(1/3, 2/3)$, the set F_2 is obtained from F_1 by removing the "middle" intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ of both parts of F_1 , and so on. The set F_i consists of 2^i disjoint intervals of length $1/3^i$.



Fig. 2

The subspace C of the real line is called the *Cantor set*. Since C is a closed subset of I , the Cantor set is compact.

The Cartesian product $Q^n \subset R^n$ is zero-dimensional, because it is a countable space. The Cartesian product $P^n \subset R^n$ is also zero-dimensional; the proof is left to the reader (cf. Theorem 1.3.6). $\square$

As shown in the following theorem, zero-dimensional separable metric spaces have a separation property which is much stronger than the property described in Proposition 1.2.3.

1.2.6. The first separation theorem for dimension 0. *If X is a zero-dimensional separable metric space, then for every pair A, B of disjoint closed subsets of X the empty set is a partition between A and B , i.e., there exists an open-and-closed set $U \subset X$ such that $A \subset U$ and $B \subset X \setminus U$.*

Proof. For every $x \in X$ there exists an open-and-closed set $W_x \subset X$ such that $x \in W_x$ and

$$(1) \quad \text{either } A \cap W_x = \emptyset \quad \text{or} \quad B \cap W_x = \emptyset.$$

The open cover $\{W_x\}_{x \in X}$ of the space X has a countable subcover $\{W_{x_i}\}_{i=1}^{\infty}$. The sets

$$U_i = W_{x_i} \setminus \bigcup_{j < i} W_{x_j} \subset W_{x_i}, \quad \text{where } i = 1, 2, \dots,$$

are open and constitute a cover of the space X . Let us define

$$U = \bigcup \{U_i: A \cap U_i \neq \emptyset\} \quad \text{and} \quad W = \bigcup \{U_i: A \cap U_i = \emptyset\};$$

obviously, $A \subset U$ and it follows from (1) that $B \subset W$. Since the sets U_i are pairwise disjoint, $W = X \setminus U$, which implies that the set U is open-and-closed and that $B \subset X \setminus U$. $\square$

1.2.7. Remark. It follows from the above proof that in Theorem 1.2.6 the assumption that X is a separable metric space can be replaced by the weaker assumption that X is a *Lindelöf space*, i.e., a regular space which has the property that every open cover of X has a countable subcover.

Now we are going to prove the second separation theorem, which is still stronger than Theorem 1.2.6. Let us recall that two subsets A and B of a topological space X are *separated* if $A \cap \bar{B} = \emptyset = \bar{A} \cap B$. One easily sees that the sets A and B are separated if and only if they are disjoint and open (or—equivalently—closed) in their union $A \cup B$, i.e., if $A \cap B = \emptyset$ and the empty set is a partition between A and B in the subspace $A \cup B$ of X . In particular, two disjoint open sets, and also two disjoint closed sets, are separated.

The second separation theorem will be deduced from two lemmas; with a view to further applications, the second lemma is formulated in a more general way than needed in this section.

1.2.8. Lemma. *For every pair A, B of separated sets in a metric space X there exist open sets $U, W \subset X$ such that*

$$(2) \quad A \subset U, \quad B \subset W \quad \text{and} \quad U \cap W = \emptyset.$$

Proof. Let ϱ be the metric on the space X and let $f(x) = \varrho(x, A)$ and $g(x) = \varrho(x, B)$ denote the distance of the point $x \in X$ from A and B , respectively. Since the functions f and g are continuous, the sets

$$U = \{x \in X: f(x) - g(x) < 0\} \quad \text{and} \quad W = \{x \in X: f(x) - g(x) > 0\}$$

are open. The inclusions in (2) follow from the equalities $f^{-1}(0) = \bar{A}$ and $g^{-1}(0) = \bar{B}$; the equality $U \cap W = \emptyset$ follows directly from the definition of U and W . $\square$

1.2.9. Lemma. *Let M be a subspace of a metric space X and A, B a pair of disjoint closed subsets of X . For every partition L' in the space M between $M \cap \bar{V}_1$ and $M \cap \bar{V}_2$, where V_1, V_2 are open subsets of X such that $A \subset V_1$, $B \subset V_2$ and $\bar{V}_1 \cap \bar{V}_2 = \emptyset$, there exists a partition L in the space X between A and B which satisfies the inclusion $M \cap L \subset L'$.*

If M is a closed subspace of a metric space X and A, B a pair of disjoint closed subsets of X , then for every partition L' in the space M between $M \cap A$ and $M \cap B$ there exists a partition L in the space X between A and B which satisfies the inclusion $M \cap L \subset L'$.

Proof. Let U', W' be open subsets of M satisfying the conditions

$$M \cap \bar{V}_1 \subset U', \quad M \cap \bar{V}_2 \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad M \setminus L' = U' \cup W'.$$

Observe that

$$(3) \quad A \cap \bar{W}' = \emptyset = B \cap \bar{U}'.$$

Indeed, since $V_1 \cap W' = M \cap V_1 \cap W' \subset U' \cap W' = \emptyset$ and since the set V_1 is open, we have $V_1 \cap \bar{W}' = \emptyset$, which implies that $A \cap \bar{W}' = \emptyset$; by symmetry of assumptions, also $B \cap \bar{U}' = \emptyset$.

The sets U' and W' are disjoint and open in their union $U' \cup W'$, and thus they are separated, i.e.,

$$(4) \quad U' \cap \bar{W}' = \emptyset = \bar{U}' \cap W'.$$

It follows from (3) and (4) that the sets $A \cup U'$ and $B \cup W'$ are also separated. Hence, by Lemma 1.2.8, there exist open sets $U, W \subset X$ such that

$$A \cup U' \subset U, \quad B \cup W' \subset W \quad \text{and} \quad U \cap W = \emptyset.$$

The set $L = X \setminus (U \cup W)$ is a partition in the space X between A and B . Since

$$M \cap L = M \setminus (U \cup W) \subset M \setminus (U' \cup W') = L',$$

the first part of the lemma is established.

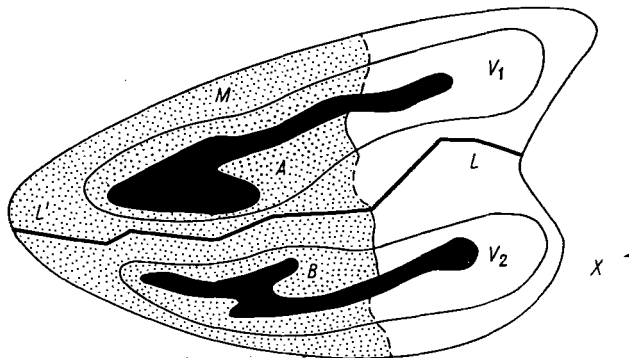


Fig. 3

To prove the second part, consider open subsets U_1, W_1 of the space M satisfying the conditions

$$M \cap A \subset U_1, \quad M \cap B \subset W_1, \quad U_1 \cap W_1 = \emptyset \quad \text{and} \quad M \setminus L' = U_1 \cup W_1.$$

Since $A \cap (M \setminus U_1) = \emptyset$, $B \cap (M \setminus W_1) = \emptyset$ and $A \cap B = \emptyset$, there exist open sets $V_1, V_2 \subset X$ such that

$$A \subset V_1 \subset \bar{V}_1 \subset X \setminus (M \setminus U_1), \quad B \subset V_2 \subset \bar{V}_2 \subset X \setminus (M \setminus W_1)$$

$$\text{and} \quad \bar{V}_1 \cap \bar{V}_2 = \emptyset.$$

Obviously, L' is a partition in the space M between $M \cap \bar{V}_1$ and $M \cap \bar{V}_2$, so that the partition L exists by the first part of the lemma. $\square$

1.2.10. Remark. In the proof of Lemma 1.2.9 only the fact that Lemma 1.2.8 holds in metric spaces was applied; as the latter lemma holds in hereditarily normal spaces (see Theorem 2.1.1), Lemma 1.2.9 also holds in hereditarily normal spaces.

1.2.11. The second separation theorem for dimension 0. If X is an arbitrary metric space and Z a zero-dimensional separable subspace of X , then for

every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $L \cap Z = \emptyset$.

Proof. Consider open sets $V_1, V_2 \subset X$ such that $A \subset V_1$, $B \subset V_2$ and $\overline{V_1} \cap \overline{V_2} = \emptyset$. By virtue of Theorem 1.2.6, the empty set is a partition in the space Z between $Z \cap \overline{V_1}$ and $Z \cap \overline{V_2}$. Applying the first part of Lemma 1.2.9 we obtain the required partition L . $\square$

The last theorem yields a characterization of zero-dimensional subspaces in terms of neighbourhoods in the whole space (cf. the proof of Proposition 1.1.4):

1.2.12. Proposition. *A separable subspace M of an arbitrary metric space X is zero-dimensional if and only if M is non-empty and for every point $x \in M$ (or—equivalently—for every point $x \in X$) and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $M \cap \text{Fr } U = \emptyset$. $\square$*

Proposition 1.2.12 and Lemma 1.1.5 imply

1.2.13. Proposition. *A subspace M of a separable metric space X is zero-dimensional if and only if M is non-empty and X has a countable base $\mathcal{B}$ such that $M \cap \text{Fr } U = \emptyset$ for every $U \in \mathcal{B}$. $\square$*

It is natural to ask at this point whether every zero-dimensional subspace of a given space can be enlarged to a “better” zero-dimensional subspace. The example of the subspace of the real line consisting of rational numbers shows that, generally, zero-dimensional subspaces cannot be enlarged to closed zero-dimensional subspaces; however, as shown in the next theorem, they can always be enlarged to zero-dimensional G_δ -sets. Let us recall that G_δ -sets are defined as countable intersections of open sets, and F_σ -sets as their complements, i.e., countable unions of closed sets.

1.2.14. The enlargement theorem for dimension 0. *For every zero-dimensional separable subspace Z of an arbitrary metric space X there exists a G_δ -set Z^* in X such that $Z \subset Z^*$ and the subspace Z^* of the space X is zero-dimensional.*

Proof. Since every closed subset of a metric space is a G_δ -set and a G_δ -set in a subspace which itself is a G_δ -set is a G_δ -set in the space, one can assume

that $\bar{Z} = X$. Thus X is separable and, by virtue of Proposition 1.2.13, has a countable base $\mathcal{B}$ such that $Z \cap \text{Fr } U = \emptyset$ for every $U \in \mathcal{B}$. The union $F = \bigcup \{\text{Fr } U : U \in \mathcal{B}\}$ is an F_σ -set, and its complement $Z^* = X \setminus F$ is a G_δ -set which contains the set Z . From Proposition 1.2.13 it follows that Z^* is zero-dimensional. $\square$

As the reader has undoubtedly observed, Theorem 1.2.6 states that in separable metric spaces two properties, viz., the property that the empty set is a partition between any disjoint closed sets A , B , and the property that the empty set is a partition between every point x and each closed set B such that $x \notin B$, are equivalent. The question arises whether the property that the empty set is a partition between any distinct points x , y is still the same property. As shown in the following example, the answer to this question is negative (cf. the notion of a totally disconnected space discussed in Section 1.4).

1.2.15. Erdős' example. Let us recall that *Hilbert space* H consists of all infinite sequences $\{x_i\}$ of real numbers such that the series $\sum_{i=1}^{\infty} x_i^2$ is convergent. For every point $x = \{x_i\} \in H$ the *norm* $\|x\|$ of the point x is the number $\sqrt{\sum_{i=1}^{\infty} x_i^2}$, and the distance between $x = \{x_i\}$ and $y = \{y_i\}$ is defined by

$$\varrho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

i.e., is equal to the norm of the difference $x - y$. The function ϱ is a metric on H , and H is a separable metric space.

We shall show that, in the subspace H_0 of the space H consisting of the points $\{x_i\} \in H$ such that x_i is rational for every i , the empty set is a partition between any distinct points x , y , and yet H_0 is not zero-dimensional.

Let us consider a pair $x = \{x_i\}$, $y = \{y_i\}$ of distinct points of H_0 . There exists a natural number i_0 such that $x_{i_0} \neq y_{i_0}$; without loss of generality one can assume that $x_{i_0} < y_{i_0}$. Take an irrational number t such that $x_{i_0} < t < y_{i_0}$ and define

$$U = \{ \{z_i\} \in H_0 : z_{i_0} < t \}.$$

One can easily verify that the set U is open-and-closed in H_0 . As $x \in U \subset H_0 \setminus \{y\}$, the empty set is a partition between x and y .

Now, let $x_0 \in H_0$ be the sequence whose terms are all equal to zero and let $V = B(x_0, 1) = \{x \in H_0 : \|x\| < 1\}$. We shall show that for every neighbourhood U of the point x_0 which is contained in the neighbourhood V of x_0 we have $\text{Fr } U \neq \emptyset$.

We shall define inductively a sequence $a_1, a_2, \dots$ of rational numbers such that

$$(5) \quad x_k = (a_1, a_2, \dots, a_k, 0, 0, \dots) \in U \quad \text{and} \quad \varrho(x_k, H_0 \setminus U) \leq 1/k$$

for $k = 1, 2, \dots$. Conditions (5) are satisfied for $k = 1$ if we let $a_1 = 0$. Suppose that the rational numbers $a_1, a_2, \dots, a_{m-1}$ are already defined and conditions (5) are satisfied for $k \leq m-1$. The sequence

$$x_i^m = (a_1, a_2, \dots, a_{m-1}, i/m, 0, 0, \dots)$$

is an element of H_0 for $i = 1, 2, \dots, m$. As $x_0^m = x_{m-1} \in U$ and $x_m^m \notin U$ because $\|x_m^m\| \geq 1$, there exists an $i_0 < m$ such that $x_{i_0}^m \in U$ and $x_{i_0+1}^m \notin U$. One easily sees that conditions (5) are satisfied for $k = m$ if we let $a_m = i_0/m$. Thus the sequence $a_1, a_2, \dots$ is defined. From the first part of

(5) it follows that $\sum_{i=1}^k a_i^2 < 1$ for $k = 1, 2, \dots$, so that $\sum_{i=1}^{\infty} a_i^2 \leq 1$. Hence

$a = \{a_i\}$ is a point of H_0 and $a \in \bar{U}$. On the other hand, from the second part of (5) it follows that $a \in \overline{H_0 \setminus U}$, so that $a \in \text{Fr } U$ and $\text{Fr } U \neq \emptyset$.

Thus we have shown that there is no open-and-closed set $U \subset H_0$ such that $x_0 \in U \subset V$, and this implies that the space H_0 is not zero-dimensional.

To conclude, let us observe that the only open-and-closed bounded subset of the space H_0 is the empty set. Indeed, if there existed a non-empty open-and-closed bounded set $W \subset H_0$, then by a suitable translation and contraction of W we could obtain an open-and-closed set $U \subset H_0$ such that $x_0 \in U \subset V$. $\square$

Historical and bibliographic notes

Zero-dimensional spaces were defined by Sierpiński in [1921], before dimension theory was originated. Sierpiński's objective was to compare a few classes of metric spaces which are highly disconnected; a similar comparison will be drawn in Section 1.4. The theorems in the present

section are all special cases of theorems which will be proved in Section 1.5 for an arbitrary dimension n ; they were at once established in this more general form. Theorem 1.2.6 was proved for compact metric spaces by Menger in [1924] and by Urysohn in [1926] and was extended to separable metric spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]; the generalization stated in Remark 1.2.7 was obtained by Vedenisoff in [1939]. Theorem 1.2.11 was proved for compact metric spaces by Menger in [1924] and was extended to separable metric spaces by Hurewicz in [1927]. Theorem 1.2.14 was obtained by Tumarkin in [1926] (announcement [1925]). The space in Example 1.2.15 was described by Erdős in [1940]; the first example of a space with similar properties was given by Sierpiński in [1921].

Problems

1.2.A. Let M be a subspace of a metric space X and let x be a point of M . Prove that $\text{ind}_x M = 0$ if and only if there exists a base $\{U_i\}_{i=1}^\infty$ for the space X at the point x such that $M \cap \text{Fr } U_i = \emptyset$ for $i = 1, 2, \dots$

1.2.B. Show that a subspace M of a metric space X is zero-dimensional if and only if M is non-empty and for every point $x \in M$ and each neighbourhood V of the point x in the space X there exists an open set $U \subset V$ such that $x \in U \subset V$ and $M \cap \text{Fr } U = \emptyset$ (cf. Proposition 1.2.12 and Problem 4.1.C).

1.2.C. Show by an example that in the second part of Lemma 1.2.9 the assumption that the subspace M is closed cannot be omitted.

1.2.D. Check that every countable compact space has isolated points. Note that locally compact countable spaces have the same property.

Hint. Arrange all points of the space into a sequence $x_1, x_2, \dots$ and, assuming that none of the points x_i is isolated, define a decreasing sequence $F_1 \supset F_2 \supset \dots$ of non-empty closed sets such that $x_i \notin F_i$ for $i = 1, 2, \dots$

One can equally well apply the Baire category theorem.

Remark. Every completely metrizable space with no isolated points contains a subspace homeomorphic to the Cantor set and thus is of cardinality at least $\mathfrak{c}$ (see [GT], Problem 4.5.5(a)).

1.2.E (Kuratowski [1932a]). Define a mapping f of the Cantor set C to the interval $[-1, 1]$ by letting

$$f(0) = 0 \quad \text{and} \quad f(x) = \frac{(-1)^{i_1}}{2} + \frac{(-1)^{i_2}}{2^2} + \dots$$

$$\text{for each } x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \in C \setminus \{0\},$$

where $i_1 < i_2 < \dots$ is the sequence (finite or infinite) of all i 's such that $x_i = 1$; let C_0 denote the set of all points $x \in C \setminus \{0\}$ for which the sequence $i_1, i_2, \dots$ is infinite and let $C_1 = C \setminus C_0$.

(a) Observe that the set C_1 consists of the number 0 and the right end-points of the intervals $(1/3, 2/3)$, $(1/9, 2/9)$, $(7/9, 8/9)$, ... removed from I to obtain the Cantor set; note that C_1 is a countable set.

(b) Check that the function f is continuous at all points of C_0 .

(c) Verify that for every $x = \frac{2}{3^{i_1}} + \frac{2}{3^{i_2}} + \dots + \frac{2}{3^{i_k}} \in C_1$, the upper limit $\overline{f(x)}$ and the lower limit $\underline{f(x)}$ of the function f at the point x are equal to $f(x) + \frac{1}{2^k}$ and $f(x) - \frac{1}{2^k}$, respectively; observe that the function f is discontinuous at all points of C_1 .

(d) For each point $x \in C_1$ define two sequences, $\{x'_n\}$ and $\{x''_n\}$, of points in C_1 such that

$$\lim x'_n = x, \quad \lim \overline{f(x'_n)} = \overline{f(x)}, \quad \lim \underline{f(x'_n)} = f(x)$$

and

$$\lim x''_n = x, \quad \lim \underline{f(x''_n)} = \underline{f(x)}, \quad \lim \overline{f(x''_n)} = f(x).$$

(e) Consider the graph $K = \{(x, f(x)): x \in C\}$ of the function f and prove that $\text{ind}_{(x, f(x))} K = 1$ for every $x \in C_1$.

Hint. Prove that the space $D = K \cup \bigcup_{x \in C_1} (\{x\} \times [\underline{f(x)}, \overline{f(x)}]) \subset R^2$ is compact. Assuming that $\text{ind}_{(x_0, f(x_0))} K = 0$ for an $x_0 \in C_1$, show that there exists a partition L in the space D between $(x_0, f(x_0))$ and $(x_0, f(x_0))$ such that $L \cap K = \emptyset$. Show that the set M consisting of all points $x \in C_1$ such that L is a partition between either $(x, f(x))$ and $(x, \underline{f(x)})$ or $(x, f(x))$ and $(x, \overline{f(x)})$ is contained in the projection of L onto C and contains an isolated point (see Problem 1.2.D); deduce a contradiction of (d).

(f) Observe that $\text{ind}_{(x, f(x))} K = 1$ for every $x \in C_0$.

(g) Check that the empty set is a partition between any distinct points of the space K .

(h) Note that the space K is completely metrizable.

Hint. Check that K is a G_δ -set in D and apply Lemma 1.3.12.

1.3. The sum, Cartesian product, universal space, compactification and embedding theorems for dimension 0

The theorems enumerated in the title of the section belong to the most important results of dimension theory. For the time being we shall only prove their special cases pertaining to zero-dimensional spaces. We begin with the sum theorem.

1.3.1. The sum theorem for dimension 0. *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed zero-dimensional subspaces, then X is zero-dimensional.*

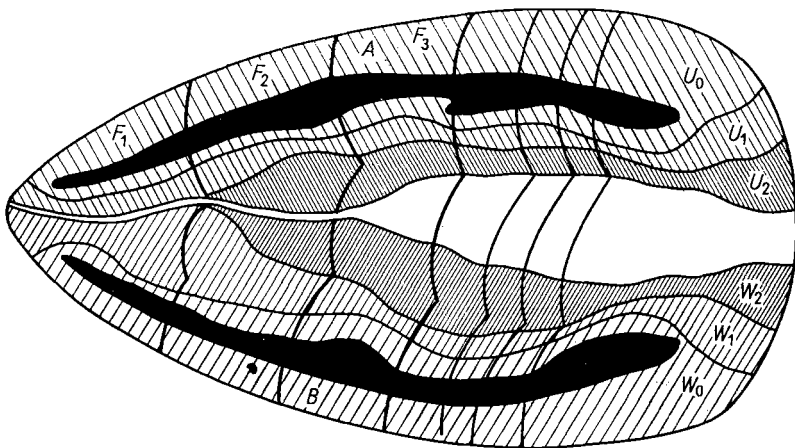


Fig. 4

Proof. Consider a pair A, B of disjoint closed subsets of the space X . We shall prove that there exist open sets $U, W \subset X$ such that

(1) $A \subset U, \quad B \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad X = U \cup W,$
i.e., that the empty set is a partition between A and B .

Let U_0, W_0 be open subsets of X such that

(2) $A \subset U_0, \quad B \subset W_0 \quad \text{and} \quad \overline{U_0} \cap \overline{W_0} = \emptyset.$

We shall define inductively two sequences $U_0, U_1, U_2, \dots$ and $W_0, W_1, W_2, \dots$ of open subsets of X satisfying for $i = 0, 1, 2, \dots$ the conditions:

$$(3) \quad U_{i-1} \subset U_i, \quad W_{i-1} \subset W_i \quad \text{if} \quad i \geq 1 \quad \text{and} \quad \bar{U}_i \cap \bar{W}_i = \emptyset.$$

$$(4) \quad F_i \subset U_i \cup W_i, \quad \text{where} \quad F_0 = \emptyset.$$

Clearly, the sets U_0, W_0 defined above satisfy both conditions for $i = 0$. Assume that the sets U_i, W_i satisfying (3) and (4) are defined for all $i < k$.

The sets $\bar{U}_{k-1} \cap F_k$ and $\bar{W}_{k-1} \cap F_k$ are closed and disjoint; since the space F_k is zero-dimensional, by virtue of Theorem 1.2.6 there exists an open-and-closed subset V of F_k such that

$$(5) \quad \bar{U}_{k-1} \cap F_k \subset V \quad \text{and} \quad \bar{W}_{k-1} \cap F_k \subset F_k \setminus V.$$

The set F_k being closed in X , the sets V and $F_k \setminus V$ are also closed in X ; from (5) it follows that

$$(\bar{U}_{k-1} \cup V) \cap [\bar{W}_{k-1} \cup (F_k \setminus V)] = (V \cap \bar{W}_{k-1}) \cup [\bar{U}_{k-1} \cap (F_k \setminus V)] = \emptyset,$$

so that there exist open sets $U_k, W_k \subset X$ satisfying

$$\bar{U}_{k-1} \cup V \subset U_k, \quad \bar{W}_{k-1} \cup (F_k \setminus V) \subset W_k \quad \text{and} \quad \bar{U}_k \cap \bar{W}_k = \emptyset.$$

The sets U_k, W_k satisfy (3) and (4) for $i = k$; thus, the construction of the sequences $U_0, U_1, U_2, \dots$ and $W_0, W_1, W_2, \dots$ is completed. It follows from (2), (3) and (4) that the unions $U = \bigcup_{i=0}^{\infty} U_i$ and $W = \bigcup_{i=0}^{\infty} W_i$ satisfy (1). $\square$

1.3.2. Remark. Undoubtedly, the reader has noted that in the proof of Theorem 1.3.1 only the normality of the space X and the fact that the empty set is a partition between each pair of disjoint closed subsets of the space F_k were applied. Hence we have proved that if a normal space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces with the property that for $i = 1, 2, \dots$ the empty set is a partition between each pair of disjoint closed subsets of the space F_i , then the empty set is a partition between any disjoint closed subsets of the space X .

From Theorem 1.3.1 several corollaries follow:

1.3.3. Corollary. *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \dots$ of zero-dimensional subspaces, where F_i is an F_σ -set for $i = 1, 2, \dots$, then X is zero-dimensional. $\square$*

1.3.4. Corollary. *If a separable metric space X can be represented as the union of two zero-dimensional subspaces A and B , one of them closed, then X is zero-dimensional.*

Proof. Let us suppose that $A = \bar{A}$; the open set $X \setminus A \subset B$ is zero-dimensional by virtue of the subspace theorem. Since every open subset of a metric space is an F_σ -set and since $X = A \cup (X \setminus A)$, to complete the proof it suffices to apply Corollary 1.3.3. $\square$

1.3.5. Corollary. *If by adjoining a finite number of points to a zero-dimensional separable metric space one obtains a metric space, then the space obtained is zero-dimensional and separable.* $\square$

In connection with the last corollary, let us note that by adjoining countably many points to the space of irrational numbers one can obtain the real line, i.e., a space of positive dimension.

We shall now prove the Cartesian product theorem.

1.3.6. The Cartesian product theorem for dimension 0. *The Cartesian product $X = \prod_{i=1}^{\infty} X_i$ of a countable family $\{X_i\}_{i=1}^{\infty}$ of regular spaces is zero-dimensional if and only if all spaces X_i are zero-dimensional.*

Proof. If $X \neq \emptyset$, then each space X_i is homeomorphic to a subspace of X , so that if X is zero-dimensional, then all spaces X_i are zero-dimensional. To prove the reverse implication, it is enough to consider for $i = 1, 2, \dots$ a base $\mathcal{B}_i$ for the space X_i consisting of open-and-closed sets and observe that the sets of the form $U_1 \times U_2 \times \dots \times U_k \times \prod_{i=k+1}^{\infty} X_i$, where $U_i \in \mathcal{B}_i$ for $i \leq k$ and $k = 1, 2, \dots$, constitute a base for X and are open-and-closed in X . $\square$

Theorems 1.1.2 and 1.3.6 yield

1.3.7. Corollary. *The limit of an inverse sequence $\{X_i, \pi_j^i\}$ of zero-dimensional spaces is either zero-dimensional or empty.* $\square$

The sum and Cartesian product theorems allow us to increase our stock of zero-dimensional spaces.

1.3.8. Examples. For every pair k, n of integers satisfying $0 \leq k \leq n \leq 1$ denote by Q_k^n the subspace of Euclidean n -space R^n consisting of all points

which have exactly k rational coordinates. We shall prove that Q_k^n is a zero-dimensional space.

For each choice of k distinct natural numbers $i_1, i_2, \dots, i_k$ not larger than n and each choice of k rational numbers $r_1, r_2, \dots, r_k$, the Cartesian product $\prod_{i=1}^n R_i$, where $R_{i_j} = \{r_j\}$ for $j = 1, 2, \dots, k$ and $R_i = R$ for $i \neq i_j$, is a closed subspace of R^n . Hence, $Q_k^n \cap \prod_{i=1}^n R_i$ is a closed subspace of Q_k^n . Since the space $Q_k^n \cap \prod_{i=1}^n R_i$ is homeomorphic to the subspace of R^{n-k} consisting of all points with irrational coordinates, it follows from Example 1.2.5 and Theorem 1.3.6 that it is a zero-dimensional space. Theorem 1.3.1 implies that the space Q_k^n is zero-dimensional, because the family of all subspaces of the form $Q_k^n \cap \prod_{i=1}^{\infty} R_i$ is countable and its union is equal to the whole of Q_k^n .

It also follows from Theorem 1.3.6 that the space $Q^{\aleph_0}$, which is the Cartesian product of $\aleph_0$ copies of the space of rational numbers, and the space $P^{\aleph_0}$, which is the Cartesian product of $\aleph_0$ copies of the space of irrational numbers, are zero-dimensional. $\square$

1.3.9. Definition. We say that a topological space X is *universal* for a class $\mathcal{K}$ of topological spaces if X belongs to $\mathcal{K}$ and every space in the class $\mathcal{K}$ is homeomorphic to a subspace of the space X .

We are now going to prove that the Cantor set C and the space P of irrational numbers are universal spaces for the class of all zero-dimensional separable metric spaces; in the proof we shall apply the fact that both C and P can be represented as countable Cartesian products.

1.3.10. Proposition. *The Cantor set C is homeomorphic to the Cartesian product $D^{\aleph_0} = \prod_{i=1}^{\infty} D_i$, where D_i , for $i = 1, 2, \dots$, is the two-point discrete space $D = \{0, 1\}$.*

Proof. As one readily verifies, for each point $x \in C$ the representation in the form $x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i}$, where x_i 's are equal to 0 or 1, is unique. Hence,

letting

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \quad \text{for } \{x_i\} \in D^{\aleph_0},$$

we define a one-to-one mapping of $D^{\aleph_0}$ onto C . Since the function $f_i: D^{\aleph_0} \rightarrow I$ defined by $f_i(\{x_i\}) = \frac{2x_i}{3^i}$ is continuous for $i = 1, 2, \dots$ and the series $\sum_{i=1}^{\infty} f_i$ is uniformly convergent to f , the latter function is continuous. It follows from the compactness of $D^{\aleph_0}$ that f is a homeomorphism. $\square$

The proof of the counterpart of Proposition 1.3.10 for the space of irrational numbers requires some calculation to remedy the lack of compactness. Let us recall that if X is a metric space, then by a *metric on the space X* we mean any metric on the set X which is equivalent to the original metric on X , i.e., induces the same convergence as the original metric.

1.3.11. Lemma. *Let ϱ be an arbitrary metric on the space P of irrational numbers and ε a positive number. For every non-empty open set $U \subset P$ there exists an infinite sequence $F_1, F_2, \dots$ of pairwise disjoint non-empty open-and-closed subsets of P such that $U = \bigcup_{i=1}^{\infty} F_i$ and the diameters with respect to ϱ of all sets F_i are less than ε .*

Proof. Consider an interval $(a, b) \subset R$ with rational end-points such that $(a, b) \cap P \subset U$ and divide it into $\aleph_0$ pairwise disjoint non-empty intervals $(a_1, b_1), (a_2, b_2), \dots$ with rational end-points. Thus we have $(a, b) \cap P = \bigcup_{i=1}^{\infty} A_i$, where $A_i = (a_i, b_i) \cap P \neq \emptyset$, $a_i, b_i \in Q$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$; in addition let $A_0 = U \setminus (a, b)$. The sets $A_0, A_1, A_2, \dots$ are open in P and by virtue of Proposition 1.2.4 for $i = 0, 1, 2, \dots$ there exist in P open-and-closed sets $A_{i,1}, A_{i,2}, \dots$, all of diameter less than ε , such that $A_i = \bigcup_{j=1}^{\infty} A_{i,j}$; letting $B_{i,j} = A_{i,j} \setminus \bigcup_{k < j} A_{i,k}$ for $j = 1, 2, \dots$, we obtain pairwise disjoint open-and-closed subsets of P whose union is equal to A_i . To complete the proof it suffices to arrange all non-empty sets $B_{i,j}$ into a simple sequence $F_1, F_2, \dots$ $\square$

The next lemma is an important theorem on complete spaces. In con-

sideration of further applications it is stated in full generality and not merely for the subspace P of the real line R .

1.3.12. Lemma. *Every G_δ -set X in a completely metrizable space X_0 is completely metrizable.*

Proof. Let ϱ be a complete metric on the space X_0 and let $X = \bigcap_{i=1}^{\infty} G_i$, where G_i is open in X_0 for $i = 1, 2, \dots$. Define

$$F_i = X_0 \setminus G_i \quad \text{and} \quad f_i(x) = 1/\varrho(x, F_i) \quad \text{for } x \in X \text{ and } i = 1, 2, \dots;$$

the functions $f_1, f_2, \dots$ from X to the real line R are continuous. One readily sees that the formula $f(x) = (x, f_1(x), f_2(x), \dots)$ defines a homeo-

morphic embedding $f: X \rightarrow \prod_{i=0}^{\infty} X_i$, where $X_i = R$ for $i \geq 1$. Since the

Cartesian product of countably many completely metrizable spaces and a closed subspace of a completely metrizable space are completely metrizable, to complete the proof it suffices to show that $f(X)$ is a closed subset

of $\prod_{i=0}^{\infty} X_i$. We shall show that every point $x = \{x_i\} \in \prod_{i=0}^{\infty} X_i \setminus f(X)$ has a neighbourhood V contained in the complement of $f(X)$.

We first consider the case where $x_0 \in X$. As $x \notin f(X)$, there exists a $k > 0$ such that $x_k \neq f_k(x_0)$. Let U_1 and U_2 be disjoint neighbourhoods of x_k and $f_k(x_0)$ in the real line. The functions f_k being continuous, there exists a neighbourhood $U_0 \subset X_0$ of the point x_0 such that $f_k(U_0 \cap X) \subset U_2$. One easily checks that

$$(6) \quad x = \{x_i\} \in V = p_0^{-1}(U_0) \cap p_k^{-1}(U_1) \subset \prod_{i=0}^{\infty} X_i \setminus f(X),$$

where p_i denotes the projection of $\prod_{i=0}^{\infty} X_i$ onto X_i .

Now, consider the case where $x_0 \notin X$; thus we have $x_0 \in F_k$ for a $k > 0$. Take a positive number r such that $x_k + 1 < 1/r$ and let $U_0 = B(x_0, r)$ and $U_1 = \{x \in R: x < x_k + 1\}$. One easily checks that formula (6) also holds in this case. $\square$

1.3.13. Proposition. *The space of irrational numbers P is homeomorphic to the Cartesian product $N^{\aleph_0} = \prod_{i=1}^{\infty} N_i$, where N_i , for $i = 1, 2, \dots$, is the discrete space of natural numbers N .*

Proof. By virtue of Lemma 1.3.12, there exists a complete metric ρ on the space P . Applying Lemma 1.3.11, for every sequence $k_1, k_2, \dots, k_i$ of natural numbers define an open-and-closed subset $F_{k_1 k_2 \dots k_i}$ of P such that

$$(7) \quad P = \bigcup_{k=1}^{\infty} F_k \quad \text{and} \quad F_{k_1 k_2 \dots k_i} = \bigcup_{k=1}^{\infty} F_{k_1 k_2 \dots k_i k}.$$

$$(8) \quad F_{k_1 k_2 \dots k_i} \neq \emptyset \quad \text{and} \quad \delta(F_{k_1 k_2 \dots k_i}) < 1/i.$$

$$(9) \quad F_{k_1 k_2 \dots k_i} \cap F_{m_1 m_2 \dots m_i} = \emptyset \quad \text{whenever} \quad (k_1, k_2, \dots, k_i) \neq (m_1, m_2, \dots, m_i).$$

It follows from (7) and (8) that for every $\{k_i\} \in N^{\aleph_0}$ the subsets $F_{k_1}, F_{k_1 k_2}, F_{k_1 k_2 k_3}, \dots$ of the space P form a decreasing sequence of non-empty closed sets whose diameters converge to zero; hence, by virtue of the Cantor theorem, the set $\bigcap_{i=1}^{\infty} F_{k_1 k_2 \dots k_i}$ contains exactly one point, which we shall denote by $f(\{k_i\})$. Conditions (7)–(9) imply that by assigning $f(\{k_i\})$ to $\{k_i\} \in N^{\aleph_0}$ one defines a one-to-one mapping of $N^{\aleph_0}$ onto P . Since, as one readily verifies,

$$f(\{k_1\} \times \{k_2\} \times \dots \times \{k_j\} \times \prod_{i=j+1}^{\infty} N_i) = F_{k_1 k_2 \dots k_j},$$

the mapping f is a homeomorphism, because the sets on the left-hand side of the last equality form a base for $N^{\aleph_0}$ and the sets on the right-hand side form a base for P . $\square$

1.3.14. Corollary. *The Cantor set is homeomorphic to a subspace of the space of irrational numbers.* $\square$

Now we are ready to prove the universality of C and P .

1.3.15. The universal space theorem for dimension 0. *The Cantor set and the space of irrational numbers are universal spaces for the class of all zero-dimensional separable metric spaces.*

Proof. By virtue of Corollary 1.3.14, it suffices to show that for every zero-dimensional separable metric space X there exists a homeomorphic embedding $f: X \rightarrow D^{\aleph_0}$.

It follows from Proposition 1.2.4 that the space X has a countable base $\mathcal{B} = \{U_i\}_{i=1}^{\infty}$ consisting of open-and-closed sets. For $i = 1, 2, \dots$ define a mapping $f_i: X \rightarrow D_i = D$ by letting

$$f_i(x) = \begin{cases} 1 & \text{for } x \in U_i, \\ 0 & \text{for } x \in X \setminus U_i. \end{cases}$$

Consider the mapping $f: X \rightarrow D^{\aleph_0}$ defined by the formula $f(x) = (f_1(x), f_2(x), \dots)$. Since for every natural number k we have

$$f(U_k) = f(X) \cap \{ \{x_i\} \in D^{\aleph_0} : x_k = 1 \},$$

the mapping f is a homeomorphic embedding. $\square$

The universal space theorem implies the compactification theorem and the embedding theorem.

1.3.16. The compactification theorem for dimension 0. *For every zero-dimensional separable metric space X there exists a zero-dimensional compactification $\tilde{X}$, i.e., a zero-dimensional compact metric space $\tilde{X}$ which contains a dense subspace homeomorphic to X .*

Proof. Let $f: X \rightarrow C$ be a homeomorphic embedding of X in the Cantor set C . Since the Cantor set is compact, so is its closed subspace $\tilde{X} = f(\overline{X})$; the space $\tilde{X}$ is zero-dimensional by virtue of the subspace theorem. $\square$

1.3.17. The embedding theorem for dimension 0. *Every zero-dimensional separable metric space is embeddable in the real line R .* $\square$

Let us conclude this section by observing that in the theory of zero-dimensional spaces the key role is played by four theorems, viz., the separation theorems, the sum theorem and the universal space theorem. All the remaining results either are elementary or easily follow from one of the four cited theorems. As the reader shall see later, the situation changes when we pass to higher dimensions.

1.3.18. Remark. In Theorems 1.3.6, 1.3.15 and 1.3.16 countability is not essential. In the same way one proves that the Cartesian product $X = \prod_{s \in S} X_s$ of a family $\{X_s\}_{s \in S}$ of regular spaces satisfies the equality $\text{ind} X = 0$ if and only if $\text{ind} X_s = 0$ for every $s \in S$, and that every regular space X satisfying $\text{ind} X = 0$ is embeddable in the Cantor cube D^m (i.e., the Cartesian product of m copies of the two-point discrete space D), which implies that X has a compactification $\tilde{X} \subset D^m$ such that $\text{ind} \tilde{X} = 0$; the cardinal number m is the cardinality of a base for the space X consisting of open-and-closed sets.

Historical and bibliographic notes

The theorems in the present section are all special cases of theorems which will be proved in Sections 1.5, 1.7 and 1.11 for an arbitrary dimension n . Theorem 1.3.1 was proved for compact spaces (and for an arbitrary dimension n) by Menger in [1924] and by Urysohn in [1926] (announcement in [1922]); it was extended to separable metric spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]. Theorem 1.3.6 was established by Kuratowski in [1933] and Theorem 1.3.15 by Sierpiński in [1921].

Problems

1.3.A. (a) Prove that if a separable metric space X can be represented as the union of a sequence $F_0, F_1, F_2, \dots$ of closed subspaces such that $\text{ind} F_i = 0$ for $i = 1, 2, \dots$, then $\text{ind}_x X = 0$ for every point $x \in F_0$ such that $\text{ind}_x F_0 = 0$.

(b) Give an example of a separable metric space X which can be represented as the union of two closed subspaces F_1 and F_2 in such a way that for a point $x \in F_1 \cap F_2$ we have $\text{ind}_x F_1 = \text{ind}_x F_2 = 0$ and yet $\text{ind}_x X > 0$.

1.3.B. Note that Theorem 1.3.1 for a subspace X of the real line is a consequence of the Baire category theorem. Deduce from Theorem 1.3.1 the Baire category theorem for the real line.

1.3.C. (a) (implicitly, Sierpiński [1928]) Prove that every non-empty closed subset A of a zero dimensional separable metric space X is a *retract* of X , i.e., that there exists a continuous mapping $r: X \rightarrow A$ such that $r(x) = x$ for every $x \in A$.

Hint. Represent the complement $X \setminus A$ as the union of a sequence $F_1, F_2, \dots$ of pairwise disjoint open-and-closed sets such that $\lim \delta(F_i) = 0$. For $i = 1, 2, \dots$ choose a point $x_i \in A$ such that $\varrho(x_i, F_i) < \varrho(A, F_i) + 1/i$ and let $r(x) = x_i$ for $x \in F_i$.

(b) Note that if a non-empty regular space X has the property that every non-empty closed set $A \subset X$ is a retract of X , then X is zero-dimensional.

Remark. The characterization of zero-dimensional spaces contained in (a) and (b) generalizes to higher dimensions; see Problem 4.1.F.

1.3.D (Alexandroff [1927a] (announcement [1925]), Hausdorff [1927]). Check that letting

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{x_i}{2^i} \quad \text{for } \{x_i\} \in D^{\aleph_0},$$

one defines a continuous mapping of the Cantor set $D^{\aleph_0}$ onto the closed interval I . Verify that if y is an end-point of one of the intervals removed from I to obtain the Cantor set, then $|f^{-1}(y)| = 2$, and otherwise $|f^{-1}(y)| = 1$.

Define a continuous mapping of the Cantor set $D^{\aleph_0}$ onto the Hilbert cube $I^{\aleph_0}$ and—applying Problem 1.3.C together with the fact that $I^{\aleph_0}$ is a universal space for the class of all separable metric spaces—show that every non-empty compact metric space is a continuous image of the Cantor set. Deduce that every non-empty separable metric space is an image of a zero-dimensional separable metric space under a one-to-one continuous mapping.

1.3.E. (a) (Mazurkiewicz [1917]) Prove that every G_δ -set which is dense, and whose complement is also dense, in a completely metrizable separable zero-dimensional space is homeomorphic to the space of irrational numbers.

Hint. Modify the proof of Proposition 1.3.13.

(b) (Alexandroff and Urysohn [1928]) Show that every completely metrizable separable zero-dimensional space which does not contain any non-empty compact open subspace is homeomorphic to the space of irrational numbers.

Hint. Apply (a).

(c) Note that the subspace of the Cantor set C consisting of all points which are not end-points of intervals removed from I to obtain the Cantor set is homeomorphic to the space of irrational numbers.

1.3.F (Brouwer [1910]). Prove that every zero-dimensional compact metric space with no isolated points is homeomorphic to the Cantor set.

Hint. Modify the construction in the proof of Proposition 1.3.13 in such a way that the sets $F_{k_1 k_2 \dots k_i}$ will be defined for $k_1 \leq m_1, k_2 \leq m_2, \dots, k_i \leq m_i$, where $m_1, m_2, \dots$ is a sequence of powers of the number 2.

1.3.G. (a) (Brouwer [1913a]; implicitly, Fréchet [1910]) Prove that for any two countable dense subsets A, B of the real line R there exists a homeomorphism $f: R \rightarrow R$ such that $f(A) = B$ (see Problem 1.8.D).

Hint. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$; define inductively a function f from A to R by letting $f(a_1) = b_1$ and taking as $f(a_i)$ an element of B , with the smallest possible index, such that the conditions $a_j < a_k$ and $f(a_j) < f(a_k)$ are equivalent for $j, k \leq i$. Extend f over R and prove that this extension is a homeomorphism.

(b) (Fréchet [1910]) Prove that the space of rational numbers Q is a universal space for the class of all countable metric spaces.

Hint. Observe that countable metric spaces are embeddable in the real line; for a countable $X \subset R$ apply (a) to the sets $X \cup Q$ and Q .

1.3.H. (a) Prove that if A_1, A_2, B_1 and B_2 are countable dense subsets of the real line R satisfying the condition $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, then there exists a homeomorphism $f: R \rightarrow R$ such that $f(A_1) = B_1$ and $f(A_2) = B_2$.

Hint. See Problem 1.3.G(a).

(b) Show that for any two countable dense subsets A, B of the space of irrational numbers P there exists a homeomorphism $f: P \rightarrow P$ such that $f(A) = B$.

(c) Show that for any two countable dense subsets A, B of the Cantor set C there exists a homeomorphism $f: C \rightarrow C$ such that $f(A) = B$.

Hint. Observe that for every countable set $A \subset C$ there exists a homeomorphism $g: C \rightarrow C$ such that the set $g(A)$ is disjoint from the set consisting of the end-points of all intervals removed from I to obtain the Cantor set.

(d) (Sierpiński [1920a] (announcement [1915])) Prove that every countable metric space dense in itself is homeomorphic to the space of rational numbers.

Hint. By virtue of (b) it suffices to show that every countable metric space X dense in itself is homeomorphic to a dense subspace of the space of irrational numbers P . To that end, embed X in P , consider the closure $\bar{X} \subset P$, remove in an appropriate way $\aleph_0$ points from $\bar{X} \setminus X$, and apply Problem 1.3.E(a).

One can equally well use Problem 1.3.F and apply (c).

(e) Note that the subspace of the Cantor set C consisting of the end-points of all intervals removed from I to obtain the Cantor set is homeomorphic to the space of rational numbers.

1.4. Various kinds of disconnectedness

We shall now compare the class of zero-dimensional spaces with three other classes of highly disconnected spaces. It will follow from this comparison that none of these three classes satisfies the counterparts of the theorems proved for zero-dimensional spaces in Sections 1.2 and 1.3. Hence, zero-dimensional spaces form the nicest class of highly disconnected spaces. The dimension functions that one could define inductively, in the way the function ind is defined, starting at the zero-level with another class of highly disconnected spaces instead of the class of zero-dimensional spaces, would not lead to a dimension theory as rich and harmonious as the theory based on the dimension function ind developed in this chapter.

1.4.1. Definition. A topological space X is called *totally disconnected* if for every pair x, y of distinct points of X there exists an open-and-closed set $U \subset X$ such that $x \in U$ and $y \in X \setminus U$, i.e., if the empty set is a partition between any distinct points x, y of the space X .

Clearly, every zero-dimensional space is totally disconnected.

Totally disconnected spaces are characterized by the property that their quasi-components are one-point sets. Let us recall that quasi-components of a topological space X are defined as the minimal non-empty intersections of open-and-closed subsets of X , i.e., a non-empty set $K \subset X$ is a *quasi-component* of the space X if K can be represented as the intersection of open-and-closed sets and for every open-and-closed set $U \subset X$ such that $K \cap U \neq \emptyset$ we have $K \subset U$. The quasi-components of a space X constitute a decomposition of X into pairwise disjoint closed subsets.

1.4.2. Definition. A topological space X is called *hereditarily disconnected* if X does not contain any connected subspace of cardinality larger than one.

Every totally disconnected space is hereditarily disconnected. Indeed, if X is a totally disconnected space, then for each subspace $M \subset X$ which contains at least two distinct points x, y the sets $M \cap U$ and $M \setminus U$, where U is an open-and-closed subset of X such that $x \in U$ and $y \in X \setminus U$, form a decomposition of the space M into two non-empty disjoint open subsets, so that the space M is not connected.

Hereditarily disconnected spaces are characterized by the property that their components are one-point sets. Let us recall that components

of a topological space X are defined as the maximal non-empty connected subsets of X , i.e., a non-empty set $S \subset X$ is a *component* of the space X if S is connected and for every connected set $A \subset X$ such that $S \subset A$ we have $S = A$. The components of a space X constitute a decomposition of X into pairwise disjoint closed subsets.

1.4.3. Definition. A topological space X is called *punctiform*, or *discontinuous*, if X does not contain any continuum of cardinality larger than one.

Clearly, every hereditarily disconnected space is punctiform and every compact punctiform space is hereditarily disconnected. As shown in Example 1.4.8 below, there exist connected punctiform spaces of cardinality larger than one.

The reader can easily verify that the above three classes of spaces are closed with respect to the subspace operation.

We shall now show that in the realm of non-empty locally compact spaces the three classes under consideration coincide with the class of zero-dimensional spaces.

1.4.4. Lemma. *In every compact space quasi-components and components coincide.*

Proof. To begin with, we shall prove that in an arbitrary topological space X quasi-components contain the components. Consider a component S of the space X . Let x be a point in S and K the quasi-component of the space X which contains the point x ; we shall show that $S \subset K$. Take an open-and-closed set $U \subset X$ which contains x . Since the sets $S \cap U$ and $S \setminus U$ are open in S and disjoint and since $S \cap U \neq \emptyset$, it follows from the connectedness of S that $S \setminus U = \emptyset$, i.e., that $S \subset U$. The set K being the intersection of all open-and-closed subsets of X which contain x , we have $S \subset K$.

To complete the proof it suffices to show that the quasi-components of a compact space are connected. Let us consider the decomposition of a quasi-component K of a compact space X into two disjoint closed sets A, B and let us assume that $A \neq \emptyset$. By the normality of compact spaces there exist open sets $V, W \subset X$ such that

$$A \subset V, \quad B \subset W \quad \text{and} \quad V \cap W = \emptyset.$$

Denote by $\mathcal{U}$ a family of open-and-closed subsets of X satisfying $\bigcap \mathcal{U} = K$.

Since $\bigcap \mathcal{U} \subset V \cup W$, the family $\mathcal{F} = \{U \setminus (V \cup W) : U \in \mathcal{U}\}$ of closed subsets of X has an empty intersection. It follows from the compactness of X that a finite subfamily of $\mathcal{F}$ also has an empty intersection, i.e., that there exists a finite number of sets $U_1, U_2, \dots, U_k \in \mathcal{U}$ satisfying

$$U = U_1 \cap U_2 \cap \dots \cap U_k \subset V \cup W.$$

The set U is open-and-closed. Since

$$\overline{V \cap U} \subset \overline{V} \cap U = \overline{V} \cap (V \cup W) \cap U = V \cap U,$$

the set $V \cap U$ is also open-and-closed. From the relation $\emptyset \neq A \subset V \cap U$ it follows that $K \subset V \cap U$, and so $B \subset K \cap W \subset V \cap U \cap W = \emptyset$, which proves that the quasi-component K is connected. $\square$

1.4.5. Theorem. *Zero-dimensionality, total disconnectedness, hereditary disconnectedness and punctiformness are equivalent in the realm of non-empty locally compact spaces.*

Proof. It suffices to prove that every non-empty locally compact punctiform space is zero-dimensional. Consider a point $x \in X$ and a neighbourhood $V \subset X$ of the point x . By the local compactness of the space X the point x has a neighbourhood $W \subset X$ such that the closure $\overline{W}$ is compact. The subspace $M = \overline{V \cap W}$ of the space X is compact and punctiform, so that it is hereditarily disconnected. By virtue of Lemma 1.4.4, the component $\{x\}$ of the space M can be represented as the intersection of a family $\mathcal{U}$ of open-and-closed subsets of M . It follows from the compactness of M that there exists a finite number of sets $U_1, U_2, \dots, U_k \in \mathcal{U}$ such that the intersection $U = U_1 \cap U_2 \cap \dots \cap U_k$ is disjoint from the set $M \setminus (V \cap W)$. The set U is closed in M , and thus it is closed in X ; on the other hand, the set U is open in $V \cap W$, so that U is an open-and-closed subset of X . As $x \in U \subset V$, the space X is zero-dimensional. $\square$

We shall now describe three subspaces of the plane which exhibit the difference between the classes of zero-dimensional, totally disconnected, hereditarily disconnected and punctiform spaces. They are all closely related to the space H_0 described in Example 1.2.15, which is itself a totally disconnected non zero-dimensional space.

1.4.6. Example. One readily checks that by assigning to every point $\{x_i\}$ of the space H_0 described in Example 1.2.15 the same point $\{x_i\}$ in the Cartesian product $Q^{\aleph_0}$ of $\aleph_0$ copies of the space of rational numbers one

defines a one-to-one continuous mapping of H_0 to $Q^{\aleph_0}$. Hence, by virtue of Theorems 1.3.6 and 1.3.15, there exists a one-to-one continuous mapping $f: H_0 \rightarrow C$ of the space H_0 to the Cantor set C . Since the Cantor set is homogeneous (i.e., for every pair x, y of distinct points of C there exists a homeomorphism of C onto itself which transforms x to y), one can suppose that $f(x_0) = 0$, where $x_0 \in H_0$ is the sequence whose terms are all equal to zero. Letting

$$h(x) = \left(\frac{f(x)}{\max(1, \|x\|)}, \frac{\|x\|}{1 + \|x\|} \right) \quad \text{for } x \in H_0,$$

one defines a continuous mapping $h: H_0 \rightarrow I^2$. One can prove that h is a homeomorphic embedding (see Problem 1.4.B(a)). This implies that the subspace $X = h(H_0)$ of the plane is totally disconnected but is not zero-dimensional; however, a direct proof of these properties of the space X is simpler.

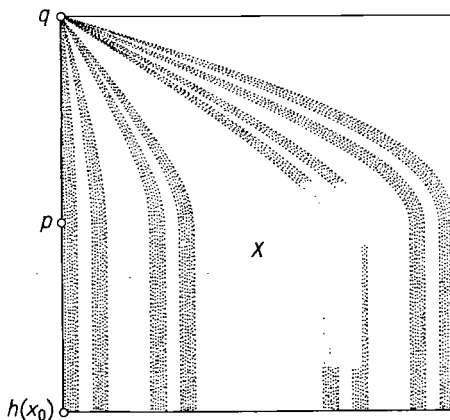


Fig. 5

To begin with, let us observe that by letting

$$G(y_1, y_2) = y_1 \cdot \max\left(1, \frac{y_2}{1 - y_2}\right) \quad \text{for } 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 < 1$$

one defines a continuous mapping $G: I \times [0, 1) \rightarrow R$ which to the point $h(x) \in X$ assigns the point $f(x) \in C$, so that the restriction $g = G|X: X \rightarrow C$ is a one-to-one continuous mapping of the space X to the Cantor set C . From the existence of such a mapping it follows that X is a totally disconnected space. Indeed, for every pair x, y of distinct points of X there

exists an open-and-closed set $V \subset C$ such that $g(x) \in V$ and $g(y) \in C \setminus V$; the inverse image $U = g^{-1}(V)$ is an open-and-closed subset of X such that $x \in U$ and $y \in X \setminus U$.

Now, consider an open-and-closed set $U \subset X$ such that $U \subset (I \times [0, 1/2)) \cap X$. The inverse image $h^{-1}(U) \subset H_0$ is an open-and-closed bounded subset of H_0 and thus, by virtue of the final observation in Example 1.2.15, it is empty. Hence, there is no open-and-closed set $U \subset X$ such that $h(x_0) = (0, 0) \in U \subset (I \cap [0, 1/2)) \cap X$, which shows that the space X is not zero-dimensional. $\square$

The following two examples are: a space $Y \subset I^2$ which is hereditarily disconnected but is not totally disconnected and a space $Z \subset I^2$ which is punctiform and connected; in both examples we shall use the notation introduced in Example 1.4.6.

1.4.7. Example. We shall show that the subspace of the plane $Y = X \cup \{p\}$, where $p = (0, 1/2)$, is hereditarily disconnected but is not totally disconnected.

Consider a connected subspace A of the space Y . As $G(p) = 0 \in C$, the image $G(A)$ is a connected subspace of the Cantor set and thus contains at most one point. It follows that A is contained in a fibre of the mapping $G|Y$. Since all fibres of $G|Y$ are at most of cardinality 2, the set A either is empty or consists of exactly one point, and this implies that the space Y is totally disconnected.

Consider now an open-and-closed set $U \subset Y$ such that $p \in U$. There exists a number $a \in I \setminus C$ such that $([0, a) \times \{1/2\}) \cap Y \subset U$. One readily sees that the set

$$V = ([0, a) \times [0, 1/2)) \cap Y \setminus U = ([0, a) \times [0, 1/2)) \cap Y \setminus U$$

is open-and-closed in X . The inverse image $h^{-1}(V) \subset H_0$ is an open-and-closed bounded subset of H_0 and thus is empty. Hence the set V is also empty, which implies that $h(x_0) = (0, 0) \in U$. Thus for the pair $x = p$, $y = h(x_0)$ of distinct points of Y there exists no open-and-closed set $U \subset X$ such that $x \in U$ and $y \in Y \setminus U$, i.e., the space Y is not totally disconnected. $\square$

1.4.8. Example. We shall show that the subspace of the plane $Z = X \cup \{q\}$, where $q = (0, 1)$, is punctiform and connected.

Consider a continuum $A \subset Z$. The difference $A \setminus \{q\} \subset X$ is an F_σ -set in A , and so $A \setminus \{q\} = \bigcup_{i=1}^{\infty} A_i$, where A_i is compact for $i = 1, 2, \dots$. For

every i the restriction $g_i = g|_{A_i}: A_i \rightarrow g_i(A_i) \subset C$ is a homeomorphism, because it is a one-to-one continuous mapping defined on a compact space. Since $\text{ind } g_i(A_i) \leq 0$, we have $\text{ind } A_i \leq 0$ for $i = 1, 2, \dots$ and—by virtue of Theorems 1.1.2 and 1.3.1— $\text{ind } A \leq \text{ind}(\{q\} \cup \bigcup_{i=1}^{\infty} A_i) = 0$. Hence the set A either is empty or consists of exactly one point, and this implies that the space Z is punctiform.

Consider now an open-and-closed set $U \subset Z$ such that $q \in U$. One readily sees that there exists a number $a \in (0, 1)$ such that $(I \times (a, 1)) \cap Z \subset U$. The inverse image $h^{-1}(V) \subset H_0$, where $V = Z \setminus U$, is an open-and-closed bounded subset of H_0 and thus is empty. Hence $U = Z$, i.e., the space Z is connected. $\square$

In the table on p. 37 the basic properties of countable, zero-dimensional, totally disconnected, hereditarily disconnected, and punctiform spaces are compared; a plus means that a theorem holds in the corresponding class, a minus that it does not hold. Formal statements of the results in the table together with hints how to obtain them can be found in the problems below.

Historical and bibliographic notes

Totally disconnected spaces were introduced by Sierpiński in [1921], hereditarily disconnected spaces—by Hausdorff in [1914], and punctiform spaces—by Janiszewski in [1912]. Theorem 1.4.5 was proved (for compact metric spaces) by Menger in [1923] and by Urysohn in [1925]. The first example of a totally disconnected space which is not zero-dimensional was given by Sierpiński in [1921]; Sierpiński's space is a completely metrizable subspace of the plane. The first example of a hereditarily disconnected space which is not totally disconnected was also given by Sierpiński in [1921]; this space is also a completely metrizable subspace of the plane. Example 1.4.7 is a simplified version of an example described by Roberts in [1956]. The first example of a punctiform space which is not hereditarily disconnected was described by Sierpiński in [1920]; this space is a connected subspace of the plane. An example of a completely metrizable punctiform and connected subspace of the plane was given by Mazurkiewicz in [1920]. A simple modification that leads from spaces described in Examples 1.4.6–1.4.8 to similar spaces which are, moreover, completely metrizable is sketched in Problem 1.4.B.

	Enlargement to a G_δ -set	Countable sum	Finite sum	Adjoining one point	Countable Cartesian product	Finite Cartesian product	Universal space	Compactifi- cation	Embedding in the real line	Embedding in an Euclid- ean space
Countable spaces		+	+	+		+	+		+	+
Zero-dimensional spaces	+	+	+	+	+	+	+	+	+	+
Totally disconnected spaces					+	+				
Hereditarily discon- nected spaces					+	+				
Punctiform spaces		+	+	+	+	+				

Problems

1.4.A. (a) Observe that zero-dimensionality, total disconnectedness, hereditary disconnectedness and punctiformness are equivalent in the realm of regular spaces which can be represented as countable unions of compact subspaces.

(b) Note that zero-dimensionality, total disconnectedness, hereditary disconnectedness and punctiformness are equivalent in the realm of subspaces of the real line. Deduce that there exist totally disconnected spaces which are not embeddable in the real line.

1.4.B. (a) (Roberts [1956]) Prove that the mapping $h: H_0 \rightarrow I^2$ defined in Example 1.4.6 is a homeomorphic embedding.

Hint. Prove that in Hilbert space H a sequence of points $x^1, x^2, \dots$, where $x^m = \{x_i^m\}$ for $m = 1, 2, \dots$, converges to a point $x = \{x_i\}$ if and only if the sequence $\|x^m\|$ converges to $\|x\|$ and the sequence $x_i^1, x_i^2, \dots$ converges to x_i for $i = 1, 2, \dots$ (cf. Example 1.5.17).

(b) Give examples of completely metrizable spaces $X_1, Y_1, Z_1 \subset I^2$ such that X_1 is totally disconnected but is not zero-dimensional, Y_1 is hereditarily disconnected but is not totally disconnected, and Z_1 is punctiform and connected.

Hint. Consider the subspace H_1 of Hilbert space H consisting of the points $\{x_i\} \in H$ such that x_i is irrational for every i and suitably modify the constructions in Examples 1.4.6–1.4.8. When proving that X_1 is completely metrizable, apply Lemma 1.3.12. In the proof of complete metrizability of Y_1 and Z_1 use the fact that every completely metrizable subspace of I^2 is a G_δ -set (see [GT], Theorem 4.3.24).

1.4.C. (a) (Knaster and Kuratowski [1921]) Let C be the Cantor set on the interval $[0, 1] \times \{0\} \subset R^2$; denote by Q the set of the end-points of all intervals removed from $[0, 1] \times \{0\}$ to obtain the Cantor set and let $P = C \setminus Q$. Join every point $c \in C$ to the point $q = (1/2, 1/2) \in R^2$ by a segment L_c and denote by F_c the set of all points $(x, y) \in L_c$, where y is rational if $c \in Q$ and y is irrational if $c \in P$. The subspace $F = \bigcup_{c \in C} F_c$ of the plane is called the *Knaster–Kuratowski fan*.

Prove that the Knaster–Kuratowski fan is connected and punctiform.

Hint. Let $r_1, r_2, \dots$ be the sequence of all rational numbers in the interval $[0, 1/2]$ and let $P_i \subset R^2$ be the horizontal line $y = r_i$. Suppose

that $F = (F \cap A) \cup (F \cap B)$, where the sets A and B are closed in R^2 , $F \cap A \cap B = \emptyset$ and $q \in A$. Consider the sets $K_i = \{c \in C: A \cap B \cap L_c \cap P_i \neq \emptyset\}$, check that they are closed and that $\bigcup_{i=1}^{\infty} K_i \subset P$. Show that $B \cap L_c = \emptyset$ for every point $c \in P \setminus \bigcup_{i=1}^{\infty} K_i$ and apply the Baire category theorem to prove that $F \cap B = \emptyset$.

(b) (Knaster and Kuratowski [1921]) Prove that the space $F \setminus \{q\}$ is hereditarily disconnected but is not totally disconnected.

(c) (E. Pol [1978a]) Prove that every completely metrizable space X which contains a subspace homeomorphic to $F \setminus \{q\}$ also contains a subspace homeomorphic to the closed unit interval.

Hint. Apply the Lavrentieff theorem (see [GT], Theorem 4.3.21) to reduce the problem to the case where X is a G_δ -set in the plane and $F \setminus \{q\} \subset X$. Consider the set $P \cap \bigcap_{i=1}^{\infty} p(X \cap P_i)$, where p is the projection from q onto C and P_i is defined in the hint to part (a).

(d) Observe that modifying the construction of the space F by taking all points $(x, y) \in L_c$, where y is irrational if $c \in Q$ and y is rational if $c \in P$, one obtains a zero-dimensional space.

1.4.D. Let K and D be the spaces discussed in Problem 1.2.E.

(a) Show that if $\text{ind}_z K = 1$, then there exists a point $p \in D \setminus K$ such that the empty set is not a partition in the space $K' = K \cup \{p\}$ between z and p .

Hint. See the hint to Problem 1.2.E(e).

(b) Prove that the space K' defined in (a) is hereditarily disconnected but is not totally disconnected.

(c) Verify that the space K' is completely metrizable.

1.4.E. (a) Note that by adjoining a point to a totally disconnected space one can obtain a space which is not totally disconnected. Deduce that the counterpart of the sum theorem does not hold either for totally disconnected spaces or for hereditarily disconnected spaces, even in the case where the space is represented as the union of two closed subspaces. Show that the counterpart of the sum theorem holds for punctiform spaces.

(b) Check that the Cartesian product of a family of totally disconnected, hereditarily disconnected, or punctiform spaces is a space in the same class.

(c) Note that the counterpart of the compactification theorem does not hold for any of the four classes of spaces in the table on p. 37 distinct from the class of zero-dimensional spaces.

1.4.F (Hilgers [1937]). (a) Let Z be a topological space and T a subspace of Z whose cardinality is equal to c . Prove that for every family $\mathcal{G}$ of subspaces of the Cartesian product $Z \times Z$ such that $|\mathcal{G}| \leq c$ there exists a set $H \subset Z \times Z$ satisfying the following conditions:

- (1) *The projection of $Z \times Z$ onto the first axis maps H in a one-to-one way onto the subspace T .*
- (2) *If $H \subset G$ for a $G \in \mathcal{G}$, then G contains a set homeomorphic to Z .*

Hint. Let φ be an arbitrary transformation of T onto $\mathcal{G}$; define a mapping f of T to Z by letting $f(t)$ be a point $z \in Z$ such that $(t, z) \in (\{t\} \times Z) \setminus \varphi(t)$ if such points exist, and an arbitrary point $z \in Z$ otherwise. Consider the set $H = \{(t, f(t)) : t \in T\}$, i.e., the graph of the mapping f .

(b) Applying Theorem 1.5.11 and the equality $\text{ind } R^n = n$ (see Theorem 1.8.2), for every natural number n define a separable metric space X such that $\text{ind } X = n$ and X can be mapped by a continuous and one-to-one mapping onto a zero-dimensional space. Observe that X is totally disconnected and deduce that there exists a totally disconnected separable metric space which cannot be embedded in a Euclidean space.

Hint. Consider the space $Z = R^n$, a subspace $T \subset Z$ homeomorphic to the Cantor set, and the family $\mathcal{G}$ of all G_δ -sets in the Cartesian product $R^n \times R^n$; then apply (a).

Remark. The first example of a totally disconnected separable metric space of an arbitrary dimension $n \geq 1$ was given by Mazurkiewicz in [1927]; Mazurkiewicz's spaces are completely metrizable. Clearly, such spaces do not contain any compact subspace of positive dimension.

1.4.G (R. Pol [1973]). (a) Prove that every separable metric space X which for each punctiform separable metric space Y contains a subspace homeomorphic to Y contains also a subspace homeomorphic to the Hilbert cube.

Hint. One can assume that $X \subset I^{\aleph_0}$. Consider the space $Z = I^{\aleph_0}$, a subspace $T \subset Z$ homeomorphic to the Cantor set, and the family $\mathcal{G}$ consisting of all sets of the form $f^{-1}(X)$, where f is a continuous mapping defined on a G_δ -set in the Cartesian product $I^{\aleph_0} \times I^{\aleph_0}$ and taking values

in $I^{\aleph_0}$; then apply Problem 1.4.F(a) and the Lavrentieff theorem (see [GT], Theorem 4.3.21).

(b) Define a totally disconnected separable metric space X with the property that every completely metrizable separable space which contains a subspace homeomorphic to X also contains a subspace homeomorphic to the Hilbert cube.

Hint. Consider the space $Z = I^{\aleph_0}$, a subspace $T \subset Z$ homeomorphic to the Cantor set and the family $\mathcal{G}$ of all G_δ -sets in the Cartesian product $I^{\aleph_0} \times I^{\aleph_0}$; then apply Problem 1.4.F(a) and the Lavrentieff theorem (see [GT], Theorem 4.3.21).

1.5. The sum, decomposition, addition, enlargement, separation and Cartesian product theorems

We begin with some observations on the dimension of subspaces. The subspace theorem established in Section 1.1 states that for every subspace M of a regular space X we have $\text{ind} M \leq \text{ind} X$. In this context it is natural to ask whether among the subspaces of a space X such that $\text{ind} X = n$ one can find subspaces of all intermediate dimensions between 0 and $n-1$. As shown in the next theorem, the answer is positive and there even exist closed subspaces of intermediate dimensions.

1.5.1. Theorem. *If X is a regular space and $\text{ind} X = n \geq 1$, then for $k = 0, 1, \dots, n-1$ the space X contains a closed subspace M such that $\text{ind} M = k$.*

Proof. It is enough to show that X contains a closed subspace M such that $\text{ind} M = n-1$. As $\text{ind} X > n-1$, there exist a point $x \in X$ and a neighbourhood $V \subset X$ of the point x such that for every open set $U \subset X$ satisfying the condition $x \in U \subset V$ we have $\text{ind Fr } U > n-2$. On the other hand, as $\text{ind} X \leq n$, there exists an open set $U \subset X$ satisfying the above condition and such that $\text{ind Fr } U \leq n-1$. The closed subspace $M = \text{Fr } U$ of the space X has the required property. $\square$

The situation is quite different in spaces of dimension ∞ . In Example 1.8.21 we shall describe, applying the continuum hypothesis, a separable metric space of dimension ∞ whose finite-dimensional subspaces are all countable (it turns out that the existence of such a space is equivalent to the continuum hypothesis). Let us observe that spaces with the above

property are rather peculiar; in particular, no such space is completely metrizable, because every uncountable completely metrizable separable space contains a subspace homeomorphic to the Cantor set (see [GT], Problems 1.7.11 and 4.5.5). There also exist compact metric spaces of dimension ∞ whose finite-dimensional subspaces are all zero-dimensional, but examples of such spaces are very complicated.

We now pass to the sum theorem.

1.5.2. Lemma. *If a separable metric space X can be represented as the union of two subspaces Y and Z such that $\text{ind} Y \leq n-1$ and $\text{ind} Z \leq 0$, then $\text{ind} X \leq n$.*

Proof. Consider a point $x \in X$ and a neighbourhood $V \subset X$ of the point x . By virtue of Theorem 1.2.11, there exist disjoint open sets $U, W \subset X$ such that $x \in U$, $X \setminus V \subset W$ and $[X \setminus (U \cup W)] \cap Z = \emptyset$. Clearly, $x \in U \subset V$; as $\text{Fr} U \subset [X \setminus (U \cup W)] \subset X \setminus Z \subset Y$, we have $\text{ind Fr} U \leq n-1$. Hence $\text{ind} X \leq n$. $\square$

1.5.3. The sum theorem. *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces such that $\text{ind} F_i \leq n$ for $i = 1, 2, \dots$, then $\text{ind} X \leq n$.*

Proof. We shall apply induction with respect to the number n . For $n = 0$ the theorem is already proved. Assume that the theorem holds for dimensions less than n and consider a space $X = \bigcup_{i=1}^{\infty} F_i$, where F_i is closed and $\text{ind} F_i \leq n \geq 1$ for $i = 1, 2, \dots$. Applying Theorem 1.1.6, choose for $i = 1, 2, \dots$, a countable base $\mathcal{B}_i$ for the space F_i such that $\text{ind Fr} U \leq n-1$ for every $U \in \mathcal{B}_i$, where Fr denotes the boundary operator in the space F_i . By the inductive assumption the subspace $Y = \bigcup \{\text{Fr} U: U \in \bigcup_{i=1}^{\infty} \mathcal{B}_i\}$ of the space X satisfies the inequality $\text{ind} Y \leq n-1$. Now, Proposition 1.2.13 implies that for $i = 1, 2, \dots$ the subspace $Z_i = F_i \setminus Y$ of the space F_i satisfies the inequality $\text{ind} Z_i \leq 0$; hence, by the sum theorem for dimension 0, the subspace $Z = \bigcup_{i=1}^{\infty} Z_i = X \setminus Y$ of the space X also satisfies the inequality $\text{ind} Z \leq 0$, because it follows from the relation $Z_i = F_i \setminus Y = F_i \cap Z$ that all the Z_i 's are closed in Z . Thus by virtue of Lemma 1.5.2 we have $\text{ind} X \leq n$. $\square$

As in the case of zero-dimensional spaces, the sum theorem implies three corollaries:

1.5.4. Corollary. *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \dots$ of subspaces such that $\text{ind} F_i \leq n$ and F_i is an F_σ -set for $i = 1, 2, \dots$, then $\text{ind} X \leq n$. $\square$*

1.5.5. Corollary. *If a separable metric space X can be represented as the union of two subspaces A and B , one of them closed, such that $\text{ind} A \leq n$ and $\text{ind} B \leq n$, then $\text{ind} X \leq n$. $\square$*

1.5.6. Corollary. *If by adjoining a finite numbers of points to a separable metric space X such that $\text{ind} X \leq n$ one obtains a metric space Y , then the space Y satisfies the inequality $\text{ind} Y \leq n$ and is separable. $\square$*

Let us observe that the sum theorem plays a key role in the dimension theory of separable metric spaces. Indeed, all the remaining results in this section follow either from the sum theorem or from one of the decomposition theorems which are easy consequences of the sum theorem.

Applying the sum theorem, one readily shows that the condition in Lemma 1.5.2 characterizes separable metric spaces of dimension $\leq n$:

1.5.7. The first decomposition theorem. *A separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if X can be represented as the union of two subspaces Y and Z such that $\text{ind} Y \leq n-1$ and $\text{ind} Z \leq 0$.*

Proof. Consider a separable metric space X such that $\text{ind} X \leq n \geq 0$. By virtue of Theorem 1.1.6, the space X has a countable base $\mathcal{B}$ such that $\text{ind} \text{Fr } U \leq n-1$ for every $U \in \mathcal{B}$. It follows from the sum theorem that the subspace $Y = \bigcup \{\text{Fr } U : U \in \mathcal{B}\}$ has dimension $\leq n-1$ and from Proposition 1.2.13 that the subspace $Z = X \setminus Y$ has dimension ≤ 0 . To complete the proof it suffices to apply Lemma 1.5.2. $\square$

From the first decomposition theorem we obtain by an easy induction

1.5.8. The second decomposition theorem. *A separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if X can be represented*

as the union of $n+1$ subspaces $Z_1, Z_2, \dots, Z_{n+1}$ such that $\text{ind} Z_i \leq 0$ for $i = 1, 2, \dots, n+1$. $\square$

Let us at once explain that, as will be shown in Section 1.8 (see Theorem 1.8.20), the Hilbert cube cannot be represented as a countable union of zero-dimensional subspaces, or—equivalently—of finite-dimensional subspaces. Hence, the second decomposition theorem does not extend to separable metric spaces of dimension ∞ .

1.5.9. Examples. For every point x in the real line R or in the circle S^1 and each neighbourhood V of the point x there exists a neighbourhood U of x such that $U \subset V$ and the boundary $\text{Fr} U$ is a two-point set. Hence, $\text{ind} R \leq 1$ and $\text{ind} S^1 \leq 1$. Since $\text{ind} I > 0$ by virtue of Example 1.2.5, the subspace theorem implies that $\text{ind} R = \text{ind} S^1 = \text{ind} I = 1$.

For every point x in Euclidean n -space R^n or in the n -sphere S^n and each neighbourhood V of the point x there exists a neighbourhood U of x such that $U \subset V$ and the boundary $\text{Fr} U$ is homeomorphic to S^{n-1} . Hence, as shown by an inductive argument, $\text{ind} R^n \leq n$, $\text{ind} S^n \leq n$ and $\text{ind} I^n \leq n$ for every natural number n .

The small inductive dimension of R^n , S^n and I^n is indeed equal to n , but the proof of this fact is much more difficult than the above evaluations; it will be given in Section 1.8. The equality $\text{ind} R^n = n$ is of utmost importance for dimension theory. In a sense, it justifies the definition of the dimension function by showing that this definition yields a notion conforming to geometric intuition. The fact that $\text{ind} R^n = n$ is sometimes called the fundamental theorem of dimension theory.

The decomposition of R^n into $n+1$ zero-dimensional subspaces following from the second decomposition theorem can be defined—according to Example 1.3.8—by the equality

$$R^n = Q_0^n \cup Q_1^n \cup \dots \cup Q_n^n.$$

The last formula yields another proof of the inequality $\text{ind} R^n \leq n$.

For every pair k, n of integers satisfying $0 \leq k \leq n \leq 1$ define

$$N_k^n = Q_0^n \cup Q_1^n \cup \dots \cup Q_k^n \quad \text{and} \quad L_k^n = Q_k^n \cup Q_{k+1}^n \cup \dots \cup Q_n^n;$$

thus N_k^n is the subspace of Euclidean n -space R^n consisting of all points which have at most k rational coordinates and L_k^n is the subspace of R^n consisting of all points which have at least k rational coordinates. From the second decomposition theorem it follows that $\text{ind} N_k^n \leq k$ and $\text{ind} L_k^n \leq n-k$; we shall show in Section 1.8, applying the equality $\text{ind} R^n = n$,

that $\text{ind} N_k^n = k$ and $\text{ind} L_k^n = n - k$. The space $N_n^{2n+1} \subset R^{2n+1}$ will play a particularly important role in the sequel: it turns out to be a universal space for the class of all separable metric spaces of dimension $\leq n$ (see Theorem 1.11.5). $\square$

We shall now state further consequences of the sum and decomposition theorems. Let us begin with the addition theorem, which follows immediately from the second decomposition theorem.

1.5.10. The addition theorem. *For every pair X, Y of separable subspaces of a metric space we have*

$$\text{ind}(X \cup Y) \leq \text{ind} X + \text{ind} Y + 1. \quad \square$$

1.5.11. The enlargement theorem. *For every separable subspace M of an arbitrary metric space X satisfying the inequality $\text{ind} M \leq n$ there exists a G_δ -set M^* in X such that $M \subset M^*$ and $\text{ind} M^* \leq n$.*

Proof. By the second decomposition theorem $M = Z_1 \cup Z_2 \cup \dots \cup Z_{n+1}$, where $\text{ind} Z_i \leq 0$ for $i = 1, 2, \dots, n+1$. Applying Theorem 1.2.14, enlarge each Z_i to a G_δ -set Z_i^* in X such that $\text{ind} Z_i^* \leq 0$. The union $M^* = Z_1^* \cup Z_2^* \cup \dots \cup Z_{n+1}^*$ has the required properties. $\square$

1.5.12. The first separation theorem. *If X is a separable metric space such that $\text{ind} X \leq n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{ind} L \leq n - 1$.*

Proof. By the first decomposition theorem $X = Y \cup Z$, where $\text{ind} Y \leq n - 1$ and $\text{ind} Z \leq 0$. Applying Theorem 1.2.11, we obtain a partition L between A and B such that $L \cap Z = \emptyset$. As $L \subset X \setminus Z \subset Y$, we have $\text{ind} L \leq n - 1$ by the subspace theorem. $\square$

In a similar way, applying the first decomposition theorem to the subspace M , we obtain

1.5.13. The second separation theorem. *If X is an arbitrary metric space and M is a separable subspace of X such that $\text{ind} M \leq n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{ind}(L \cap M) \leq n - 1$. $\square$*

Clearly, the first separation theorem is a special case of the second separation theorem. On the other hand, the second separation theorem easily follows (cf. the proof of Theorem 1.2.11) from the first separation theorem and Lemma 1.2.9, which is an elementary topological fact; hence, both separation theorems are in a sense equivalent.

The second separation theorem yields a characterization of the dimension of subspaces in terms of neighbourhoods in the whole space, which generalizes Proposition 1.2.12:

1.5.14. Proposition. *A separable subspace M of an arbitrary metric space X satisfies the inequality $\text{ind } M \leq n \geq 0$ if and only if for every point $x \in M$ (or—equivalently—for every point $x \in X$) and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\text{ind}(M \cap \text{Fr } U) \leq n-1$. $\square$*

Proposition 1.5.14 and Lemma 1.1.5 imply the following generalization of Proposition 1.2.13.

1.5.15. Proposition. *A subspace M of a separable metric space X satisfies the inequality $\text{ind } M \leq n \geq 0$ if and only if X has a countable base $\mathcal{B}$ such that $\text{ind}(M \cap \text{Fr } U) \leq n-1$ for every $U \in \mathcal{B}$. $\square$*

The general Cartesian product theorem reads as follows:

1.5.16. The Cartesian product theorem. *For every pair X, Y of separable metric spaces of which at least one is non-empty we have*

$$\text{ind}(X \times Y) \leq \text{ind } X + \text{ind } Y.$$

Proof. The theorem is obvious if one of the spaces has dimension ∞ , and so we can suppose that $k(X, Y) = \text{ind } X + \text{ind } Y$ is finite. We shall apply induction with respect to that number. If $k(X, Y) = -1$, then either $X = \emptyset$ or $Y = \emptyset$, and our inequality holds. Assume that the inequality is proved for every pair of separable metric spaces the sum of the dimensions of which is less than $k \geq 0$ and consider separable metric spaces X and Y such that $\text{ind } X = n \geq 0$, $\text{ind } Y = m \geq 0$ and $n + m = k$. Let (x, y) be a point of $X \times Y$ and $W \subset X \times Y$ a neighbourhood of (x, y) . There exist neighbourhoods $U', U \subset X$ of the point x and $V, V' \subset Y$ of the point y such that $U' \times V' \subset W$, $U \subset U'$, $V \subset V'$, $\text{ind Fr } U \leq n-1$ and $\text{ind Fr } V \leq m-1$. Since

$$\text{Fr}(U \times V) \subset (X \times \text{Fr } V) \cup (\text{Fr } U \times Y),$$

by virtue of the inductive assumption and the sum theorem we have $\text{indFr}(U \times V) \leq k-1$. Hence $\text{ind}(X \times Y) \leq k$ and the proof is completed. $\square$

The inequality in the Cartesian product theorem cannot be replaced by an equality. In the next example we shall describe a separable metric space X such that $\text{ind}X = 1$, and yet $\text{ind}(X \times X) = 1$, because X is homeomorphic to the square $X \times X$ (cf. the remark to Problem 1.5.C). There exist even compact metrizable spaces the dimension of the Cartesian product of which is less than the sum of their dimensions, but they are more complicated, and in checking their properties one has to apply the methods of algebraic topology; let us note that such spaces are necessarily of dimension ≥ 2 , because for every pair X, Y of compact metrizable spaces X, Y such that $\text{ind}Y = 1$ we have $\text{ind}(X \times Y) = \text{ind}X + 1 = \text{ind}X + \text{ind}Y$ (see Problem 1.9.E(b)).

1.5.17. Example. We shall show that the space H_0 defined in Example 1.2.15 has the required properties.

To establish the equality $\text{ind}H_0 = 1$ it is enough to prove that $\text{ind}H_0 \leq 1$. Since every point $x \in H_0$ can be transformed by a suitable translation to the point $x_0 \in H_0$, it suffices to show that for every natural number n the boundary $F_n = \{x \in H_0: \|x\| = 1/n\}$ of the $1/n$ -ball $U_n = \{x \in H_0: \|x\| < 1/n\}$ about x_0 is zero-dimensional. This, however, is a consequence of the final paragraph of Example 1.3.8 and the fact that the mapping $h: F_n \rightarrow h(F_n) \subset Q^{\aleph_0}$ defined by letting $h(\{x_i\}) = \{x_i\}$ is a homeomorphism, which, in its turn, is implied by the fact that both in $Q^{\aleph_0}$ and in F_n a sequence $x^1 = \{x_i^1\}, x^2 = \{x_i^2\}, \dots$ converges to $x = \{x_i\}$ if and only if the sequence $x_i^1, x_i^2, \dots$ converges to x_i for $i = 1, 2, \dots$. The last equivalence is well known to hold in the Cartesian product $Q^{\aleph_0}$; it does not generally hold in Hilbert space H , but it does hold in the subspace F_n , because all points of F_n have the same norm (cf. the hint to Problem 1.4.B(a)).

Now, to show that H_0 is homeomorphic to the square $H_0 \times H_0$ it suffices to note that by assigning to a point $(x, y) = (\{x_i\}, \{y_i\}) \in H_0 \times H_0$ the point $(x_1, y_1, x_2, y_2, \dots) \in H_0$ one defines a homeomorphism of $H_0 \times H_0$ onto H_0 .

To conclude, let us observe that the Cartesian products $H_0^2, H_0^3, \dots$ are all homeomorphic to H_0 , and thus are one-dimensional spaces. Since the countable Cartesian product $X^{\aleph_0}$ is homeomorphic to the limit

of an inverse sequence consisting of finite Cartesian products X^i , and since the limit of an inverse sequence consisting of separable metric spaces of dimension $\leq n$ is itself of dimension $\leq n$ (see Theorem 1.13.4), the countable Cartesian product H_0^∞ is also one-dimensional. Let us add that every infinite Cartesian product of compact metric spaces of finite positive dimension is infinite-dimensional. Indeed, every compact metric space of finite dimension > 0 contains a compact subspace of dimension one, and—as noted above—multiplying by such spaces raises the dimension by one. $\square$

Historical and bibliographic notes

An example of a compact metric space of dimension ∞ whose finite-dimensional subspaces are all zero-dimensional was announced by Walsh in [1978]. In [1965] Henderson constructed a compact metric space of dimension ∞ whose finite-dimensional closed subspaces are all zero-dimensional (simpler, but still very difficult examples of such spaces were given by Henderson in [1967] and by Zarelua in [1972]). Theorem 1.5.3 was proved for compact metric spaces by Menger in [1924] and by Urysohn in [1926] (announcement in [1922]) and was extended to separable metric spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz [1927] (the latter gave the simple proof reproduced here). Theorems 1.5.7, 1.5.8 and 1.5.10 were established for compact metric spaces by Urysohn in [1926] (announcement in [1922]) and were extended to separable metric spaces by Tumarkin and Hurewicz in the above quoted papers. Theorem 1.5.11 was proved by Tumarkin in [1926] (announcement in [1925]). As the reader will see in the next section, Theorem 1.5.12 states, in substance, that for every separable metric space X we have the equality $\text{ind} X = \text{Ind} X$. For compact metric spaces this equality was announced by Brouwer in [1924] while he was discussing relationships between his *Dimensionsgrad* and the small inductive dimension ind (Brouwer commented that the equality was also known to Urysohn); the proof was given by Menger in [1924] and by Urysohn in [1926]. For separable metric spaces, the equality of ind and Ind was established by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]. Theorem 1.5.13 was proved by Menger in [1924] for compact metric spaces, and extended by Hurewicz in [1927] to separable metric spaces. Theorem 1.5.16

was obtained by Menger in [1928]. Example 1.5.17 was described by Erdős in [1940]; let us add that Anderson and Keisler described in [1967] a space $K(n) \subset R^n$, where $n = 2, 3, \dots$, such that $\text{ind} K(n) = \text{ind}[K(n)]^{\aleph_0} = n - 1$.

The first example of two-dimensional compact metric spaces X and Y such that the Cartesian product $X \times Y$ is three-dimensional was given by Pontrjagin in [1930]; examples of such spaces can be found in Kodama [1970]. No “geometric” characterization of compact metric spaces satisfying the equality $\text{ind}(X \times Y) = \text{ind} X + \text{ind} Y$ is known; in particular, it is an open question if this equality holds for all absolute neighbourhood retracts (as shown by Borsuk in [1936], it holds if X and Y are absolute neighbourhood retracts satisfying condition (A)). The question is connected with the problem of delineating the class of spaces in which the small inductive dimension ind coincides with the cohomological dimension dim_{Z_p} with respect to the group Z_p of integers modulo p (cf. the final part of Section 1.9), because for every pair X, Y of locally compact spaces and every prime number p we have $\text{dim}_{Z_p}(X \times Y) = \text{dim}_{Z_p} X + \text{dim}_{Z_p} Y$.

Problems

1.5.A (de Groot and Nagata [1969]; announcement Hurewicz [1928]). Prove that if a completely metrizable separable space X of dimension ∞ can be represented as the union of countably many finite-dimensional subspaces, then for $n = 0, 1, 2, \dots$ the space X contains a closed subspace M such that $\text{ind} M = n$.

Hint. Let $X = \bigcup_{i=1}^{\infty} Z_i$, where $\text{ind} Z_i = 0$. Assuming that X does not contain any closed subspace of dimension n and applying Theorem 1.5.1, define a point whose all sufficiently small neighbourhoods have infinite-dimensional boundaries; consider such a boundary F_1 satisfying $\delta(F_1) < 1$ and $F_1 \cap Z_1 = \emptyset$. Iterating this procedure obtain a contradiction to the Cantor theorem.

1.5.B. (a) Observe that if a separable metric space X can be represented as the union of a family $\{F_s\}_{s \in S}$ of closed subspaces such that every point $x \in X$ has a neighbourhood which meets at most countably many sets F_s and $\text{ind} F_s \leq n$ for every $s \in S$, then $\text{ind} X \leq n$.

(b) Prove that if a separable metric space X can be represented as the union of a sequence $F_0, F_1, F_2, \dots$ of closed subspaces such that $\text{ind} F_i \leq n$ for $i = 1, 2, \dots$, then $\text{ind}_x X \leq n$ for every point $x \in F_0$ such that $\text{ind}_x F_0 \leq n$.

1.5.C. Let X be a separable metric space such that $\text{ind} X = n \geq 1$; the set $\{x \in X: \text{ind}_x X = n\}$ is called the *dimensional kernel* of the space X .

(a) (Menger [1924], Urysohn [1926]) Check that the dimensional kernel is an F_σ -set.

(b) (Menger [1926]) Show that the dimensional kernel of a separable metric space X such that $\text{ind} X = n \geq 1$ has dimension $\geq n-1$.

Hint. Represent the complement of the kernel as the union of two subspaces Y and Z such that $\text{ind} Y \leq n-2$, $\text{ind} Z \leq 0$ and Y is an F_σ -set in X .

(c) (Menger [1926]) Prove that the dimensional kernel of a compact metric space X such that $\text{ind} X = n \geq 1$ has dimension n at each point (cf. Theorem 1.9.8).

Hint. Suppose that for a point x of the dimensional kernel M the inequality $\text{ind}_x M \leq n-1$ holds, and for every positive number ε define a neighbourhood U of the point x in the space X such that $\delta(U) < \varepsilon$ and $\text{ind} \text{Fr} U \leq n-2$. To that end take a neighbourhood U_0 of the point x in the space X such that $\delta(U_0) < \varepsilon/2$ and $\text{ind}(M \cap \text{Fr} U_0) \leq n-2$. Then enlarge $M \cap \text{Fr} U_0$ to an $(n-2)$ -dimensional G_δ -set M^* in X . Let $U = U_0$, if $\text{Fr} U_0 \subset M^*$, and if $\text{Fr} U_0 \setminus M^* \neq \emptyset$, define a countable family $\{U_i\}_{i=1}^\infty$ of open subsets of X such that

$$\delta(U_i) < \varepsilon/2, \quad \text{ind} \text{Fr} U_i \leq n-2, \quad U_i \cap (\text{Fr} U_0 \setminus M^*) \neq \emptyset$$

for $i = 1, 2, \dots$, and

$$\text{Fr} U_0 \setminus M^* \subset \bigcup_{i=1}^\infty U_i \subset \overline{\bigcup_{i=1}^\infty U_i} \subset \text{Fr} U_0 \cup \bigcup_{i=1}^\infty \overline{U_i},$$

and let $U = \bigcup_{i=1}^\infty U_i$; note that $\text{Fr} U \subset (\text{Fr} U_0 \setminus \bigcup_{i=1}^\infty U_i) \cup \bigcup_{i=1}^\infty \text{Fr} U_i$.

Remark. A separable metric space X such that $\text{ind} X = n \geq 1$ and the dimensional kernel of X has dimension $n-1$ is called a *weakly n -dimensional* space. Clearly, a weakly n -dimensional space contains no compact subspace of dimension n . The space K described in Problem 1.2.E is weakly one-dimensional; the first example of such a space was given by Sierpiński in [1921]. First examples of weakly n -dimensional spaces for $n = 2, 3, \dots$ were given by Mazurkiewicz in [1929]; Mazurkiewicz's spaces are completely metrizable. A simpler construction of weakly n -dimensional spaces

for $n = 1, 2, \dots$ was described by Tomaszewski in [1979], where it is also shown that if X is a weakly n -dimensional space and Y is a weakly m -dimensional space, then $\text{ind}(X \times Y) \leq n + m - 1 = \text{ind} X + \text{ind} Y - 1$.

1.5.D. Prove that a subspace M of a metric space X satisfies the inequality $\text{ind} M \leq n \geq 0$ if and only if for every point $x \in M$ and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\text{ind}(M \cap \text{Fr } U) \leq n - 1$ (cf. Proposition 1.5.14 and Problem 4.1.C).

Hint. Apply Lemma 1.2.9.

1.5.E (Menger [1928]). (a) Show that if $\text{ind}_x X \leq 0$ and $\text{ind}_y Y \leq 0$, then $\text{ind}_{(x,y)}(X \times Y) \leq 0$.

(b) Applying the equality $\text{ind} I^2 = 2$, give an example of two subspaces X and Y of the real line such that for some points $x \in X$ and $y \in Y$ we have $\text{ind}_x X = 0$ and $\text{ind}_y Y = 1$, and yet $\text{ind}_{(x,y)}(X \times Y) = 2$.

1.5.F. Give an example of a completely metrizable separable space X such that $\text{ind} X = 1$ and X is homeomorphic to the square $X \times X$.

1.6. Definitions of the large inductive dimension and the covering dimension.

Metric dimension

The first separation theorem, established in the preceding section, suggests a modification in the definition of the small inductive dimension consisting in replacing the point x by a closed set A . In this way we are led to the notion of the large inductive dimension Ind , defined for all normal spaces. Both dimensions coincide in the realm of separable metric spaces. They diverge, however, in the class of metric spaces and also in the class of compact spaces; let us make clear at once that examples in point are very difficult and will not be discussed in this book. The theory of the dimension function Ind will be developed in Chapter 2, and in Chapter 4 it will be shown that in the realm of all metric spaces this theory is quite similar to the theory of the dimension function ind in separable metric spaces. In the present chapter, the large inductive dimension Ind , just as the covering dimension dim discussed later in this section, will play an auxiliary role: introducing these dimension functions leads to a simplification of the theory.

We pass to the formal definition of the dimension Ind .

1.6.1. Definition. To every normal space X one assigns the *large inductive dimension* of X , denoted by $\text{Ind}X$, which is an integer larger than or equal to -1 or the "infinite number ∞ "; the definition of the dimension function Ind consists in the following conditions:

(BČ1) $\text{Ind}X = -1$ if and only if $X = \emptyset$;

(BČ2) $\text{Ind}X \leq n$, where $n = 0, 1, \dots$, if for every closed set $A \subset X$ and each open set $V \subset X$ which contains the set A there exists an open set $U \subset X$ such that

$$A \subset U \subset V \quad \text{and} \quad \text{IndFr } U \leq n-1;$$

(BČ3) $\text{Ind}X = n$ if $\text{Ind}X \leq n$ and $\text{Ind}X > n-1$;

(BČ4) $\text{Ind}X = \infty$ if $\text{Ind}X > n$ for $n = -1, 0, 1, \dots$

The large inductive dimension Ind is also called the *Brouwer-Čech dimension*.

Applying induction with respect to $\text{Ind}X$, one can easily verify that whenever normal spaces X and Y are homeomorphic, then $\text{Ind}X = \text{Ind}Y$, i.e., the large inductive dimension is a topological invariant.

Modifying slightly the proof of Proposition 1.1.4, one obtains

1.6.2. Proposition. A normal space X satisfies the inequality $\text{Ind}X \leq n \geq 0$ if and only if for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{Ind}L \leq n-1$. $\square$

Applying induction with respect to $\text{Ind}X$, one can easily prove the following theorem, which justifies the names of the small and the large inductive dimensions.

1.6.3. Theorem. For every normal space X we have $\text{ind}X \leq \text{Ind}X$. $\square$

Both dimensions coincide in the realm of separable metric spaces.

1.6.4. Theorem. For every separable metric space X we have $\text{ind } X = \text{Ind } X$.

Proof. It suffices to show that $\text{Ind}X \leq \text{ind}X$; clearly, one can suppose that $\text{ind}X < \infty$. We shall apply induction with respect to $\text{ind}X$. The inequality holds if $\text{ind}X = 1$. Assume that the inequality is proved for all

separable metric spaces of small inductive dimension less than $n \geq 0$ and consider a separable metric space X such that $\text{ind} X = n$. Let A, B be a pair of disjoint closed subsets of X . By virtue of the first separation theorem, there exists a partition L between A and B such that $\text{ind} L \leq n-1$. It follows from the inductive assumption that $\text{Ind} L \leq n-1$, so that $\text{Ind} X \leq n$ by Proposition 1.6.2. Hence $\text{Ind} X \leq \text{ind} X$ and the proof is completed. $\square$

Using the dimension function Ind one can reformulate Theorem 1.2.6 in the following form (cf. Remark 1.2.7):

1.6.5. Theorem. *For every Lindelöf space X the conditions $\text{ind} X = 0$ and $\text{Ind} X = 0$ are equivalent.* $\square$

Besides the inductive dimensions ind and Ind , in dimension theory one studies another dimension function, namely the covering dimension dim defined for all normal spaces. In the following section we shall prove that the dimensions ind and dim coincide in the realm of separable metric spaces. Later on the reader will see that they diverge in the class of metric spaces and also in the class of compact spaces (see Remark 4.1.6 and Example 3.1.31). On the other hand, the dimensions Ind and dim coincide in the realm of all metric spaces (see Theorem 4.1.3) and diverge in the class of compact spaces (see Example 3.1.31). The reason why we introduce the covering dimension now is that this notion comes out in a natural way in proofs of the compactification, embedding and universal space theorems.

Let us sum up. There are three ways of defining the dimension of separable metric spaces. They are all equivalent and equally natural, but they are based on different geometric properties of spaces. Outside the class of separable metric spaces the dimensions ind , Ind , and dim diverge and three different dimension theories arise, all poorer than the dimension theory of separable metric spaces. The dimensions Ind and dim lead to much more interesting results than the dimension ind ; as a matter of fact, the latter is practically of no importance outside the class of separable metric spaces. Finally, in the dimension theory of separable metric spaces some theorems depend—roughly speaking—on the dimension ind , and other ones depend on the dimension dim ; so far we have discussed theorems of the first group, in the subsequent sections we shall discuss those of the second group.

In the definition of the covering dimension the notion of the order of a family of sets will be applied.

1.6.6. Definition. Let X be a set and $\mathcal{A}$ a family of subsets of X . By the *order* of the family $\mathcal{A}$ we mean the largest integer n such that the family $\mathcal{A}$ contains $n+1$ sets with a non-empty intersection; if no such integer exists, we say that the family $\mathcal{A}$ has order ∞ . The order of a family $\mathcal{A}$ is denoted by $\text{ord } \mathcal{A}$.

Thus, if the order of a family $\mathcal{A} = \{A_s\}_{s \in S}$ equals n , then for each $n+2$ distinct indexes $s_1, s_2, \dots, s_{n+2} \in S$ we have $A_{s_1} \cap A_{s_2} \cap \dots \cap A_{s_{n+2}} = \emptyset$. In particular, a family of order -1 consists of the empty set alone, and a family of order 0 consists of pairwise disjoint sets which are not all empty.

Let us recall that a cover $\mathcal{B}$ is a *refinement* of another cover $\mathcal{A}$ of the same space, in other words $\mathcal{B}$ *refines* $\mathcal{A}$, if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $B \subset A$. Clearly, every subcover $\mathcal{A}_0$ of $\mathcal{A}$ is a refinement of $\mathcal{A}$.

1.6.7. Definition. To every normal space X one assigns the *covering dimension* of X , denoted by $\dim X$, which is an integer larger than or equal to -1 or the "infinite number ∞ "; the definition of the dimension function $\dim$ consists in the following conditions:

- (ČL1) $\dim X \leq n$, where $n = -1, 0, 1, \dots$, if every finite open cover of the space X has a finite open refinement of order $\leq n$;
- (ČL2) $\dim X = n$ if $\dim X \leq n$ and $\dim X > n-1$;
- (ČL3) $\dim X = \infty$ if $\dim X > n$ for $n = -1, 0, 1, \dots$

The covering dimension $\dim$ is also called the *Čech-Lebesgue dimension*.

One readily sees that whenever normal spaces X and Y are homeomorphic, then $\dim X = \dim Y$, i.e., the covering dimension is a topological invariant. Clearly, $\dim X = -1$ if and only if $X = \emptyset$.

The next proposition contains two useful characterizations of the covering dimension; in the second one the notion of a shrinking is used.

1.6.8. Definition. By a *shrinking* of the cover $\{A_s\}_{s \in S}$ of a topological space X we mean any cover $\{B_s\}_{s \in S}$ of the space X such that $B_s \subset A_s$ for every $s \in S$. A shrinking is *open* (*closed*) if all its members are open (closed) subsets of the space X .

Clearly, every shrinking $\mathcal{B}$ of a cover $\mathcal{A}$ is a refinement of $\mathcal{A}$ and satisfies the inequality $\text{ord } \mathcal{B} \leq \text{ord } \mathcal{A}$.

1.6.9. Proposition. *For every normal space X the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\dim X \leq n$.*
- (b) *Every finite open cover of the space X has an open refinement of order $\leq n$.*
- (c) *Every finite open cover of the space X has an open shrinking of order $\leq n$.*

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a) are obvious. Consider a normal space X which satisfies (b). Let $\{U_i\}_{i=1}^k$ be a finite open cover of the space X and $\mathcal{V}$ an open refinement of this cover such that $\text{ord } \mathcal{V} \leq n$. For every $V \in \mathcal{V}$ choose an $i(V) \leq k$ such that $V \subset U_{i(V)}$ and define $V_i = \bigcup \{V: i(V) = i\}$. One readily verifies that $\{V_i\}_{i=1}^k$ is a shrinking of $\{U_i\}_{i=1}^k$ and has order $\leq n$, so that (b) $\Rightarrow$ (c). $\square$

We shall now show that when checking the inequality $\dim X \leq n$ it suffices to consider $(n+2)$ -element covers.

1.6.10. Theorem. *A normal space X satisfies the inequality $\dim X \leq n$ if and only if every $(n+2)$ -element open cover $\{U_i\}_{i=1}^{n+2}$ of the space X has an open shrinking $\{W_i\}_{i=1}^{n+2}$ of order $\leq n$, i.e., such that $\bigcap_{i=1}^{n+2} W_i = \emptyset$.*

Proof. It suffices to show that every normal space X such that $\dim X > n$ has an $(n+2)$ -element open cover $\{U_i\}_{i=1}^{n+2}$ with the property that each open shrinking $\{W_i\}_{i=1}^{n+2}$ of $\{U_i\}_{i=1}^{n+2}$ satisfies the condition $\bigcap_{i=1}^{n+2} W_i \neq \emptyset$.

Since $\dim X > n$, by virtue of Proposition 1.6.9 there exists an open cover $\mathcal{V} = \{V_i\}_{i=1}^k$ of the space X which has no open shrinking of order $\leq n$. Moreover, one can assume—replacing, if necessary, $\mathcal{V}$ by a suitable shrinking—that if $\mathcal{V}' = \{V'_i\}_{i=1}^k$ is an open shrinking of $\{V_i\}_{i=1}^k$, then

$$(1) \quad V'_1 \cap V'_2 \cap \dots \cap V'_{i_m} \neq \emptyset \quad \text{whenever} \quad V_1 \cap V_2 \cap \dots \cap V_{i_m} \neq \emptyset,$$

where $i_1, i_2, \dots, i_m$ is a sequence of natural numbers less than or equal to k . Indeed, if $\mathcal{V}$ has an open shrinking $\mathcal{V}'$ which does not satisfy (1), one replaces $\mathcal{V}$ by $\mathcal{V}'$ and one continues this procedure until an open

cover with the required property is obtained; as the number of intersections in (1) is finite, the process will come to an end after finitely many steps. Since $\text{ord } \mathcal{V} > n$, rearranging if need be the members of $\mathcal{V}$, we have

$$(2) \quad \bigcap_{i=1}^{n+2} V_i \neq \emptyset.$$

We shall show that the $(n+2)$ -element open cover $\{U_i\}_{i=1}^{n+2}$ of the space X , where $U_i = V_i$ for $i \leq n+1$ and $U_{n+2} = \bigcup_{i=n+2}^k V_i$, has the required property. Consider an open shrinking $\{W_i\}_{i=1}^{n+2}$ of $\{U_i\}_{i=1}^{n+2}$. The cover

$$\{W_1, W_2, \dots, W_{n+1}, W_{n+2} \cap V_{n+2}, W_{n+2} \cap V_{n+3}, \dots, W_{n+2} \cap V_k\}$$

of the space X is an open shrinking of $\mathcal{V}$, so that by (1) and (2) we have

$$\bigcap_{i=1}^{n+2} W_i \supset \left(\bigcap_{i=1}^{n+1} W_i \right) \cap (W_{n+2} \cap V_{n+2}) \neq \emptyset. \quad \square$$

Let us note that the last theorem immediately yields

1.6.11. Theorem. *For every normal space X the conditions $\text{Ind } X = 0$ and $\dim X = 0$ are equivalent. $\square$*

In the realm of compact metric spaces, the covering dimension can be characterized in terms of a metric, viz., by the condition that the space has finite covers of order $\leq n$ by open sets of arbitrarily small diameter. Let us recall that the *mesh* of a family $\mathcal{A}$ of subsets of a metric space X , denoted by $\text{mesh } \mathcal{A}$, is defined as the least upper bound of the diameters of all members of $\mathcal{A}$, i.e.,

$$\text{mesh } \mathcal{A} = \sup \{ \delta(A) : A \in \mathcal{A} \};$$

the mesh is either a non-negative real number or the "infinite number" ∞ .

1.6.12. Theorem. *For every compact metric space X the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\dim X \leq n$.*
- (b) *For every metric ρ on the space X and for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh } \mathcal{U} < \varepsilon$ and $\text{ord } \mathcal{U} \leq n$.*
- (c) *There exists a metric ρ on the space X with the property that for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh } \mathcal{U} < \varepsilon$ and $\text{ord } \mathcal{U} \leq n$.*

Proof. Let X be a compact metric space satisfying $\text{ind} X \leq n$; consider a metric ρ on the space X and a positive number ε . By the compactness of X , the open cover $\{B(x, \varepsilon/3)\}_{x \in X}$ has a finite open subcover; by applying (ČL1) to this subcover we obtain a finite open cover $\mathcal{U}$ such that $\text{mesh} \mathcal{U} < \varepsilon$ and $\text{ord} \mathcal{U} \leq n$. Hence (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) being obvious, to conclude the proof it suffices to show that (c) $\Rightarrow$ (a). Let ρ be a metric on the space X which has the property stated in (c). Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space X and denote by ε a Lebesgue number for the cover $\{U_i\}_{i=1}^k$, i.e., a positive number such that every subset of X which has diameter less than ε is contained in one of the sets U_i . The cover $\mathcal{U}$ in condition (c) is a refinement of $\{U_i\}_{i=1}^k$, so that $\dim X \leq n$. $\square$

The attempts to extend the last theorem to separable metric spaces led to the notion of the metric dimension, with a discussion of which we shall conclude this section. Before that, let us briefly comment upon conditions (b) and (c) in separable metric spaces. To begin with, observe that if for a metric ρ on a space X and for every positive number ε there exists a finite cover of the space X with mesh less than ε , then the metric ρ is totally bounded. Hence, when passing to separable metric spaces, we have to replace condition (b) by the following condition:

(b') *For every totally bounded metric ρ on the space X and for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh} \mathcal{U} < \varepsilon$ and $\text{ord} \mathcal{U} \leq n$.*

Now, one proves that conditions (a) and (b') are equivalent for every separable metric space X (see Problem 1.6.B), whereas conditions (a) and (c) are generally not equivalent for such spaces (see Example 1.10.23).

To every metric space (X, ρ) one assigns the *metric dimension* of (X, ρ) , denoted by $\mu \dim(X, \rho)$, which is an integer larger than or equal to -1 or the "infinite number" ∞ ; the definition follows the pattern of the definition of the covering dimension $\dim$ except that condition (ČL1) is replaced by the condition that for every positive number ε there exists an open cover $\mathcal{U}$ of the space X such that $\text{mesh} \mathcal{U} < \varepsilon$ and $\text{ord} \mathcal{U} \leq n$. Clearly, if (X, ρ) is a compact space, then $\mu \dim(X, \rho) = \dim X$.

From the discussion in the penultimate paragraph it follows that in the realm of separable metric spaces the metric dimension is not a topological invariant; in general, the number $\mu \dim(X, \rho)$ depends upon the metric ρ on the space X . Nevertheless, a theory of the metric dimension $\mu \dim$

can be developed, which shows a resemblance to the theory of the covering dimension $\dim$. Let us observe that—as the reader can readily verify—if the metrics ϱ and σ on a space X are uniformly equivalent, i.e., the identity mapping of (X, ϱ) to (X, σ) and also the identity mapping of (X, σ) to (X, ϱ) are uniformly continuous, then $\mu\dim(X, \varrho) = \mu\dim(X, \sigma)$; this means that the metric dimension is a uniform invariant.

Let us also note that in the definition of the metric dimension one has to consider an arbitrary open cover $\mathcal{U}$, because the restriction to finite open covers would imply the restriction of the definition to totally bounded spaces. However, in the case where (X, ϱ) is a totally bounded metric space, the restriction to finite open covers yields an equivalent definition (see Problem 1.6.C).

From Problem 1.7.E below it follows that for every separable metric space (X, ϱ) we have $\mu\dim(X, \varrho) \leq \dim X$; by virtue of an important characterization of the covering dimension, to be established in Chapter 3 (see Theorem 3.2.1), this inequality extends to all metric spaces. On the other hand, for every metric space (X, ϱ) we have $\dim X \leq 2\mu\dim(X, \varrho)$; a proof is sketched out in the hint to Problem 1.6.D.

We shall return briefly to the metric dimension in Section 1.10, where a characterization of this dimension function in terms of mappings to polyhedra will be given (see Problem 1.10.L).

Historical and bibliographic notes

As we have already observed in the notes to Section 1.1, the notion of the large inductive dimension Ind is related to Brouwer's notion of *Dimensionsgrad*. A formal definition of the dimension function Ind in the class of normal spaces was first given by Čech in [1931], which was a short announcement of results in his paper [1932] devoted to a study of the large inductive dimension. Theorem 1.6.4 is a restatement of Theorem 1.5.12, its history is described in the notes to Section 1.5. The covering dimension $\dim$ was formally introduced and discussed in Čech's paper [1933]; it is related to a property of covers of the n -cube I^n discovered by Lebesgue in [1911] (see the notes to Section 1.1). Theorem 1.6.10 was proved by Hemmingsen in [1946], and Theorem 1.6.11 by Vedenisoff in [1939].

The notion of the metric dimension was introduced by Alexandroff around 1930. As a definition he used the characterization given here in

Problem 1.10.L, which is connected with his famous theorem on ε -translations to polyhedra (see Theorem 1.10.19). Alexandroff's question whether the metric dimension coincides with the covering dimension in the realm of separable metric spaces was solved in the negative by Sitnikov in [1953] (see Example 1.10.23). This last paper called the topologists' attention back to the notion of the metric dimension. The basic properties of the dimension $\mu\dim$ were established by Smirnov in [1956] and by Egorov in [1959]. Besides the notion of the metric dimension discussed in this section, which is a natural geometric notion with a sound intuitive background, a few other metric dimension functions have recently been studied; they are all obtained by replacing topological conditions by the corresponding metrical ones in various characterizations of the dimension $\dim$. A discussion of this topic can be found in Nagami's book [1970].

Problems

1.6.A. Give a direct proof of Theorem 1.6.11.

1.6.B (Hurewicz [1930]). Prove that a separable metric space X satisfies the inequality $\dim X \leq n$ if and only if for every totally bounded metric ϱ on the space X and for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh } \mathcal{U} < \varepsilon$ and $\text{ord } \mathcal{U} \leq n$.

Hint. For a finite open cover $\{U_i\}_{i=1}^k$ of the space X define a metric ϱ on the space X with the property that every subset of X which has diameter less than 1 is contained in one of the sets U_i . To that end define continuous functions $f_1, f_2, \dots, f_k$ from X to I such that $f_i(X \setminus U_i) \subset \{0\}$ for $i = 1, 2, \dots$

$\dots, k$ and $\sum_{i=1}^k f_i^{-1}(1) = X$; observe that by adding to the original distance of x and y the sum $\sum_{i=1}^k |f_i(x) - f_i(y)|$ one obtains a metric on the space X .

1.6.C (Egorov [1959]). Prove that a totally bounded metric space (X, ϱ) satisfies the inequality $\mu\dim(X, \varrho) \leq n$ if and only if for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh } \mathcal{U} < \varepsilon$ and $\text{ord } \mathcal{U} \leq n$.

Hint. Consider a finite subset $\{x_1, x_2, \dots, x_k\}$ of the space X with the property that for every point $x \in X$ there exists an $i \leq k$ such that $\varrho(x, x_i) < \varepsilon/4$ and observe that every subset of X which has diameter less than $\varepsilon/4$ is contained in a member of the cover $\{B(x_i, \varepsilon/2)\}_{i=1}^k$.

1.6.D (Katětov [1958]). Prove that for every metric space (X, ρ) we have $\dim X \leq 2\mu \dim(X, \rho)$.

Hint. For $i = 1, 2, \dots$ choose an open cover $\mathcal{U}_i$ of the space X such that $\text{mesh } \mathcal{U}_i < 1/3^i$ and $\text{ord } \mathcal{U}_i \leq n = \mu \dim(X, \rho)$. Consider a finite open cover $\{U_j\}_{j=1}^k$ of the space X and denote by $\mathcal{V}_i$ the family of all sets $U \in \mathcal{U}_i$ such that $\rho(U, X \setminus U_j) \geq 1/3^i$ for a certain $j \leq k$. Check that the sets $V_1, V_2, \dots$, where $V_i = \bigcup \mathcal{V}_i$, form a cover of the space X and that $\bar{V}_i \subset V_{i+1}$ for $i = 1, 2, \dots$. Consider the sets $F_i = \bar{V}_{i-1} \cup \bar{C}_i$, where $V_0 = \emptyset$ and C_i consists of all points of X which belong to $n+1$ members of $\mathcal{V}_i$, and the families $\mathcal{W}_i = \{V \setminus F_{i-1} : V \in \mathcal{V}_i\}$, where $F_0 = \emptyset$; define $\mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ and $W_i = \bigcup \mathcal{W}_i$. Show that $\bar{C}_i \subset V_i$ for $i = 1, 2, \dots$ and deduce that $\mathcal{W}$ is a cover of the space X ; observe that it is a refinement of $\{U_j\}_{j=1}^k$. Check that $W_i \cap W_m = \emptyset$ whenever $m \geq i+2$ and deduce that $\text{ord } \mathcal{W} \leq 2n$.

Remark. The evaluation of $\dim X$ in Problem 1.6.D cannot be improved (see Example 1.10.23 and Problem 1.10.J).

1.7. The compactification and coincidence theorems. Characterization of dimension in terms of partitions

The compactification theorem belongs to the group of theorems depending on the dimension $\dim$, and in its proof covers are used in an essential way; accordingly, we formulate this theorem in terms of the covering dimension $\dim$. The compactification theorem is an important step towards the proof of the coincidence theorem, which states that the dimensions ind , Ind and $\dim$ coincide in the realm of separable metric spaces.

We begin with introducing three simple operations on covers.

If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k$ are covers of a topological space X , then the family of all intersections $A_1 \cap A_2 \cap \dots \cap A_k$, where $A_i \in \mathcal{A}_i$ for $i = 1, 2, \dots, k$, is a cover of the space X . We denote this cover by $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \dots \wedge \mathcal{A}_k$; obviously, it is a refinement of $\mathcal{A}_i$ for $i = 1, 2, \dots, k$.

If $f: X \rightarrow Y$ is a continuous mapping of a topological space X to a topological space Y and $\mathcal{A}$ is a cover of the space Y , then the family of all inverse images $f^{-1}(A)$, where $A \in \mathcal{A}$, is a cover of the space X . We denote this cover by $f^{-1}(\mathcal{A})$.

If M is a subspace of a topological space X and $\mathcal{A}$ is a cover of the space X , then the family of all intersections $M \cap A$, where $A \in \mathcal{A}$, is a cover of the subspace M . We denote this cover by $\mathcal{A}|M$.

One readily sees that the above operations applied to open (closed) covers yield open (closed) covers, and applied to finite covers yield finite covers.

1.7.1. Lemma. *Let (X, ϱ) be a totally bounded metric space such that $\dim X \leq n$. For every finite sequence $f_1, f_2, \dots, f_k$ of continuous functions from X to I and for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh } \mathcal{U} < \varepsilon$, $\text{ord } \mathcal{U} \leq n$, and $|f_i(x) - f_i(y)| < \varepsilon$ for $i = 1, 2, \dots, k$ whenever x and y belong to the same member of $\mathcal{U}$.*

Proof. Let $\mathcal{V}$ be a finite open cover of the space X such that $\text{mesh } \mathcal{V} < \varepsilon$ and let $\mathcal{W}$ be a finite open cover of the interval I such that $\text{mesh } \mathcal{W} < \varepsilon$. The reader can readily check that any finite open refinement $\mathcal{U}$ of the cover $\mathcal{V} \wedge f_1^{-1}(\mathcal{W}) \wedge f_2^{-1}(\mathcal{W}) \wedge \dots \wedge f_k^{-1}(\mathcal{W})$ such that $\text{ord } \mathcal{U} \leq n$ has the required properties. $\square$

1.7.2. The compactification theorem. *For every separable metric space X there exists a dimension preserving compactification, i.e., a compact metric space $\tilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequality $\dim \tilde{X} \leq \dim X$.*

More exactly, for every totally bounded metric ϱ on the space X there exists an equivalent metric $\tilde{\varrho}$ on X such that $\varrho(x, y) \leq \tilde{\varrho}(x, y)$ for $x, y \in X$ and the completion $\tilde{X}$ of the space X with respect to the metric $\tilde{\varrho}$ is a compactification of X which satisfies the inequality $\dim \tilde{X} \leq \dim X$.

Proof. We can suppose that $\dim X = n < \infty$. For $m = 1, 2, \dots$ we shall define a finite open cover $\mathcal{U}_m = \{U_{m,k}\}_{k=1}^{k_m}$ of the space X such that $\text{mesh } \mathcal{U}_m < 1/2^m$, $\text{ord } \mathcal{U}_m \leq n$, and

$$(1) \quad |f_{i,j}(x) - f_{i,j}(y)| < 1/2^m \quad \text{for } i < m \text{ and } j = 1, 2, \dots, k_i$$

whenever x and y belong to the same member of $\mathcal{U}_m$, where

$$(2) \quad f_{i,j}(x) = \frac{\varrho(x, X \setminus U_{i,j})}{\sum_{l=1}^{k_i} \varrho(x, X \setminus U_{i,l})} \quad \text{for } x \in X.$$

To obtain $\mathcal{U}_1$ it suffices to apply the total boundedness of ϱ and the inequality $\dim X \leq n$. When the covers $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_{m-1}$ are defined, it suffices to apply Lemma 1.7.1 to obtain $\mathcal{U}_m$.

Let us now arrange all pairs (i, j) , where $i = 1, 2, \dots$ and $j = 1, 2, \dots, k_i$, into a simple infinite sequence and denote by $n(i, j)$ the place of (i, j) in this sequence. Define a new metric $\tilde{\varrho}$ on the set X by letting

$$(3) \quad \tilde{\varrho}(x, y) = \varrho(x, y) + \sum_{n(i,j)=1}^{\infty} \frac{1}{2^{n(i,j)}} |f_{i,j}(x) - f_{i,j}(y)|;$$

one readily verifies that the metric ϱ and $\tilde{\varrho}$ are equivalent.

We shall show that the sequence $\widetilde{\text{mesh}} \mathcal{U}_m$, where $\widetilde{\text{mesh}}$ denotes the mesh with respect to the metric $\tilde{\varrho}$, converges to zero. Consider an arbitrary positive number ε . Let N be a natural number satisfying the inequality $1/2^N < \varepsilon/3$, and M a natural number such that $M \geq N$,

$$\frac{1}{2^M} \sum_{i=1}^N \frac{1}{2^i} < \varepsilon/3,$$

and $M > i$ whenever $n(i, j) \leq N$. From (1) and (3) it follows by a simple computation that $\widetilde{\text{mesh}} \mathcal{U}_m < \varepsilon$ if $m \geq M$, i.e., $\lim \widetilde{\text{mesh}} \mathcal{U}_m = 0$. In particular, the space $(X, \tilde{\varrho})$ is totally bounded, so that the completion $\tilde{X}$ of the space X with respect to the metric $\tilde{\varrho}$ is a compactification of X .

Since the functions $f_{m,k}: X \rightarrow I$ are uniformly continuous with respect to $\tilde{\varrho}$, they can be extended to continuous functions $\tilde{f}_{m,k}: \tilde{X} \rightarrow I$. From (2) it follows that $\sum_{k=1}^{k_m} \tilde{f}_{m,k}(x) = 1$ for every $x \in X$. Hence $\sum_{k=1}^{k_m} \tilde{f}_{m,k}(x) = 1$ for every $x \in \tilde{X}$, which implies that the family $\tilde{\mathcal{U}}_m = \{\tilde{U}_{m,k}\}_{k=1}^{k_m}$, where $\tilde{U}_{m,k} = \tilde{f}_{m,k}^{-1}((0, 1])$, is an open cover of the space $\tilde{X}$ for $m = 1, 2, \dots$. Now, by the density of X in $\tilde{X}$ and since $X \cap \tilde{U}_{m,k} = U_{m,k}$ for $m = 1, 2, \dots$ and $k = 1, 2, \dots, k_m$, we have $\widetilde{\text{mesh}} \tilde{\mathcal{U}}_m = 0$ and $\text{ord} \tilde{\mathcal{U}}_m \leq n$, so that $\dim \tilde{X} \leq n = \dim X$ by virtue of Theorem 1.6.12. $\square$

A variant of the above proof of the compactification theorem is outlined in Problem 1.7.B.

Let us observe that from the equality $\text{ind} \tilde{X} = \dim \tilde{X}$, which is a consequence of Lemmas 1.7.4 and 1.7.6, and from Theorem 1.1.2 and Lemma 1.7.4 it follows that $\dim \tilde{X} = \dim X$ in the compactification theorem.

We now pass to the coincidence theorem.

1.7.3. Lemma. *Let X be a metric space and M a subspace of X . For every family $\{F_i\}_{i=1}^k$ of pairwise-disjoint closed subsets of M there exists a family $\{W_i\}_{i=1}^k$ of pairwise-disjoint open subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \dots, k$.*

Proof. The sets

$$W_i = \bigcap_{j \neq i} \{x \in X: \varrho(x, F_i) < \varrho(x, F_j)\}$$

have all the required properties. $\square$

1.7.4. Lemma. *For every separable metric space X we have $\dim X \leq \operatorname{ind} X$.*

Proof. We can suppose that $\operatorname{ind} X < \infty$. If $\operatorname{ind} X = -1$, we clearly have $\dim X \leq \operatorname{ind} X$. Consider the case where $\operatorname{ind} X = 0$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of the space X . By virtue of Proposition 1.2.4, the cover $\mathcal{U}$ has a refinement $\{V_i\}_{i=1}^\infty$ consisting of open-and-closed subsets of X . The sets

$$\begin{aligned} W_1 &= V_1, & W_2 &= V_2 \setminus W_1, & \dots, \\ W_i &= V_i \setminus (W_1 \cup W_2 \cup \dots \cup W_{i-1}), & \dots \end{aligned}$$

are open-and-closed and pairwise disjoint, and form a cover of the space X which refines the cover $\mathcal{U}$. From Proposition 1.6.9 it follows that $\dim X \leq 0$, so that again $\dim X \leq \operatorname{ind} X$.

Now, consider the case where $\operatorname{ind} X = n > 0$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of the space X . By virtue of the second decomposition theorem

$$X = Z_1 \cup Z_2 \cup \dots \cup Z_{n+1}, \quad \text{where } \operatorname{ind} Z_j \leq 0 \text{ for } j = 1, 2, \dots, n+1.$$

It follows from the already proved special case of our lemma that $\dim Z_j \leq 0$ for $j = 1, 2, \dots, n+1$. Hence the cover $\mathcal{U}|_{Z_j}$ of the space Z_j has a shrinking $\{F_{j,i}\}_{i=1}^k$ consisting of pairwise disjoint open-and-closed subsets of Z_j . Applying Lemma 1.7.3, we obtain a family $\{W_{j,i}\}_{i=1}^k$ of pairwise disjoint open subsets of X such that $F_{j,i} \subset W_{j,i}$ for $i = 1, 2, \dots, k$. The sets $V_{j,i} = W_{j,i} \cap U_i$, where $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n+1$, form an open cover of the space X which refines the cover $\mathcal{U}$; the order of this cover is not larger than n , because any $n+2$ sets $V_{j,i}$ include at least two with the same index j , and each two of such sets have an empty intersection. Thus $\dim X \leq n$, i.e., $\dim X \leq \operatorname{ind} X$. $\square$

Let us observe that in the case where $\operatorname{ind} X = 0$ the inequality $\dim X \leq \operatorname{ind} X$ follows from Theorems 1.6.4 and 1.6.11, but the argument given above is much simpler.

1.7.5. Remark. It follows from the second part of the last proof that for every separable subspace M of a metric space X such that $\operatorname{ind} M \leq n$

and for every family $\{U_i\}_{i=1}^k$ of open subsets of X such that $M \subset \bigcup_{i=1}^k U_i$ there exists a finite family $\mathcal{V}$ of open subsets of X satisfying the conditions $M \subset \bigcup \mathcal{V}$ and $\text{ord } \mathcal{V} \leq n$ and such that each of its members is contained in a set U_i .

1.7.6. Lemma. *For every compact metric space X we have $\text{ind } X \leq \dim X$.*

Proof. We can suppose that $\dim X < \infty$. We shall apply induction with respect to $\dim X$. If $\dim X = -1$, we clearly have $\text{ind } X \leq \dim X$. Assume that our inequality holds for all compact metric spaces with covering dimension $\leq n-1$ and consider a compact metric space X such that $\dim X = n \geq 0$, a point $x \in X$, and a closed set B such that $x \notin B$. It suffices to define open sets $K, M \subset X$ which, together with the set $L = X \setminus (K \cup M)$, satisfy the conditions

$$(4) \quad x \in K, \quad B \subset M, \quad K \cap M = \emptyset \quad \text{and} \quad \dim L \leq n-1;$$

indeed, the set L is then a partition between x and B , and $\text{ind } L \leq n-1$ by virtue of the inductive assumption. To that end we shall define two sequences $K_0, K_1, K_2, \dots$ and $M_0, M_1, M_2, \dots$ of closed subsets of X satisfying for $i = 1, 2, \dots$ the following conditions:

$$(5) \quad x \in K_{i-1} \subset \text{Int } K_i, \quad B \subset M_{i-1} \subset \text{Int } M_i \quad \text{and} \quad K_i \cap M_i = \emptyset.$$

$$(6) \quad \text{The set } L_i = X \setminus (K_i \cup M_i) \text{ has a finite open cover with mesh } < 1/i \text{ and order } \leq n-1.$$

Let $K_0 = \{x\}$, $M_0 = B$, and assume that the sets K_i, M_i are already defined for $i < j$ and satisfy (5) and (6) for $0 < i < j$. Since in a compact metric space the distance of two disjoint closed sets is positive, there exists a finite open cover $\mathcal{U}_j$ of the space X such that $\text{mesh } \mathcal{U}_j < \min(1/j, \varrho(K_{j-1}, M_{j-1}))$ and $\text{ord } \mathcal{U}_j \leq n$. Let $K_j = X \setminus H_j$ and $M_j = X \setminus G_j$, where

$$G_j = \bigcup \{U \in \mathcal{U}_j: U \cap \bar{M}_{j-1} = \emptyset\}$$

and

$$H_j = \bigcup \{U \in \mathcal{U}_j: \bar{U} \cap M_{j-1} \neq \emptyset\}.$$

As the closure of no member of $\mathcal{U}_j$ meets both K_{j-1} and M_{j-1} , it follows from the definitions of G_j and H_j that

$$\bar{G}_j \cap M_{j-1} = \emptyset = \bar{H}_j \cap K_{j-1},$$

which implies that $K_{j-1} \subset X \setminus \bar{H}_j = \text{Int } K_j$ and $M_{j-1} \subset X \setminus \bar{G}_j = \text{Int } M_j$; moreover, $K_j \cap M_j = \emptyset$, because $G_j \cup H_j = X$. Thus, condition (5) is

satisfied for $i = j$. The family $\mathcal{W}_j = \{U \cap L_j: U \in \mathcal{U}_j \text{ and } \bar{U} \cap M_{j-1} \neq \emptyset\}$ is an open cover of the set $L_j = X \setminus (K_j \cup M_j) = G_j \cap H_j$ and $\text{mesh } \mathcal{W}_j < 1/j$. Since every point $x \in L_j \subset G_j$ belongs to at least one set $U \in \mathcal{U}_j$ such that $\bar{U} \cap M_{j-1} = \emptyset$, the order of $\mathcal{W}_j$ is not larger than $n-1$. Thus condition (6) is also satisfied for $i = j$, so that the construction of the sequences $K_0, K_1, K_2, \dots$ and $M_0, M_1, M_2, \dots$ is completed.

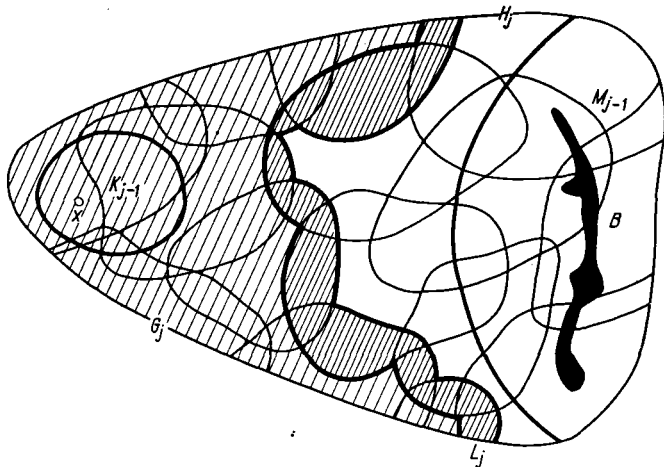


Fig. 6

The open sets $K = \bigcup_{i=0}^{\infty} K_i$ and $M = \bigcup_{i=0}^{\infty} M_i$ are disjoint and contain x and B respectively, and the set $L = X \setminus (K \cup M) = \bigcup_{i=1}^{\infty} L_i$ satisfies, by virtue of (6) and Theorem 1.6.12, the inequality $\dim L \leq n-1$; hence conditions (4) hold and the proof of the theorem is completed. $\square$

1.7.7. The coincidence theorem. For every separable metric space X we have $\text{ind } X = \text{Ind } X = \dim X$.

Proof. By virtue of Theorem 1.6.4 and Lemma 1.7.4 it suffices to show that $\text{ind } X \leq \dim X$. Apply the compactification theorem to obtain a compactification $\tilde{X}$ of the space X such that $\dim \tilde{X} \leq \dim X$. It follows from Lemma 1.7.6 that $\text{ind } \tilde{X} \leq \dim \tilde{X}$, so that $\text{ind } X \leq \dim X$ by virtue of the subspace theorem. $\square$

Let us now make some comments on the last theorem. To prove that for every separable metric space X we have $\text{ind } X = \dim X$ one has to establish two inequalities: $\dim X \leq \text{ind } X$ and $\text{ind } X \leq \dim X$. The proof

of the former is fairly easy, and the proof of the latter much more difficult. Usually, we prove the inequality $\text{ind} X \leq \dim X$ for compact metric spaces, which greatly simplifies the arguments, and then apply the compactification theorem to extend the inequality over all separable metric spaces. In all the existing proofs of the inequality $\text{ind} X \leq \dim X$ one can detect an auxiliary integer-valued invariant $d(X)$ for which the inequalities $\text{ind} X \leq d(X)$ and $d(X) \leq \dim X$ are established. In our proof the invariant $d(X)$, defined for every compact metric space X , was the smallest integer $n \geq 0$ such that the space X has finite open covers of order $\leq n$ with arbitrarily small meshes; the inequality $d(X) \leq \dim X$ was trivial and the proof reduced to showing that $\text{ind} X \leq d(X)$. In Section 1.11 we shall, incidentally, give another proof of the inequality $\text{ind} X \leq \dim X$ for compact metric spaces. In that proof the invariant $d(X)$ will be the smallest integer $n \geq 0$ such that X is embeddable in the space $N_n^{2n+1} \subset R^{2n+1}$; since $\text{ind} X \leq d(X)$ by virtue of Example 1.5.9, the proof will reduce to showing that $d(X) \leq \dim X$. One more proof of the inequality $\text{ind} X \leq \dim X$ is sketched out in Problem 1.7.D.

Let us also note that in Section 4.1 the proof of Lemma 1.7.6 will reappear almost verbatim as part of the proof that the inequality $\text{Ind} X \leq \dim X$ holds for every metric space X .

We conclude this section with a characterization of dimension stated in terms of partitions. At the basis of this characterization of dimension lies an interesting geometric property of the n -cube I^n ; viz., the property that if L_i is a partition between the pair of opposite faces $A_i = \{\{x_j\} \in I^n: x_i = 0\}$ and $B_i = \{\{x_j\} \in I^n: x_i = 1\}$ of I^n for $i = 1, 2, \dots, n$, then $\bigcap_{i=1}^n L_i \neq \emptyset$; this property is closely related to the fixed-point property of I^n (see Theorem 1.8.1 and Problem 1.8.B). Let us note that the importance of the theorem on partitions consists in the fact that it provides an internal characterization of n -dimensional spaces which, in effect, is but a reformulation of an important external characterization of such spaces, namely of the characterization in terms of mappings to the n -sphere S^n (see Theorem 1.9.3), which in turn is very close to the cohomological characterization of dimension (see the final part of Section 1.9). The theorem on partitions will also be used in the proof of the fundamental equality $\text{ind} R^n = n$ (see Theorem 1.8.2).

Let us begin with an auxiliary theorem which will often be used in our study of the covering dimension.

1.7.8. Theorem. *Every finite open cover $\{U_i\}_{i=1}^k$ of a normal space X has a closed shrinking $\{F_i\}_{i=1}^k$.*

Proof. We shall apply induction with respect to k beginning with $k = 2$. In this case, applying the definition of normality to disjoint closed subsets $A = X \setminus U_1$ and $B = X \setminus U_2$ of the space X , we obtain disjoint open sets $V_1, V_2 \subset X$ such that $A \subset V_1$ and $B \subset V_2$. The sets $F_1 = X \setminus V_1$ and $F_2 = X \setminus V_2$ form the required shrinking. Assume that the theorem is proved for every natural number $k < m \geq 3$ and consider an m -element open cover $\{U_i\}_{i=1}^m$ of the space X . Define

$$U'_i = U_i \text{ for } i \leq m-2 \quad \text{and} \quad U'_{m-1} = U_{m-1} \cup U_{m-2};$$

applying the theorem to the cover $\{U'_i\}_{i=1}^{m-1}$ we obtain a closed shrinking $\{F'_i\}_{i=1}^{m-1}$. The closed subspace F'_{m-1} of the space X is normal; applying the theorem again, this time to the two-element open cover $\{F'_{m-1} \cap U_{m-1}, F'_{m-1} \cap U_m\}$ of the space F'_{m-1} , we obtain a closed cover $\{F_{m-1}, F_m\}$ of F'_{m-1} such that $F_{m-1} \subset U_{m-1}$ and $F_m \subset U_m$. One readily checks that the family $\{F_i\}_{i=1}^m$, where $F_i = F'_i$ for $i \leq m-2$, is the required shrinking of the cover $\{U_i\}_{i=1}^m$ of the space X . $\square$

1.7.9. Theorem on partitions. *A separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.*

Proof. If $\text{ind} X \leq n \geq 0$, then—applying the second separation theorem—we can define, one by one, partitions $L_1, L_2, \dots, L_{n+1}$ such that $\text{ind}(L_1 \cap L_2 \cap \dots \cap L_i) \leq n-i$ for $i = 1, 2, \dots, n+1$; clearly $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

We shall now show that if X satisfies the condition in the theorem, then $\dim X \leq n$ which, by the coincidence theorem, will complete the proof. We are going to apply Theorem 1.6.10; consider thus an $(n+2)$ -element open cover $\{U_i\}_{i=1}^{n+2}$ of the space X . By Theorem 1.7.8, the cover $\{U_i\}_{i=1}^{n+2}$ has a closed shrinking $\{B_i\}_{i=1}^{n+2}$; let $A_i = X \setminus U_i$ for $i = 1, 2, \dots, n+1$. The sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ consists of $n+1$ pairs of disjoint closed subsets of X . Hence, there exist closed sets $L_1, L_2, \dots, L_{n+1}$

such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Let us consider open sets $V_i, W_i \subset X$ such that

$$(7) \quad A_i \subset V_i, \quad B_i \subset W_i, \quad V_i \cap W_i = \emptyset \quad \text{and} \quad X \setminus L_i = V_i \cup W_i$$

for $i = 1, 2, \dots, n+1$. Observe that

$$(8) \quad \bigcup_{i=1}^{n+1} V_i \cup \bigcup_{i=1}^{n+1} W_i = \bigcup_{i=1}^{n+1} (V_i \cup W_i) = \bigcup_{i=1}^{n+1} (X \setminus L_i) = X \setminus \bigcap_{i=1}^{n+1} L_i = X.$$

From (8), (7) and the inclusion $B_{n+2} \subset U_{n+2}$ it follows that

$$\bigcup_{i=1}^{n+1} W_i \cup [U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i] = [\bigcup_{i=1}^{n+1} W_i \cup U_{n+2}] \cap [\bigcup_{i=1}^{n+1} W_i \cup \bigcup_{i=1}^{n+1} V_i] \supset \bigcup_{i=1}^{n+2} B_i = X,$$

so that the family $\{W_i\}_{i=1}^{n+2}$, where $W_{n+2} = U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i$, is an open shrinking of the cover $\{U_i\}_{i=1}^{n+2}$. It follows from (7) that

$$\bigcap_{i=1}^{n+2} W_i = \bigcap_{i=1}^{n+1} W_i \cap [U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i] \subset \bigcap_{i=1}^{n+1} W_i \cap \bigcap_{i=1}^{n+1} V_i = \emptyset,$$

therefore $\dim X \leq n$ by virtue of Theorem 1.6.10. $\square$

1.7.10. Remark. Let us note that in the second part of the above proof only the normality of the space X was applied; hence, we have shown that if for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of a normal space X there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$, then $\dim X \leq n$.

Let us call the reader's attention to the structure of the last proof. We started with the inequality $\text{ind} X \leq n$, showed that this inequality implies a property of the space X , then proved that this property implies the inequality $\dim X \leq n$, and, finally, applied the coincidence theorem. Thus, we incidentally gave another proof of Lemma 1.7.4; obviously, the original proof of Lemma 1.7.4 is more perspicuous. The proof of Theorem 1.9.3 below has a similar structure. The conditions in both theorems characterize the covering dimension $\dim$ in the realm of normal spaces; however, in the realm of separable metric spaces the proofs are considerably shortened by applying the coincidence theorem (cf. Theorems 3.2.6 and 3.2.10).

Historical and bibliographic notes

Theorem 1.7.2 was established by Hurewicz in [1927b] (another proof in [1930]). This theorem permitted or facilitated the extension of many theorems of dimension theory from compact metric spaces to separable metric spaces; it was an important achievement in the development of the theory. Lemma 1.7.4 was proved for metric compact spaces by Menger in [1924] and by Urysohn in [1926] (announcement in [1922]) and was extended to separable metric spaces by Hurewicz in [1927b]. Lemma 1.7.6 was obtained by Urysohn in [1926]. The equality $\text{ind} X = \dim X$ in Theorem 1.7.7 was established by Hurewicz in [1927b], the history of the equality $\text{ind} X = \text{Ind} X$ is described in the notes to Section 1.5. Theorem 1.7.9 was proved by Eilenberg and Otto in [1938].

Problems

1.7.A. Observe that the enlargement theorem readily follows from the compactification theorem and the Lavrentieff theorem (see [GT], Theorem 4.3.21).

1.7.B. Let X be a separable metric space such that $\dim X = n$ and $f_{i,j}: X \rightarrow I$, where $i = 1, 2, \dots$ and $j = 1, 2, \dots, k_i$, the functions defined in the proof of Theorem 1.7.2. Show that by assigning to every point $x \in X$ the point $f(x)$ in the Hilbert cube $I^{\aleph_0}$, the $n(i,j)$ -th coordinate of which is equal to $f_{i,j}(x)$, one defines a homeomorphic embedding $f: X \rightarrow I^{\aleph_0}$ such that $\dim \overline{f(X)} \leq n$.

1.7.C. (a) (Engelking [1960], Forge [1961]) Show that for every separable metric space X and for every sequence of continuous functions $f_1, f_2, \dots$, where $f_i: X \rightarrow I$ for $i = 1, 2, \dots$, there exists a compactification $\tilde{X}$ of the space X such that $\dim \tilde{X} \leq \dim X$ and each f_i is extendable to a continuous function $\tilde{f}_i: \tilde{X} \rightarrow I$.

(b) (Engelking [1960]; for $n = 0$, de Groot and McDowell [1960]) Prove that for every separable metric space X and for every sequence of continuous mappings $g_1, g_2, \dots$, where $g_i: X \rightarrow X$ for $i = 1, 2, \dots$, there exists a compactification $\tilde{X}$ of the space X such that $\dim \tilde{X} \leq \dim X$ and each g_i is extendable to a continuous mapping $\tilde{g}_i: \tilde{X} \rightarrow \tilde{X}$.

Hint. One can assume that $g_1 = \text{id}_X$ and that for every pair i, j there exists a k such that $g_j g_i = g_k$. As in the proof of Theorem 1.7.2, for m

$= 1, 2, \dots$ define a finite open cover $\mathcal{U}_m$ which moreover is a refinement of $g_1^{-1}(\mathcal{U}_{m-1}) \wedge g_2^{-1}(\mathcal{U}_{m-1}) \wedge \dots \wedge g_{m-1}^{-1}(\mathcal{U}_{m-1})$, where $\mathcal{U}_0 = \{X\}$. Consider the metric $\tilde{q}$ on the space X defined by letting

$$\tilde{q}(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho(g_i(x), g_i(y)) + \sum_{n(i,j,k)=1}^{\infty} \frac{1}{2^{n(i,j,k)}} |f_{i,j}g_k(x) - f_{i,j}g_k(y)|,$$

where $n(i, j, k)$ is the place of the triple (i, j, k) in an infinite sequence consisting of all triples of natural numbers.

1.7.D. (a) (Kuratowski [1932]) Prove that for every compact metric space X with no isolated point such that $0 \leq \dim X \leq n$ there exists a continuous mapping $f: C \rightarrow X$ of the Cantor set C onto the space X with fibres of cardinality at most $n+1$. Deduce the topological characterization of the Cantor set stated in Problem 1.3.F.

Hint. Observe that the subspace $\Phi = \{f \in X^C: f(C) = X\}$ of the function space X^C is non-empty and closed in X^C , and hence is completely metrizable (see Problem 1.3.D and the beginning of Section 1.11). For $k = 1, 2, \dots$ consider the subset Ψ_k of Φ consisting of all functions $f \in X^C$ which have the property that for some $n+2$ points $x_1, x_2, \dots, x_{n+2}$ of the Cantor set C such that $|x_i - x_j| > 1/k$ whenever $i \neq j$ we have the equality $f(x_1) = f(x_2) = \dots = f(x_{n+2})$. Prove that the sets Ψ_k are closed and nowhere dense in Φ ; then apply the Baire category theorem. When proving that Ψ_k is nowhere dense, observe that every finite open cover of the space X consisting of non-empty sets has a closed shrinking of order $\leq n$ consisting of non-empty sets; then apply Problem 1.3.D.

(b) (Kuratowski [1932]) Prove that for every compact metric space X such that $\dim X \leq n \geq 0$ there exists a continuous mapping $f: A \rightarrow X$ of a closed subspace A of the Cantor set C onto the space X with fibres of cardinality at most $n+1$.

Hint. Applying the definition of $\dim$, prove that $\dim(X \times C) \leq n$.

(c) (Hurewicz [1926]) Prove that if for a compact metric space X there exists a continuous mapping $f: A \rightarrow X$ of a closed subspace A of the Cantor set C onto the space X with fibres of cardinality at most $n+1$, then $\text{ind } X \leq n$ (cf. Theorem 1.12.2).

Hint. Apply induction with respect to n . Assuming that (c) holds for every $n < m$, consider a continuous mapping $f: A \rightarrow X$ of a closed subspace A of the Cantor set C onto the space X with fibres of cardinality at most $m+1$, a point $x \in X$, and a neighbourhood $V \subset X$ of the point x ; then take an open-and-closed set $W \subset A$ such that $f^{-1}(y) \subset W \subset f^{-1}(V)$ and

show that the open set $U = X \setminus f(A \setminus W) \subset X$ satisfies the conditions $y \in U \subset V$ and $\text{Fr } U \subset f(W) \cap f(A \setminus W)$.

(d) (Kuratowski [1932]) Observe that (b) and (c) imply that for every compact metric space X we have $\text{ind } X \leq \dim X$.

Remark. Hurewicz proved in [1926] that a separable metric space X satisfies the inequality $\text{ind } X \leq n \geq 0$ if and only if there exists a closed mapping (see the beginning of Section 1.12) $f: Z \rightarrow X$ of a zero-dimensional separable metric space Z onto the space X with fibres of cardinality at most $n + 1$. This fact can be deduced from the compactification theorem, the coincidence theorem, part (b) of the present problem, and the hint to part (c); obviously, Hurewicz's original proof was a direct one.

Let us also note that, as proved by Nagata in [1960] (announcement by Hurewicz in [1928]), a separable metric space X is a continuous image of a zero-dimensional separable metric space under a closed mapping with finite fibres if and only if X can be represented as the union of countably many finite-dimensional subspaces.

1.7.E. Show that a separable metric space X satisfies the inequality $\dim X \leq n$ if and only if every open cover of the space X has an open refinement of order $\leq n$ or—equivalently—if every open cover of the space X has an open shrinking of order $\leq n$ (cf. Proposition 3.2.2).

Hint. When proving that every open cover of a separable metric space X satisfying $\dim X \leq n$ has an open refinement of order $\leq n$, first reduce the problem to countable covers, then consider the special case where $n = 0$, and finally apply the second decomposition theorem and Lemma 1.2.8.

1.8. Dimensional properties of Euclidean spaces and the Hilbert cube. Infinite-dimensional spaces

The main result in this section is the fundamental theorem of dimension theory, which states that Euclidean n -space R^n has dimension n , i.e., that $\text{ind } R^n = \text{Ind } R^n = \dim R^n = n$ for $n = 1, 2, \dots$. This theorem justifies the definitions of our three dimension functions, because any dimension function assigning to R^n a number distinct from n would contradict the intuitive notion of dimension and thus would not be acceptable. It readily follows from the evaluation of dimensions of Euclidean n -space R^n that the dimensions ind , Ind and $\dim$ of the n -cube I^n and the n -sphere

S^n are also equal to n . The spaces R^n , I^n and S^n are our first examples of spaces of dimension larger than one; so far we have not shown that such spaces exist.

The proof that Euclidean n -space R^n has dimension n requires a deeper insight into the structure of this space; by the nature of things, some combinatorial or algebraic arguments must appear in it. In this book, instead of producing a complete proof of the fundamental theorem of dimension theory, we shall deduce this result from the Brouwer fixed-point theorem, which states that for every continuous mapping $g: I^n \rightarrow I^n$ there exists a point $x \in I^n$ such that $g(x) = x$. The latter theorem is certainly well known to the reader; let us note that it is closely related to the "geometric versions" of the fundamental theorem of dimension theory, i.e., to Theorems 1.8.1 and 1.8.15 (see Problem 1.8.B).

We begin with a theorem reflecting an interesting geometric property of the n -cube I^n .

1.8.1. Theorem. *Let A_i and B_i , where $i = 1, 2, \dots, n$, be the subsets of the n -cube I^n defined by the conditions*

$$A_i = \{\{x_j\} \in I^n: x_i = 0\} \quad \text{and} \quad B_i = \{\{x_j\} \in I^n: x_i = 1\},$$

i.e., the pairs of opposite faces of I^n . If L_i is a partition between A_i and B_i for $i = 1, 2, \dots, n$, then $\bigcap_{i=1}^n L_i \neq \emptyset$.

Proof. Let us consider open sets $U_i, W_i \subset I^n$ such that $A_i \subset U_i, B_i \subset W_i$, $U_i \cap W_i = \emptyset$ and $I^n \setminus L_i = U_i \cup W_i$ for $i = 1, 2, \dots, n$. Since $(I^n \setminus W_i) \cap (I^n \setminus U_i) = I^n \setminus (U_i \cup W_i) = L_i$, the formulas

$$(1) \quad f_i(x) = \begin{cases} \frac{1}{2} \frac{\varrho(x, L_i)}{\varrho(x, L_i) + \varrho(x, A_i)} + \frac{1}{2} & \text{for } x \in I^n \setminus W_i, \\ -\frac{1}{2} \frac{\varrho(x, L_i)}{\varrho(x, L_i) + \varrho(x, B_i)} + \frac{1}{2} & \text{for } x \in I^n \setminus U_i \end{cases}$$

define for $i = 1, 2, \dots, n$ a continuous function $f_i: I^n \rightarrow I$. Clearly, we have

$$(2) \quad f_i^{-1}(1/2) = L_i, \quad f_i(A_i) = \{1\} \quad \text{and} \quad f_i(B_i) = \{0\}.$$

Assume that $\bigcap_{i=1}^n L_i = \emptyset$; it follows from the first part of (2) that the continuous mapping $f: I^n \rightarrow I^n$ defined by letting $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$ for $x \in I^n$ does not assume the value $a = (1/2, 1/2, \dots$

$\dots, 1/2) \in I^n$. The composition $g: I^n \rightarrow I^n$ of the mapping f and the projection p of $I^n \setminus \{a\}$ from the point a onto the boundary of I^n , i.e., onto the set $B = \bigcup_{i=1}^n (A_i \cup B_i)$, satisfies the inclusion $g(I^n) \subset B$; by the second and the third part of (2), we have $g(A_i) \subset B_i$ and $g(B_i) \subset A_i$. The last three inclusions show that $g(x) \neq x$ for every $x \in I^n$, which contradicts the Brouwer fixed-point theorem. Hence $\bigcap_{i=1}^n L_i \neq \emptyset$. $\square$

1.8.2. The fundamental theorem of dimension theory. *For every natural number n we have*

$$\text{ind } R^n = \text{Ind } R^n = \dim R^n = n.$$

Proof. By virtue of the inequality $\text{ind } R^n \leq n$ established in Example 1.5.9 and by the coincidence theorem, it suffices to show that $\text{ind } R^n \geq n$; the latter inequality follows immediately from Theorem 1.8.1 and the theorem on partitions. $\square$

1.8.3. Corollary. *For every natural number n we have*

$$\text{ind } I^n = \text{Ind } I^n = \dim I^n = n = \dim S^n = \text{Ind } S^n = \text{ind } S^n. \quad \square$$

1.8.4. Corollary. *For the Hilbert cube $I^{\aleph_0}$ we have*

$$\text{ind } I^{\aleph_0} = \text{Ind } I^{\aleph_0} = \dim I^{\aleph_0} = \infty. \quad \square$$

1.8.5. Theorem. *For the subspace N_k^n of Euclidean n -space R^n consisting of all points which have at most k rational coordinates and the subspace L_k^n of R^n consisting of all points which have at least k rational coordinates we have*

$$\text{ind } N_k^n = k \quad \text{and} \quad \text{ind } L_k^n = n - k.$$

Proof. In Example 1.5.9 we have shown that $\text{ind } N_k^n \leq k$ and $\text{ind } L_k^n \leq n - k$. Since $R^n = N_k^n \cup L_{k+1}^n = N_{k-1}^n \cup L_k^n$, the reverse inclusions follow from the equality $\text{ind } R^n = n$ and the addition theorem. $\square$

Looking closer at the above proof of the fundamental theorem of dimension theory we can see that first the inequalities $\text{ind } R^n \leq n$ and $\text{Ind } R^n \geq n$ were established, then—by applying Theorem 1.6.4—it was deduced that $\text{ind } R^n = \text{Ind } R^n = n$, and finally Lemmas 1.7.4 and 1.7.6 were applied to obtain the equality $\dim R^n = n$. The equality $\dim I^n = n$

can be also obtained in a direct way. Theorem 1.7.8 and Theorem 1.8.15 below imply that $\dim I^n \geq n$, and the reverse inequality is obtained by defining a finite open cover of I^n which has order $\leq n$ and arbitrarily small mesh. This can be done by dividing I^n into small “bricks” and then enlarging the bricks to open sets in such a way that the order of the family does not increase; Fig. 7 illustrates the procedure in the case of $n = 2$.

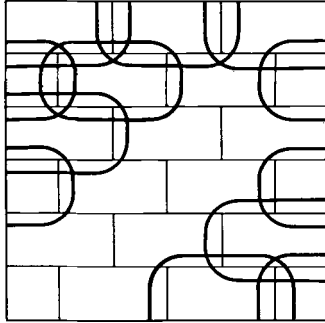


Fig. 7

Our next task is to characterize n -dimensional subspaces of Euclidean n -space R^n ; we shall show that they are subsets of R^n which have a non-empty interior. We start with an auxiliary geometric result, interesting in itself, which states that if a subset C of R^n has an empty interior, then C is homeomorphic to a subset D of R^n contained together with its closure in N_{n-1}^n ; in particular, each such subset is homeomorphic to a nowhere dense subset of R^n .

1.8.6. Lemma. *If a subset C of R^n has an empty interior and is dense in R^n , then every subset D of R^n which is homeomorphic to C has an empty interior.*

Proof. Let $f: D \rightarrow C$ be a homeomorphism of D onto C . Suppose that $\text{Int } D \neq \emptyset$ and consider a non-empty open subset V of R^n such that the closure $\bar{V}$ of V in R^n is compact and contained in D . The image $f(V)$ is open in C and its closure in C equals $f(\bar{V})$; the last set being compact, the closure $\overline{f(V)}$ of $f(V)$ in R^n also equals $f(\bar{V})$. Now, let U be an open subset of R^n such that $f(V) = C \cap U$. As the set C is dense in R^n , we have $\emptyset \neq U \subset \bar{U} = \bar{C} \cap \bar{U} = \overline{f(V)} = f(\bar{V}) = f(V) \subset C$, which contradicts the assumption that $\text{Int } C = \emptyset$. $\square$

1.8.7. Lemma. *Let A be a subset of R^n and a an arbitrary point of $R^n \setminus A$. For every positive number r there exists a homeomorphism $f: A \rightarrow f(A) \subset R^n$ such that for all $x, y \in A$ we have*

$$\varrho(x, f(x)) \leq r, \quad \varrho(x, y) \leq \varrho(f(x), f(y)) \quad \text{and} \quad B(a, r) \cap f(A) = \emptyset.$$

Proof. It suffices to assign to every $x \in A$ the point $f(x)$ situated on the ray starting from a and passing through x , which satisfies the condition $\varrho(a, f(x)) = \varrho(a, x) + r$. $\square$

1.8.8. Theorem. *For every subset C of R^n which has an empty interior there exists a subset D of R^n homeomorphic to C and such that $\overline{D} \subset N_{n-1}^n$.*

Proof. The set C can easily be enlarged to a dense subset of R^n which has an empty interior, and it suffices to prove the theorem for this larger set. Hence, without loss of generality, one can assume that the set C is dense in R^n . Let us arrange all points of the complement $R^n \setminus N_{n-1}^n$, i.e., all points of R^n with rational coordinates, into a sequence $a_1, a_2, \dots$. We shall define inductively a sequence $C_0, C_1, C_2, \dots$ of subsets of R^n , where $C_0 = C$, and a sequence of homeomorphisms $f_0, f_1, f_2, \dots$, where $f_i: C_i \rightarrow C_{i+1}$, satisfying the following conditions:

$$(3) \quad \varrho(x, f_i(x)) \leq 1/3^{i+1} \quad \text{for every } x \in C_i.$$

$$(4) \quad \varrho(x, y) \leq \varrho(f_i(x), f_i(y)) \quad \text{for all } x, y \in C_i.$$

$$(5) \quad B(a_i, 1/4 \cdot 3^i) \cap C_{i+1} = \emptyset \quad \text{for } i \geq 1.$$

Conditions (3)–(5) are satisfied for $i = 0$ if we let $C_1 = C$ and $f_0 = \text{id}_C$. Assume that the sets $C_0, C_1, C_2, \dots, C_k$ and the homeomorphisms $f_0, f_1, f_2, \dots, f_{k-1}$ are defined and satisfy (3)–(5) for $i < k \geq 1$. The set $C_k = f_{k-1}(C_{k-1})$ is homeomorphic to $C_0 = C$, so that $\text{Int } C_k = \emptyset$ by virtue of Lemma 1.8.6; thus there exists a point $a \in B(a_k, 1/4 \cdot 3^{k+1}) \setminus C_k$. One readily checks that Lemma 1.8.7 applied to the set $A = C_k$, the point a , and the number $r = 1/3^{k+1}$ yields a homeomorphism $f_k = f: C_k \rightarrow f(C_k)$ which together with the set $C_{k+1} = f(C_k)$ satisfies conditions (3)–(5) for $i = k$. Thus our construction is completed.

For $i = 1, 2, \dots$ consider the composition $h_i = g_i f_{i-1} \dots f_1 f_0: C \rightarrow R^n$, where g_i denotes the embedding of C_i in R^n . By virtue of (3), we have for $k \geq 1$

$$\begin{aligned} (6) \quad & \varrho(h_i(x), h_{i+k}(x)) \\ & \leq \varrho(h_i(x), h_{i+1}(x)) + \varrho(h_{i+1}(x), h_{i+2}(x)) + \dots + \varrho(h_{i+k-1}(x), h_{i+k}(x)) \\ & \leq 1/3^{i+1} + 1/3^{i+2} + \dots + 1/3^{i+k} < 1/2 \cdot 3^i \quad \text{for every } x \in C; \end{aligned}$$

hence, for every $x \in C$ the sequence $h_1(x), h_2(x), \dots$ is a Cauchy sequence and thus converges to a point $h(x) \in R^n$. In this way a mapping h of C to R^n is defined. Passing in (6) to the limit with respect to k , we see that the sequence of mappings $h_1, h_2, \dots$ uniformly converges to the mapping h , so that h is a continuous mapping. Condition (4) implies that $\varrho(x, y) \leq \varrho(h_i(x), h_i(y))$ for all $x, y \in C$ and $i = 1, 2, \dots$. Hence, $\varrho(x, y) \leq \varrho(h(x), h(y))$ for all $x, y \in C$, which implies that h is a one-to-one mapping and that the inverse mapping h^{-1} is continuous, i.e., that h is a homeomorphism of the set C onto the set $D = h(C) \subset R^n$.

Now, consider a point $a_i \in R^n \setminus N_{n-1}^n$ and an arbitrary point $z \in D$; let $h^{-1}(z) = x \in C$. It follows from (5) that

$$(7) \quad \varrho(a_i, h_{i+1}(x)) \geq 1/4 \cdot 3^i.$$

By virtue of (6) we have $\varrho(h_{i+1}(x), h_{i+k}(x)) < 1/2 \cdot 3^{i+1}$ for $k = 1, 2, \dots$, so that

$$(8) \quad \varrho(h_{i+1}(x), z) \leq 1/2 \cdot 3^{i+1}.$$

Relations (7) and (8) yield the inequality $\varrho(a_i, z) \geq 1/4 \cdot 3^{i+1}$; hence $B(a_i, 1/4 \cdot 3^{i+1}) \cap D = \emptyset$, which implies that $a_i \notin \bar{D}$. Thus the inclusion $\bar{D} \subset N_{n-1}^n$ is established. $\square$

Theorem 1.8.8 yields

1.8.9. Theorem. *Every subset of R^n which has an empty interior is homeomorphic to a nowhere dense subset of R^n .* $\square$

1.8.10. Theorem. *A subspace M of Euclidean n -space R^n satisfies the condition $\text{ind } M = n$ if and only if the interior of M in R^n is non-empty.*

Proof. If $\text{Int } M \neq \emptyset$, then M contains a subspace homeomorphic to I^n , so that $\text{ind } M = n$. On the other hand, if $\text{Int } M = \emptyset$, then—as follows from Theorem 1.8.8—the space M is homeomorphic to a subspace of N_{n-1}^n , so that $\text{ind } M \leq n-1$ by virtue of Theorem 1.8.5. $\square$

1.8.11. Corollary. *A subspace M of the n -cube I^n , or the n -sphere S^n , satisfies the condition $\text{ind } M = n$ if and only if the interior of M is non-empty.* $\square$

1.8.12. Theorem. *Let X be Euclidean n -space R^n , the n -cube I^n , or the n -sphere S^n . If a set $F \subset X$ is the boundary of a non-empty open subset of X which is not dense in X , then $\text{ind } F = n-1$.*

Proof. Consider first a subspace F of S^n , and let $F = \text{Fr } U$, where U is an open subset of S^n such that $U \neq \emptyset \neq S^n \setminus \overline{U}$. From Corollary 1.8.11 it follows that $\text{ind } F \leq n-1$; suppose that $\text{ind } F \leq n-2$. Let x be a point of U . Since $\overline{U} \neq S^n$, for every positive number ε one can readily define a homeomorphism f of the n -sphere S^n onto itself such that $f(x) = x$ and $f(\overline{U}) \subset B(x, \varepsilon)$. Hence, the point x has arbitrarily small neighbourhoods with $(n-2)$ -dimensional boundaries. This contradiction of Theorem 1.8.2 shows that $\text{ind } F = n-1$.

Since R^n is homeomorphic to the open subspace $S^n \setminus \{a\}$ of S^n and for every open set $U \subset R^n$ the boundary of U in R^n and the boundary of its counterpart in S^n differ topologically by at most one point, the validity of our theorem for subspaces of R^n follows from its validity for subspaces of S^n and Corollary 1.5.6.

The case of $X = I^n$ is left to the reader. $\square$

Let us recall that a set $T \subset X$ *separates* a topological space X if T separates X between a pair of points x and y , i.e., if the complement $X \setminus T$ is disconnected.

1.8.13. Theorem. *Let X be Euclidean n -space R^n , the n -cube I^n , or the n -sphere S^n . If a closed subset L of X satisfies the inequality $\text{ind } L \leq n-2$, then L does not separate the space X , i.e., the complement $X \setminus L$ is connected.*

Proof. Suppose that $X \setminus L = U \cup V$, where U, V are non-empty disjoint open sets. Clearly $\text{Fr } U \subset L$, so that $\text{ind } \text{Fr } U \leq n-2$, which contradicts Theorem 1.8.12. $\square$

The last theorem leads to the notion of a Cantor-manifold, which will be discussed in the following section.

We shall now pass to a study of deeper dimensional properties of Euclidean spaces. We aim at Mazurkiewicz's theorem, which is much stronger than Theorem 1.8.13 and states that if a subset M of a region $G \subset R^n$ (i.e., of a connected open subspace G of R^n) satisfies the inequality $\text{ind } M \leq n-2$, then M does not cut G (cf. Problems 1.8.E and 1.8.F). Let us recall that a set $T \subset X$ *cuts* a topological space X if T cuts X between a pair of points x and y , i.e., if each continuum $C \subset X$ which contains x and y meets the set T . We start with Lebesgue's covering theorem, reflecting an interesting geometric property of the n -cube I^n , which is closely related to the equality $\dim I^n = n$. As stated in the notes to Section 1.1, this theorem played an important role in the formation of dimension theory.

1.8.14. Lemma. Let A_i, B_i , where $i = 1, 2, \dots, n$, be the pairs of opposite faces of I^n . If $I^n = L'_0 \supset L'_1 \supset \dots \supset L'_n$ is a descending sequence of closed sets such that L'_i is a partition in L'_{i-1} between $L'_{i-1} \cap A_i$ and $L'_{i-1} \cap B_i$ for $i = 1, 2, \dots, n$, then $L'_n \neq \emptyset$.

Proof. By virtue of the second part of Lemma 1.2.9, for $i = 2, 3, \dots, n$ there exists a partition L_i in I^n between A_i and B_i such that

$$(9) \quad L'_{i-1} \cap L_i \subset L'_i \text{ for } i = 2, 3, \dots, n; \text{ let, additionally, } L_1 = L'_1.$$

From (9) we obtain, one by one, the inclusions $L_1 \subset L'_1, L_1 \cap L_2 \subset L'_2, \dots, L_1 \cap L_2 \cap \dots \cap L_n \subset L'_n$, so that $L'_n \neq \emptyset$ by Theorem 1.8.1. $\square$

1.8.15. Lebesgue's covering theorem. If $\mathcal{F}$ is a finite closed cover of the n -cube I^n no member of which meets two opposite faces of I^n , then $\text{ord } \mathcal{F} \geq n$.

Proof. Let A_i, B_i , where $i = 1, 2, \dots, n$, be the pairs of opposite faces of I^n . For $i = 1, 2, \dots, n$ define

$$\mathcal{F}_i = \{F \in \mathcal{F} : F \cap A_i \neq \emptyset\}$$

and consider the families

$$\mathcal{K}_1 = \mathcal{F}_1, \quad \mathcal{K}_2 = \mathcal{F}_2 \setminus \mathcal{K}_1, \quad \dots, \quad \mathcal{K}_n = \mathcal{F}_n \setminus (\mathcal{K}_1 \cup \dots \cup \mathcal{K}_{n-1})$$

and

$$\mathcal{K}_{n+1} = \mathcal{F} \setminus (\mathcal{K}_1 \cup \mathcal{K}_2 \cup \dots \cup \mathcal{K}_n).$$

It follows from the assumptions of the theorem that the closed sets $K_i = \bigcup \mathcal{K}_i$ satisfy the inclusions

$$(10) \quad A_i \subset I^n \setminus (K_{i+1} \cup \dots \cup K_{n+1}) \quad \text{and} \quad B_i \subset I^n \setminus K_i \\ \text{for } i = 1, 2, \dots, n.$$

The sets $L'_1, L'_2, \dots, L'_n$, where

$$L'_i = K_1 \cap K_2 \cap \dots \cap K_i \cap (K_{i+1} \cup K_{i+2} \cup \dots \cup K_{n+1}),$$

are closed and form a descending sequence. Observe that $I^n \setminus L'_i = U_i \cup W_i$, where

$$U_i = I^n \setminus (K_{i+1} \cup K_{i+2} \cup \dots \cup K_{n+1})$$

and

$$W_i = I^n \setminus (K_1 \cap K_2 \cap \dots \cap K_i).$$

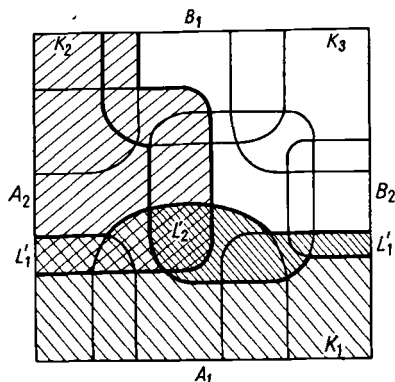


Fig. 8

Inclusions (10) yield

$$L'_{i-1} \cap A_i \subset L'_{i-1} \cap U_i \quad \text{and} \quad L'_{i-1} \cap B_i \subset L'_{i-1} \cap W_i$$

for $i = 1, 2, \dots, n$, where $L'_0 = I^n$. One readily checks that $(L'_{i-1} \cap U_i) \cap (L'_{i-1} \cap W_i) = \emptyset$; moreover,

$$L'_{i-1} \setminus [(L'_{i-1} \cap U_i) \cup (L'_{i-1} \cap W_i)] = L'_{i-1} \setminus (U_i \cup W_i) = L'_{i-1} \cap L'_i = L'_i,$$

so that L'_i is a partition in L'_{i-1} between $L'_{i-1} \cap A_i$ and $L'_{i-1} \cap B_i$ for $i = 1, 2, \dots, n$. By virtue of Lemma 1.8.14, we have $L'_n = K_1 \cap K_2 \cap \dots \cap K_{n+1} \neq \emptyset$. Since every member of the cover $\mathcal{F}$ belongs to exactly one family $\mathcal{K}_i$, the last inequality implies that $\text{ord } \mathcal{F} \geq n$. $\square$

If L is a partition in I^n between A_1 and B_1 , then letting $L'_1 = L$ and defining, as in the above proof, the families $\mathcal{F}_i, \mathcal{K}_i$ and the sets K_i, L'_i for $i = 2, 3, \dots, n$, we obtain the following generalization of Lebesgue's covering theorem:

1.8.16. Theorem. *If L is a partition between a pair of opposite faces of the n -cube I^n and $\mathcal{F}$ is a finite closed cover of L no member of which meets two opposite faces of I^n , then $\text{ord } \mathcal{F} \geq n-1$. $\square$*

We are ready now to prove Mazurkiewicz's theorem. The theorem will be preceded by two lemmas; the second lemma is a special case of the theorem, to which the general situation will be reduced by a relatively simple argument.

1.8.17. Lemma. *Let M be a subspace of a totally bounded metric space X . If there exists a positive number ε with the property that every finite family $\mathcal{V}$ of open subsets of X such that $\text{mesh } \mathcal{V} < \varepsilon$ and $M \subset \bigcup \mathcal{V}$ satisfies the inequality $\text{ord } \mathcal{V} \geq n \geq 0$, then $\text{ind } M \geq n$.*

Proof. Suppose that $\text{ind } M \leq n-1$ and consider a finite cover $\{U_i\}_{i=1}^k$ of the space X by $\varepsilon/3$ -balls. By virtue of Remark 1.7.5, there exists a finite family $\mathcal{V}$ of open subsets of X such that $\text{mesh } \mathcal{V} \leq 2\varepsilon/3 < \varepsilon$, $M \subset \bigcup \mathcal{V}$ and $\text{ord } \mathcal{V} \leq n-1$, which is impossible. Hence $\text{ind } M \geq n$. $\square$

1.8.18. Lemma. *Let A, B be a pair of opposite faces of the n -cube I^n . If M is a subspace of $X = I^n \setminus (A \cup B)$ which meets every continuum $C \subset I^n$ such that $A \cup B \subset C$, then $\text{ind } M \geq n-1$.*

Proof. Consider a finite family $\mathcal{V}$ of open subsets of X such that $\text{mesh } \mathcal{V} < 1$ and $M \subset V = \bigcup \mathcal{V}$; obviously, $A \cup B \subset I^n \setminus V$. Let S_A and S_B be the components of the compact space $I^n \setminus V$ which contain A and B , respectively. Since S_A and S_B are both continua and $A \cup B \subset S_A \cup S_B \subset I^n \setminus M$, it follows from the assumption about M that $S_A \cap S_B = \emptyset$. By virtue of Lemma 1.4.4 there exist disjoint open-and-closed subsets U_1, W_1 of $I^n \setminus V$ such that

$$(11) \quad A \subset S_A \subset U_1, \quad B \subset S_B \subset W_1 \quad \text{and} \quad U_1 \cup W_1 = I^n \setminus V.$$

The sets U_1, W_1 being closed in I^n , there exists open sets $U, W \subset I^n$ such that

$$U_1 \subset U, \quad W_1 \subset W \quad \text{and} \quad U \cap W = \emptyset.$$

By virtue of (11), the set $L = I^n \setminus (U \cup W)$ is contained in V and is a partition in I^n between A and B . From Theorem 1.7.8 it follows that the cover $\mathcal{V}|L$ of the space L has a closed shrinking $\mathcal{F}$. As $\text{mesh } \mathcal{V} < 1$, no member of $\mathcal{F}$ meets two opposite faces of I^n , and Theorem 1.8.16 implies that $\text{ord } \mathcal{F} \geq n-1$. Since $\text{ord } \mathcal{V} \geq \text{ord } \mathcal{F}$, it follows from Lemma 1.8.17 that $\text{ind } M \geq n-1$. $\square$

1.8.19. Mazurkiewicz's theorem. *If a subset M of a region $G \subset R^n$ satisfies the inequality $\text{ind } M \leq n-2$, then M does not cut G , i.e., for every pair of points $x, y \in G \setminus M$ there exists a continuum $C \subset G \setminus M$ which contains x and y .*

Proof. Let us start with the special case where $G = R^n$. Consider a pair of points $x, y \in R^n \setminus M$ and denote by K the closed ball in R^n whose centre coincides with the centre of the segment with end-points x and y and whose radius is equal to $\frac{1}{2}\varrho(x, y)$. Let $f: I^n \rightarrow K$ be a continuous mapping of I^n onto K which maps a pair A, B of opposite faces of I^n to x and y , respectively, and has the property that the restriction $g = f|_{[I^n \setminus (A \cup B)]}$ is a homeomorphism of $I^n \setminus (A \cup B)$ onto $K \setminus \{x, y\}$; such a mapping can easily be obtained by an application of Problem 1.8.A and Lemma 1.8.7. The set $M' = f^{-1}(K \cap M) \subset I^n \setminus (A \cup B)$ satisfies the inequality $\text{ind } M' \leq n-2$. Hence by Lemma 1.8.18 there exists a continuum $C' \subset I^n$ such that $A \cup B \subset C'$ and $C' \cap M' = \emptyset$. The set $C = f(C') \subset R^n \setminus M$ is a continuum which contains x and y , so that Mazurkiewicz's theorem is proved for $G = R^n$.

Now, consider an arbitrary region $G \subset R^n$ and a pair of points $x, y \in G \setminus M$. Let $B_1, B_2, \dots, B_k$ be a sequence of open balls in R^n such that $x \in B_1, y \in B_k, B_i \subset G$ for $i = 1, 2, \dots, k$ and $B_i \cap B_{i+1} \neq \emptyset$ for $i \leq k-1$. The existence of such a sequence follows from the connectedness of G , because the set of all points in G that can be joined to the point x in a similar way is open-and-closed. Since the set M has an empty interior, for $i = 1, 2, \dots, k-1$ there exists a point $z_i \in B_i \cap B_{i+1} \setminus M$; let, additionally, $z_0 = x$ and $z_k = y$. By the special case of Mazurkiewicz's theorem established above, for $i = 1, 2, \dots, k$ there exists a continuum $C_i \subset B_i \setminus M$ which contains z_{i-1} and z_i . The union $C = C_1 \cup C_2 \cup \dots \cup C_k \subset G \setminus M$ is a continuum which contains x and y . $\square$

We conclude this section with a theorem and an example announced in Section 1.5 (cf. Problem 1.8.G).

1.8.20. Theorem. *The Hilbert cube $I^{\aleph_0}$ cannot be represented as a countable union of finite-dimensional subspaces.*

Proof. By virtue of the second decomposition theorem it is enough to show that for every sequence $Z_1, Z_2, \dots$ of zero-dimensional subspaces

of $I^{\aleph_0}$ we have $\bigcup_{i=1}^{\infty} Z_i \neq I^{\aleph_0}$. Let

$$A_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 0\} \quad \text{and} \quad B_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 1\}$$

for $i = 1, 2, \dots$. Applying Theorem 1.2.11, we can find a partition L_i between A_i and B_i such that

$$(12) \quad L_i \cap Z_i = \emptyset \quad \text{for } i = 1, 2, \dots$$

Define $I_n = \{\{x_j\} \in I^{\aleph_0} : x_j = 0 \text{ for } j > n\}$; the intersection $L_i \cap I_n$ is a partition in I_n between $A_i \cap I_n$ and $B_i \cap I_n$ for $i = 1, 2, \dots, n$. From Theorem 1.8.1 it follows that $\bigcap_{i=1}^n L_i \supset \bigcap_{i=1}^n L_i \cap I_n \neq \emptyset$. Hence, the family $\{L_i\}_{i=1}^\infty$ of closed subsets of $I^{\aleph_0}$ has the finite intersection property. The space $I^{\aleph_0}$ being compact, $\bigcap_{i=1}^\infty L_i \neq \emptyset$. It follows from (12) that $\bigcap_{i=1}^\infty L_i \subset I^{\aleph_0} \setminus \bigcup_{i=1}^\infty Z_i$, so that $\bigcup_{i=1}^\infty Z_i \neq I^{\aleph_0}$. $\square$

1.8.21. Example. We shall now describe, applying the *continuum hypothesis*, i.e., the equality $\aleph_1 = \mathfrak{c}$, a space $X \subset I^{\aleph_0}$ of dimension ∞ whose finite-dimensional subspaces are all countable.

One readily checks that the family of all G_δ -sets in $I^{\aleph_0}$ has cardinality $\mathfrak{c}$; let us arrange—applying the continuum hypothesis—all zero-dimensional members of this family into a transfinite sequence

$$Z_1, Z_2, \dots, Z_\alpha, \dots, \alpha < \omega_1,$$

where ω_1 denotes the first uncountable ordinal number, i.e., the initial number of cardinality $\aleph_1$. As all one-point subsets of $I^{\aleph_0}$ are among the sets Z_α , we have

$$(13) \quad \bigcup_{\alpha < \omega_1} Z_\alpha = I^{\aleph_0}.$$

From Theorem 1.8.20 it follows that $\bigcup_{\alpha < \gamma} Z_\alpha \neq I^{\aleph_0}$ for every $\gamma < \omega_1$, so that by virtue of (13) there exists an uncountable set Γ of countable ordinal numbers such that $Z_\gamma \setminus \bigcup_{\alpha < \gamma} Z_\alpha \neq \emptyset$ for $\gamma \in \Gamma$. Let us choose a point $x_\gamma \in Z_\gamma \setminus \bigcup_{\alpha < \gamma} Z_\alpha$ for every $\gamma \in \Gamma$ and consider the subspace $X = \bigcup_{\gamma \in \Gamma} \{x_\gamma\}$ of $I^{\aleph_0}$. By Theorem 1.2.14, for each zero-dimensional subspace $Z \subset X \subset I^{\aleph_0}$ there exists a zero-dimensional G_δ -set $Z^* \subset I^{\aleph_0}$ such that $Z \subset Z^*$, i.e., there exists an index $\alpha < \omega_1$ such that $Z \subset Z_\alpha$. Thus we have $Z \subset \bigcup_{\gamma \leq \alpha} \{x_\gamma\}$, which implies that the subspace Z is countable. It follows from the second decomposition theorem that each finite-dimensional subspace of X is also countable. As the space X itself is uncountable, $\text{ind } X = \infty$.

Conversely, the existence of a space X with the above properties implies the continuum hypothesis. Indeed, since every non-empty metric space of cardinality $< \mathfrak{c}$ is zero-dimensional (see Example 1.2.5), $|X| \geq \mathfrak{c}$ and

every subset of X which has cardinality $< \mathfrak{c}$ is countable; considering a set $A \subset X$ such that $|A| = \aleph_1$, we conclude that $\aleph_1 = \mathfrak{c}$. As the continuum hypothesis is independent of the standard axioms of set theory, the existence of a metric space X such that $\text{ind} X = \infty$ and all finite-dimensional subspaces of X are countable is also independent of the standard axioms of set theory. $\square$

In the light of Theorem 1.8.20 we see that among all infinite-dimensional separable metric spaces there are “weakly” infinite-dimensional spaces, for example the union $\bigcup_{n=1}^{\infty} X_n$, where X_n is homeomorphic to the n -cube I^n and $X_n \cap X_m = \emptyset$ whenever $n \neq m$, and “strongly” infinite-dimensional spaces, for example the Hilbert cube $I^{\aleph_0}$. The classification of infinite-dimensional spaces into spaces which can be represented as countable unions of finite-dimensional spaces (such spaces are called *countable-dimensional*) and spaces which cannot be represented in such a way, suggested by Theorem 1.8.20, is not the only possible classification into “weakly” and “strongly” infinite-dimensional spaces; several classifications of this kind are defined and studied in the literature. It is also possible to extend the inductive definition of ind and Ind from natural numbers to ordinal numbers; in this way one obtains the transfinite small inductive dimension trind and the transfinite large inductive dimension trInd , which satisfy the inequality $\text{trind} X \leq \text{trInd} X$. It turns out, however, that there exist separable metric spaces, even countable-dimensional spaces, with trind larger than any given ordinal number. E.g., the subspace X of the Hilbert cube $I^{\aleph_0}$ consisting of all points which have at most finitely many coordinates distinct from zero does not satisfy the inequality $\text{trind} X \leq \alpha$, and, *a fortiori*, the inequality $\text{trInd} X \leq \alpha$, for any ordinal number α ; so that for the space X neither trind nor trInd is defined. One can also prove that for the union $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is homeomorphic to the n -cube I^n and $X_n \cap X_m = \emptyset$ whenever $n \neq m$, trInd is not defined, although, as can easily be seen, $\text{trind} X = \omega_0$. Finally, there exist compact metric spaces for which both trind and trInd are defined but are distinct ordinal numbers.

The theory dealing with transfinite dimensions and different notions of “weak” infinite-dimensionality now forms quite an extensive part of dimension theory. It includes various characterizations of “weakly” infinite-dimensional spaces (see, e.g., the remark to Problem 1.7.D), variations on addition, sum, enlargement, compactification and universal space

theorems, and also many interesting examples. Let us note that in this domain several natural and interesting questions are still unanswered. In consideration of the elementary character of the present book we confine ourselves to calling the reader's attention to these topics; in the notes below we list the most important items of the bibliography of infinite-dimensional spaces.

Historical and bibliographic notes

A proof of the Brouwer fixed-point theorem can be found in the appendix to Section 7.3 of [GT]. Theorem 1.8.1 is implicit in Eilenberg and Otto's paper [1938]. As already mentioned in the notes to Section 1.1, the equality $\dim R^n = n$ (more exactly, Theorem 1.8.15) was discovered by Lebesgue in [1911] and proved by Brouwer in [1913]; the gap in Lebesgue's original outline of proof, given in [1911], was filled in his later paper [1921]. Brouwer's paper [1913] contains also a proof of the equality $\text{Ind } R^n = n$, which is reduced to the equality $\dim R^n = n$. The equality $\text{ind } R^n = n$ was established—also by a reduction to the equality $\dim R^n = n$ and an application of Lebesgue's result—by Menger in [1924] and by Urysohn in [1925] (announcement in [1922]). Theorem 1.8.9 was given by Sierpiński in [1922]; Theorem 1.8.8 is obtained by a small modification of Sierpiński's proof. Theorems 1.8.10, 1.8.12 and 1.8.13 were obtained by Menger in [1924] and by Urysohn in [1925] (announcement in [1922]). Urysohn in his proof of Theorem 1.8.10 applied an earlier result of Fréchet and Brouwer (see Problem 1.8.D), and Menger showed that if a subspace M of R^n has an empty interior, then for every point $x \in R^n$ and each positive number ε there exists a neighbourhood $U \subset R^n$ of the point x such that $\delta(U) < \varepsilon$, the boundary $\text{Fr } U$ is homeomorphic to S^{n-1} and the intersection $M \cap \text{Fr } U$ has an empty interior in the space $\text{Fr } U$. Theorem 1.8.16 was proved in Lebesgue's paper [1921]. Mazurkiewicz established Theorem 1.8.19 in [1929]. Theorem 1.8.20 and Example 1.8.21 were given by Hurewicz in [1928] and [1932], respectively.

Hurewicz was the first to hint, in [1928], at the possibility of a classification of infinite-dimensional spaces; he defined there countably-dimensional spaces. A comprehensive exposition of the theory of infinite-dimensional spaces can be found in Alexandroff and Pasynkov's book [1973]; some information is contained in Nagata's book [1965]. Further results were obtained by Arhangel'skii in [1963], Lelek in [1965], Nagami in [1965], Nagami and Roberts in [1965], Schurle in [1969], Shmueli in [1972], and Ljusksemburg in [1973] and [1973a].

Problems

1.8.A. Let us recall that a subset A of Euclidean n -space R^n is *convex* if for each pair x, y of points of A the segment with end-points x and y is contained in A .

Show that every convex compact subspace $A \subset R^n$ which has a non-empty interior is homeomorphic to the n -ball B^n and its boundary $\text{Fr}A$ is homeomorphic to the $(n-1)$ -sphere S^{n-1} . Note that, in particular, I^n and B^n , and also $\text{Fr}I^n$ and S^{n-1} , are homeomorphic to each other for $n = 1, 2, \dots$

Hint. Consider a point $x \in \text{Int}A$ and prove that every ray starting from x meets $\text{Fr}A$ at exactly one point.

1.8.B. Show that the Brouwer fixed-point theorem follows easily both from Theorem 1.8.1 and from Theorem 1.8.15.

Hint. The Brouwer fixed-point theorem is equivalent to the statement that S^{n-1} is not a retract of B^n .

1.8.C (Nöbeling [1932]). Prove that if the projection of a compact subspace X of R^m onto the Cartesian product of each n coordinate axes of R^m , where $0 < n < m$, has an empty interior, then the subspace X is embeddable in N_{n-1}^m . Deduce that if an F_σ -set $X \subset R^m$ satisfies the inequality $\text{ind}X \geq n > 0$, then there exists a set of n coordinate axes of R^m such that the projection of X onto the Cartesian product of these axes has dimension n (the assumption that X is an F_σ -set cannot be omitted; see Example 1.10.23).

Hint. Let $H_1, H_2, \dots$ be the sequence of all linear $(m-n)$ -varieties in R^m defined by conditions of the form $x_{i_1} = r_1, x_{i_2} = r_2, \dots, x_{i_n} = r_n$, where $1 \leq i_1 < i_2 < \dots < i_n \leq m$ and $r_1, r_2, \dots, r_n$ are arbitrary rational numbers. Observe that for $i = 1, 2, \dots$ the set $A_i \subset R^m$ of all points $a \in R^m$ such that $\{x+a: x \in X\} \cap H_i \neq \emptyset$ is closed and has an empty interior.

1.8.D (Brouwer [1913a]; implicitly, Fréchet [1910]). Prove that for any two countable dense subsets A, B of Euclidean n -space R^n there exists a homeomorphism $f: R^n \rightarrow R^n$ such that $f(A) = B$. Note that this result yields Theorem 1.8.10.

Hint. A set $A \subset R^n$ is in general position with respect to the coordinate axes if for every pair of distinct points $x = \{x_i\}, y = \{y_i\} \in A$ the difference $x_i - y_i$ does not vanish for $i = 1, 2, \dots, n$. Prove first that for every countable set $A \subset R^n$ there exists a homeomorphism of R^n onto itself

that maps A onto a set in general position with respect to the coordinate axes. Then show that the elements of two countably infinite sets in general position with respect to the coordinate axes can be arranged into two sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$, where $x_j = \{x_i^j\}$ and $y_j = \{y_i^j\}$ for $j = 1, 2, \dots$, in such a way that the differences $x_i^j - x_i^k$ and $y_i^j - y_i^k$ have the same sign for $j, k = 1, 2, \dots, j \neq k$, and $i = 1, 2, \dots, n$ (cf. the hint to Problem 1.3.G(a)).

Remark. It was proved by Fort in [1962] that the Hilbert cube also has the above strong homogeneity property; more general results were obtained by Bessaga and Pełczyński in [1970] and by Bennett in [1972].

1.8.E. (a) Deduce from Theorem 1.8.13 that if a subset M of a region $G \subset R^n$ satisfies the inequality $\text{ind } M \leq n-2$, then M does not separate G , i.e., the complement $G \setminus M$ is connected.

Hint. Observe first that every set T which separates a topological space X between a pair of points x and y contains a partition between x and y . Then reduce the problem to the special case where $G = R^n$ (cf. the proof of Theorem 1.8.19).

(b) Give an example of a one-dimensional subset of the plane R^2 which does not separate any region $G \subset R^2$.

Remark. Sitnikov gave in [1954] an example of a two-dimensional subset of R^3 which does not separate any region $G \subset R^3$. On the other hand, every $(n-1)$ -dimensional closed subset of R^n does separate a region $G \subset R^n$ (see Alexandroff [1930], Frankl and Pontrjagin [1930], and Frankl [1930]).

1.8.F. Give an example of a subset of the plane R^2 which cuts R^2 but does not separate it.

1.8.G (Smirnov [1958]). Show that every non-empty separable metric space can be represented as the union of an increasing transfinite sequence of type ω_1 consisting of zero-dimensional subspaces.

Hint. Show that the interval I can be represented in this way, and apply the universality of the Hilbert cube for the class of all separable metric spaces.

Remark. Related results can be found in Arhangel'skii's paper [1963] and Nagami's paper [1965].

1.9. Characterization of dimension in terms of mappings to spheres. Cantor-manifolds. Cohomological dimension

In the preceding sections we chiefly studied the internal properties of n -dimensional spaces. In the present one, and the next two to follow, we shall study the external properties; more exactly, we shall discuss the relations of n -dimensional spaces to standard spaces such as spheres, polyhedra and Euclidean spaces.

We begin with a characterization of dimension in terms of extending mappings to spheres from a closed subspace over the whole space.

Let us recall that a continuous mapping $f: A \rightarrow Y$ defined on a subspace A of a space X is *continuously extendable* over X if there exists a continuous mapping $F: X \rightarrow Y$ such that $F|A = f$, i.e. $F(x) = f(x)$ for every $x \in X$; the mapping F is called a *continuous extension* of f . One of the most important results on extending mappings is the *Tietze–Urysohn theorem*, which states that every continuous function from a closed subspace A of a normal space X to the closed interval I is continuously extendable over X . *Urysohn's lemma* is a special case of this theorem; it states that for every pair A, B of disjoint closed subsets of a normal space X there exists a continuous function $f: X \rightarrow I$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. When X is a metric space, the function f in Urysohn's lemma can be obtained in a particularly simple way, viz., by defining

$$f(x) = \frac{\varrho(x, A)}{\varrho(x, A) + \varrho(x, B)} \quad \text{for } x \in X.$$

The Tietze–Urysohn theorem implies that for every continuous mapping $f: A \rightarrow I^n$ of a closed subspace A of a normal space X to the n -cube I^n there exists a continuous extension $F: X \rightarrow I^n$ over X ; indeed, it suffices to choose for $i = 1, 2, \dots, n$ a continuous extension $F_i: X \rightarrow I$ of the composition $p_i f$, where $p_i: I^n \rightarrow I$ is the projection of I^n onto the i -th coordinate axis, and define $F(x) = (F_1(x), F_2(x), \dots, F_n(x))$.

On the other hand, for every continuous mapping $f: A \rightarrow S^n$ of a closed subspace A of a normal space X to the n -sphere S^n there exists an open set $U \subset X$ containing A and such that f is continuously extendable over U . Indeed, the two spaces being homeomorphic (see Problem 1.8.A), one can replace the sphere S^n by the boundary S^n_I of the $(n+1)$ -cube I^{n+1} in R^{n+1} and, applying the above observation, obtain a continuous mapping $F_0: X \rightarrow I^{n+1}$ such that $F_0(x) = f(x)$ for every $x \in A$; now, the mapping f is continuously extendable over the open set $U = F_0^{-1}(I^{n+1} \setminus \{a\}) \subset X$,

where $a = (1/2, 1/2, \dots, 1/2)$; the composition F of the restriction $F_0|U$ and the projection p of $I^{n+1} \setminus \{a\}$ from the point a to S_I^n is a continuous extension of f over U .

It is well known (cf. the hint to Problem 1.8.B) that in general continuous mappings to the n -sphere S^n are not continuously extendable from a closed subspace to the whole space. We shall now show that the extendability of such mappings depends exclusively on the dimension of the complement of the subspace under consideration.

1.9.1. Lemma. *Let X be a separable metric space and A a closed subspace of X such that $\text{ind}(X \setminus A) \leq n \geq 0$. For every pair A_1, A_2 of closed subsets of A such that $A_1 \cup A_2 = A$ there exist closed subspaces X_1, X_2 of the space X which satisfy the conditions*

- (1) $X = X_1 \cup X_2, \quad A_1 \subset X_1, \quad A_2 \subset X_2, \quad A_1 \cap X_2 = A_1 \cap A_2 = X_1 \cap A_2$
and
(2) $\text{ind}[(X_1 \cap X_2) \setminus (A_1 \cap A_2)] \leq n - 1.$

Proof. The sets $A_1 \setminus A_2 = A_1 \setminus (A_1 \cap A_2)$ and $A_2 \setminus A_1 = A_2 \setminus (A_1 \cap A_2)$ are disjoint and closed in the subspace $X \setminus (A_1 \cap A_2)$ of the space X . By virtue of the second separation theorem, there exists a partition L in $X \setminus (A_1 \cap A_2)$ between $A_1 \setminus A_2$ and $A_2 \setminus A_1$ such that $\text{ind}[L \cap (X \setminus A)] \leq n - 1$; since $L \cap A = [L \cap (A_1 \setminus A_2)] \cup [(L \cap (A_2 \setminus A_1))] \cup [L \cap (A_1 \cap A_2)] = \emptyset$, the last inequality can be rewritten as $\text{ind} L \leq n - 1$.

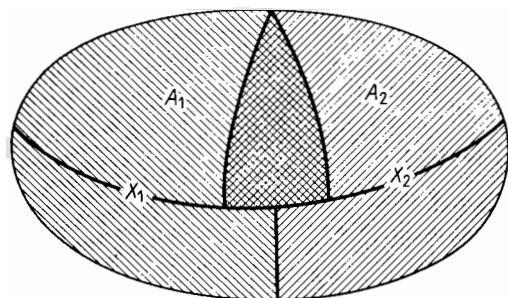


Fig. 9

Consider a pair of sets $U, V \subset X \setminus (A_1 \cap A_2)$ which are open in $X \setminus (A_1 \cap A_2)$ and satisfy the conditions
 $A_1 \setminus A_2 \subset U, \quad A_2 \setminus A_1 \subset V, \quad U \cap V = \emptyset$ and $[X \setminus (A_1 \cap A_2)] \setminus L = U \cup V.$

The sets U and V are open in X ; we shall verify that their complements $X_1 = X \setminus V$ and $X_2 = X \setminus U$ satisfy (1) and (2).

First of all,

$$X_1 \cup X_2 = (X \setminus V) \cup (X \setminus U) = X \setminus (U \cap V) = X.$$

Next,

$$A_1 = (A_1 \setminus A_2) \cup (A_1 \cap A_2) \subset U \cup [X \setminus (U \cup V)] \subset X \setminus V = X_1,$$

and—similarly— $A_2 \subset X_2$. Then,

$$A_1 \cap A_2 \subset A_1 \cap X_2 = A_1 \setminus U \subset A_1 \setminus (A_1 \setminus A_2) = A_1 \cap A_2,$$

so that $A_1 \cap A_2 = A_1 \cap X_2$, and—similarly— $A_1 \cap A_2 = X_1 \cap A_2$. Finally, since $X_1 \cap X_2 = X \setminus (U \cup V) = L \cup (A_1 \cap A_2)$, we have $[(X_1 \cap X_2) \setminus (A_1 \cap A_2)] = L$, so that $\text{ind}[(X_1 \cap X_2) \setminus (A_1 \cap A_2)] \leq n-1$. $\square$

1.9.2. Theorem. *If X is a separable metric space and A a closed subspace of X such that $\text{ind}(X \setminus A) \leq n \geq 0$, then for every continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X .*

Proof. We shall apply induction with respect to n . When $n = 0$, the mapping f takes values in the two-point space $S^0 = \{-1, 1\}$ and it follows from Lemma 1.9.1, applied to the sets $A_1 = f^{-1}(1)$ and $A_2 = f^{-1}(-1)$, that there exist closed subspaces X_1, X_2 of the space X such that $X = X_1 \cup X_2$, $A_1 \subset X_1$, $A_2 \subset X_2$ and $X_1 \cap X_2 = \emptyset$. The mapping $F: X \rightarrow S^0$ defined by letting

$$F(x) = \begin{cases} 1 & \text{for } x \in X_1, \\ -1 & \text{for } x \in X_2 \end{cases}$$

is then the required continuous extension of f .

Consider now an $n \geq 1$ and assume that the theorem holds for continuous mappings to the $(n-1)$ -sphere S^{n-1} . Let $f: A \rightarrow S^n$ be a continuous mapping defined on a closed subspace A of a separable metric space X such that $\text{ind}(X \setminus A) \leq n$. Denote by S_+^n and S_-^n the upper and the lower hemisphere of S^n , respectively. Let $A_1 = f^{-1}(S_+^n)$ and $A_2 = f^{-1}(S_-^n)$; since $S_+^n \cap S_-^n = S^{n-1}$, the restriction $g = f|_{A_1 \cap A_2}$ maps $A_1 \cap A_2$ to S^{n-1} . Applying Lemma 1.9.1, we obtain closed subspaces $X_1, X_2 \subset X$ which satisfy conditions (1) and (2). From (2) and the inductive assumption it follows that the mapping $g: A_1 \cap A_2 \rightarrow S^{n-1}$ has a continuous extension $G: X_1 \cap X_2 \rightarrow S^{n-1}$ over the subspace $X_1 \cap X_2$ of the space X . Since $A_1 \cap X_2$

$= A_1 \cap A_2 = X_1 \cap A_2$, the formulas

$$f_1(x) = \begin{cases} f(x) & \text{for } x \in A_1, \\ G(x) & \text{for } x \in X_1 \cap X_2, \end{cases} \quad f_2(x) = \begin{cases} f(x) & \text{for } x \in A_2, \\ G(x) & \text{for } x \in X_1 \cap X_2 \end{cases}$$

define continuous mappings

$$f_1: A_1 \cup (X_1 \cap X_2) \rightarrow S_+^n \quad \text{and} \quad f_2: A_2 \cup (X_1 \cap X_2) \rightarrow S_-^n.$$

The hemispheres S_+^n and S_-^n being homeomorphic to I^n , it follows from the Tietze–Urysohn theorem that f_1 and f_2 are continuously extendable to mappings $F_1: X_1 \rightarrow S_+^n$ and $F_2: X_2 \rightarrow S_-^n$. Letting

$$F(x) = \begin{cases} F_1(x) & \text{for } x \in X_1, \\ F_2(x) & \text{for } x \in X_2, \end{cases}$$

we define a continuous extension $F: X \rightarrow S^n$ of the mapping f . $\square$

1.9.3. Theorem on extending mappings to spheres. *A separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if for every closed subspace A of the space X and each continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X .*

Proof. By virtue of Theorem 1.9.2, it suffices to show that extendability of mappings implies the inequality $\text{ind} X \leq n$. We shall apply the theorem on partitions.

Let $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ be a sequence of $n+1$ pairs of disjoint closed subsets of X . Define $A = \bigcup_{i=1}^{n+1} (A_i \cup B_i)$ and, for $i = 1, 2, \dots, n+1$, consider a continuous function $f_i: A \rightarrow I$ such that

$$f_i(A_i) \subset \{0\} \quad \text{and} \quad f_i(B_i) \subset \{1\}.$$

Letting $f(x) = (f_1(x), f_2(x), \dots, f_{n+1}(x))$ for $x \in A$, we define a continuous mapping $f: A \rightarrow S_I^n$ of A to the boundary S_I^n of the $(n+1)$ -cube I^{n+1} in R^{n+1} . As the space S_I^n is homeomorphic to S^n , the mapping f has a continuous extension $F: X \rightarrow S_I^n$ over X . The composition $F_i: X \rightarrow I$ of F and the projection of S_I^n onto the i -th coordinate axis is an extension of f_i for $i = 1, 2, \dots, n+1$, so that the set $L_i = F_i^{-1}(1/2)$ is a partition between A_i and B_i . Since $\bigcap_{i=1}^{n+1} L_i = \emptyset$, we have $\text{ind} X \leq n$ by virtue of the theorem on partitions. $\square$

1.9.4. Remark. Let us note that in the second part of the above proof only the normality of the space X was applied; hence, we have shown that if a normal space X has the property that for every closed subspace A of X and each continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X , then for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

The characterization of dimension in terms of mappings to spheres will now be applied to establish the Cantor-manifold theorem; another important application of this characterization will be given in Section 1.12 (see Theorem 1.12.4).

1.9.5. Definition. A compact metric space X such that $\text{ind} X = n \geq 1$ is an n -dimensional Cantor-manifold if no closed subset L of X satisfying the inequality $\text{ind} L \leq n-2$ separates the space X , i.e., if for every such set the complement $X \setminus L$ is connected.

Clearly, every one-dimensional metric continuum is a one-dimensional Cantor-manifold, and from Theorem 1.8.13 it follows that for every $n \geq 1$ the n -cube I^n and the n -sphere S^n are n -dimensional Cantor-manifolds. On the other hand, for every $n \geq 2$ the union of two copies of the n -cube I^n with exactly one point in common is an example of an n -dimensional compact metric space which is not a Cantor-manifold. Let us observe that if X is an n -dimensional Cantor-manifold, then the dimension of X at each point $x \in X$ (see Problem 1.1.B) is equal to n , but—as shown by the last example—a compact metric space of dimension n at each point need not be a Cantor-manifold.

As we have already observed (see remarks to Problems 1.4.F(b) and 1.5.C), for every $n \geq 1$ there exist separable metric spaces of dimension n which contain no compact subspace of dimension n , and, *a fortiori*, no n -dimensional Cantor-manifold. However, every compact metric space of dimension $n \geq 1$ does contain an n -dimensional Cantor-manifold. The proof of this important theorem is preceded by two lemmas.

Let us recall that continuous mappings f_0, f_1 of a topological space X to a topological space Y are *homotopic* if there exists a continuous mapping $F: X \times I \rightarrow Y$ such that $F(x, i) = f_i(x)$ for $i = 0, 1$ and $x \in X$; the mapping F is called a *homotopy* between f_0 and f_1 .

1.9.6. Lemma. *Let $f, g: X \rightarrow S^n$ be continuous mappings of a separable metric space X to the n -sphere S^n . If the set*

$$D(f, g) = \{x \in X: f(x) \neq g(x)\}$$

satisfies the inequality $\text{ind } D(f, g) \leq n-1$, then the mappings f and g are homotopic.

Proof. Consider the space $Y = X \times I$, the closed set $A = (X \times \{0, 1\}) \cup (X \setminus D(f, g)) \times I \subset Y$ and the continuous mapping $h: A \rightarrow S^n$ defined by letting

$$h(x, 0) = f(x), \quad h(x, 1) = g(x) \quad \text{for } x \in X$$

and

$$h(x, t) = f(x) = g(x) \quad \text{for } x \in X \setminus D(f, g) \text{ and } t \in I.$$

As $Y \setminus A \subset D(f, g) \times I$, we have $\text{ind}(Y \setminus A) \leq n$, and by virtue of Theorem 1.9.2 there exists a continuous extension $H: Y \rightarrow S^n$ of the mapping $h: A \rightarrow S^n$; clearly, H is a homotopy between f and g . $\square$

The next lemma describes an important property of mappings to spheres; it is called the *Borsuk homotopy extension theorem*.

1.9.7. Lemma. *Let X be a topological space such that the Cartesian product $X \times I$ is normal (in particular, a metric or a compact space), A a closed subspace of X , and $f, g: A \rightarrow S^n$ a pair of homotopic continuous mappings of A to the n -sphere S^n . If f is continuously extendable over X , then g is also continuously extendable over X ; moreover, for every extension $F: X \rightarrow S^n$ of f there exists an extension $G: X \rightarrow S^n$ of g such that F and G are homotopic.*

Proof. Let $h: A \times I \rightarrow S^n$ be a homotopy between f and g . The continuous mapping $h': (X \times \{0\}) \cup (A \times I) \rightarrow S^n$ defined by letting

$$h'(x, 0) = F(x) \quad \text{for } x \in X$$

and

$$h'(x, t) = h(x, t) \quad \text{for } x \in A \text{ and } t \in I$$

can be extended to a continuous mapping $H': U \rightarrow S^n$ defined on an open set $U \subset X \times I$ which contains $(X \times \{0\}) \cup (A \times I)$. It follows from the compactness of I and the definition of the Cartesian product topology that every point $a \in A$ has a neighbourhood $V_a \subset X$ such that $V_a \times I \subset U$; the union $V = \bigcup_{a \in A} V_a$ is an open subset of X such that $A \times I \subset V \times I \subset U$.

The space X , being homeomorphic to the closed subspace $X \times \{0\}$ of the normal space $X \times I$, is itself normal, so that by Urysohn's lemma there

exists a continuous function $u: X \rightarrow I$ such that

$$u(X \setminus V) \subset \{0\} \quad \text{and} \quad u(A) \subset \{1\}.$$

One readily verifies that the formula

$$H(x, t) = H'(x, t \cdot u(x)) \quad \text{for } (x, t) \in X \times I$$

defines a continuous mapping $H: X \times I \rightarrow S^n$ such that $H(x, 0) = F(x)$ for $x \in X$ and $H(x, 1) = g(x)$ for $x \in A$. Hence, the mapping $G: X \rightarrow S^n$ defined by letting $G(x) = H(x, 1)$ for $x \in X$ is a continuous extension of g , and H is a homotopy between F and G . $\square$

1.9.8. The Cantor-manifold theorem. *Every compact metric space X such that $\text{ind} X = n \geq 1$ contains an n -dimensional Cantor-manifold.*

Proof. By virtue of Theorem 1.9.3, there exists a closed subspace A of the space X and a continuous mapping $f: A \rightarrow S^{n-1}$ which cannot be continuously extended over X . Denote by $\mathcal{C}$ the family consisting of all closed sets $C \subset X$ such that the mapping f cannot be continuously extended over $A \cup C$, and define an order $\leq$ in the family $\mathcal{C}$ by letting $C_1 \leq C_2$ whenever $C_2 \subset C_1$; since $X \in \mathcal{C}$, the family $\mathcal{C}$ is non-empty.

Now, consider a subfamily $\mathcal{C}_0$ of $\mathcal{C}$ which is linearly ordered by $\leq$. We shall show that the closed set $C_0 = \bigcap \mathcal{C}_0$ is a member of $\mathcal{C}$. Assume the contrary, i.e., that the mapping f is continuously extendable over $A \cup C_0$. There exists then an open set $U \subset X$ containing $A \cup C_0$ such that f is continuously extendable over U . Since $C_0 = \bigcap \mathcal{C}_0 \subset U$, from the compactness of X follows the existence of a finite number of sets $C_1, C_2, \dots, C_k \in \mathcal{C}_0$ such that $C_1 \cap C_2 \cap \dots \cap C_k \subset U$. The family $\mathcal{C}_0$ being linearly ordered by $\leq$, there exists an $i_0 \leq k$ such that $C_{i_0} \subset C_i$ for every $i \leq k$. Hence, $C_{i_0} \subset U$ and the function f is continuously extendable over $A \cup C_{i_0} \subset U$; but this is impossible, because C_{i_0} is a member of $\mathcal{C}$. The contradiction shows that $C_0 \in \mathcal{C}$; clearly $C \leq C_0$ for every $C \in \mathcal{C}_0$. Applying the Kuratowski–Zorn lemma, we obtain a maximal set $M \in \mathcal{C}$, i.e., a closed set $M \subset X$ such that the mapping f cannot be continuously extended over $A \cup M$, and yet for every proper closed subset M' of M it is continuously extendable over $A \cup M'$. We shall show that M is an n -dimensional Cantor-manifold.

As $\text{ind} M \leq \text{ind} X = n$, it suffices to show that if $M = M_1 \cup M_2$, where M_1 and M_2 are proper closed subsets of M , then $\text{ind}(M_1 \cap M_2) \geq n-1$. Assume that there exist proper closed subsets M_1, M_2 of the set M such that $M = M_1 \cup M_2$ and $\text{ind}(M_1 \cap M_2) \leq n-2$; consider the sets $A_1 = A \cup M_1$

and $A_2 = A \cup M_2$. For $i = 1, 2$ the mapping f can be extended to a continuous mapping $f_i: A_i \rightarrow S^{n-1}$. The restrictions $f_1|B$ and $f_2|B$ of f_1 and f_2 to the set $B = A \cup (M_1 \cap M_2)$ differ on a subset of $M_1 \cap M_2$, so that by virtue of Lemma 1.9.6 they are homotopic. Since $f_2|B$ is continuously extendable over $A \cup M_2$, it follows from Lemma 1.9.7 that $f_1|B$ is also extendable to a continuous mapping $f'_1: A \cup M_2 \rightarrow S^{n-1}$. From the equality $(A \cup M_1) \cap (A \cup M_2) = B$ it follows that letting

$$F(x) = \begin{cases} f_1(x) & \text{for } x \in A \cup M_1, \\ f'_1(x) & \text{for } x \in A \cup M_2 \end{cases}$$

we define a continuous mapping $F: A \cup M \rightarrow S^{n-1}$ which extends the mapping f . This contradiction concludes the proof. $\square$

1.9.9. Corollary. *Every n -dimensional compact metric space X has an n -dimensional component.* $\square$

The characterization of dimension in terms of mappings to spheres leads, via Hopf's extension theorem, to the cohomological characterization of dimension. We shall describe briefly this process.

It is a well-known fact that the Čech cohomology groups with coefficients in the group of integers, based on all open covers, satisfy the Eilenberg–Steenrod axioms. In particular, for every topological space X and every closed subset A of the space X the cohomology sequence

$$H^0(X, A) \xrightarrow{j^*} \dots \xrightarrow{\delta} H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta} H^{n+1}(X, A) \xrightarrow{j^*} \dots$$

of the pair (X, A) is an exact sequence, and for every continuous mapping $f: X \rightarrow S^n$ an element $f^*(s^n)$ of the cohomology group $H^n(X)$ is defined, the image of a fixed generator of the group $H^n(S^n)$ under the induced homomorphism $f^*: H^n(S^n) \rightarrow H^n(X)$. Hopf's extension theorem, which was mentioned above, states that

(H) *A continuous mapping $f: A \rightarrow S^n$ defined on a closed subspace A of a paracompact space X such that $\dim X \leq n+1$ is continuously extendable over X if and only if $f^*(s^n) \in i^*(H^n(X))$.*

It follows from the definition of cohomology groups that if X is a paracompact space such that $\dim X \leq n$, then $H^{n+1}(X, A) = 0$ for every closed set $A \subset X$ (cf. Proposition 3.2.2). On the other hand, if a finite-dimensional separable metric space X satisfies the inequality $\dim X > n$, then by virtue

of Theorem 1.5.1 there exists a closed subspace X' of X such that $\dim X' = n+1$, and by virtue of Theorem 1.9.3 there exists a closed subspace A of X' and a continuous mapping $f: A \rightarrow S^n$ which cannot be continuously extended over X' ; from Hopf's extension theorem and the elementary properties of cohomology groups it follows that $f^*(s^n) \notin i^*(H^n(X))$, so that $H^{n+1}(X, A) \neq 0$ by the exactness of the cohomology sequence of the pair (X, A) . Thus we obtain the cohomological characterization of dimension, which states that a finite-dimensional separable metric space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if $H^{n+i}(X, A) = 0$ for every closed set $A \subset X$ and for $i = 1, 2, \dots$. This characterization remains valid in the class of all paracompact spaces (cf. Theorem 3.2.10). Let us observe that for a metric space X it would be enough to consider the group $H^{n+1}(X, A)$, but for arbitrary paracompact spaces it is necessary to take into account all groups $H^{n+i}(X, A)$, because no counterpart of Theorem 1.5.1 holds in paracompact spaces for the covering dimension $\dim$. Let us also note that it is an open question whether the assumption of finite-dimensionality of X in the cohomological characterization of dimension is essential.

The cohomological characterization of dimension was the point of departure for cohomological dimension theory, a section of dimension theory on the border-line of point-set and algebraic topology, which studies the notion of the cohomological dimension with respect to an abelian group. For a fixed abelian group $G \neq 0$, to every topological space X one assigns the *cohomological dimension of X with respect to G* , denoted by $\dim_G X$, which is an integer larger than or equal to -1 or the "infinite number" ∞ ; the definition of the dimension function $\dim_G$ consists in the following conditions:

- (CD1) $\dim_G X = -1$ if and only if $X = \emptyset$;
- (CD2) $\dim_G X \leq n$, where $n = 0, 1, \dots$, if $H^{n+i}(X, A; G) = 0$ for every closed set $A \subset X$ and for $i = 1, 2, \dots$;
- (CD3) $\dim_G X = n$ if $\dim_G X \leq n$ and $\dim_G X > n-1$;
- (CD4) $\dim_G X = \infty$ if $\dim_G X > n$ for $n = -1, 0, 1, \dots$

One proves that the functions $\dim_G$, although distinct for different groups G , have many properties of dimension. In particular, for every abelian group $G \neq 0$ and every natural number n we have $\dim_G R^n = \dim_G I^n = \dim_G S^n = n$. One also proves that, under suitable assumptions on the space X and the group G , cohomological dimensions satisfy the counter-

parts of the subspace, sum, Cartesian product and compactification theorems. In the proofs of these theorems methods of algebraic topology and homological algebra are largely applied.

To conclude, let us note that the dimension of compact spaces can also be characterized in terms of Čech homology groups with coefficients in the group R_1 of real numbers modulo 1. One proves that a finite-dimensional compact space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if $H_{n+i}(X, A; R_1) = 0$ for every closed set $A \subset X$ and for $i = 1, 2, \dots$. In the proof, the exactness of the homology sequence and the homological counterpart of Hopf's extension theorem are used; since compactness is crucial for the validity of both these results, when passing to larger classes of spaces one has to replace homology with cohomology.

Historical and bibliographic notes

Theorem 1.9.2 was established by Hurewicz in [1935a]. The same paper contains Theorem 1.9.3 for compact metric spaces. It is a restatement of an earlier result of Alexandroff (see Problem 1.9.A); Hurewicz's contribution was to find a clever formulation and a simple proof (cf. Problem 1.9.C). The extension of Theorem 1.9.3 to separable metric spaces was given by Hurewicz and Wallman in [1941]. The notion of a Cantor-manifold was introduced by Urysohn in [1925]; Theorem 1.9.8 was proved independently by Hurewicz and Menger in [1928] and by Tumarkin in [1928]. The original proofs were much more involved than the one presented here, discovered by Hurewicz in [1937] and by Kuratowski in [1937] (a similar proof was given by Freudenthal in [1937]). Homological dimension theory was originated by Alexandroff in [1932]. Complete proofs of homological and cohomological characterizations of dimension in the realm of compact metric spaces can be found in Hurewicz and Wallman's book [1941]. A comprehensive exposition of cohomological dimension theory was developed by Kuz'minov in [1968] and by Kodama in [1970].

Problems

1.9.A (Alexandroff [1932] for compact metric spaces). A continuous mapping $f: X \rightarrow B^{n+1}$ of a topological space X to the $(n+1)$ -ball in R^{n+1} is *essential* if there is no continuous mapping $g: X \rightarrow B^{n+1}$ such that

$g|f^{-1}(S^n) = f|f^{-1}(S^n)$ and $B^{n+1} \setminus g(X) \neq \emptyset$; essential mappings with values in sets homeomorphic to balls are defined in a similar way.

Show that a separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if no continuous mapping $f: X \rightarrow B^{n+1}$ is essential.

1.9.B (Hurewicz and Wallman [1941]). A point $y \in f(X)$ is an *unstable value* of a continuous mapping $f: X \rightarrow Y$ of a topological space X to a metric space Y if for every positive number ε there exists a continuous mapping $g: X \rightarrow Y$ such that $\varrho(f(x), g(x)) < \varepsilon$ for every $x \in X$ and $y \notin g(X)$; otherwise, $y \in f(X)$ is a *stable value* of f .

(a) Show that $y \in f(X)$ is an unstable value of a continuous mapping $f: X \rightarrow I^n$ if and only if for every neighbourhood U of the point y there exists a continuous mapping $g: X \rightarrow I^n$ such that $g(x) = f(x)$ whenever $f(x) \notin U$, $g(x) \in U$ whenever $f(x) \in U$, and $y \notin g(X)$.

(b) Show that a separable metric space X satisfies the inequality $\text{ind} X \leq n \geq 0$ if and only if for every continuous mapping $f: X \rightarrow I^{n+1}$ all points in $f(X)$ are unstable values of f . Observe that instead of I^{n+1} one can consider S^{n+1} or R^{n+1} .

1.9.C (Hurewicz [1935a]). (a) Let Y be a complete metric space and Z a totally bounded metric space. Prove that for every open set $U \subset Y \times Z$ which is dense in $Y \times Z$ there exists a set $A \subset Y$ which is dense in Y and such that for every $a \in A$ the set $\{z \in Z: (a, z) \in U\}$ is dense in Z .

(b) Prove that if X is a compact metric space and the set $\{f \in (R^{n+1})^X: f(x) \neq (0, 0, \dots, 0) \text{ for every } x \in X\}$ is dense in the function space $(R^{n+1})^X$, then $\text{ind} X \leq n$ (see the beginning of Section 1.11).

Hint. Apply induction with respect to n ; use the equality $(R^{n+1})^X = (R^n)^X \times R^X$ and part (a).

(c) Apply part (b) to show that if for every closed subspace A of a compact metric space X and each continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X , then $\text{ind} X \leq n$.

1.9.D (Borsuk [1937]; for a compact metric space X , Eilenberg [1936]). Prove that for every continuous mapping $f: A \rightarrow S^k$ defined on a closed subspace A of a separable metric space X such that $\text{ind}(X \setminus A) \leq n$, where $0 \leq k \leq n$, there exists a closed subspace B of the space X such that $A \cap B = \emptyset$, $\text{ind} B \leq n - k - 1$, and the mapping f has a continuous extension $F: X \setminus B \rightarrow S^k$ over $X \setminus B$.

Hint. Apply induction with respect to $k + n$; modify the proof of Theorem 1.9.2.

1.9.E. (a) Let $p: R \rightarrow S^1$ be the continuous mapping of the real line to the circle defined by letting $p(t) = e^{2\pi it}$. Applying the fact that for every continuous mapping $g: X \rightarrow S^1$ of a metric space X to S^1 which is homotopic to a constant mapping there exists a continuous mapping $\tilde{g}: X \rightarrow R$ such that $g = p\tilde{g}$ (see Spanier [1966], p. 67), show that for every one-dimensional compact metric space Z there exists a closed set $M \subset Z$ such that the quotient space Z/M can be mapped onto S^1 by a mapping which is not homotopic to the constant mapping.

Hint. Let M be a closed subspace of the space Z with the property that there exists a continuous mapping $f: M \rightarrow S^0$ which cannot be continuously extended over Z , and let $F: Z \rightarrow I$ be a continuous mapping such that $F(x) = f(x)$ for every $x \in M$. Consider the quotient space Z/M and the continuous mapping $g: Z/M \rightarrow I/S^0 = S^1$ induced by F .

(b) (Hurewicz [1935]) Applying the fact that for every n -dimensional compact metric space Z , where $n = 1, 2, \dots$, there exists a closed set $M \subset Z$ such that the quotient space Z/M can be mapped onto S^n by a mapping which is not homotopic to the constant mapping (see Dowker [1947], p. 235), show that for every compact metric space X and every one-dimensional separable metric space Y we have $\text{ind}(X \times Y) = \text{ind} X + 1 = \text{ind} X + \text{ind} Y$.

Hint. Let $\text{ind} X = n$; reduce the problem to the special case where there exists a continuous mapping $g: X \rightarrow S^n$ which is not homotopic to the constant mapping. Then consider a pair A, B of disjoint closed subsets of Y such that the empty set is not a partition between A and B and, assuming that $\text{ind}(X \times Y) \leq n$, extend the mapping $f: X \times (A \cup B) \rightarrow S^n$ defined by letting

$$f(x, y) = (1, 0, 0, \dots, 0) \text{ for } y \in A \quad \text{and} \quad f(x, y) = g(x) \text{ for } y \in B$$

to a continuous mapping $F: X \times Y \rightarrow S^n$; examine the set of those $y \in Y$ for which the restriction $F|(X \times \{y\})$ is homotopic to the constant mapping.

1.9.F (Kuratowski and Otto [1939]). Let $\{X_s\}_{s \in S}$ be a family of n -dimensional Cantor-manifolds contained in an n -dimensional compact metric space X . Prove that if there exists an $s_0 \in S$ such that the inequality $\text{ind}(X_s \cap X_{s_0}) \geq n-1$ holds for all $s \in S$, then the subspace $\bigcup_{s \in S} X_s$ of X is an n -dimensional Cantor-manifold.

Remark. As proved by Borsuk in [1951], the Cartesian product of two Cantor-manifolds is not necessarily a Cantor-manifold.

1.9.G. (a) (Anderson and Keisler [1967]) Let X be an open subspace of an n -dimensional Cantor-manifold. Prove that if a subset M of X meets every continuum $C \subset X$ which has cardinality larger than one, then $\text{ind } M \geq n-1$ (cf. Theorem 1.8.19).

Hint. Assuming that $\text{ind } M \leq n-2$, define inductively a decreasing sequence $X_1 \supset X_2 \supset \dots \supset X_{n-1}$ of subsets of X such that X_k is an $(n-k)$ -dimensional Cantor-manifold and $\text{ind}(M \cap X_k) \leq n-k-2$ for $k = 1, 2, \dots, n-1$.

(b) Give an example of a two-dimensional Cantor-manifold of which a one-point subset is a cut.

1.9.H. (a) (Alexandroff [1932]) Show that every n -dimensional Cantor-manifold contained in an n -dimensional compact metric space X can be enlarged to a maximal Cantor-manifold contained in X , i.e., to a Cantor manifold which is not a proper subset of another Cantor manifold contained in X ; maximal n -dimensional Cantor-manifolds contained in an n -dimensional compact metric space X are called *dimensional components* of X . Check that dimensional components of a one-dimensional compact metric space coincide with components of the space X which have cardinality larger than one. Note that the union of all dimensional components of a compact metric space X is not necessarily equal to X . Observe that the intersection of two dimensional components of an n -dimensional compact metric space has dimension $\leq n-2$.

(b) (Mazurkiewicz [1933]) Prove that if A is a dimensional component of an n -dimensional compact metric space, and B is the union of all the remaining dimensional components of X , then $\text{ind}(A \cap B) \leq n-2$.

Hint. Define a decreasing transfinite sequence $X = F_1 \supset F_2 \supset \dots \supset F_\alpha \supset \dots$, $\alpha < \omega_1$ of closed sets containing A such that if $F_\alpha \setminus A \neq \emptyset$, then $F_\alpha = F_{\alpha+1} \cup \overline{(F_\alpha \setminus F_{\alpha+1})}$, where $\text{ind}[F_{\alpha+1} \cap \overline{(F_\alpha \setminus F_{\alpha+1})}] \leq n-2$, and $F_\lambda = \bigcap_{\alpha < \lambda} F_\alpha$ for every limit number $\lambda < \omega_1$. Applying the fact that there exists an $\alpha_0 < \omega_1$ such that $F_\alpha = F_{\alpha_0}$ for every $\alpha \geq \alpha_0$ (see [GT], Problem 3.12.7(b)), show that $F_{\alpha_0} = A$ and $X = A \cup \bigcup_{\alpha < \alpha_0} (F_\alpha \setminus F_{\alpha+1})$.

1.10. Characterization of dimension in terms of mappings to polyhedra

The characterization of dimension which is the object of the present section will be formulated in terms of mappings with “arbitrarily small fibres” to polyhedra of geometric dimension $\leq n$.

We begin by recalling the notions of a simplex, a simplicial complex, and a polyhedron.

Let $p = \{p_i\}$ and $q = \{q_i\}$ be points of Euclidean m -space R^m . The *sum* $p+q$ of the points p and q and the *product* λp of the point p by the real number λ are defined by the formulas

$$p+q = r = \{r_i\}, \quad \text{where } r_i = p_i + q_i,$$

and

$$\lambda p = s = \{s_i\}, \quad \text{where } s_i = \lambda p_i.$$

The point of R^m which has all coordinates equal to zero, i.e., the *origin* of R^m , will be denoted by the symbol 0 ; the distance $\varrho(0, x)$ will be denoted by $\|x\|$. One can readily verify that

$$\|p-q\| = \varrho(p, q), \quad \|\lambda p\| = |\lambda| \cdot \|p\| \quad \text{and} \quad \|p+q\| \leq \|p\| + \|q\|;$$

the last inequality is a reformulation of the triangle inequality in R^m .

A finite system of points $p_0, p_1, \dots, p_k \in R^m$ is *linearly independent* if for each sequence $\lambda_0, \lambda_1, \dots, \lambda_k$ of $k+1$ real numbers the conditions

$$\lambda_0 p_0 + \lambda_1 p_1 + \dots + \lambda_k p_k = 0 \quad \text{and} \quad \lambda_0 + \lambda_1 + \dots + \lambda_k = 0$$

imply that $\lambda_i = 0$ for $i = 0, 1, \dots, k$.

Let $p_0, p_1, \dots, p_n$ be a linearly independent system of $n+1$ points in R^m ; the subset of R^m consisting of all points

$$(1) \quad p = \lambda_0 p_0 + \lambda_1 p_1 + \dots + \lambda_n p_n,$$

where

$$(2) \quad \lambda_0 + \lambda_1 + \dots + \lambda_n = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for } i = 0, 1, \dots, n,$$

is called an *n-simplex* spanned by the points $p_0, p_1, \dots, p_n$ and is denoted by $p_0 p_1 \dots p_n$. Clearly, the simplex $p_0 p_1 \dots p_n$ does not depend on the ordering of points $p_0, p_1, \dots, p_n$, it depends on the set $\{p_0, p_1, \dots, p_n\}$ only. One proves that the simplex $p_0 p_1 \dots p_n$ determines the points $p_0, p_1, \dots, p_n$ and can be characterized as the smallest convex subset of R^m which contains these points; one also proves that the diameter of the simplex $p_0 p_1 \dots p_n$ is equal to the diameter of the set $\{p_0, p_1, \dots, p_n\}$ (see Problem 1.10.A).

Consider an n -simplex $p_0 p_1 \dots p_n \subset R^m$; for each choice of $k+1$ distinct non-negative integers $i_0, i_1, \dots, i_k$ not larger than n , where $0 \leq k \leq n$, the points $p_{i_0}, p_{i_1}, \dots, p_{i_k}$ form a linearly independent system, so that the k -simplex $p_{i_0} p_{i_1} \dots p_{i_k}$ is well defined. Every simplex of this form is called a *k-face* of the simplex $p_0 p_1 \dots p_n$; 0-faces $p_0, p_1, \dots, p_m$ are also called *vertices* of the simplex $p_0 p_1 \dots p_n$. One easily sees that the k -face $p_{i_0} p_{i_1} \dots p_{i_k}$

consists of all points of form (1) satisfying (2) and such that

$$\lambda_i = 0 \quad \text{whenever } i \neq i_j \text{ for } j = 0, 1, \dots, k.$$

To denote the fact that $S_0 = p_{i_0}p_{i_1} \dots p_{i_k}$ is a face of $S = p_0p_1 \dots p_n$ we write symbolically $S_0 \leq S$. For each n -simplex S the union of all k -faces of S with $k < n$ is called the *boundary* of the simplex S ; the complement of the boundary is called the *interior* of the simplex S .

It follows from the linear independence of the set of vertices, that every point p of the simplex spanned by the points $p_0, p_1, \dots, p_n \in R^m$ can be represented in the form (1), under conditions (2), in a unique way. The coefficients $\lambda_0, \lambda_1, \dots, \lambda_n$ in (1) are called the *barycentric coordinates* of the point p ; they will also be denoted by $\lambda_0(p), \lambda_1(p), \dots, \lambda_n(p)$.

One readily checks that every simplex $S = p_0p_1 \dots p_n \subset R^m$ is a compact subspace of R^m and that the barycentric coordinates $\lambda_0, \lambda_1, \dots, \lambda_n$ are continuous functions from S to I . This implies, in particular, that any two n -simplexes are homeomorphic. Hence (see Problems 1.8.A and 1.10.A(b)), every n -simplex is homeomorphic to the n -cube I^n . It follows from Corollary 1.8.3, that for every n -simplex S we have $\text{ind } S = \text{Ind } S = \dim S = n$.

A *simplicial complex*, or, briefly, a *complex*, is an arbitrary finite family $\mathcal{K}$ of simplexes in a Euclidean space such that if $S \in \mathcal{K}$ and $S_0 \leq S$ then $S_0 \in \mathcal{K}$, and if $S_1, S_2 \in \mathcal{K}$ then the intersection $S_1 \cap S_2$ either is empty or is a face of both S_1 and S_2 ; all 0-simplexes that belong to $\mathcal{K}$ are called *vertices* of the complex $\mathcal{K}$. Every subfamily $\mathcal{K}_0$ of a complex $\mathcal{K}$ which itself is a complex, i.e. which together with a simplex $S \in \mathcal{K}$ contains all faces of S , is called a *subcomplex* of $\mathcal{K}$.

Let $\mathcal{K}$ be a simplicial complex consisting of simplexes in Euclidean m -space R^m . The union $|\mathcal{K}| = \bigcup \{S: S \in \mathcal{K}\} \subset R^m$ is the *underlying polyhedron* of the complex $\mathcal{K}$; it is a compact subspace of R^m . By a *polyhedron* we mean a subspace of a Euclidean space which is the underlying polyhedron $|\mathcal{K}|$ of a simplicial complex $\mathcal{K}$; clearly, the representation of a polyhedron as the underlying polyhedron of a complex is not unique. One can prove (see Problem 1.10.B) that for every non-empty polyhedron $K = |\mathcal{K}|$, the largest integer n such that the complex $\mathcal{K}$ contains n -simplexes does not depend on the complex $\mathcal{K}$ but on the polyhedron K only. The number n is called the *geometric dimension* of the polyhedron K ; the geometric dimension of an empty polyhedron is equal to -1 . From the sum theorem and the fact that an n -simplex is an n -dimensional space it follows that the geometric dimension of a polyhedron coincides

with its topological dimensions ind , Ind and dim . Let us note that from the same two premises it follows that the integer n discussed above depends on the polyhedron K only. However, since one of the aspects of the characterization of dimension in terms of mappings to polyhedra is that it reduces the topological notion of dimension to the elementary notion of geometric dimension, it is not devoid of importance that the correctness of the definition of geometric dimension can be checked in an elementary way. In the sequel we shall abbreviate "geometric dimension" to "dimension" and call a polyhedron whose geometric dimension equals n an n -dimensional polyhedron.

Every point p of the underlying polyhedron $|\mathcal{K}| = K$ of a simplicial complex $\mathcal{K}$ with vertices $p_0, p_1, \dots, p_k$ can be represented in the form

$$(3) \quad p = \lambda_0 p_0 + \lambda_1 p_1 + \dots + \lambda_k p_k,$$

where $\lambda_0 + \lambda_1 + \dots + \lambda_k = 1$ and $\lambda_i \geq 0$ for $i = 0, 1, \dots, k$; moreover, if $p \in p_{i_0} p_{i_1} \dots p_{i_l} \in \mathcal{K}$, then $\lambda_i = 0$ whenever $i \neq i_j$ for $j = 0, 1, \dots, l$. It follows from the definition of a simplicial complex that the above representation is unique, so that the coefficients in (3) can be written as $\lambda_0(p), \lambda_1(p), \dots, \lambda_k(p)$; by the continuity of barycentric coordinates, $\lambda_0, \lambda_1, \dots, \lambda_k$ are continuous functions from K to I . For every vertex p_i of the complex $\mathcal{K}$ the *star* of p_i is a subset of the underlying polyhedron defined by

$$\text{St}_{\mathcal{K}}(p_i) = |\mathcal{K}| \setminus \bigcup \{S \in \mathcal{K} : p_i \notin S\};$$

one readily checks that $\text{St}_{\mathcal{K}}(p_i) = \{p \in |\mathcal{K}| : \lambda_i(p) > 0\}$, so that the stars of vertices of $\mathcal{K}$ are open subsets of $|\mathcal{K}|$.

We are now going to prepare tools that will be applied later in this section to prove theorems on mappings to polyhedra and in the next one to prove general embedding and universal space theorems.

1.10.1. Definition. A finite system of points $p_1, p_2, \dots, p_k \in R^m$ is in *general position* if for each sequence $i_0 < i_1 < \dots < i_l \leq k$ of $l+1$ natural numbers, where $l \leq m$, the system $p_{i_0}, p_{i_1}, \dots, p_{i_l}$ is linearly independent.

Since no system of $m+2$ points in R^m is linearly independent, general position means the minimum of bonds between points in a system.

1.10.2. Theorem. For every finite system of points $q_1, q_2, \dots, q_k \in R^m$ and every positive number α there exists a system of points $p_1, p_2, \dots, p_k \in R^m$ in general position such that $\varrho(p_i, q_i) < \alpha$ for $i = 1, 2, \dots, k$. If, moreover, a system of points $r_1, r_2, \dots, r_l \in R^m$ in general position is given, one can

chose the points $p_1, p_2, \dots, p_k$ in such a way that the whole system $r_1, r_2, \dots, r_l, p_1, p_2, \dots, p_k$ is in general position.

Proof. The points $p_1, p_2, \dots, p_k$ will be defined by induction. Let $p_1 = q_1$; assume that for a $j \leq k$ the points $p_1, p_2, \dots, p_{j-1}$ in general position are defined in such a way that $\varrho(p_i, q_i) < \alpha$ for $i = 1, 2, \dots, j-1$. For every system of points $p_{i_0}, p_{i_1}, \dots, p_{i_n}$, where $i_0 < i_1 < \dots < i_n \leq j-1$ and $n \leq m-1$, the linear n -variety determined by these points is nowhere dense in R^m ; hence the union of all such linear varieties is also nowhere dense in R^m and there exists a point p_j outside this union such that $\varrho(p_j, q_j) < \alpha$. Clearly, the system of points $p_1, p_2, \dots, p_j$ is in general position; thus the first part of the theorem is proved. The proof of the second part is much the same, only—when defining the point p_j —one has to consider all systems of $n \leq m-1$ points in the set $\{r_1, r_2, \dots, r_l, p_1, p_2, \dots, p_{j-1}\}$. $\square$

1.10.3. Definition. Let X be a topological space and $\mathcal{U} = \{U_i\}_{i=1}^k$ a finite open cover of X . By a *nerve* of the cover $\mathcal{U}$ we understand an arbitrary simplicial complex $\mathcal{N}(\mathcal{U})$ which has the property that its vertices can be arranged into a sequence $p_1, p_2, \dots, p_k$ in such a way that

$$p_{i_0} p_{i_1} \dots p_{i_m} \in \mathcal{N}(\mathcal{U}) \quad \text{if and only if} \quad U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_m} \neq \emptyset.$$

When discussing a nerve $\mathcal{N}(\mathcal{U})$ of a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$, we shall always assume that the vertices $p_1, p_2, \dots, p_k$ of $\mathcal{N}(\mathcal{U})$ are arranged in such a way that the above equivalence holds. The underlying polyhedron of the nerve $\mathcal{N}(\mathcal{U})$ will be denoted by $N(\mathcal{U})$. One readily sees that if $\text{ord } \mathcal{U} \leq n$, then the dimension of the polyhedron $N(\mathcal{U})$ is not larger than n . Note that every finite open cover has a nerve, although not uniquely determined (cf. Problem 1.10.C). Indeed, for a given open cover $\{U_i\}_{i=1}^k$ one can, for example, consider an arbitrary linearly independent system $p_1, p_2, \dots, p_k$ of k points in R^{k-1} and define $\mathcal{N}(\mathcal{U})$ as the simplicial complex consisting of all faces $p_{i_0} p_{i_1} \dots p_{i_m}$ of the simplex $p_1 p_2 \dots p_k$ such that $U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_m} \neq \emptyset$. It turns out, however, that nerves can be defined in a more economical way.

1.10.4. Theorem. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of a topological space X . If $\text{ord } \mathcal{U} \leq n$, then there exists a nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$ consisting of simplexes contained in Euclidean $(2n+1)$ -space R^{2n+1} . If, moreover, a linear n -variety $H \subset R^{2n+1}$, a system $q_1, q_2, \dots, q_k$ of points in R^{2n+1} , and a positive number $\alpha > 0$ are given, one can choose the nerve

$\mathcal{N}(\mathcal{U})$ in such a way that $H \cap N(\mathcal{U}) = \emptyset$ and the vertices $p_1, p_2, \dots, p_k$ of $\mathcal{N}(\mathcal{U})$ satisfy the inequality $\varrho(p_i, q_i) < \alpha$ for $i = 1, 2, \dots, k$.

Proof. Let $r_1, r_2, \dots, r_{n+1}$ be a linearly independent system of points in R^{2n+1} which spans the linear variety H . By virtue of Theorem 1.10.2, there exists a system of points $p_1, p_2, \dots, p_k \in R^{2n+1}$ such that $\varrho(p_i, q_i) < \alpha$ for $i = 1, 2, \dots, k$ and the system $r_1, r_2, \dots, r_{n+1}, p_1, p_2, \dots, p_k$ is in general position. Since $\text{ord } \mathcal{U} \leq n$, for each sequence $i_0 < i_1 < \dots < i_l \leq k$ of $l+1$ natural numbers such that $U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_l} \neq \emptyset$ we have $l \leq n < 2n+1$, so that the points $p_{i_0}, p_{i_1}, \dots, p_{i_l}$ form a linearly independent system and the simplex $p_{i_0} p_{i_1} \dots p_{i_l} \subset R^{2n+1}$ is well defined. Let us denote by $\mathcal{N}(\mathcal{U})$ the family of all simplexes $p_{i_0} p_{i_1} \dots p_{i_l} \subset R^{2n+1}$ obtained in this way. To complete the proof it suffices to show that $\mathcal{N}(\mathcal{U})$ is a simplicial complex and that $H \cap N(\mathcal{U}) = \emptyset$. Since each face of a simplex in $\mathcal{N}(\mathcal{U})$ is in $\mathcal{N}(\mathcal{U})$, to show that $\mathcal{N}(\mathcal{U})$ is a complex it remains to check that if $S_1 = p_{i_0} p_{i_1} \dots p_{i_l}$ and $S_2 = p_{j_0} p_{j_1} \dots p_{j_m}$ belong to $\mathcal{N}(\mathcal{U})$ and the intersection $S_1 \cap S_2$ is non-empty, then $S_1 \cap S_2$ is a face of both S_1 and S_2 . Obviously, it is enough to prove that if $p \in S_1 \cap S_2$ and in the representation $p = \sum_{s=0}^l \lambda_s p_{i_s}$, where $\sum_{s=0}^l \lambda_s = 1$, we have $\lambda_{s_0} > 0$, then $p_{i_{s_0}}$ is a vertex of

S_2 . Since $p \in S_2$, $p = \sum_{t=0}^m \lambda'_t p_{j_t}$, where $\sum_{t=0}^m \lambda'_t = 1$; we have

$$\sum_{s=0}^l \lambda_s p_{i_s} - \sum_{t=0}^m \lambda'_t p_{j_t} = 0 \quad \text{and} \quad \sum_{s=0}^l \lambda_s - \sum_{t=0}^m \lambda'_t = 0.$$

As observed before, $l \leq n$; similarly, $m \leq n$. Hence, the points $p_{i_0}, p_{i_1}, \dots, p_{i_l}, p_{j_0}, p_{j_1}, \dots, p_{j_m}$ form a linearly independent system, because its cardinality is not larger than $l+m+2 \leq 2n+2$ and the whole system $r_1, r_2, \dots, r_{n+1}, p_1, p_2, \dots, p_k$ is in general position. It follows that $p_{i_{s_0}}$ occurs among the points $p_{j_0}, p_{j_1}, \dots, p_{j_m}$, i.e., $p_{i_{s_0}}$ is a vertex of S_2 . In a similar way one shows that all simplexes in $\mathcal{N}(\mathcal{U})$ are disjoint from the linear n -variety H . $\square$

For each finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of a metric space X and a sequence of points $p_1, p_2, \dots, p_k$ in Euclidean m -space R^m a continuous mapping of X to R^m is defined in a natural way.

1.10.5. Definition. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of a metric space X and $p_1, p_2, \dots, p_k$ a sequence of points in Euclidean m -space R^m .

The continuous mapping $\kappa: X \rightarrow R^m$ defined by letting

$$(4) \quad \kappa(x) = \kappa_1(x)p_1 + \kappa_2(x)p_2 + \dots + \kappa_k(x)p_k,$$

where

$$(5) \quad \kappa_i(x) = \frac{\varrho(x, X \setminus U_i)}{\sum_{j=1}^k \varrho(x, X \setminus U_j)} \quad \text{for } i = 1, 2, \dots, k,$$

is called the κ -mapping determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \dots, p_k$. Let us observe that the denominator in (5) does not vanish, because $\varrho(x, X \setminus U_j) > 0$ whenever $x \in U_j$. Let us also note that $\sum_{i=1}^k \kappa_i(x) = 1$ for every $x \in X$.

As explained in the following theorem, all continuous mappings of a metric space to a Euclidean space can be approximated by κ -mappings.

1.10.6. Theorem. *Let $f: X \rightarrow R^m$ be a continuous mapping of a metric space X to Euclidean m -space R^m and let δ be a positive number. If $\mathcal{U} = \{U_i\}_{i=1}^k$ is a finite open cover of the space X and $p_1, p_2, \dots, p_k \in R^m$ is a sequence of points such that*

$$(6) \quad \delta(\{p_i\} \cup f(U_i)) < \delta \quad \text{for } i = 1, 2, \dots, k,$$

then the κ -mapping $\kappa: X \rightarrow R^m$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \dots, p_k$ has the property that $\varrho(f(x), \kappa(x)) < \delta$ for every $x \in X$.

Proof. Consider a point $x \in X$ and let $U_{i_0}, U_{i_1}, \dots, U_{i_l}$ be all members of the cover $\mathcal{U}$ that contain the point x . By virtue of (5), $\kappa_i(x) = 0$ whenever $i \neq i_j$ for $j = 0, 1, \dots, l$. It follows from (6) that $\|f(x) - p_{i_j}\| = \varrho(f(x), p_{i_j}) < \delta$ for $j = 0, 1, \dots, l$; applying (4), we obtain

$$\begin{aligned} \varrho(f(x), \kappa(x)) &= \|f(x) - \kappa(x)\| = \left\| \sum_{j=0}^l \kappa_{i_j}(x)f(x) - \sum_{j=0}^l \kappa_{i_j}(x)p_{i_j} \right\| \\ &\leq \sum_{j=0}^l \kappa_{i_j}(x) \|f(x) - p_{i_j}\| < \sum_{j=0}^l \kappa_{i_j}(x) \delta = \delta \sum_{j=0}^l \kappa_{i_j}(x) = \delta. \quad \square \end{aligned}$$

The notion of a κ -mapping determined by a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of a metric space and points $p_1, p_2, \dots, p_k \in R^m$ proves particularly useful in the case where the points $p_1, p_2, \dots, p_k$ are the vertices of a nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$.

1.10.7. Theorem. If $\mathcal{U} = \{U_i\}_{i=1}^k$ is a finite open cover of a metric space X and $\mathcal{N}(\mathcal{U})$ a nerve of $\mathcal{U}$ with vertices $p_1, p_2, \dots, p_k \in R^m$, then the κ -mapping $\kappa: X \rightarrow R^m$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \dots, p_k$ satisfies the conditions

$$(7) \quad \kappa(X) \subset N(\mathcal{U})$$

and

$$(8) \quad \kappa^{-1}(\text{St}_{\mathcal{N}(\mathcal{U})}(p_i)) = U_i \quad \text{for } i = 1, 2, \dots, k.$$

Proof. Consider a point $x \in X$ and let $U_{i_0}, U_{i_1}, \dots, U_{i_l}$ be all members of the cover $\mathcal{U}$ that contain the point x . Since $U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_l} \neq \emptyset$, the simplex $p_{i_0} p_{i_1} \dots p_{i_l}$ belongs to the nerve $\mathcal{N}(\mathcal{U})$. It follows from (5) and (4) that $\kappa(x) \in p_{i_0} p_{i_1} \dots p_{i_l}$, so that (7) is satisfied.

Since the representation of points of $N(\mathcal{U})$ in form (3) is unique, we have $\lambda_i(\kappa(x)) = \kappa_i(x)$ for $i = 1, 2, \dots, k$. Thus

$$\begin{aligned} \kappa^{-1}(\text{St}_{\mathcal{N}(\mathcal{U})}(p_i)) &= \{x \in X: \kappa(x) \in \text{St}_{\mathcal{N}(\mathcal{U})}(p_i)\} = \{x \in X: \lambda_i(\kappa(x)) > 0\} \\ &= \{x \in X: \kappa_i(x) > 0\} = U_i \quad \text{for } i = 1, 2, \dots, k, \end{aligned}$$

i.e., (8) is also satisfied. $\square$

1.10.8. Definition. Let $\mathcal{E}$ be an open cover of a topological space X and $f: X \rightarrow Y$ a continuous mapping of X to a topological space Y ; we say that f is an $\mathcal{E}$ -mapping if there exists an open cover $\mathcal{U}$ of the space Y such that the cover $f^{-1}(\mathcal{U})$ is a refinement of $\mathcal{E}$.

1.10.9. Definition. Let ε be a positive number and $f: X \rightarrow Y$ a continuous mapping of a metric space X to a topological space Y ; we say that f is an ε -mapping if $\delta(f^{-1}(y)) < \varepsilon$ for every $y \in Y$. Obviously, if $\mathcal{E}$ is an open cover of a metric space and $\text{mesh } \mathcal{E} < \varepsilon$, then each $\mathcal{E}$ -mapping is an ε -mapping.

1.10.10. Theorem. If X is a compact metric space, then for every open cover $\mathcal{E}$ of the space X there exists a positive number ε such that each ε -mapping of X to a Hausdorff space is an $\mathcal{E}$ -mapping.

Proof. Let ε be a Lebesgue number for the cover $\mathcal{E}$. Consider an ε -mapping $f: X \rightarrow Y$ of X to a Hausdorff space Y . For every $y \in Y$ there exists a $V_y \in \mathcal{E}$ such that $f^{-1}(y) \subset V_y$. Since every continuous mapping of a compact space to a Hausdorff space maps closed sets onto closed sets, the set U_y

$= Y \setminus f(X \setminus V_y)$ is open for every $y \in Y$; one readily checks that $y \in U_y$ and $f^{-1}(U_y) \subset V_y$. Hence $\mathcal{U} = \{U_y\}_{y \in Y}$ is an open cover of the space Y such that the cover $f^{-1}(\mathcal{U})$ is a refinement of $\mathcal{E}$, i.e., f is an $\mathcal{E}$ -mapping. $\square$

1.10.11. Theorem. *If for every finite open cover $\mathcal{E}$ of a normal space X there exists an $\mathcal{E}$ -mapping $f: X \rightarrow Y$ of X to a compact space Y such that $\dim Y \leq n$, then $\dim X \leq n$.*

Proof. Consider an arbitrary finite open cover $\mathcal{E}$ of the space X and an $\mathcal{E}$ -mapping $f: X \rightarrow Y$ of X to a compact space Y such that $\dim Y \leq n$. Let $\mathcal{U}$ be an open cover of the space Y such that the cover $f^{-1}(\mathcal{U})$ is a refinement of $\mathcal{E}$. As the space Y is compact, $\mathcal{U}$ has a finite open refinement $\mathcal{V}$ which, by virtue of the inequality $\dim Y \leq n$, has in its turn a finite open refinement $\mathcal{W}$ such that $\text{ord } \mathcal{W} \leq n$. Now, the cover $f^{-1}(\mathcal{W})$ of the space X is a finite open cover of order $\leq n$ which refines $\mathcal{E}$, so that $\dim X \leq n$. $\square$

Let us observe that if the space Y in the last theorem is a polyhedron, one can slightly modify the above proof so as to use only the geometric dimension of Y . Indeed, in this case the existence of the refinement $\mathcal{W}$ of the cover $\mathcal{V}$ such that $\text{ord } \mathcal{W} \leq n$ follows from the elementary fact that every polyhedron of geometric dimension $\leq n$ has finite covers of order $\leq n$ by open sets of arbitrarily small diameter.

Theorems 1.10.10 and 1.10.11 imply

1.10.12. Theorem. *If X is a compact metric space and for every positive number ε there exists an ε -mapping $f: X \rightarrow Y$ of X to a compact space Y such that $\dim Y \leq n$, then $\dim X \leq n$.* $\square$

We are now ready to characterize dimension in terms of mappings to polyhedra.

1.10.13. Theorem on $\mathcal{E}$ -mappings. *A metric space X satisfies the inequality $\dim X \leq n$ if and only if for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$ -mapping of X to a polyhedron of dimension $\leq n$.*

Proof. The theorem is obvious if $\dim X = -1$. Consider a metric space X such that $0 \leq \dim X \leq n$ and a finite open cover $\mathcal{E}$ of the space X . Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be an open refinement of $\mathcal{E}$ such that $\text{ord } \mathcal{U} \leq n$ and let $\mathcal{N}(\mathcal{U})$ be a nerve of $\mathcal{U}$ with vertices $p_1, p_2, \dots, p_k \in R^m$. From Theorem 1.10.7 it follows that the κ -mapping $\kappa: X \rightarrow N(\mathcal{U})$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \dots, p_k$ is an $\mathcal{E}$ -mapping, because $\{\text{St}_{\mathcal{N}(\mathcal{U})}(p_i)\}_{i=1}^k$

is an open cover of the underlying polyhedron $N(\mathcal{U})$. The inequality $\text{ord}\mathcal{U} \leq n$ implies that $N(\mathcal{U})$ has dimension $\leq n$. To complete the proof it suffices to apply Theorem 1.10.11. $\square$

Let us note that, referring in the above proof to Theorem 1.10.11, we used the coincidence of the geometric dimension and the topological dimension of polyhedra. The observation following Theorem 1.10.11 shows how this can be eliminated.

1.10.14. Theorem on ε -mappings. *A compact metric space X satisfies the inequality $\dim X \leq n$ if and only if for every positive number ε there exists an ε -mapping of X to a polyhedron of dimension $\leq n$.*

Proof. Consider a compact metric space X such that $\dim X \leq n$ and a positive number ε . Let $\mathcal{E}$ be a finite open refinement of the open cover $\{B(x, \varepsilon/2)\}_{x \in X}$ of the space X . By virtue of the theorem on $\mathcal{E}$ -mappings, there exists an $\mathcal{E}$ -mapping of X to a polyhedron of dimension $\leq n$; one readily checks that this is an ε -mapping. To complete the proof it suffices to apply Theorem 1.10.12. $\square$

The comment following the theorem on $\mathcal{E}$ -mappings applies as well to the above proof. Let us also observe that in the theorem on ε -mappings the assumption of compactness is essential; indeed, there exist separable metric spaces of dimension larger than one that can be mapped to the interval I by a one-to-one continuous mapping (see Problem 1.4.F(b)).

The reader has undoubtedly noted that in the proofs of Theorems 1.10.13 and 1.10.14 only the existence of a nerve was applied and not the much stronger Theorem 1.10.4. The latter theorem will be applied in the next section, where it will prove to be, together with Theorem 1.10.6, the core of the proofs of the embedding and the universal space theorems.

We shall now show that the theorems on $\mathcal{E}$ -mappings and on ε -mappings can be somewhat strengthened, viz., that the existence of mappings onto polyhedra can be established. In the proof we shall apply the following auxiliary theorem, which states that every subset A of the underlying polyhedron $|\mathcal{K}|$ of a complex $\mathcal{K}$ can be swept out of interiors of all simplexes in $\mathcal{K}$ which are not contained in A .

1.10.15. The sweeping out theorem. *For every simplicial complex $\mathcal{K}$ and each subspace A of the underlying polyhedron $|\mathcal{K}|$ there exist a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ and a continuous mapping $f: A \rightarrow |\mathcal{K}_0|$ such that $f(A) = |\mathcal{K}_0|$ and $f(A \cap S) \subset S$ for every $S \in \mathcal{K}$.*

Proof. Let K be the collection of all subcomplexes $\mathcal{K}'$ of $\mathcal{K}$ for which there exists a continuous mapping $f: A \rightarrow |\mathcal{K}'|$ such that

$$(9) \quad f(A \cap S) \subset S \quad \text{for every } S \in \mathcal{K}.$$

As the embedding $f: A \rightarrow |\mathcal{K}|$ of the subspace A in $|\mathcal{K}|$ satisfies (9), the complex $\mathcal{K}$ itself belongs to the collection K . Hence the collection K is non-empty and, being finite, contains a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ such that no proper subcomplex of $\mathcal{K}_0$ belongs to K . Consider a continuous mapping $f: A \rightarrow |\mathcal{K}_0|$ which satisfies (9); we shall show that $f(A) = |\mathcal{K}_0|$.

Suppose that there exists a point $x_0 \in |\mathcal{K}_0| \setminus f(A)$. Let $S_0 = p_0 p_1 \dots p_k$ be the intersection of all simplexes in $\mathcal{K}_0$ which contain the point x_0 . Define

$$\mathcal{S}_0 = \{S \in \mathcal{K}_0: S_0 \leq S\}$$

and

$$\mathcal{B}_0 = \{T \in \mathcal{K}_0: T \not\leq S_0 \text{ and } T \leq S \text{ for an } S \in \mathcal{S}_0\};$$

clearly, $\mathcal{K}_1 = \mathcal{K}_0 \setminus \mathcal{S}_0$ and $\mathcal{B}_0$ are subcomplexes of $\mathcal{K}_0$. The set $G = \bigcap_{i=0}^k \text{St}_{\mathcal{K}_0}(p_i)$ is open in $|\mathcal{K}_0|$; one readily checks that

$$\overline{G} = \bigcup \mathcal{S}_0 = G \cup B_0, \quad \text{where } B_0 = |\mathcal{B}_0|.$$

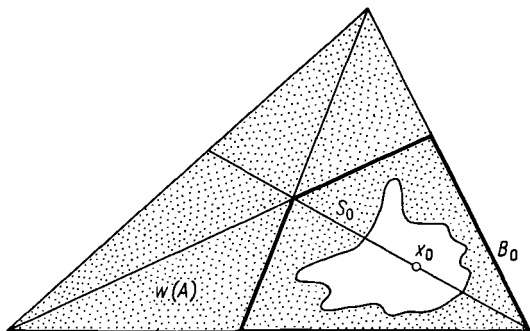


Fig. 10

Let p denote the projection of $f(A) \cap \overline{G}$ from the point x_0 onto B_0 . The restriction

$$(10) \quad p|_{f(A) \cap S}: f(A) \cap S \rightarrow B_0 \cap S$$

is continuous for every $S \in \mathcal{S}_0$; thus $p: f(A) \cap \overline{G} \rightarrow B_0$ is a continuous mapping. Since $p(x) = x$ for $x \in f(A) \cap B_0$, the formula

$$g(x) = \begin{cases} p(x) & \text{for } x \in f(A) \cap \overline{G}, \\ x & \text{for } x \in f(A) \setminus \overline{G} \end{cases}$$

defines a continuous mapping $g: f(A) \rightarrow |\mathcal{K}_1|$; let $f_1 = gf: A \rightarrow |\mathcal{K}_1|$. It follows from (10) that $f_1(A \cap S) \subset S$ for every $S \in \mathcal{K}$; hence $\mathcal{K}_1 \in \mathcal{K}$, which contradicts the definition of $\mathcal{K}_0$. The contradiction shows that $f(A) = |\mathcal{K}_0|$. $\square$

1.10.16. Theorem. *A metric (compact metric) space X satisfies the inequality $\dim X \leq n$ if and only if for every finite open cover $\mathcal{E}$ of the space X (for every positive number ε) there exists an $\mathcal{E}$ -mapping (an ε -mapping) of X onto a polyhedron of dimension $\leq n$.*

Proof. It suffices to show that if a metric space X satisfies the inequality $0 \leq \dim X \leq n$, then for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$ -mapping of X onto a polyhedron of dimension $\leq n$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be an open refinement of $\mathcal{E}$ such that $\text{ord } \mathcal{U} \leq n$ and let $\kappa: X \rightarrow |\mathcal{K}|$ be the κ -mapping of X to the underlying polyhedron of a nerve $\mathcal{K} = \mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$. By virtue of the sweeping out theorem, there exists a continuous mapping f of the subspace $A = \kappa(X)$ of $|\mathcal{K}|$ onto the underlying polyhedron $|\mathcal{K}_0|$ of a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ which satisfies (9). Let us note that, for every $x \in A$ and every vertex p_i of $\mathcal{K}_0$, if $f(x) \in \text{St}_{\mathcal{K}_0}(p_i)$, then $x \in \text{St}_{\mathcal{K}}(p_i)$, i.e.,

$$(11) \quad f^{-1}(\text{St}_{\mathcal{K}_0}(p_i)) \subset \text{St}_{\mathcal{K}}(p_i).$$

Indeed, if $x \notin \text{St}_{\mathcal{K}}(p_i)$, then the intersection S of all simplexes in $\mathcal{K}$ that contain the point x does not contain p_i ; since $f(x) \in S$ by virtue of (9), it follows that $S \in \mathcal{K}_0$, and thus $f(x) \notin \text{St}_{\mathcal{K}_0}(p_i)$. Inclusions (8) and (11) show that the composition $f\kappa: X \rightarrow |\mathcal{K}_0|$ is an $\mathcal{E}$ -mapping of X onto $|\mathcal{K}_0|$. $\square$

The final part of this section will be devoted to subspaces of Euclidean spaces. We start by introducing the notion of an ε -translation.

1.10.17. Definition. Let ε be a positive number, A, B subspaces of a metric space X , and $f: A \rightarrow B$ a continuous mapping of A to B ; we say that f is an ε -translation if $\varrho(x, f(x)) < \varepsilon$ for every $x \in A$. Obviously, each ε -translation is a 3ε -mapping and an ε -translation defined on a compact subspace of X is a 2ε -mapping.

For subspaces of Euclidean spaces it is much more interesting to discuss ε -translations to polyhedra rather than ε -mappings. Clearly, every subspace of R^m which can be mapped to a polyhedron by an ε -translation is bounded. It turns out that for bounded subspaces of Euclidean spaces we have a theorem on ε -translations which strictly parallels the theorem on ε -map-

pings. In the proof of that theorem we shall apply the elementary fact that, for every finite family $\mathcal{S}$ of simplexes in R^m , the union $L = \bigcup \mathcal{S}$ is a polyhedron whose dimension is equal to the largest integer n such that $\mathcal{S}$ contains an n -simplex; more exactly, we shall assume that there exists a simplicial complex $\mathcal{L}$ such that $L = |\mathcal{L}|$ and every simplex of $\mathcal{L}$ is contained in a member of $\mathcal{S}$. A proof of this fact is outlined in Problem 1.10.I.

1.10.18. Theorem on ε -translations. *If X is a bounded subspace of Euclidean m -space R^m , and X satisfies the inequality $\dim X \leq n$, then for every positive number ε there exists an ε -translation $f: X \rightarrow K$ of X onto a polyhedron $K \subset R^m$ of dimension $\leq n$.*

Proof. Without loss of generality one can suppose that $0 \leq n \leq m$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of the space X such that $\text{mesh } \mathcal{U} < \varepsilon/4$ and $U_i \neq \emptyset$ for $i = 1, 2, \dots, k$; since $\dim X \leq n$, one can assume—replacing $\mathcal{U}$, if necessary, by a refinement—that $\text{ord } \mathcal{U} \leq n$. For $i = 1, 2, \dots, k$ choose a point $q_i \in U_i$ and apply Theorem 1.10.2 to obtain a system of points $p_1, p_2, \dots, p_k \in R^m$ in general position such that

$$(12) \quad \delta(\{p_i\} \cup U_i) < \varepsilon/4 \quad \text{for } i = 1, 2, \dots, k.$$

Since $\text{ord } \mathcal{U} \leq n$, for each sequence $i_0 < i_1 < \dots < i_l \leq k$ of $l+1$ natural numbers such that $U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_l} \neq \emptyset$ we have $l \leq n \leq m$, so that—the system $p_1, p_2, \dots, p_k$ being in general position—the simplex $p_{i_0} p_{i_1} \dots p_{i_l} \subset R^m$ is well defined; by virtue of (12), $\delta(p_{i_0} p_{i_1} \dots p_{i_l}) = \delta(\{p_{i_0}, p_{i_1}, \dots, p_{i_l}\}) < \varepsilon/2$. Let $\mathcal{S}$ be the family of all simplexes obtained in this way; clearly, $\text{mesh } \mathcal{S} < \varepsilon/2$. The union $L = \bigcup \mathcal{S}$ is a polyhedron of dimension $\leq n$; moreover, one can assume that $L = |\mathcal{L}|$, where $\mathcal{L}$ is a simplicial complex every simplex of which is contained in a member of $\mathcal{S}$, so that we have $\text{mesh } \mathcal{L} < \varepsilon/2$. It follows from (12) and Theorem 1.10.6 applied to the embedding of X in R^m that the κ -mapping $\kappa: X \rightarrow R^m$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \dots, p_k$ has the property that

$$(13) \quad \varrho(x, \kappa(x)) < \varepsilon/4 \quad \text{for every } x \in X.$$

Moreover, as one easily sees, $A = \kappa(X) \subset L$. By virtue of the sweeping out theorem, there exists a continuous mapping $g: A \rightarrow K$ of A onto a polyhedron $K \subset L$ of dimension $\leq n$ such that

$$(14) \quad \varrho(\kappa(x), g\kappa(x)) < \varepsilon/2 \quad \text{for every } x \in X.$$

It follows from (13) and (14) that the composition $f = g\kappa: X \rightarrow K$ is the required ε -translation. $\square$

Theorems 1.10.12 and 1.10.18 yield the following

1.10.19. Theorem. *A compact subspace X of Euclidean m -space R^m satisfies the inequality $\dim X \leq n$ if and only if for every positive number ε there exists an ε -translation $f: X \rightarrow K$ of X onto a polyhedron $K \subset R^m$ of dimension $\leq n$. $\square$*

We shall now describe a two-dimensional subspace X of the cube I^3 such that for every positive number ε there exists an ε -translation $f: X \rightarrow K$ of X to a polyhedron $K \subset R^3$ of dimension ≤ 1 . In this way it will be proved that in the theorem on ε -translations the assumption of compactness is essential. Let us at once note that the space X has finite open covers of order ≤ 1 by sets with arbitrarily small diameters (cf. Theorem 1.6.12).

We start with a lemma on decompositions of continua, which will be applied to evaluate the dimension of the space X . The lemma states an important topological fact and is known as *Sierpiński's theorem*; it is preceded by two technical lemmas.

1.10.20. Lemma. *If A is a closed subspace of a continuum X such that $\emptyset \neq A \neq X$, then for every component C of the space A we have $C \cap \text{Fr} A \neq \emptyset$, where $\text{Fr} A$ is the boundary of A in X .*

Proof. Assume that $C \cap \text{Fr} A = \emptyset$ and consider the family $\{U_s\}_{s \in S}$ of all open-and-closed subsets of A which contain the component C ; it follows from Lemma 1.4.4 that $\bigcap_{s \in S} U_s = C$. The subspace $F = \text{Fr} A$ of the space X is compact, and the family $\{F \setminus U_s\}_{s \in S}$ is an open cover of F ; thus there exists a finite number of indexes $s_1, s_2, \dots, s_k \in S$ such that $F \subset \bigcup_{i=1}^k (F \setminus U_{s_i}) = F \setminus \bigcap_{i=1}^k U_{s_i}$. The set $U = \bigcap_{i=1}^k U_{s_i}$ is disjoint from F , i.e., $U \subset \text{Int} A$; being open-and-closed in A , the set U is open-and-closed in the continuum X . Now, $\emptyset \neq C \subset U$, so that $U = X$, and thus $\text{Fr} A = \emptyset$, which contradicts the connectedness of X . $\square$

1.10.21. Lemma. *If a continuum X is represented as the union of a sequence $X_1, X_2, \dots$ of pairwise disjoint closed sets of which at least two are non-empty, then for every natural number i there exists a continuum $C \subset X$ such that $C \cap X_i = \emptyset$ and at least two sets in the sequence $C \cap X_1, C \cap X_2, \dots$ are non-empty.*

Proof. If $X_i = \emptyset$, we let $C = X$; hence, we can assume that $X_i \neq \emptyset$. Consider a natural number $j \neq i$ such that $X_j \neq \emptyset$ and a pair $U, V \subset X$ of disjoint open sets such that $X_i \subset U$ and $X_j \subset V$. Let x be a point in X_j and C the component of the space $\bar{V}$ which contains the point x . Clearly, C is a continuum, $C \cap X_i = \emptyset$ and $C \cap X_j \neq \emptyset$. Since, by virtue of Lemma 1.10.20, $C \cap \text{Fr } \bar{V} \neq \emptyset$, and since $X_j \subset \text{Int } \bar{V}$, there exists a natural number $k \neq j$ such that $C \cap X_k \neq \emptyset$. $\square$

1.10.22. Lemma. *If a continuum X is represented as the union of a sequence $X_1, X_2, \dots$ of pairwise disjoint closed sets, then at most one of the sets X_i is non-empty.*

Proof. Assume that $X = \bigcup_{i=1}^{\infty} X_i$, where the sets X_i are closed, $X_i \cap X_j = \emptyset$ whenever $i \neq j$, and at least two of the sets X_i are non-empty. From Lemma 1.10.21 it follows that there exists a decreasing sequence $C_1 \supset C_2 \supset \dots$ of non-empty continua contained in X such that $C_i \cap X_i = \emptyset$ for $i = 1, 2, \dots$. Thus $\bigcap_{i=1}^{\infty} C_i = (\bigcap_{i=1}^{\infty} C_i) \cap (\bigcup_{i=1}^{\infty} X_i) = \emptyset$, which contradicts the compactness of X . $\square$

1.10.23. Sitnikov's example. For every natural number i consider the family of all planes in R^3 determined by equations of the form $x_j = z/i$, where $j = 1, 2, 3$ and z is an arbitrary integer. The planes yield a decomposition of R^3 into congruent cubes whose edges have length $1/i$ and whose interiors are pairwise disjoint; denote by $\mathcal{K}_i$ the family of all the cubes thus obtained and by A_i the union of all the edges of cubes in $\mathcal{K}_i$.

Let $B_1 = A_1$ and for $i = 2, 3, \dots$ translate the set A_i as a rigid body to obtain a set B_i disjoint from the union $B_1 \cup B_2 \cup \dots \cup B_{i-1}$ of all the sets previously defined. Clearly, the sets B_i are closed in R^3 and the union $B = \bigcup_{i=1}^{\infty} B_i$ is dense in R^3 . We shall show that the subspace $X = I^3 \setminus B$ of the cube I^3 is two-dimensional and yet for every positive number ε there exists an ε -translation $f: X \rightarrow K$ of X to a polyhedron $K \subset R^3$ of dimension ≤ 1 .

Since the set $I^3 \setminus X = I^3 \cap B$ is dense in I^3 , it follows from Corollary 1.8.11 that $\text{ind } X \leq 2$. Assume that $\text{ind } X \leq 1$ and consider arbitrary points

x and y in the interior G of I^3 in R^3 such that $x \in B_2$ and $y \in B_3$. By virtue of Mazurkiewicz's theorem there exists a continuum $C \subset G \setminus X = G \cap B$ which contains x and y ; the continuum C is the union of the sequence $C \cap B_1, C \cap B_2, \dots$ of pairwise disjoint closed sets of which at least two are non-empty. The contradiction of Sierpiński's theorem shows that $\text{ind} X = 2$.

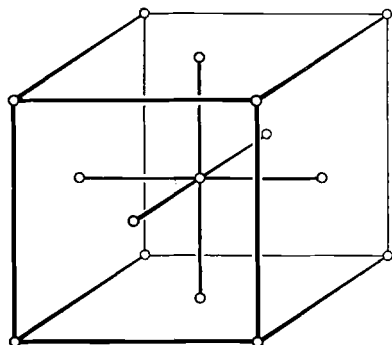


Fig. 11

Now, let ε be an arbitrary positive number. Consider a natural number i such that $4/i < \varepsilon$ and translate the set $X \cup B_i$ as a rigid body to make the joints of B_i coincide with the centres of the cubes in $\mathcal{K}_i$. One can assume that the translation $g: X \cup B_i \rightarrow R^3$ does not shift points by more than $2/i$, i.e., that $\varrho(x, g(x)) \leq 2/i$ for $x \in X$; clearly $g(X) \subset R^3 \setminus g(B_i)$. Consider now a fixed cube $T \in \mathcal{K}_i$. One readily checks that the projection from the centre of T onto the boundary of T maps the set $T \cap g(X)$ onto a subset of the boundary which does not contain the centres of the faces of T . On the boundary of T the projection coincides with the identity mapping, so that by performing such projections simultaneously on all the cubes in $\mathcal{K}_i$ we obtain a continuous mapping p_1 that sends $g(X)$ to the union of all faces of cubes in $\mathcal{K}_i$ in such a way that the centres of the faces do not belong to $p_1 g(X)$. Now, consider a fixed face S of T . The projection from the centre of S onto the boundary of S maps the set $S \cap p_1 g(X)$ onto a subset of A_i and coincides with the identity mapping on the boundary of S . By performing such projections simultaneously on all faces of cubes in $\mathcal{K}_i$ we obtain a continuous mapping p_2 that sends $p_1 g(X)$ to A_i . Since the points $g(x)$ and $p_2 p_1 g(x)$ lie in the same cube of $\mathcal{K}_i$, we have $\varrho(g(x), p_2 p_1 g(x)) < 2/i$. Hence $\varrho(x, p_2 p_1 g(x)) < 4/i < \varepsilon$ for every $x \in X$ and the set $p_2 p_1 g(X)$ is contained in a one-dimensional polyhedron $K \subset A_i$;

thus the required ε -translation $f: X \rightarrow K$ is defined by letting $f(x) = p_2 p_1 g(x)$ for every $x \in X$.

Though $\dim X = \text{ind} X = 2$, the space X has, for every positive number ε , a finite open cover $\mathcal{U}$ such that $\text{mesh} \mathcal{U} < \varepsilon$ and $\text{ord} \mathcal{U} \leq 1$. Indeed, if $f: X \rightarrow K$ is an $\varepsilon/3$ -translation of X to a polyhedron $K \subset R^3$ of dimension ≤ 1 , then the family $\mathcal{U} = f^{-1}(\mathcal{V})$, where $\mathcal{V}$ is a finite open cover of K such that $\text{mesh} \mathcal{V} < \varepsilon/3$ and $\text{ord} \mathcal{V} \leq 1$, has the required properties. $\square$

Historical and bibliographic notes

Nerves of covers were introduced and studied by Alexandroff in [1927], and κ -mappings—by Kuratowski in [1933a]. The discovery of these two notions was a turning point in the development of dimension theory, and even of the whole of topology; it made possible the combining of the point-set methods of general topology and the combinatorial methods of traditional algebraic topology. Theorem 1.10.13 was proved by Kuratowski in [1933a] and Theorem 1.10.14 by Alexandroff in [1928]. Theorem 1.10.15 for compact subspaces of polyhedra was obtained by Alexandroff in [1928] (cf. Problem 1.10.H); the extension to arbitrary subspaces was given by Kuratowski in [1933a]. Theorems 1.10.18 and 1.10.19 were proved by Alexandroff in [1928] (announcement in [1926]). Example 1.10.23 was described by Sitnikov in [1953].

Problems

1.10.A. (a) Check that the diameter of the simplex $p_0 p_1 \dots p_n$ is equal to the diameter of the set $\{p_0, p_1, \dots, p_n\}$ of its vertices.

(b) Prove that the simplex $p_0 p_1 \dots p_n$ is the smallest convex set which contains the points $p_0, p_1, \dots, p_n$.

(c) Show that every simplex determines its vertices, i.e., that if $p_0 p_1 \dots p_n = q_0 q_1 \dots q_n$, then $\{p_0, p_1, \dots, p_n\} = \{q_0, q_1, \dots, q_n\}$.

Hint. For every point x of a simplex S which is not a vertex of S the set $S \setminus \{x\}$ is not convex.

1.10.B. Prove that if $\mathcal{K}$ and $\mathcal{L}$ are simplicial complexes such that $|\mathcal{K}| = |\mathcal{L}|$ and $\mathcal{K}$ contains an n -simplex, then $\mathcal{L}$ also contains an n -simplex.

Hint. If $m < n$, then every m -simplex contained in an n -simplex S is nowhere dense in S .

1.10.C. Show that if the simplicial complexes $\mathcal{N}_1$ and $\mathcal{N}_2$ are nerves of the same finite open cover, then the underlying polyhedra $|\mathcal{N}_1|$ and $|\mathcal{N}_2|$ are homeomorphic.

1.10.D. Applying the fact that the one-dimensional polyhedron in Fig. 14 on p. 127 cannot be embedded in the plane (see Example 1.11.8), define a finite open cover $\mathcal{U}$ of the interval I such that $\text{ord}\mathcal{U} = 1$ and yet $\mathcal{U}$ has no nerve in R^2 .

1.10.E. (a) Observe that if for every finite open cover $\mathcal{E}$ of a normal space X there exists an $\mathcal{E}$ -mapping $f: X \rightarrow K$ to a zero-dimensional polyhedron such that the cover $\{f^{-1}(y)\}_{y \in Y}$ is a refinement of $\mathcal{E}$, then $\dim X \leq 0$.

(b) For every natural number $n \geq 2$ define a separable metric space X such that $\dim X = n$ and for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$ -mapping $f: X \rightarrow I$ such that the cover $\{f^{-1}(t)\}_{t \in I}$ is a refinement of $\mathcal{E}$.

Hint. Apply Problem 1.4.F(b).

1.10.F (Alexandroff [1932]). (a) Let $f: X \rightarrow B^{n+1}$ be an essential mapping of a compact metric space X to the $(n+1)$ -ball (cf. Problem 1.9.A) and ε a positive number less than 1. Show that if a continuous mapping $g: X \rightarrow B^{n+1}$ satisfies $\varrho(f(x), g(x)) < \varepsilon$ for every $x \in f^{-1}(S^n)$, then the image $g(X)$ contains the ball of radius $1-\varepsilon$ concentric with B^{n+1} .

(b) Apply (a) and Theorem 1.10.6 to show that if a compact metric space X satisfies the inequality $\dim X \leq n \geq 0$, then no continuous mapping $f: X \rightarrow B^{n+1}$ is essential (see Problem 1.9.A).

(c) Deduce from the theorem on $\mathcal{E}$ -mappings that if a compact metric space X satisfies the equality $\dim X = n \geq 0$, then there exists an essential mapping $f: X \rightarrow B^n$.

1.10.G (Alexandroff [1928a], Chogoshvili [1938]). (a) Prove that a compact subspace X of Euclidean m -space R^m satisfies the inequality $\dim X \leq n$ if and only if for every polyhedron $K \subset R^m$ of dimension $m-n-1$ and every positive number ε there exists an ε -translation $f: X \rightarrow R^m$ such that $f(X) \cap K = \emptyset$, or—equivalently—if and only if for every linear $(m-n-1)$ -variety $H \subset R^m$ and every positive number ε there exists an ε -translation $f: X \rightarrow R^m$ such that $f(X) \cap H = \emptyset$.

Hint. Show that if X is a compact subspace of R^m and for every linear $(m-n-1)$ -variety $H \subset R^m$ and every positive number ε there exists an ε -translation $f: X \rightarrow R^m$ such that $f(X) \cap H = \emptyset$, then X is embeddable in N_n^m .

(b) Show that in (a) the assumption of compactness of X is essential.

(c) Applying the fact that in R^3 there exists an *Antoine set*, i.e., a subspace A homeomorphic to the Cantor set such that for a circle $S \subset R^3 \setminus A$ the embedding of S in $R^3 \setminus A$ is not homotopic to the constant mapping of S to a point of $R^3 \setminus A$ (see Rushing [1973], p. 71), show that in (a) the words “every polyhedron $K \subset R^m$ ” cannot be replaced by “every compact subspace K of R^m ”.

Hint. Consider the disk bounded by S .

1.10.H. Note that in the case where A is a compact subspace of $|\mathcal{X}|$, the sweeping out theorem can be proved in a simpler way.

Hint. Sweep out A consecutively from the interiors of all simplexes in $\mathcal{X}$ which are not contained in A starting with the simplexes of highest dimension.

1.10.I. Prove that the union and the intersection of a finite family of polyhedra contained in a Euclidean space also are polyhedra.

Hint. A bounded subset of R^m which can be represented as the intersection of a finite family of half-spaces in R^m is a geometric cell. Define the interior and the boundary of a geometric cell and the geometric dimension of a geometric cell. Prove that polyhedra can be defined as finite unions of geometric cells. To this end, by analogy to the notion of a simplicial complex, introduce the notion of a cellular complex, observe that every polyhedron can be represented as the union of all cells in a cellular complex, and—applying induction with respect to the maximal geometric dimension of cells—prove that every cellular complex has a subdivision which is a simplicial complex (all the details are worked out in Alexandroff and Hopf’s book [1935], pp. 124ff.).

1.10.J (Sitnikov [1953]). Modify the construction in Example 1.10.23 to obtain, for every natural number $n \geq 4$, an $(n-1)$ -dimensional subspace X of the n -cube I^n such that for every positive number ε there exists an ε -translation $f: X \rightarrow K$ of X to a polyhedron $K \subset R^n$ of dimension $\leq k$, where $k = n/2$ if n is even, and $k = (n-1)/2$ if n is odd.

1.10.K. Prove that if X is a bounded subspace of Euclidean m -space R^m and X satisfies the inequality $\dim X \leq n$, then for every positive number ε there exists a finite open cover $\mathcal{U}$ of the space X such that $\text{mesh } \mathcal{U} < \varepsilon$, $\text{ord } \mathcal{U} \leq n$, and $\mathcal{U}$ has a nerve in R^m .

1.10.L (Smirnov [1956]). Show that a bounded subspace X of Euclidean m -space R^m satisfies the inequality $\mu \dim(X, \varrho) \leq n$, where $\mu \dim$ is the metric dimension defined in Section 1.6 and ϱ is the natural metric on R^m , if and only if for every positive number ε there exists an ε -translation $f: X \rightarrow K$ of X onto a polyhedron $K \subset R^m$ of dimension $\leq n$.

Hint. Apply Problem 1.6.C.

1.11. The embedding and universal space theorems

The considerations in the present section largely apply the notion of a function space. Let us recall that if (X, σ) is a non-empty compact metric space and (Y, ϱ) an arbitrary metric space, then by letting

$$\hat{\varrho}(f, g) = \sup_{x \in X} \varrho(f(x), g(x)) \quad \text{for } f, g \in Y^X$$

we define a metric $\hat{\varrho}$ on the set Y^X of all continuous mappings of X to Y ; the metric space $(Y^X, \hat{\varrho})$ obtained in this way is a *function space*. One readily shows that if (X, σ) is a compact metric space and (Y, ϱ) is a complete metric space, then the function space $(Y^X, \hat{\varrho})$ is complete. As the reader will see, this simple observation leads, via the Baire category theorem, to important applications of function spaces. We begin with three lemmas on function spaces.

1.11.1. Lemma. *For every positive number ε the set of all ε -mappings is open in the function space Y^X .*

Proof. Consider an ε -mapping $f: X \rightarrow Y$. The closed subspace

$$A = \{(x, x') \in X \times X: \sigma(x, x') \geq \varepsilon\}$$

of the Cartesian product $X \times X$ is compact, so that, since for each pair $(x, x') \in A$ the inequality $\varrho(f(x), f(x')) > 0$ holds, there exists a positive number δ such that

$$(1) \quad \varrho(f(x), f(x')) \geq \delta \quad \text{for each } (x, x') \in A.$$

To complete the proof it suffices to observe that every continuous mapping $g: X \rightarrow Y$ such that $\hat{\varrho}(f, g) < \delta/2$ is an ε -mapping. Indeed, if $g(x) = g(x')$, then $\varrho(f(x), f(x')) < \delta$ which implies by virtue of (1) that $(x, x') \notin A$, i.e., $\sigma(x, x') < \varepsilon$; thus $\delta(g^{-1}(y)) < \varepsilon$ for every $y \in Y$. $\square$

1.11.2. Lemma. *For every closed set $F \subset Y$ the set $\{f \in Y^X: f(X) \cap F = \emptyset\}$ is open in the function space Y^X .*

Proof. Consider a continuous mapping $f: X \rightarrow Y$ such that $f(X) \cap F = \emptyset$. As $f(X)$ is a compact subspace of Y , the distance $\delta = \varrho(f(X), F)$ is positive. To complete the proof it suffices to note that for every continuous mapping $g: X \rightarrow Y$ such that $\hat{\varrho}(f, g) < \delta$ the relation $g(X) \cap F = \emptyset$ holds. $\square$

1.11.3. Lemma. *If X is a compact metric space such that $0 \leq \dim X \leq n$ and H is a linear n -variety in R^{2n+1} , then for every positive number ε the set of all ε -mappings of X to R^{2n+1} whose values miss H is dense in the function space $(R^{2n+1})^X$; in particular, the set of all ε -mappings of X to R^{2n+1} is dense in the function space $(R^{2n+1})^X$.*

Proof. Consider an arbitrary continuous mapping $f: X \rightarrow Y$ and a positive number δ . It follows from the compactness of X that the mapping f is uniformly continuous; thus there exists a positive number η such that $\delta(f(A)) < \delta$ whenever $\delta(A) < \eta$. By virtue of Theorem 1.6.12 there exists a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space X such that $\text{mesh } \mathcal{U} < \min(\varepsilon, \eta)$, $\text{ord } \mathcal{U} \leq n$ and the sets U_i are non-empty; obviously, $\delta(f(U_i)) < \delta$ for $i = 1, 2, \dots, k$. Choose a point $q_i \in f(U_i)$ for $i = 1, 2, \dots, k$, and apply Theorem 1.10.4 to obtain a nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$, consisting of simplexes contained in R^{2n+1} and such that $H \cap \mathcal{N}(\mathcal{U}) = \emptyset$ and the vertices $p_1, p_2, \dots, p_k$ of $\mathcal{N}(\mathcal{U})$ satisfy the inequality $\varrho(p_i, q_i) < \alpha$ for $i = 1, 2, \dots, k$, where $\alpha = \min\{\delta - \delta(f(U_i)): i = 1, 2, \dots, k\}$. We have

$$\delta(\{p_i\} \cup f(U_i)) \leq \varrho(p_i, q_i) + \delta(f(U_i)) < \alpha + \delta(f(U_i)) \leq \delta$$

for $i = 1, 2, \dots, k$.

From Theorem 1.10.6 it follows that the κ -mapping $\kappa: X \rightarrow R^{2n+1}$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \dots, p_k$ satisfies the inequality $\hat{\varrho}(f, \kappa) < \delta$. To conclude the proof it suffices to observe that by virtue of Theorem 1.10.7 we have $\kappa(X) \subset \mathcal{N}(\mathcal{U})$ and $\kappa^{-1}(\text{St}_{\mathcal{N}(\mathcal{U})}(p_i)) = U_i$ for $i = 1, 2, \dots, k$, so that $\kappa(X) \cap H = \emptyset$ and κ is an ε -mapping. $\square$

The above lemmas will be applied in the proof of the embedding theorem and in the proof of the first universal space theorem.

1.11.4. The embedding theorem. *Every separable metric space X such that $0 \leq \dim X \leq n$ is embeddable in Euclidean $(2n+1)$ -space R^{2n+1} ; if, moreover, the space X is compact, then all homeomorphic embeddings of X in R^{2n+1} form a G_δ -set dense in the function space $(R^{2n+1})^X$.*

Proof. First, consider a compact metric space X such that $0 \leq \dim X \leq n$. Let Φ_i denote for $i = 1, 2, \dots$ the subset of the function space $(R^{2n+1})^X$ consisting of all $(1/i)$ -mappings; it follows from Lemmas 1.11.1 and 1.11.3 that the sets Φ_i are open and dense in $(R^{2n+1})^X$. By virtue of the Baire category theorem, the G_δ -set $\Phi = \bigcap_{i=1}^{\infty} \Phi_i$ is dense in the function space $(R^{2n+1})^X$. Since the space X is compact, a continuous mapping $f: X \rightarrow R^{2n+1}$ is a homeomorphic embedding if and only if it is a one-to-one mapping, i.e., if f is an ε -mapping for every positive number ε . Thus Φ is the set of all homeomorphic embeddings of X in R^{2n+1} .

Now, consider a separable metric space X such that $0 \leq \dim X \leq n$. By virtue of the compactification theorem, there exists a compact metric space $\tilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequality $\dim \tilde{X} \leq n$. As established above, $\tilde{X}$ is embeddable in R^{2n+1} , and so X is also embeddable in R^{2n+1} . $\square$

We shall show in Example 1.11.8 below that the exponent $2n+1$ in the embedding theorem cannot be lowered.

1.11.5. The first universal space theorem. *The subspace N_n^{2n+1} of Euclidean $(2n+1)$ -space R^{2n+1} consisting of all points which have at most n rational coordinates is a universal space for the class of all separable metric spaces whose covering dimension is not larger than n .*

Proof. It follows from Theorem 1.8.5 and the coincidence theorem that $\dim N_n^{2n+1} = n$; hence—by virtue of the compactification theorem—it suffices to prove that every compact metric space X such that $0 \leq \dim X \leq n$ is embeddable in N_n^{2n+1} .

The complement $R^{2n+1} \setminus N_n^{2n+1} = L_{n+1}^{2n+1}$ can be represented as the union of a countable family of linear n -varieties in R^{2n+1} , viz., of all sets defined by conditions of the form $x_{i_1} = r_1, x_{i_2} = r_2, \dots, x_{i_{n+1}} = r_{n+1}$, where $1 \leq i_1 < i_2 < \dots < i_{n+1} \leq 2n+1$ and $r_1, r_2, \dots, r_{n+1}$ are arbitrary

rational numbers. Arrange all linear n -varieties in the family under consideration into a sequence $H_1, H_2, \dots$ and for $i = 1, 2, \dots$ denote by Φ_i the subset of the function space $(R^{2n+1})^X$ consisting of all $(1/i)$ -mappings whose values miss H_i . It follows from Lemmas 1.11.1–1.11.3 that the intersection $\Phi = \bigcap_{i=1}^{\infty} \Phi_i$ is a G_δ -set dense in the function space $(R^{2n+1})^X$, and from the compactness of X it follows that Φ consists of homeomorphic embeddings. $\square$

Let us observe that the last paragraph incidentally yields another proof of the inequality $\text{ind} X \leq \dim X$ for compact metric spaces; as announced in Section 1.7, the auxiliary invariant $d(X)$ is the smallest integer $n \geq 0$ such that X is embeddable in the space N_n^{2n+1} .

The space N_n^{2n+1} is called *Nöbeling's universal n -dimensional space*. Obviously, Nöbeling's universal 0-dimensional space is the space of irrational numbers; thus the last theorem extends to higher dimensions our earlier result that the space of irrational numbers is universal for the class of all zero-dimensional separable metric spaces. As the reader remembers, the Cantor set is another universal space for the same class of spaces. We shall now describe the n -dimensional counterpart of the latter universal space, i.e., *Menger's universal n -dimensional space* M_n^{2n+1} . The construction will be carried out under more general circumstances: for every pair n, m of integers satisfying $0 \leq n \leq m \geq 1$ we shall define a compact subspace M_n^m of Euclidean m -space R^m .

For $i = 0, 1, 2, \dots$ let $\mathcal{K}_i$ be the family of 3^{mi} congruent cubes obtained by dividing the m -cube I^m by all linear $(m-1)$ -varieties in R^m determined by equations of the form $x_j = k/3^i$, where $j = 1, 2, \dots, m$ and $0 \leq k \leq 3^i$. For every family $\mathcal{K}$ of cubes let

$$|\mathcal{K}| = \bigcup \{K: K \in \mathcal{K}\} \quad \text{and} \quad \mathcal{S}_n(\mathcal{K}) = \bigcup \{\mathcal{S}_n(K): K \in \mathcal{K}\},$$

where $\mathcal{S}_n(K)$ is the family of all faces of K which have dimension $\leq n$; moreover, for $\mathcal{K} \subset \mathcal{K}_i$ let

$$\mathcal{K}' = \{K \in \mathcal{K}_{i+1}: K \subset |\mathcal{K}|\}.$$

For every pair n, m of integers satisfying $0 \leq n \leq m \geq 1$ define inductively a sequence $\mathcal{F}_0, \mathcal{F}_1, \dots$ of finite collections of cubes, where $\mathcal{F}_i \subset \mathcal{K}_i$ for $i = 0, 1, 2, \dots$, and a decreasing sequence $F_0 \supset F_1 \supset F_2 \supset \dots$ of closed subsets of I^m by letting $\mathcal{F}_0 = \{I^m\}$, $F_0 = |\mathcal{F}_0| = I^m$ and

$$F_i = |\mathcal{F}_i|, \quad \text{where } \mathcal{F}_i = \{K \in \mathcal{F}'_{i-1}: K \cap |\mathcal{S}_n(\mathcal{F}_{i-1})| \neq \emptyset\},$$

for $i = 1, 2, \dots$. The intersection

$$M_n^m = \bigcap_{i=0}^{\infty} F_i \subset I^m$$

is a compact subspace of I^m . One readily sees that the construction of the Cantor set described in Example 1.2.5 is a special case of the above construction, viz., the case where $m = 1$ and $n = 0$; thus the space M_0^1 is the Cantor set. One proves that the spaces M_0^m are all homeomorphic to the Cantor set (see Problem 1.11.D(a)); obviously, $M_m^m = I^m$ for $m = 1, 2, \dots$. In Figs. 12 and 13 the first three steps in constructing M_1^2 and M_1^3 are exhibited.

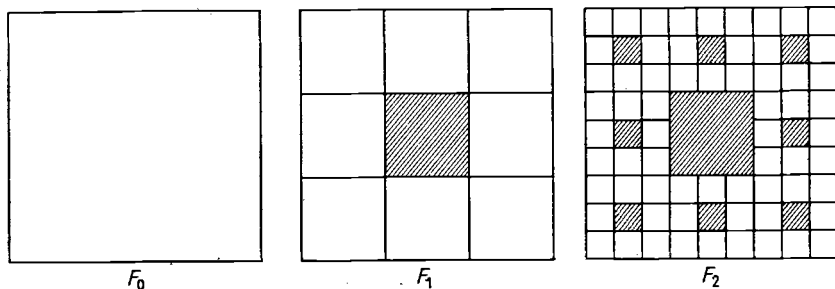


Fig. 12

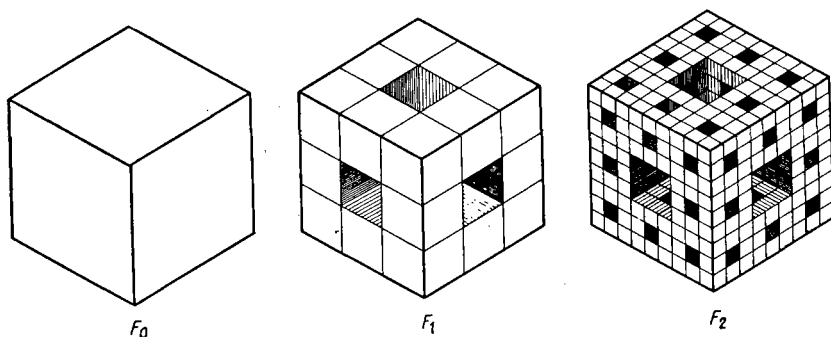


Fig. 13

Let us observe that $\dim M_n^m = n$. Indeed, the inequality $\dim M_n^m \geq n$ follows directly from the inclusion $|\mathcal{S}_n(\mathcal{F}_0)| \subset M_n^m$ and the reverse inequality is a consequence of Theorem 1.10.12, because the set F_i , and, *a fortiori*, the set M_n^m , can be translated by a $(\sqrt{m}/2 \cdot 3^{i-1})$ -mapping to the space $|\mathcal{S}_n(\mathcal{F}_{i-1})|$ which has dimension $\leq n$. The construction of such a mapping is left to the reader; it should be defined separately on each set of the form $F_i \cap K$, where $K \in \mathcal{F}_{i-1}$.

To prove that the space M_n^{n+1} is a universal space for the class of all separable metric spaces of dimension $\leq n$, rather laborious computations are necessary; some of them will be left to the reader. Since the same argumentation yields an interesting theorem exhibiting a relationship between the spaces N_n^m and M_n^m , we shall prove this theorem, and then deduce from it the universality property of M_n^{2n+1} .

1.11.6. Theorem. *Every compact subspace X of the space N_n^m is embeddable in the space M_n^m .*

Proof. We shall consider on I^m the metric σ defined by letting

$$\sigma(x, y) = \max\{|x_j - y_j| : j = 1, 2, \dots, m\},$$

$$\text{where } x = \{x_j\} \text{ and } y = \{y_j\};$$

obviously, the metric σ is equivalent to the natural metric on I^m . Let us note that if two points x and y are contained in the same cube $K \in \mathcal{K}_i$, then $\sigma(x, y) \leq 1/3^i$. For $i = 1, 2, \dots$ denote by S_i the subset of I^m consisting of all points which have, at least $n+1$ coordinates of the form $(k/3^i) + 1/2 \cdot 3^i$, where $0 \leq k \leq 3^{i-1}$, and by T_i the subset of I^m consisting of all points which have at least $m-n$ coordinates of the form $k/3^i$, where $0 \leq k \leq 3^i$; obviously, $T_i = |\mathcal{S}_n(\mathcal{K}_i)|$. One readily checks that for $i = 0, 1, 2, \dots$

$$(2) \quad \sigma(S_i, T_i) = 1/2 \cdot 3^i$$

and

$$(3) \quad F_{i+1} = F_i \setminus B(S_i, 1/2 \cdot 3^{i+1}),$$

where $B(A, r)$ denotes the r -ball about A with respect to the metric σ .

The proof consists in defining by induction a sequence $f_0, f_1, f_2, \dots$ of homeomorphisms of I^m onto itself which transform the intersection $I^m \cap N_n^m$ to N_n^m and map X consecutively to $F_0, F_1, F_2, \dots$, and which uniformly converge to a homeomorphism that maps X to the intersection

$$\bigcap_{i=0}^{\infty} F_i = M_n^m. \text{ In the inductive step one observes that the set } f_i(X) \text{ is dis-}$$

joint from S_i , and one modifies f_i to f_{i+1} by sweeping out the set $f_i(X)$ from the ball $B(S_i, 1/2 \cdot 3^{i+1})$. The modification of f_i to f_{i+1} is performed separately on each coordinate axis and is described by a piecewise linear homeomorphism h_i of I onto itself.

We shall now define inductively a sequence $h_0, h_1, h_2, \dots$ of homeomorphisms of I onto itself and a sequence $f_0, f_1, f_2, \dots$ of homeomorphisms of I^m onto itself such that for $i = 0, 1, 2, \dots$ the following conditions will be satisfied:

(4) The interval I can be divided into finitely many closed intervals with pairwise disjoint interiors in such a way that on each of these intervals h_i is a linear function with slope $\geq 2/3$.

$$(5) \quad h_i(k/2 \cdot 3^i) = k/2 \cdot 3^i \quad \text{for } 0 \leq k \leq 2 \cdot 3^i.$$

$$(6) \quad f_{i+1} = H_i f_i, \text{ where } H_i((x_1, x_2, \dots, x_m)) = (h_i(x_1), h_i(x_2), \dots, h_i(x_m)).$$

$$(7) \quad f_i(I^m \cap N_n^m) \subset N_n^m.$$

$$(8) \quad f_i(X) \subset F_i.$$

Without loss of generality one can assume that $X \subset I^m$. Hence, if we let $f_0 = \text{id}_{I^m}$, conditions (7) and (8) are satisfied for $i = 0$. Assume that the homeomorphisms $f_0, f_1, \dots, f_i$ and $h_0, h_1, \dots, h_{i-1}$ with all the required properties are already defined. We shall first define a homeomorphism h_i which satisfies (4) and (5) and then show that the homeomorphism f_{i+1} defined in (6) satisfies (7) and (8) with i replaced by $i+1$.

The relations $f_i(X) \subset f_i(I \cap N_n^m) \subset N_n^m$ and $S_i \cap N_n^m = \emptyset$ imply that $f_i(X) \cap S_i = \emptyset$; since $f_i(X)$ is a compact space, there exists a positive rational number ε such that

$$0 < \varepsilon < \min \left(\sigma(f_i(X), S_i), \frac{1}{2 \cdot 3^i} - \frac{1}{3^{i+1}} \right).$$

Consider the division of the interval I into closed intervals with pairwise disjoint interiors determined by the points

$$0 = b_0 < a_0 < b_1 < a_1 < \dots < a_{3^i-1} < b_{3^i} < a_{3^i} = 1,$$

where

$$a_k = \frac{k}{3^i} + \frac{1}{2 \cdot 3^i} - \varepsilon \quad \text{and} \quad b_{k+1} = \frac{k}{3^i} + \frac{1}{2 \cdot 3^i} + \varepsilon$$

for $k = 0, 1, \dots, 3^i - 1$. The reader can easily check that the functions $g'_0, g'_1, \dots, g'_{3^i}$, where

$$g'_k(t) = \frac{1}{3^{i+1}} \left(\frac{1}{2 \cdot 3^i} - \varepsilon \right)^{-1} \left(t - \frac{k}{3^i} \right) + \frac{k}{3^i} \quad \text{for } b_k \leq t \leq a_k,$$

and the functions $g_0'', g_1'', \dots, g_{3^i-1}'',$ where

$$g_k''(t) = \varepsilon^{-1} \left(\frac{1}{2 \cdot 3^i} - \frac{1}{3^{i+1}} \right) \left[t - \left(\frac{k}{3^i} + \frac{1}{2 \cdot 3^i} \right) \right] + \frac{k}{3^i} + \frac{1}{2 \cdot 3^i}$$

for $a_k \leq t \leq b_{k+1},$

satisfy the equalities:

$$(9) \quad g_k'(a_k) = g_k''(a_k) = \frac{k}{3^i} + \frac{1}{3^{i+1}},$$

$$g_k''(b_{k+1}) = g_{k+1}'(b_{k+1}) = \frac{k}{3^i} + \frac{1}{3^i} - \frac{1}{3^{i+1}}$$

for $k = 0, 1, \dots, 3^i - 1.$ As $g_0'(0) = 0$ and $g_{3^i-1}'(1) = 1,$ the functions g_k' and g_k'' determine a homeomorphism h_i of I onto itself which, as one easily checks, satisfies (4) and (5).

Consider now the homeomorphism f_{i+1} defined in (6). Since ε is a rational number, the homeomorphism h_i transforms each rational number in I into a rational number, so that by virtue of (7) we have $f_{i+1}(I^m \cap N_n^m) \subset N_n^m.$

Let x be a point in X ; by virtue of (8), $f_i(x) = (t_1, t_2, \dots, t_m) \in F_i.$ Since $\sigma(f_i(X), S_i) > \varepsilon,$ there exist $m-n$ coordinates of the point $(t_1, t_2, \dots, t_m),$ say $t_{j_1}, t_{j_2}, \dots, t_{j_{m-n}},$ and $m-n$ non-negative integers $k_1, k_2, \dots, k_{m-n} \leq 3^i$ such that

$$\left| t_{j_l} - \frac{k_l}{3^i} \right| < \frac{1}{2 \cdot 3^i} - \varepsilon \quad \text{for } l = 1, 2, \dots, m-n.$$

The last inequality means that $b_{k_l} \leq t_{j_l} \leq a_{k_l},$ which together with (9) yield

$$\frac{k_l}{3^i} - \frac{1}{3^{i+1}} \leq h_i(t_{j_l}) \leq \frac{k_l}{3^i} + \frac{1}{3^{i+1}} \quad \text{for } l = 1, 2, \dots, m-n,$$

so that

$$(10) \quad f_{i+1}(x) = (h_i(t_1), h_i(t_2), \dots, h_i(t_m)) \in B(T_i, 1/3^{i+1}).$$

From (2) and (10) it follows that

$$(11) \quad f_{i+1}(x) \notin B(S_i, 1/2 \cdot 3^{i+1}).$$

Since, by virtue of (5), the homeomorphism H_i maps each cube in $\mathcal{K}_i$ onto itself, $f_{i+1}(x) \in F_i;$ the last relation together with (11) and (3) show that $f_{i+1}(x) \in F_{i+1}.$ Hence we have $f_{i+1}(X) \subset F_{i+1}.$

It remains to check that the sequence of homeomorphisms $f_0, f_1, f_2, \dots$ uniformly converges to a homeomorphism f of I^m onto itself; the inclusion $f(X) \subset M_n^m$ will then follow from (8). This amounts to checking that the sequence $h_0, h_1 h_0, h_2 h_1 h_0, \dots$ of homeomorphisms of I onto itself uniformly converges to a homeomorphism of I onto itself. The last fact can be deduced from (4) and (5) by a straightforward computation, which we leave to the reader. $\square$

1.11.7. The second universal space theorem. *The compact subspace M_n^{2n+1} of Euclidean $(2n+1)$ -space R^{2n+1} is a universal space for the class of all separable metric spaces whose covering dimension is not larger than n .*

Proof. As observed above, $\dim M_n^{2n+1} = n$; hence—by virtue of the compactification theorem—it suffices to prove that every compact metric space X such that $\dim X \leq n$ is embeddable in M_n^{2n+1} . This is, however, an immediate consequence of Theorems 1.11.5 and 1.11.6. $\square$

In connection with the above universal space theorems one can ask whether there exists a universal space for the class of all subspaces of R^m which have dimension $\leq n$. It is an old hypothesis that M_n^m is such a space. Quite recently, it was proved that M_n^m is indeed a universal space for the class of all compact subspaces of R^m which have dimension $\leq n$; it is a very deep and difficult result. Hence, the hypothesis on the universality of M_n^m is now reduced to the question whether every subspace X of Euclidean m -space R^m has a dimension preserving compactification embeddable in R^m . Let us observe that in some special cases the universality of the space M_n^m can be deduced from our earlier results. Indeed, the universality of M_0^m and M_m^m is obvious, and the universality of M_n^{2n+1} is a consequence of Theorem 1.11.7; finally, Theorems 1.8.10, 1.8.8, and 1.11.6, together with the simple observation that each closed subset of N_{m-1}^m has a compactification embeddable in N_{m-1}^m , imply that the space M_{m-1}^m is universal for the class of all subspaces of R^m which have dimension $\leq m-1$ (a direct proof of this fact is outlined in the hint to Problem 1.11.D(c)).

The considerations of the preceding paragraph imply, in particular, that the space N_n^m is embeddable in the space M_n^m for $n = 0$, $n = m$, $m = 2n+1$ and $n = m-1$; the problem whether N_n^m is always embeddable in M_n^m is still open. On the other hand, the space M_n^m is embeddable in the space N_n^m for every pair of integers n, m satisfying $0 \leq n \leq m \geq 1$ (see Problem 1.11.E). Hence the question if N_n^m is a universal space for the

class of all subspaces of R^m which have dimension $\leq n$ is a weaker version of the problem of the universality of the space M_n^m .

To conclude, let us note that from the above discussion of special cases it follows that for every pair of integers n, m satisfying $0 \leq n \leq m \leq 3$ and $m \geq 1$, the spaces M_n^m and N_n^m are universal for the class of all subspaces of R^m which have dimension $\leq n$; in particular, for every such pair n, m , the space N_n^m is embeddable in the space M_n^m and each n -dimensional subspace of R^m has an n -dimensional compactification embeddable in R^m .

We close this section by showing that the exponent $2n+1$ in the embedding theorem cannot be lowered.

1.11.8. Example. Let K_1 be the union of all 1-faces of the 4-simplex $p_0p_1p_2p_3p_4$ (see Fig. 14). Applying the Jordan curve theorem, which states that every simple closed curve (i.e., a set homeomorphic to S^1) in the plane R^2 separates R^2 into two regions, we shall show that the one-dimensional polyhedron K_1 cannot be embedded in R^2 .

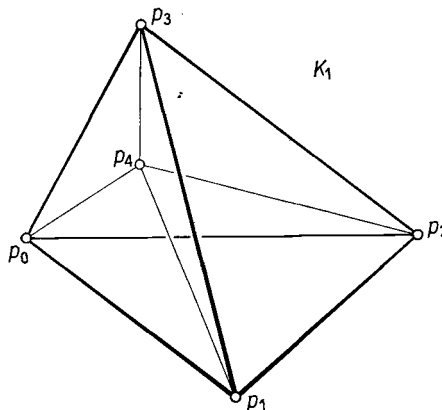


Fig. 14

Assume that there exists a homeomorphic embedding $f: K_1 \rightarrow R^2$ and define $a_i = f(p_i)$ for $0 \leq i \leq 4$. It follows from the Jordan curve theorem that the simple closed curve $S_1 = f(p_0p_1 \cup p_1p_2 \cup p_2p_0)$ separates R^2 into two regions. One can suppose that the point a_3 belongs to the bounded component U of $R^2 \setminus S_1$, otherwise one should replace the homeomorphic embedding f by the composition of f and a suitable inversion. Since the points a_3 and a_4 can be joined in $f(K_1)$ by an arc (i.e., a set homeomorphic to I) disjoint to S_1 , we have $a_4 \in U$. The Jordan curve theorem implies that the set $U \setminus T$, where $T = f(p_0p_3 \cup p_1p_3 \cup p_2p_3)$, can be repre-

sented as the union of three pairwise disjoint regions U_0, U_1, U_2 such that $a_i \notin U_i$ for $i = 0, 1, 2$. As $a_4 \notin T$, we have $a_4 \in U \setminus T$ and without loss of generality we can suppose that $a_4 \in U_0$. Now, the set $f(p_0 p_4)$ is an arc joining the points a_0, a_4 and disjoint to the simple closed curve $S_2 = f(p_1 p_2 \cup p_1 p_3 \cup p_2 p_3)$. Since the points a_0 and a_4 are contained in distinct components of $R^2 \setminus S_2$, the assumption that K_1 is embeddable in R^2 yields a contradiction. $\square$

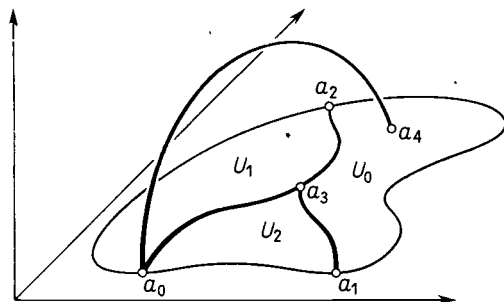


Fig. 15

One can prove that the union K_n of all n -faces of a $(2n+2)$ -simplex cannot be embedded in R^{2n} for any natural number n ; a proof of this fact, based on the Borsuk–Ulam antipodal theorem, is outlined in the hint to Problem 1.11.F.

Historical and bibliographic notes

The first part of Theorem 1.11.4 was formulated, for compact metric spaces, by Menger in [1926] and was proved there for $n = 1$. In [1928] Menger again proved the theorem for $n = 1$ and hinted at the modifications in the proof that should permit us to obtain the theorem in full generality. For an arbitrary n the first part of the embedding theorem was proved simultaneously by Nöbeling in [1931], Pontrjagin and Tolstowa in [1931], and Lefschetz in [1931]; the three proofs consisted in constructing a sequence of continuous mappings uniformly converging to a homeomorphic embedding and were rather involved. The present proof was given by Hurewicz in [1933] (announcement in [1931]); application of function spaces yielded the stronger result about the set of all homeomorphic embeddings. The consideration of function spaces and resorting to the Baire category theorem (proofs by category method) proved very useful in the dimension theory of separable metric spaces. This idea, originated by Hurewicz in [1931], was repeatedly exploited by Hurewicz, Kuratowski

(cf. Problem 1.7.D) and their followers. Theorem 1.11.5 was established by Nöbeling in [1931]. The spaces M_n^m were introduced by Menger in [1926]. They are generalizations of the Cantor set and of *Sierpiński's universal curve*, i.e., the space M_1^2 described by Sierpiński in [1916], where it was also proved that M_1^2 is a universal space for the class of all compact subspaces of the plane which have an empty interior (in [1922] Sierpiński observed that the assumption of compactness is not essential). In [1926] Menger proved that the space M_1^3 is universal for the class of all compact metric spaces of dimension ≤ 1 , observed that Sierpiński's argument yields the universality of M_{m-1}^m for the class of all compact subspaces of R^m which have dimension $\leq m-1$, announced Theorem 1.11.7 for compact spaces and put forward the hypothesis that the space M_n^m is universal for the class of all compact subspaces of R^m which have dimension $\leq n$. Theorem 1.11.6 was proved by Bothe in [1963]. In [1931] Lefschetz defined for every pair n, m of integers satisfying $0 \leq n \leq m \leq 1$ a compact subspace S_n^m of R^m which is very much like M_n^m (the difference consists in considering simplexes rather than cubes) and proved that S_n^{2n+1} is a universal space for the class of all separable metric spaces whose covering dimension is not larger than n ; he also proved there that the space S_n^m is embeddable in the space N_n^m . It is a general belief that the spaces M_n^m and S_n^m are homeomorphic, but no proof was ever produced. Hence, Lefschetz is considered to be the author of Theorem 1.11.7, although—formally—the theorem was first proved by Bothe in [1963]. Let us mention, by way of digression, that M_n^m and S_n^m are obviously homeomorphic if $n = 0$ or $n = m$, that M_1^2 is homeomorphic to S_1^2 and M_1^3 is homeomorphic to S_1^3 by virtue of topological characterizations of M_1^2 and M_1^3 given by Whyburn in [1958] and by Anderson in [1958], respectively, and finally that, as proved by Cannon in [1973], M_{m-1}^m and S_{m-1}^m are homeomorphic if $m \neq 4$. The theorem stating that M_n^m is a universal space for the class of all compact subspaces of R^m which have dimension $\leq n$ was proved by Štan'ko in [1971]. A proof of the Jordan curve theorem can be found in Kuratowski's book [1968], p. 510.

Problems

1.11.A (Kuratowski [1937a], Hurewicz and Wallman [1941]). (a) Check that if (X, σ) is a non-empty metric space and (Y, ϱ) is a compact metric space then the function $\hat{\varrho}$ defined at the beginning of this section is

a metric on the set Y^X of all continuous mappings of X to Y and the function space $(Y^X, \hat{\rho})$ is complete. Verify that for every open cover $\mathcal{E}$ of the space X the set of all $\mathcal{E}$ -mappings is open in the function space Y^X .

(b) Show that for every separable metric space X there exists a sequence $\mathcal{E}_1, \mathcal{E}_2, \dots$ of finite open covers of X such that every continuous mapping $f: X \rightarrow Y$ of X onto a topological space Y which is an $\mathcal{E}_i$ -mapping for $i = 1, 2, \dots$ is a homeomorphism.

Hint. Consider a countable base $\mathcal{B}$ for the space X and arrange into a sequence all covers of the form $\{W, X \setminus \bar{U}\}$, where $U, W \in \mathcal{B}$ and $\bar{U} \subset W$.

(c) Prove that for every finite open cover $\mathcal{E}$ of a separable metric space X such that $0 \leq \dim X \leq n$ the set of all $\mathcal{E}$ -mappings of X to I^{2n+1} is dense in the function space $(I^{2n+1})^X$.

(d) Deduce from (a), (b) and (c) that every separable metric space X such that $0 \leq \dim X \leq n$ is embeddable in the $(2n+1)$ -cube I^{2n+1} ; observe that the set of all homeomorphic embeddings of X in I^{2n+1} contains a G_δ -set dense in the function space $(I^{2n+1})^X$.

Remark. As opposed to the case where X is a compact space, the set of all homeomorphic embeddings of a separable metric space X such that $0 \leq \dim X \leq n$ in I^{2n+1} is generally not a G_δ -set (see Roberts [1948]).

1.11.B (Kuratowski [1937a], Hurewicz and Wallman [1941]). (a) Check that for every closed subset F of a compact metric space Y the set $\{f \in Y^X: f(X) \cap F = \emptyset\}$ is open in the function space Y^X .

(b) Prove that for every finite open cover $\mathcal{E}$ of a separable metric space X such that $0 \leq \dim X \leq n$ and every linear n -variety H in R^{2n+1} the set of all $\mathcal{E}$ -mappings of X to I^{2n+1} whose values miss H is dense in the function space $(I^{2n+1})^X$.

(c) Prove that for every separable metric space X such that $\dim X \leq n \geq 0$ there exists a homeomorphic embedding $f: X \rightarrow I^{2n+1}$ of X in I^{2n+1} which satisfies the inclusion $\bar{f}(X) \subset N_n^{2n+1}$. Observe that this fact implies the compactification theorem.

1.11.C. (a) Show that for every continuous mapping $f: X \rightarrow Y$ of a separable metric space X to a separable metric space Y and for every compact metric space $\tilde{Y}$ that contains Y there exists a compact metric space $\tilde{X}$ that contains X as a dense subset and a continuous mapping $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ such that $\tilde{f}|_X = f$ (cf. Lemma 1.13.3).

Hint. Consider the completion $\tilde{X}$ of the space X with respect to the metric $\tilde{\varrho}$ defined by letting $\tilde{\varrho}(x, y) = \varrho(x, y) + \sigma(f(x), f(y))$, where ϱ is a totally bounded metric on the space X and σ is an arbitrary metric on the space $\tilde{Y}$.

(b) Let $S = \{X_i, \pi_i^j\}$ be an inverse sequence of completely metrizable separable spaces. Prove that if $\pi_i^{i+1}(X_{i+1})$ is a dense subset of X_i for $i = 1, 2, \dots$, then $\varprojlim S \neq \emptyset$.

Hint. Apply (a) to define an inverse sequence $\tilde{S} = \{\tilde{X}_i, \tilde{\pi}_i^j\}$ of compact metric spaces such that X_i is a dense subset of $\tilde{X}_i$ and $\tilde{\pi}_i^{i+1}|_{X_{i+1}} = \pi_i^{i+1}$ for $i = 1, 2, \dots$. Show that for every i the inverse image $G_i = \tilde{\pi}_i^{-1}(X_i) \subset \tilde{X} = \varprojlim \tilde{S}$, where $\tilde{\pi}_i: \tilde{X} \rightarrow \tilde{X}_i$ denotes the projection, is a G_δ -set dense in the space $\tilde{X}$; use the fact that $\tilde{\pi}_i$ maps $\tilde{X}$ onto $\tilde{X}_i$ (see [GT], Corollary 3.2.15).

(c) Prove that for every continuous mapping $f: X \rightarrow R^{2n+1}$ of a separable metric space X such that $\dim X \leq n$ to R^{2n+1} and for every positive number ε there exists a homeomorphic embedding $g: X \rightarrow R^{2n+1}$ such such that $\varrho(f(x), g(x)) < \varepsilon$ for every $x \in X$.

Hint. Let $Z = R^{2n+1} \cup \{\omega\}$ be the one-point compactification of Euclidean $(2n+1)$ -space R^{2n+1} . Define a compact metric space $\tilde{X}$ that contains X as a dense subset and satisfies the inequality $\dim \tilde{X} \leq n$, and a continuous mapping $\tilde{f}: \tilde{X} \rightarrow Z$ such that $\tilde{f}|_X = f$. For $i = 1, 2, \dots$ let $X_i = \tilde{f}^{-1}(B_i) \subset \tilde{X}$, where $B_i = \overline{B(0, i)} \subset R^{2n+1}$; assume that $X_i \neq \emptyset$ and consider the subspace Φ_i of the function space $(R^{2n+1})^{X_i}$ consisting of all homeomorphic embeddings $g: X_i \rightarrow R^{2n+1}$ such that $\varrho(g(x), \tilde{f}(x)) < \varepsilon$ for every $x \in X_i$. Apply (b) to the inverse sequence $S = \{\Phi_i, \pi_i^j\}$, where $\pi_i^j(g) = g|_{X_j}$ for every pair i, j of natural numbers satisfying $j \leq i$.

1.11.D. (a) Show that for every natural number $m \geq 1$ the space M_0^m is homeomorphic to the Cantor set.

Hint. Apply Problem 1.3.F.

(b) (Sierpiński [1916] and [1922]) Prove that M_1^2 is a universal space for the class of all subspaces of the plane which have dimension ≤ 1 .

Hint. For a compact one-dimensional subspace X of the plane define a subspace of R^2 which contains X and is homeomorphic to M_1^2 . To this end, consider a rectangle containing X and remove from it smaller rectangles disjoint from X in the same way as one removes squares from I^2 to obtain M_1^2 (see Fig. 16, where X has the shape of the letter a). Apply Theorem 1.8.9 to extend the result to all one-dimensional subspaces of the plane.

(c) (Menger [1926] for compact spaces) Prove that the space M_{m-1}^m is a universal space for the class of all subspaces of R^m which have dimen-

sion $\leq m-1$.

Hint. Modify the construction described in the hint to part (b).

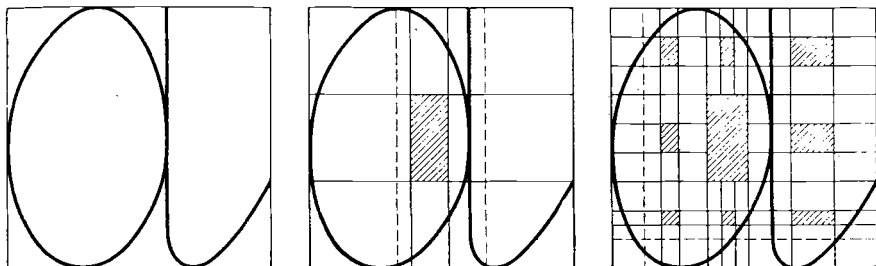


Fig. 16

1.11.E. Prove that the space M_n^m is embeddable in the space N_n^m for every pair of integers m, n satisfying $0 \leq n \leq m \geq 1$.

Hint. Apply Problem 1.8.C.

1.11.F (Flores [1935]). (a) Let $p_0, p_1, \dots, p_{2n+2}$ be the vertices of a regular $(2n+2)$ -simplex T^{2n+2} inscribed in the $(2n+1)$ -sphere $S^{2n+1} \subset R^{2n+2}$ and let $q_i = -p_i$ for $i = 0, 1, \dots, 2n+2$. Check that for each sequence $i_0 < i_1 < \dots < i_{2n+1} \leq 2n+2$ of $2n+2$ non-negative integers, the system of points $p_{i_0}, p_{i_1}, \dots, p_{i_n}, q_{i_{n+1}}, \dots, q_{i_{2n+1}} \in R^{2n+2}$ is linearly independent and the linear $(2n+1)$ -variety in R^{2n+2} spanned by these points does not contain the origin. Denote by S_1^{2n+1} the union of all $(2n+1)$ -simplexes of the form $p_{i_0}p_{i_1} \dots p_{i_n}q_{i_{n+1}} \dots q_{i_{2n+1}}$ and show that the projection p of S_1^{2n+1} from the origin onto S^{2n+1} is a homeomorphism. Observe that $p(-x) = -p(x)$ for every $x \in S_1^{2n+1}$.

(b) Let $\mathcal{K}_n$ be the family of all faces of T^{2n+2} which have dimension $\leq n$ and let $K_n = |\mathcal{K}_n|$. Consider the cone $C(K_n)$ over K_n with vertex at the origin, i.e., the subset of R^{2n+2} consisting of all points of the form tx , where $x \in K_n$ and $0 \leq t \leq 1$, and the subspace S_2^{2n+1} of the Cartesian product $C(K_n) \times C(K_n)$ consisting of all pairs (x, ty) and (tx, y) , where $0 \leq t \leq 1$, $x \in T_1$, $y \in T_2$ and T_1, T_2 are disjoint members of $\mathcal{K}_n$. Show that by mapping, in a linear way, every segment with end-points $(x, 0)$ and (x, y) contained in S_2^{2n+1} onto the segment with end-points x and $\frac{1}{2}(x-y)$ contained in S_1^{2n+1} , and every segment with end-points $(0, y)$ and (x, y) contained in S_2^{2n+1} onto the segment with end-points $-y$ and $\frac{1}{2}(x-y)$ contained in S_1^{2n+1} , one obtains a homeomorphism h of S_2^{2n+1} onto S_1^{2n+1} . Observe that $h(x, y) = -h(y, x)$.

(c) Applying the Borsuk-Ulam antipodal theorem, i.e., the fact that

for every continuous mapping $g: S^n \rightarrow R^n$ there exists a point $x \in S^n$ such that $g(x) = g(-x)$ (see Spanier [1966], p. 266), prove that the n -dimensional polyhedron K_n defined in (b) cannot be embedded in R^{2n} for any $n \geq 1$.

Hint. Assume that there exists a homeomorphic embedding $f: K_n \rightarrow R^{2n}$. Observe that f determines a homeomorphic embedding $f_1: C(K_n) \rightarrow R^{2n+1}$, consider the mapping $f_2: S_2^{2n+1} \rightarrow R^{2n+1}$ defined by letting $f_2(x, y) = f_1(y) - f_1(x)$ and the composition $g = f_2 h^{-1} p^{-1}: S^{2n+1} \rightarrow R^{2n+1}$.

1.12. Dimension and mappings

We shall now study the relations between the dimensions of the domain and the range of a continuous mapping. Let us begin with the observation that since one-to-one continuous mappings onto can arbitrarily raise or lower the dimension (see Problems 1.3.C and 1.4.F(b)), to obtain sound results we have to restrict ourselves to special classes of mappings. We find that for closed mappings and open mappings many interesting results can be obtained.

Let us recall that a continuous mapping $f: X \rightarrow Y$ is *closed* (*open*) if for every closed (open) set $A \subset X$, the image $f(A)$ is closed (open) in Y . One readily checks that if $f: X \rightarrow Y$ is a closed (an open) mapping, then for every closed (open) subset A of X the restriction $f|_A: A \rightarrow f(A) \subset Y$ is a closed (an open) mapping; similarly, if $f: X \rightarrow Y$ is a closed (an open) mapping, then for an arbitrary subset B of Y the restriction $f_B: f^{-1}(B) \rightarrow B$ is a closed (an open) mapping. Clearly, a mapping $f: X \rightarrow Y$ is closed if and only if $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$, so that each continuous mapping of a compact space to a Hausdorff space is closed.

We shall first discuss closed mappings and begin with the theorem on dimension-raising mappings. In the lemma to this theorem the notion of a network appears; a family $\mathcal{N}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and each neighbourhood U of the point x there exists an $M \in \mathcal{N}$ such that $x \in M \subset U$. The definition of a network imitates the definition of a base, only one does not require the members of a network to be open sets. Clearly, every base for a topological space is a network for that space. The family of all one-point subsets is another example of a network.

1.12.1. Lemma. *A separable metric space X satisfies the inequality $\text{ind } X \leq n \geq 0$ if and only if X has a countable network $\mathcal{N}$ such that $\text{ind } \text{Fr } M \leq n-1$ for every $M \in \mathcal{N}$.*

Proof. By virtue of Theorem 1.1.6, it suffices to show that if a separable metric space X has a network $\mathcal{N} = \{M_i\}_{i=1}^{\infty}$ such that $\text{ind Fr } M_i \leq n-1$ for $i = 1, 2, \dots$, then $\text{ind } X \leq n$. Let $Y = \bigcup_{i=1}^{\infty} \text{Fr } M_i$ and $Z = X \setminus Y$.

It follows from the sum theorem that $\text{ind } Y \leq n-1$; we shall show that $\text{ind } Z \leq 0$. For an arbitrary point $x \in Z$ and a neighbourhood $V \subset X$ of the point x there exists an $M_i \in \mathcal{N}$ such that $x \in M_i \subset V$. Since $x \in X \setminus Y \subset X \setminus \text{Fr } M_i$, we have $x \in U = \text{Int } M_i \subset V$. The inclusion $\text{Fr } \text{Int } M_i \subset \text{Fr } M_i$ implies that $Z \cap \text{Fr } U = \emptyset$ and thus we have $\text{ind } Z \leq 0$. The inequality $\text{ind } X \leq n$ now follows from Lemma 1.5.2. $\square$

1.12.2. Theorem on dimension-raising mappings. *If $f: X \rightarrow Y$ is a closed mapping of a separable metric space X onto a separable metric space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $\text{ind } Y \leq \text{ind } X - (k-1)$.*

Proof. We can suppose that $0 \leq \text{ind } X < \infty$. We shall apply induction with respect to the number $n+k$, where $n = \text{ind } X$. If $n+k = 1$, we have $k = 1$, so that f is a homeomorphism and the theorem holds. Assume that the theorem holds whenever $n+k < m \geq 2$ and consider a closed mapping $f: X \rightarrow Y$ such that $f(X) = Y$ and $n+k = m$.

Let $\mathcal{B}$ be a countable base for X such that $\text{ind Fr } U \leq n-1$ for every $U \in \mathcal{B}$. Consider an arbitrary $U \in \mathcal{B}$; by the closedness of f we have

$$(1) \quad \begin{aligned} \text{Fr } f(U) &= \overline{f(U)} \cap \overline{Y \setminus f(U)} \subset f(\overline{U}) \cap f(X \setminus U) \\ &= [f(U) \cup f(\text{Fr } U)] \cap f(X \setminus U) \subset f(\text{Fr } U) \cup B, \end{aligned}$$

where $B = f(U) \cap f(X \setminus U)$. Since the restriction $f|_{\text{Fr } U}: \text{Fr } U \rightarrow f(\text{Fr } U)$ is a closed mapping, it follows from the inductive assumption that

$$\text{ind } f(\text{Fr } U) \leq (n-1) + (k-1) = n+k-2.$$

Assume that $B \neq \emptyset$. Consider the restriction $f_B: f^{-1}(B) \rightarrow B$ and the restriction $f' = f_B|_{(X \setminus U) \cap f^{-1}(B)}: (X \setminus U) \cap f^{-1}(B) \rightarrow B$; both f_B and f' are closed, and the fibres of f' all have cardinality $\leq k-1$, because $f^{-1}(y) \cap U \neq \emptyset$ for every $y \in B$. It follows from the inductive assumption that

$$\text{ind } B \leq n + (k-1) - 1 = n+k-2.$$

As U is an F_σ -set in X , both $f(U)$ and B are F_σ -sets in Y ; applying Corollary 1.5.4, we obtain the inequality $\text{ind } [f(\text{Fr } U) \cup B] \leq n+k-2$. From the last inequality and from (1) it follows that $\text{ind Fr } f(U) \leq n+k-2$ for

every $U \in \mathcal{B}$; the same inequality holds if $B = \emptyset$. One readily checks that the family $\mathcal{N} = \{f(U): U \in \mathcal{B}\}$ is a network for the space Y , so that $\text{ind } Y \leq n+k-1 = \text{ind } X+k-1$ by virtue of Lemma 1.12.1. $\square$

We now pass to the theorem on dimension-lowering mappings. The theorem will be preceded by a lemma which, roughly speaking, shows that in condition (MU2), in the definition of the dimension function ind , points can be replaced by closed sets of small dimension.

1.12.3. Lemma. *If a separable metric space X has a closed cover $\{A_s\}_{s \in S}$ such that $\text{ind } A_s \leq m \geq 0$ for each $s \in S$ and if for every $s \in S$ and each open set $V \subset X$ that contains A_s there exists an open set $U \subset X$ such that*

$$A_s \subset U \subset \bar{U} \subset V \quad \text{and} \quad \text{ind Fr } U \leq m-1,$$

then $\text{ind } X \leq m$.

Proof. By virtue of Theorem 1.9.3 it suffices to show that for every closed subspace A of the space X and each continuous mapping $f: A \rightarrow S^m$ there exists a continuous extension $F: X \rightarrow S^m$ of f over X . It follows from Theorem 1.9.2 that for each $s \in S$ the mapping f is continuously extendable over $A \cup A_s$, so that there exists an open set $V_s \subset X$ containing $A \cup A_s$ such that f is continuously extendable over V_s . Consider an open set $U_s \subset X$ satisfying

$$(2) \quad A_s \subset U_s \subset \bar{U}_s \subset V_s \quad \text{and} \quad \text{ind Fr } U_s \leq m-1;$$

obviously, f is continuously extendable over $A \cup \bar{U}_s$. The open cover $\{U_s\}_{s \in S}$ of the space X has a countable subcover $\{U_{s_j}\}_{j=1}^\infty$. We shall inductively define a sequence $F_1, F_2, \dots$ of continuous mappings, where $F_i: A \cup \bigcup_{j=1}^i \bar{U}_{s_j} \rightarrow S^m$, such that

$$(3) \quad F_i|_{(A \cup \bigcup_{j=1}^{i-1} \bar{U}_{s_j})} = F_{i-1} \quad \text{for } i > 1.$$

Let F_1 be an arbitrary continuous extension of f over $A \cup \bar{U}_{s_1}$. Assume that the mappings F_i satisfying (3) are defined for $i < k$. The set $A \cup \bigcup_{j=1}^k \bar{U}_{s_j}$ can be represented as the union of two closed sets

$$A' = A \cup \bigcup_{j=1}^{k-1} \bar{U}_{s_j} \quad \text{and} \quad A'' = A \cup (\bar{U}_{s_k} \setminus \bigcup_{j=1}^{k-1} U_{s_j}).$$

The mapping f is extendable to a continuous mapping $f'': A'' \rightarrow S^m$ and

$$D = \{x \in A' \cap A'': F_{k-1}(x) \neq f''(x)\} \subset \bigcup_{j=1}^{k-1} \bar{U}_{s_j} \cap (X \setminus \bigcup_{j=1}^{k-1} U_{s_j}) \subset \bigcup_{j=1}^{k-1} \text{Fr } U_{s_j},$$

so that $\text{ind } D \leq m-1$ by virtue of (2) and the sum theorem. It follows from Lemma 1.9.6 that the mappings $F_{k-1}|_{A' \cap A''}$ and $f''|_{A' \cap A''}$ are homotopic. Since the mapping $f''|_{A' \cap A''}$ is continuously extendable over A'' , it follows from Lemma 1.9.7 that the mapping $F_{k-1}|_{A' \cap A''}$ is extendable to a continuous mapping $F'': A'' \rightarrow S^m$.

Letting

$$F_k(x) = \begin{cases} F_{k-1}(x) & \text{for } x \in A', \\ F''(x) & \text{for } x \in A'', \end{cases}$$

we define a continuous mapping F_k of $A' \cup A'' = A \cup \bigcup_{j=1}^k \bar{U}_{s_j}$ to S^m , which satisfies (3) for $i = k$.

As $X = \bigcup_{i=1}^{\infty} U_{s_i}$, the formula

$$F(x) = F_i(x) \quad \text{for } x \in U_{s_i}$$

defines a continuous mapping $F: X \rightarrow S^m$, which is the required extension of f over X . $\square$

1.12.4. Theorem on dimension-lowering mappings. *If $f: X \rightarrow Y$ is a closed mapping of a separable metric space X to a separable metric space Y and there exists an integer $k \geq 0$ such that $\text{ind } f^{-1}(y) \leq k$ for every $y \in Y$, then $\text{ind } X \leq \text{ind } Y + k$.*

Proof. We can suppose that $\text{ind } Y < \infty$. We shall apply induction with respect to $n = \text{ind } Y$. If $n = -1$, we have $Y = \emptyset$ and $X = \emptyset$, and so the theorem holds. Assume that the theorem holds for closed mappings to spaces of dimension less than $n \geq 0$ and consider a closed mapping $f: X \rightarrow Y$ to a space Y such that $\text{ind } Y = n$.

We shall show that the closed cover $\{f^{-1}(y)\}_{y \in Y}$ of the space X satisfies the conditions of Lemma 1.12.3 for $m = n + k$. Clearly, $\text{ind } f^{-1}(y) \leq k \leq m$ for each $y \in Y$. Consider now an $y \in Y$ and an open set $V \subset X$ which contains $f^{-1}(y)$. The set $W = Y \setminus f(X \setminus V)$ is a neighbourhood of the point y ; since $\text{ind } Y = n$, there exists an open set $U' \subset Y$ such that

$$y \in U' \subset \bar{U}' \subset W \quad \text{and} \quad \text{ind Fr } U' \leq n-1.$$

Applying the inductive assumption to the restriction $f_{\text{Fr } U'}: f^{-1}(\text{Fr } U') \rightarrow \text{Fr } U'$, we obtain the inequality $\text{ind } f^{-1}(\text{Fr } U') \leq n+k-1$. The open set $U = f^{-1}(U')$ satisfies the conditions

$$f^{-1}(y) \subset U \subset \bar{U} = \overline{f^{-1}(U')} \subset f^{-1}(\bar{U}') \subset f^{-1}(W) \subset V$$

and

$$\begin{aligned} \text{Fr } U &= \text{Fr } f^{-1}(U') = \overline{f^{-1}(U')} \setminus f^{-1}(U') \\ &\subset f^{-1}(\bar{U}') \setminus f^{-1}(U') \subset f^{-1}(\text{Fr } U'), \end{aligned}$$

so that $\text{ind } \text{Fr } U \leq n+k-1$. Lemma 1.12.3 now implies that $\text{ind } X \leq n+k = \text{ind } Y+k$. $\square$

Let us note that in Theorem 1.12.4 the assumption that f is a closed mapping cannot be replaced by the assumption that f is open (see Problem 1.12.C). On the other hand, Theorem 1.12.2 holds for open mappings as well; we shall show below that even more is true: open mappings with finite fibres do not change dimension. We shall also show that open mappings with countable fibres defined on locally compact spaces do not change dimension. In the proofs of both theorems the following lemma will be applied; we recall that the symbol A^a denotes the set of all accumulation points of the set A , i.e., the set of all points x such that $x \in \overline{A \setminus \{x\}}$.

1.12.5. Lemma. *Let $f: X \rightarrow Y$ be an open mapping of a metric space X onto a metric space Y . For every base $\mathcal{B} = \{U_s\}_{s \in S}$ for the space X there exists a family $\{A_s\}_{s \in S}$ of subsets of X such that $A_s \subset U_s$ for each $s \in S$ and*

- (i) A_s and $f(A_s)$ are F_σ -sets in X and Y , respectively,
- (ii) $f|_{A_s}: A_s \rightarrow f(A_s)$ is a homeomorphism,
- (iii) $X = \left(\bigcup_{s \in S} A_s\right) \cup \left(\bigcup_{y \in Y} [f^{-1}(y)]^a\right)$.

Proof. For each $s \in S$ let

$$A_s = \{x \in U_s: U_s \cap f^{-1}f(x) = \{x\}\}.$$

Observe first that the set $f(U_s) \setminus f(A_s)$ is open in Y . Indeed, for every point $y \in f(U_s) \setminus f(A_s)$ there exist two distinct points $x_1, x_2 \in U_s$ such that $f(x_1) = f(x_2) = y$, and—as one readily checks—the open set $V = f(W_1) \cap f(W_2)$, where W_1, W_2 are disjoint open subsets of U_s which contain x_1 and x_2 , respectively, contains the point y and is contained in the set $f(U_s) \setminus f(A_s)$. As $f(A_s) \subset f(U_s)$,

$$f(A_s) = f(U_s) \setminus [f(U_s) \setminus f(A_s)],$$

so that the set $f(A_s)$, being the intersection of an open set and a closed set, is an F_σ -set in Y . From the obvious equality

$$(4) \quad A_s = U_s \cap f^{-1}f(A_s)$$

it follows that A_s is an F_σ -set in X ; thus (i) is proved.

The restriction $f|U_s: U_s \rightarrow f(U_s)$ is an open mapping, and so is its restriction $(f|U_s)|_{f(A_s)}: U_s \cap f^{-1}f(A_s) \rightarrow f(A_s)$. By virtue of (4) the last mapping coincides with $f|A_s: A_s \rightarrow f(A_s)$. Thus the mapping $f|A_s: A_s \rightarrow f(A_s)$ is open and, by the definition of A_s , one-to-one, i.e., we have (ii).

To prove (iii), it suffices to observe that if $x \notin [f^{-1}f(x)]^d$, then there exists a member U_s of the base $\mathcal{B}$ such that $U_s \cap f^{-1}(x) = \{x\}$, and thus $x \in A_s$. $\square$

1.12.6. Theorem. *If $f: X \rightarrow Y$ is an open mapping of a separable metric space X onto a separable metric space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point, then $\text{ind} Y \leq \text{ind} X$.*

Proof. Let $\{U_i\}_{i=1}^\infty$ be a countable base for the space X ; consider a family $\{A_i\}_{i=1}^\infty$ of subsets of X which satisfy (i)–(iii) in Lemma 1.12.5. By the assumption on the fibres, $Y = \bigcup_{i=1}^\infty f(A_i)$, and since the subspaces $A_i \subset X$ and $f(A_i) \subset Y$ are homeomorphic, $\text{ind} f(A_i) = \text{ind} A_i \leq \text{ind} X$ for $i = 1, 2, \dots$. Hence we have $\text{ind} Y \leq \text{ind} X$ by virtue of Corollary 1.5.4. $\square$

1.12.7. Theorem. *If $f: X \rightarrow Y$ is an open mapping of a separable metric space X onto a separable metric space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X , then $\text{ind} X = \text{ind} Y$.*

Proof. Let $\{U_i\}_{i=1}^\infty$ be a countable base for the space X ; consider a family $\{A_i\}_{i=1}^\infty$ of subsets of X which satisfy (i)–(iii) in Lemma 1.12.5. By the assumption on the fibres, $X = \bigcup_{i=1}^\infty A_i$, and since the subspaces $A_i \subset X$ and $f(A_i) \subset Y$ are homeomorphic, $\text{ind} A_i = \text{ind} f(A_i) \leq \text{ind} Y$ for $i = 1, 2, \dots$. Hence we have $\text{ind} X \leq \text{ind} Y$ by virtue of Corollary 1.5.4; to complete the proof it suffices to apply Theorem 1.12.6. $\square$

1.12.8. Alexandroff's theorem. *If $f: X \rightarrow Y$ is an open mapping of a separable locally compact metric space X onto a separable metric space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\text{ind} X = \text{ind} Y$.*

Proof. Since each closed subspace of a locally compact space is locally compact, the fibres of f are locally compact. It follows from the Baire category theorem and complete metrizability of locally compact spaces (or from Problem 1.2.D) that for every $y \in Y$ the fibre $f^{-1}(y)$, being countable, has an isolated point. Hence we have $\text{ind } Y \leq \text{ind } X$ by Theorem 1.12.6.

Let $\{F_i\}_{i=1}^\infty$ be a countable cover of X consisting of compact subspaces. For every i the restriction $f|F_i: F_i \rightarrow f(F_i)$ is a closed mapping with zero-dimensional fibres, so that $\text{ind } F_i \leq \text{ind } f(F_i) \leq \text{ind } Y$ by virtue of the theorem on dimension-lowering mappings. Hence we have $\text{ind } X \leq \text{ind } Y$ by virtue of the sum theorem. $\square$

It is possible to define open mappings with countable fibres which arbitrarily raise or lower dimension (see Problems 1.12.E and 1.12.F). Open mappings with zero-dimensional fibres defined on compact spaces can also arbitrarily raise dimension, but examples of such mappings are very complicated and will not be discussed here. Let us note, however, that the space M_1^3 defined in the last section can be mapped onto every locally connected metric continuum by an open mapping whose fibres are all homeomorphic to the Cantor set.

We conclude this section with a discussion of *open-and-closed* mappings, i.e., mappings which are both open and closed. It follows from the last paragraph that open-and-closed mappings with zero-dimensional fibres can arbitrarily raise dimension. We shall now show that open-and-closed mappings with countable fibres preserve dimension.

1.12.9. Vainštein's lemma. *If $f: X \rightarrow Y$ is a closed mapping of a metric space X onto a metric space Y , then for every $y \in Y$ the boundary $\text{Fr } f^{-1}(y)$ of the fibre $f^{-1}(y)$ is a compact subspace of X .*

Proof. It suffices to show that every countably infinite subset $A = \{x_1, x_2, \dots\}$ of the boundary $\text{Fr } f^{-1}(y)$ of an arbitrary fibre $f^{-1}(y)$ has an accumulation point. Let $\{V_i\}_{i=1}^\infty$ be a base for the space Y at the point y . For $i = 1, 2, \dots$ choose a point $x'_i \in f^{-1}(V_i) \setminus f^{-1}(y)$ satisfying $\varrho(x_i, x'_i) < 1/i$; such a choice is possible, because the intersection $B(x_i, 1/i) \cap f^{-1}(V_i)$ is a neighbourhood of x_i . Consider the set $B = \{x'_1, x'_2, \dots\} \subset X$. We have $y \in \overline{f(B)} \setminus f(B)$ so that $\overline{B} \neq B$, i.e., $B^a \neq \emptyset$. Now, since $\varrho(x_i, x'_i) < 1/i$ for $i = 1, 2, \dots$, $A^a = B^a \neq \emptyset$ and the proof is completed. $\square$

1.12.10. Theorem. *If $f: X \rightarrow Y$ is an open-and-closed mapping of a separable metric space X onto a separable metric space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\text{ind} X = \text{ind} Y$.*

Proof. Let us note first that Theorem 1.12.4 yields the inequality $\text{ind} X \leq \text{ind} Y$.

Now, let $Y_0 = \{y \in Y: \text{Int} f^{-1}(y) \neq \emptyset\}$, $Y_1 = Y \setminus Y_0$, and $X_1 = f^{-1}(Y_1)$. The mapping f being open, for every $y \in Y_0$ the one-point set $\{y\} = f(\text{Int} f^{-1}(y))$ is open, which implies that the set Y_0 is open in Y and $\text{ind} Y_0 \leq 0$. The restriction $f_1 = f|_{X_1}: X_1 \rightarrow Y_1$ is also an open mapping, and since $f_1^{-1}(y) = \text{Fr} f^{-1}(y) \subset f^{-1}(y)$, the fibres of f_1 are countable and compact, the latter by virtue of Vainštein's lemma; in particular, for every $y \in Y$ the fibre $f_1^{-1}(y)$ has an isolated point. It then follows from Theorem 1.12.6 that $\text{ind} Y_1 \leq \text{ind} X_1 \leq \text{ind} X$; as $Y = Y_0 \cup Y_1$, Corollary 1.5.5 yields the inequality $\text{ind} Y \leq \text{ind} X$. $\square$

Historical and bibliographic notes

Theorem 1.12.2 was proved by Hurewicz in [1927a]. The same paper contains Theorem 1.12.4 for continuous mappings defined on compact metric spaces; the extension of this theorem to separable metric spaces was given by Hurewicz and Wallman in [1941]. Theorem 1.12.6 was first stated by Taĭmanov in [1955]; however, it is implicitly contained in Alexandroff's paper [1936]. Theorem 1.12.7 was proved by Hodel in [1963] (for the special case of a mapping with finite fibres it was proved earlier by Nagami, namely in [1960]). Alexandroff established Theorem 1.12.8 in [1936]. The first example of a dimension-raising open mapping with zero-dimensional fibres defined on a compact metric space was described by Kolmogoroff in [1937]. Keldyš defined in [1954] an open mapping with zero-dimensional fibres which maps a one-dimensional compact metric space onto the square I^2 ; a detailed description of Keldyš' example can be found in Alexandroff and Pasynkov's book [1973]. The fact that M_1^3 can be mapped onto every locally connected metric continuum by an open mapping whose fibres are all homeomorphic to the Cantor set was established by Wilson in [1972]. Theorem 1.12.10 was given by Vainštein in [1949]. Lelek's paper [1971] contains a comprehensive discussion

of the topic of the present section and a good bibliography. We shall return to this subject in Section 4.3, where two theorems of a more special character will be proved (see Theorems 4.3.9 and 4.3.12).

Problems

1.12.A (Hurewicz [1926]). Prove that if $f: X \rightarrow Y$ is a closed mapping of a separable metric space X onto a separable metric space Y and $|f^{-1}(y)| = k < \infty$ for every $y \in Y$, then $\text{ind} X = \text{ind} Y$.

Hint. Consider a countable base $\mathcal{B}$ for the space X and the family of all intersections $\bigcap_{i=1}^k f(U_i)$, where $U_i \in \mathcal{B}$ for $i = 1, 2, \dots, k$ and $\bar{U}_i \cap \bar{U}_j = \emptyset$ whenever $i \neq j$.

1.12.B (Hurewicz [1937]). Observe that, under the additional hypothesis that X is a finite-dimensional compact space, the theorem on dimension-lowering mappings is a direct consequence of the Cantor-manifold theorem.

Hint. Assume that X is a Cantor-manifold and apply induction with respect to $\text{ind} Y$.

1.12.C. Give an example of an open mapping with zero-dimensional fibres which maps a one-dimensional separable metric space onto the Cantor set.

Hint. Use the Knaster-Kuratowski fan.

Remark. It follows from Problem 1.12.F that there even exist such mappings with countable fibres (cf. Problem 1.12.G(b)).

1.12.D. (a) Observe that in Theorem 1.12.2 the inequality $|f^{-1}(y)| \leq k$ can be replaced by the weaker inequality $|\text{Fr} f^{-1}(y)| \leq k$.

Hint. Consider the restriction $f|X_1$, where X_1 is obtained by adjoining to the union $\bigcup_{y \in Y} \text{Fr} f^{-1}(y)$ one point from each fibre $f^{-1}(y)$ which has an empty boundary.

(b) Observe that in Theorem 1.12.6 it suffices to assume that for every $y \in Y$ the boundary $\text{Fr} f^{-1}(y)$ either has an isolated point or is empty.

1.12.E (Hausdorff [1934]). Show that every separable metric space X can be represented as the image of a subspace Z of the space P of irrational numbers under an open mapping.

Hint. Consider a countable base $\{U_k\}_{k=1}^\infty$ for the space X and the subspace Z of the Cartesian product $N^{\aleph_0} = \prod_{i=1}^\infty N_i$, where $N_i = N$ for $i = 1, 2, \dots$, consisting of all points $\{k_i\}$ such that the family $\{U_{k_i}\}_{i=1}^\infty$ is a base for X at a point x ; assign the point x to the point $\{k_i\}$.

1.12.F (Roberts [1947]). Show that for every open mapping $f: X \rightarrow Y$ of a separable metric space X onto a separable metric space Y there exists a set $X_1 \subset X$ such that the restriction $f|X_1$ is an open mapping of X_1 onto Y and has countable fibres.

Hint. Consider a countable base $\{U_i\}_{i=1}^\infty$ for the space X and choose one point from each non-empty intersection of the form $U_i \cap f^{-1}(y)$, where $y \in Y$.

1.12.G. (a) Observe that if $f: X \rightarrow Y$ is an open mapping of a complete separable metric space X onto a separable metric space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\text{ind } Y \leq \text{ind } X$.

(b) Give an example of an open mapping with countable fibres which maps a one-dimensional complete separable metric space onto the Cantor set.

Hint. For every sequence $i_1, i_2, \dots, i_m$ consisting of zeros and ones, let

$$C(i_1, i_2, \dots, i_m) = \left\{ x \in C: \sum_{k=1}^m \frac{2i_k}{3^k} \leq x \leq \sum_{k=1}^m \frac{2i_k}{3^k} + \frac{1}{3^m} \right\},$$

and let $C'(i_1, i_2, \dots, i_m)$ be a subspace of I which is homeomorphic to the Cantor set and is contained in the interval

$$\left\{ x \in I: \sum_{k=1}^m \frac{2i_k}{3^k} + \frac{1}{3^{m+1}} < x < \sum_{k=1}^m \frac{2i_k}{3^k} + \frac{2}{3^{m+1}} \right\}$$

removed from I in the process of constructing the Cantor set. Consider a countable set $\{a_1, a_2, \dots\}$ dense in the interval $[-1, 1]$ and define

$$X = K \cup \bigcup_{m=1}^\infty \bigcup_{i_1, \dots, i_m} \bigcup_{k=1}^m [C'(i_1, i_2, \dots, i_m) \times \{a_k\}] \subset X \times [-1, 1],$$

where K is the space in Problem 1.2.E; show that the mapping $f: X \rightarrow C$, where $f|K$ is the projection of K onto C and $f|[C'(i_1, i_2, \dots, i_m) \times \{a_k\}]$ is an arbitrary homeomorphism of $C'(i_1, i_2, \dots, i_m) \times \{a_k\}$ onto $C(i_1, i_2, \dots, i_m)$, has the required properties.

1.13. Dimension and inverse sequences of polyhedra

Besides theorems on ε -mappings and ε -translations there is one more characterization of dimension in the realm of compact metric spaces through their relations to polyhedra. It has been discovered that the class of compact metric spaces which have dimension $\leq n$ coincides with the class of spaces which are homeomorphic to the limits of inverse sequences of polyhedra which have dimension $\leq n$. We shall deduce this characterization from two theorems which we are going to prove: the theorem on expansion in an inverse sequence and the theorem on the dimension of the limit of an inverse sequence.

As the subject of this section is more specific than our previous considerations, we shall assume here that the reader is familiar with the basic definitions and theorems in the theory of inverse systems (see, e.g., [GT], pp. 135–140, 188 and 189). We shall only recall that an inverse sequence $\{X_i, \pi_j^i\}$ is an inverse system $\{X_i, \pi_j^i, N\}$ where N is the set of natural numbers directed by its natural order.

In Lemma 1.13.1 and in the proof of Theorem 1.13.2 below the symbol T^m will denote the m -simplex in R^{m+1} spanned by the points $p_1 = (1, 0, \dots, 0)$, $p_2 = (0, 1, \dots, 0)$, $\dots$, $p_{m+1} = (0, 0, \dots, 1)$. We shall consider on T^m the metric σ defined by letting

$$\sigma(x, y) = \|x - y\|, \quad \text{where } \|z\| = \sum_{i=1}^{m+1} |\lambda_i| \text{ for } z = (\lambda_1, \lambda_2, \dots, \lambda_{m+1}) \in R^{m+1};$$

obviously, the metric σ is equivalent to the natural metric on T^m .

1.13.1. Lemma. *Let a set $A_j \subset \{1, 2, \dots, n+1\}$ be given for $j = 1, 2, \dots, l$ and let π be the mapping of T^{l-1} to $T^n = p_1 p_2 \dots p_{n+1}$ defined by the formula*

$$\pi((\lambda_1, \lambda_2, \dots, \lambda_l)) = \sum_{j=1}^l \frac{\lambda_j}{n_j} \sum_{i \in A_j} p_i, \quad \text{where } n_j = |A_j|.$$

If $A_1 \cap A_2 \cap \dots \cap A_l \neq \emptyset$, then $\sigma(\pi(x), \pi(y)) \leq \frac{n}{n+1} \sigma(x, y)$ for all $x, y \in T^{l-1}$.

Proof. Consider arbitrary points $x = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $y = (\mu_1, \mu_2, \dots, \mu_l)$ in T^{l-1} . Let $\alpha_j = \lambda_j - \mu_j$ for $j = 1, 2, \dots, l$, $B = \{j: \alpha_j \geq 0\}$ and

$C = \{j: \alpha_j < 0\}$. From the equality $\sum_{j=1}^l \alpha_j = 0$ it follows that $\sum_{j \in B} |\alpha_j| = \sum_{j \in C} |\alpha_j| = \frac{1}{2} \sigma(x, y)$. One easily checks that

$$\left| \sum_{j=1}^l \frac{\alpha_j}{n_j} \right| = \sum_{j=1}^l \frac{|\alpha_j|}{n_j} - 2 \min(|b|, |c|), \quad \text{where } b = \sum_{j \in B} \frac{\alpha_j}{n_j}, \quad c = \sum_{j \in C} \frac{\alpha_j}{n_j}.$$

Since $n_j \leq n+1$ for $j = 1, 2, \dots, l$, we have

$$|b| = \sum_{j \in B} \frac{|\alpha_j|}{n_j} \geq \frac{1}{n+1} \sum_{j \in B} |\alpha_j| = \frac{1}{2} \frac{\sigma(x, y)}{n+1};$$

similarly $|c| \geq \frac{1}{2} \frac{\sigma(x, y)}{n+1}$, so that $\min(|b|, |c|) \geq \frac{1}{2} \frac{\sigma(x, y)}{n+1}$. Choosing arbitrarily an $i_0 \in A_1 \cap A_2 \cap \dots \cap A_l$ and letting $A'_j = A_j \setminus \{i_0\}$ for $j = 1, 2, \dots, l$ we have

$$\begin{aligned} \sigma(\pi(x), \pi(y)) &= \left| \sum_{j=1}^l \frac{\alpha_j}{n_j} \sum_{i \in A_j} p_i \right| = \left| \left(\sum_{j=1}^l \frac{\alpha_j}{n_j} \right) p_{i_0} + \sum_{j=1}^l \frac{\alpha_j}{n_j} \sum_{i \in A'_j} p_i \right| \\ &\leq \left| \sum_{j=1}^l \frac{\alpha_j}{n_j} \right| + \sum_{j=1}^l \frac{|\alpha_j|(n_j-1)}{n_j} = \sum_{j=1}^l \frac{|\alpha_j|}{n_j} + \sum_{j=1}^l \frac{|\alpha_j|(n_j-1)}{n_j} - 2 \min(|b|, |c|) \\ &= \sum_{j=1}^l |\alpha_j| - 2 \min(|b|, |c|) = \sigma(x, y) - 2 \min(|b|, |c|) \\ &\leq \sigma(x, y) - \frac{\sigma(x, y)}{n+1} = \frac{n}{n+1} \sigma(x, y). \quad \square \end{aligned}$$

1.13.2. Theorem on expansion in an inverse sequence. *For every compact metric space X such that $\dim X \leq n$ there exists an inverse sequence $\{K_i, \pi_j^i\}$ consisting of polyhedra of dimension $\leq n$ whose limit is homeomorphic to X ; moreover, one can assume that, for $i = 1, 2, \dots$, K_i is the underlying polyhedron of a nerve $\mathcal{K}_i$ of a finite open cover of the space X , and that for every $j \leq i$ the bonding mapping π_j^i is linear on each simplex in $\mathcal{K}_i$.*

Proof. We can suppose that $\dim X \geq 0$. Consider a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of finite open covers of the space X , where $\mathcal{U}_i = \{U_{i,k}\}_{k=1}^{k_m}$ and $U_{i,k} \neq \emptyset$ for $k = 1, 2, \dots, k_m$, such that

$$(1) \quad \text{ord } \mathcal{U}_i \leq n$$

and

$$(2) \quad \text{mesh } \mathcal{U}_{i+1} \leq \min(1/i + 1, \varepsilon_i/2),$$

where ε_i is a Lebesgue number for the cover $\mathcal{U}_i$. It follows from (2) that

$$(3) \quad \text{if } U_{i+1,j_1} \cap U_{i+1,j_2} \cap \dots \cap U_{i+1,j_l} \neq \emptyset, \text{ then there exists a } k \leq k_i \text{ such that } U_{i+1,j_m} \subset U_{i,k} \text{ for } m = 1, 2, \dots, l.$$

For $i = 1, 2, \dots$ let $\mathcal{K}_i = \mathcal{N}(\mathcal{U}_i)$ be a nerve of the cover $\mathcal{U}_i$ consisting of faces of the simplex T^{k_i-1} . By virtue of (1), the underlying polyhedra $K_i = |\mathcal{K}_i|$ all have dimension $\leq n$.

We shall now define continuous mappings $\pi_i^{i+1}: K_{i+1} \rightarrow K_i$ for $i = 1, 2, \dots$. Let $p_{i+1,j}$ be a vertex of the complex $\mathcal{K}_{i+1}$. Consider the member $U_{i+1,j}$ of the cover $\mathcal{U}_{i+1}$ which corresponds to $p_{i+1,j}$. By virtue of (3) the family

$$\mathcal{U}_{i,j} = \{U \in \mathcal{U}_i: U_{i+1,j} \subset U\}$$

is non-empty. Since $\bigcap \mathcal{U}_{i,j} \neq \emptyset$, the vertices of $\mathcal{K}_i$ which correspond to the members of $\mathcal{U}_{i,j}$ span a simplex $S_{i,j} \in \mathcal{K}_i$; we let

$$(4) \quad \pi_i^{i+1}(p_{i+1,j}) = b(S_{i,j}),$$

where $b(S)$ denotes the barycentre of S .

We shall prove that

$$(5) \quad \text{for every simplex } S \in \mathcal{K}_{i+1}, \text{ the images of vertices of } S \text{ under } \pi_i^{i+1} \text{ are contained in a simplex } T \in \mathcal{K}_i.$$

Indeed, if $S = p_{i+1,j_1} p_{i+1,j_2} \dots p_{i+1,j_l}$, then

$$\emptyset \neq U_{i+1,j_1} \cap U_{i+1,j_2} \cap \dots \cap U_{i+1,j_l} \subset \bigcap \mathcal{U}_{i,j_1} \cap \bigcap \mathcal{U}_{i,j_2} \cap \dots \cap \bigcap \mathcal{U}_{i,j_l},$$

so that the vertices of $\mathcal{K}_i$ which correspond to the members of the union $\mathcal{U}_{i,j_1} \cup \mathcal{U}_{i,j_2} \cup \dots \cup \mathcal{U}_{i,j_l}$ span a simplex $T \in \mathcal{K}_i$ which contains the point $\pi_i^{i+1}(p_{i+1,j_m})$ for $m = 1, 2, \dots, l$.

It follows from (5) that the mapping π_i^{i+1} defined on the set of all vertices of $\mathcal{K}_{i+1}$ can be extended over each simplex $S = p_{i+1,j_1} p_{i+1,j_2} \dots p_{i+1,j_l} \in \mathcal{K}_{i+1}$ by letting

$$\pi_i^{i+1}\left(\sum_{m=1}^l \lambda_{j_m} p_{i+1,j_m}\right) = \sum_{j=1}^l \lambda_{j_m} \pi_i^{i+1}(p_{i+1,j_m});$$

in this way a continuous mapping $\pi_i^{i+1}: K_{i+1} \rightarrow K_i$ is defined for $i = 1, 2, \dots$. One easily checks that by defining, for every pair i, j of natural numbers satisfying $j \leq i$,

$$\pi_j^i = \pi_j^{j+1} \pi_{j+1}^{j+2} \dots \pi_{i-1}^i \quad \text{if } j < i \quad \text{and} \quad \pi_i^i = \text{id}_{K_i},$$

one obtains continuous mappings $\pi_j^i: K_i \rightarrow K_j$ linear on each simplex in $\mathcal{K}_i$.

Thus the inverse sequence $S = \{K_i, \pi_j^i\}$ is defined. It remains to show that the space X is homeomorphic to the limit K of this inverse sequence.

By virtue of (3) and (5) we can apply Lemma 1.13.1 to restrictions of π_i^{i+1} to simplexes in $\mathcal{K}_{i+1}$, so that

$$(6) \quad \delta(\pi_i^{i+1}(F)) \leq \frac{n}{n+1} \delta(F) \quad \text{for every } F \subset S \in \mathcal{K}_{i+1}.$$

Let us observe that for every choice of a point y_i in K_i for $i = 1, 2, \dots$

(7) if $\emptyset \neq F_i = \bar{F}_i \subset \bigcup \{S \in \mathcal{K}_i: y_i \in S\}$ and $\pi_i^{i+1}(F_{i+1}) \subset F_i$ for $i = 1, 2, \dots$, then the limit L of the inverse sequence $\{F_i, \pi_j^i|F_i\}$ is a one-point set.

Indeed, the set L is non-empty as the limit of an inverse sequence of non-empty compact spaces; since for $j = 1, 2, \dots$ $\pi_j(L) \subset \bigcap_{i=j}^{\infty} \pi_j^i(F_i)$, where $\pi_j: K \rightarrow K_i$ is the projection, and since—by virtue of (6) and the inequality $\delta(F_i \cap S) \leq \delta(S) \leq 2 - \delta(\pi_j^i(F_i)) \leq 4(n/n+1)^{i-j}$ whenever $j \leq i$, the sets $\pi_j(L)$ are all one-point sets, which implies that L is a one-point set.

Now, consider a point $x \in X$; for every natural number i let

$$\mathcal{U}_i(x) = \{U \in \mathcal{U}_i: x \in U\}$$

and denote by $K_i(x)$ the simplex in $\mathcal{K}_i$ spanned by the vertices of $\mathcal{K}_i$ which correspond to the members of $\mathcal{U}_i(x)$. Let us note that

$$(8) \quad \pi_i^{i+1}(K_{i+1}(x)) \subset K_i(x) \quad \text{for } i = 1, 2, \dots$$

Indeed, if $U_{i+1,j} \in \mathcal{U}_{i+1}(x)$, then $\mathcal{U}_{i,j} \subset \mathcal{U}_i(x)$, so that the images of vertices of $K_{i+1}(x)$ under π_i^{i+1} are contained in the simplex $K_i(x)$, and this implies (8). It follows from (8) that $K(x) = \{K_i(x), \pi_j^i|K_i(x)\}$ is an inverse sequence. By virtue of (7) the limit of this inverse sequence contains exactly one point; let it be denoted by $f(x)$. Clearly, $f(x) \in K$; a mapping f of X to K is thus defined.

We shall prove that f is a continuous mapping. Obviously, it is enough to show that for every natural number j the composition $f_j = \pi_j f$ is continuous. Consider a point $x_0 \in X$, a positive number ε and a natural number i such that $i \geq j$ and $(n/n+1)^{i-j} < \varepsilon/2$. The set $U = \bigcap \mathcal{U}_i(x_0)$ is a neighbourhood of x_0 and for every $x \in U$ the inclusion $\mathcal{U}_i(x_0) \subset \mathcal{U}_i(x)$ holds,

so that $K_i(x_0) \subset K_i(x)$ and

$$f_j(x_0) \in \pi_j^i(K_i(x_0)) \subset \pi_j^i(K_i(x)) \quad \text{for } x \in U.$$

Since by virtue of (6)

$$\delta(\pi_j^i(K_i(x))) \leq 2(n/n+1)^{i-j} < \varepsilon$$

and since $f_j(x) \in \pi_j^i(K_i(x))$,

$$\sigma(f_j(x_0), f_j(x)) < \varepsilon \quad \text{for } x \in U,$$

i.e., the mapping f_j is continuous.

As for each pair x, x' of distinct points of X there exists —by virtue of (2)—a natural number i such that $\mathcal{U}_i(x) \cap \mathcal{U}_i(x') = \emptyset$, i.e., $K_i(x) \cap K_i(x') = \emptyset$, the mapping $f: X \rightarrow K$ is one-to-one. As X is a compact space, to complete the proof it suffices to show that $f(X) = K$.

Consider a point $y = \{y_i\} \in K$. For $i = 1, 2, \dots$ let

$$A_i = \bigcup \{S \in \mathcal{K}_i : y_i \in S\} \subset K_i \quad \text{and} \quad B_i = f_i^{-1}(A_i) \subset X.$$

The sets B_i are non-empty. Indeed, if $S_i = p_{i,j_1} p_{i,j_2} \dots p_{i,j_l}$ is a maximal simplex in $\mathcal{K}_i$ which contains y_i , and x_i is a point in $U_{i,j_1} \cap U_{i,j_2} \cap \dots \cap U_{i,j_l}$, then $K_i(x_i) = S_i \subset A_i$, which implies that $x_i \in B_i$.

Since the image of each simplex $S \in \mathcal{K}_{i+1}$ under π_i^{i+1} is contained in a simplex $T \in \mathcal{K}_i$, and since $\pi_i^{i+1}(y_{i+1}) = y_i$,

$$(9) \quad \pi_i^{i+1}(A_{i+1}) \subset A_i \quad \text{for } i = 1, 2, \dots$$

The last inclusion implies that

$$\begin{aligned} B_{i+1} &= f_{i+1}^{-1}(A_{i+1}) \subset f_{i+1}^{-1}(\pi_i^{i+1})^{-1}(A_i) \\ &= (\pi_i^{i+1} f_{i+1})^{-1}(A_i) = f_i^{-1}(A_i) = B_i \end{aligned}$$

for $i = 1, 2, \dots$, so that, by the compactness of X , there exists a point $x \in \bigcap_{i=1}^{\infty} B_i$. Let $F_i = A_i \cap K_i(x)$ for $i = 1, 2, \dots$; as $f_i(x) \in F_i$, the sets F_i are non-empty. By virtue of (9) and (8)

$$\begin{aligned} \pi_i^{i+1}(F_{i+1}) &= \pi_i^{i+1}(A_{i+1} \cap K_{i+1}(x)) \subset \pi_i^{i+1}(A_{i+1}) \cap \pi_i^{i+1}(K_{i+1}(x)) \\ &\subset A_i \cap K_i(x) = F_i \quad \text{for } i = 1, 2, \dots, \end{aligned}$$

so that $\{F_i, \pi_j^i|F_i\}$ is an inverse sequence; by (7), the limit of it is a one-point set $\{z\} \subset K$. From (9) and (7) it follows that $\{A_i, \pi_j^i|A_i\}$ is an inverse sequence whose limit is the one-point set $\{y\}$. By the definition

of f , the one-point set $\{f(x)\}$ is the limit of the inverse sequence $K(x)$. Since $\{z\} \subset \{y\} \cap \{f(x)\}$, we have $f(x) = y$. $\square$

Let us note that another proof of Theorem 1.13.2 (see the hint to Problem 1.13.G(a)) leads to an inverse sequence $\{K_i, \pi_j^i\}$, where, for $i = 1, 2, \dots$, K_i is the underlying polyhedron of a nerve $\mathcal{K}_i$ of a finite open cover of the space X , and for every $j \leq i$ the bonding mapping π_j^i is a *quasi-simplicial* mapping of K_i onto K_j , i.e., it is the linear extension of a simplicial mapping of a barycentric subdivision of $\mathcal{K}_i$ onto a barycentric subdivision of $\mathcal{K}_j$ (cf. Problem 1.13.C).

Let us also note that from Problem 1.10.K it follows that every compact subspace X of Euclidean m -space R^m such that $\dim X \leq n \geq 0$ can be represented as the limit of an inverse sequence consisting of polyhedra of dimension $\leq n$ which are all contained in R^m . The converse does not hold; simple examples show that the limit of an inverse sequence of polyhedra contained in R^m need not be embeddable in R^m (see Problem 1.13.B).

We now pass to the theorem on the dimension of the limit of an inverse sequence. In the proof we shall apply the following lemma, a slight strengthening of the compactification theorem, which allows the reduction of the problem to the special case where the inverse sequence consists of compact spaces.

1.13.3. Lemma. *For every continuous mapping $f: X \rightarrow Z$ of a separable metric space X to a compact metric space Z there exists a compact metric space $\tilde{X}$ containing X as a dense subspace and such that $\dim \tilde{X} \leq \dim X$ and f is extendable to a continuous mapping $\tilde{f}: \tilde{X} \rightarrow Z$.*

Proof. Let ϱ_0 be a totally bounded metric on the space X and σ an arbitrary metric on the space Z . One easily checks that the formula

$$\varrho(x, y) = \varrho_0(x, y) + \sigma(f(x), f(y)) \quad \text{for } x, y \in X$$

defines a metric equivalent to ϱ_0 and such that the mapping f is uniformly continuous with respect to ϱ and σ . The metric ϱ is totally bounded. Indeed, for every positive number ε there exists a finite cover $\mathcal{A}$ of the space (X, ϱ_0) such that $\text{mesh } \mathcal{A} < \varepsilon/2$ and a finite cover $\mathcal{B}$ of the space (Y, σ) such that $\text{mesh } \mathcal{B} < \varepsilon/2$; one can readily check that the mesh of the cover $\mathcal{A} \wedge f^{-1}(\mathcal{B})$ of the space (X, ϱ) is less than ε . It follows from the second part of the compactification theorem that on X there exists a metric $\tilde{\varrho}$ equivalent to ϱ such that $\varrho(x, y) \leq \tilde{\varrho}(x, y)$ for $x, y \in X$, and that the completion $\tilde{X}$ of the space X with respect to $\tilde{\varrho}$ is a compact space such that $\dim \tilde{X}$

$\leq \dim X$. Obviously, the mapping f is uniformly continuous with respect to $\tilde{g}$ and σ , which implies that f is extendable to a continuous mapping $\tilde{f}: \tilde{X} \rightarrow Z$. $\square$

1.13.4. Theorem on the dimension of the limit of an inverse sequence. *If the inverse sequence $S = \{X_i, \pi_i^j\}$ consists of separable metric spaces X_i such that $\dim X_i \leq n$ for $i = 1, 2, \dots$, then the limit $X = \varprojlim S$ satisfies the inequality $\dim X \leq n$.*

Proof. Let $\tilde{X}_1$ be a compact metric space which contains X_1 as a dense subspace and such that $\dim \tilde{X}_1 \leq n$. An inductive construction applying Lemma 1.13.3 yields for $i = 2, 3, \dots$ a compact metric space $\tilde{X}_i$ which contains X_i as a dense subspace and such that $\dim \tilde{X}_i \leq n$ and $\pi_{i-1}^i: X_i \rightarrow \tilde{X}_{i-1}$ is extendable to a continuous mapping $\tilde{\pi}_{i-1}^i: \tilde{X}_i \rightarrow \tilde{X}_{i-1}$. Letting

$$\tilde{\pi}_j^i = \tilde{\pi}_{j+1}^{j+1} \tilde{\pi}_{j+2}^{j+2} \dots \tilde{\pi}_{i-1}^{i-1} \quad \text{for } j < i \quad \text{and} \quad \tilde{\pi}_i^i = \text{id}_{\tilde{X}_i},$$

one obtains an inverse sequence $\tilde{S} = \{\tilde{X}_i, \tilde{\pi}_i^j\}$ consisting of compact metric spaces $\tilde{X}_i$ such that $\dim \tilde{X}_i \leq n$ for $i = 1, 2, \dots$. As $\varprojlim S \subset \varprojlim \tilde{S}$, it suffices to prove the theorem under the additional assumption that all spaces X_i are compact.

Let $\mathcal{U}$ be a finite open cover of the space X . Since the family of all sets $\pi_i^{-1}(U_i)$, where $\pi_i: X \rightarrow X_i$ is the projection and U_i is an open subset of X_i , is a base for the space X , and since the space X is compact, as is the limit of an inverse sequence of compact spaces, the cover $\mathcal{U}$ has a finite refinement $\{\pi_{i_k}^{-1}(U_{i_k})\}_{k=1}^m$, where $i_1 \leq i_2 \leq \dots \leq i_m$ and U_{i_k} is an open subset of X_{i_k} for $k = 1, 2, \dots, m$. The family $\{\pi_{i_m}^{-1}(V_k)\}_{k=1}^m$, where $V_k = (\pi_{i_k}^{i_m})^{-1}(U_{i_k})$, is also a refinement of $\mathcal{U}$. Consider a finite open refinement $\mathcal{V}$ of the cover $\{V_k\}_{k=1}^m$ of the space $M = \bigcup_{k=1}^m V_k \subset X_{i_m}$ such that $\text{ord } \mathcal{V} \leq n$; clearly, $\pi_{i_m}^{-1}(\mathcal{V})$ is a finite open refinement of $\mathcal{U}$ and has order $\leq n$. $\square$

Theorems 1.13.2 and 1.13.4 yield the characterization of dimension which was announced at the beginning of the present section.

1.13.5. Theorem on inverse sequences. *A compact metric space X satisfies the inequality $\dim X \leq n$ if and only if X is homeomorphic to the limit of an inverse sequence consisting of polyhedra of dimension $\leq n$.* $\square$

One readily sees that the theorem on ε -mappings is an easy consequence of the theorem on expansion in an inverse sequence (see Problem 1.13.A).

The study of relations between these two theorems led to the notion of a Π -like space. Let Π be a family of polyhedra; we say that a compact metric space X is Π -like if for every positive number ε there exists an ε -mapping of X onto a polyhedron $K \in \Pi$. The main theorem on Π -like spaces states that if either Π is a family of connected polyhedra (more generally, the space X below is a continuum) or Π is a hereditary family of polyhedra (i.e., together with the underlying polyhedron of a simplicial complex $\mathcal{K}$ contains the underlying polyhedra of all subcomplexes of $\mathcal{K}$), then a compact metric space X is Π -like if and only if X is homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where $K_i \in \Pi$ for $i = 1, 2, \dots$ and π_j^i maps X_i onto X_j for every $j \leq i$. Obviously, this is a generalization of the theorem on inverse sequences, because the family Π of all polyhedra of dimension $\leq n$ is a hereditary family of polyhedra and, for this family, the class of Π -like spaces coincides with the class of all compact metric spaces whose covering dimension is not larger than n . Various families Π yield interesting classes of compact metric spaces. Thus, for the family Π consisting of the interval I alone, one obtains the class of *snake-like continua*. Clearly, each snake-like continuum is one-dimensional; one proves that snake-like continua are embeddable in the plane (see the remark to Problem 1.13.B) and that there exists a universal space for the class of all snake-like continua. In a more general setting, one can prove that if Π is a hereditary family of polyhedra which, moreover, is additive (i.e., together with each pair K, L of polyhedra contains the disjoint sum of K and L), then there exists a universal space for the class of all Π -like spaces. Obviously, this implies the existence of a universal space for the class of all compact metric spaces whose covering dimension is not larger than n . In a less direct way, this also implies the existence of a universal snake-like continuum.

Historical and bibliographic notes

Theorem 1.13.2 was proved by Freudenthal in [1937]. Freudenthal's proof leads to an inverse sequence of polyhedra with quasi-simplicial mappings onto which, moreover, are irreducible (see Problem 1.13.F); the simpler proof given here was outlined by Isbell in [1959]. Theorem 1.13.4 was proved by Nagami in [1959]; for compact metric spaces it is implicitly contained in Freudenthal's paper [1937]. The notion of a Π -like space was introduced by Mardešić and Segal in [1963]. In the same paper Mardešić and Segal proved that if Π is a family of connected polyhedra, then a compact metric space is Π -like if and only if it is homeomorphic

to the limit of an inverse sequence of polyhedra in the family Π ; Pasynkov in [1966] showed that this is likewise true under the assumption that Π is a hereditary family of polyhedra. It was also shown by Pasynkov in [1966] that if Π is a hereditary and additive family of polyhedra, then there exists a universal space for the family of all Π -like spaces; a somewhat stronger result was obtained by McCord in [1966]. Snake-like continua were introduced by Bing in [1951]; among other things, Bing proved that each snake-like continuum is embeddable in the plane. The existence of a universal space for the class of all snake-like continua was established by Shori in [1965].

Problems

1.13.A. Let $X \neq \emptyset$ be the limit of an inverse sequence $\{X_i, \pi_j^i\}$ of compact metric spaces and let $\varepsilon_i = \text{mesh}(\{\pi_i^{-1}(x)\}_{x \in X_i})$, where $\pi_i: X \rightarrow X_i$ is the projection. Show that the sequence $\varepsilon_1, \varepsilon_2, \dots$ converges to zero.

1.13.B. Define an inverse sequence $S = \{X_i, \pi_j^i\}$, where $X_i = I$ for $i = 1, 2, \dots$ and π_j^i maps X_i onto X_j for every $j \leq i$, such that the limit of S cannot be embedded in the real line.

Remark. As shown by Isbell in [1959], the limit of an inverse sequence of compact subspaces of R^m is embeddable in R^{2^m} .

1.13.C (Isbell [1964]). Let $S = \{|\mathcal{K}_i|, \pi_j^i\}$ be an inverse sequence consisting of polyhedra of dimension ≥ 1 , where for $j \leq i$ the bonding mapping π_j^i maps $|\mathcal{K}_i|$ onto $|\mathcal{K}_j|$ and is the linear extension of a simplicial mapping of $\mathcal{K}_i$ to $\mathcal{K}_j$. Prove that if the limit of S contains more than one point, then it contains a subspace homeomorphic to the interval I . Applying the fact that there exist one-dimensional continua with no subspace homeomorphic to the interval I (see, Kuratowski [1968], p. 206), observe that there exist one-dimensional compact metric spaces which are not homeomorphic to the limit of an inverse sequence $\{|\mathcal{K}_i|, \pi_j^i\}$ of one-dimensional polyhedra, where for $j \leq i$ the bonding mapping π_j^i is the linear extension of a simplicial mapping of $\mathcal{K}_i$ to $\mathcal{K}_j$.

1.13.D. Define an inverse sequence $S = \{X_i, \pi_j^i\}$ of one-dimensional compact metric spaces, where for $j \leq i$ the bonding mapping π_j^i maps X_i onto X_j , such that the limit of S is homeomorphic to the Cantor set.

1.13.E (Mardešić and Segal [1963]). Let Π be the family consisting of all polyhedra which are unions of a one-simplex and a finite number of zero-simplexes. Observe that the Cantor set is a Π -like space, and yet is not homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where $K_i \in \Pi$ for $i = 1, 2, \dots$ and π_j^i maps X_i onto X_j for every $j \leq i$.

1.13.F (Freudenthal [1937]). Let $f: X \rightarrow |\mathcal{K}|$ be a continuous mapping of a topological space X to the underlying polyhedron of a simplicial complex $\mathcal{K}$. A continuous mapping $g: X \rightarrow |\mathcal{K}|$ is a *modification* of f if, for every $S \in \mathcal{K}$, $g(x) \in S$ whenever $f(x) \in S$; the mapping f is *irreducible* if there is no modification $g: X \rightarrow |\mathcal{K}|$ of f such that $f(X) \setminus g(X) \neq \emptyset$.

(a) Show that if F is a closed subset of X and $g': F \rightarrow |\mathcal{K}|$ is a modification of the restriction $f|_F: F \rightarrow |\mathcal{K}|$ of a continuous mapping $f: X \rightarrow |\mathcal{K}|$, then there exists a modification $g: X \rightarrow |\mathcal{K}|$ of f such that $g|_F = g'$.

(b) Check that if $f: X \rightarrow |\mathcal{K}|$ is an irreducible mapping, then for every subcomplex $\mathcal{K}_0$ of the complex $\mathcal{K}$ the restriction $f|_{\mathcal{K}_0}: f^{-1}(|\mathcal{K}_0|) \rightarrow |\mathcal{K}_0|$ also is irreducible.

(c) Let $\mathcal{S}$ be the simplicial complex consisting of all faces of a simplex and let $\mathcal{S}_0$ be the subcomplex of $\mathcal{S}$ consisting of all proper faces of the simplex under consideration. Prove that if for a continuous mapping $f: X \rightarrow |\mathcal{S}|$ there exists a modification $g': X \rightarrow |\mathcal{S}|$ of f such that an interior point p of $|\mathcal{S}|$, i.e., a point $p \in |\mathcal{S}| \setminus |\mathcal{S}_0|$, does not belong to $g'(X)$, then there also exists a modification $g: X \rightarrow |\mathcal{S}|$ of f such that $g|_{f^{-1}(|\mathcal{S}_0|)} = f|_{f^{-1}(|\mathcal{S}_0|)}$ and $p \in |\mathcal{S}| \setminus g(X)$.

Hint. There exists an open set $U \subset X$ containing $f^{-1}(|\mathcal{S}_0|)$ and such that the segment with end-points $f(x)$ and $g(x)$ does not contain the point p for any $x \in U$.

(d) Show that a continuous mapping $f: X \rightarrow |\mathcal{K}|$ is irreducible if and only if, for every $S \in \mathcal{K}$ such that an interior point of S belongs to $f(X)$, the restriction $f_S: f^{-1}(S) \rightarrow S$ is an essential mapping (see Problem 1.9.A).

(e) Prove that for every continuous mapping $f: X \rightarrow |\mathcal{K}|$ there exist an irreducible modification $g: X \rightarrow |\mathcal{K}|$ of f and a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ such that $g(X) = |\mathcal{K}_0|$.

Hint. To begin with, apply (c) to observe that if for a simplex $S \in \mathcal{K}$ the restriction $f_S: f^{-1}(S) \rightarrow S$ is essential, then for every face T of S the restriction $f_T: f^{-1}(T) \rightarrow T$ also is essential; then modify the mapping f on all simplexes for which the corresponding restriction is not essential, beginning with the simplexes of the highest dimension.

1.13.G. (a) (Pasyukov [1966]) Prove that if Π is a hereditary family of polyhedra, then a compact metric space X is Π -like if and only if X is homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where $K_i \in \Pi$ for $i = 1, 2, \dots$ and π_j^i is a quasi-simplicial mapping of K_i onto K_j for every $j \leq i$.

Hint. Apply Problem 1.13.F(e) to define a simplicial complex $\mathcal{K}_1$, a barycentric subdivision $\mathcal{P}_1$ of $\mathcal{K}_1$, and an irreducible mapping $f_1: X \rightarrow |\mathcal{P}_1|$ of X onto $|\mathcal{P}_1|$ in such a way that $|\mathcal{K}_1| \in \Pi$ and the inverse images of stars of vertices of $\mathcal{P}_1$ under f_1 all have diameters less than $1/2$.

Assume that for each $j < i$ a simplicial complex $\mathcal{K}_j$, a barycentric subdivision $\mathcal{P}_j$ of $\mathcal{K}_j$ and an irreducible mapping $f_j: X \rightarrow |\mathcal{P}_j|$ of X onto $|\mathcal{P}_j|$ are defined in such a way that $|\mathcal{K}_j| \in \Pi$ and the inverse images of stars of vertices of $\mathcal{P}_j$ under f_j all have diameters less than $1/2^j$; assume, moreover, that for each pair k, j of integers satisfying $k \leq j < i$ a quasi-simplicial mapping π_k^j of $|\mathcal{P}_j|$ onto $|\mathcal{P}_k|$ is defined in such a way that $\pi_k^k = \text{id}_{|\mathcal{P}_k|}$, $\pi_k^l \pi_l^j = \pi_k^j$ whenever $l \leq k$, and $\varrho(\pi_k^{j-1} f_{j-1}(x), \pi_k^j f_j(x)) < 1/2^j$ for $x \in X$ and $k \leq j-1$.

Let $\mathcal{P}$ be a barycentric subdivision of $\mathcal{P}_{i-1}$ such that $\text{mesh}(\{\pi_k^{i-1}(S) : S \in \mathcal{P}\}) < 1/2^i$ for every $k \leq i-1$; consider the cover $\mathcal{U}$ of the space X consisting of inverse images of stars of vertices of $\mathcal{P}$ under f_{i-1} and let $\delta_i = \min(\varepsilon_i, 1/2^i)$, where ε_i is a Lebesgue number for the cover $\mathcal{U}$. Define a simplicial complex $\mathcal{K}_i$, a barycentric subdivision $\mathcal{P}_i$ of $\mathcal{K}_i$ and an irreducible mapping $f_i: X \rightarrow |\mathcal{P}_i|$ of X onto $|\mathcal{P}_i|$ in such a way that $|\mathcal{K}_i| \in \Pi$ and the inverse images of stars of vertices of $\mathcal{P}_i$ all have diameters less than δ_i . Observe that by assigning to each vertex $p \in \mathcal{P}_i$ a vertex $q \in \mathcal{P}$ such that $f_i^{-1}(\text{St}_{\mathcal{P}_i}(p)) \subset f_{i-1}^{-1}(\text{St}_{\mathcal{P}}(q))$ one defines a simplicial mapping π_{i-1}^i of $\mathcal{P}_i$ to $\mathcal{P}$, and extend π_{i-1}^i to a quasi-simplicial mapping $\pi_{i-1}^i: |\mathcal{P}_i| \rightarrow |\mathcal{P}_{i-1}|$. Check that if $f_{i-1}(x) \in S \in \mathcal{P}$, then $\pi_{i-1}^i f_i(x) \in S$; deduce that $\pi_{i-1}^i(|\mathcal{P}_i|) = |\mathcal{P}_{i-1}|$ and that $\varrho(\pi_k^{i-1} f_{i-1}(x), \pi_k^i f_i(x)) < 1/2^i$ for $x \in X$ and $k \leq i-1$, where $\pi_k^i = \pi_k^{i-1} \pi_{i-1}^i$.

Note that for every natural number k the sequence of compositions $\pi_k^{k+1} f_{k+1}, \pi_k^{k+2} f_{k+2}, \dots$ uniformly converges to a continuous mapping $g_k: X \rightarrow |\mathcal{P}_k|$ and that $\pi_k^{k+1} g_{k+1} = g_k$. Check that X is homeomorphic to the limit of the inverse sequence $\{|\mathcal{K}_i|, \pi_j^i\}$.

(b) (Freudenthal [1937]) Prove that a compact metric space X satisfies the inequality $\dim X \leq n$ if and only if X is homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where K_i is a polyhedron of dimension $\leq n$ for $i = 1, 2, \dots$ and π_j^i is a quasi-simplicial mapping of K_i onto K_j for every $j \leq i$.

1.14. Dimension and axioms

Since the origin of dimension theory attempts have been made to characterize dimension functions by a few simple properties that could serve as a basis for an axiomatization of the theory. However, no satisfactory set of axioms has been proposed so far; the main drawbacks are that the axioms include properties which are either somewhat artificial or too close to the definition of dimension and that no part of dimension theory, no matter how small, can be deduced from the axioms. As the problem of axiomatization of dimension theory is of secondary importance, we shall confine ourselves to a rather sketchy discussion of this topic.

We shall consider a class $\mathcal{K}$ of topological spaces, which together with each space X contains all closed subspaces of X , and a function d defined on $\mathcal{K}$, having values which are integers larger than or equal to -1 or the "infinite number" ∞ , and such that $d(X) = d(Y)$ for each pair of homeomorphic spaces $X, Y \in \mathcal{K}$. By assuming that the function f satisfies some simple conditions which are known to be satisfied by the function $\dim$ we shall obtain three sets of axioms for dimension theory.

We begin with *Alexandroff's axioms*; in this instance $\mathcal{K}$ is the class of all compact subspaces of Euclidean spaces and the function d satisfies the following conditions:

- (A1) $d(\emptyset) = -1$, $d(\{0\}) = 0$ and $d(I^n) = n$ for $n = 1, 2, \dots$
- (A2) If a space $X \in \mathcal{K}$ is represented as the union of two closed subspaces X_1 and X_2 , then $d(X) = \max(d(X_1), d(X_2))$.
- (A3) For every space $X \in \mathcal{K}$ there exists a positive number ε such that if $f: X \rightarrow Y$ is an ε -mapping of X onto a space $Y \in \mathcal{K}$, then $d(X) \leq d(Y)$.
- (A4) For every space $X \in \mathcal{K}$ of cardinality larger than one there exists a closed set $L \subset X$ separating X and such that $d(L) < d(X)$.

1.14.1. Theorem. *The covering dimension $\dim$ is the only function d which satisfies conditions (A1)–(A4) in the class $\mathcal{K}$ of all compact subspaces of Euclidean spaces.*

Proof. Clearly, the function $d = \dim$ satisfies conditions (A1)–(A4). Consider now a function d which satisfies (A1)–(A4). It follows from (A1) and (A2) that if K is a polyhedron, then $d(K) = \dim K$, so that by virtue of (A3) and the theorem on ε -mappings, $d(X) \leq \dim X$ for every $X \in \mathcal{K}$.

Assume that there exists a space $X \in \mathcal{K}$ such that $d(X) < \dim X$. Let $d(X) = k$ and $\dim X = n$; it follows from (A1) that $n \geq 1$. Without loss

of generality one can suppose that each $Y \in \mathcal{K}$ such that $d(Y) < k$ satisfies the equality $d(Y) = \dim Y$. By virtue of Theorems 1.9.8 and 1.7.7, the space X contains an n -dimensional Cantor manifold M . From (A2) it follows that $d(M) \leq d(X) < n = \dim M$; since $d(M) \neq \dim M$, we have $d(M) = k$. Now apply (A4) to obtain a closed set $L \subset M$ separating M and such that $d(L) < d(M)$. Thus we have $\dim L = d(L) < k < n$, so that $\dim L \leq n-2$, which contradicts the definition of a Cantor manifold. Hence $d(X) = \dim X$ for every $X \in \mathcal{K}$. $\square$

It turns out that by replacing condition (A2) with

(A2') *If a space $X \in \mathcal{K}$ is represented as the union of a sequence $X_1, X_2, \dots$ of closed subspaces, then $d(X) = \sup\{d(X_i): i = 1, 2, \dots\}$*

one obtains a set of conditions which characterizes the covering dimension $\dim$ in the class $\mathcal{K}$ of all subspaces of Euclidean spaces (see Problem 1.14.C).

We now pass to *Nishiura's axioms*; in this instance $\mathcal{K}$ is the class of all separable metric spaces and the function d satisfies the following conditions:

(N1) $d(\{0\}) = 0$.

(N2) *If Y is a subspace of a space $X \in \mathcal{K}$, then $d(Y) \leq d(X)$.*

(N3) *If a space $X \in \mathcal{K}$ is represented as the union of a sequence $X_1, X_2, \dots$ of closed subspaces, then $d(X) = \sup\{d(X_i): i = 1, 2, \dots\}$.*

(N4) *If a space $X \in \mathcal{K}$ is represented as the union of two subspaces X_1 and X_2 , then $d(X) \leq d(X_1) + d(X_2) + 1$.*

(N5) *For every space $X \in \mathcal{K}$ there exists a compactification $\tilde{X} \in \mathcal{K}$ such that $d(\tilde{X}) = d(X)$.*

(N6) *If a non-empty space $X \in \mathcal{K}$ satisfies the inequality $d(X) < \infty$, then for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset V$ such that*

$$x \in U \subset V \quad \text{and} \quad d(\text{Fr } U) \leq d(X) - 1.$$

One can prove that the covering dimension $\dim$ is the only function d which satisfies conditions (N1)–(N6) in the class $\mathcal{K}$ of all separable metric spaces (see Problem 1.14.B).

We conclude with *Menger's axioms*, chronologically the earliest set of axioms for dimension theory; in this instance $\mathcal{K}$ is the class of all subspaces of Euclidean m -space R^m and the function d satisfies the following conditions:

- (M1) $d(\emptyset) = -1$, $d(\{0\}) = 0$ and $d(R^n) = n$ for $n = 1, 2, \dots, m$.
 (M2) If Y is a subspace of a space $X \in \mathcal{K}$, then $d(Y) \leq d(X)$.
 (M3) If a space $X \in \mathcal{K}$ is represented as the union of a sequence $X_1, X_2, \dots$ of closed subspaces, then $d(X) \leq \sup\{d(X_i): i = 1, 2, \dots\}$.
 (M4) For every space $X \in \mathcal{K}$ there exists a compactification $\tilde{X} \in \mathcal{K}$ such that $d(\tilde{X}) = d(X)$.

Menger put forward the hypothesis that for every natural number m the covering dimension $\dim$ is the only function d which satisfies conditions (M1)–(M4) in the class of all subspaces of Euclidean m -space R^m and showed that the hypothesis is valid for $m \leq 2$ (see Problem 1.14.D). The problem whether the hypothesis is valid for $m > 2$ is still open. Let us recall (cf. the discussion in the final part of Section 1.11) that for $m > 3$ it is not even known whether the function $d = \dim$ satisfies condition (M4). Clearly, the covering dimension $\dim$ satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of Euclidean spaces. We find, however, that $\dim$ is not the only function with this property; each cohomological dimension $\dim_G$ with respect to a finitely generated abelian group G also satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of Euclidean spaces.

Historical and bibliographic notes

Theorem 1.14.1 was proved by Alexandroff in [1932]. Ščepin announced in [1972] that by replacing condition (A2) by the stronger condition (A2') one obtains a set of axioms that characterizes the covering dimension $\dim$ in the class of all subspaces of Euclidean spaces; the proof was published in Alexandroff and Pasynkov's book [1973], where a stronger result, also due to Ščepin, is announced, viz., that the same set of axioms characterizes $\dim$ in the class of all metric spaces whose covering dimension is finite. Nishiura proved in [1966] that the axioms (N1)–(N6) characterize the covering dimension $\dim$ in the class of all separable metric spaces. Sakai in [1968] and Aarts in [1971] modified Nishiura's axioms to obtain a set of axioms that characterizes $\dim$ in the class of all metric spaces. The fact that conditions (M1)–(M4) characterize the covering dimension $\dim$ in the class of all subspaces of the plane, and also in the class of all subspaces of the real line, was established by Menger in [1929]. The theorem that each cohomological dimension with respect to a finitely generated

abelian group satisfies conditions (M1)–(M4) in the class of all subspaces of Euclidean spaces was proved by I. Švedov; the proof was first published in Kuz'minov's paper [1968]. A set of axioms characterizing the covering dimension $\dim$ in the class of all (not necessarily metric) compact spaces whose dimension $\dim$ is finite was given by Lokucievskii in [1973].

Problems

1.14.A (Ščepin, cited in Alexandroff and Pasynkov [1973]; announcement Ščepin [1972]). Verify that the axioms (A1)–(A4) are independent.

1.14.B (Nishiura [1966]). (a) Prove that the covering dimension $\dim$ is the only function d which satisfies conditions (N1)–(N6) in the class $\mathcal{K}$ of all separable metric spaces.

(b) Verify that the axioms (N1)–(N6) are independent.

Hint. To verify that (N2) is independent of the remaining axioms, observe that every separable metric space which is not compact has an infinite-dimensional compactification.

1.14.C (Ščepin, cited in Alexandroff and Pasynkov [1973]; announcement Ščepin [1972]). (a) Show that for every separable metric space X such that $\dim X = n \geq 0$ there exists a separable metric space X_*

$= \{x, y\} \cup \bigcup_{i=1}^{\infty} X_i$, where for $i = 1, 2, \dots$ X_i is a closed subspace of X_*

homeomorphic to X , with the property that no closed set $L \subset X_*$ satisfying the inequality $\dim L \leq n-2$ separates the space X_* between x and y .

(b) Show that for every separable metric space X such that $\dim X = n \geq 0$ there exists a separable metric space $X^* = \bigcup_{i=1}^{\infty} X_i$,

where, for $i = 1, 2, \dots$, X_i is a closed subspace of X^* homeomorphic to X , with the property that no closed set $L \subset X^*$ satisfying the inequality $\dim L \leq n-2$ separates the space X^* .

(c) Prove that the covering dimension $\dim$ is the only function d which satisfies conditions (A1), (A2'), (A3) and (A4) in the class $\mathcal{K}$ of all subspaces of Euclidean spaces.

(d) Verify that (A2') is independent of the axioms (A1)–(A4).

1.14.D. (a) (Menger [1929]) Check that the covering dimension $\dim$ is the only function d which satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of the real line.

(b) (Kuratowski and Menger [1930]) Applying the Denjoy–Riesz theorem, i.e., the fact that every zero-dimensional compact subspace of the plane is contained in an arc $L \subset R^2$ (see Kuratowski [1968], p. 539), and the Moore theorem, i.e., the fact that if there exists a continuous mapping $f: S^2 \rightarrow X$ of the two-sphere onto a space X such that the fibres of f are connected and do not separate S^2 , then the space X is homeomorphic to S^2 (see Kuratowski [1968], p. 533), prove that every zero-dimensional F_σ -set in the plane is contained in the union of a sequence of arcs which are pairwise disjoint and have diameters converging to zero.

Hint. Show that if a zero-dimensional compact set A is contained in an open set $U \subset S^2$, then for every $\varepsilon > 0$ there exists a sequence $L_1, L_2, \dots$ of arcs such that $L_i \cap L_j = \emptyset$ whenever $i \neq j$, $\lim \delta(L_i) = 0$, $\delta(L_i) < \varepsilon$ for $i = 1, 2, \dots$, and $A \subset \bigcup_{i=1}^{\infty} L_i \subset \overline{\bigcup_{i=1}^{\infty} L_i} \subset U$. Consider first the case where the set U is connected, observe that no component of $S^2 \setminus U$ separates S^2 and apply the Moore theorem.

(c) (Menger [1929], Kuratowski and Menger [1930]) Applying (b) and the Moore theorem, prove that the covering dimension $\dim$ is the only function d which satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of the plane.

Hint. Let $\mathcal{D}_k = \{X \subset R^2: d(X) \leq k\}$ for $k = -1, 0, 1, 2$; it suffices to show that if the family $\mathcal{D}_k$ contains an n -dimensional space, then $\mathcal{D}_k$ contains all n -dimensional subspaces of the plane. Only the case of $n = 1$ and $k = 0, 1$ is non-trivial. Let $X \in \mathcal{D}_k$ be a one-dimensional space; by virtue of Corollary 1.9.9 and condition (M4) one can assume that X is a continuum. Place a copy of the continuum X in a square K in such a way that the four corners of K are the only points of the boundary of K which belong to X . Then divide K into 9 congruent squares and place in the same way a copy of X in each of these smaller squares. Continue the procedure, dividing K consecutively into smaller and smaller squares, and consider the union X^* of countably many smaller and smaller copies of X placed in these squares. Prove that $d(X^*) = k$ and $\text{ind}(K \setminus X^*) = 0$. Apply the Lavrentieff theorem (see [GT], Theorem 4.3.21) and condition (M4) to obtain a G_δ -set Y such that $X^* \subset Y \subset K$ and $d(Y) = k$. Then apply (b) to the zero-dimensional F_σ -set $\text{Int } K \setminus Y$ and denote by

$L_1, L_2, \dots$ the arcs in (b). Deduce from the Moore theorem that the space obtained from R^2 by identifying to points the set $R^2 \setminus \text{Int } K$ and each of the arcs L_i is homeomorphic to S^2 . Show that the space Y contains a subspace homeomorphic to the complement of a countable dense subset of R^2 and deduce from Problem 1.8.D that Y is a universal space for the class of all one-dimensional subspaces of the plane.

(d) (Menger [1929]) Verify that the axioms (M1)–(M4) are independent in the class $\mathcal{K}$ of all subspaces of the plane.

CHAPTER 2

THE LARGE INDUCTIVE DIMENSION

Outside the class of separable metric spaces the dimensions ind , Ind and dim generally do not coincide. Nevertheless, a number of theorems established in Chapter 1 extend beyond this class of spaces. In larger classes they hold either for the dimension Ind , or for the dimension dim , or else for both Ind and dim . The dimension ind is practically of no importance outside the class of separable metric spaces and from now on will reappear here only occasionally. Slightly exaggerating, one could say that ind is a satisfactory dimension function only when it is equal to Ind . Thus, for general spaces we have two separate dimension theories: the theory of the large inductive dimension Ind and the theory of the covering dimension dim . They are both poorer and less harmonious than the dimension theory of separable metric spaces, yet they contain many interesting theorems and shed light on classical dimension theory. It should be noted that, while the dimension dim behaves properly in the class of all normal spaces, the dimension Ind does so only in the more restricted class of strongly hereditarily normal spaces. The present chapter and the next are devoted to a closer study of Ind and of dim , respectively. In the final chapter it will be proved that the dimensions Ind and dim coincide in the class of all metrizable spaces and that a dimension theory can be developed in that class which is by no means inferior to the dimension theory of separable metric spaces.

Section 2.1 contains supplementary information about hereditarily normal spaces and an investigation of the class of strongly hereditarily normal spaces.

In Section 2.2 those rare theorems on the dimension Ind are proved which hold either in all normal spaces or in all hereditarily normal spaces. In the final part of the section two important examples are described, showing that neither the subspace theorem nor the sum theorem holds for the dimension Ind in the class of all normal spaces.

Section 2.3 is crucial for the present chapter; we develop in it a dimension theory for Ind in the class of strongly hereditarily normal spaces. The main results of this theory are the subspace theorem and a group of sum theorems.

The last section is devoted to a study of the relations between ind and Ind and to the Cartesian product theorems for the dimension Ind .

2.1. Hereditarily normal and strongly hereditarily normal spaces

The large inductive dimension Ind is defined for all normal spaces (see Definition 1.6.1). It turns out however, that in such an extensive class of spaces the dimension Ind develops some pathological properties. As the reader will see in Section 2.2, there exist a compact space Z and a normal subspace X of Z such that $\text{Ind}X > \text{Ind}Z$. There also exists a compact space X with $\text{Ind}X = 2$ which can be represented as the union of two closed subspaces F_1 and F_2 such that $\text{Ind}F_1 = \text{Ind}F_2 = 1$. Finally, there exist compact spaces X and Y such that $\text{Ind}(X \times Y) > \text{Ind}X + \text{Ind}Y$. Besides, since a subspace of a normal space is not necessarily normal (see [GT], Example 2.3.36 or 3.2.7), it may happen that the dimension Ind is defined for a space X , and yet is not defined for a subspace M of X . From all the adduced phenomena one gathers that to develop a satisfactory theory of the large inductive dimension Ind one has to restrict the class of spaces under consideration. As the spaces in all the examples cited above are not hereditarily normal, we might expect that no such pathological phenomena can occur in the class of hereditarily normal spaces. Still, as has recently been shown, the dimension Ind is not monotonic in the latter class, so that a further restriction of the class of spaces is necessary.

A natural class of spaces where a satisfactory theory of the large inductive dimension can be developed is the class of strongly hereditarily normal spaces; it is contained in the class of all hereditarily normal spaces and constitutes a common extension of the class of perfectly normal spaces and the class of hereditarily paracompact spaces. The present section is devoted to a study of the topological properties of hereditarily normal spaces and strongly hereditarily normal spaces.

Let us recall that a space X is *hereditarily normal* if every subspace of X is normal. We begin with two simple characterizations of hereditarily

normal spaces; in the second one appears the notion of separated subsets of a topological space introduced in Section 1.2.

2.1.1. Theorem. *For every T_1 -space X the following conditions are equivalent:*

- (a) *The space X is hereditarily normal.*
- (b) *Every open subspace of X is normal.*
- (c) *For every pair A, B of separated sets in X there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious. We shall show that (b) $\Rightarrow$ (c). Consider a pair A, B of separated sets in a space X which satisfies (b) and let $M = X \setminus (\bar{A} \cap \bar{B})$. Obviously, M is an open subspace of the space X and $A, B \subset M$. The closures of A and B in M are disjoint, so that by the normality of M there exist sets $U, V \subset M$ open in M and such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The subspace M being open in X , the sets U, V are open in X , so that the space X satisfies (c).

To complete the proof it remains to show that (c) $\Rightarrow$ (a). Consider an arbitrary subspace M of a space X which satisfies (c) and a pair A, B of disjoint closed subsets of M . Clearly A and B are separated in X , so that there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The intersections $M \cap U$ and $M \cap V$ are open in M and disjoint, and contain A and B , respectively, which means that the space X satisfies (a). $\square$

Let us recall that a family $\{A_s\}_{s \in S}$ of subsets of a topological space X is *point-finite* (*point-countable*) if for every point $x \in X$ the set $\{s \in S: x \in A_s\}$ is finite (countable).

2.1.2. Definition. A topological space X is called *strongly hereditarily normal* if X is a T_1 -space and for every pair A, B of separated sets in X there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$ and U and V can be represented as the union of a point-finite family of open F_σ -sets in X .

Obviously, every strongly hereditarily normal space is hereditarily normal; moreover, every subspace M of a strongly hereditarily normal space is strongly hereditarily normal, because any sets A, B separated in M are also separated in X .

Besides hereditary normality one considers another strengthening of normality, namely perfect normality. Let us recall that a space X is

perfectly normal if X is a normal space and every open subset of X is an F_σ -set in X . Perfect normality is a hereditary property; in particular, every perfectly normal space is hereditarily normal (cf. Lemma 3.1.22 below). The latter fact and definition 2.1.2 imply

2.1.3. Theorem. *Every perfectly normal space is strongly hereditarily normal.* $\square$

2.1.4. Theorem. *Every hereditarily paracompact space is strongly hereditarily normal.*

Proof. Since every paracompact space is normal (see [GT], Theorem 5.1.5), every hereditarily paracompact space is hereditarily normal. Thus to complete the proof it suffices to show that every open subset U of a hereditarily paracompact space X can be represented as the union of a point-finite family of open F_σ -sets in X .

For every $x \in U$ consider a neighbourhood U_x of the point x such that $\overline{U_x} \subset U$. The family $\mathcal{U} = \{U_x\}_{x \in U}$ is an open cover of the subspace U of X . Since the space U is paracompact, there exists a locally finite partition of unity $\{f_s\}_{s \in S}$ on U subordinated to $\mathcal{U}$ (see [GT], Theorem 5.1.9). For every $s \in S$ the set $U_s = f_s^{-1}((0, 1])$ is an open F_σ -set in U . As U is an open subspace of X , the set U_s is also open in X . Furthermore, $U_s \subset U_x \subset \overline{U_x} \subset U$ for a certain $x \in U$, so that U_s is an F_σ -set in the closed subspace $\overline{U_x}$ of X which implies that U_s is an F_σ -set in X . Finally, the family $\{U_s\}_{s \in S}$ is point-finite and $U = \bigcup_{s \in S} U_s$. $\square$

We shall now slightly generalize the last theorem. Let us recall that a topological space X is *weakly paracompact*¹⁾ if X is a Hausdorff space and every open cover of the space X has a point-finite open refinement. Weakly paracompact spaces are not necessarily normal (see [GT], Example 5.3.4 or Exercise 5.3.B(b)).

2.1.5. Theorem. *Every hereditarily weakly paracompact hereditarily normal space is strongly hereditarily normal.*

Proof. It suffices to show that every open subset U of a hereditarily weakly paracompact hereditarily normal space X can be represented as the union of a point-finite family of open F_σ -sets in X .

¹⁾ The terms *metacompact* and *point-paracompact* are also used.

For every $x \in U$ consider a neighbourhood U_x of the point x such that $\overline{U_x} \subset U$. The family $\mathcal{U} = \{U_x\}_{x \in U}$ is an open cover of the subspace U of X . Since the space U is weakly paracompact, the cover $\mathcal{U}$ has a point-finite open refinement $\mathcal{V} = \{V_s\}_{s \in S}$. The cover $\mathcal{V}$, as any point-finite open cover of a normal space (see [GT], Theorem 1.5.18), has a closed shrinking $\{F_s\}_{s \in S}$. By virtue of Urysohn's lemma for every $s \in S$ there exists a continuous function $f_s: U \rightarrow I$ such that $f_s(U \setminus V_s) \subset \{0\}$ and $f_s(F_s) \subset \{1\}$. For every $s \in S$ the set $U_s = f_s^{-1}((0, 1])$ is an open F_σ -set in U . Obviously, the set U_s is open in X . Furthermore, $U_s \subset V_s \subset U_x \subset \overline{U_x} \subset U$ for a certain $x \in U$, so that U_s is an F_σ -set in X . Finally, the family $\{U_s\}_{s \in S}$ is point-finite and $U = \bigcup_{s \in S} U_s$. $\square$

We conclude this section with two examples: an example of a compact strongly hereditarily normal space which is neither perfectly normal nor hereditarily weakly paracompact and an example of a compact hereditarily normal space which is not strongly hereditarily normal.

2.1.6. Example. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1 . The set W is well-ordered by the natural order $<$. Consider on W the topology obtained by taking as a base all sets of the form

$$(1) \quad (\alpha, \omega_1] = \{x: \alpha < x\}, \quad [0, \beta) = \{x: x < \beta\} \\ \text{and} \quad (\alpha, \beta) = \{x: \alpha < x < \beta\},$$

where $\alpha < \beta \leq \omega_1$. One easily sees that W is a Hausdorff space. We shall show that W is compact.

Let $\{U_s\}_{s \in S}$ be an open cover of the space W and let A consist of all $\alpha \in W$ such that the set $[0, \alpha] = \{x: x \leq \alpha\}$ is contained in the union of finitely many members of the cover under consideration. It suffices to show that $W \setminus A = \emptyset$.

Assume that $W \setminus A \neq \emptyset$ and denote by x_0 the smallest element of this set. Choose an $s_0 \in S$ such that $x_0 \in U_{s_0}$; since $x_0 > 0$, there exists an $x < x_0$ such that $(x, x_0] \subset U_{s_0}$. By the definition of x_0 , the point x belongs to A , so that $[0, x] \subset \bigcup_{i=1}^k U_{s_i}$. It follows that $[0, x_0] \subset \bigcup_{i=0}^k U_{s_i}$, and we have a contradiction.

We shall now prove that every open subspace U of W is normal, i.e., that the space W is hereditarily normal. We shall say that a set $C \subset W$ is *convex* if $(\alpha, \beta) \subset C$ whenever $\alpha, \beta \in C$. One readily sees that the union of a family of convex sets is convex provided that the intersection of the

family is non-empty. Hence, considering for each point $x \in U$ the union of all convex subsets of U which contain that point, we obtain a decomposition of the set U into pairwise disjoint convex sets $\{U_s\}_{s \in S}$, which will be called the *convex components* of the set U . The set U being open, all U_s 's are open subsets of W , so that $U = \bigoplus_{s \in S} U_s$, where the symbol $\bigoplus$ denotes the sum of topological spaces (see [GT], Section 2.2). Since the set W is well-ordered by $<$, every open and convex proper subset of W is of form (1). Thus, to prove that U is normal it suffices to show that all subspaces of form (1) are normal. The subspaces of the form $(\alpha, \omega_1] = [\alpha + 1, \omega_1]$, where $\alpha < \omega_1$, are normal as closed subspaces of the normal space W . The subspaces of the form $[0, \beta)$ and (α, β) , where $\beta < \omega_1$, are regular second-countable spaces and thus are metrizable (see [GT], Theorem 4.2.9) and, *a fortiori*, normal. It remains to prove that the subspace $W_0 = W \setminus \{\omega_1\}$ of the space W is normal.

We shall show more, viz., that for every pair A, B of disjoint closed subsets of W_0 the closures $\overline{A}$ and $\overline{B}$ of A and B in the space W are disjoint. This follows from the fact that ω_1 belongs to at most one of the sets $\overline{A}$ and $\overline{B}$. Indeed, if we had $\omega_1 \in \overline{A} \cap \overline{B}$, we could define inductively two sequences, $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$, of countable ordinal numbers satisfying

$$\alpha_i < \beta_i < \alpha_{i+1}, \quad \alpha_i \in A, \quad \beta_i \in B \quad \text{for } i = 1, 2, \dots;$$

then the smallest ordinal number γ larger than all α_i 's and β_i 's would belong to $\overline{A} \cap \overline{B}$, which is impossible, because $\gamma < \omega_1$ since the set W_0 contains no countable cofinal subset. Let us recall that a subset K of an ordered set X is *cofinal* in X if for every $\alpha \in X$ there exists a $\beta \in K$ such that $\alpha \leq \beta$; a set $K \subset X$ is *bounded* in X , if it is not cofinal in X .

Since the space W is hereditarily normal, in proving that W is a strongly hereditarily normal space it suffices to consider a pair of open separated sets, i.e., disjoint open sets $U, V \subset X$. Let $U = \bigoplus_{s \in S} U_s$ and $V = \bigoplus_{t \in T} V_t$ be the decompositions of U and V into convex components. If all the convex components U_s and V_t are bounded in W , they are countable and, *a fortiori*, they are F_σ -sets in W . So, in this case U and V can themselves be represented as the union of a point-finite family of open F_σ -sets in W . On the other hand, if one of the convex components, say the convex component U_{s_0} , of the set U is cofinal in W , then the set V is bounded in W . In this case the sets U and V are contained, respectively, in disjoint open sets $U \cup \{\omega_1\}$ and V which can be represented as the union of a point-finite family of open F_σ -sets in W .

Every closed subset of W which is contained in W_0 is bounded in W , so that every F_σ -set in W which is contained in W_0 also is bounded in W . It follows that the open subset W_0 of the space W is not an F_σ -set in W . Thus the space W is not perfectly normal.

To prove that the space W is not hereditarily weakly paracompact it is enough to show that if $\mathcal{U} = \{U_s\}_{s \in S}$ is an open cover of the subspace W_0 of W and all U_s 's are bounded in W_0 , then $\mathcal{U}$ is not point-finite. Suppose that $\mathcal{U}$ is point-finite. Hence for every $x \in W_0$ the set $\text{St}(x, \mathcal{U}) = \bigcup \{U_s : x \in U_s\}$, i.e., the *star of the point x with respect to the cover $\mathcal{U}$* , is bounded; thus one can define inductively an increasing sequence $\alpha_1 < \alpha_2 < \dots$ of countable ordinal numbers satisfying

$$(2) \quad \alpha_{i+1} \notin \text{St}(\alpha_i, \mathcal{U}) \quad \text{for } i = 1, 2, \dots$$

The smallest ordinal number γ larger than all α_i 's belongs to a member U_{s_0} of the cover $\mathcal{U}$. The set U_{s_0} is open and thus contains almost all α_i 's. This contradiction of (2) shows that the cover $\mathcal{U}$ is not point-finite. $\square$

We now turn to the example of a compact hereditarily normal space which is not strongly hereditarily normal.

2.1.7. Example. Let W' be a topological space homeomorphic to the space W described in Example 2.1.6 and such that $W' \cap W = \emptyset$, and let W'_0 and ω'_1 denote the counterparts of W_0 and ω_1 in W' . The sum $W \oplus W'$ is hereditarily normal and so is the space X obtained from $W \oplus W'$ by identifying the points ω_1 and ω'_1 , i.e., the quotient space determined by the decomposition of $W \oplus W'$ into the set $\{\omega_1, \omega'_1\}$ and all one-point sets $\{x\}$ with $x \in W_0 \cup W'_0$. However, the space X is not strongly hereditarily normal, because for its separated subsets $q(W_0)$ and $q(W'_0)$, where $q: W \oplus W' \rightarrow X$ is the natural quotient mapping, there exist no disjoint open sets U, V , which can be represented as the union of a point-finite family of open F_σ -sets in X , such that $q(W_0) \subset U$ and $q(W'_0) \subset V$. Indeed, the only disjoint open sets $U, V \subset X$ that contain $q(W_0)$ and $q(W'_0)$, respectively, are $U = q(W_0)$ and $V = q(W'_0)$. Now, if $q(W_0)$ could be represented as the union of a point-finite family of open F_σ -sets in X , the subspace W_0 of W would have a point-finite open cover by sets bounded in W_0 , whereas, by the last part of Example 2.1.6, no such cover exists. $\square$

Historical and bibliographic notes

Various restrictions of the class of normal spaces to a class where a satisfactory theory of the large inductive dimension Ind can be developed

have been proposed more than once. The first step was made by Čech in [1932], who developed the theory of the large inductive dimension in the class of perfectly normal spaces. Then, Dowker in [1953] introduced the class of totally normal spaces and extended the theory to that class. Let us recall here that a topological space X is *totally normal*, if X is a normal space and every open subset U of X can be represented as the union of a locally finite in U family of open F_σ -sets in X . Clearly, every perfectly normal space is totally normal and so is every hereditarily paracompact space; the fact that every totally normal space is hereditarily normal is by no means obvious (see Problem 2.1.C). Subsequently, Pasynkov in [1967] defined *Dowker spaces* as hereditarily normal spaces in which every open set can be represented as the union of a point-finite family of open F_σ -sets, and announced extensions of some theorems on Ind to this class of spaces. Proofs of the announced theorems, together with further results, were published by Lifanov and Pasynkov in [1970]. Finally, Nishiura in [1977] introduced the class of super normal spaces and correspondingly extended the theory of the large inductive dimension; according to Nishiura, a topological space X is called *super normal* if X is a T_1 -space and for every pair A, B of separated sets in X there exist open sets $U, V \subset X$ such that $A \subset U, B \subset V, U \cap V = \emptyset$ and U and V can be represented as the union of a locally finite, in U and V respectively, family of open F_σ -sets in X . Our class of strongly hereditarily normal spaces is obtained by amalgamating the ideas of Pasynkov and Nishiura. Let us add that in the process of extending the theory of the large inductive dimension from the class of totally normal spaces to the larger classes mentioned above, only slight modifications in Dowker's original arguments were necessary.

Problems

2.1.A. Deduce from Urysohn's lemma that a subset A of a normal space X is an open F_σ -set (a closed G_δ -set) if and only if there exists a continuous function $f: X \rightarrow I$ such that $A = f^{-1}((0, 1])$ (such that $A = f^{-1}(0)$).

2.1.B. (a) Show that a topological space X is strongly hereditarily normal if and only if X is hereditarily normal and every open domain in X can be represented as the union of a point-finite family of open F_σ -sets in X . Let us recall that a subset U of a topological space X is an open domain in X if $U = \text{Int } \bar{U}$.

(b) Applying the fact that for every pair U, V of disjoint open sets

in the Cantor cube $D^c = \prod_{t \in I} D_t$, where $D_t = D$ for $t \in I$, there exists a countable set $I_0 \subset I$ such that the projections of U and V onto $\prod_{t \in I_0} D_t$ are disjoint (see [GT], Problem 2.7.12 (b)), show that every open domain in D^c is an F_σ -set. Observe, using Remark 1.3.18, that the Cantor cube D^c is not hereditarily normal.

2.1.C (Dowker [1953]). (a) Show that if a space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed subspaces each of which is normal, then X is a normal space.

Hint. Map the sum $\bigoplus_{s \in S} F_s$ onto X and apply the fact that normality is an invariant of closed mappings.

(b) Show that if a space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed normal subspaces such that $F_i \subset \text{Int} F_{i+1}$ for $i = 1, 2, \dots$, then X is a normal space.

Hint. Note that the family $\{A_i\}_{i=1}^\infty$, where $A_1 = F_1$ and $A_i = F_i \setminus \text{Int} F_{i-1}$ for $i > 1$, is locally finite.

(c) Show that every totally normal space is hereditarily normal and deduce that every subspace of a totally normal space is totally normal.

Hint. Apply (a), (b) and Problem 2.1.A.

2.1.D. (a) Prove that every paracompact totally normal space is hereditarily paracompact.

(b) Prove that every weakly paracompact Dowker space is hereditarily weakly paracompact.

2.1.E. (a) Prove that a T_1 -space X is normal if and only if for every closed set $F \subset X$ and each open set $W \subset X$ that contains F there exists a sequence $W_1, W_2, \dots$ of open subsets of X such that $F \subset \bigcup_{i=1}^\infty W_i$ and $\overline{W_i} \subset W$ for $i = 1, 2, \dots$

Hint. Let A and B be disjoint closed subsets of a T_1 -space X which has the property under consideration. Define sequences $W_1, W_2, \dots$ and $V_1, V_2, \dots$ of open subsets of X such that

$$A \subset \bigcup_{i=1}^\infty W_i, \quad B \subset \bigcup_{i=1}^\infty V_i \quad \text{and} \quad B \cap \overline{W_i} = \emptyset = A \cap \overline{V_i} \quad \text{for } i = 1, 2, \dots$$

Verify that the sets $U = \bigcup_{i=1}^\infty G_i$ and $V = \bigcup_{i=1}^\infty H_i$, where $G_i = W_i \setminus \bigcup_{j \leq i} \overline{V_j}$

and $H_i = V_i \setminus \bigcup_{j \leq i} \overline{W}_j$, satisfy the conditions $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

(b) Prove that a T_1 -space X is perfectly normal if and only if for every open set $W \subset X$ there exists a sequence $W_1, W_2, \dots$ of open subsets of X such that $W = \bigcup_{i=1}^{\infty} W_i$ and $\overline{W}_i \subset W$ for $i = 1, 2, \dots$

2.2. Basic properties of the dimension Ind in normal and hereditarily normal spaces

Among the theorems of dimension theory established in Chapter 1 only a few are valid for the dimension Ind in normal or hereditarily normal spaces. As noted above, the dimension Ind is not monotonic in hereditarily normal spaces and the sum theorem for the dimension Ind does not hold in normal spaces, so that one cannot think of developing a dimension theory for Ind in those classes of spaces. In the following section such a theory will be developed in the more restricted class of strongly hereditarily normal spaces. In the present section we merely clear the ground for the considerations of the next one.

The definition of the large inductive dimension Ind was stated in Section 1.6; let us recall that $\text{Ind}X = -1$ if and only if $X = \emptyset$, and that a normal space X satisfies the inequality $\text{Ind}X \leq n \geq 0$ if and only if for every closed set $A \subset X$ and each open set $V \subset X$ which contains the set A there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\text{IndFr}U \leq n-1$. In other words, $\text{Ind}X \leq n \geq 0$ if and only if for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{Ind}L \leq n-1$.

Since normality is not a hereditary property, it may happen that the dimension Ind is defined for a space X and yet is not defined for a subspace $M \subset X$. Still, normality being hereditary with respect to closed subsets, Ind M is defined for every closed subspace $M \subset X$. Moreover in much the same way as Theorem 1.1.2 one obtains

2.2.1. Theorem. *For every closed subspace M of a normal space X we have $\text{Ind}M \leq \text{Ind}X$. $\square$*

The counterpart of Theorem 1.5.1 reads as follows

2.2.2. Theorem. *If X is a normal space and $\text{Ind} X = n \geq 1$, then for $k = 0, 1, \dots, n-1$ the space X contains a closed subspace M such that $\text{Ind} M = k$. $\square$*

In Example 2.2.11 we shall define a compact space Z which contains a normal subspace X such that $\text{Ind} X > \text{Ind} Z$. Hence, in Theorem 2.2.1 the assumption that M is a closed subspace of X cannot be replaced by the weaker assumption that $\text{Ind} M$ is defined. Recently, a much stronger result was obtained: one defined a hereditarily normal space X such that $\text{Ind} X = \dim X = 0$, and yet X contains, for every natural number n , a subspace A_n with $\text{Ind} A_n = \dim A_n = n$. The latter example, however, is too difficult to be described in this book.

We shall now show that for subspaces of a fixed hereditarily normal space X monotonicity of the dimension Ind is equivalent to its being monotonic with respect to open subspaces.

2.2.3. Proposition. *For every hereditarily normal space X the following conditions are equivalent:*

- (a) *For each subspace $Y \subset X$ and every subspace M of Y we have $\text{Ind} M \leq \text{Ind} Y$.*
- (b) *For each subspace $Y \subset X$ and every open subspace U of Y we have $\text{Ind} U \leq \text{Ind} Y$.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious. Suppose that X satisfies (b). Condition (a) is satisfied if $\text{Ind} Y = \infty$, so that it suffices to consider subspaces $Y \subset X$ with $\text{Ind} Y < \infty$. We shall apply induction with respect to $\text{Ind} Y$ to show that $\text{Ind} M \leq \text{Ind} Y$ whenever $M \subset Y$. Clearly, the inequality holds if $\text{Ind} Y = -1$. Assume that the inequality is proved for all subspaces of X the large inductive dimension of which does not exceed $n-1 \geq -1$ and consider a subspace $Y \subset X$ with $\text{Ind} Y = n$ and an arbitrary subspace M of Y . Let A and B be disjoint closed subsets of M . As $U = Y \setminus (\overline{A \cap B})$ is an open subspace of Y , we have $\text{Ind} U \leq \text{Ind} Y = n$ by virtue of (b). The intersections $U \cap \overline{A}$ and $U \cap \overline{B}$ are disjoint closed subsets of U ; therefore there exists a partition L in U between $U \cap \overline{A}$ and $U \cap \overline{B}$ such that $\text{Ind} L \leq n-1$. Since $M \subset U$, $U \cap \overline{A} \cap M = A$, and $U \cap \overline{B} \cap M = B$, the set $L \cap M$ is a partition in M between A and B . By the inductive assumption $\text{Ind}(L \cap M) \leq \text{Ind} L \leq n-1$, so that $\text{Ind} M \leq n = \text{Ind} Y$. Thus X satisfies condition (a). $\square$

The separation and addition theorems for the dimension Ind hold

in hereditarily normal spaces. From Lemma 1.2.9, Remark 1.2.10 and Theorem 2.2.1 one easily obtains the following theorem (cf. the proof of Theorem 1.2.11):

2.2.4. The separation theorem for Ind. *If X is a hereditarily normal space and M is a subspace of X such that $\text{Ind } M \leq n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{Ind}(L \cap M) \leq n - 1$. $\square$*

2.2.5. The addition theorem for Ind. *For every pair X, Y of subspaces of a hereditarily normal space we have*

$$\text{Ind}(X \cup Y) \leq \text{Ind } X + \text{Ind } Y + 1.$$

Proof. The theorem is obvious if one of the subspaces has dimension ∞ , so that we can suppose that $m(X, Y) = \text{Ind } X + \text{Ind } Y$ is finite. We shall apply induction with respect to that number. If $m(X, Y) = -2$, then $X = \emptyset = Y$ and our inequality holds. Assume that the inequality is proved for every pair of subspaces the sum of the large inductive dimensions of which is less than $n \geq -1$ and consider subspaces X and Y such that $m(X, Y) = n$; clearly, we can suppose that $\text{Ind } X \geq 0$. Let A and B be disjoint closed subsets of $X \cup Y$. By virtue of the separation theorem there exists a partition L in $X \cup Y$ between A and B such that $\text{Ind}(L \cap X) \leq \text{Ind } X - 1$. Since $m(L \cap X, L \cap Y) \leq \text{Ind } X + \text{Ind } Y - 1 = n - 1$, we have $\text{Ind } L \leq n$ by the inductive assumption. This implies that $\text{Ind}(X \cup Y) \leq n + 1 = \text{Ind } X + \text{Ind } Y + 1$. $\square$

The addition theorem yields.

2.2.6. Corollary. *If a hereditarily normal space X can be represented as the union of $n + 1$ subspaces $Z_1, Z_2, \dots, Z_{n+1}$ such that $\text{Ind } Z_i \leq 0$ for $i = 1, 2, \dots, n + 1$, then $\text{Ind } X \leq n$. $\square$*

Let us note that it is an open problem whether every normal (or even hereditarily normal) space X with $\text{Ind } X \leq n$ can be represented as the union of $n + 1$ subspaces the large inductive dimension of which does not exceed zero.

Remark 1.3.2 implies that Theorem 1.3.1 can be restated as follows:

2.2.7. Theorem. *If a normal space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces such that $\text{Ind } F_i \leq 0$ for $i = 1, 2, \dots$, then $\text{Ind } X \leq 0$. $\square$*

Example 2.2.13 below shows that in normal spaces the dimension Ind satisfies only the sum theorem for dimension 0, i.e., Theorem 2.2.7. It is an open problem whether the situation improves in hereditarily normal spaces.

A very weak version of the sum theorem for the dimension Ind in normal spaces reads as follows:

2.2.8. Proposition. *Let $\{X_s\}_{s \in S}$ be a family of normal spaces and let $X = \bigoplus_{s \in S} X_s$. The inequality $\text{Ind} X \leq n$ holds if and only if $\text{Ind} X_s \leq n$ for every $s \in S$. $\square$*

Let us note that a few, rather specialized, results related to Theorems 2.2.5 and 2.2.7 are stated in Problem 2.2.C.

The status of the Cartesian product theorem for Ind in normal spaces is similar to that of the sum theorem. There exist compact spaces X and Y such that $\text{ind} X = \text{Ind} X = 1$, $\text{ind} Y = \text{Ind} Y = 2$, and yet $\text{Ind}(X \times Y) \geq \text{ind}(X \times Y) \geq 4$ as well as a normal space Z , whose square $Z \times Z$ is also normal, such that $\text{Ind} Z = 0$ and yet $\text{Ind}(Z \times Z) > 0$. The descriptions of these examples are very difficult and cannot be reproduced in this book.

We now turn to a discussion of dimension preserving compactifications. As the reader certainly knows, among the compactifications of a completely regular space X a particular role is played by the Čech–Stone compactification βX (see [GT], Section 3.6), which can be characterized by the property that every continuous function $f: X \rightarrow I$ is continuously extendable over βX (we assume here that X is actually a subspace of βX). In the realm of normal spaces the Čech–Stone compactification can also be characterized by the property that every pair of disjoint closed subsets of X has disjoint closures in βX . Hence, for every closed subspace M of a normal space X the closure $\overline{M}$ of M in βX is the Čech–Stone compactification of the space M . We shall show that the Čech–Stone compactification preserves the dimension Ind.

2.2.9. Theorem. *For every normal space X we have $\text{Ind} \beta X = \text{Ind} X$.*

Proof. To begin with, we shall prove that $\text{Ind} X \leq \text{Ind} \beta X$. The inequality is obvious if $\text{Ind} \beta X = \infty$, so that we can suppose that $\text{Ind} \beta X < \infty$. We shall apply induction with respect to $\text{Ind} \beta X$. If $\text{Ind} \beta X = -1$, then $\beta X = \emptyset = X$ and our inequality holds. Assume that the inequality holds for all normal spaces the dimension Ind of the Čech–Stone compactification of which is less than $n \geq 0$ and consider a normal space X such

that $\text{Ind} \beta X = n$. Let A and B be disjoint closed subsets of X . The sets $\bar{A}$ and $\bar{B}$, where the bar denotes the closure operator in βX , are disjoint, so that there exists a partition L in βX between $\bar{A}$ and $\bar{B}$ such that $\text{Ind} L \leq n-1$. Clearly, $L_0 = L \cap X$ is a partition in X between A and B . Since $\beta L_0 = \bar{L}_0 \subset L$, it follows from Theorem 2.2.1 and the inductive assumption that $\text{Ind} L_0 \leq n-1$, so that $\text{Ind} X \leq n = \text{Ind} \beta X$.

Now, we shall prove that $\text{Ind} \beta X \leq \text{Ind} X$. As in the first part of the proof, we shall suppose that $\text{Ind} X < \infty$ and apply induction with respect to $\text{Ind} X$. Our equality holds if $\text{Ind} X = -1$. Assume that it is proved for all normal spaces the dimension Ind of which is less than $n \geq 0$ and consider a normal space X such that $\text{Ind} X = n$. Let A and B be disjoint closed subsets of βX . There exist open sets $V_1, V_2 \subset \beta X$ such that

$$A \subset V_1, \quad B \subset V_2 \quad \text{and} \quad \bar{V}_1 \cap \bar{V}_2 = \emptyset.$$

The sets $A_0 = X \cap \bar{V}_1$ and $B_0 = X \cap \bar{V}_2$ are closed in X and disjoint, so that there exists a partition L_0 in X between A_0 and B_0 such that $\text{Ind} L_0 \leq n-1$. Let U_0, W_0 be open subsets of X satisfying

$$A_0 \subset U_0, \quad B_0 \subset W_0, \quad U_0 \cap W_0 = \emptyset \quad \text{and} \quad X \setminus L_0 = U_0 \cup W_0.$$

We shall show that

$$(1) \quad \bar{U}_0 \cap \bar{W}_0 \subset \bar{L}_0.$$

Consider a point $x \in \bar{U}_0 \cap \bar{W}_0$ and an arbitrary neighbourhood $G \subset \beta X$ of the point x . Let $H \subset \beta X$ be an open set such that $x \in H \subset \bar{H} \subset G$. One easily sees that

$$x \in \bar{W}_0 \cap H \subset \overline{W_0 \cap H} \quad \text{and} \quad x \in \bar{U}_0 \cap H \subset \overline{U_0 \cap H};$$

therefore, by the above-mentioned characterization of the Čech–Stone compactification in the realm of normal spaces, the closures of $W_0 \cap H$ and $U_0 \cap H$ in X intersect. Hence

$$\emptyset \neq (W_0 \cup L_0) \cap (U_0 \cup L_0) \cap \bar{H} = [(U_0 \cap W_0) \cup L_0] \cap \bar{H} = L_0 \cap \bar{H} \subset L_0 \cap G,$$

which shows that every neighbourhood of x meets L_0 , i.e., that $x \in \bar{L}_0$. Thus inclusion (1) is established.

By virtue of (1) the sets

$$U = \beta X \setminus (\bar{W}_0 \cup \bar{L}_0) \quad \text{and} \quad W = \beta X \setminus (\bar{U}_0 \cup \bar{L}_0)$$

satisfy

$$(2) \quad \begin{aligned} U \cup W &= \beta X \setminus [(\bar{W}_0 \cup \bar{L}_0) \cap (\bar{U}_0 \cup \bar{L}_0)] = \beta X \setminus [(\bar{W}_0 \cap \bar{U}_0) \cup \bar{L}_0] \\ &= \beta X \setminus \bar{L}_0; \end{aligned}$$

on the other hand,

$$(3) \quad \begin{aligned} U \cap W &= \beta X \setminus [(\bar{W}_0 \cup \bar{L}_0) \cup (\bar{U}_0 \cup \bar{L}_0)] = \beta X \setminus (\bar{U}_0 \cup \bar{W}_0 \cup \bar{L}_0) \\ &= \beta X \setminus \bar{X} = \emptyset. \end{aligned}$$

Since the sets $A_0 = X \cap \bar{V}_1$ and $W_0 \cup L_0$ are closed in X and disjoint, we have $\bar{V}_1 \cap (\beta X \setminus U) = X \cap \bar{V}_1 \cap (\bar{W}_0 \cup \bar{L}_0) = \emptyset$, i.e., $\bar{V}_1 \subset U$; similarly, $\bar{V}_2 \subset W$. Thus $A \subset U$ and $B \subset W$, which together with (2) and (3) shows that $\bar{L}_0$ is a partition in βX between A and B . As $\bar{L}_0 = \beta L_0$, by the inductive assumption $\text{Ind } \bar{L}_0 \leq \text{Ind } L_0 \leq n-1$, so that $\text{Ind } \beta X \leq n = \text{Ind } X$. $\square$

2.2.10. Corollary. *For every normal space X and a dense normal subspace $M \subset X$ which has the property that every continuous function $f: M \rightarrow I$ is continuously extendable over X we have $\text{Ind } M = \text{Ind } X$.*

In other words, $\text{Ind } Y = \text{Ind } X$ for every normal space X and every normal subspace Y of βX which contains X .

Proof. From the extendability of every continuous function $f: M \rightarrow I$ over X it follows that $\beta M = \beta X$. $\square$

Let us observe that the counterpart of Theorem 2.2.9 for the dimension ind does not hold (see Problem 2.2.E).

As the Čech-Stone compactification generally raises the weight of spaces, it is natural to ask if there exist compactifications preserving both the dimension Ind and weight. One proves that for every normal space X there exists a compactification $\tilde{X}$ such that $\text{Ind } \tilde{X} \leq \text{Ind } X$ and $w(\tilde{X}) = w(X)$, where w denotes the *weight* of a space, i.e., the infimum of the cardinalities of the bases for that space. The construction of such a compactification is rather difficult and will not be given here. Let us note that even more is true, viz., that for every integer $n \geq 0$ and every cardinal number $m \geq \aleph_0$ there exists a compact space B_m^n such that $\text{Ind } B_m^n = n$, $w(B_m^n) = m$, and every normal space X which satisfies the conditions $\text{Ind } X \leq n$ and $w(X) \leq m$ is homeomorphic to a subspace of the space B_m^n . In other words, there exists a compact universal space for the class of all normal spaces whose large inductive dimension is not larger than n and whose weight is not larger than m .

We conclude this section with the counter-examples announced above. The first is a compact space Z with $\text{Ind } Z = 0$ which contains a normal subspace X such that $\text{Ind } X > 0$.

2.2.11. Dowker's example. Let Q denote the subspace of the interval I consisting of all rational numbers in I (clearly, Q is homeomorphic to the space of rational numbers). By letting

$$xEy \quad \text{if and only if} \quad |x - y| \in Q$$

we define an equivalence relation E on the set I . Since each equivalence class of E is countable, the family of all equivalence classes has cardinality c . Let us choose from it a subfamily of cardinality $\aleph_1$ which does not contain the equivalence class Q and let us arrange the members of this subfamily into a transfinite sequence $Q_0, Q_1, \dots, Q_\alpha, \dots, \alpha < \omega_1$.

For every $\gamma < \omega_1$ the set $P_\gamma = I \setminus \bigcup_{\alpha \geq \gamma} Q_\alpha$ satisfies the equality $\text{ind } P_\gamma = 0$; indeed $Q \subset P_\gamma$, and since the sets Q_α are dense in I , the set P_γ contains no interval. Let W be the space of all ordinal numbers $\leq \omega_1$ and let $W_0 = W \setminus \{\omega_1\}$ (see Example 2.1.6). For every $\alpha < \omega_1$ the subspace $M_\alpha = [0, \alpha] = [0, \alpha + 1)$ is open-and-closed in W . Consider the Cartesian product $W \times I$ and its subspaces

$$X_\alpha = \bigcup_{\gamma \leq \alpha} (\{\gamma\} \times P_\gamma), \quad X = \bigcup_{\alpha < \omega_1} X_\alpha \quad \text{and} \quad X^* = X \cup (\{\omega_1\} \times I).$$

As noted in Example 2.1.6, the subspaces M_α are metrizable. Being countable, they are zero-dimensional, so that $\text{ind}(M_\alpha \times P_\alpha) = 0$ by virtue of Theorem 1.3.6. Since for every $\alpha < \omega_1$ the set X_α is open-and-closed in X and $\text{ind } X_\alpha = 0$ in view of the inclusion $X_\alpha \subset M_\alpha \times P_\alpha$, we have $\text{ind } X = 0$. From Remark 1.3.18 and Theorem 1.6.5 it follows that there exists a compact space Z with $\text{Ind } Z = 0$ which contains X as a subspace.

We shall show now that the space X^* is normal which is the first step towards establishing normality of the space X . As X^* is a subspace of $W \times I$, it is a T_1 -space. Consider a pair A, B of disjoint closed subsets of X^* . The sets

$$F_1 = \{x \in I: (\omega_1, x) \in A\} \quad \text{and} \quad F_2 = \{x \in I: (\omega_1, x) \in B\}$$

are disjoint and closed in I so that there exist open sets $H_1, H_2 \subset I$ such that $F_1 \subset H_1$, $F_2 \subset H_2$ and $H_1 \cap H_2 = \emptyset$. For every $x \in I \setminus H_1$ there exists a neighbourhood $U_x \subset I$ of the point x and an ordinal number $\alpha(x) < \omega_1$ such that $[(W \setminus M_{\alpha(x)}) \times U_x] \cap A = \emptyset$. The set $I \setminus H_1$ being closed in I , we can choose a finite number of points $x_1, x_2, \dots, x_k \in I \setminus H_1$ such that $I \setminus H_1 \subset \bigcup_{i=1}^k U_{x_i}$. For $\alpha_1 = \max(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_k)) < \omega_1$ we have $[(W \setminus M_{\alpha_1}) \times (I \setminus H_1)] \cap A = \emptyset$, i.e.,

$$(4) \quad A \cap (X^* \setminus X_{\alpha_1}) \subset (W \setminus M_{\alpha_1}) \times H_1.$$

In a similar way we determine an $\alpha_2 < \omega_1$ such that

$$(5) \quad B \cap (X^* \setminus X_{\alpha_2}) \subset (W \setminus M_{\alpha_2}) \times H_2;$$

without loss of generality we can assume that $\alpha_1 \leq \alpha_2$. The sets $A \cap X_{\alpha_2}$ and $B \cap X_{\alpha_2}$ are disjoint and closed in X_{α_2} , so that there exist open sets $U_1, V_1 \subset X_{\alpha_2}$ such that

$$(6) \quad A \cap X_{\alpha_2} \subset U_1, \quad B \cap X_{\alpha_2} \subset V_1 \quad \text{and} \quad U_1 \cap V_1 = \emptyset;$$

since X_{α_2} is an open subset of X^* , the sets U_1 and V_1 are open in X^* . The sets $U = U_1 \cup \{(W \setminus M_{\alpha_2}) \times H_1\} \cap X^*$ and $V = V_1 \cup \{(W \setminus M_{\alpha_2}) \times H_2\} \cap X^*$ are open in X^* . By virtue of (4)–(6)

$$A \subset U, \quad B \subset V \quad \text{and} \quad U \cap V = \emptyset,$$

so that the space X^* is normal.

To prove that the space X is normal, it suffices to show that, for every pair A, B of disjoint closed subsets of X , the closures $\bar{A}$ and $\bar{B}$ of A and B in the space X^* are disjoint. Suppose that there exists a point $(\omega_1, x) \in \bar{A} \cap \bar{B}$. It follows from the definition of P_γ that there exists an $\alpha_0 < \omega_1$ such that $x \in P_\alpha$ for every $\alpha \geq \alpha_0$. We can readily define by induction two sequences $\alpha_1, \alpha_2, \dots$ and $\beta_1, \beta_2, \dots$ of countable ordinal numbers and two sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ of real numbers in I satisfying

$$\alpha_0 < \alpha_i < \beta_i < \alpha_{i+1}, \quad |x - x_i| < 1/i, \quad |x - y_i| < 1/i, \\ (\alpha_i, x_i) \in A, \quad (\beta_i, y_i) \in B$$

for $i = 1, 2, \dots$. Now, for the smallest ordinal number γ larger than all α_i 's we have $(\gamma, x) \in A \cap B$, which is impossible. Thus $\bar{A} \cap \bar{B} = \emptyset$ and X is a normal space.

It remains to show that $\text{Ind} X > 0$. Assume that $\text{Ind} X = 0$ and consider the pair $W_0 \times \{0\}, W_0 \times \{1\}$ of disjoint closed subsets of the space X . Then there exists an open-and-closed set $U \subset X$ such that $W_0 \times \{0\} \subset U$ and $W_0 \times \{1\} \subset X \setminus U$. As established in the preceding paragraph, the closures $\bar{A}$ and $\bar{B}$ of the sets $A = U$ and $B = X \setminus U$ in the space X^* are disjoint. Since $X = A \cup B$ and $\bar{X} = X^*$, we have $\bar{A} \cup \bar{B} = X^*$. The sets $A_1 = \{x \in I: (\omega_1, x) \in \bar{A}\}$ and $B_1 = \{x \in I: (\omega_1, x) \in \bar{B}\}$ are disjoint and closed in I ; moreover, $A_1 \cup B_1 = I$ and $A_1 \neq \emptyset \neq B_1$, because $0 \in A_1$ and $1 \in B_1$. Thus our assumption contradicts the connectedness of the interval I . Hence $\text{Ind} X > 0$. $\square$

We now turn to the counterexample to the sum theorem for Ind in normal spaces. To begin with, we shall describe two auxiliary spaces L and L_0 , which are known as the long segment and the long line.

2.2.12. Example. Let W_0 be the set of all countable ordinal numbers and let $L_0 = W_0 \times [0, 1)$. Define a linear order $<$ in the set L_0 by letting $(\alpha_1, t_1) < (\alpha_2, t_2)$ whenever $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$ and $t_1 < t_2$. Furthermore, let $L = L_0 \cup \{\omega_1\}$ and extend the linear order $<$ over L by letting $x < \omega_1$ for every $x \in L_0$. By assigning to each $\alpha \in W_0$ the point $(\alpha, 0) \in L_0$ we define a one-to-one mapping of W_0 onto the set $W_0 \times \{0\} \subset L_0$; in the sequel we shall identify α with $(\alpha, 0)$ and we shall consider W_0 as a subset of L_0 . The set W of all ordinal numbers $\leq \omega_1$ is a subset of L .

Consider on L the topology obtained by taking as a base all sets of the form

$$(x_0, \omega_1] = \{x: x_0 < x\}, \quad [0, x_1) = \{x: x < x_1\}$$

$$\text{and} \quad (x_0, x_1) = \{x: x_0 < x < x_1\},$$

where $x_0, x_1 \in L$ and $x_0 < x_1$. One easily sees that L is a connected compact space and that the subspace topology on $W \subset L$ coincides with the topology on W defined in Example 2.1.6. For every $x_0 \in L_0$ the closed subspace $[0, x_0] = \{x: x \leq x_0\}$ of L has a countable base, viz., the family of all sets $((\alpha_0, t_0), (\alpha_1, t_1))$, where $\alpha_0, \alpha_1 \leq x_0$ and t_0, t_1 are rational numbers in I , and thus is a compact metric space. The space L is called the *long segment*; the subspace L_0 of L is called the *long line*. $\square$

We shall now describe a compact space X with $\text{Ind} X \geq 2$ which contain closed subspaces F_1, F_2 such that $F_1 \cup F_2 = X$ and $\text{Ind} F_1 = \text{Ind} F_2 = 1$.

2.2.13. Lokucievskii's example. Let L be the long segment and C the Cantor set. The subspace of the Cantor set consisting of the end-points of all intervals removed from I to obtain the Cantor set will be denoted by Q (cf. Problem 1.3.H(e)). Since L and C are compact, the Cartesian product $L \times C$ is a compact space. We shall now establish a property of open subsets of $L \times C$ which will prove crucial for the evaluation of $\text{Ind} X$. Namely

- (7) for every open set $U \subset L \times C$ such that $U \cap (\{\omega_1\} \times C) \neq \emptyset$ and $t_U = \sup\{t: (\omega_1, t) \in U\}$ belongs to $(C \setminus Q) \setminus \{1\}$ either
- (i) there exists an $x' \in L_0$ such that $(x', \omega_1] \times \{t_U\} \subset \text{Fr } U$
 - or
 - (ii) there exist a $t' \in (t_U, 1]$ and a set $L' \subset L_0$ cofinal in L_0 such that $L' \times ([t_U, t'] \cap C) \subset U$.

Consider a sequence $t_1, t_2, \dots$ converging to t_U and such that $(\omega_1, t_i) \in U$ for $i = 1, 2, \dots$. The set U being open, there exists a sequence $\alpha_1, \alpha_2, \dots$ of countable ordinal numbers such that $(\alpha_i, \omega_1] \times t_i \subset U$ for $i = 1, 2, \dots$

For the smallest ordinal number $\alpha_0 \in W_0 \subset L_0$ larger than all α_i 's we have $(\alpha_0, \omega_1] \times \{t_v\} \subset \bar{U}$. If (i) does not hold, then there exists a set $L_1 \subset L_0$ cofinal in L_0 and such that $L_1 \times \{t_v\} \subset U$. For every $x \in L_1$ we can find a $t_x \in (t_v, 1]$ satisfying $\{x\} \times ([t_v, t_x] \cap C) \subset U$. Clearly, there exists a natural number k such that the set $L(k) = \{x \in L_1: t_x - t_v > 1/k\}$ is cofinal in L_0 , and thus (ii) holds with $t' = t_v + 1/k$ and $L' = L(k)$.

Let $f: C \rightarrow I$ be the continuous mapping of C onto I consisting in matching the end-points of each interval removed from I to obtain the Cantor set (see Problem 1.3.D) and let E be the equivalence relation on the space $L \times C$ corresponding to the decomposition of $L \times C$ into one-point subsets of $L_0 \times C$ and the sets $\{\omega_1\} \times f^{-1}(t)$, where $t \in I$. Since the equivalence relation E is closed, the quotient space $Y = (L \times C)/E$ is compact (see [GT], Theorem 3.2.11). The image of the set $\{\omega_1\} \times C \subset L \times C$ under the natural quotient mapping will be denoted by I and will be identified with the interval $[0, 1]$; the image of $\{\omega_1\} \times Q$ will be denoted by K .

We shall prove that $\text{ind } Y = \text{Ind } Y = 1$. To begin with, let us observe that $\text{ind } Y \leq 1$. Indeed, every neighbourhood of a point $x \in Y \setminus I$ contains a neighbourhood of this point with a boundary homeomorphic to $C \oplus C$ and every neighbourhood of a point $x \in I$ contains a neighbourhood of this point with a boundary homeomorphic to $C \oplus B$, where B is a discrete space of cardinality ≤ 2 ; the last fact is a consequence of the density of K in I . Now we shall show that $\text{Ind } Y \leq 1$. Consider a closed set $A \subset Y$ and an open set $V \subset Y$ which contains the set A . For every $x \in A$ there exists a neighbourhood U_x such that $U_x \subset V$ and $\text{Fr } U_x$ is a zero-dimensional compact metric space. The subspace A of Y being compact, we can choose a finite number of points $x_1, x_2, \dots, x_k$ such that $A \subset U = \bigcup_{i=1}^k U_{x_i} \subset V$.

The subspace $F = \bigcup_{i=1}^k \text{Fr } U_{x_i}$ of Y is normal, so that by virtue of Theorems 1.6.4 and 2.2.7 we have $\text{Ind } F = 0$. Since $\text{Fr } U \subset F$, it follows from Theorem 2.2.1 that $\text{Ind } \text{Fr } U \leq 0$. Hence $\text{Ind } Y \leq 1$; as the space Y contains the interval I , we have $\text{ind } Y = \text{Ind } Y = 1$.

Let us note that (7) yields

- (8) for every open set $U \subset Y$ such that $U \cap I \neq \emptyset$ and $\sup\{t: t \in U \cap I\}$ belongs to $(I \setminus K) \setminus \{1\}$ we have $\text{ind } \text{Fr } U \geq 1$.

Consider two disjoint copies Y_1 and Y_2 of the space Y , and denote by I_i and K_i the counterparts of I and K in Y_i for $i = 1, 2$. Let $K_3 \subset I_2 \setminus K_2$ be a countable set dense in I_2 which does not contain the end-points of I_2 .

It follows from Problem 1.3.G(a) that there exists a homeomorphism $h: I_1 \rightarrow I_2$ such that $h(K_1) = K_3 \subset I_2 \setminus K_2$. Let S be the equivalence relation on the space $Y_1 \oplus Y_2$ corresponding to the decomposition of $Y_1 \oplus Y_2$ into one-point subsets of $(Y_1 \setminus I_1) \cup (Y_2 \setminus I_2)$ and the sets $\{x, h(x)\}$, where $x \in I_1$. Since the equivalence relation S is closed, the quotient space $X = (Y_1 \oplus Y_2)/S$ is compact. Roughly speaking, the space X is obtained by matching Y_1 with Y_2 along I_1 and I_2 in such a way that no points of K_1 and K_2 are matched to each other. For $i = 1, 2$ the image F_i of the set Y_i under the natural quotient mapping $q: Y_1 \oplus Y_2 \rightarrow X$ is closed in X and homeomorphic to Y , so that $\text{Ind} F_1 = \text{Ind} F_2 = 1$; moreover $F_1 \cup F_2 = X$.

It remains to show that $\text{Ind} X \geq 2$. Consider the mid-point x of the interval obtained by matching I_1 with I_2 and a neighbourhood $V \subset X$ of the point x which does not contain the end-points of this interval. From (8) it follows that for every open set $U \subset X$ such that $x \in U \subset V$ we have $\text{ind} \text{Fr} U \geq 1$, because $\sup\{t: t \in q^{-1}(U) \cap I_i\}$ belongs to $I_i \setminus K_i$ either for $i = 1$ or for $i = 2$. Thus $\text{Ind} X \geq \text{ind} X \geq 2$; one can show that $\text{Ind} X = \text{ind} X = 2$ (see Problem 2.2.C(c)). $\square$

Historical and bibliographic notes

Theorem 2.2.1 was noted by Čech in [1932]. An example of a hereditarily normal space X with $\text{Ind} X = \dim X = 0$ which for every natural number n contains a subspace A_n with $\text{Ind} A_n = \dim A_n = n$ was described by E. Pol and R. Pol in [1979] (in [1977] the same authors gave an example to show that Ind and $\dim$ are not monotonic in hereditarily normal spaces); under additional set theoretic assumptions such an example was defined by Filippov in [1973]. Theorem 2.2.3 was proved by Dowker in [1953]. Theorem 2.2.5 was given by Smirnov in [1951]. Theorem 2.2.7 is implicit in Čech's paper [1933] (cf. Theorems 1.6.11 and 3.1.8); it was first formulated by Vedenissov in [1939]. An example of compact spaces X and Y such that $\text{ind} X = \text{Ind} X = 1$, $\text{ind} Y = \text{Ind} Y = 2$ and $\text{Ind}(X \times Y) \geq \text{ind}(X \times Y) \geq 4$ was described by Filippov in [1972] and an example of a normal space Z , whose square $Z \times Z$ is also normal, such that $\text{Ind} Z = 0$ and yet $\text{Ind}(Z \times Z) > 0$ was given by Wage in [1977]. In his original construction Wage applied the continuum hypothesis; Przymusiński in [1977] noted that the continuum hypothesis can be avoided by a modification of Wage's construction. Theorem 2.2.9 was established by

Vedenisoff in [1939]. The existence of compactifications preserving both the dimension Ind and the weight of normal spaces and the corresponding universal space theorem were announced by Pasynkov in [1971]; proofs were given in Alexandroff and Pasynkov's book [1973]. Example 2.2.11 was described by Dowker in [1955], and Example 2.2.13—by Lokucievskii in [1949].

Problems

2.2.A (Urysohn [1925]). (a) Prove that a subspace M of a hereditarily normal space X satisfies the inequality $\text{ind} M \leq n \geq 0$ if and only if for every point $x \in M$ and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\text{ind}(M \cap \text{Fr } U) \leq n - 1$.

(b) Show that for every pair X, Y of subspaces of a hereditarily normal space we have

$$\text{ind}(X \cup Y) \leq \text{ind} X + \text{ind} Y + 1.$$

2.2.B (Aarts and Nishiura [1971]). Prove that for every continuous mapping $f: A \rightarrow S^k$ defined on a closed subspace A of a hereditarily normal space X such that $\text{Ind}(X \setminus A) \leq n$, where $0 \leq k \leq n$, there exists a closed subspace B of the space X such that $A \cap B = \emptyset$, $\text{Ind} B \leq n - k - 1$, and the mapping f has a continuous extension $F: X \setminus B \rightarrow S^k$ over $X \setminus B$.

Hint. See Problem 1.9.D.

2.2.C. (a) Show that if a normal space X can be represented as the union of two subspaces M and F such that M is normal and non-empty and F is closed in X , then for every pair A, B of disjoint closed subsets of X there exist a partition L in X between A and B and a partition L' in M between $M \cap A$ and $M \cap B$ such that $L \setminus F = L' \setminus F$ and $\text{Ind} L' \leq \text{Ind} M - 1$.

(b) Deduce from (a) that if a normal space X can be represented as the union of two subspaces X_1 and F such that X_1 is normal and non-empty, F is closed, and $\text{Ind} F = 0$, then $\text{Ind} X \leq \text{Ind} X_1$.

(c) Show that if a normal space can be represented as the union of two closed subspaces F_1 and F_2 , then $\text{Ind} X \leq \text{Ind} F_1 + \text{Ind} F_2$.

(d) Show that if a normal space can be represented as the union of two normal subspaces X and Y of which one is either closed or open, then $\text{Ind}(X \cup Y) \leq \text{Ind} X + \text{Ind} Y + 1$.

(e) Show that if a hereditarily normal space X contains a closed subspace F such that $\text{Ind} F \leq n$ and $\text{Ind}(X \setminus F) \leq n$, then $\text{Ind} X \leq n$ (cf. Theorem 2.3.1).

2.2.D. Check that if X is a metric space and a dense subspace $M \subset X$ has the property that every continuous function $f: M \rightarrow I$ is continuously extendable over X , then $M = X$.

2.2.E. Note that for the space X described in Example 2.2.11 we have $\text{nd} \beta X \neq \text{ind} X$.

2.2.F (Dowker [1955]). (a) Prove that the space X described in Example 2.2.11 satisfies the equality $\text{Ind} X = 1$.

(b) Show that to the space X described in Example 2.2.11 one point can be adjoined either in such a way that one obtains a normal space X_1 with $\text{ind} X_1 > 0$ or in such a way that one obtains a normal space X_2 with $\text{Ind} X_2 = 0$.

2.2.G (Smirnov [1958]). Modify the construction of the space X in Example 2.2.11 to obtain a compact space Z with $\text{Ind} Z = 0$ which contains a normal subspace Y such that $\text{Ind} Y = \infty$.

Hint. Replace the interval by the Hilbert cube and the sets P_γ by the Cartesian products $P_\gamma^{\aleph_0}$.

2.2.H. (a) Show that the long segment L is a strongly hereditarily normal space.

(b) Prove that for every point x_0 in the long line L_0 the subspace $[0, x_0]$ of L is homeomorphic to the interval I .

Hint. Define a countable dense subset of $[0, x_0]$ which is ordered similarly to the set of all rational numbers in I .

(c) Check that for the subspace W of the long segment L there exists no G_δ -set W^* in L such that $W \subset W^*$ and $\text{Ind} W^* = \text{Ind} W$.

2.3. Basic properties of the dimension Ind in strongly hereditarily normal spaces

Strongly hereditarily normal spaces constitute a relatively wide class of spaces where two of the most important theorems of dimension theory, viz., the subspace theorem and the countable sum theorem, hold for the

dimension Ind. Both theorems are tightly connected; we shall prove them simultaneously. For more clarity the proof is divided into several lemmas.

Let us consider the following properties of a hereditarily normal space X :

- (μ_n) For each subspace $Y \subset X$ and every open subspace U of Y , if $\text{Ind } Y \leq n$, then $\text{Ind } U \leq n$.
- (σ_n) For each subspace $Y \subset X$ and every sequence $F_1, F_2, \dots$ of closed subspaces of Y such that $Y = \bigcup_{i=1}^{\infty} F_i$, if $\text{Ind } F_i \leq n$ for $i = 1, 2, \dots$, then $\text{Ind } Y \leq n$.

Clearly, every hereditarily normal space X has property (μ_{-1}), and thus to prove that all strongly hereditarily normal spaces have properties (μ_n) and (σ_n) for $n = -1, 0, 1, \dots$, it suffices to show that the implications (μ_{n-1}) $\Rightarrow$ (μ_n) and (μ_n) $\Rightarrow$ (σ_n) hold for every strongly hereditarily normal space X . The second implication holds for all hereditarily normal spaces; it will be deduced from the following version of the sum theorem.

2.3.1. Theorem. *If a hereditarily normal space X can be represented as the union of a sequence $K_1, K_2, \dots$ of pairwise disjoint subspaces such that $\text{Ind } K_i \leq n$ and the union $\bigcup_{j \leq i} K_j$ is closed for $i = 1, 2, \dots$, then $\text{Ind } X \leq n$.*

Proof. We shall apply induction with respect to the number n . For $n = -1$ the theorem is obvious. Assume that the corresponding statements hold for dimensions less than n and consider a hereditarily normal space X which satisfies the assumptions of our theorem. Let $F_i = \bigcup_{j \leq i} K_j$ for $i = 1, 2, \dots$

Consider a pair A, B of disjoint closed subsets of X . Let U_0, V_0 be open subsets of X such that

$$(1) \quad A \subset U_0, \quad B \subset V_0 \quad \text{and} \quad \bar{U}_0 \cap \bar{V}_0 = \emptyset,$$

and let $K_0 = F_0 = L_0 = \emptyset$. We shall define inductively two sequences $U_0, U_1, \dots$ and $V_0, V_1, \dots$ of open subsets of X and a sequence $L_0, L_1, \dots$ of subsets of X satisfying for $i = 0, 1, \dots$ the following conditions:

$$(2) \quad L_i \subset K_i \quad \text{and} \quad \text{Ind } L_i \leq n-1.$$

$$(3) \quad \text{The set } E_i = \bigcup_{j \leq i} L_j \text{ is closed.}$$

$$(4) \quad F_i \subset U_i \cup V_i \cup E_i.$$

$$(5) \quad U_i \cup V_i \subset X \setminus E_i \quad \text{and} \quad \bar{U}_i \cap \bar{V}_i \subset E_i.$$

$$(6) \quad U_{i-1} \subset U_i \quad \text{and} \quad V_{i-1} \subset V_i \quad \text{for } i > 0.$$

The sets U_0, V_0 and L_0 satisfying (2)–(6) for $i = 0$ are already defined. Let us assume that the sets U_i, V_i and L_i satisfying (2)–(6) are defined for $i < m > 0$. The sets $\bar{U}_{m-1} \cap K_m$ and $\bar{V}_{m-1} \cap K_m$ are closed in K_m and disjoint, because by virtue of (5) and (2) with $i = m-1$

$$\bar{U}_{m-1} \cap \bar{V}_{m-1} \subset E_{m-1} \subset \bigcup_{j < m} K_j$$

and the last set is disjoint from K_m . Since $\text{Ind } K_m \leq n$, there exists a partition L_m in K_m between $\bar{U}_{m-1} \cap K_m$ and $\bar{V}_{m-1} \cap K_m$ which satisfies (2) with $i = m$; there also exist open subsets G, H of K_m such that

$$(7) \quad \bar{U}_{m-1} \cap K_m \subset G, \quad \bar{V}_{m-1} \cap K_m \subset H, \quad G \cap H = \emptyset$$

and $K_m \setminus L_m = G \cup H.$

From (7) it follows that $L_m \cap (U_{m-1} \cup V_{m-1}) = \emptyset$; the union $U_{m-1} \cup V_{m-1}$ being open in X , we have $\bar{L}_m \cap (U_{m-1} \cup V_{m-1}) = \emptyset$. Since L_m is closed in K_m and F_m is closed in X , the last equality and (4) with $i = m-1$ yield the inclusion

$$\bar{L}_m \subset L_m \cup [F_{m-1} \setminus (U_{m-1} \cup V_{m-1})] \subset E_m,$$

which shows that (3) holds for $i = m$.

Since $G \cap V_{m-1} = \emptyset = H \cap U_{m-1}$, we have $\bar{G} \cap V_{m-1} = \emptyset = \bar{H} \cap U_{m-1}$, which implies that

$$(8) \quad \bar{G} \cap \bar{H} \subset L_m \cup [F_{m-1} \setminus (U_{m-1} \cup V_{m-1})] \subset E_m,$$

because $\bar{G} \cap \bar{H} \subset \bar{K}_m \subset \bar{F}_m = F_m$. The same equalities $\bar{G} \cap V_{m-1} = \emptyset = \bar{H} \cap U_{m-1}$ together with (7), the equality $U_{m-1} \cap V_{m-1} = \emptyset$, which is a consequence of (5) with $i = m-1$, and (4) with $i = m-1$ yield

$$(9) \quad \bar{G} \cap \bar{V}_{m-1} = [\bar{G} \cap (\bar{V}_{m-1} \setminus V_{m-1})] \setminus K_m \subset F_{m-1} \cap (\bar{V}_{m-1} \setminus V_{m-1}) \subset E_{m-1}$$

and

$$(10) \quad \bar{H} \cap \bar{U}_{m-1} = [\bar{H} \cap (\bar{U}_{m-1} \setminus U_{m-1})] \setminus K_m \subset F_{m-1} \cap (\bar{U}_{m-1} \setminus U_{m-1}) \subset E_{m-1}.$$

Relations (8)–(10) and the second part of (5) with $i = m-1$ show that the sets $(\bar{U}_{m-1} \cup \bar{G}) \setminus E_m$ and $(\bar{V}_{m-1} \cup \bar{H}) \setminus E_m$ are disjoint. As these sets are closed in $X \setminus E_m$, from the hereditary normality of X it follows that

there exist disjoint sets U_m, V_m open in $X \setminus E_m$, and consequently open in X , such that

$$(\bar{U}_{m-1} \cup \bar{G}) \setminus E_m \subset U_m, \quad (\bar{V}_{m-1} \cup \bar{H}) \setminus E_m \subset V_m,$$

and (5) is satisfied for $i = m$. The last two inclusions imply that (4) and (6) also are satisfied for $i = m$, because $L_m \cap (U_{m-1} \cup V_{m-1}) = \emptyset$ by virtue of (7). Hence the construction of the sets U_i, V_i and L_i satisfying (2)–(6) for $i = 1, 2, \dots$ is completed.

Let $U = \bigcup_{i=1}^{\infty} U_i$, $V = \bigcup_{i=1}^{\infty} V_i$, and $L = \bigcup_{i=1}^{\infty} L_i$; it follows from (4) that $X = U \cup V \cup L$. The sets U and V are open and, by virtue of (6) and the equality $U_i \cap V_i = \emptyset$, which is a consequence of (5), disjoint. From (5), (6) and the inclusion $E_{i-1} \subset E_i$ it follows that $U \cap L = V \cap L = \emptyset$, so that $X \setminus L = U \cup V$, which together with (1) shows that L is a partition between A and B . By virtue of (2) and (3) the inductive assumption can be applied to the space L and the sequence $L_1, L_2, \dots$; thus $\text{Ind } L \leq n-1$, which shows that $\text{Ind } X \leq n$. $\square$

2.3.2. Corollary. *If a hereditarily normal space X has property (μ_n) , then it also has property (σ_n) .*

Proof. Consider a subspace $Y \subset X$ and a sequence $F_1, F_2, \dots$ of closed subspaces of Y such that $Y = \bigcup_{i=1}^{\infty} F_i$ and $\text{Ind } F_i \leq n$ for $i = 1, 2, \dots$. By virtue of (μ_n) the set $K_i = F_i \setminus \bigcup_{j < i} F_j$ satisfies the inequality $\text{Ind } K_i \leq n$ for $i = 1, 2, \dots$. Applying Theorem 2.3.1 to the space Y and the sequence $K_1, K_2, \dots$, we obtain the inequality $\text{Ind } Y \leq n$. $\square$

We now turn to the implication $(\mu_{n-1}) \Rightarrow (\mu_n)$. To begin with, we shall establish a simple property of point-finite open covers, which will be applied here and in the following chapter.

2.3.3. Lemma. *Let $\{U_s\}_{s \in S}$ be a point-finite open cover of a topological space X . For $i = 1, 2, \dots$ denote by K_i the set of all points of the space X which belong to exactly i members of the cover $\{U_s\}_{s \in S}$ and by $\mathcal{T}_i$ the family of all subsets of S that have exactly i elements. Then*

$$(11) \quad X = \bigcup_{i=1}^{\infty} K_i, \quad K_i \cap K_j = \emptyset \text{ whenever } i \neq j, \text{ and}$$

the union $F_i = \bigcup_{j \leq i} K_j$ is closed for $i = 1, 2, \dots$;

moreover,

$$(12) \quad K_i = \bigcup_{T \in \mathcal{T}_i} K_T, \text{ where the sets } K_T, \text{ defined by letting } K_T = K_i \cap \bigcap_{s \in T} U_s, \\ \text{for } T \in \mathcal{T}_i, \text{ are open in } K_i \text{ and pairwise disjoint.}$$

Proof. The first two equalities in (11) follow directly from the definition of the sets K_i . If $x \notin F_i$, then $x \in U_{s_1} \cap U_{s_2} \cap \dots \cap U_{s_{i+1}} \subset X \setminus F_i$, where $s_1, s_2, \dots, s_{i+1}$ are distinct elements of S , so that the sets F_i are closed. To establish (12) it suffices to note that $K_T \subset K_i$ for $T \in \mathcal{T}_i$ and that whenever T, T' are distinct members of $\mathcal{T}_i$, then $K_T \cap K_{T'} = \emptyset$, because the union $T \cup T'$ contains at least $i+1$ elements of S . $\square$

We shall now prove a lemma which, together with Corollary 2.3.2, yields the implication $(\mu_{n-1}) \Rightarrow (\mu_n)$ for every perfectly normal space X ; afterwards, we shall deduce from this lemma that the implication $(\mu_{n-1}) \Rightarrow (\mu_n)$ holds for every strongly hereditarily normal space X .

2.3.4. Lemma. *If a hereditarily normal space X has property (σ_{n-1}) and if $\text{Ind } X \leq n$, then $\text{Ind } U \leq n$ for every open subspace U of X which can be represented as the union of a point-finite family of open F_σ -sets in X .*

Proof. Let us first consider the special case where U is an open F_σ -set in X . Then there exists a continuous function $f: X \rightarrow I$ such that $U = f^{-1}((0, 1])$ (see Problem 2.1.A). The sets $B_i = f^{-1}([1/i, 1])$ are closed in X and satisfy

$$(13) \quad B_i \subset \text{Int } B_{i+1}, \quad \text{Ind } B_i \leq n \quad \text{for } i = 1, 2, \dots \quad \text{and} \quad U = \bigcup_{i=1}^{\infty} B_i.$$

Consider an arbitrary set $A \subset U$ which is closed in U and an open set $V \subset U$ which contains the set A . For $i = 1, 2, \dots$ let

$$(14) \quad A_i = A \cap (B_i \setminus \text{Int } B_{i-1}) \quad \text{and} \quad V_i = V \cap (\text{Int } B_{i+1} \setminus B_{i-2}),$$

where $B_{-1} = B_0 = \emptyset$; clearly, $A_i \subset V_i \subset B_{i+1}$, A_i is closed in B_{i+1} , and V_i is open in B_{i+1} . Since $\text{Ind } B_{i+1} \leq n$, there exists an open subset U_i of B_{i+1} such that

$$(15) \quad A_i \subset U_i \subset \bar{U}_i \subset V_i \quad \text{and} \quad \text{Ind Fr } U_i \leq n-1,$$

where the closure and boundary operators refer to the space U . Indeed, as B_{i+1} is a closed subset of U and $U_i \subset \text{Int } B_{i+1}$, the closure and the boundary of U_i in B_{i+1} and in U coincide. From the inclusion $\bar{U}_i \subset U \setminus B_{i-2}$

it follows that the family $\{\bar{U}_i\}_{i=1}^\infty$ is locally finite in the space U , so that for the set $U_0 = \bigcup_{i=1}^\infty U_i$ we have $\bar{U}_0 = \bigcup_{i=1}^\infty \bar{U}_i$ and $\text{Fr } U_0 \subset \bigcup_{i=1}^\infty \text{Fr } U_i$. Applying (13)–(15) and (σ_{n-1}) , we obtain

$$A = A \cap \bigcup_{i=1}^\infty B_i \subset \bigcup_{i=1}^\infty A_i \subset U_0 \subset \bar{U}_0 = \bigcup_{i=1}^\infty \bar{U}_i \subset V$$

$$\text{and } \text{Ind Fr } U_0 \leq n-1,$$

which shows that $\text{Ind } U \leq n$.

Let us now pass to the general situation and consider a point-finite family $\{U_s\}_{s \in S}$ of open F_σ -sets in X such that $U = \bigcup_{s \in S} U_s$. Applying Lemma 2.3.3 to the cover $\{U_s\}_{s \in S}$ of the space U , we obtain the sets K_i which satisfy (11) and (12) with $X = U$. Since for each $T \in \mathcal{T}_i$ we have $K_T = K_i \cap \bigcap_{s \in T} U_s = F_i \cap \bigcap_{s \in T} U_s$, the sets K_T are F_σ -sets in X .

The subspace $X_i = (X \setminus U) \cup F_i$ is closed in X , so that $\text{Ind } X_i \leq n$. Since the set $K_i = F_i \setminus F_{i-1} = X_i \cap (U \setminus F_{i-1})$ is open in X_i , the sets K_T are open F_σ -sets in X_i . Applying the already established special case of our theorem to the space X_i and the set K_T , we conclude that $\text{Ind } K_T \leq n$ for every $T \in \mathcal{T}_i$. Thus $\text{Ind } K_i \leq n$ for $i = 1, 2, \dots$ by virtue of (12) and Proposition 2.2.8. Theorem 2.3.1 and (11) imply that $\text{Ind } U \leq n$. $\square$

2.3.5. Lemma. *If a strongly hereditarily normal space X has property (μ_{n-1}) , then it also has property (μ_n) .*

Proof. Since strong hereditary normality and property (μ_{n-1}) are both hereditary properties, it suffices to show that if $\text{Ind } X \leq n$, then $\text{Ind } U \leq n$ for every open subspace U of X . Consider a pair A, B of disjoint closed subsets of the space U . Let $U' = X \setminus (\bar{A} \cap \bar{B})$, $A' = U' \cap \bar{A}$ and $B' = U' \cap \bar{B}$. Obviously, $U \subset U'$, $A \subset A'$, $B \subset B'$ and the sets A', B' are disjoint and closed in U' . It is enough to show that there exists a partition L' in U' between A' and B' such that $\text{Ind } L' \leq n-1$, because then $L = U \cap L'$ will be a partition in U between A and B satisfying, by virtue of (μ_{n-1}) , the inequality $\text{Ind } L \leq n-1$. Since $\bar{A'} = \bar{A}$ and $\bar{B'} = \bar{B}$, we have $U' = X \setminus (\bar{A'} \cap \bar{B'})$. Thus without loss of generality we can suppose that $U = X \setminus (\bar{A} \cap \bar{B})$ and define the required partition in U .

As the sets A and B are separated in X , there exist disjoint open sets $V, W \subset X$ such that $A \subset V$, $B \subset W$ and V and W can be represented

as the union of a point-finite family of open F_σ -sets in X . From the equality $\bar{V} \cap W = \emptyset = V \cap \bar{W}$ it follows that $\bar{A} \cap W = \emptyset = V \cap \bar{B}$, so that $(\bar{A} \cap \bar{B}) \cap (V \cup W) = \emptyset$, i.e., $V \cup W \subset U$. The space U being normal there exists an open set $O \subset U$ such that $A \subset O \subset U \cap \bar{O} \subset V$. The sets A and $F = V \setminus O$ are disjoint and closed in V ; since $\text{Ind } V \leq n$ by virtue of Corollary 2.3.2 and Lemma 2.3.4, there exists a partition L in V between A and F such that $\text{Ind } L \leq n-1$. Thus there exist open sets $G, H \subset X$ such that

$$A \subset G, \quad F \subset H, \quad G \cap H = \emptyset \quad \text{and} \quad V \setminus L = G \cup H.$$

The set $H \cup (U \setminus \bar{O})$ is open in X , disjoint from G , and contains B , because $B \subset W \subset U \setminus V \subset U \setminus \bar{O}$; moreover,

$$G \cup [H \cup (U \setminus \bar{O})] = (V \setminus L) \cup (U \setminus \bar{O}) = U \setminus L.$$

Thus L is a partition in U between A and B . $\square$

From Corollary 2.3.2 and Lemma 2.3.5 it follows that all strongly hereditarily normal spaces have properties (μ_n) and (σ_n) for $n = -1, 0, 1, \dots$, which, together with Theorems 2.2.3 and 2.3.1, yields the following two results.

2.3.6. The subspace theorem for Ind. *For every subspace M of a strongly hereditarily normal space X we have $\text{Ind } M \leq \text{Ind } X$. $\square$*

2.3.7. Proposition. *If a strongly hereditarily normal space X can be represented as the union of a sequence $K_1, K_2, \dots$ of subspaces such that $\text{Ind } K_i \leq n$ and the union $\bigcup_{j \leq i} K_j$ is closed for $i = 1, 2, \dots$, then $\text{Ind } X \leq n$. $\square$*

The last proposition yields

2.3.8. The countable sum theorem for Ind. *If a strongly hereditarily normal space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces such that $\text{Ind } F_i \leq n$ for $i = 1, 2, \dots$, then $\text{Ind } X \leq n$. $\square$*

In the dimension theory of general spaces there occur also sum theorems of a different kind, where instead of countable covers one considers locally finite ones. Such theorems are not discussed in the classical dimension theory of separable metric spaces, because—as is easy to verify—every locally finite cover of a separable metric space is countable, and thus in such spaces the locally finite sum theorem is only a particular case of the countable sum theorem. Exactly as in the case of the countable sum

theorem, the locally finite sum theorem for Ind in strongly hereditarily normal spaces shall be deduced from a version of this theorem which holds in all hereditarily normal spaces. Let us note that in countably paracompact strongly hereditarily normal spaces (in particular, in perfectly normal spaces and in hereditarily weakly paracompact hereditarily normal spaces; cf. [GT], Sections 5.2 and 5.3), the locally finite sum theorem is an easy consequence of Theorems 2.3.1, 2.3.6 and 2.3.8 (see Problem 2.3.B).

2.3.9. Theorem. *If a hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of pairwise disjoint subspaces such that $\text{Ind } K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\text{Ind } X \leq n$.*

Proof. We shall apply induction with respect to the number n . For $n = -1$ the theorem is obvious. Assume that the corresponding statements hold for dimensions less than n and consider a hereditarily normal space X which satisfies the assumptions of our theorem. Let $F_\alpha = \bigcup_{\beta \leq \alpha} K_\beta$ for $\alpha < \xi$.

Consider a pair A, B of disjoint closed subsets of X . We shall define inductively three transfinite sequences $U_1, U_2, \dots, U_\alpha, \dots, \alpha < \xi$, $V_1, V_2, \dots, V_\alpha, \dots, \alpha < \xi$, and $L_1, L_2, \dots, L_\alpha, \dots, \alpha < \xi$ of subsets of X satisfying for $\alpha < \xi$ the following conditions:

$$(16) \quad L_\alpha \subset K_\alpha \quad \text{and} \quad \text{Ind } L_\alpha \leq n-1.$$

$$(17) \quad \text{The sets } U_\alpha \text{ and } V_\alpha \text{ are open in } F_\alpha \text{ and } U_\alpha \cup V_\alpha = F_\alpha \setminus E_\alpha, \\ \text{where } E_\alpha = \bigcup_{\beta \leq \alpha} L_\beta.$$

$$(18) \quad A \cap F_\alpha \subset U_\alpha, \quad B \cap F_\alpha \subset V_\alpha \quad \text{and} \quad U_\alpha \cap V_\alpha = \emptyset.$$

$$(19) \quad U_\beta = F_\beta \cap U_\alpha, \quad V_\beta = F_\beta \cap V_\alpha \quad \text{for} \quad \beta < \alpha.$$

Since $F_1 = K_1$ and $\text{Ind } K_1 \leq n$, there exist sets U_1, V_1 and L_1 satisfying (16)–(19) for $\alpha = 1$. Let us assume that the sets U_α, V_α and L_α satisfying (16)–(19) are defined for $\alpha < \gamma < 1$. Let

$$F'_\gamma = \bigcup_{\alpha < \gamma} K_\alpha, \quad U'_\gamma = \bigcup_{\alpha < \gamma} U_\alpha, \quad V'_\gamma = \bigcup_{\alpha < \gamma} V_\alpha \quad \text{and} \quad E'_\gamma = \bigcup_{\alpha < \gamma} L_\alpha;$$

note that if $\gamma = \gamma_0 + 1$, then the sets defined above are equal to $F_{\gamma_0}, U_{\gamma_0}, V_{\gamma_0}$ and E_{γ_0} , respectively. From conditions (17), (18) and (19) with $\alpha < \gamma$

it follows that

$$(20) \quad A \cap F'_\gamma \subset U'_\gamma, \quad B \cap F'_\gamma \subset V'_\gamma, \quad U'_\gamma \cap V'_\gamma = \emptyset \\ \text{and} \quad F'_\gamma \setminus E'_\gamma = U'_\gamma \cup V'_\gamma.$$

We shall show that the sets $U'_\gamma, V'_\gamma \subset F'_\gamma$ are open in F'_γ ; clearly, it suffices to consider the case where γ is a limit number.

Consider an arbitrary point $x \in U'_\gamma$. Since the family $\{K_\alpha\}_{\alpha < \gamma}$ is locally finite, there exist a neighbourhood $W \subset X$ of the point x and an ordinal number $\alpha_0 < \gamma$ such that $W \cap K_\alpha = \emptyset$ whenever $\alpha_0 < \alpha < \gamma$, i.e., such that $W \cap F'_\gamma = W \cap F_{\alpha_0}$. The set U_{α_0} being open in F_{α_0} , there exists an open set $W' \subset X$ such that $U_{\alpha_0} = F_{\alpha_0} \cap W'$. From (19) it follows that $F_{\alpha_0} \cap U'_\gamma = U_{\alpha_0}$, so that

$$x \in W \cap U'_\gamma = W \cap F'_\gamma \cap U'_\gamma = W \cap F_{\alpha_0} \cap U'_\gamma = W \cap U_{\alpha_0} \subset W \cap W',$$

i.e., the set $W \cap W'$ is a neighbourhood of x in X . As

$$F'_\gamma \cap W \cap W' = W \cap F_{\alpha_0} \cap W' = W \cap U_{\alpha_0} \subset U_{\alpha_0} \subset U'_\gamma,$$

the set U'_γ is open in F'_γ . From symmetry of our assumptions it follows that the set V'_γ is also open in F'_γ . By virtue of (20) the set E'_γ is closed in F'_γ , which implies that E'_γ is closed in X , and $(A \cup B) \cap E'_\gamma = \emptyset$. The sets $A \cup U'_\gamma$ and $B \cup V'_\gamma$ are disjoint and closed in $X \setminus E'_\gamma$. From the hereditary normality of X it follows that there exist disjoint sets $G, H \subset X \setminus E'_\gamma$ open in $X \setminus E'_\gamma$, and consequently open in X , such that

$$(21) \quad A \cup U'_\gamma \subset G, \quad B \cup V'_\gamma \subset H \quad \text{and} \quad \bar{G} \cap \bar{H} \subset E'_\gamma.$$

By virtue of the last inclusion in (21) and by equality $E'_\gamma \cap K_\gamma = \emptyset$, the sets $G \cap K_\gamma$ and $H \cap K_\gamma$ have disjoint closures in K_γ . Hence there exists a partition L_γ in K_γ between $G \cap K_\gamma$ and $H \cap K_\gamma$ which satisfies (16) with $\alpha = \gamma$; there also exist open sets $G', H' \subset K_\gamma$ such that

$$(22) \quad G \cap K_\gamma \subset G', \quad H \cap K_\gamma \subset H', \quad G' \cap H' = \emptyset \\ \text{and} \quad K_\gamma \setminus L_\gamma = G' \cup H'.$$

Since the set $K_\gamma = F_\gamma \setminus F'_\gamma$ is open in F_γ , the sets G' and H' are open in F_γ . From (20), (21) and (22) it follows that the sets

$$U_\gamma = (G \cap F_\gamma) \cup G' \quad \text{and} \quad V_\gamma = (H \cap F_\gamma) \cup H'$$

satisfy conditions (17)–(19) with $\alpha = \gamma$. Hence the construction of the sets U_α, V_α and L_α satisfying (16)–(19) for $\alpha < \xi$ is completed.

It follows from (20) that the set $L = E'_\xi$ is a partition in $F'_\xi = X$ between A and B . Applying the inductive assumption to the space L and the se-

quence $L_1, L_2, \dots, L_\alpha, \dots, \alpha < \xi$, we obtain the inequality $\text{Ind} L \leq n-1$, which shows that $\text{Ind} X \leq n$. $\square$

Theorems 2.3.9 and 2.3.6 yield

2.3.10. The locally finite sum theorem for Ind. *If a strongly hereditarily normal space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind} F_s \leq n$ for $s \in S$, then $\text{Ind} X \leq n$.* $\square$

The following two theorems are common generalizations of the countable and the locally finite sum theorems. Let us recall that a family $\{A_s\}_{s \in S}$ of subsets of a topological space X is *locally countable* if for every point $x \in X$ there exists a neighbourhood U such that the set $\{s \in S: U \cap A_s \neq \emptyset\}$ is countable.

2.3.11. Theorem. *If a strongly hereditarily normal space X can be represented as the union of a σ -locally finite family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind} F_s \leq n$ for $s \in S$, then $\text{Ind} X \leq n$.*

Proof. The family $\{F_s\}_{s \in S}$ decomposes into countably many locally finite families the union of each of which is closed in X and—by virtue of the locally finite sum theorem—has dimension not larger than n . To complete the proof it suffices to apply the countable sum theorem. $\square$

2.3.12. Theorem. *If a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \dots, F_\alpha, \dots, \alpha < \xi$ of closed subspaces such that $\text{Ind} F_\alpha \leq n$ and the family $\{F_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, and the family $\{F_\alpha\}_{\alpha < \xi}$ is locally countable, then $\text{Ind} X \leq n$.*

Proof. If the set of all ordinal numbers less than ξ contains a countable cofinal subset, then the family $\{F_\alpha\}_{\alpha < \xi}$ is σ -locally-finite and the theorem follows from Theorem 2.3.11.

Assume now that the set of all ordinal numbers less than ξ contains no countable cofinal subset. To complete the proof it suffices to show that under this assumption the family $\{F_\alpha\}_{\alpha < \xi}$ is locally finite. Consider an arbitrary point $x \in X$ and a neighbourhood U of this point such that the set $\{\alpha < \xi: U \cap F_\alpha \neq \emptyset\}$ is countable. By our assumption there exists an $\alpha_0 < \xi$ such that $U \cap F_\alpha = \emptyset$ for $\alpha \geq \alpha_0$. The family $\{F_\alpha\}_{\alpha < \alpha_0}$ being locally finite, there exists a neighbourhood V of the point x such that the set $\{\alpha < \alpha_0: V \cap F_\alpha \neq \emptyset\}$ is finite. The intersection $W = U \cap V$ is

a neighbourhood of the point x which meets only finitely many sets F_α . $\square$

The next result is still another sum theorem. It will yield two further sum theorems which hold in the class of weakly paracompact strongly hereditarily normal spaces. Let us observe that in this class Theorem 2.3.13 generalizes the locally finite sum theorem (see Problem 2.3.F) and Theorem 2.3.15 generalizes both the countable and the locally finite sum theorems.

2.3.13. The point-finite sum theorem for Ind. *If a strongly hereditarily normal space X can be represented as the union of a family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind} F_s \leq n$ for $s \in S$, and if there exists a point-finite open cover $\{U_s\}_{s \in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then $\text{Ind} X \leq n$.*

Proof. Consider the decomposition of the space X described in Lemma 2.3.3. From the definition of the sets K_τ it follows that $K_\tau \subset \bigcup_{s \in T} F_s$ for $T \in \mathcal{T}_i$, so that $\text{Ind} K_\tau \leq n$ by virtue of Theorems 2.3.8 and 2.3.6. Theorem 2.2.8 and (12) imply that $\text{Ind} K_i \leq n$ for $i = 1, 2, \dots$. To complete the proof it suffices to apply Theorem 2.3.1. $\square$

2.3.14. Theorem. *If a weakly paracompact strongly hereditarily normal space X can be represented as the union of a family $\{U_s\}_{s \in S}$ of open subspaces such that $\text{Ind} U_s \leq n$ for $s \in S$, then $\text{Ind} X \leq n$.*

Proof. The space X being weakly paracompact, one can assume that the cover $\{U_s\}_{s \in S}$ is point-finite, and thus has a closed shrinking $\{F_s\}_{s \in S}$ (see [GT], Theorem 1.5.18). To complete the proof it suffices to apply Theorems 2.2.1 and 2.3.13. $\square$

2.3.15. Theorem. *If a weakly paracompact strongly hereditarily normal space X can be represented as the union of a locally countable family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind} F_s \leq n$ for $s \in S$, then $\text{Ind} X \leq n$.*

Proof. For every point $x \in X$ there exist a neighbourhood U_x and a countable set $S(x) \subset S$ such that $U_x \cap F_s = \emptyset$ for $s \in S \setminus S(x)$. From the last relation it follows that $U_x \subset \bigcup \{F_s : s \in S(x)\}$, so that by virtue of Theorems 2.3.8 and 2.3.6 we have $\text{Ind} U_x \leq n$ for $x \in X$. To complete the proof it suffices to apply Theorem 2.3.14. $\square$

We close this section with a characterization of the dimension Ind in strongly hereditarily normal spaces which will be applied in Section 2.4.

2.3.16. Lemma. *If for a pair A, B of disjoint closed subsets of a topological space X there exists a σ -locally finite open cover $\mathcal{V}$ of the space X which has the property that for every $V \in \mathcal{V}$ either $\bar{V} \cap A = \emptyset$ or $\bar{V} \cap B = \emptyset$, then there exists a partition L between A and B such that*

$$(23) \quad L \subset \bigcup \{ \text{Fr } V : V \in \mathcal{V} \}.$$

Proof. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where the families $\mathcal{V}_i$ are locally finite. For $i = 1, 2, \dots$ define $\mathcal{W}_i = \{V \in \mathcal{V}_i : \bar{V} \cap A = \emptyset\}$ and $\mathcal{U}_i = \mathcal{V}_i \setminus \mathcal{W}_i$; consider the sets

$$U_i = \bigcup \mathcal{U}_i \quad \text{and} \quad W_i = \bigcup \mathcal{W}_i.$$

The family $\mathcal{V}_i$ being locally finite,

$$(24) \quad \bar{U}_i \cap B = \emptyset = \bar{W}_i \cap A \quad \text{for } i = 1, 2, \dots;$$

moreover

$$(25) \quad A \subset \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad B \subset \bigcup_{i=1}^{\infty} W_i.$$

Let

$$(26) \quad G_i = U_i \setminus \bigcup_{j < i} \bar{W}_j \quad \text{and} \quad H_i = W_i \setminus \bigcup_{j \leq i} \bar{U}_j.$$

From (24)–(26) it follows that the open sets $U = \bigcup_{i=1}^{\infty} G_i$ and $W = \bigcup_{i=1}^{\infty} H_i$ satisfy the conditions $A \subset U$, $B \subset W$ and $U \cap W = \emptyset$. From the local finiteness of $\mathcal{V}_i$ it follows that $\text{Fr } U_i \cup \text{Fr } W_i \subset \bigcup \{ \text{Fr } V : V \in \mathcal{V}_i \}$, so that to complete the proof it suffices to show that the partition $L = X \setminus (U \cup W)$ between A and B satisfies the inclusion

$$(27) \quad L \subset \bigcup_{i=1}^{\infty} \text{Fr } U_i \cup \bigcup_{i=1}^{\infty} \text{Fr } W_i.$$

Consider a point $x \in L$ and denote by F the first element of the sequence $\bar{U}_1, \bar{W}_1, \bar{U}_2, \bar{W}_2, \dots$ that contains the point x . If $F = \bar{U}_i$, then $x \in \text{Fr } U_i = \bar{U}_i \setminus U_i$, because $x \notin G_i$ and $x \notin \bar{W}_j$ for $j < i$. On the other hand, if

$F = \overline{W}_i$, then $x \in \text{Fr } \overline{W}_i = \overline{W}_i \setminus W_i$, because $x \notin H_i$ and $x \notin \overline{U}_j$ for $j \leq i$.

Hence in both cases $x \in \bigcup_{i=1}^{\infty} \text{Fr } U_i \cup \bigcup_{i=1}^{\infty} \text{Fr } W_i$. $\square$

2.3.17. Theorem. *For every strongly hereditarily normal space X and each integer $n \geq 0$ the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\text{Ind } X \leq n$.*
- (b) *Every locally finite open cover of the space X has a locally finite open refinement $\mathcal{V}$ such that $\text{Ind Fr } V \leq n-1$ for $V \in \mathcal{V}$.*
- (c) *Every two-element open cover of the space X has a σ -locally finite open refinement $\mathcal{V}$ such that $\text{Ind Fr } V \leq n-1$ for every $V \in \mathcal{V}$.*

Proof. Let $\{U_s\}_{s \in S}$ be a locally finite open cover of a normal space X such that $\text{Ind } X \leq n \geq 0$. Consider a closed shrinking $\{F_s\}_{s \in S}$ of the cover $\{U_s\}_{s \in S}$ (see [GT], Theorem 1.5.18) and for every $s \in S$ choose an open set $V_s \subset X$ such that

$$F_s \subset V_s \subset U_s \quad \text{and} \quad \text{Ind Fr } V_s \leq n-1.$$

Clearly, the family $\mathcal{V} = \{V_s\}_{s \in S}$ is a locally finite open refinement of the cover $\{U_s\}_{s \in S}$. Hence (a) $\Rightarrow$ (b) for every normal space X .

The implication (b) $\Rightarrow$ (c) being obvious, to complete the proof it suffices to show that (c) $\Rightarrow$ (a). Let X be a strongly hereditarily normal space which satisfies (c). Consider a pair A, B of disjoint closed subsets of X . There exist open sets $U, W \subset X$ such that $A \subset U$, $B \subset W$ and $\overline{U} \cap \overline{W} = \emptyset$. The two-element open cover $\{X \setminus \overline{U}, X \setminus \overline{W}\}$ of the space X has a σ -locally finite open refinement $\mathcal{V}$ such that $\text{Ind Fr } V \leq n-1$ for every $V \in \mathcal{V}$. Since for every $V \in \mathcal{V}$ either $\overline{V} \cap A = \emptyset$ or $\overline{V} \cap B = \emptyset$, by virtue of Lemma 2.3.16 there exists a partition L between A and B such that $L \subset \bigcup \{\text{Fr } V : V \in \mathcal{V}\}$. From Theorem 2.3.11 it follows that $\text{Ind } L \leq n-1$, so that $\text{Ind } X \leq n$. $\square$

Historical and bibliographic notes

The study of relations between properties (μ_n) and (σ_n) was originated by Dowker in [1953]. Dowker's paper contains Theorem 2.3.1, Corollary 2.3.2, Lemma 2.3.4 with "point-finite" replaced by "locally finite", as well as theorems 2.3.6, 2.3.7 and 2.3.8 for totally normal spaces (Theorems

2.3.6 and 2.3.8 for perfectly normal spaces were established by Čech in [1932]). Lemma 2.3.3 and the present version of Lemma 2.3.4 were given by Lifanov and Pasynkov in [1970]; the same paper contains Theorems 2.3.6, 2.3.7 and 2.3.8 for Dowker spaces (announced in Pasynkov [1967]). The last three theorems were extended to super normal spaces by Nishiura in [1977]. Theorem 2.3.9 was established by Lifanov and Pasynkov in [1970]. Theorem 2.3.10 was proved by Kimura in [1967] for totally normal spaces (under the additional assumption of countable paracompactness in [1963]) and was extended to super normal spaces by Nishiura in [1977]. Theorems 2.3.12, 2.3.14 and 2.3.15 were proved by Lifanov and Pasynkov in [1970] for Dowker spaces; for totally normal spaces, Theorem 2.3.14 was given by Dowker in [1955], and Theorem 2.3.16—by Kimura in [1967] (implicitly). Theorem 2.3.17 was proved by Nagami in [1969] and [1960a], respectively for totally normal spaces and hereditarily paracompact spaces.

Problems

2.3.A (Smirnov [1951]). Let X be a normal space with the property that for each closed subspace $Y \subset X$ and every finite sequence $F_1, F_2, \dots, F_k$ of closed subspaces of Y such that $Y = \bigcup_{i=1}^k F_i$, if $\text{Ind} F_i \leq n$ for $i = 1, 2, \dots, k$, then $\text{Ind} Y \leq n$. Prove that $\text{ind} \beta X = \text{Ind} X$.

Deduce that for every strongly hereditarily normal space X we have $\text{ind} \beta X = \text{Ind} X$.

2.3.B. Show that under the additional hypothesis that X is a countably paracompact space Theorem 2.3.10 is an easy consequence of Theorems 2.3.1, 2.3.6 and 2.3.8.

Hint. For $i = 1, 2, \dots$ denote by K_i the set of all points of the space X which belong to exactly i members of the cover $\{F_s\}_{s \in S}$ and by $\mathcal{T}_i$ the family of all subsets of S that have exactly i elements. Note that $K_i = \bigcup_{T \in \mathcal{T}_i} K_T$, where $K_T = K_i \cap \bigcap_{s \in T} F_s$. Show that the sets $U_i = K_1 \cup K_2 \cup \dots \cup K_i$ are open in X and that $\text{Ind} U_i \leq n$ for $i = 1, 2, \dots$. Apply the fact that the open cover $\{U_i\}_{i=1}^\infty$ of the space X has a closed shrinking (see [GT], Theorem 5.2.3).

2.3.C. (a) Observe that Theorem 2.3.7 is a consequence of Theorem 2.3.8 and Problem 2.2.C(e).

(b) Deduce from Theorem 2.3.9 that if a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of subspaces such that $\text{Ind} K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\text{Ind} X \leq n$. Observe that the latter fact is also a consequence of Theorem 2.3.10 and Problem 2.2.C(e).

2.3.D (Lifanov and Pasynkov [1970]). Prove that if a hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of pairwise disjoint subspaces such that $\text{Ind} K_\alpha \leq n$, the union $\bigcup_{\beta < \alpha} K_\beta$ is closed and the family $\{K_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally countable, then $\text{Ind} X \leq n$.

2.3.E. (a) (Nagata [1965]) Prove that if a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \dots, F_\alpha, \dots, \alpha < \xi$ of closed subspaces such that $\text{Ind} F_\alpha \leq n$ for $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \dots, U_\alpha, \dots, \alpha < \xi$ of open subsets of X such that $F_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, then $\text{Ind} X \leq n$.

Hint. Observe that if the set of all ordinal numbers less than ξ contains no countable cofinal subset, then the family $\{U_\alpha\}_{\alpha < \xi}$ is point-finite.

(b) (Nagata [1967]) Prove that if a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of subspaces such that $\text{Ind} K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \dots, U_\alpha, \dots, \alpha < \xi$ of open subsets of X such that $K_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, then $\text{Ind} X \leq n$.

2.3.F (Smith and Krajewski [1971]). Prove that for every locally finite family $\{F_s\}_{s \in S}$ of closed subsets of a weakly paracompact space X there exists a point-finite family $\{U_s\}_{s \in S}$ of open sets such that $F_s \subset U_s$ for $s \in S$.

Hint. For every point $x \in X$ choose a neighbourhood U_x which meets only finitely many sets F_s . Consider a point-finite open refinement $\mathcal{V}$ of the cover $\{U_x\}_{x \in X}$ and let $U_s = \bigcup \{V \in \mathcal{V} : V \cap F_s \neq \emptyset\}$.

2.4. Relations between the dimensions ind and Ind . Cartesian product theorems for the dimension Ind

As shown in Section 1.6, for every separable metric space X we have $\text{ind}X = \text{Ind}X$. In the present section this equality will be extended to a somewhat larger class of spaces. However, we should make it clear at once that the class of all spaces whose small inductive and large inductive dimensions coincide is rather restricted. Indeed, there exists a first-countable compact space X such that $\text{ind}X = 2$ and $\text{Ind}X = 3$, as well as a completely metrizable space X , known as *Roy's space*, such that $\text{ind}X = 0$ and $\text{Ind}X = \dim X = 1$. The description of the above two spaces is quite complicated and the computation of their dimensions is rather difficult, so that they will not be reproduced in this book. The only space with non-coinciding inductive dimensions discussed here in detail is the space X described in Example 2.2.11, which satisfies the equalities $\text{ind}X = 0$ and $\text{Ind}X = 1$.

Let us recall that a topological space X is *strongly paracompact*¹⁾ if X is a Hausdorff space and every open cover of the space X has a star-finite open refinement (a family of sets $\mathcal{A}$ is *star-finite* if every set $A \in \mathcal{A}$ meets only finitely many members of $\mathcal{A}$). It immediately follows from the definitions that every strongly paracompact space is paracompact. One proves that every Lindelöf space is strongly paracompact (see [GT], Corollary 5.3.11).

In the considerations of this section we shall apply the fact that star-finite covers decompose in a natural way into countable families of sets. Let us recall that the *component* of a member A_0 of a family of sets $\mathcal{A}$ is the subfamily $\mathcal{A}_0 \subset \mathcal{A}$ consisting of all sets $A \in \mathcal{A}$ for which there exists a finite sequence $A_1, A_2, \dots, A_k$ of members of $\mathcal{A}$ such that $A_k = A$ and $A_i \cap A_{i-1} \neq \emptyset$ for $i = 1, 2, \dots, k$. The components of two distinct members of $\mathcal{A}$ either coincide or are disjoint, so that

$$\mathcal{A} = \bigcup_{s \in S} \mathcal{A}_s, \quad \text{where } \mathcal{A}_s \cap \mathcal{A}_{s'} = \emptyset \text{ for } s \neq s'$$

and for every $s \in S$ the family $\mathcal{A}_s$ is the component of a member of $\mathcal{A}$. The representation of $\mathcal{A}$ as the union of the families $\mathcal{A}_s$ will be called the decomposition of $\mathcal{A}$ into components; clearly $(\bigcup \mathcal{A}_s) \cap (\bigcup \mathcal{A}_{s'}) = \emptyset$ for $s \neq s'$. Let us observe that all components of a star-finite family $\mathcal{A}$ are countable. Indeed, the component $\mathcal{A}_0$ of the set $A_0 \in \mathcal{A}$ can be rep-

¹⁾ The term *hypocompact* is also used.

resented as the union of families $\mathcal{A}_1, \mathcal{A}_2, \dots$, where $\mathcal{A}_k$ consists of all sets $A \in \mathcal{A}$ for which there exist a sequence $A_1, A_2, \dots, A_k$ of k members of $\mathcal{A}$ such that $A_k = A$ and $A_i \cap A_{i-1} \neq \emptyset$ for $i = 1, 2, \dots, k$, and—as can easily be verified—each of the families $\mathcal{A}_k$ is countable.

2.4.1. Lemma. *If X is a strongly paracompact space such that $\text{ind} X \leq n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B which can be represented as the union of a sequence $L_1, L_2, \dots$ of closed subspaces of X such that $\text{ind} L_i \leq n-1$ for $i = 1, 2, \dots$*

Proof. For every point $x \in X$ there exists a neighbourhood U_x such that $\text{ind Fr } U_x \leq n-1$ and either $\bar{U}_x \cap A = \emptyset$ or $\bar{U}_x \cap B = \emptyset$. Consider a star-finite open refinement $\mathcal{U}$ of the cover $\{U_x\}_{x \in X}$ of the space X . Let $\mathcal{U} = \bigcup_{s \in S} \mathcal{U}_s$ be the decomposition of the cover $\mathcal{U}$ into components; as the cover $\mathcal{U}$ is star-finite, the components are countable, i.e., $\mathcal{U}_s = \{U_{s,i}\}_{i=1}^\infty$ for $s \in S$. Let $X_s = \bigcup_{i=1}^\infty U_{s,i}$ for every $s \in S$; the sets X_s are open and pairwise disjoint, so that $X = \bigoplus_{s \in S} X_s$. For each set $U_{s,i}$ choose a point $x(s,i) \in X$ such that $U_{s,i} \subset U_{x(s,i)}$ and let $V_{s,i} = U_{x(s,i)} \cap X_s$; clearly, $X_s = \bigcup_{i=1}^\infty V_{s,i}$ and we have either $\bar{V}_{s,i} \cap A = \emptyset$ or $\bar{V}_{s,i} \cap B = \emptyset$. Since for $i = 1, 2, \dots$ the family $\{V_{s,i}\}_{s \in S}$ is discrete, the family $\mathcal{V}$ consisting of all the sets $V_{s,i}$ is a σ -locally finite open cover of the space X . By virtue of Lemma 2.3.16 there exists a partition L between A and B such that $L \subset \bigcup_{i=1}^\infty F_i$, where $F_i = \bigcup_{s \in S} \text{Fr } V_{s,i}$. Let $L_i = L \cap F_i$ for $i = 1, 2, \dots$. To complete the proof it suffices to show that L_i is closed in X and $\text{ind} L_i \leq n-1$ for $i = 1, 2, \dots$. Since $\text{Fr } V_{s,i} \subset X_s$ for $s \in S$, $F_i = \bigoplus_{s \in S} \text{Fr } V_{s,i}$ and F_i is a closed subset of X ; thus L_i is closed in X for $i = 1, 2, \dots$. On the other hand, $\text{Fr } V_{s,i} \subset \text{Fr } U_{x(s,i)} \cup \text{Fr } X_s = \text{Fr } U_{x(s,i)}$, so that $\text{ind Fr } V_{s,i} \leq n-1$ for $s \in S$; thus $\text{ind} F_i \leq n-1$ and $\text{ind} L_i \leq n-1$ for $i = 1, 2, \dots$ $\square$

From Lemma 2.4.1 we immediately obtain the following generalization of Theorem 1.6.5.

2.4.2. Theorem. *For every strongly paracompact space X the conditions $\text{ind} X = 0$ and $\text{Ind} X = 0$ are equivalent.* $\square$

The next theorem shows that one step forward is possible.

2.4.3. Theorem. *For every strongly paracompact space X the conditions $\text{ind}X = 1$ and $\text{Ind}X = 1$ are equivalent.*

Proof. In view of Theorem 2.4.2 it suffices to show that for every strongly paracompact space X such that $\text{ind}X = 1$ we have $\text{Ind}X = 1$. Consider a pair A, B of disjoint closed subsets of the space X . By virtue of Lemma 2.4.1 there exists a partition L between A and B such that $L = \bigcup_{i=1}^{\infty} L_i$, where L_i is closed in X and $\text{ind}L_i \leq 0$ for $i = 1, 2, \dots$. As every closed subspace of a strongly paracompact space is strongly paracompact, it follows from Theorem 2.4.2 that $\text{Ind}L_i \leq 0$ for $i = 1, 2, \dots$, so that $\text{Ind}L \leq 0$ by virtue of Theorem 2.2.7. Thus $\text{Ind}X \leq 1$, and from the inequality $\text{ind}X \leq \text{Ind}X$ it follows that $\text{Ind}X = 1$. $\square$

2.4.4. Theorem. *For every strongly paracompact strongly hereditarily normal space X we have $\text{ind}X = \text{Ind}X$.*

Proof. It suffices to show that $\text{Ind}X \leq \text{ind}X$; clearly, one can suppose that $\text{ind}X < \infty$. We shall apply induction with respect to $\text{ind}X$. The inequality holds if $\text{ind}X = -1$. Assume that the inequality is proved for all strongly paracompact strongly hereditarily normal spaces of small inductive dimension less than $n \geq 0$ and consider a strongly paracompact strongly hereditarily normal space X such that $\text{ind}X = n$. Let A, B be a pair of disjoint closed subsets of X . By virtue of Lemma 2.4.1 and by the inductive assumption there exists a partition L between A and B such that $L = \bigcup_{i=1}^{\infty} L_i$, where L_i is closed in X and $\text{Ind}L_i \leq n-1$ for $i = 1, 2, \dots$. By virtue of Theorem 2.3.8 we have $\text{Ind}L \leq n$, so that $\text{Ind}X \leq n = \text{ind}X$. $\square$

Let us note that in the realm of Lindelöf spaces Lemma 2.4.1 is an immediate consequence of Lemma 2.3.16, so that the fact that Theorems 2.4.2, 2.4.3 and 2.4.4 hold for Lindelöf spaces can be obtained independently of the above-mentioned result that every strongly paracompact space is a Lindelöf space.

Let us also note that an obvious modification of the proof of Theorem 2.4.4 shows that $\text{ind}X = \text{Ind}X$ for every strongly paracompact space X in each closed subspace of which the countable sum theorem holds either for ind or for Ind .

In the problems listed at the end of this section Theorems 2.4.2, 2.4.3 and 2.4.4 are extended to some classes of spaces larger than the class of strongly paracompact spaces. However, the definitions of these classes of spaces are a little less natural than the definition of strongly paracompact spaces and, to some degree, are inspired by the methods of proofs.

We now turn to a study of the behaviour of the dimension Ind under Cartesian multiplication. Let us begin with recalling that, as stated in Section 2.2, there exist compact spaces X and Y such that $\text{Ind}(X \times Y) > \text{Ind}X + \text{Ind}Y$. Thus, one can see that the Cartesian product theorem for Ind , i.e., the inequality $\text{Ind}(X \times Y) \leq \text{Ind}X + \text{Ind}Y$, requires rather strong assumptions on X and Y . Several theorems of this type were proved under various assumptions. Usually, one assumes that X and Y have some properties related to paracompactness and that the Cartesian product $X \times Y$ is totally normal; generally, this last assumption can be weakened to the assumption that $X \times Y$ is strongly hereditarily normal or to the assumption that the finite sum theorem for Ind holds in closed subspaces of X and Y , i.e., that, for every closed subspace Z of either X or Y , $\text{Ind}Z \leq n$ whenever Z can be represented as the union of a finite number of closed subspaces $F_1, F_2, \dots, F_k$ such that $\text{Ind}F_i \leq n$ for $i = 1, 2, \dots, k$. Among the Cartesian product theorems for Ind so far discovered there is no strongest result. We shall quote two such theorems, which are relatively strong. Thus, the inequality $\text{Ind}(X \times Y) \leq \text{Ind}X + \text{Ind}Y$ holds for every pair X, Y of normal spaces of which at least one is non-empty provided that either

- (i) the finite sum theorem for Ind holds in closed subspaces of X and Y the Cartesian product $X \times Y$ is normal and one of the factors is compact (more generally: the projection onto one of the factors is a closed mapping),

or

- (ii) the finite sum theorem for Ind holds in closed subspaces of X and Y , the Cartesian product $X \times Y$ is normal, one of the factors is metrizable and the other is countably paracompact (more generally: one of the factors can be mapped to a metrizable space by a perfect mapping and the other is countably paracompact).

Let us observe that the proofs of Cartesian product theorems for Ind are fairly difficult; yet the difficulties are chiefly connected with the complicated structure of subsets of Cartesian products, so that—roughly speaking—they are of “topological” rather than “dimensional” nature. As

a sample we shall prove here two Cartesian product theorems for Ind which are among the simplest.

Arguing as in the proof of Theorem 1.5.16, we obtain

2.4.5. Lemma. *For every pair X, Y of regular spaces of which at least one is non-empty such that the finite sum theorem for ind holds in closed subspaces of the Cartesian product $X \times Y$ we have*

$$\text{ind}(X \times Y) \leq \text{ind} X + \text{ind} Y. \quad \square$$

Theorem 2.4.4 and Lemma 2.4.5 yield

2.4.6. Theorem. *For every pair X, Y of normal spaces of which at least one is non-empty such that the Cartesian product $X \times Y$ is strongly paracompact and strongly hereditarily normal we have*

$$\text{Ind}(X \times Y) \leq \text{Ind} X + \text{Ind} Y. \quad \square$$

2.4.7. Lemma. *The Cartesian product $X \times Y$ of a perfectly normal space X and a metrizable space Y is perfectly normal.*

Proof. As $X \times Y$ is a T_1 -space it suffices to show (see Problem 2.1.E (b)) that for every open set $W \subset X \times Y$ there exists a sequence $W_1, W_2, \dots$

of open subsets of $X \times Y$ such that $W = \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for $i = 1, 2, \dots$

Let $\mathcal{V}$ be a base for the space Y which can be represented as the union of locally finite families $\mathcal{V}_1, \mathcal{V}_2, \dots$ (see [GT], Corollary 4.4.4). Consider the family $\mathcal{W}$ of all sets $U \times V$, where U is an open subset of X and $V \in \mathcal{V}$, such that $\overline{U} \times \overline{V} \subset W$; clearly $W = \bigcup \mathcal{W}$. For every $V \in \mathcal{V}$ the union $U(V) = \bigcup \{U: U \times V \in \mathcal{W}\}$ is an open F_σ -set in X , therefore $U(V) = \bigcup_{j=1}^{\infty} U_j(V)$, where $U_j(V)$ is open and $\overline{U_j(V)} \subset U(V)$ for $j = 1, 2, \dots$

The family $\mathcal{W}_{j,k} = \{U_j(V) \times V: V \in \mathcal{V}_k\}$ is locally finite in $X \times Y$ for $j, k = 1, 2, \dots$, so that the set $W_{j,k} = \bigcup \mathcal{W}_{j,k}$ satisfies the inclusion

$\overline{W}_{j,k} \subset W$. Since $\bigcup_{j,k=1}^{\infty} W_{j,k} = \bigcup \mathcal{W} = W$, to complete the proof it suffices to arrange all the sets $W_{j,k}$ into a sequence $W_1, W_2, \dots$ $\square$

2.4.8. Theorem. *For every perfectly normal space X and every metrizable space Y of which at least one is non-empty we have*

$$\text{Ind}(X \times Y) \leq \text{Ind}X + \text{Ind}Y.$$

Proof. The theorem is obvious if the dimension Ind of one of the spaces X , Y equals ∞ , so that we can suppose that $k(X, Y) = \text{Ind}X + \text{Ind}Y$ is finite. We shall apply induction with respect to that number. If $k(X, Y) = -1$, then our inequality holds. Assume that the inequality is proved for every perfectly normal space and every metrizable space such that at least one of them is non-empty and the sum of large inductive dimensions of which is less than $k \geq 0$, and consider a perfectly normal space X and a metrizable space Y such that $\text{Ind}X = n \geq 0$, $\text{Ind}Y = m \geq 0$ and $n + m = k$. Let ϱ be a metric on the space Y . Since every metrizable space is paracompact, the space Y has for $i = 1, 2, \dots$ a locally finite open cover $\mathcal{U}_i$ such that $\text{mesh}\mathcal{U}_i < 1/i$. By virtue of Theorem 2.3.17 the cover $\mathcal{U}_i$ has a locally finite open refinement $\mathcal{V}_i$ such that $\text{IndFr}V \leq m - 1$ for every $V \in \mathcal{V}_i$. Obviously, the family $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a base for the space Y .

Consider now an arbitrary two-element open cover $\{G, H\}$ of the Cartesian product $X \times Y$; let $\mathcal{W}$ be a refinement of the cover $\{G, H\}$ consisting of sets of the form $U \times V$, where U is an open subset of X and $V \in \mathcal{V}$. For every $V \in \mathcal{V}$ the unions

$$G(V) = \bigcup \{U: U \times V \in \mathcal{W} \text{ and } U \times V \subset G\} \quad \text{and}$$

$$H(V) = \bigcup \{U: U \times V \in \mathcal{W} \text{ and } U \times V \subset H\}$$

are open F_σ -sets in X therefore

$$G(V) = \bigcup_{j=1}^{\infty} G_j(V) \quad \text{and} \quad H(V) = \bigcup_{j=1}^{\infty} H_j(V),$$

where $G_j(V)$, $H_j(V)$ are open and

$$\text{IndFr}G_j(V) \leq n - 1 \quad \text{and} \quad \text{IndFr}H_j(V) \leq n - 1 \quad \text{for } j = 1, 2, \dots$$

The family

$$\mathcal{W}_{j,i} = \{G_j(V) \times V: V \in \mathcal{V}_i\} \cup \{H_j(V) \times V: V \in \mathcal{V}_i\}$$

is locally finite in $X \times Y$ for $j, i = 1, 2, \dots$, so that the union $\bigcup_{j,i=1}^{\infty} \mathcal{W}_{j,i}$ is a σ -locally finite open refinement of the cover $\{G, H\}$.

From Theorem 2.1.3 and Lemma 2.4.7 it follows that the Cartesian product $X \times Y$ is strongly hereditarily normal. Thus, by virtue of Theorem 2.3.17, to complete the proof it suffices to show that $\text{IndFr } W \leq k-1$ for every $W \in \mathcal{W}_{j,i}$ and $j, i = 1, 2, \dots$. The last inequality is a consequence of the inclusions

$$\text{Fr}(G_j(V) \times V) \subset (X \times \text{Fr } V) \cup (\text{Fr } G_j(V) \times Y)$$

and

$$\text{Fr}(H_j(V) \times V) \subset (X \times \text{Fr } V) \cup (\text{Fr } H_j(V) \times Y),$$

the inductive assumption, and Theorem 2.3.8. $\square$

Historical and bibliographic notes

The first example of a compact space X such that $\text{ind } X \neq \text{Ind } X$ was defined by Filippov in [1969]; the example is discussed in detail in Filippov's paper [1970b] and in Pears' book [1975]. Simpler, but still quite complicated examples of such spaces were described by Filippov in [1970], Pasynkov and Lifanov in [1970], and Pasynkov in [1970]; these last spaces, moreover, are first-countable. The example of a completely metrizable space X such that $\text{ind } X = 0$ and $\text{Ind } X = \dim X = 1$ was outlined by Roy in [1962]; a detailed discussion of this example is contained in Roy's paper [1968] and Pears' book [1975]. The first example of a normal space with non-coinciding inductive dimensions was given by Smirnov in [1951]. Theorems 2.4.2 and 2.4.3 were proved by Vedenisoff in [1939] under the stronger assumption that X is a Lindelöf space. Theorems 2.4.4 and 2.4.6 were given by Katuta in [1966] with strong hereditary normality replaced by total normality; an important special case of Theorem 2.4.4, viz., the equality $\text{ind } X = \text{Ind } X$ for every strongly paracompact metrizable space X , was proved by Morita in [1950a] (see also notes to Theorem 1.5.13). The fact that the inequality $\text{Ind}(X \times Y) \leq \text{Ind } X + \text{Ind } Y$ holds for every pair X, Y of spaces satisfying either (i) or (ii) was proved by Filippov in [1979] (announcement [1973]); part (ii) was announced independently by Pasynkov in [1973]. In the original formulation of (ii) Filippov and Pasynkov assume that the Cartesian product $X \times Y$ is normal and countably paracompact; since Rudin and Starbird established in [1975] that if the Cartesian product of a metrizable space and a countably paracompact space is normal, then it is also countably paracompact, the as-

sumption of the countable paracompactness of $X \times Y$ can be dropped. Theorem 2.4.8 was obtained by Kimura in [1963]; under the additional assumption that X is a paracompact space it was proved by Nagami in [1960a]. Further information on Cartesian product theorems for Ind can be found in Kimura [1963], Nagata [1967], Lifanov [1968], Nagami [1969], Pasyukov [1969], van Dalen [1972] and Pears [1975].

Problems

2.4.A. (a) (Nagata [1957]) Prove that the Cartesian product of the open unit interval $(0, 1)$ and the Baire space $B(\aleph_1)$ (see Example 4.1.23) is not strongly paracompact. Deduce that the class of all strongly paracompact metrizable spaces is not hereditary with respect to F_σ -sets and is not finitely multiplicative. Show that a metrizable space which can be represented as the union of a locally finite countable family of strongly paracompact closed subspaces is not necessarily strongly paracompact.

(b) Show that every paracompact space X such that $\text{Ind} X = 0$ is strongly paracompact.

(c) Show that the Cartesian product $X \times Y$ of a strongly paracompact space X and a compact space Y is strongly paracompact.

(d) (Morita [1954]) Prove that the Cartesian product $I^{\aleph_0} \times B(m)$ is a universal space for the class of all strongly paracompact metrizable spaces whose weight is not larger than $m \geq \aleph_0$.

2.4.B (Zarelua [1963] (announcement [1961])). A topological space X is called *completely paracompact* if X is a regular space and for every open cover $\mathcal{U}$ of the space X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of star-finite open covers of X such that the union $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ contains a refinement of $\mathcal{U}$.

(a) Observe that every completely paracompact space is paracompact and that every strongly paracompact space is completely paracompact. Show that in the realm of connected spaces complete paracompactness is equivalent to the Lindelöf property. Prove that complete paracompactness is hereditary with respect to F_σ -sets.

(b) Prove that a metrizable space is completely paracompact if and only if it has a base which can be represented as the union of countably many star-finite covers. Deduce that the class of completely paracompact metrizable spaces is hereditary and countably multiplicative. Show that

a metrizable space which can be represented as the union of a countable family of completely paracompact closed subspaces is not necessarily completely paracompact.

(c) Prove that the Cartesian product $I^{\aleph_0} \times B(m)$ is a universal space for the class of all completely paracompact metrizable spaces the weight of which is not larger than $m \geq \aleph_0$.

(d) Give an example of a completely paracompact metrizable space which cannot be represented as the union of a σ -locally finite family of strongly paracompact closed subspaces.

Hint. Consider the Cartesian product $R^{\aleph_0} \times B(\aleph_1)$.

(e) Check that Theorems 2.4.2, 2.4.3 and 2.4.4 remain valid for completely paracompact spaces.

2.4.C. (a) (Nagami [1969]) A topological space X is called σ -totally paracompact if X is a regular space and for every base $\mathcal{B}$ for the space X there exists a σ -locally finite open cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$ one can find a $U \in \mathcal{B}$ satisfying $V \subset U$ and $\text{Fr } V \subset \text{Fr } U$.

Show that the class of σ -totally paracompact spaces is hereditary with respect to F_σ -sets. Check that Theorems 2.4.2, 2.4.3 and 2.4.4 remain valid for σ -totally paracompact spaces.

(b) (Zarelua [1963] (announcement [1961])) Observe that every σ -totally paracompact space is paracompact and that every completely paracompact space is σ -totally paracompact.

2.4.D. (a) (Fitzpatrick and Ford [1967]) A topological space X is *order totally paracompact* if X is a regular space and for every base $\mathcal{B}$ for the space X there exists an open cover $\{V_s\}_{s \in S}$, where the set S is linearly ordered by a relation $<$, such that for every $s_0 \in S$ the family $\{V_{s_0} \cap V_s\}_{s < s_0}$ is locally finite in the space V_{s_0} and for each $s \in S$ one can find a $U \in \mathcal{B}$ satisfying $V_s \subset U$ and $\text{Fr } V_s \subset \text{Fr } U$.

Show that the class of order totally paracompact spaces is hereditary with respect to closed subspaces.

(b) Prove that Theorem 2.4.2 remains valid for order totally paracompact spaces.

Hint. Check that if for a pair A, B of disjoint closed subsets of a topological space X there exists an open cover $\{V_s\}_{s \in S}$ of X , where the set S is linearly ordered by a relation $<$, such that for every $s_0 \in S$ either $\bar{V}_{s_0} \cap A = \emptyset$ or $\bar{V}_{s_0} \cap B = \emptyset$ and the family $\{V_{s_0} \cap V_s\}_{s < s_0}$ is locally finite in the

space V_{s_0} , then there exists a partition L between A and B such that $L \subset \bigcup_{s \in S} \text{Fr} V_s$.

Remark. It is not known if Theorems 2.4.3 and 2.4.4 remain valid for order totally paracompact spaces.

(c) (Fitzpatrick and Ford [1967]) Prove that for every order totally paracompact metrizable space X we have $\text{ind} X = \text{Ind} X$.

Hint. Apply the hint to (b); use the fact that X has a σ -locally finite base.

(d) (Katuta [1967]) Prove that a regular space X is paracompact if and only if every open cover $\mathcal{U}$ of X has an open refinement $\{V_s\}_{s \in S}$, where the set S is linearly ordered by a relation $<$, such that for every $s \in S_0$ the family $\{V_{s_0} \cap V_s\}_{s < s_0}$ is locally finite in the space V_{s_0} .

Hint. To prove that the space X is paracompact it suffices to show that every open cover of X has a locally finite refinement consisting of arbitrary sets (see [GT], Theorem 5.1.11).

(e) Deduce from (d) that every order totally paracompact space is paracompact and note that every σ -totally paracompact space is order totally paracompact.

2.4.E. Show that if a metrizable space X can be represented as the union of a locally countable family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{ind} F_s = \text{Ind} F_s$ for $s \in S$, then $\text{ind} X = \text{Ind} X$.

CHAPTER 3

THE COVERING DIMENSION

Chapter 2 was devoted to an examination of the question which results of the classical dimension theory of separable metric spaces hold in more general classes of spaces for the large inductive dimension Ind . In the present chapter, the covering dimension dim becomes the subject of similar considerations. It will be seen that outside the class of separable metric spaces the dimension dim behaves somewhat better than the dimension Ind , i.e., that for the dimension dim a larger number of theorems of the classical theory can be extended to topological spaces and that the extensions hold under weaker assumptions.

Section 3.1 is primarily devoted to the question of monotonicity of dim in general spaces and to a study of sum theorems for dim . We also discuss the relations between the covering dimension dim and the inductive dimensions ind and Ind ; in particular, we prove that $\text{dim} X \leq \text{Ind} X$ for every normal space X .

Section 3.2 begins with several characterizations of the dimension dim in normal spaces. They include generalizations of three important theorems of the classical theory of dimension, namely the theorems on partitions, on extending mappings to spheres and on $\mathcal{C}$ -mappings. The final part of the section is devoted to a discussion of Cartesian product theorems for the covering dimension.

In Section 3.3 we establish the compactification and the universal space theorems for dim and characterize compact spaces whose covering dimension does not exceed n as limits of inverse systems of compact metric spaces of dimension $\leq n$.

We shall return briefly to the topic of this chapter in Section 4.3, where some information on relations between the covering dimension of the domain and the range of a closed mapping will be given.

3.1. Basic properties of the dimension $\dim$ in normal spaces. Relations between the dimensions ind , Ind and $\dim$

The definition of the covering dimension $\dim$ was stated in Section 1.6; let us recall that a normal space X satisfies the inequality $\dim X \leq n$ if and only if every finite open cover of the space X has a finite open refinement of order $\leq n$. We shall begin this section with a characterization of the dimension $\dim$ which demonstrates that instead of finite open refinements of order $\leq n$ one can equally well consider finite closed refinements of order $\leq n$. The characterization is preceded by an auxiliary theorem on swellings of finite families of closed subsets of normal spaces which will be applied in its proof.

3.1.1. Definition. By a *swelling* of the family $\{A_s\}_{s \in S}$ of subsets of a topological space X we mean any family $\{B_s\}_{s \in S}$ of subsets of the space X such that $A_s \subset B_s$ for every $s \in S$ and for every finite set of indices $s_1, s_2, \dots, s_m \in S$

$$A_{s_1} \cap A_{s_2} \cap \dots \cap A_{s_m} \neq \emptyset \quad \text{if and only if} \quad B_{s_1} \cap B_{s_2} \cap \dots \cap B_{s_m} \neq \emptyset;$$

a swelling is *open* if all its members are open subsets of the space X .

Clearly, every swelling $\mathcal{B}$ of a family $\mathcal{A}$ satisfies the equality $\text{ord} \mathcal{B} = \text{ord} \mathcal{A}$.

The following theorem is, in a sense, dual to Theorem 1.7.8, which deals with shrinkings of finite open covers of normal spaces.

3.1.2. Theorem. Every finite family $\{F_i\}_{i=1}^k$ of closed subsets of a normal space X has an open swelling $\{U_i\}_{i=1}^k$. If, moreover, a family $\{V_i\}_{i=1}^k$ of open subsets of X satisfying $F_i \subset V_i$ for $i = 1, 2, \dots, k$ is given, then the swelling can be defined in such a way that $\bar{U}_i \subset V_i$ for $i = 1, 2, \dots, k$.

Proof. The union E_1 of all intersections $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_m}$ satisfying the equality $F_1 \cap F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_m} = \emptyset$ is a closed set disjoint from F_1 , so that there exist an open set U_1 such that $F_1 \subset U_1$ and $\bar{U}_1 \cap E_1 = \emptyset$. One readily sees that the family $\{\bar{U}_1, F_1, \dots, F_k\}$ is a swelling of the family $\{F_i\}_{i=1}^k$.

Assume that for $i = 1, 2, \dots, n-1$ an open set U_i is defined in such a way that $F_i \subset U_i$ and the family $\{\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{n-1}, F_n, \dots, F_k\}$ is a swelling of the family $\{F_i\}_{i=1}^k$. The union E_n of all intersections of members of the family $\{\bar{U}_1, \bar{U}_2, \dots, \bar{U}_{n-1}, F_n, \dots, F_k\}$ which are disjoint from F_n

is a closed set disjoint from F_n , so that there exists an open set U_n such that $F_n \subset U_n$ and $\bar{U}_n \cap E_n = \emptyset$. The family $\{\bar{U}_1, \bar{U}_2, \dots, \bar{U}_n, F_{n+1}, \dots, F_k\}$ is a swelling of the family $\{F_i\}_{i=1}^k$. In this way we obtain open sets $U_1, U_2, \dots, U_k$ such that $F_i \subset U_i$ for $i = 1, 2, \dots, k$ and the family $\{\bar{U}_i\}_{i=1}^k$ is a swelling of the family $\{F_i\}_{i=1}^k$. Clearly, the family $\{U_i\}_{i=1}^k$ is the required open swelling. The second part of the theorem is obvious. $\square$

Theorems 1.7.8 and 3.1.2 yield

3.1.3. Proposition. *For every normal space X the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\dim X \leq n$.*
- (b) *Every finite open cover of the space X has a closed shrinking of order $\leq n$.*
- (c) *Every finite open cover of the space X has a finite closed refinement of order $\leq n$.* $\square$

Since normality is not a hereditary property, it may happen that the dimension $\dim$ is defined for a space X and yet is not defined for a subspace $M \subset X$. Still, normality being hereditary with respect to closed subsets, $\text{Ind } M$ is defined for every closed subspace $M \subset X$. Moreover, the following counterpart of Theorem 2.2.1 holds.

3.1.4. Theorem. *For every closed subspace M of a normal space X we have $\dim M \leq \dim X$.*

Proof. The theorem is obvious if $\dim X = \infty$, so that we can assume that $\dim X = n < \infty$. Consider a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space M . For $i = 1, 2, \dots, k$ let W_i be an open subset of X such that $U_i = M \cap W_i$. The family $\{X \setminus M\} \cup \{W_i\}_{i=1}^k$ is an open cover of the space X and, since $\dim X \leq n$, it has a finite open refinement $\mathcal{V}$ of order $\leq n$. One easily sees that the family $\mathcal{V}|M$ is a finite open cover of the space M , refines $\mathcal{U}$ and has order $\leq n$, so that $\dim M \leq n = \dim X$. $\square$

From Theorem 1.6.11 it follows that the compact space Z and its normal subspace X defined in Example 2.2.11 satisfy the relation $0 < \dim X > \dim Z = 0$. Hence, in Theorem 3.1.4 the assumption that M is a closed subspace of X cannot be replaced by the weaker assumption that $\dim M$ is defined. In Section 2.2 an even stronger example is cited which shows that the dimension $\dim$ is not monotonic in hereditarily normal spaces.

Let us observe that the theorem on subspaces of intermediate dimensions, i.e., the counterpart of Theorems 1.5.1 and 2.2.2, does not hold for the dimension $\dim$ in normal spaces. More exactly, for every natural number $n \geq 0$ there exists a compact space X_n such that $\dim X_n = n$ and for every closed subspace $M \subset X_n$ we have either $\dim M \leq 0$ or $\dim M = n$. The description of spaces X_n and verification of their properties are too difficult to be reproduced in this book.

We shall now show that for a fixed hereditarily normal space X the monotonicity of the dimension $\dim$ is equivalent to its being monotonic with respect to open subspaces.

3.1.5. Proposition. *For every hereditarily normal space X the following conditions are equivalent:*

- (a) *For every subspace M of X we have $\dim M \leq \dim X$.*
- (b) *For every open subspace U of X we have $\dim U \leq \dim X$.*

Proof. The implication (a) $\Rightarrow$ (b) is obvious. Suppose that X satisfies (b). Condition (a) is satisfied if $\dim X = \infty$, so that we can assume that $\dim X = n < \infty$. Consider a subspace M of X and a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space M . For $i = 1, 2, \dots, k$ let W_i be an open subset of X such that $U_i = M \cap W_i$. Since for the open subspace $W = \bigcup_{i=1}^k W_i$ of X we have $\dim W \leq n$, the cover $\{W_i\}_{i=1}^k$ of the space W has a finite open refinement $\mathcal{V}$ of order $\leq n$. One easily sees that the family $\mathcal{V}|M$ is a finite open cover of the space M , refines $\mathcal{U}$ and has order $\leq n$, so that $\dim M \leq n = \dim X$. Thus X satisfies condition (a). $\square$

We shall return to the question of the monotonicity of $\dim$ later in this section and we shall show that the dimension $\dim$ is monotonic in the class of strongly hereditarily normal spaces; the proof of this important fact depends on the countable sum theorem for $\dim$.

We now turn to a study of sum theorems for the covering dimension.

3.1.6. Lemma. *If a normal space X can be represented as the union of a sequence $K_1, K_2, \dots$ of subspaces such that $\dim Z \leq n$ for every closed subspace Z of the space X contained in a set K_i and the union $\bigcup_{j \leq i} K_j$ is closed for $i = 1, 2, \dots$, then $\dim X \leq n$.*

Proof. Consider a finite open cover $\mathcal{U} = \{U_j\}_{j=1}^k$ of the space X . We shall define inductively a sequence $\mathcal{U}_0, \mathcal{U}_1, \dots$ of open covers of the space X , where $\mathcal{U}_i = \{U_{i,j}\}_{j=1}^k$, satisfying the conditions:

- (1) $\bar{U}_{i,j} \subset U_{i-1,j}$ if $i \geq 1$ and $U_{0,j} \subset U_j$ for $j = 1, 2, \dots, k$.
- (2) $\text{ord}(\{F_i \cap \bar{U}_{i,j}\}_{j=1}^k) \leq n$, where $F_i = K_1 \cup K_2 \cup \dots \cup K_i$
if $i \geq 1$ and $F_0 = \emptyset$.

Both conditions are satisfied for $i = 0$ if we define $U_{0,j} = U_j$ for $j = 1, 2, \dots, k$. Assume that the coverings $\mathcal{U}_i$ satisfying (1) and (2) are defined for all $i < m \geq 1$. Consider the set $A = \bigcup_{T \in \mathcal{T}} \bigcap_{j \in T} \bar{U}_{m-1,j}$, where $\mathcal{T}$ is the family of all subsets of the set $1, 2, \dots, k$ which have exactly $n+2$ elements. From (2) with $i = m-1$ it follows that $A \cap F_{m-1} = \emptyset$. The intersection $Z = A \cap F_m$ is a closed subspace of the space X and is contained in $F_m \setminus F_{m-1} \subset K_m$, so that $\dim Z \leq n$. By virtue of Proposition 1.6.9 the cover $\{Z \cap U_{m-1,j}\}_{j=1}^k$ of the space Z has an open shrinking $\{V_j\}_{j=1}^k$ of order $\leq n$. One readily observes that the family $\{W_j\}_{j=1}^k$, where $W_j = (U_{m-1,j} \setminus Z) \cup V_j \subset U_{m-1,j}$, is an open cover of the space X and that $\text{ord}(\{F_m \cap W_j\}_{j=1}^k) \leq n$. By Theorem 1.7.8 and the normality of X there exists an open shrinking $\mathcal{U}_m = \{U_{m,j}\}_{j=1}^k$ of the cover $\{W_j\}_{j=1}^k$ such that $\bar{U}_{m,j} \subset W_j$ for $j = 1, 2, \dots, k$. Clearly, the cover $\mathcal{U}_m$ satisfies conditions (1) and (2) with $i = m$. Hence the construction of the covers $\mathcal{U}_i$ satisfying (1) and (2) for $i = 0, 1, \dots$ is completed.

For every point $x \in X$ there exists an $j(x) \leq k$ such that x belongs to infinitely many sets $U_{i,j(x)}$; it follows from (1) that $x \in \bigcap_{i=1}^{\infty} U_{i,j(x)}$.

Applying (1) and (2) we readily see that the family $\{\bigcap_{i=1}^{\infty} \bar{U}_{i,j}\}_{j=1}^k$ is a closed shrinking of the cover $\{U_j\}_{j=1}^k$ and has order $\leq n$. Therefore we have $\dim X \leq n$ by virtue of Proposition 3.1.3. $\square$

From Theorem 3.1.4 and Lemma 3.1.6 we obtain

3.1.7. Proposition. *If a normal space X can be represented as the union of a sequence $K_1, K_2, \dots$ of normal subspaces such that $\dim K_i \leq n$ and the union $\bigcup_{j \leq i} K_j$ is closed for $i = 1, 2, \dots$, then $\dim X \leq n$. $\square$*

The last proposition yields

3.1.8. The countable sum theorem for dim. *If a normal space X can be represented as the union of a sequence $F_1, F_2, \dots$ of closed subspaces such that $\dim F_i \leq n$ for $i = 1, 2, \dots$, then $\dim X \leq n$. $\square$*

We now pass to the locally finite sum theorem for dim. The theorem will be deduced from a lemma which is formulated here in a more general form in view of an application in the following section.

3.1.9. Lemma. *Let $\mathcal{U} = \{U_s\}_{s \in S}$ be an open cover of a normal space X . If the space X has a locally finite closed cover $\mathcal{F}$ each member of which has covering dimension not larger than n and meets only finitely many sets U_s , then the cover $\mathcal{U}$ has an open shrinking of order $\leq n$.*

Proof. Let us adjoin the set $F_0 = \emptyset$ to the cover $\mathcal{F}$ and let us arrange the members of this cover into a transfinite sequence $F_0, F_1, \dots, F_\alpha, \dots$, $\alpha \leq \xi$ of type $\xi + 1$. We shall define inductively a transfinite sequence $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_\alpha, \dots$, $\alpha \leq \xi$ of open covers of the space X , where $\mathcal{U}_\alpha = \{U_{\alpha,s}\}_{s \in S}$, satisfying the conditions:

- (3) $U_{\alpha,s} \subset U_{\beta,s}$ if $\alpha > \beta \geq 0$ and $U_{0,s} \subset U_s$ for $s \in S$.
- (4) $\text{ord}(\{F_\alpha \cap U_{\alpha,s}\}_{s \in S}) \leq n$.
- (5) $U_{\beta,s} \setminus U_{\alpha,s} \subset \bigcup_{\beta \leq \gamma < \alpha} F_\gamma$ for $\beta < \alpha$ and $s \in S$.

All conditions are satisfied for $\alpha = 0$ if we define $U_{0,s} = U_s$ for $s \in S$. Assume that the coverings $\mathcal{U}_\alpha$ satisfying (3)–(5) are defined for all $\alpha < \alpha_0 \geq 1$. To begin with, we shall show that the family $\mathcal{U}'_{\alpha_0} = \{U'_{\alpha_0,s}\}_{s \in S}$, where

$$U'_{\alpha_0,s} = \bigcap_{\alpha < \alpha_0} U_{\alpha,s} \quad \text{for } s \in S,$$

is an open cover of the space X . This is clear if $\alpha_0 = \alpha_1 + 1$, because then $U'_{\alpha_0} = U_{\alpha_1}$; thus, we can assume that α_0 is a limit number.

Consider an arbitrary point $x \in X$. Since the family $\mathcal{F}$ is locally finite, there exist a neighbourhood $U \subset X$ of the point x and an ordinal number $\beta < \alpha_0$ such that $U \cap F_\gamma = \emptyset$ whenever $\beta \leq \gamma < \alpha_0$. The family $\mathcal{U}_\beta$ being a cover of X , there exists an $s \in S$ such that $x \in U_{\beta,s}$. It follows from (5) that $x \in U_{\alpha,s}$ whenever $\beta < \alpha < \alpha_0$, so that $x \in U'_{\alpha_0,s}$. Hence $\mathcal{U}'_{\alpha_0}$ is a cover of the space X .

To show that the sets $U'_{\alpha_0,s}$ are open it suffices to consider, for an arbitrary point $x \in U'_{\alpha_0,s}$, a neighbourhood $U \subset X$ of the point x and an ordinal number $\beta < \alpha_0$ such that $U \cap F_\gamma = \emptyset$ whenever $\beta \leq \gamma < \alpha_0$,

to choose a set $U_{\beta,s}$ which contains x , and to observe that $x \in U \cap U_{\beta,s} \subset U'_{\alpha_0,s}$ by virtue of (5).

The open cover $\{F_{\alpha_0} \cap U'_{\alpha_0,s}\}_{s \in S}$ of the subspace $F_{\alpha_0} \subset X$ has an open shrinking $\{V_s\}_{s \in S}$ of order $\leq n$, because—by (3) and the assumption on the family $\mathcal{F}$ —only finitely many members of that cover are non-empty. One readily observes that the family $\mathcal{U}_{\alpha_0} = \{U_{\alpha_0,s}\}_{s \in S}$, where $U_{\alpha_0,s} = (U'_{\alpha_0,s} \setminus F_{\alpha_0}) \cup V_s$, is an open cover of the space X satisfying conditions (3)–(5) with $\alpha = \alpha_0$. Hence the construction of the covers $\mathcal{U}_\alpha$ satisfying (3)–(5) for $\alpha \leq \xi$ is completed.

Now, it follows from (4) that $\text{ord } \mathcal{U}_\xi \leq n$; as $\mathcal{U}_\xi$ is, by virtue of (3), a shrinking of $\mathcal{U}$, the lemma is established. $\square$

Lemma 3.1.9 yields

3.1.10. The locally finite sum theorem for $\dim$. *If a normal space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, then $\dim X \leq n$. $\square$*

The following two theorems are common generalizations of the countable and the locally finite sum theorems. The proofs, parallel to the proofs of Theorems 2.3.11 and 2.3.12, are left to the reader.

3.1.11. Theorem. *If a normal space X can be represented as the union of a σ -locally finite family $\{F_s\}_{s \in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, then $\dim X \leq n$. $\square$*

3.1.12. Theorem. *If a normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \dots, F_\alpha, \dots, \alpha < \xi$ of closed subspaces such that $\dim F_\alpha \leq n$ and the family $\{F_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, and the family $\{F_\alpha\}_{\alpha < \xi}$ is locally countable, then $\dim X \leq n$. $\square$*

The next result is the point-finite sum theorem for $\dim$. It will yield two further sum theorems which hold in the class of weakly paracompact normal spaces. Let us note that in this last class Theorem 3.1.13. generalizes the locally finite sum theorem (see Problem 2.3.F) and Theorem 3.1.15 generalizes both the countable and the locally finite sum theorems.

3.1.13. The point-finite sum theorem for $\dim$. *If a normal space X can be represented as the union of a family $\{F_s\}_{s \in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, and if there exists a point-finite open cover $\{U_s\}_{s \in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then $\dim X \leq n$.*

Proof. Consider the decomposition of the space X described in Lemma 2.3.3, i.e., let

$$(6) \quad X = \bigcup_{i=1}^{\infty} K_i, \text{ where the union } \bigcup_{j \leq i} K_j \text{ is closed for } i = 1, 2, \dots$$

and

$$(7) \quad K_i = \bigcup_{T \in \mathcal{T}_i} K_T, \text{ where the sets } K_T = K_i \cap \bigcap_{s \in T} U_s \text{ are open in } K_i$$

and pairwise disjoint.

Let Z be a closed subspace of the space X contained in a set K_i . It follows from (7) that for every $T \in \mathcal{T}_i$ the set $Z \cap K_T$ is closed in X . Since $Z \cap K_T \subset \bigcup_{s \in T} F_s$, by virtue of Theorems 3.1.8 and 3.1.4 we have $\dim(Z \cap K_T) \leq n$ for every $T \in \mathcal{T}_i$; and this implies that $\dim Z \leq n$. In view of (6), to conclude the proof it suffices to apply Lemma 3.1.6. $\square$

3.1.14. Theorem. *If a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s \in S}$ of open subspaces such that $\dim \bar{U}_s \leq n$ for $s \in S$, then $\dim X \leq n$.*

Proof. The space X being weakly paracompact, one can assume that the cover $\{U_s\}_{s \in S}$ is point-finite and thus has a closed shrinking $\{F_s\}_{s \in S}$ (see [GT], Theorem 1.5.18). To complete the proof it suffices to apply Theorems 3.1.4 and 3.1.13. $\square$

A variant of the last theorem which closely parallels Theorem 2.3.14 is given in Problem 3.1.D(a).

3.1.15. Theorem. *If a weakly paracompact normal space X can be represented as the union of a locally countable family $\{F_s\}_{s \in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, then $\dim X \leq n$.* $\square$

Proof. For every point $x \in X$ there exist a neighbourhood U_x and a countable set $S(x) \subset S$ such that $\bar{U}_x \cap F_s = \emptyset$ for $s \in S \setminus S(x)$. From this relation it follows that $\bar{U}_x \subset \bigcup \{F_s : s \in S(x)\}$, so that by virtue of Theorems 3.1.4 and 3.1.8 we have $\dim \bar{U}_x \leq n$ for $x \in X$. To complete the proof it suffices to apply Theorem 3.1.14. $\square$

We now pass to the addition theorems for $\dim$.

3.1.16. Proposition. *If a normal space X can be represented as the union of two subspaces A and B such that A is normal, $\dim A \leq n$, and $\dim Z \leq m$ for every closed subspace Z of the space X contained in B , then $\dim X \leq n + m + 1$.*

Proof. By virtue of Lemma 3.1.6 it suffices to show that $\dim \bar{A} \leq n + m + 1$. Consider a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space $\bar{A}$. Since $\dim A \leq n$, there exists a family $\mathcal{V} = \{V_i\}_{i=1}^k$ of open subsets of $\bar{A}$ such that

$$A \cap V_i \subset A \cap U_i \quad \text{for } i = 1, 2, \dots, k, \quad A \subset \bigcup_{i=1}^k V_i$$

$$\text{and} \quad \text{ord}(\{A \cap V_i\}_{i=1}^k) \leq n;$$

without loss of generality one can assume that $V_i \subset U_i$ for $i = 1, 2, \dots, k$.

The set $Z = \bar{A} \setminus \bigcup_{i=1}^k V_i$ is closed in X and contained in B , so that $\dim Z \leq m$, which implies that there exists a closed cover $\{F_i\}_{i=1}^k$ of Z such that $F_i \subset Z \cap U_i$ for $i = 1, 2, \dots, k$ and $\text{ord}(\{F_i\}_{i=1}^k) \leq m$. By virtue of Theorem 3.1.2 the family $\{F_i\}_{i=1}^k$ of closed subsets of $\bar{A}$ has an open swelling $\mathcal{W} = \{W_i\}_{i=1}^k$ in the space $\bar{A}$ such that $W_i \subset U_i$ for $i = 1, 2, \dots, k$. Thus

$$Z = \bar{A} \setminus \bigcup_{i=1}^k V_i \subset \bigcup_{i=1}^k W_i \quad \text{and} \quad \text{ord}(\{W_i\}_{i=1}^k) \leq m.$$

The union $\mathcal{V} \cup \mathcal{W}$ covers the space $\bar{A}$ and refines the cover $\mathcal{U}$. As $\text{ord}(\mathcal{V} \cup \mathcal{W}) \leq n + m + 1$, $\dim \bar{A} \leq n + m + 1$. $\square$

Theorem 3.1.16 implies

3.1.17. The addition theorem for $\dim$. *For every pair X, Y of subspaces of a hereditarily normal space we have*

$$\dim(X \cup Y) \leq \dim X + \dim Y + 1. \quad \square$$

The addition theorem yields

3.1.18. Corollary. *If a hereditarily normal space X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \dots, Z_{n+1}$ such that $\dim Z_i \leq 0$ for $i = 1, 2, \dots, n+1$, then $\dim X \leq n$. $\square$*

Let us note that the implication in the last corollary cannot be reversed even in the class of compact perfectly normal spaces. Indeed, applying the continuum hypothesis one can define a compact perfectly normal space X such that $\dim X = 1$ and $\text{ind} X = \text{Ind} X = 2$; now, if the space X

could be represented as the union $Z_1 \cup Z_2$, where $\dim Z_1 = \dim Z_2 = 0$, then by virtue of Theorems 1.6.11 and 2.2.5 we would have $\text{Ind} X = 1$. The description of the space X is too complicated to be discussed in this book.

Let us return to the question of the monotonicity of $\dim$. We are now able to prove

3.1.19. The subspace theorem for $\dim$. *For every subspace M of a strongly hereditarily normal space X we have $\dim M \leq \dim X$.*

Proof. We can assume that $\dim X = n < \infty$. To begin with, let us observe that from Theorems 3.1.13, 3.1.8 and 3.1.4 it follows that $\dim U \leq n$ for every open subspace U of X which can be represented as the union of a point-finite family of open F_σ -sets in X .

Now, consider an arbitrary subspace M of X and a finite open cover $\{U_i\}_{i=1}^k$ of the space M ; let $\{F_i\}_{i=1}^k$ be a closed shrinking of this last cover. For $i = 1, 2, \dots, k$ the sets F_i and $M \setminus U_i$ are separated in X , so that there exists an open set $V_i \subset X$ such that $F_i \subset V_i$, $M \cap V_i \subset U_i$, and V_i can be represented as the union of a point-finite family of open F_σ -sets in X . By the above observation the set $V = \bigcup_{i=1}^k V_i$ satisfies the inequality $\dim V \leq n$; therefore there exists an open shrinking $\{W_i\}_{i=1}^k$ of the cover $\{V_i\}_{i=1}^k$ of the space V such that $\text{ord}(\{W_i\}_{i=1}^k) \leq n$. Since $M = \bigcup_{i=1}^k F_i \subset \bigcup_{i=1}^k V_i = V$ and $M \cap V_i \subset U_i$, the family $\{M \cap W_i\}_{i=1}^k$ is an open shrinking of the cover $\{U_i\}_{i=1}^k$. As $\text{ord}(\{M \cap W_i\}_{i=1}^k) \leq n$, $\dim M \leq n$. $\square$

The last theorem together with Theorems 2.1.3 and 2.1.5 yield the following two corollaries, which can also be deduced directly from Proposition 3.1.5 and Theorem 3.1.8 and 3.1.14.

3.1.20. Corollary. *For every subspace M of a perfectly normal space X we have $\dim M \leq \dim X$. $\square$*

3.1.21. Corollary. *For every subspace M of a hereditarily weakly paracompact hereditarily normal space X we have $\dim M \leq \dim X$. $\square$*

We shall now establish a theorem on the monotonicity of $\dim$ which slightly differs in nature from our previous theorems of this type: here,

the assumptions are about the internal topological properties of the subspace M rather than the position of the subspace M in the space X .

3.1.22. Lemma. *Every subspace M of a normal space X which is an F_σ -set in X is normal.*

Proof. It suffices to show (see Problem 2.1.E(a) or [GT], Lemma 1.5.14) that for every closed set $F \subset M$ and each open set $W \subset M$ there exists a sequence $W_1, W_2, \dots$ of open subsets of M such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $M \cap \overline{W_i} \subset W$ for $i = 1, 2, \dots$, where the bar denotes the closure operator in X . Let $F_1, F_2, \dots$ be closed subsets of X such that $M = \bigcup_{i=1}^{\infty} F_i$ and U an open subset of X such that $W = M \cap U$. The space X being normal, for $i = 1, 2, \dots$ there exists an open set $U_i \subset X$ such that $F_i \cap F_i \subset U_i \subset \overline{U_i} \subset U$. The sets $W_i = M \cap U_i$ satisfy all the required conditions. $\square$

3.1.23. Theorem. *For every strongly paracompact subspace M of a normal space X we have $\dim M \leq \dim X$.*

Proof. We can assume that $\dim X = n < \infty$. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space M . For $i = 1, 2, \dots, k$ let W_i be an open subset of X such that $U_i = M \cap W_i$ and let $W = \bigcup_{i=1}^k W_i$. For every point $x \in M$ choose a neighbourhood V_x of the point x in X such that $x \in V_x \subset \overline{V_x} \subset W$. The open cover $\{M \cap V_x\}_{x \in M}$ of the space M has a star-finite open refinement $\mathcal{V}$. Consider the decomposition $\{\mathcal{V}_s\}_{s \in S}$ of the cover $\mathcal{V}$ into components; as was established in Section 2.4, the components $\mathcal{V}_s$ are countable and the sets $V_s = \bigcup \mathcal{V}_s$ are pairwise disjoint. Let F_s be the union of the closures in X of all sets which belong to the family $\mathcal{V}_s$. Clearly $V_s \subset F_s \subset W$, and by virtue of Lemma 3.1.22 the subspace F_s of X is normal; from the countable sum theorem it follows that $\dim F_s \leq n$. Thus, for every $s \in S$ there exists an open cover $\{W_{s,i}\}_{i=1}^k$ of the space F_s such that $W_{s,i} \subset W_i$ for $i = 1, 2, \dots, k$ and $\text{ord}(\{W_{s,i}\}_{i=1}^k) \leq n$. One readily sees that the family $\{V_i\}_{i=1}^k$, where $V_i = M \cap \bigcup_{s \in S} (V_s \cap W_{s,i})$, is an open shrinking of the cover $\{U_i\}_{i=1}^k$ of the space M . As $\text{ord}(\{V_i\}_{i=1}^k) \leq n$, $\dim M \leq n = \dim X$. $\square$

Applying Theorem 3.1.14 we can strengthen the last theorem as follows (we recall that a space X is *locally strongly paracompact* if every point of X has a neighbourhood whose closure is strongly paracompact).

3.1.24. Theorem. *For every weakly paracompact locally strongly paracompact normal subspace M of a normal space X we have $\dim M \leq \dim X$. $\square$*

Let us observe that in Theorem 3.1.23 the assumption that X is strongly paracompact cannot be replaced by the weaker assumption that X is paracompact (cf. Problem 3.2.G(b)). Indeed, by virtue of Remark 1.3.18, Roy's space X , cited in Section 2.4, which satisfies $\text{ind} X = 0$ and $\dim X = 1$, is embeddable in a Cantor cube $D^{\mathfrak{m}}$; since $\dim D^{\mathfrak{m}} = 0$ (see Theorems 1.6.5 and 1.6.11), we have $\dim X > \dim D^{\mathfrak{m}}$.

We shall now show that the Čech–Stone compactification preserve the dimension $\dim$. In Section 3.3 it will be proved that for every normal space there exist compactifications preserving both the dimension $\dim$ and the weight.

3.1.25. Theorem. *For every normal space X we have $\dim \beta X = \dim X$.*

Proof. To begin with, we shall prove that $\dim X \leq \dim \beta X$. The inequality is obvious if $\dim \beta X = \infty$, so that we can suppose that $\dim \beta X = n < \infty$. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space X . By virtue of Theorem 1.7.8 there exists a closed shrinking $\{F_i\}_{i=1}^k$ of the cover $\{U_i\}_{i=1}^k$, and by Urysohn's lemma for $i = 1, 2, \dots, k$ there exists a continuous function $f_i: X \rightarrow I$ such that

$$(8) \quad f_i(X \setminus U_i) \subset \{0\} \quad \text{and} \quad f_i(F_i) \subset \{1\};$$

let $\bar{f}_i: \beta X \rightarrow I$ be the continuous extension of f_i over βX . By virtue of (8) the family $\{W_i\}_{i=1}^k$, where $W_i = \bar{f}_i^{-1}((0, 1])$, is an open cover of the space βX and

$$(9) \quad X \cap W_i \subset U_i \quad \text{for } i = 1, 2, \dots, k.$$

Since $\dim \beta X \leq n$, the cover $\{W_i\}_{i=1}^k$ has an open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$. From (9) it follows that the family $\{X \cap V_i\}_{i=1}^k$ is an open shrinking of the cover $\{U_i\}_{i=1}^k$ of the space X . As $\text{ord}(\{X \cap V_i\}_{i=1}^k) \leq n$, $\dim X \leq n$.

Now, we shall prove that $\dim \beta X \leq \dim X$. As in the first part of the proof, we shall suppose that $\dim X = n < \infty$. Consider a finite open cover

$\{U_i\}_{i=1}^k$ of the space βX . By virtue of Theorem 1.7.8 there exists an open shrinking $\{W_i\}_{i=1}^k$ of the cover $\{U_i\}_{i=1}^k$ such that $\bar{W}_i \subset U_i$ for $i = 1, 2, \dots, k$. Since $\dim X = n$, the cover $\{X \cap W_i\}_{i=1}^k$ of the space X has an open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$ which, in turn, has a closed shrinking $\{E_i\}_{i=1}^k$. For $i = 1, 2, \dots, k$ let $F_i = \bar{E}_i$, where the bar denotes the closure operator in βX . The family $\{F_i\}_{i=1}^k$ is a cover of the space βX ; since $F_i = \bar{E}_i \subset \bar{V}_i \subset \bar{X \cap W_i} = \bar{W}_i \subset U_i$ for $i = 1, 2, \dots$, it is a closed shrinking of the cover $\{U_i\}_{i=1}^k$. To complete the proof it suffices to show that $\text{ord}(\{F_i\}_{i=1}^k) \leq n$.

For $i = 1, 2, \dots, k$ define a continuous function $f_i: X \rightarrow I$ such that $f_i(E_i) \subset \{0\}$ and $f_i(X \setminus V_i) \subset \{1\}$; let $\bar{f}_i: \beta X \rightarrow I$ be the continuous extension of f_i over βX . Consider an arbitrary subfamily $\{F_{i_1}, F_{i_2}, \dots, F_{i_m}\}$ of the cover $\{F_i\}_{i=1}^k$ such that $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_m} \neq \emptyset$. Let $\bar{f} = \max(\bar{f}_{i_1}, \bar{f}_{i_2}, \dots, \bar{f}_{i_m})$. The set $U = \bar{f}^{-1}([0, 1))$ is open in βX , and since $F_{i_1} \cap F_{i_2} \cap \dots \cap F_{i_m} \subset U$, we have $U \neq \emptyset$, which implies that $U \cap X \neq \emptyset$. One readily checks that $U \cap X \subset V_{i_1} \cap V_{i_2} \cap \dots \cap V_{i_m}$; thus the relation $\text{ord}(\{V_i\}_{i=1}^k) \leq n$ yields the inequality $m \leq n + 1$, which shows that $\text{ord}(\{F_i\}_{i=1}^k) \leq n$. $\square$

3.1.26. Corollary. *For every normal space X and a dense normal subspace $M \subset X$ which has the property that every continuous function $f: M \rightarrow I$ is continuously extendable over X we have $\dim M = \dim X$.*

In other words, $\dim Y = \dim X$ for every normal space X and every normal subspace Y of βX which contains X .

Proof. From the extendability of every continuous function $f: M \rightarrow I$ over X it follows that $\beta M = \beta X$. $\square$

The final part of this section is devoted to a study of relations between ind, Ind and dim.

Let us begin with reminding the reader that in Section 1.7, when proving that the equality $\text{ind} X = \dim X$ holds for every separable metric space X , we ascertained that the proof of the inequality $\dim X \leq \text{ind} X$ was much easier than the proof of the reverse inequality. We shall now show that the inequality $\dim X \leq \text{ind} X$ holds for every strongly paracompact space X and that the related inequality $\dim X \leq \text{Ind} X$ holds for every normal space X . Both results will be deduced from a common lemma.

3.1.27. Lemma. *If for every pair A, B of disjoint closed subsets of a normal space X there exists a partition L between A and B such that $\dim L \leq n - 1$, then $\dim X \leq n$.*

Proof. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space X ; let $\{F_i\}_{i=1}^k$ be a closed shrinking of $\{U_i\}_{i=1}^k$. For $i = 1, 2, \dots, k$ there exists a partition L_i between F_i and $X \setminus U_i$ such that $\dim L_i \leq n-1$; let W_i be an open set satisfying $F_i \subset W_i \subset U_i$ and $\text{Fr } W_i \subset L_i$. Since $L = \bigcup_{i=1}^k L_i$ is a normal space, $\dim L \leq n-1$ by virtue of Theorem 3.1.8. By shrinking the open cover $\{L \cap U_i\}_{i=1}^k$ of the space L to a closed cover of order $\leq n$ and then swelling the latter cover in conformity with Theorem 3.1.2, we obtain a family $\{V_i\}_{i=1}^k$ of open subsets of X such that $\bar{V}_i \subset U_i$ for $i = 1, 2, \dots, k$,

$$(10) \quad L \subset V = \bigcup_{i=1}^k V_i \quad \text{and} \quad \text{ord}(\{\bar{V}_i\}_{i=1}^k) \leq n-1.$$

The sets $\bar{V}_1, \bar{V}_2, \dots, \bar{V}_k$ together with the sets $Z_1, Z_2, \dots, Z_k$, where

$$Z_i = \bar{W}_i \setminus (V \cup \bigcup_{j < i} W_j),$$

constitute a closed refinement of $\{U_i\}_{i=1}^k$. To complete the proof it suffices to observe that this refinement has order $\leq n$, which, however, follows immediately from the second part of (10) and the fact that for $j < i \leq k$ we have

$$Z_j \cap Z_i \subset \bar{W}_j \cap [\bar{W}_i \setminus (V \cup W_j)] \subset (X \setminus V) \cap \text{Fr } W_j = \emptyset. \quad \square$$

From the last lemma, by applying induction with respect to $\text{Ind } X$, we obtain

3.1.28. Theorem. *For every normal space X we have $\dim X \leq \text{Ind } X$. $\square$*

In the next chapter we shall show that for every metrizable space X we have the equality $\text{Ind } X = \dim X$. From example 3.1.31 below it follows that the equality does not hold in compact spaces. The commentary to Corollary 3.1.18 above shows that it does not hold in perfectly normal compact spaces either.

From Lemmas 3.1.27, 2.4.1 and Theorem 3.1.8, by applying induction with respect to $\text{ind } X$, we obtain the following

3.1.29. Theorem. *For every strongly paracompact space X we have $\dim X \leq \text{ind } X$. $\square$*

As shown by Roy's space cited above, in the last theorem the assumption that X is strongly paracompact cannot be replaced by the weaker

assumption that X is paracompact (cf. Problem 3.1.F). The same example shows that the equality $\text{ind}X = \dim X$ does not hold in metric spaces. That it does not hold in compact spaces either is shown in Example 3.1.31.

To conclude, let us observe that Theorems 2.4.2 and 1.6.11 yield the following

3.1.30. Theorem. *For every strongly paracompact space X the conditions $\text{ind}X = 0$, $\text{Ind}X = 0$ and $\dim X = 0$ are equivalent.* $\square$

3.1.31. Example. In Example 2.2.13 we described a compact space X with $\text{Ind}X \geq \text{ind}X \geq 2$ which contains closed subspaces F_1, F_2 such that $F_1 \cup F_2 = X$ and $\text{Ind}F_1 = \text{Ind}F_2 = 1$. From Theorems 3.1.28, 3.1.8 and 1.6.11 it follows that $\dim X = 1$. Thus X is a compact space such that $\dim X \neq \text{Ind}X$ and $\dim X \neq \text{ind}X$. $\square$

Historical and bibliographic notes

Proposition 3.1.3 and Theorem 3.1.4 were given by Čech in [1933]. It was proved by Fedorčuk in [1973] that for every natural number $n \geq 2$ there exists a compact space X_n such that $\dim X_n = n$ and for every closed subspace $M \subset X_n$ we have either $\dim M \leq 0$ or $\dim M = n$. Theorem 3.1.5 and Lemma 3.1.6 in the case where $X = K_1 \cup K_2$ (cf. Problem 3.1.B (a)) were obtained by Dowker in [1955]. Theorem 3.1.8 was established by Čech in [1933], and Theorem 3.1.10 independently by Morita in [1950a] and Katětov in [1952]; Lemma 3.1.9 appeared in Katětov's paper [1952]. Theorem 3.1.14 was proved by Dowker in [1955] and by Nagami in [1955] for paracompact spaces; Theorem 3.1.15, also for paracompact spaces, was given by Nagami in [1955]. Proposition 3.1.16 was established by Zarelua in [1963a]; its particular case formulated as Theorem 3.1.17 was obtained by Smirnov in [1951]. An example of a compact perfectly normal space X such that $\dim X = 1$ and $\text{ind}X = \text{Ind}X = 2$ was described under the assumption of the continuum hypothesis by Filippov in [1970a]; the first example of such a space was constructed under the joint assumption of the continuum hypothesis and the existence of a Souslin space (see [GT], Remark to Problem 2.7.9(f)) by Lifanov and Filippov in [1970]. Theorem 3.1.19 for totally normal spaces was proved by Dowker in [1955]; it was extended to Dowker spaces by Lifanov and Pasynkov in [1970].

(announcement in Pašynkov [1967]) and to super normal spaces by Nishiura in [1977]. Corollary 3.1.20 was established by Čech in [1933]. Theorem 3.1.23 was proved by Morita in [1953]; its strengthening stated as Theorem 3.1.24 was given by Lifanov and Pašynkov in [1970] (an intermediate result was obtained by Pupko in [1961]). Theorem 3.1.25 was established by Wallman in [1938]. Theorem 3.1.28 was proved by Vedenisoff in [1941] and Theorem 3.1.29 by Morita in [1950a] (the latter theorem for Lindelöf spaces was obtained independently by Morita in [1950] and by Smirnov in [1951]; for compact spaces it was proved by Alexandroff in [1941]). Example 3.1.31 was given by Lokucievskii in [1949]; the first example of a compact space X such that $\dim X \neq \operatorname{ind} X$ was described by Lunc in [1949].

As observed by Katětov in [1950], the definition of the covering dimension $\dim$ can be slightly modified so as to lead to a notion of dimension which behaves relatively well in completely regular spaces. The modification consists in replacing condition (ČL1) by

(ČL1') $\dim X \leq n$, where $n = -1, 0, 1, \dots$, if every finite functionally open cover of the space X has a finite functionally open refinement of order $\leq n$.

Let us recall that a subset A of a topological space X is *functionally open* (*functionally closed*)¹⁾ if there exists a continuous function $f: X \rightarrow I$ such that $A = f^{-1}((0, 1])$ (such that $A = f^{-1}(0)$); a family of subsets of a topological space is *functionally open* (*closed*) if all its members are functionally open (closed) sets. As noted in Problem 2.1.A, in normal spaces functionally open (closed) sets coincide with open F_σ -sets (closed G_δ -sets). From Theorem 3.1.2 and Proposition 3.1.3 it easily follows that the modified definition of $\dim$ is equivalent to the original one in the realm of normal spaces. When applied to completely regular spaces, the modified definition yields the *covering dimension for completely regular spaces*, which is also denoted by $\dim$. Some theorems proved in this section hold in the larger class of completely regular spaces for the covering dimension thus extended (see Problems 3.1.H, 3.1.I, and 3.2.H–3.2.K).

Occasionally, the covering dimension for completely regular spaces, or even for larger classes of spaces, was defined just by conditions (ČL1)–(ČL3) with no modifications. The dimension function obtained in this way satisfies the counterparts of a few theorems proved in this section

¹⁾ The term *cozero-set* (*zero-set*) is also used.

(see Ostrand [1971]). It seems, however, that such a definition is not quite consistent with the geometrical conception of dimension. The notion obtained displays some undesirable features of a separation axiom; for example, one can easily check that every T_1 -space whose dimension in the sense discussed equals zero is normal.

Problems

3.1.A (Morita [1950a]). Prove that if for a family $\{F_s\}_{s \in S}$ of closed subsets of a normal space X there exists a locally finite family $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s$ for every $s \in S$, then the family $\{F_s\}_{s \in S}$ has an open swelling $\{U_s\}_{s \in S}$ such that $\bar{U}_s \subset V_s$ for $s \in S$ (cf. Problem 4.2.B(a)).

Hint. Apply transfinite induction; modify the proof of Theorem 3.1.2.

Remark. For every locally finite family $\{F_s\}_{s \in S}$ of closed subsets of a countably paracompact collectionwise normal space X there exists a locally finite family of open subsets of X such that $F_s \subset V_s$ for every $s \in S$ (see [GT], Problem 5.5.18(a)).

3.1.B. (a) Observe that Lemma 3.1.6 is a consequence of its special case, viz., the case where $X = K_1 \cup K_2$, and Theorem 3.1.8.

(b) Prove that if a normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of subspaces such that $\dim Z \leq n$ for every closed subspace Z of the space X contained in a set K_α , the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\dim X \leq n$.

Hint. Apply transfinite induction; use Lemma 3.1.6 and Theorem 3.1.10.

(c) Prove that if a normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of normal subspaces such that $\dim K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\dim X \leq n$.

3.1.C. (a) (Nagata [1965]) Prove that if a normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \dots, F_\alpha, \dots, \alpha < \xi$ of closed subspaces such that $\dim F_\alpha \leq n$ for $\alpha < \xi$, and if there exists

a transfinite sequence $U_1, U_2, \dots, U_\alpha, \dots, \alpha < \xi$ of open subsets of X such that $F_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, then $\dim X \leq n$.

Hint. See the hint to Problem 2.3.E(a).

(b) Prove that if a normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of subspaces such that $\dim Z \leq n$ for every closed subspace Z of the space X contained in a set K_α and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \dots, U_\alpha, \dots, \alpha < \xi$ of open subsets of X such that $K_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta < \alpha}$ is locally finite for $\alpha < \xi$, then $\dim X \leq n$.

3.1.D. (a) Show that if a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s \in S}$ of normal open subspaces such that $\dim U_s \leq n$ for $s \in S$, then $\dim X \leq n$.

(b) Show, applying Problem 3.1.I(a) below, that the assumption of normality in (a), as well as in Proposition 3.1.7, can be omitted if by $\dim$ one understands the covering dimension for completely regular spaces as defined in the notes to this section.

3.1.E (Dowker [1955]). Note that the space X described in Example 2.2.11 satisfies the equality $\dim X = 1$.

3.1.F (Zarelua [1963] (announcement [1961])). Show that for every completely paracompact space X we have $\dim X \leq \text{ind} X$ (see Problem 2.4.B).

3.1.G. (a) Check that the union and the intersection of finitely many functionally open (closed) sets are functionally open (closed). Show that the union (intersection) of countably many functionally open (closed) sets is functionally open (closed). Prove that for every pair A, B of disjoint functionally closed subsets of a topological space X there exists a continuous function $f: X \rightarrow I$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

(b) Prove that every finite functionally open cover $\{U_i\}_{i=1}^k$ of a topological space X has shrinkings $\{F_i\}_{i=1}^k$ and $\{W_i\}_{i=1}^k$, which are, respectively, functionally closed and functionally open, and such that $F_i \subset W_i \subset \overline{W_i} \subset U_i$ for $i = 1, 2, \dots, k$.

(c) Prove that every finite family $\{F_i\}_{i=1}^k$ of functionally closed subsets of a topological space X has a functionally open swelling $\{U_i\}_{i=1}^k$. Observe

that if, moreover, a family $\{V_i\}_{i=1}^k$ of functionally open subsets of X satisfying $F_i \subset V_i$ for $i = 1, 2, \dots, k$ is given, then the swelling can be defined in such a way that $\bar{U}_i \subset V_i$ for $i = 1, 2, \dots, k$.

3.1.H. Show that for every completely regular space X the following conditions are equivalent (see the notes to this section):

- (i) *The space X satisfies the inequality $\dim X \leq n$.*
- (ii) *Every finite functionally open cover of the space X has a functionally open shrinking of order $\leq n$.*
- (iii) *Every finite functionally open cover of the space X has a functionally closed shrinking of order $\leq n$.*
- (iv) *Every finite functionally open cover of the space X has a finite functionally closed refinement of order $\leq n$.*

3.1.I (Katětov [1950]). (a) Show that if a subspace M of a completely regular space X has the property that every continuous function $f: M \rightarrow I$ is continuously extendable over X , then $\dim M \leq \dim X$.

(b) Prove that for every completely regular space X we have $\dim \beta X = \dim X$.

(c) Deduce from (b) that if a completely regular space X can be represented as the union of a sequence $A_1, A_2, \dots$ of subspaces such that $\dim A_i \leq n$ and every continuous function $f: A_i \rightarrow I$ is continuously extendable over X for $i = 1, 2, \dots$, then $\dim X \leq n$.

Remark. Terasawa defined in [1977] a completely regular space X with $\dim X > 0$ which can be represented as the union of a functionally closed subspace F with $\dim F = 0$ and an open discrete subspace of cardinality $\aleph_0$. As shown by E. Pol in [1978] (announcement in [1976]), there exists a completely regular space X with $\dim X > 0$ which can be represented as the union of two functionally closed subspaces F_1 and F_2 such that $\dim F_1 = \dim F_2 = 0$. It is an open problem whether every completely regular space X which can be represented as the union of a locally finite family $\{A_s\}_{s \in S}$ of subspaces such that $\dim A_s \leq n$ and every continuous function $f: A_s \rightarrow I$ is continuously extendable over X for $s \in S$ satisfies the inequality $\dim X \leq n$.

3.2. Characterizations of the dimension $\dim$ in normal spaces. Cartesian product theorems for the dimension $\dim$

In this section several characterizations of the covering dimension in normal spaces are established. They split into two groups. The first consists of characterizations which are outwardly close to the definition of $\dim$. The second is made up of generalizations of three important theorems proved in Chapter 1 for separable metric spaces, viz., of the theorems on partitions, on extending mappings to spheres, and on $\mathcal{E}$ -mappings. In the final part of the section we apply one of the characterizations of the first group to obtain two theorems on the dimension $\dim$ of Cartesian products.

3.2.1. Dowker's theorem. *For every normal space X the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\dim X \leq n$.*
- (b) *Every locally finite open cover of the space X has an open shrinking of order $\leq n$.*
- (c) *Every locally finite open cover of the space X has an open refinement of order $\leq n$.*

Proof. We shall show first that (a) $\Rightarrow$ (b). Consider a normal space X such that $\dim X \leq n$. Let $\mathcal{U} = \{U_s\}_{s \in S}$ be an arbitrary locally finite open cover of the space X . Denote by $\mathcal{T}$ the family of all non-empty finite subsets of S and for every $T \in \mathcal{T}$ define

$$F_T = \bigcap_{s \in T} \bar{U}_s \cap \bigcap_{s \notin T} (X \setminus U_s);$$

by virtue of Theorem 3.1.4 $\dim F_T \leq n$. The family $\mathcal{F} = \{F_T\}_{T \in \mathcal{T}}$ is a closed cover of X each member of which meets only finitely many sets U_s . The cover $\mathcal{U}$ being locally finite, for every point $x \in X$ there exists a neighbourhood U and a finite set $S_0 \subset S$ such that $U \cap U_s = \emptyset$, and consequently $U \cap \bar{U}_s = \emptyset$, for $s \in S \setminus S_0$. From the definition of the sets F_T it follows that if $U \cap F_T \neq \emptyset$, then $T \subset S_0$; thus the cover $\mathcal{F}$ is locally finite. By virtue of Lemma 3.1.9 the cover $\mathcal{U}$ has an open shrinking of order $\leq n$, so that the space X satisfies condition (b).

To complete the proof it suffices to observe that the implication (b) $\Rightarrow$ (c) is obvious and the implication (c) $\Rightarrow$ (a) follows from Proposition 1.6.9. $\square$

The last theorem yields

3.2.2. Proposition. *For every paracompact space X the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\dim X \leq n$.*
- (b) *Every open cover of the space X has an open shrinking of order $\leq n$*
- (c) *Every open cover of the space X has an open refinement of order $\leq n$. $\square$*

Conditions (a) and (c) in the last proposition are not equivalent in the realm of all normal spaces. Indeed, each cover of finite order is point-finite, so that every space X which satisfies (c) is weakly paracompact, whereas there exist normal spaces which are not weakly paracompact (see Example 2.1.6). One can show that conditions (a) and (c) are not equivalent even in the realm of all weakly paracompact spaces but the example is more difficult (see [GT], Problem 5.5.3(c)). On the other hand, conditions (b) and (c) are clearly equivalent for every normal space X .

Obviously, every family of sets which can be represented as the union of $n+1$ families of order ≤ 0 , i.e., consisting of pairwise disjoint sets has order $\leq n$. We are now going to strengthen Theorem 3.2.1 by proving that every locally finite open cover of a normal space X such that $\dim X \leq n$ has an open shrinking of this last form. This result is in a sense a substitute for the decomposition theorem for $\dim$.

3.2.3. Lemma. *For every locally finite open cover $\mathcal{U} = \{U_s\}_{s \in S}$ of a normal space X such that $\text{ord } \mathcal{U} \leq n \geq 0$ there exists an open cover $\mathcal{V}$ of the space X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{n+1}$, where $\mathcal{V}_i = \{V_{i,s}\}_{s \in S}$, such that $\text{ord } \mathcal{V}_i \leq 0$ and $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \dots, n+1$.*

Proof. We shall apply induction with respect to n . The lemma is obvious if $n = 0$. Assume that the lemma is proved for all normal spaces and all locally finite open covers of order $< n \geq 1$ and consider a normal space X and a locally finite open cover $\mathcal{U} = \{U_s\}_{s \in S}$ of X such that $\text{ord } \mathcal{U} \leq n$. Let $\{W_s\}_{s \in S}$ and $\{F_s\}_{s \in S}$ be, respectively, an open and a closed shrinking of $\mathcal{U}$ such that $W_s \subset F_s$ for every $s \in S$ (cf. [GT], Theorem 1.5.18). Denote by $\mathcal{T}$ the family of all subsets of S that have exactly $n+1$ elements and for every $T \in \mathcal{T}$ define

$$U_T = \bigcap_{s \in T} U_s, \quad W_T = \bigcap_{s \in T} W_s \quad \text{and} \quad F_T = \bigcap_{s \in T} F_s.$$

From the local finiteness of $\mathcal{U}$ it follows that the family $\{F_T\}_{T \in \mathcal{T}}$ is locally finite. The inequality $\text{ord} \mathcal{U} \leq n$ implies that $U_T \cap U_{T'} = \emptyset$ whenever $T \neq T'$.

For every $T \in \mathcal{T}$ choose arbitrarily an $s(T) \in T$ and let

$$V_{n+1,s} = \bigcup \{U_T : s(T) = s\} \quad \text{for } s \in S.$$

The family $\mathcal{V}_{n+1} = \{V_{n+1,s}\}_{s \in S}$ has order ≤ 0 and $V_{n+1,s} \subset U_s$ for $s \in S$; moreover

$$(1) \quad \bigcup_{s \in S} V_{n+1,s} = \bigcup_{T \in \mathcal{T}} U_T.$$

Consider now two subspaces of the space X

$$Y = X \setminus \bigcup_{T \in \mathcal{T}} W_T \quad \text{and} \quad Z = X \setminus \bigcup_{T \in \mathcal{T}} F_T;$$

the subspace Y is closed in X , the subspace Z is open in X , and $Z \subset Y$. From the definition of the sets W_T it follows that the locally finite open cover $\{Y \cap W_s\}_{s \in S}$ of the space Y has order $\leq n-1$. Since Y is a normal space, by virtue of the inductive assumption there exists an open cover $\mathcal{V}'$ of the space Y which can be represented as the union of n families $\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_n$, where $\mathcal{V}'_i = \{V'_{i,s}\}_{s \in S}$, such that $\text{ord} \mathcal{V}'_i \leq 0$ and $V'_{i,s} \subset Y \cap W_s \subset U_s$ for $s \in S$ and $i = 1, 2, \dots, n$.

For $i = 1, 2, \dots, n$ let $\mathcal{V}_i = \{V_{i,s}\}_{s \in S}$, where $V_{i,s} = Z \cap V'_{i,s}$. One readily sees that the sets $V_{i,s}$ are open in X , that $\text{ord} \mathcal{V}_i \leq 0$ for $i = 1, 2, \dots, n$ and that $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \dots, n$; moreover, $Z \subset \bigcup_{i=1}^n \bigcup_{s \in S} V_{i,s}$.

Since $F_T \subset U_T$ for $T \in \mathcal{T}$, we have

$$X \setminus Z = \bigcup_{T \in \mathcal{T}} F_T \subset \bigcup_{T \in \mathcal{T}} U_T,$$

so that by virtue of (1), the union $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ is an open cover of the space X . $\square$

Theorem 3.2.1 and Lemma 3.2.3 yield

3.2.4. Ostrand's theorem. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every locally finite open cover $\mathcal{U} = \{U_s\}_{s \in S}$ of the space X there exists an open cover $\mathcal{V}$ of the space X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{n+1}$, where $\mathcal{V}_i = \{V_{i,s}\}_{s \in S}$, such that $\text{ord} \mathcal{V}_i \leq 0$ and $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \dots, n+1$. $\square$*

The next theorem is a strong version of the theorem on partitions.

3.2.5. Morita's theorem. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every locally finite family $\{U_s\}_{s \in S}$ of open subsets of X and every family $\{F_s\}_{s \in S}$ of closed subsets of X such that $F_s \subset U_s$ for $s \in S$ there exist families $\{W_s\}_{s \in S}$ and $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s \subset \bar{V}_s \subset W_s \subset \bar{W}_s \subset U_s$ for $s \in S$ and $\text{ord}(\{\bar{W}_s \setminus V_s\}_{s \in S}) \leq n-1$.*

Proof. First we shall prove that every normal space X with $\dim X \leq n$ satisfies the condition in the theorem. Consider a locally finite family $\{U_s\}_{s \in S}$ of open subsets of X and a family $\{F_s\}_{s \in S}$ of closed subsets of X such that $F_s \subset U_s$ for $s \in S$. Denote by $\mathcal{T}$ the family of all finite subsets of S , define for every non-empty $T \in \mathcal{T}$

$$(2) \quad G_T = \bigcap_{s \in T} U_s \cap \bigcap_{s \notin T} (X \setminus F_s),$$

and let $G_\emptyset = X \setminus \bigcup_{s \in S} F_s$. The family $\{G_T\}_{T \in \mathcal{T}}$ is a locally finite open cover of the space X . By virtue of Dowker's theorem (cf. Remark 3.2.7) the cover $\{G_T\}_{T \in \mathcal{T}}$ has an open shrinking $\{H_T\}_{T \in \mathcal{T}}$ of order $\leq n$, and, in turn, the cover $\{H_T\}_{T \in \mathcal{T}}$ has a closed shrinking $\{A_T\}_{T \in \mathcal{T}}$. From (2) it follows that

$$(3) \quad \text{if } A_T \cap F_s \neq \emptyset, \text{ then } s \in T.$$

For every $T \in \mathcal{T}$ and each $s \in T$ we can define open sets $W_T(s)$ and $V_T(s)$ such that

$$(4) \quad A_T \subset V_T(s) \subset \bar{V}_T(s) \subset W_T(s) \subset \bar{W}_T(s) \subset H_T \quad \text{for } s \in T$$

and

$$(5) \quad \text{if } s, s' \in T \text{ and } s' \neq s, \text{ then either } \bar{W}_T(s) \subset V_T(s') \text{ or } \bar{W}_T(s') \subset V_T(s).$$

Moreover, define $W_T(s) = V_T(s) = \emptyset$ for $s \in S \setminus T$.

Let

$$W_s = \bigcup_{T \in \mathcal{T}} W_T(s) \quad \text{and} \quad V_s = \bigcup_{T \in \mathcal{T}} V_T(s) \quad \text{for } s \in S.$$

The sets W_s and V_s are open; from the local finiteness of the cover $\{H_T\}_{T \in \mathcal{T}}$ it follows that $\bar{V}_s \subset W_s$ and $\bar{W}_s \subset U_s$, because (2) implies that $\bar{W}_T(s) \subset U_s$. Now, for every point $x \in F_s$ there exists a $T \in \mathcal{T}$ such that $x \in A_T$; as $s \in T$ by virtue of (3), it follows from (4) that $x \in V_T(s) \subset V_s$, so that $F_s \subset V_s$. To complete the first part of the proof it suffices to show that $\text{ord}(\{\bar{W}_s \setminus V_s\}_{s \in S}) \leq n-1$.

Consider a sequence $s_1, s_2, \dots, s_k$ of distinct elements of the set S such that $\bigcap_{i=1}^k (\overline{W_{s_i}} \setminus V_{s_i}) \neq \emptyset$. Let x be a point in the last intersection. There exist sets $T_1, T_2, \dots, T_k \in \mathcal{T}$ such that $s_i \in T_i$ for $i = 1, 2, \dots, k$ and $x \in \bigcap_{i=1}^k (\overline{W_{T_i}(s_i)} \setminus V_{T_i}(s_i))$; by virtue of (5) $T_i \neq T_j$ for $i \neq j$. From (4) it follows that $x \notin \bigcup_{i=1}^k A_{T_i}$, and since $\{A_T\}_{T \in \mathcal{T}}$ is a cover of X , there exists a $T_0 \in \mathcal{T}$ such that $x \in A_{T_0} \subset H_{T_0}$; clearly $T_0 \neq T_i$ for $i = 1, 2, \dots, k$. Thus $x \in \bigcap_{i=0}^k H_{T_i}$, and as $\text{ord}(\{H_T\}_{T \in \mathcal{T}}) \leq n$ we have $k \leq n$, which shows that $\text{ord}(\{\overline{W_s} \setminus V_s\}_{s \in S}) \leq n-1$.

Consider now a normal space X which satisfies the condition in the theorem. By virtue of Remark 1.7.10, to prove that $\dim X \leq n$ it suffices to show that for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Define

$$U_i = X \setminus B_i \quad \text{and} \quad F_i = A_i \quad \text{for } i = 1, 2, \dots, n+1.$$

Applying the condition in the theorem to $S = \{1, 2, \dots, n+1\}$ and the sets U_i and F_i defined above, we obtain a family $\{V_i\}_{i=1}^{n+1}$ of open subsets of X such that

$$(6) \quad A_i \subset V_i \subset \overline{V_i} \subset X \setminus B_i \quad \text{for } i = 1, 2, \dots, n+1 \quad \text{and} \\ \text{ord}(\{\overline{V_i} \setminus V_i\}_{i=1}^{n+1}) \leq n-1.$$

By virtue of the first part of (6) the set $L_i = \overline{V_i} \setminus V_i$ is a partition between A_i and B_i for $i = 1, 2, \dots, n+1$; the second part of (6) means that $\bigcap_{i=1}^{n+1} L_i = \emptyset$. $\square$

Theorem 3.2.5 and Remark 1.7.10 yield

3.2.6. Theorem on partitions. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. $\square$*

3.2.7. Remark. Let us note that if the family $\{U_s\}_{s \in S}$ in Morita's theorem is finite then in the proof of this theorem the cover $\{H_\tau\}_{\tau \in \mathcal{T}}$ can be obtained by applying Theorem 1.7.8 rather than Dowker's theorem. In particular, the theorem on partitions can be proved without resorting to Dowker's theorem.

We now turn to the theorem on extending mappings to spheres.

3.2.8. Lemma. *Let $f, g: X \rightarrow S^n_I$ be continuous mappings of a topological space X to the boundary S^n_I of the $(n+1)$ -cube I^{n+1} in R^{n+1} . If for every point $x \in X$ the points $f(x)$ and $g(x)$ belong to the same face of I^{n+1} , then the mappings f and g are homotopic.*

Proof. By assigning to every point $x \in X$ and each number $t \in I$ the point $h(x, t)$ which divides the interval with end-points $f(x)$ and $g(x)$ in the ratio of t to $1-t$ one defines a homotopy $h: X \times I \rightarrow S^n_I$ between f and g . $\square$

3.2.9. Theorem. *If X is a normal space and A is a closed subspace of X such that $\dim Z \leq n \geq 0$ for every closed subspace Z of the space X contained in $X \setminus A$, then for every continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X .*

Proof. There exists an open set $W \subset X$ containing A and such that f has a continuous extension $\bar{f}: W \rightarrow S^n$ over W . Consider an open set V such that $A \subset V \subset \bar{V} \subset W$ and let $Z = X \setminus V$ and $B = Z \cap \bar{V}$. To show that f is continuously extendable over X it suffices to prove that for the mapping $h = \bar{f}|B: B \rightarrow S^n$ there exists a continuous extension $H: Z \rightarrow S^n$ over Z . Indeed, the mapping $F: X \rightarrow S^n$ defined by letting

$$F(x) = \bar{f}(x) \text{ for } x \in \bar{V} \quad \text{and} \quad F(x) = H(x) \text{ for } x \in Z$$

will then be a continuous extension of f over X . Since $\dim Z \leq n$, the above observation shows that with no loss of generality one can assume that the space X satisfies the inequality $\dim X \leq n$. Moreover, one can assume that the space X is compact, because $\dim \beta X \leq n$ by Theorem 3.1.25 and the mapping f is continuously extendable over the closure of the set A in βX , this last closure being the Čech-Stone compactification βA of the space A (see [GT], Corollary 3.6.8). Finally, instead of mappings to the n -sphere S^n one can consider mappings to the boundary S^n_I of the $(n+1)$ -cube I^{n+1} in R^{n+1} , which is homeomorphic to S^n .

Consider therefore a continuous mapping $f: A \rightarrow S_I^n$ defined on a closed subspace A of a compact space X such that $\dim X \leq n$. For $i = 1, 2, \dots, n+1$ denote by $p_i: I^{n+1} \rightarrow I$ the projection of the $(n+1)$ -cube I^{n+1} onto the i -th coordinate axis and let

$$f_i = p_i f: A \rightarrow I, \quad A_i = f_i^{-1}(0) \quad \text{and} \quad B_i = f_i^{-1}(1);$$

obviously

$$(7) \quad A = \bigcup_{i=1}^{n+1} A_i \cup \bigcup_{i=1}^{n+1} B_i.$$

By virtue of Theorem 3.2.6 there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Consequently, there exist open sets $U_i, W_i \subset X$, where $i = 1, 2, \dots, n+1$, such that

$$A_i \subset U_i, \quad B_i \subset W_i, \quad U_i \cap W_i = \emptyset \quad \text{and} \quad X \setminus L_i = U_i \cup W_i.$$

From Theorem 3.1.2 it follows that there exist open sets $V_1, V_2, \dots, V_{n+1}$ satisfying

$$L_i \subset V_i \subset X \setminus (A_i \cup B_i) \quad \text{for } i = 1, 2, \dots, n+1 \quad \text{and} \quad \bigcap_{i=1}^{n+1} V_i = \emptyset.$$

By virtue of Urysohn's lemma for $i = 1, 2, \dots, n+1$ there exist continuous functions

$$g'_i: U_i \cup L_i \rightarrow [0, 1/2] \quad \text{and} \quad g''_i: W_i \cup L_i \rightarrow [1/2, 1]$$

such that

$$g'_i(U_i \setminus V_i) \subset \{0\}, \quad g'_i(L_i) \subset \{1/2\}$$

and

$$g''_i(W_i \setminus V_i) \subset \{1\}, \quad g''_i(L_i) \subset \{1/2\}.$$

Letting

$$g_i(x) = \begin{cases} g'_i(x) & \text{for } x \in U_i \cup L_i, \\ g''_i(x) & \text{for } x \in W_i \cup L_i, \end{cases}$$

we define continuous functions $g_i: X \rightarrow I$ such that

$$(8) \quad g_i(A_i) \subset \{0\}, \quad g_i(B_i) \subset \{1\} \quad \text{and} \quad g_i^{-1}(1/2) \subset V_i$$

for $i = 1, 2, \dots, n+1$.

Since $\bigcap_{i=1}^{n+1} V_i = \emptyset$, the continuous mapping $g: X \rightarrow I^{n+1}$ defined by letting $g(x) = (g_1(x), g_2(x), \dots, g_{n+1}(x))$ does not assume the value $a = (1/2, 1/2, \dots, 1/2) \in I^{n+1}$. The composition of the mapping g and the projection

p of $I^{n+1} \setminus \{a\}$ from the point a onto the boundary S_I^n of I^{n+1} is a continuous mapping $G: X \rightarrow S_I^n$. From (7) and (8) it follows that $g(A) \subset S_I^n$, so that $G|A = g|A$, i.e., the mapping G is a continuous extension of the restriction $g|A$ over X . Now, for every point $x \in A$ there exists an $i \leq n+1$ such that either $f_i(x) = g_i(x) = 0$ or $f_i(x) = g_i(x) = 1$, so that for every point $x \in A$ the points $f(x)$ and $g(x)$ belong to the same face of I^{n+1} . From Lemma 3.2.8 it follows that the mappings f and $g|A$ are homotopic, and Lemma 1.9.7 implies that there exists a continuous extension $F: X \rightarrow S_I^n$ of the mapping f over X . $\square$

Theorem 3.2.9, Remark 1.9.4 and Theorem 3.2.6 yield the following

3.2.10. Theorem on extending mappings to spheres. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every closed subspace A of the space X and each continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X .* $\square$

The last characterization of the covering dimension to be established in this section uses the notion of an $\mathcal{E}$ -mapping (see Definition 1.10.8).

3.2.11. Theorem on $\mathcal{E}$ -mappings. *A normal space X satisfies the inequality $\dim X \leq n$ if and only if for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$ -mapping of X to a polyhedron of dimension $\leq n$.*

Proof. By virtue of Theorem 1.10.11, it suffices to show that for every finite open cover $\mathcal{E} = \{U_i\}_{i=1}^k$ of a normal space X with $0 \leq \dim X \leq n$ there exists an $\mathcal{E}$ -mapping of X to a polyhedron of dimension $\leq n$.

Consider an open shrinking $\mathcal{V} = \{V_i\}_{i=1}^k$ of the cover $\mathcal{E}$ such that $\text{ord } \mathcal{V} \leq n$ and a closed shrinking $\{F_i\}_{i=1}^k$ of $\mathcal{V}$. Let $\mathcal{N}(\mathcal{V})$ be a nerve of $\mathcal{V}$ with vertices $p_1, p_2, \dots, p_k \in R^m$. By Urysohn's lemma for $i = 1, 2, \dots, k$ there exists a continuous function $f_i: X \rightarrow I$ such that $f_i(X \setminus V_i) \subset \{0\}$ and $f_i(F_i) \subset \{1\}$. One readily checks (cf. the proof of Theorem 1.10.7) that the formula

$$\kappa(x) = \kappa_1(x)p_1 + \kappa_2(x)p_2 + \dots + \kappa_k(x)p_k,$$

where

$$\kappa_i(x) = \frac{f_i(x)}{f_1(x) + f_2(x) + \dots + f_k(x)} \quad \text{for } i = 1, 2, \dots, k,$$

defines a continuous mapping $\kappa: X \rightarrow N(\mathcal{V})$ of X to the underlying polyhedron of the nerve $\mathcal{N}(\mathcal{V})$ which satisfies the conditions

$$\kappa^{-1}(\text{St}_{\mathcal{N}(\mathcal{V})}(p_i)) = V_i \subset U_i \quad \text{for } i = 1, 2, \dots, k.$$

As $\{\text{St}_{\mathcal{N}(\mathcal{V})}(p_i)\}_{i=1}^k$ is an open cover of $N(\mathcal{V})$, the mapping κ is an $\mathcal{E}$ -mapping. The inequality $\text{ord } \mathcal{V} \leq n$ implies that $N(\mathcal{V})$ has dimension $\leq n$. $\square$

Applying Theorem 1.10.15 and arguing as in the proof of Theorem 1.10.16, one obtains the following strengthening of the theorem on $\mathcal{E}$ -mappings.

3.2.12. Theorem. *A normal space X satisfies the inequality $\dim X \leq n$ if and only if for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$ -mapping of X onto a polyhedron of dimension $\leq n$. $\square$*

We now turn to a study of the behaviour of the dimension $\dim$ under Cartesian multiplication. First of all, let us recall that in Section 2.2 we cited an example of a normal space Z , whose square $Z \times Z$ is also normal, such that $\text{Ind } Z = 0$ and yet $\text{Ind}(Z \times Z) > 0$; in view of Theorem 1.6.11, this example shows that the inequality $\dim(X \times Y) \leq \dim X + \dim Y$ does not hold under the sole assumption of the normality of the Cartesian product $X \times Y$. Just as in the case of the dimension Ind , several theorems determining conditions for that inequality have been discovered, but there is no strongest result among them. We shall quote two such theorems which are relatively strong. Thus, the inequality $\dim(X \times Y) \leq \dim X + \dim Y$ holds for every pair X, Y of normal spaces of which at least one is non-empty provided that either

- (i) the Cartesian product $X \times Y$ is normal and one of the factors is compact (more generally: the projection onto one of the factors is a closed mapping),

or

- (ii) the Cartesian product $X \times Y$ is normal, one of the factors is metrizable and the other is countably paracompact (more generally: one of the factors can be mapped to a metrizable space by a perfect mapping and the other is countably paracompact).

Let us add that the proofs of Cartesian product theorems for $\dim$ are fairly difficult. As a sample we shall prove here two Cartesian product theorems for $\dim$ which are among the simplest.

3.2.13. Theorem. *For every pair X, Y of compact spaces of which at least one is non-empty we have*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

Proof. We can assume that $\dim X = n$ and $\dim Y = m$, where n and m are non-negative integers. Consider an arbitrary sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+m+1}, B_{n+m+1})$ of $n+m+1$ pairs of disjoint closed subsets of the Cartesian product $X \times Y$. The sets A_i being compact, for $i = 1, 2, \dots, n+m+1$ there exist closed sets $E_{i,j} \subset X$ and $F_{i,j} \subset Y$ and open sets $U_{i,j} \subset X$ and $V_{i,j} \subset Y$, where $j = 1, 2, \dots, k_i$, such that

$$(9) \quad A_i \subset \bigcup_{j=1}^{k_i} (E_{i,j} \times F_{i,j})$$

$$\text{and } E_{i,j} \times F_{i,j} \subset U_{i,j} \times V_{i,j} \subset (X \times Y) \setminus B_i.$$

By virtue of Theorem 3.2.5 there exist open sets $G_{i,j} \subset X$ and $H_{i,j} \subset Y$ such that

$$(10) \quad E_{i,j} \subset G_{i,j} \subset \bar{G}_{i,j} \subset U_{i,j} \quad \text{and} \quad F_{i,j} \subset H_{i,j} \subset \bar{H}_{i,j} \subset V_{i,j}$$

and

$$(11) \quad \text{ord}(\{\text{Fr } G_{i,j}\}_{(i,j) \in S}) \leq n-1 \quad \text{and} \quad \text{ord}(\{\text{Fr } H_{i,j}\}_{(i,j) \in S}) \leq m-1,$$

where S denotes the set of all pairs (i, j) with $i = 1, 2, \dots, n+m+1$ and $j = 1, 2, \dots, k_i$. For $i = 1, 2, \dots, n+m+1$ let $W_i = \bigcup_{j=1}^{k_i} (G_{i,j} \times H_{i,j})$. From (9) and (10) it follows that $A_i \subset W_i \subset \bar{W}_i \subset (X \times Y) \setminus B_i$, so that the set $L_i = \text{Fr } W_i$ is a partition between A_i and B_i . Consider the family

$$\mathcal{C}_i = \{\text{Fr } G_{i,j} \times \bar{H}_{i,j}\}_{j=1}^{k_i} \cup \{\bar{G}_{i,j} \times \text{Fr } H_{i,j}\}_{j=1}^{k_i}$$

of subsets of $X \times Y$, denote by C_i the union $\bigcup \mathcal{C}_i$, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_{n+m+1}$. Since $L_i \subset \bigcup_{j=1}^{k_i} \text{Fr}(G_{i,j} \times H_{i,j}) \subset C_i$, by virtue of Theorem

3.2.6 to complete the proof it suffices to show that $\bigcap_{i=1}^{n+m+1} C_i = \emptyset$. Now,

the last equality follows from the inequality $\text{ord } \mathcal{C} \leq n+m+1$, which in turn is a consequence of (11), because among each $n+m+1$ members of the family $\mathcal{C}$ there are either $n+1$ sets of the form $\text{Fr } G_{i,j} \times \overline{H}_{i,j}$ or $m+1$ sets of the form $\overline{G}_{i,j} \times \text{Fr } H_{i,j}$. $\square$

3.2.14. Theorem. *For every pair X, Y of normal spaces of which at least one is non-empty such that the Cartesian product $X \times Y$ is strongly paracompact we have*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

Proof. From Theorems 3.2.13 and 3.1.25 it follows that

$$\dim(\beta X \times \beta Y) \leq \dim \beta X + \dim \beta Y = \dim X + \dim Y.$$

To complete the proof it suffices to apply Theorem 3.1.23. $\square$

Historical and bibliographic notes

Theorem 3.2.1 and Proposition 3.2.2 were established by Dowker in [1947]. Lemma 3.2.3 and Theorem 3.2.4 were proved by Ostrand in [1971]; special cases of Theorem 3.2.4 were obtained earlier by Ostrand in [1965] (for metrizable spaces) and by French in [1970] (for collectionwise normal spaces). Theorem 3.2.5 was proved by Morita in [1950a], and Theorem 3.2.6—by Hemmingsen in [1946]. Theorem 3.2.10 was obtained independently by Hemmingsen in [1946], Alexandroff in [1947], and Dowker in [1947]; for compact spaces it was proved by Alexandroff in [1940] and by Morita in [1940]. Theorem 3.2.9, which is a simple consequence of Theorem 3.2.10, was noted by Alexandroff in [1947]. As stated in the notes to Section 1.10, Theorems 3.2.11 and 3.2.12 for metric spaces were established by Kuratowski in [1933a] (who generalized a characterization of dimension of compact metric spaces discovered by Alexandroff in [1928]); Kuratowski's proof extends without substantial changes to normal spaces. The fact that the inequality $\dim(X \times Y) \leq \dim X + \dim Y$ holds for every pair X, Y of spaces satisfying either (i) or (ii) was proved by Filippov in [1979] (announcement [1973]); (ii) part was announced independently by Pasyukov in [1973]). In the original formulation of (ii) one assumes that the Cartesian product $X \times Y$ is normal and countably

paracompact (cf. the commentary to (ii) in the notes to Section 2.4). Theorem 3.2.13 was proved by Hemmingsen in [1946], and Theorem 3.2.14—by Morita in [1953]. Further information on Cartesian product theorems for $\dim$ can be found in Morita [1953], Kodama [1969] (a simplified proof in Engelking [1973]), Nagami [1970], and Pears [1975].

Problems

3.2.A. Show that a normal space X satisfies the inequality $\dim X \leq n$ if and only if every locally finite open cover of the space X has a locally finite closed refinement of order $\leq n$ or—equivalently—if every locally finite open cover of the space X has a closed shrinking of order $\leq n$.

3.2.B (Ostrand [1971]; for collectionwise normal spaces French [1970]; for metrizable spaces Ostrand [1965]). Prove that a normal space X satisfies the inequality $\dim X \leq n$ if and only if for every locally finite open cover $\mathcal{U} = \{U_s\}_{s \in S}$ of the space X there exists such a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of discrete families of open subsets of X , where $\mathcal{V}_i = \{V_{i,s}\}_{s \in S}$, that $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \dots$ and the union of each $n+1$ families $\mathcal{V}_i$ constitutes a cover of the space X . Show that for every locally finite open cover $\mathcal{U} = \{U_s\}_{s \in S}$ of a normal space X such that $\dim X \leq n$ besides the sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ with the above properties there also exists such a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of discrete families of open subsets of X , where $\mathcal{W}_i = \{W_{i,s}\}_{s \in S}$, that $\bar{V}_{i,s} \subset W_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \dots$

Hint. Define the families $\mathcal{V}_i$ and $\mathcal{W}_i$ for $i \leq n+1$ by applying Theorem 3.2.4, then define inductively the families $\mathcal{V}_i$ and $\mathcal{W}_i$ for $i > n+1$. Assume that the families $\mathcal{V}_i$ and $\mathcal{W}_i$ are defined for $i \leq m-1 \geq n+1$, denote by $\mathcal{T}$ the family of all subsets of the set $\{1, 2, \dots, m-1\}$ which have exactly n elements and, for every $T \in \mathcal{T}$, define $F_T = X \setminus \bigcup_{i \in T} V_i$, where $V_i = \bigcup_{s \in S} V_{i,s}$. Consider families $\{V_T\}_{T \in \mathcal{T}}$ and $\{W_T\}_{T \in \mathcal{T}}$ of open subsets of X such that

$$F_T \subset V_T \subset \bar{V}_T \subset W_T \text{ for } T \in \mathcal{T} \quad \text{and} \quad \bar{W}_T \cap \bar{W}_{T'} = \emptyset \text{ for } T \neq T',$$

for every $T \in \mathcal{T}$ choose arbitrarily an $i(T) \leq m-1$ such that $i(T) \notin T$, note that $F_T \subset V_{i(T)}$ and let

$$V_{m,s} = \bigcup_{T \in \mathcal{T}} V_T \cap V_{i(T),s} \quad \text{and} \quad W_{m,s} = \bigcup_{T \in \mathcal{T}} W_T \cap W_{i(T),s}.$$

(a) Let $f, g: X \rightarrow S^n$ be continuous mappings of a compact space X to the n -sphere S^n . Show that if $\dim Z \leq n-1$ for every closed subspace Z of the space X contained in the set $D(f, g) = \{x \in X: f(x) \neq g(x)\}$, then the mappings f and g are homotopic.

(b) Prove that every compact space X such that $\dim X = n \geq 1$ contains an n -dimensional Cantor-manifold.

Hint. See the proof of Theorem 1.9.8.

3.2.G (Zolotarev [1975]). (a) Show that a normal subspace M of a normal space X satisfies the inequality $\dim M \leq n$ if and only if for every open set $U \subset X$ which contains the set M there exist a normal space Z , a normal subspace $A \subset Z$ satisfying $\dim A \leq n$ and continuous mappings $f: M \rightarrow Z$ and $g: Z \rightarrow X$ such that $f(M) \subset A \subset g^{-1}(U)$ and $gf(x) = x$ for $x \in M$.

(b) Prove that for every completely paracompact subspace M of a normal space X we have $\dim M \leq \dim X$ (see Problem 2.4.B).

Hint. One can assume that $\dim X = n < \infty$ and X is a compact space (see Theorem 3.1.25). Consider an open set $U \subset X$ which contains the set M and for every point $x \in M$ choose a neighbourhood V_x such that $x \in V_x \subset \bar{V}_x \subset U$. Let $\mathcal{V}_1, \mathcal{V}_2, \dots$ be a sequence of star-finite open covers of the space M such that the union $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ contains a refinement $\mathcal{V}$ of the cover $\{M \cap V_x\}_{x \in M}$. For $i = 1, 2, \dots$ consider the decomposition $\{\mathcal{V}_s\}_{s \in S}$ of the cover $\mathcal{V}_i$ into the components where $S_i \cap S_j = \emptyset$ whenever $i \neq j$; for every $x \in M$ denote by $f_i(x)$ the unique $s \in S_i$ such that $x \in \bigcup \mathcal{V}_s$, and for each $s \in S_i$ let $\mathcal{V}_s \cap \mathcal{V} = \{V_{s,j}\}_{j=1}^{\infty}$. Note that the space $Z = X \times \prod_{i=1}^{\infty} S_i$, where S_i has the discrete topology, is normal, and apply Problem 2.4.A and Theorem 3.2.14 to show that $\dim Z \leq n$. Apply (a) to the continuous mapping $f: M \rightarrow Z$ defined by letting $f(x) = (x, f_1(x), f_2(x), \dots)$ for $x \in M$, the projection $g: Z \rightarrow X$ and the set $A = \bigcup_{i,j=1}^{\infty} \bigcup_{s \in S_i} \overline{f(V_{s,j})} \subset Z$.

3.2.H. (a) Prove that every locally finite functionally open cover $\{U_s\}_{s \in S}$ of a topological space X has such shrinkings $\{F_s\}_{s \in S}$ and $\{W_s\}_{s \in S}$, respectively functionally closed and functionally open, that $F_s \subset W_s \subset \bar{W}_s \subset U_s$ for $s \in S$.

Hint. For every $s \in S$ choose a continuous function $f_s: X \rightarrow I$ such that $U_s = f_s^{-1}((0, 1])$ and consider the function $f: X \rightarrow I$ defined by letting $f(x) = \sup_{s \in S} f_s(x)$ for $x \in X$.

(b) (Pasyukov [1965]) Prove that a Tychonoff space X satisfies the inequality $\dim X \leq n$ if and only if every locally finite functionally open cover of the space X has a functionally open refinement of order $\leq n$ (see notes to Section 3.1).

Hint. Let $\{U_s\}_{s \in S}$ be a locally finite functionally open cover of the space X . Consider a functionally closed shrinking $\{F_s\}_{s \in S}$ of $\{U_s\}_{s \in S}$ and continuous functions $f_s: X \rightarrow I$ such that $f_s(X \setminus U_s) \subset \{0\}$ and $f_s(F_s) \subset \{1\}$. Verify that the formula $\varrho(x, y) = \sum_{s \in S} |f_s(x) - f_s(y)|$ defines a pseudometric on the set X and consider the metric space (Y, ϱ) obtained by identifying each pair x, y of points in X such that $\varrho(x, y) = 0$. Check that by letting $f(x) = [x]$ one defines a continuous mapping of X to Y and consider the mapping $F: \beta X \rightarrow \beta Y$ such that $F(x) = f(x)$ for $x \in X$. Apply the fact that every Hausdorff space which can be mapped onto a paracompact space by a perfect mapping is itself paracompact (see [GT], Theorem 5.1.35) and use Problem 3.1.1(b) and Theorem 3.2.1.

3.2.I. (a) Observe that if M is a functionally open subspace of a space X , then a set $A \subset M$ is functionally open in X if and only if it is functionally open in M . Give an example of a functionally closed subspace M of a completely regular space X and of a set $A \subset M$ which is functionally closed in M and yet is not functionally closed in X .

(b) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every finite functionally open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space X there exists a functionally open cover $\mathcal{V}$ of the space X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_{n+1}$, where $\mathcal{V}_j = \{V_{j,i}\}_{i=1}^k$, such that $\text{ord } \mathcal{V}_j \leq 0$ and $V_{j,i} \subset U_i$ for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n+1$.

3.2.J. (a) Prove that a completely regular space X satisfies the inequality $\dim X \leq n$ if and only if every $(n+2)$ -element functionally open cover $\{U_i\}_{i=1}^{n+2}$ of the space X has a functionally open shrinking $\{W_i\}_{i=1}^{n+2}$ of order $\leq n$, i.e., such that $\bigcap_{i=1}^{n+2} W_i = \emptyset$.

Hint. See the proof of Theorem 1.6.10.

(b) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every finite family $\{U_i\}_{i=1}^k$ of functionally open subsets of X and every family $\{F_i\}_{i=1}^k$ of functionally closed subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \dots, k$ there exist such families $\{E_i\}_{i=1}^k$

and $\{V_i\}_{i=1}^k$ of functionally closed and functionally open subsets of X , respectively, that $F_i \subset V_i \subset E_i \subset U_i$ for $i = 1, 2, \dots, k$ and $\text{ord}(\{E_i \setminus V_i\}_{i=1}^k) \leq n-1$.

(c) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint functionally closed subsets of X there exist functionally closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

Hint. Observe that if L is a functionally closed subset of a space X and $U, W \subset X$ are disjoint open sets satisfying the equality $X \setminus L = U \cup W$, then U and W are functionally open.

3.2.K. (a) (Smirnov [1956a]) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every closed subspace A of the space X and each continuous mapping $f: A \rightarrow S^n$ which is the restriction of a continuous mapping $\bar{f}: X \rightarrow B^{n+1}$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X .

(b) Prove that a completely regular space X satisfies the inequality $\dim X \leq n$ if and only if for every finite functionally open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$ -mapping of X to a polyhedron of dimension $\leq n$.

3.3. The compactification and the universal space theorems for the dimension dim. The dimension dim and inverse systems of compact spaces

In the proofs of the compactification and the universal space theorems we shall apply Mardešić's factorization theorem, which asserts that every continuous mapping $f: X \rightarrow Y$ of a compact space X to a compact space Y can be let through an intermediate space Z such that $\dim Z \leq \dim X$ and $w(Z) \leq w(Y)$. We start with a simple lemma on normal spaces. Let us recall that a cover $\mathcal{B}$ is a *star refinement* of another cover $\mathcal{A}$ of the same space if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $\text{St}(B, \mathcal{B}) \subset A$, where $\text{St}(B, \mathcal{B})$ denotes the *star of the set B with respect to the cover $\mathcal{B}$* , i.e., the set $\bigcup \{B' \in \mathcal{B} : B \cap B' \neq \emptyset\}$.

3.3.1. Lemma. Every finite open cover of a normal space has a finite open star refinement.

Proof. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of a normal space X . Consider a closed shrinking $\{F_i\}_{i=1}^k$ of $\{U_i\}_{i=1}^k$. By Urysohn's lemma, for $i = 1, 2, \dots, k$ there exists a continuous function $f_i: X \rightarrow I$ such that $f_i(X \setminus U_i) \subset \{0\}$ and $f_i(F_i) \subset \{1\}$. Define $I_0 = [0, 1/2)$, $I_1 = (1/4, 3/4)$ and $I_2 = (1/2, 1]$. The family $\mathcal{V}$ of all sets

$$(1) \quad V_{j_1, j_2, \dots, j_k} = f_1^{-1}(I_{j_1}) \cap f_2^{-1}(I_{j_2}) \cap \dots \cap f_k^{-1}(I_{j_k}),$$

where $j_i \in \{0, 1, 2\}$ for $i = 1, 2, \dots, k$, is a finite open cover of the space X . For every non-empty $V = V_{m_1, m_2, \dots, m_k} \in \mathcal{V}$ there exists an $i \leq k$ such that $V \cap F_i \neq \emptyset$, and we clearly have $m_i = 2$. If a set of form (1) intersects V , then j_i is equal to 1 or 2, so that $\text{St}(V, \mathcal{V}) \subset U_i$. Thus $\mathcal{V}$ is a star refinement of $\mathcal{U}$. $\square$

3.3.2. Mardešić's factorization theorem. *For every continuous mapping $f: X \rightarrow Y$ of a compact space X to a compact space Y there exist a compact space Z and continuous mappings $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.*

Proof. If $\dim X = \infty$ or $w(Y) < \aleph_0$, then $Z = f(X)$, $g = f$ and $h = \text{id}_Z$ satisfy the theorem. Thus one can suppose that $\dim X = n < \infty$ and $w(Y) = m \geq \aleph_0$. We shall define inductively a sequence $W_0, W_1, \dots$ of classes of finite open covers of the space X . Consider a base $\mathcal{B}$ for the space Y such that $|\mathcal{B}| = m$ and denote by W_0 the class of all finite covers of the space X by members of the family $f^{-1}(\mathcal{B})$. Clearly, for each $x, y \in X$

(2) if $f(x) \neq f(y)$, then there exists a $\mathcal{W} \in W_0$ such that $y \notin \text{St}(x, \mathcal{W})$.

Assume now that the classes W_i are defined for all $i < k$. By virtue of Lemma 3.3.1 and the inequality $\dim X \leq n$, for each pair of covers $\mathcal{W}, \mathcal{W}' \in W_{k-1}$ we can choose a finite open star refinement of $\mathcal{W} \wedge \mathcal{W}'$ which has order n ; let W_k be the class of all covers of the space X thus obtained. In this way the sequence $W_0, W_1, \dots$ is defined. Since $|W_0| \leq m$, we have $|W_i| \leq m$ for $i = 1, 2, \dots$. Let $W = \bigcup_{i=1}^{\infty} W_i$; clearly, $|W| \leq m$.

For $x, y \in X$ define

(3) xEy if and only if for every cover $\mathcal{W} \in W$ there exists a set $U \in \mathcal{W}$ such that $x, y \in U$.

We shall show that E is an equivalence relation on the space X . It follows directly from the definition that the relation E is reflexive and symmetric, so that it remains to show that if xEy and yEz , then xEz . To this end, it suffices to note that

- (4) if $\mathcal{W}'$ is a star refinement of $\mathcal{W}$, $y \in \text{St}(x, \mathcal{W}')$ and $z \in \text{St}(y, \mathcal{W}')$, then $z \in \text{St}(x, \mathcal{W})$,

because for every $\mathcal{W} \in \mathcal{W}$ there exists a $\mathcal{W}' \in \mathcal{W}$ which is a star refinement of $\mathcal{W}$. The relation E determines a decomposition of the space X into equivalence classes; from (3) it follows that

$$(5) \quad [x] = \bigcap_{\mathcal{W} \in \mathcal{W}} \text{St}(x, \mathcal{W}) \quad \text{for } x \in X,$$

where $[x]$ denotes the equivalence class that contains x .

We shall show now that the equivalence relation E is closed. Thus, we have to show that for every open set $U \subset X$ the union of all equivalence classes that are contained in U is an open subset of X . In view of a subsequent application, we shall show a little more, viz., that

- (6) for every open set $U \subset X$ and each equivalence class $[x] \subset U$ there exist an open set $V \subset X$ and a cover $\mathcal{W} \in \mathcal{W}$ such that $[x] \subset V \subset \bigcup_{y \in V} [y] \subset \text{St}(V, \mathcal{W}) \subset U$.

To begin with, let us note that if $\mathcal{W}'$ is a star refinement of $\mathcal{W}$, then $\overline{\text{St}(x, \mathcal{W}')} \subset \text{St}(x, \mathcal{W})$, so that from (5) it follows that $[x] = \bigcap_{\mathcal{W} \in \mathcal{W}} \overline{\text{St}(x, \mathcal{W})}$.

Now, the space X being compact, there exists a finite number of covers

$\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_k \in \mathcal{W}$ such that $\bigcap_{i=1}^k \text{St}(x, \mathcal{W}_i) \subset U$. Consider a cover

$\mathcal{W}_0 \in \mathcal{W}$ which refines all the covers $\mathcal{W}_i$ and a star refinement $\mathcal{W}' \in \mathcal{W}$ of the cover $\mathcal{W}_0$. Clearly, the open set $V = \text{St}(x, \mathcal{W}')$ satisfies the relation $[x] \subset V \subset \bigcup_{y \in V} [y]$; the penultimate inclusion in (6) follows from

(5), and the last inclusion is a consequence of the relation $\text{St}(x, \mathcal{W}_0) \subset U$ and (4) with $\mathcal{W}' = \mathcal{W}'$ and $\mathcal{W} = \mathcal{W}_0$.

Let Z be the quotient space X/E and $g: X \rightarrow Z$ the natural quotient mapping. As E is a closed equivalence relation, the space Z is compact (see [GT], Theorem 3.2.11). By virtue of (2), $f(x) = f(y)$ whenever xEy ,

so that by letting $h([x]) = f(x)$ we define a mapping h of Z to Y ; from the relation $hg = f$ it follows that $h: Z \rightarrow Y$ is a continuous mapping.

For every open set $U \subset X$ let $U^* = Z \setminus g(X \setminus U)$ and for every $\mathcal{W} \in \mathcal{W}$ let $\mathcal{W}^* = \{W^*: W \in \mathcal{W}\}$. Clearly, $g^{-1}(U^*) \subset U$ and U^* is open in Z . If $\mathcal{W}'$ is a star refinement of $\mathcal{W}$, then for each $x \in X$ there exists a $W \in \mathcal{W}$ such that $[x] \subset \text{St}(x, \mathcal{W}') \subset W$; thus for every $\mathcal{W} \in \mathcal{W}$ the family $\mathcal{W}^*$ is a finite open cover of the space Z and $\text{ord } \mathcal{W}^* \leq n$.

Now we shall show that every finite open cover $\{U_i\}_{i=1}^k$ of the space Z has a refinement of the form $\mathcal{W}^*$, where $\mathcal{W} \in \mathcal{W}$. For each $x \in X$ there exists an $i \leq k$ such that $[x] \subset g^{-1}(U_i)$, and by virtue of (6) there exist a neighbourhood $V_x \subset X$ of the point x and a cover $\mathcal{W}(x) \in \mathcal{W}$ such that $\text{St}(V_x, \mathcal{W}(x)) \subset g^{-1}(U_i)$. The open cover $\{V_x\}_{x \in X}$ of the space X has a finite refinement $\{V_{x_i}\}_{i=1}^m$. Consider a cover $\mathcal{W} \in \mathcal{W}$ which refines all the covers $\mathcal{W}(x_i)$. For every $W \in \mathcal{W}$ there exists an $i \leq k$ such that $W \subset g^{-1}(U_i)$. Since the last inclusion implies that $W^* \subset U_i$, it follows that the cover $\mathcal{W}^*$ is a refinement of $\{U_i\}_{i=1}^k$.

Thus we have shown that $\dim Z \leq n$ and that the family $\mathcal{D} = \bigcup_{\mathcal{W} \in \mathcal{W}} \mathcal{W}^*$ is a base for the space Z . To complete the proof it suffices to observe that $w(Z) \leq |\mathcal{D}| \leq m \cdot \aleph_0 = m = w(X)$. $\square$

3.3.3. The compactification theorem for dim. *For every normal space X there exists a compactification preserving both the dimension $\dim$ and weight, i.e., a compact space $\tilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequalities $\dim \tilde{X} \leq \dim X$ and $w(\tilde{X}) \leq w(X)$.*

Proof. We can suppose that $\dim X = n < \infty$ and $w(X) = m \geq \aleph_0$. Consider a homeomorphic embedding $i: X \rightarrow I^m$ of the space X in the Tychonoff cube I^m of weight m ; let $f: \beta X \rightarrow I^m$ be the extension of i over βX . By virtue of Theorem 3.3.2 there exist a compact space $\tilde{X}$ and continuous mappings $g: \beta X \rightarrow \tilde{X}$ and $h: \tilde{X} \rightarrow I^m$ such that $\dim \tilde{X} \leq \dim \beta X = \dim X$, $w(\tilde{X}) \leq w(I^m) = m$ and $f = hg$. The composition $h_0 g_0$ of the restrictions $g_0 = g|_X: X \rightarrow g(X) \subset \tilde{X}$ and $h_0 = h|_{g(X)}: g(X) \rightarrow i(X) \subset I^m$ is a homeomorphism, so that g_0 is also a homeomorphism. Thus $\tilde{X}$ is the required compactification of the space X . $\square$

3.3.4. The universal space theorem for dim. *For every integer $n \geq 0$ and every cardinal number $m \geq \aleph_0$ there exists a compact universal space P_m^n*

for the class of all normal spaces whose covering dimension is not larger than n and whose weight is not larger than m .

Proof. Let $\{X_s\}_{s \in S}$ be the family of all normal subspaces of the Tychonoff cube I^m whose covering dimension is not larger than n , and let $i_s: X_s \rightarrow I^m$ be the embedding of X_s in I^m . Consider the sum $X = \bigoplus_{s \in S} X_s$ and the mapping $i: X \rightarrow I^m$ defined by letting $i(x) = i_s(x)$ for $x \in X_s$; let $f: \beta X \rightarrow I^m$ be the extension of i over βX . By virtue of Theorem 3.3.2 there exist a compact space P_m^n and continuous mappings $g: \beta X \rightarrow P_m^n$ and $h: P_m^n \rightarrow I^m$ such that $\dim P_m^n \leq \dim \beta X = \dim X = n$, $w(P_m^n) \leq w(I^m) = m$ and $f = hg$.

Consider now an arbitrary normal space Y such that $\dim Y \leq n$ and $w(Y) \leq m$. Since Y is embeddable in I^m , there exists an $s \in S$ such that X_s is homeomorphic to Y . The composition $h_0 g_0$ of the restrictions $g_0 = g|_{X_s}: X_s \rightarrow g(X_s) \subset P_m^n$ and $h_0 = h|_{g(X_s)}: g(X_s) \rightarrow X_s \subset I^m$ is a homeomorphism, so that g_0 is also a homeomorphism. Thus P_m^n is the required universal space. $\square$

We now turn to a study of inverse systems of compact spaces from the dimensional standpoint. To begin with, let us recall that Theorem 1.13.2 established in Chapter 1 states that for every compact metric space X such that $\dim X \leq n$ there exists an inverse sequence $\{K_i, \pi_j^i\}$ consisting of polyhedra of dimension $\leq n$ whose limit is homeomorphic to X . In Example 3.3.8 below we show that the compact space X described in Example 2.2.13 cannot be represented as the limit of an inverse system of polyhedra of dimension 1, although $\dim X = 1$. Hence, Theorem 1.13.2 in its original form does not extend to arbitrary compact spaces; yet we have the following

3.3.5. Theorem on expansion in an inverse system. *For every compact space X such that $\dim X \leq n$ there exists an inverse system $\mathcal{S} = \{X_\sigma, \pi_\sigma^\alpha, \Sigma\}$, where $|\Sigma| \leq w(X)$, consisting of metrizable compact spaces of dimension $\leq n$ whose limit is homeomorphic to X .*

Proof. We can suppose that $w(X) = m \geq \aleph_0$. Let $h: X \rightarrow I^m = \prod_{s \in S} I_s$, where $I_s = I$ for $s \in S$ and $|S| = m$, be a homeomorphic embedding of the space X in the Tychonoff cube I^m of weight m . For $i = 1, 2, \dots$ denote by Σ_i the family of all subsets of S that have exactly i elements; the union

$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$ is directed by inclusion, i.e., the relation $\leq$ defined by letting $\varrho \leq \sigma$ if and only if $\varrho \subset \sigma$, and has cardinality $m = w(X)$.

Applying induction with respect to i , we shall now define for each $\sigma \in \Sigma_i$ a metrizable compact space X_σ such that $\dim X_\sigma \leq n$ if $i > 1$ and continuous mappings $\pi_\varrho^\sigma: X_\sigma \rightarrow X_\varrho$, where $\varrho \leq \sigma$, such that

$$(7) \quad \pi_\tau^\varrho \pi_\varrho^\sigma = \pi_\tau^\sigma \text{ whenever } \tau \leq \varrho \leq \sigma \quad \text{and} \quad \pi_\sigma^\sigma = \text{id}_{X_\sigma};$$

at the same time we shall define continuous mappings $g_\sigma: X \rightarrow X_\sigma$ satisfying

$$(8) \quad g_\sigma(X) = X_\sigma \quad \text{and} \quad \pi_\varrho^\sigma g_\sigma = g_\varrho \quad \text{whenever } \varrho \leq \sigma.$$

For $i = 1$ all conditions are satisfied if for each $\sigma = \{s\} \in \Sigma_1$ we let $X_\sigma = p_s h(X) \subset I_s$, where $p_s: \prod_{s \in S} I_s \rightarrow I_s$ is the projection, $\pi_\sigma^\sigma = \text{id}_{X_\sigma}$ and $g_\sigma = p_s h$. Assume that the spaces X and the mappings π_ϱ^σ and g_σ satisfying

(7) and (8) are defined for all $\sigma \in \bigcup_{i=1}^{k-1} \Sigma_i$, where $k > 1$, and consider a set

$\sigma \in \Sigma_k$. Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be all $(k-1)$ -element subsets of σ and let $f_\sigma: X \rightarrow X_{\sigma_1} \times X_{\sigma_2} \times \dots \times X_{\sigma_k}$ be the continuous mapping defined by the formula $f_\sigma(x) = (g_{\sigma_1}(x), g_{\sigma_2}(x), \dots, g_{\sigma_k}(x))$. By virtue of Theorem 3.3.2 there exist a compact space X_σ and continuous mappings $g_\sigma: X \rightarrow X_\sigma$ and $h_\sigma: X_\sigma \rightarrow X_{\sigma_1} \times X_{\sigma_2} \times \dots \times X_{\sigma_k}$ such that $\dim X_\sigma \leq n$, $w(X_\sigma) \leq \aleph_0$, $g_\sigma(X) = X_\sigma$ and $f_\sigma = h_\sigma g_\sigma$. The last equality means that

$$(9) \quad \pi_{\sigma_i}^\sigma g_\sigma = g_{\sigma_i} \quad \text{for } i = 1, 2, \dots, k,$$

where $\pi_{\sigma_i}^\sigma: X_\sigma \rightarrow X_{\sigma_i}$ is the composition of h_σ and the projection of $X_{\sigma_1} \times \dots \times X_{\sigma_k}$ onto X_{σ_i} . As the space X_σ is compact and has a countable

weight, it is a metrizable space. For each $\varrho \in \bigcup_{i=1}^{k-1} \Sigma_i$ satisfying $\varrho \leq \sigma$ there

exists at least one $i \leq k$ such that $\varrho \leq \sigma_i \leq \sigma$. Let us observe that the composition $\pi_\varrho^{\sigma_i} \pi_{\sigma_i}^\sigma$ does not depend on the choice of a particular σ_i satisfying $\varrho \leq \sigma_i$. Indeed, if for a $j \leq k$ we also have $\varrho \leq \sigma_j$, then

$$\pi_\varrho^{\sigma_i} \pi_{\sigma_i}^\sigma g_\sigma = \pi_\varrho^{\sigma_i} g_{\sigma_i} = g_\varrho = \pi_\varrho^{\sigma_j} g_{\sigma_j} = \pi_\varrho^{\sigma_j} \pi_{\sigma_j}^\sigma g_\sigma,$$

which implies that

$$(10) \quad \pi_\varrho^{\sigma_i} \pi_{\sigma_i}^\sigma = \pi_\varrho^{\sigma_j} \pi_{\sigma_j}^\sigma,$$

because $g_\sigma(X) = X_\sigma$. In accordance with the above observation, for each

$\varrho \in \bigcup_{i=1}^{k-1} \Sigma_i$ satisfying $\varrho \leq \sigma$ we define $\pi_\varrho^\sigma = \pi_\varrho^{\sigma_i} \pi_{\sigma_i}^\sigma$ where $\varrho \leq \sigma_i \leq \sigma$. More-

over, we let $\pi_\sigma^\sigma = \text{id}_{X_\sigma}$. From (9), (10) and the inductive assumption it follows that the space X_σ and the mappings π_σ^σ and g_σ satisfy (7) and (8) for $\sigma \in \Sigma_k$. Thus, we have defined metrizable compact spaces X_σ such that $\dim X_\sigma \leq n$ if $i > 1$ and continuous mappings π_σ^σ and g_σ satisfying (7) and (8) for $\sigma \in \Sigma$ and $\rho \leq \sigma$.

It follows from (7) that $S = \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ is an inverse system; we shall show that X is homeomorphic to the limit $\varprojlim S$. In view of the second equality in (8), for every $x \in X$ the point $\{g_\sigma(x)\} \in \prod_{\sigma \in \Sigma} X_\sigma$ is a thread of S ; by assigning this thread to x we define a mapping $g: X \rightarrow \varprojlim S$. Since $\pi_\sigma g = g_\sigma$ for every $\sigma \in \Sigma$, where $\pi_\sigma: \varprojlim S \rightarrow X_\sigma$ is the projection, the mapping g is continuous. For each $\sigma = \{s\} \in \Sigma_1$ we have $\pi_\sigma g = g_\sigma = p_s h$, so that, the mapping h being one-to-one, the mapping g is also one-to-one. Finally, as $g_\sigma(X) = X_\sigma$ for every $\sigma \in \Sigma$, the mapping g maps X onto $\varprojlim S$ (see [GT], Corollary 3.2.16). Thus g is a homeomorphism of X onto $\varprojlim S$.

If $n \geq 1$, the system S satisfies all the required conditions; if $n = 0$, it has to be replaced by the system $\{X_\sigma, \pi_\sigma^\sigma, \Sigma \setminus \Sigma_1\}$, because, in general, the inequality $\dim X_\sigma \leq 0$ does not hold for $\sigma \in \Sigma_1$ (cf. Problem 3.3.A). $\square$

Let us observe that from (8) and (9) it follows that the bonding mappings in the inverse system S in Theorem 3.3.5 are mappings onto.

We shall now prove the following

3.3.6. Theorem on the dimension of the limit of an inverse system. *If the inverse system $S = \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ consists of compact spaces X_σ such that $\dim X_\sigma \leq n$ for $\sigma \in \Sigma$, then the limit $X = \varprojlim S$ satisfies the inequality $\dim X \leq n$.*

Proof. Consider a finite open cover $\mathcal{U}$ of the space X . The space X being compact (see [GT], Theorem 3.2.13), the cover $\mathcal{U}$ has a finite refinement of the form $\{\pi_{\sigma_i}^{-1}(U_i)\}_{i=1}^k$, where $\pi_{\sigma_i}: X \rightarrow X_{\sigma_i}$ is the projection and U_i is an open subset of X_{σ_i} for $i = 1, 2, \dots, k$. Let σ be an arbitrary element of Σ such that $\sigma_i \leq \sigma$ for $i = 1, 2, \dots, k$ and let $W_i = (\pi_{\sigma_i}^\sigma)^{-1}(U_i)$. One readily sees that the family $\{\pi_\sigma^{-1}(W_i)\}_{i=1}^k$ is an open refinement of the cover $\mathcal{U}$. Since $\pi_\sigma(X)$ is a closed subspace of X_σ , $\dim \pi_\sigma(X) \leq n$ and the open cover $\{\pi_\sigma(X) \cap W_i\}_{i=1}^k$ of the space $\pi_\sigma(X)$ has an open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$. The family $\{\pi_\sigma^{-1}(V_i)\}_{i=1}^k$ is an open cover of the space X which refines $\mathcal{U}$ and has order $\leq n$. Thus $\dim X \leq n$. $\square$

Theorems 3.3.5 and 3.3.6 yield the following

3.3.7. Theorem on inverse systems. *A compact space X satisfies the inequality $\dim X \leq n$ if and only if X is homeomorphic to the limit of an inverse system consisting of metrizable compact spaces of dimension $\leq n$. $\square$*

We conclude this section with the above-mentioned example of a compact space X which satisfies the equality $\dim X = 1$ and yet is not homeomorphic to the limit of an inverse system of polyhedra of dimension ≤ 1 .

3.3.8. Example. The space X described in Example 2.2.13 is compact and satisfies the relation $\dim X = 1 < \text{ind} X$ (see Example 3.1.31). Thus, to show that X has the required property it suffices to prove that for every inverse system $S = \{K_\sigma, \pi_\sigma^\varrho, \Sigma\}$ consisting of polyhedra of dimension ≤ 1 the limit $K = \varprojlim S$ satisfies the inequality $\text{ind} K \leq 1$.

Consider a point $x \in K$ and a neighbourhood $V \subset K$ of the point x . There exists a $\sigma_0 \in \Sigma$ and such a neighbourhood U_{σ_0} of the point $\pi_{\sigma_0}(x)$ in the space K_{σ_0} , where $\pi_{\sigma_0} : K \rightarrow K_{\sigma_0}$ is the projection, that the set $U = \pi_{\sigma_0}^{-1}(U_{\sigma_0})$ satisfies the relation $x \in U \subset V$. Define $\Sigma_0 = \{\sigma \in \Sigma : \sigma_0 \leq \sigma\}$ and let $U_\sigma = (\pi_{\sigma_0}^\sigma)^{-1}(U_{\sigma_0})$ and $F_\sigma = \text{Fr } U_\sigma$ for $\sigma \in \Sigma_0$. Since for each $\sigma, \varrho \in \Sigma_0$ satisfying $\varrho \leq \sigma$ we have

$$\begin{aligned} \pi_\varrho^\sigma(F_\sigma) &= \pi_\varrho^\sigma(\overline{U_\sigma \cap K_\sigma \setminus U_\sigma}) \subset \overline{\pi_\varrho^\sigma(U_\sigma) \cap \pi_\varrho^\sigma(K_\sigma \setminus U_\sigma)} \\ &= \overline{\pi_\varrho^\sigma(\pi_{\sigma_0}^\sigma)^{-1}(U_{\sigma_0}) \cap \pi_\varrho^\sigma(\pi_{\sigma_0}^\sigma)^{-1}(K_{\sigma_0} \setminus U_{\sigma_0})} \\ &\subset (\pi_{\sigma_0}^\varrho)^{-1}(U_{\sigma_0}) \cap (\pi_{\sigma_0}^\varrho)^{-1}(K_{\sigma_0} \setminus U_{\sigma_0}) = \overline{U_\varrho \cap K_\varrho \setminus U_\varrho} = F_\varrho, \end{aligned}$$

the family $S_0 = \{F_\sigma, \tilde{\pi}_\sigma^\varrho, \Sigma_0\}$, where $\tilde{\pi}_\sigma^\varrho : F_\sigma \rightarrow F_\varrho$ is defined by letting $\tilde{\pi}_\sigma^\varrho(x) = \pi_\varrho^\sigma(x)$, is an inverse system of compact spaces. Now, from Theorems 1.8.12 and 1.3.1 it follows that $\dim F_\sigma \leq 0$ for $\sigma \in \Sigma_0$ so that $\dim \varprojlim S_0 \leq 0$ by virtue of Theorem 3.3.6. One readily checks that $\pi_\sigma(\text{Fr } U) \subset F_\sigma$ for every $\sigma \in \Sigma_0$; thus $\text{Fr } U \subset \varprojlim S_0$ and by virtue of Theorem 3.1.30 we have $\text{ind Fr } U \leq 0$. Hence we have proved that $\text{ind} K \leq 1$. $\square$

Let us note in connection with the last example that there exists a compact space X with a similar property which satisfies the equality $\text{ind} X = \text{Ind} X = \dim X = 1$.

Let us also note that every compact space is homeomorphic to the limit of an inverse system consisting of polyhedra (see Problem 3.3.D) and that one can define a compact space X such that $\text{ind} X = \text{Ind} X = \dim X = 1$ which is not homeomorphic to the limit of an inverse system consisting of polyhedra (or, more generally, of locally connected metrizable compact spaces) whose bonding mappings are mappings onto.

Historical and bibliographic notes

Theorem 3.3.2 was established by Mardešić in [1960]; the present proof was given by Arhangel'skiĭ in [1967]. Theorem 3.3.3 was proved by Skljarenko in [1958]. Theorem 3.3.4 was established independently by Pasynkov in [1964] and by Zarelua in [1964]. Theorem 3.3.5 was given by Mardešić in [1960]; Theorem 3.3.6 is implicit in Freudenthal's paper [1937]. Example 3.3.8 was given independently by Pasynkov in [1958] and by Mardešić in [1960]. Both examples cited at the end of this section can be found in Pasynkov's paper [1962].

Problems

3.3.A. Observe that Theorem 3.3.5 for $n = 0$ easily follows from the fact that every normal space X such that $\dim X = 0$ is embeddable in a Cantor cube (see Remark 1.3.18 and Theorem 1.6.11).

3.3.B (Pasynkov [1962]). Let $S = \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ be an inverse system of compact spaces and let $X = \varprojlim S$. Prove that $\dim X \leq n$ if and only if for each $\varrho \in \Sigma$ and every finite open cover $\{U_i\}_{i=1}^k$ of the space X_ϱ there exists a $\sigma \in \Sigma$ satisfying $\varrho \leq \sigma$ and such that the cover $\{(\pi_\varrho^\sigma)^{-1}(U_i)\}_{i=1}^k$ of the space X_σ has a finite open refinement $\mathcal{V}$ satisfying the inequality $\text{ord}(\mathcal{V}|_{\pi_\sigma(X)}) \leq n$, where $\pi_\sigma: X \rightarrow X_\sigma$ is the projection.

3.3.C (Pasynkov [1958]). Prove that for every inverse system $S = \{K_\sigma, \pi_\sigma^\sigma, \Sigma\}$ consisting of polyhedra of dimension ≤ 1 the limit $K = \varprojlim S$ satisfies the inequality $\text{Ind } K \leq 1$.

Remark. It is not known if the number 1 in Problem 3.3.C can be replaced by an arbitrary natural number.

3.3.D (Eilenberg and Steenrod [1952]). Prove that for every compact space X there exists an inverse system $S = \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ consisting of polyhedra whose limit is homeomorphic to X .

Hint. Embed the space X in a Tychonoff cube.

CHAPTER 4

DIMENSION THEORY OF METRIZABLE SPACES

In the realm of metrizable spaces the dimensions Ind and dim coincide. Thus in metrizable spaces both the theorems which depend on the dimension Ind and the theorems which depend on the dimension dim are valid. It will appear in the course of this chapter that the dimension theory of metrizable spaces is by no means inferior to the classical dimension theory of separable metric spaces developed in the first chapter of this book.

The present chapter can be read almost directly after Chapter 1. The results of Chapter 2 are not used here, except for Lemma 2.3.16 which belongs to general topology rather than to dimension theory. From Chapter 3 we use only the beginning of Section 3.1 up to Theorem 3.1.10 and also Theorems 3.1.28, 3.1.29, 3.2.2 and 3.2.5; the last theorem is not used until Section 4.2.

In Section 4.1 the most important properties of dimension in metrizable spaces are established. We start with the Katětov–Morita theorem on the coincidence of Ind and dim and then prove the counterparts of the theorems obtained for separable metric spaces in Section 1.5.

Section 4.2 begins with two characterizations of the dimension dim in metrizable spaces, one stated in terms of special bases and the other in terms of sequences of covers. Then we discuss briefly some characterizations of dim formulated in terms of special metrics. In the second part of the section, we prove by applying an appropriate factorization theorem the existence of a universal space for the class of all metrizable spaces whose dimension is not larger than n and whose weight is not larger than m .

Section 4.3 resumes the considerations of Section 1.12. We generalize in it the theorems on dimension-raising and dimension-lowering mappings established in Chapter 1 and prove two theorems of more special character on the relations between the dimensions of the domain and the range of a closed mapping.

Let us add that the theorems on partitions and on extending mappings

to spheres established in Sections 1.7 and 1.9 extend to all normal, and, *a fortiori*, to all metrizable spaces; the proofs were given in Section 3.2 (cf. Problems 4.1.E and 4.3.B).

4.1. Basic properties of dimension in metrizable spaces

We start with one of the most important results in dimension theory, viz., with the theorem on the coincidence of the dimensions Ind and dim in metrizable spaces. In the proof we shall apply two characterizations of the dimension dim in the class of metrizable spaces which are established in Proposition 4.1.2 below (cf. Problem 4.1.A(b)).

4.1.1. Lemma. *Let X be a normal space. If there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open covers of the space X such that $\text{ord } \mathcal{W}_i \leq n$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \dots$, and the family $\{\text{St}(W, \mathcal{W}_i) : W \in \mathcal{W}_i, i = 1, 2, \dots\}$ is a base for X , then $\text{dim } X \leq n$.*

Proof. For $i = 1, 2, \dots$ let f_i^{i+1} be a mapping of $\mathcal{W}_{i+1}$ to $\mathcal{W}_i$ such that $W \subset f_i^{i+1}(W)$ for each $W \in \mathcal{W}_{i+1}$; let $f_i^k = f_i^{i+1}f_{i+1}^{i+2} \dots f_{k-1}^k$ for $i < k$ and let $f_i^i = \text{id}_{\mathcal{W}_i}$ for $i = 1, 2, \dots$. Obviously

$$(1) \quad W \subset f_i^k(W) \quad \text{for each } W \in \mathcal{W}_k \text{ and } i \leq k.$$

Consider a finite open cover $\{H_j\}_{j=1}^l$ of the space X . The sets $X_1, X_2, \dots$, where

$$(2) \quad X_k = \bigcup \{W \in \mathcal{W}_k : \text{St}(W, \mathcal{W}_k) \subset H_j \text{ for a } j \leq l\},$$

form an open cover of the space X . For $k = 1, 2, \dots$ define the subfamilies

$$\mathcal{U}_k = \{U \in \mathcal{W}_k : U \cap X_k \neq \emptyset\} \quad \text{and} \quad \mathcal{V}_k = \{V \in \mathcal{U}_k : V \cap (\bigcup_{j < k} X_j) = \emptyset\}$$

of the cover $\mathcal{W}_k$, and for every $U \in \mathcal{U}_k$ denote by $i(U)$ the largest integer $\leq k$ satisfying

$$(3) \quad f_{i(U)}^k(U) \in \mathcal{V}_{i(U)};$$

such an integer does exist because $f_1^k(U) \cap (\bigcup_{j < 1} X_j) = \emptyset$ and $f_k^k(U) \cap X_k = U \cap X_k \neq \emptyset$.

For every $V \in \mathcal{V}_i$ consider the open set

$$(4) \quad V^* = \bigcup_{k=i}^{\infty} \left[\bigcup \{U \cap X_k : U \in \mathcal{U}_k, f_i^k(U) = V \text{ and } i(U) = i\} \right];$$

as $V \cap X_i = \emptyset$, by virtue of (2) there exist a $W \in \mathcal{W}_i$ such that $V \cap W \neq \emptyset$ and a $j(V) \leq l$ satisfying $V \subset \text{St}(W, \mathcal{W}_i) \subset H_{j(V)}$. From (1) it follows that $V^* \subset V$, so that $V^* \subset H_{j(V)}$. Since $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ whenever $i \neq j$, for every $V \in \mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ the set V^* and the integer $j(V)$ are well defined.

To complete the proof it suffices to show that the family $\{V_j\}_{j=1}^l$, where $V_j = \bigcup \{V^*: V \in \mathcal{V} \text{ and } j(V) = j\} \subset H_j$, is a cover of the space X and has order $\leq n$, or—equivalently—that the family $\mathcal{V}^* = \{V^*: V \in \mathcal{V}\}$ is a cover of X and $\text{ord } \mathcal{V}^* \leq n$.

Let x be an arbitrary point of X . Consider an integer k such that

$$(5) \quad x \in X_k \setminus \bigcup_{j < k} X_j$$

and a set $U \in \mathcal{W}_k$ which contains the point x ; since $U \cap X_k \neq \emptyset$, $U \in \mathcal{U}_k$. It follows from (3) and (4) that $x \in U \cap X_k \subset (f_{i(U)}^k(U))^* \in \mathcal{V}^*$, so that $\mathcal{V}^*$ is a cover of X .

It remains to show that $\text{ord } \mathcal{V}^* \leq n$. Consider a non-empty intersection $V_1^* \cap V_2^* \cap \dots \cap V_h^*$, where $V_i \in \mathcal{V}_{m_i}$ and $V_i \neq V_j$ whenever $i \neq j$; let $x \in V_1^* \cap V_2^* \cap \dots \cap V_h^*$. From the definition of $\mathcal{V}_{m_i}$ it follows that for the integer k satisfying (5) we have $m_i \leq k$ for $i = 1, 2, \dots, h$. By (4) there exist sets $U_i \in \mathcal{U}_{k_i}$ such that $f_{m_i}^{k_i}(U_i) = V_i$, $i(U_i) = m_i$ and $x \in U_i \cap X_{k_i}$. Since $x \in X_{k_i}$ it follows from (5) that $k \leq k_i$. The sets $W_1, W_2, \dots, W_h$, where $W_i = f_{m_i}^k(U_i) \in \mathcal{W}_k$, all contain the point x , so that—as $\text{ord } \mathcal{W}_k \leq n$ —it suffices to show that $W_i \neq W_j$ whenever $i \neq j$. Let us note that the sets W_i belong to $\mathcal{U}_k$ and that $i(W_i) = i(U_i) = m_i$; hence $W_i \neq W_j$ whenever $m_i \neq m_j$. When $m_i = m_j$, we also have $W_i \neq W_j$, because then

$$f_{m_i}^k(W_i) = f_{m_i}^{k_i}(U_i) = V_i \neq V_j = f_{m_j}^k(U_j) = f_{m_j}^k(W_j). \quad \square$$

4.1.2. Proposition. *For every metrizable space X the following conditions are equivalent:*

- The space X satisfies the inequality $\dim X \leq n$.*
- For every metric ρ on the space X there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of locally finite open covers of the space X such that for $i = 1, 2, \dots$ $\text{ord } \mathcal{U}_i \leq n$, $\delta(\bar{U}) < 1/i$ for $U \in \mathcal{U}_i$, and for each $U \in \mathcal{U}_{i+1}$ the set $\bar{U}$ is contained in a $V \in \mathcal{U}_i$.*
- There exist a metric ρ on the space X and a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open covers of the space X such that for $i = 1, 2, \dots$ $\text{ord } \mathcal{W}_i \leq n$, $\delta(W) < 1/i$ for $W \in \mathcal{W}_i$, and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$.*

Proof. The implication (b) $\Rightarrow$ (c) is obvious and the implication (c) $\Rightarrow$ (a) follows from Lemma 4.1.1, so that it suffices to prove that (a) $\Rightarrow$ (b). Consider a metrizable space X such that $\dim X \leq n$ and an arbitrary metric ϱ on the space X . We shall define inductively a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X . Assume that $k = 1$ or that $k > 1$ and the covers $\mathcal{U}_i$ are defined for all $i < k$. For every point $x \in X$ there exists a neighbourhood U_x such that $\delta(U_x) < 1/k$ and the set $\bar{U}_x$ is contained in a member of $\mathcal{U}_{k-1}$ if $k > 1$. Since every metrizable space is paracompact (see [GT], Theorem 5.1.3), it follows from Proposition 3.2.2 that the open cover $\{U_x\}_{x \in X}$ of the space X has a locally finite open refinement $\mathcal{U}_k$. The sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ thus obtained satisfies all the conditions in (b), so that (a) $\Rightarrow$ (b). $\square$

4.1.3. The Katětov–Morita theorem. *For every metrizable space X we have $\text{Ind} X = \dim X$.*

Proof. In view of Theorem 3.1.28 it suffices to show that $\text{Ind} X \leq \dim X$. We can suppose that $\dim X < \infty$. We shall apply induction with respect to $\dim X$. If $\dim X = -1$, we clearly have $\text{Ind} X \leq \dim X$. Assume that our inequality holds for all metrizable spaces with covering dimension $\leq n-1$ and consider a metrizable space X such that $\dim X = n \geq 0$ and a pair A, B of disjoint closed subsets of the space X . It suffices to define open sets $K, M \subset X$ which, together with the set $L = X \setminus (K \cup M)$, satisfy the conditions

$$A \subset K, \quad B \subset M, \quad K \cap M = \emptyset \quad \text{and} \quad \dim L \leq n-1;$$

indeed, the set L is then a partition between A and B and $\text{Ind} L \leq n-1$ by virtue of the inductive assumption.

Let σ be an arbitrary metric on the space X and let $f: X \rightarrow I$ be a continuous function satisfying $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. One readily checks that the formula $\varrho(x, y) = \sigma(x, y) + |f(x) - f(y)|$ defines a metric ϱ on the space X . From now on we shall consider on X only the metric ϱ . By virtue of Proposition 4.1.2 there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of locally finite open covers of the space X such that for $i = 1, 2, \dots$ $\text{ord} \mathcal{U}_i \leq n$, $\delta(U) < 1/i$ for $U \in \mathcal{U}_i$ and for each $U \in \mathcal{U}_{i+1}$ the set $\bar{U}$ is contained in a $V \in \mathcal{U}_i$.

Let $K_0 = A$, $M_0 = B$, and for $i \geq 1$ let $K_i = X \setminus H_i$ and $M_i = X \setminus G_i$, where

$$\begin{aligned} G_i &= \bigcup \{U \in \mathcal{U}_i: \bar{U} \cap M_{i-1} = \emptyset\} \\ \text{and} \quad H_i &= \bigcup \{U \in \mathcal{U}_i: \bar{U} \cap M_{i-1} \neq \emptyset\}; \end{aligned}$$

in this way two sequences, $K_0, K_1, K_2, \dots$ and $M_0, M_1, M_2, \dots$, of sub-

sets of the space X are defined.

Let us observe that

(6) if $U \in \mathcal{U}_i$ and $\bar{U} \cap M_{i-1} \neq \emptyset$, then $\bar{U} \cap K_{i-1} = \emptyset$.

The validity of (6) for $i = 1$ follows from the definition of ϱ , because no set of diameter less than 1 meets both A and B . If $U \in \mathcal{U}_i$ where $i > 1$ and $\bar{U} \cap M_{i-1} \neq \emptyset$, then for any $V \in \mathcal{U}_{i-1}$ that contains the set $\bar{U}$ we have $V \cap M_{i-1} \neq \emptyset$, so that V is not contained in G_{i-1} ; this implies that $V \subset H_{i-1}$, which gives the equality $\bar{U} \cap K_{i-1} = \emptyset$.

From the local finiteness of $\mathcal{V}_i$, the definitions of G_i and H_i , and (6) it follows that $\bar{G}_i \cap M_{i-1} = \emptyset = \bar{H}_i \cap K_{i-1}$ for $i = 1, 2, \dots$, which implies that $K_{i-1} \subset X \setminus \bar{H}_i = \text{Int} K_i$ and $M_{i-1} \subset X \setminus \bar{G}_i = \text{Int} M_i$; moreover, as $G_i \cup H_i = X$, we have $K_i \cap M_i = \emptyset$. Hence, the sets $K = \bigcup_{i=0}^{\infty} K_i$ and $M = \bigcup_{i=0}^{\infty} M_i$ are open, disjoint and contain respectively A and B .

Let $L_i = X \setminus (K_i \cup M_i) = G_i \cap H_i$ for $i = 1, 2, \dots$; clearly $L = \bigcap_{i=1}^{\infty} L_i$.

The family $\mathcal{W}_i = \{U \cap L : U \in \mathcal{U}_i \text{ and } \bar{U} \cap M_{i-1} \neq \emptyset\}$ is, for $i = 1, 2, \dots$, an open cover of the space $L \subset H_i$ and $\text{ord } \mathcal{W}_i \leq n-1$, because each point $x \in L \subset L_i \subset G_i$ belongs to at least one $U \in \mathcal{U}_i$ satisfying $\bar{U} \cap M_{i-1} = \emptyset$. If $U \in \mathcal{U}_{i+1}$ and $\bar{U} \cap M_i \neq \emptyset$, then for any $V \in \mathcal{U}_i$ that contains $\bar{U}$ we have $V \cap M_i \neq \emptyset$, so that V is not contained in G_i , which implies that $\bar{V} \cap M_{i-1} \neq \emptyset$, i.e., that $V \cap L \in \mathcal{W}_i$. Thus $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$. Since, clearly, $\delta(W) < 1/i$ for $W \in \mathcal{W}_i$, we have $\dim L \leq n-1$ by virtue of Proposition 4.1.2. $\square$

From the coincidence of the dimensions Ind and $\dim$ in metrizable spaces it follows that some results in the dimension theory of metrizable spaces, such as the subspace and sum theorems, are particular cases of both a theorem on Ind and a theorem on $\dim$. However, the proofs of those particular cases are usually much simpler than the proofs of the corresponding general theorems. Moreover, what is more important, the number of theorems in the classical dimension theory which can be extended to metrizable spaces is larger than that of the theorems hitherto generalized in Chapters 2 and 3.

We are now going to list the counterparts of the theorems established in Section 1.5 for separable metric spaces; we shall always point out the

theorems in Chapters 2 and 3 of which those counterparts are particular cases and, when possible, supply a simpler proof. The theorems will be formulated in terms of Ind ; obviously, they could as well be formulated in terms of dim .

Let us begin, however, with a brief discussion of the status of the dimension ind in arbitrary metrizable spaces.

Theorems 2.4.4 and 4.1.3 (or 1.6.3, 3.1.29 and 4.1.3) yield the following

4.1.4. Theorem. *For every strongly paracompact metrizable space X we have $\text{ind}X = \text{Ind}X = \text{dim}X$. $\square$*

As a special case of the above theorem we obtain the following important fact, stated above as Theorem 1.7.7.

4.1.5. Theorem. *For every separable metrizable space X we have $\text{ind}X = \text{Ind}X = \text{dim}X$. $\square$*

Let us note that Theorem 4.1.5 can also be deduced directly from Theorem 4.1.3 and Lemma 1.7.4.

4.1.6. Remark. Let us state once more that there exists a completely metrizable space X , known as Roy's space, such that $\text{ind}X = 0$ and yet $\text{Ind}X = \text{dim}X = 1$. The definition of that space and the computation of its dimensions ind and Ind is too difficult to be included in this book.

As we ascertained in Chapter 2, the dimension ind develops pathological properties and is practically of no importance outside the class of separable metric spaces; suffice it to say that in metrizable spaces even the finite sum theorem for ind does not hold (see Problem 4.1.B). Therefore the dimension ind will not be discussed further in this book. It should be stressed, however, that the historical role of the small inductive dimension can hardly be overestimated. The dimension function ind was the first formal setting of the concept of dimension and a good base for the dimension theory of separable metric spaces. Besides, the dimension ind has a great intuitive appeal and yields quickly and economically the classical part of dimension theory.

We now turn to a list of the basic properties of dimension in metrizable spaces. First of all, let us observe that, since every subspace of a space X which satisfies condition (c) in Proposition 4.1.2 also satisfies this condition, from Proposition 4.1.2 and Theorem 4.1.3 we obtain the following theorem (which is a particular case of Theorems 2.3.6 and 3.1.19).

4.1.7. The subspace theorem. *For every subspace M of a metrizable space X we have $\text{Ind } M \leq \text{Ind } X$. $\square$*

The following simple theorem is a particular case of Theorem 2.2.2.

4.1.8. Theorem. *If X is a metrizable space and $\text{Ind } X = n \geq 1$, then for $k = 0, 1, \dots, n-1$ the space X contains a closed subspace M such that $\text{Ind } M = k$. $\square$*

From Theorems 2.3.8 and 2.3.10 (or 3.1.8, 3.1.10 and 4.1.3) we obtain the countable and the locally finite sum theorems.

4.1.9. The countable sum theorem. *If a metrizable space X can be represented as the union of the sequence $F_1, F_2, \dots$ of closed subspaces such that $\text{Ind } F_i \leq n$ for $i = 1, 2, \dots$, then $\text{Ind } X \leq n$. $\square$*

4.1.10. The locally finite sum theorem. *If a metrizable space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind } F_s \leq n$ for $s \in S$, then $\text{Ind } X \leq n$. $\square$*

The next theorem is a common generalization of the last two theorems; it follows from Theorem 2.3.15 (or 3.1.15) and the fact that every metrizable space is paracompact.

4.1.11. Theorem. *If a metrizable space X can be represented as the union of a locally countable family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind } F_s \leq n$ for $s \in S$, then $\text{Ind } X \leq n$. $\square$*

Another common generalization of the countable and the locally finite sum theorems is the following theorem, which is stronger than similar results in Chapters 2 and 3.

4.1.12. Theorem. *If a metrizable space X can be represented as the union of a transfinite sequence $K_1, K_2, \dots, K_\alpha, \dots, \alpha < \xi$ of subspaces such that $\text{Ind } K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for $\alpha < \xi$, then $\text{Ind } X \leq n$.*

Proof. Let ρ be a metric on the space X . For each $\alpha < \xi$ define $F_\alpha = \bigcup_{\beta < \alpha} K_\beta$ and consider the sets

$$F_{i,\alpha} = K_\alpha \setminus B(F_\alpha, 1/i) = F_{\alpha+1} \setminus B(F_\alpha, 1/i) \quad \text{for } i = 1, 2, \dots,$$

where $B(A, r)$ denotes the open r -ball about A with respect to the metric ϱ . The sets $F_{i,\alpha}$ are closed and $\text{Ind} F_{i,\alpha} \leq n$ for $\alpha < \xi$ and $i = 1, 2, \dots$. Since $F_{i,\alpha} \cap B(F_{i,\beta}, 1/i) = \emptyset$ whenever $\beta < \alpha$, the family $\{F_{i,\alpha}\}_{\alpha < \xi}$ is discrete, so that the set $F_i = \bigcup_{\alpha < \xi} F_{i,\alpha}$ is closed and satisfies the inequality $\text{Ind} F_i \leq n$ for $i = 1, 2, \dots$

It remains to show that $X = \bigcup_{i=1}^{\infty} F_i$. Consider an arbitrary point $x \in X$; let α be the smallest ordinal number less than ξ such that $x \in K_\alpha$. Since $x \notin F_\alpha$, there exists an integer i such that $F_\alpha \cap B(x, 1/i) = \emptyset$, which implies that $x \notin B(F_\alpha, 1/i)$. Thus $x \in F_{i,\alpha} \subset F_i$. $\square$

From Lemma 1.2.9 and Remark 1.2.10 one easily obtains (cf. the proof of Theorem 1.2.11) the following result, which is a particular case of Theorem 2.2.4.

4.1.13. The separation theorem. *If X is a metrizable space and M is a subspace of X such that $\text{Ind} M \leq n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{Ind}(L \cap M) \leq n-1$. $\square$*

We shall now characterize the dimension of subspaces of metrizable spaces in terms of σ -locally finite bases for the space (cf. Proposition 1.5.15); the characterization will be applied in the proofs of the decomposition, enlargement and Cartesian products theorems.

4.1.14. Proposition. *A subspace M of a metrizable space X satisfies the inequality $\text{Ind} M \leq n \geq 0$ if and only if X has a σ -locally finite base $\mathcal{B}$ such that $\text{Ind}(M \cap \text{Fr } U) \leq n-1$ for every $U \in \mathcal{B}$.*

Proof. Consider a subspace M of a metrizable space X which satisfies the inequality $\text{Ind} M \leq n \geq 0$; let ϱ be an arbitrary metric on the space X . The space X being paracompact, for $i = 1, 2, \dots$ there exists a locally finite open cover $\mathcal{V}_i = \{V_s\}_{s \in S_i}$ of the space X such that $\text{mesh } \mathcal{V}_i < 1/i$; let $\{F_s\}_{s \in S_i}$ be a closed shrinking of the cover $\mathcal{V}_i$ (see [GT], Theorem 1.5.18). By virtue of Theorem 4.1.13 for every $s \in S_i$ there exists a partition L_s between F_s and $X \setminus V_s$ such that $\text{Ind}(L_s \cap M) \leq n-1$. Consequently there exist open sets $U_s, W_s \subset X$ such that

$$F_s \subset U_s, \quad X \setminus V_s \subset W_s, \quad U_s \cap W_s = \emptyset \quad \text{and} \quad X \setminus L_s = U_s \cup W_s.$$

As $\text{Fr } U_s \subset L_s$, $\text{Ind}(M \cap \text{Fr } U_s) \leq n-1$, and as $U_s \subset X \setminus W_s \subset V_s$, $\delta(U_s) < 1/i$ for $s \in S_i$. The family $\mathcal{B}_i = \{U_s\}_{s \in S_i}$ is for $i = 1, 2, \dots$ a locally

finite open cover of the space X such that $\text{mesh } \mathcal{B}_i \leq 1/i$, so that the union $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a σ -locally finite base for the space X such that $\text{Ind}(M \cap \text{Fr } U) \leq n-1$ for every $U \in \mathcal{B}$.

Conversely, if M is a subspace of a metrizable space X and X has a σ -locally finite base $\mathcal{B}$ such that $\text{Ind}(M \cap \text{Fr } U) \leq n-1$ for every $U \in \mathcal{B}$, then the family $\{M \cap U: U \in \mathcal{B}\}$ is a σ -locally finite base for the subspace M whose members have boundaries of large inductive dimension $\leq n-1$; by virtue of Lemma 2.3.16 and Theorems 4.1.9, 4.1.10 and 4.1.7, this implies that $\text{Ind } M \leq n$. $\square$

The next theorem (cf. Theorem 1.1.6) is a particular case of Proposition 4.1.14.

4.1.15. Theorem. *A metrizable space X satisfies the inequality $\text{Ind } X \leq n \geq 0$ if and only if X has a σ -locally finite base $\mathcal{B}$ such that $\text{Ind } \text{Fr } U \leq n-1$ for every $U \in \mathcal{B}$. $\square$*

Let us note that in Proposition 4.1.14 and Theorem 4.1.15 the σ -local finiteness can be replaced by σ -discreteness (cf. [GT], Theorem 4.4.1).

4.1.16. The first decomposition theorem. *A metrizable space X satisfies the inequality $\text{Ind } X \leq n \geq 0$ if and only if X can be represented as the union of two subspaces Y and Z such that $\text{Ind } Y \leq n-1$ and $\text{Ind } Z \leq 0$.*

Proof. Consider a metrizable space X such that $\text{Ind } X \leq n \geq 0$. By virtue of Theorem 4.1.15, the space X has a σ -locally finite base $\mathcal{B}$ such that $\text{Ind } \text{Fr } U \leq n-1$ for every $U \in \mathcal{B}$. From Theorems 4.1.9 and 4.1.10 it follows that the subspace $Y = \bigcup \{\text{Fr } U: U \in \mathcal{B}\}$ satisfies the inequality $\text{Ind } Y \leq n-1$, and from Proposition 4.1.14 it follows that the subspace $Z = X \setminus Y$ satisfies the inequality $\text{Ind } Z \leq 0$.

If X is a metrizable space and $X = Y \cup Z$, where $\text{Ind } Y \leq n-1$ and $\text{Ind } Z \leq 0$, then $\text{Ind } X \leq n$ by virtue of Theorems 4.1.13 and 4.1.7. $\square$

From the first decomposition theorem we obtain by easy induction

4.1.17. The second decomposition theorem. *A metrizable space X satisfies the inequality $\text{Ind } X \leq n \geq 0$ if and only if X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \dots, Z_{n+1}$ such that $\text{Ind } Z_i \leq 0$ for $i = 1, 2, \dots, n+1$. $\square$*

From Theorem 4.1.17 immediately follows the addition theorem, which is a particular case of Theorems 2.2.5 and 3.1.17.

4.1.18. The addition theorem. *For every pair X, Y of subspaces of a metrizable space we have*

$$\text{Ind}(X \cup Y) \leq \text{Ind}X + \text{Ind}Y + 1. \quad \square$$

We now turn to the enlargement theorem.

4.1.19. The enlargement theorem. *For every subspace M of a metrizable space X satisfying the inequality $\text{Ind}M \leq n$ there exists a G_δ -set M^* in X such that $M \subset M^*$ and $\text{Ind}M^* \leq n$.*

Proof. Consider first the particular case of a subspace Z of X such that $\text{Ind}Z \leq 0$. By virtue of Proposition 4.1.14 the space X has a σ -locally finite base $\mathcal{B}$ such that $Z \cap \text{Fr}U = \emptyset$ for every $U \in \mathcal{B}$. The union $F = \bigcup \{\text{Fr}U : U \in \mathcal{B}\}$ is an F_σ -set, and its complement $Z^* = X \setminus F$ is a G_δ -set which contains the set Z . From Proposition 4.1.14 it follows that $\text{Ind}Z^* \leq 0$.

To complete the proof it suffices to use Theorem 4.1.17 and apply the particular case established above (cf. the proof of Theorem 1.5.11). $\square$

Since every compact metrizable space is separable, no non-separable metrizable space has a metrizable compactification. The next theorem is a substitute for the compactification theorem in the realm of metrizable spaces; it follows from Theorem 4.1.19, Lemma 1.3.12, and the fact that each metrizable space is homeomorphic to a subspace of a completely metrizable space (see [GT], Corollary 4.3.15).

4.1.20. The completion theorem. *For every metrizable space X there exists a completely metrizable space $\tilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the equalities $\text{Ind}\tilde{X} = \text{Ind}X$ and $w(\tilde{X}) = w(X)$. $\square$*

We now pass to the Cartesian product theorem. It is a particular case of Theorem 2.4.6; the proof of this particular case is much simpler than the proof given in Section 2.4.

4.1.21. The Cartesian product theorem. *For every pair X, Y of metrizable spaces of which at least one is non-empty we have*

$$\text{Ind}(X \times Y) \leq \text{Ind}X + \text{Ind}Y.$$

Proof. The theorem is obvious if $\text{Ind}X = \infty$ or $\text{Ind}Y = \infty$, so that we can suppose that $k(X, Y) = \text{Ind}X + \text{Ind}Y$ is finite. We shall apply induction

with respect to that number. If $k(X, Y) = -1$, then either $X = \emptyset$ or $Y = \emptyset$, and our inequality holds. Assume that the inequality is proved for every pair of metrizable spaces the sum of large inductive dimensions of which is less than $k \geq 0$ and consider metrizable spaces X and Y such that $\text{Ind} X = n \geq 0$, $\text{Ind} Y = m \geq 0$ and $n + m = k$. By virtue of Theorem 4.1.15 the space X has a base $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$, where the families $\mathcal{C}_i$ are locally finite and $\text{IndFr} U \leq n-1$ for every $U \in \mathcal{C}$. Similarly, the space Y has a base $\mathcal{D} = \bigcup_{j=1}^{\infty} \mathcal{D}_j$, where the families $\mathcal{D}_j$ are locally finite and $\text{IndFr} V \leq m-1$ for every $V \in \mathcal{D}$. For $i, j = 1, 2, \dots$ the family

$$\mathcal{B}_{i,j} = \{U \times V: U \in \mathcal{C}_i \text{ and } V \in \mathcal{D}_j\}$$

consists of open subsets of the Cartesian product $X \times Y$ and is locally finite. Since

$$\text{Fr}(U \times V) \subset (X \times \text{Fr} V) \cup (\text{Fr} U \times Y),$$

by virtue of the inductive assumption and Theorem 4.1.9 we have $\text{IndFr}(U \times V) \leq k-1$. The family $\{\mathcal{B}_{i,j}\}_{i,j=1}^{\infty}$ is a base for the Cartesian product $X \times Y$. From Theorem 4.1.15 it follows that $\text{Ind}(X \times Y) \leq k$ and the proof is completed. $\square$

We shall now prove the theorem on dimension of the limit of an inverse sequence of metrizable spaces. In consideration of the context of inverse systems and the character of the proof given below, the theorem is formulated in terms of the dimension $\dim$.

4.1.22. Theorem on dimension of the limit of an inverse sequence. *If the inverse sequence $S = \{X_i, \pi_j^i\}$ consists of metrizable spaces X_i such that $\dim X_i \leq n$ for $i = 1, 2, \dots$, then the limit $X = \varprojlim S$ satisfies the inequality $\dim X \leq n$.*

Proof. For $i = 1, 2, \dots$ consider a metric ϱ_i on the space X_i bounded by 1. On the Cartesian product $\prod_{i=1}^{\infty} X_i$ and on its subspace X we shall consider the metric ϱ defined by letting

$$(7) \quad \varrho(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \varrho_i(x_i, y_i) \quad \text{for } \{x_i\}, \{y_i\} \in \prod_{i=1}^{\infty} X_i.$$

For $i, k = 1, 2, \dots$ let $\mathcal{U}_{i,k}$ be an open cover of the space X_i such that $\text{mesh } \mathcal{U}_{i,k} < 1/2k$. We shall define inductively a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of families of sets satisfying the conditions:

- (8) The family $\mathcal{U}_i$ is an open cover of the space X_i and $\text{ord } \mathcal{U}_i \leq n$.
- (9) For each $U \in \mathcal{U}_i$, where $i > 1$, there exists a $V \in \mathcal{U}_{i-1}$ such that $U \subset (\pi_{i-1}^i)^{-1}(V)$.
- (10) For every $j \leq i$ $\text{mesh}(\{\pi_j^i(U) : U \in \mathcal{U}_i\}) < 1/2i$.

Conditions (8)–(10) are satisfied for $i = 1$ by an arbitrary open refinement $\mathcal{U}_1$ of the cover $\mathcal{U}_{1,1}$ of the space X_1 such that $\text{ord } \mathcal{U}_1 \leq n$; such a refinement exists by virtue of Proposition 3.2.2. Assume that the families $\mathcal{U}_i$ satisfying (8)–(10) are defined for all $i < k > 1$. Let $\mathcal{U}_k$ be an arbitrary open refinement of the cover

$$[(\pi_{k-1}^k)^{-1}(\mathcal{U}_{k-1})] \wedge [(\pi_1^k)^{-1}(\mathcal{U}_{1,k})] \wedge [(\pi_2^k)^{-1}(\mathcal{U}_{2,k})] \wedge \dots \wedge [(\pi_k^k)^{-1}(\mathcal{U}_{k,k})]$$

of the space X_k such that $\text{ord } \mathcal{U}_k \leq n$. From the definition it easily follows that $\mathcal{U}_k$ satisfies (8)–(10) with $i = k$; thus the construction of the families $\mathcal{U}_i$ is completed.

For $i = 1, 2, \dots$ the family $\mathcal{W}_i = \pi_i^{-1}(\mathcal{U}_i)$, where $\pi_i: X \rightarrow X_i$ denotes the projection, is an open cover of the space X . For every $W \in \mathcal{W}_{i+1}$ there exists a $U \in \mathcal{U}_{i+1}$ such that $W = \pi_{i+1}^{-1}(U)$; by virtue of (9) one can find a $V \in \mathcal{U}_i$ such that $U \subset (\pi_i^{i+1})^{-1}(V)$. Thus $W \subset \pi_{i+1}^{-1}(\pi_i^{i+1})^{-1}(V) = (\pi_i^{i+1}\pi_{i+1})^{-1}(V) = \pi_i^{-1}(V) \in \mathcal{W}_i$, which shows that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$. From (10) it follows that $\delta(W) < 1/2i + 1/2^i \leq 1/i$ for every $W \in \mathcal{W}_i$, so that $\dim X \leq n$ by virtue of Proposition 4.1.2. $\square$

We conclude this section by proving that for every cardinal number $m \geq \aleph_0$ the Baire space $B(m)$ defined below is a universal space for the class of all metrizable spaces whose large inductive dimension is not larger than 0 and whose weight is not larger than m . In the following section this result will be extended to higher dimensions; as the reader will see, the proof of the general theorem is much more difficult than that of the particular case discussed here.

We start with the definition of the Baire space.

4.1.23. Example. For $i = 1, 2, \dots$ let $X_i = D(m)$ be the discrete space of cardinality $m \geq \aleph_0$ with the metric ϱ_i defined by

$$\varrho_i(x, y) = 1 \quad \text{if } x \neq y \quad \text{and} \quad \varrho_i(x, x) = 0.$$

The Cartesian product $\prod_{i=1}^{\infty} X_i = [D(m)]^{\aleph_0}$ is a metrizable space; it is well known that formula (7) defines a metric ϱ on that space. One can readily verify that by letting

$$(11) \quad \sigma(\{x_i\}, \{y_i\}) = \begin{cases} 1/k, & \text{if } x_k \neq y_k \text{ and } x_i = y_i \text{ for } i < k, \\ 0, & \text{if } x_i = y_i \text{ for } i = 1, 2, \dots \end{cases}$$

one defines another metric on the set $\prod_{i=1}^{\infty} X_i$. The metrics ϱ and σ are equivalent.

Indeed, a sequence $\{x_i^1\}, \{x_i^2\}, \dots$ in the Cartesian product $\prod_{i=1}^{\infty} X_i$ converges to a point $\{x_i\}$ if and only if for every i there exists a $k(i)$ such that $x_i^j = x_i$ whenever $j \geq k(i)$, and the same condition is necessary and sufficient for the convergence of the sequence $\{x_i^1\}, \{x_i^2\}, \dots$ to the point $\{x_i\}$ with respect to the metric σ defined by (11).

The Cartesian product $[D(m)]^{\aleph_0}$ with the metric σ defined in (11) is called the *Baire space of weight m* and is denoted by $B(m)$. The reader can easily check that the weight of the space $B(m)$ is really equal to m . Let us note that by virtue of Proposition 1.3.13 the Baire space $B(\aleph_0)$ is homeomorphic to the space of irrational numbers.

We shall show that $\text{Ind } B(m) = 0$. Consider a pair $x = \{x_i\}$, $y = \{y_i\}$ of points of $B(m)$ and a real number r satisfying $0 < r \leq 1$. If the intersection $B(x, r) \cap B(y, r)$ is non-empty, then there exists a point $z = \{z_i\} \in B(m)$ such that $x_1 = z_1 = y_1$, $x_2 = z_2 = y_2$, $\dots$, $x_k = z_k = y_k$, where k is the integer satisfying $1/k + 1 < r \leq 1/k$; thus we have $B(x, r) = B(y, r)$. Hence in $B(m)$ two r -balls either are disjoint or coincide. In particular, for $i = 1, 2, \dots$ the family $\mathcal{B}_i = \{B(x, 1/i) : x \in B(m)\}$ is an open cover of $B(m)$ which consists of pairwise disjoint sets. The union $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a σ -locally finite base for $B(m)$ which consists of open-and-closed sets, so that $\text{Ind } B(m) \leq 0$ by virtue of Theorem 4.1.15. $\square$

4.1.24. Theorem. *For every cardinal number $m \geq \aleph_0$ the Baire space $B(m)$ is a universal space for the class of all metrizable spaces whose large inductive dimension is not larger than 0 and whose weight is not larger than m .*

Proof. By virtue of the last example it suffices to show that every metrizable space X such that $\text{Ind } X = 0$ and $w(X) = m$ is embeddable in X .

Let ϱ be an arbitrary metric on the space X and $\mathcal{B}$ an arbitrary base for X such that $|\mathcal{B}| = \mathfrak{m}$. From Theorem 1.6.11 (or 4.1.3) and Proposition 3.2.2 it follows that the cover $\{U \in \mathcal{B} : \delta(U) < 1/i\}$ of the space X has an open shrinking $\mathcal{B}_i$ consisting of pairwise disjoint sets. Adjoining to $\mathcal{B}_i$, if necessary, an appropriate number of copies of the empty set, we can assume that $\mathcal{B}_i = \{U_{i,s}\}_{s \in X_i}$, where $X_i = D(\mathfrak{m})$ is the discrete space of cardinality $\mathfrak{m}$ used in Example 4.1.23 to define the space $B(\mathfrak{m})$.

By assigning to each point $x \in X$ the element $s \in X_i$ such that $x \in U_{i,s}$ we define a continuous mapping $f_i: X \rightarrow X_i$. For every $x \in X$ and every closed set $F \subset X$ such that $x \notin F$ there exists a natural number i such that $\varrho(x, F) > 1/i$. The set $U_{i,s}$ that contains the point x is disjoint from F , so that $f_i(x) = s \notin f_i(F) = \overline{f_i(F)}$. Thus the family $\{f_i\}_{i=1}^\infty$ separates points and closed sets, which implies that the mapping $F: X \rightarrow \prod_{i=1}^\infty X_i = B(\mathfrak{m})$ defined by letting $F(x) = (f_1(x), f_2(x), \dots)$ is a homeomorphic embedding (see [GT], Theorem 2.3.20). $\square$

Since the Cartesian product of $\aleph_0$ copies of $B(\mathfrak{m})$ is homeomorphic to $B(\mathfrak{m})$, Theorem 4.1.24 yields the following

4.1.25. Theorem. *The Cartesian product $X = \prod_{i=1}^\infty X_i$ of a countable family $\{X_i\}_{i=1}^\infty$ of metrizable spaces satisfies the equality $\text{Ind} X = 0$ if and only if $\text{Ind} X_i = 0$ for $i = 1, 2, \dots$ $\square$*

Historical and bibliographic notes

Lemma 4.1.1 was established by Nagami and Roberts in [1967]. The equivalence of conditions (a) and (b) in Proposition 4.1.2 was proved by Dowker and Hurewicz in [1956]; the equivalence of conditions (a) and (c) was proved by Vopěnka in [1959]. Theorem 4.1.3 was established independently by Katětov in [1952] (announcement in [1951]) and by Morita in [1954]. Theorem 4.1.4 was given by Morita in [1950a]. References concerning Roy's space cited in Remark 4.1.6 are given in the notes to Section 2.4. Theorems 4.1.7–4.1.11, 4.1.13 and 4.1.18 are special cases of the theorems established in Chapters 2 and 3. Theorem 4.1.12 was given by Nagami in [1957]. Theorems 4.1.14 and 4.1.15 were proved by Morita in [1954]. All the remaining theorems in this section, except for Theorem 4.1.22, which was proved by Nagami in [1959], were established independently by Katětov in [1952] and by Morita in [1954].

Problems

4.1.A (Engelking [1973], Przymusiński [1974]). Let X be a metrizable space and ϱ a metric on the space X ; let $\text{ds}(X, \varrho) \leq n$ denote that the space X has a sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ with the properties stated in condition (b) in Proposition 4.1.2.

(a) Show by modifying the proof of Theorem 4.1.3, that if $\text{ds}(X, \varrho) \leq n \geq 0$ then for every pair A, B of closed subsets of X satisfying $\varrho(A, B) > 0$ there exists an open set $U \subset X$ such that $A \subset U \subset X \setminus B$ and $\text{ds}(\text{Fr } U, \varrho) \leq n-1$.

(b) Apply (a) to prove the Katětov–Morita theorem without using Lemma 4.1.1.

Hint. Observe first that if $\dim X \leq n$, then $\text{ds}(X, \varrho) \leq n$ for every metric ϱ on the space X ; then apply (a) to prove by induction that the inequality $\text{ds}(X, \varrho) \leq n \geq 0$ implies that X has a σ -locally finite base $\mathcal{B}$ such that $\dim \text{Fr } U \leq n-1$ for every $U \in \mathcal{B}$ and show that the existence of such a base implies the inequality $\text{Ind } X \leq n$.

One can also apply (a) to prove by induction that if $\text{ds}(X, \varrho) \leq n \geq 0$ then, for every closed set $A \subset X$ and each open set $V \subset X$ that contains the set A , there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\text{Ind } \text{Fr } U \leq n-1$. To this end, for $i = 1, 2, \dots$ define $A_i = B(A, 1/i)$ and $A'_i = B(X \setminus V, 1/i)$, consider open sets $W_i, W'_i \subset X$ such that $A_{i+1} \subset W_i \subset A_i$, $\text{ds}(\text{Fr } W_i, \varrho) \leq n-1$ and $A'_{i+1} \subset W'_i \subset A'_i$, $\text{ds}(\text{Fr } W'_i, \varrho) \leq n-1$, and let $U = \bigcup_{i=1}^{\infty} (W_i \setminus \overline{W'_i})$.

4.1.B (van Douwen [1973], Przymusiński [1974]). Applying the existence of a metrizable space X with the properties described in Remark 4.1.6, define a metrizable space Y with $\text{ind } Y = 1$ which can be represented as the union of two closed subspaces Y_1 and Y_2 such that $\text{ind } Y_1 = \text{ind } Y_2 = 0$ and which contains a point p such that $\text{ind}(Y \setminus \{p\}) = 0$.

Hint. Consider a pair A, B of disjoint closed subsets of X which cannot be enlarged to disjoint open-and-closed sets, and replace the set B by a point p in such a way as to obtain a metrizable space.

4.1.C. Let Y be the space considered in Problem 4.1.B and let $M = Y \setminus \{p\}$. Show that, though $\text{ind } M = 0$, there exists a neighbourhood V of the point p in the space Y such that for every open set $U \subset Y$ satisfying $p \in U \subset V$ we have $M \cap \text{Fr } U \neq \emptyset$ (cf. Propositions 1.2.12 and 1.5.14).

4.1.D (R. Pol [1979]). Prove that if a metrizable space X can be represented as the union of a family $\{F_s\}_{s \in S}$ of closed subspaces such that $\text{Ind} F_s \leq n$ for $s \in S$ and if there exists a point-countable open cover $\{U_s\}_{s \in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then $\text{Ind} X \leq n$.

Hint (Hansell [1974]). Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ be a base for the space X , where the families $\mathcal{B}_i$ are locally finite. For every non-empty $U \in \mathcal{B}$ consider a one-to-one transformation j_U of the set $\{s \in S: U \subset U_s\}$ to the integers and let $U_{s,i,j} = \bigcup \{U \in \mathcal{B}_i: U \subset U_s \text{ and } j_U(s) = j\}$. Show that the family $\{U_{s,i,j}\}_{s \in S}$ is locally finite for $i, j = 1, 2, \dots$ and $U_s = \bigcup_{i,j=1}^{\infty} U_{s,i,j}$ for $s \in S$.

4.1.E. Deduce from Theorems 4.1.3 and 4.1.13 and Remark 1.7.10 that a metrizable space X satisfies the inequality $\text{Ind} X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \dots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \dots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

4.1.F (Levšenko [1969]; for $n = 0$ Levšenko and Smirnov [1966]; for separable spaces implicitly Popruženko [1931]). (a) Prove that for every non-empty closed subset A of a metrizable space X satisfying the inequality $\text{Ind} X \leq n \geq 0$ there exists a closed subset B of the subspace $X \setminus A$ of X such that $\text{Ind} B \leq n-1$ and A is a retract of $X \setminus B$.

Hint. Consider a metric ρ on the space X and define a sequence $U_1, U_2, \dots$ of open subsets of X such that $A \subset U_i \subset B(A, 1/i)$, $\bar{U}_i \subset U_{i-1}$, where $U_0 = X$, and $\text{Ind Fr } U_i \leq n-1$ for $i = 1, 2, \dots$; consider for $i = 1, 2, \dots$ a locally finite open cover $\mathcal{U}_i$ of the subspace $\bar{U}_{i-1} \setminus U_i$ of the space X such that $\text{mesh } \mathcal{U}_i \leq 1/i$ and $\text{Ind Fr } U \leq n-1$ for every $U \in \mathcal{U}_i$. Define $B = \bigcup_{i=1}^{\infty} \text{Fr } U_i \cup \bigcup_{i=1}^{\infty} \{U \in \mathcal{U}_i\}$ and see the hint to Problem 1.3.C(a).

(b) Show that if a metrizable space X has the property that for every non-empty closed set $A \subset X$ there exists a closed subset B of the subspace $X \setminus A$ of X such that $\text{ind} B \leq n-1$ and A is a retract of $X \setminus B$, then $\text{Ind} X \leq n$.

4.1.G (Hausdorff [1934], de Groot [1956]). A metric ρ on a set X is called *non-Archimedean* if $\rho(x, z) \leq \max[\rho(x, y), \rho(y, z)]$ for all $x, y, z \in X$.

Let X be a metrizable space; show that on the space X there exists a non-Archimedean metric if and only if $\text{Ind}X \leq 0$.

Hint. There exists a non-Archimedean metric on the Baire space $B(\mathfrak{m})$.

4.2. Characterizations of dimension in metrizable spaces. The universal space theorem

In the first part of this section we shall establish four characterizations of the dimension $\dim$ in metrizable spaces formulated in terms of the existence of special bases and sequences of covers, and we shall review characterizations of $\dim$ in terms of special metrics.

We start with characterizations in terms of σ -locally finite bases with special properties. First we shall prove a lemma related to Theorem 3.2.5.

4.2.1. Lemma. *If a normal space X satisfies the inequality $\dim X \leq n \geq 0$, then for every σ -locally finite family $\{U_s\}_{s \in S}$ of open subsets of X and every family $\{F_s\}_{s \in S}$ of closed subsets of X such that $F_s \subset U_s$ for $s \in S$ there exists a family $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s \subset \bar{V}_s \subset U_s$ for $s \in S$ and $\text{ord}(\{\text{Fr } V_s\}_{s \in S}) \leq n-1$.*

Proof. Let $S = \bigcup_{i=1}^{\infty} S_i$, where $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and the family $\{U_s\}_{s \in S_i}$ is locally finite for $i = 1, 2, \dots$. Applying Theorem 3.2.5, one can easily define by induction open sets $V_{s,i}, W_{s,i} \subset X$, where $s \in T_i = S_1 \cup S_2 \cup \dots \cup S_i$ and $i = 1, 2, \dots$, satisfying the following conditions:

- (1) $F_s \subset V_{s,i} \subset \bar{V}_{s,i} \subset W_{s,i} \subset \bar{W}_{s,i} \subset U_s$ for $s \in S_i$.
- (2) $\bar{V}_{s,i-1} \subset V_{s,i} \subset \bar{V}_{s,i} \subset W_{s,i} \subset \bar{W}_{s,i} \subset W_{s,i-1}$ for $s \in T_{i-1}$ and $i > 1$.
- (3) $\text{ord}(\{\bar{W}_{s,i} \setminus V_{s,i}\}_{s \in T_i}) \leq n-1$.

For every $s \in S$ consider the open set

$$V_s = \bigcup_{j=i}^{\infty} V_{s,j}, \quad \text{where } s \in S_i.$$

Conditions (1) and (2) imply that $F_s \subset V_s \subset \bar{V}_s \subset U_s$ for $s \in S$ and that $\text{Fr } V_s = \bar{V}_s \setminus V_s \subset \bar{W}_{s,j} \setminus V_{s,j}$ for $j \geq i$, where $s \in S_i$. The last inclusion together with (3) yield the inequality $\text{ord}(\{\text{Fr } V_s\}_{s \in S}) \leq n-1$. $\square$

4.2.2. Theorem. *For every metrizable space X and each integer $n \geq 0$ the following conditions are equivalent:*

- (a) The space X satisfies the inequality $\dim X \leq n$.
 (b) The space X has a σ -locally finite base $\mathcal{B}$ such that $\text{ord}(\{\text{Fr } U : U \in \mathcal{B}\}) \leq n-1$.
 (c) The space X has a σ -locally finite base $\mathcal{B}$ such that $\dim \text{Fr } U \leq n-1$ for every $U \in \mathcal{B}$.

Proof. We shall show first that (a) $\Rightarrow$ (b). Consider a metrizable space X such that $\dim X \leq n$; let ϱ be a metric on the space X . The space X being paracompact, for $i = 1, 2, \dots$ there exists a locally finite open cover $\mathcal{U}_i = \{U_s\}_{s \in S_i}$ of the space X such that $\text{mesh } \mathcal{U}_i < 1/i$; obviously, one can assume that $S_i \cap S_j = \emptyset$ whenever $i \neq j$. Let $\{F_s\}_{s \in S_i}$ be a closed shrinking of the cover $\mathcal{U}_i$ and let $S = \bigcup_{i=1}^{\infty} S_i$. Applying Lemma 4.2.1 to the families $\{U_s\}_{s \in S}$ and $\{F_s\}_{s \in S}$ we obtain a family $\mathcal{B} = \{V_s\}_{s \in S}$ which has the properties stated in (b).

The implication (c) $\Rightarrow$ (a) follows from Lemmas 2.3.16 and 3.1.27 and the sum theorems for $\dim$.

To prove that (b) $\Rightarrow$ (c) we shall apply induction with respect to n . Condition (b) and (c) are equivalent if $n = 0$, because then they both mean that all members of $\mathcal{B}$ are open-and-closed. Assume that the implication (b) $\Rightarrow$ (c) is proved for all metrizable spaces and every $n < m$ and consider a metrizable space X which has a σ -locally finite base $\mathcal{B}$ such that $\text{ord}(\{\text{Fr } U : U \in \mathcal{B}\}) \leq m$. For every $U_0 \in \mathcal{B}$ the family $\mathcal{B}_0 = \{X_0 \cap U : U \in \mathcal{B}\}$, where $X_0 = \text{Fr } U_0$, is a σ -locally finite base for the space X_0 . Since the family of boundaries of members of $\mathcal{B}_0$ in the space X_0 has order $\leq m-1$, it follows from the inductive assumption and implication (c) $\Rightarrow$ (a) that $\dim X_0 \leq m$. Thus the space X satisfies (c) with $n = m+1$ and the proof that (b) $\Rightarrow$ (c) is completed. $\square$

Let us note that the equivalence of conditions (a) and (c) in Theorem 4.2.2 follows immediately from Theorems 4.1.3 and 4.1.15. In our proof, however, we applied Lemma 3.1.27, rather than those two theorems, in order to prepare the ground for another proof of the Katětov–Morita theorem (cf. Problem 4.2.A(a)).

We shall now introduce two topological notions which are applied in the next theorem. A sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of covers of a topological space X is called a *development* for the space X if all covers $\mathcal{W}_i$ are open and for every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that $\text{St}(x, \mathcal{W}_i) \subset U$. One easily observes that a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open covers of a topological space X is

a development for X if and only if for every point $x \in X$ each family $\{W_i\}_{i=1}^{\infty}$ of open subsets of X such that $x \in W_i \in \mathcal{W}_i$ for $i = 1, 2, \dots$ is a base for X at the point x . A sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of covers of a topological space X is called a *strong development* for the space X if all covers $\mathcal{W}_i$ are open and for every point $x \in X$ and each neighbourhood U of the point x there exist a neighbourhood V of the point x and a natural number i such that $\text{St}(V, \mathcal{W}_i) \subset U$. Clearly, every strong development is a development.

4.2.3. Theorem. *For every metrizable space X the following conditions are equivalent:*

- (a) *The space X satisfies the inequality $\dim X \leq n$.*
- (b) *The space X has a development $\mathcal{W}_1, \mathcal{W}_2, \dots$ such that $\text{ord } \mathcal{W}_i \leq n$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \dots$*
- (c) *The space X has a strong development $\mathcal{W}_1, \mathcal{W}_2, \dots$ such that $\text{ord } \mathcal{W}_i \leq n$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \dots$*

Proof. We begin with the implication (a) $\Rightarrow$ (b). Consider a metrizable space X such that $\dim X \leq n$. We shall define inductively a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open covers of X which has the properties stated in (b). Assume that $k = 1$ or that $k > 1$ and the covers $\mathcal{W}_i$ are defined for all $i < k$. The space X being paracompact, the open cover $\mathcal{W}_{k-1} \wedge \{B(x, 1/k)\}_{x \in X}$ of the space X , where $\mathcal{W}_0 = \{X\}$, has an open star refinement (see [GT], Theorem 5.1.12) which by virtue of Proposition 3.2.2 has in turn an open shrinking $\mathcal{W}_k$ such that $\text{ord } \mathcal{W}_k \leq n$. The sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ thus obtained has the required properties, so that (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) follows from the fact that if $\text{St}(x, \mathcal{W}_i) \subset U$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ then $\text{St}(V, \mathcal{W}_{i+1}) \subset U$ for any $V \in \mathcal{W}_{i+1}$ such that $x \in V$.

Finally, the implication (c) $\Rightarrow$ (a) follows from Lemma 4.1.1, because—as one readily sees by applying the definition twice—if $\mathcal{W}_1, \mathcal{W}_2, \dots$ is a strong development for a space X and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \dots$, then the family $\{\text{St}(W, \mathcal{W}_i) : W \in \mathcal{W}_i, i = 1, 2, \dots\}$ is a base for X . $\square$

Let us note that by virtue of the Nagata–Smirnov theorem (see [GT], Theorem 4.4.7), Theorem 4.2.2 holds for every regular space X if condition (a) is replaced by the following condition:

- (a') *The space X is metrizable and satisfies the inequality $\dim X \leq n$.*

Similarly, by virtue of appropriate metrization theorems (see [GT], Corollary 5.4.10 and Theorem 5.4.2), Theorem 4.2.3 holds for every T_0 -space X if condition (a) is replaced by condition (a').

As follows from Problem 4.1.G, the class of all metrizable spaces X such that $\dim X \leq 0$ can be characterized in terms of the existence of special metrics. In this connection it is natural to ask whether the class of all metrizable spaces X such that $\dim X \leq n$ can be characterized in a similar manner. Investigations in this direction led to a group of interesting theorems, which are reviewed below; the proofs of those theorems are too difficult to be reproduced in this book. Thus, a metrizable space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if on the space X there exists a metric ϱ which satisfies any of the following conditions:

- (i) For every point $x \in X$ and each positive number r we have $\dim \text{Fr } B(x, r) \leq n-1$; moreover, $\bigcup_{x \in X_0} B(x, r) = \bigcup_{x \in X_0} \overline{B(x, r)}$ for every $X_0 \subset X$.
- (ii) For every closed set $A \subset X$ and each positive number r we have $\dim \text{Fr } B(A, r) \leq n-1$.
- (iii) For every point $x \in X$, each positive number r and every sequence $y_1, y_2, \dots, y_{n+2}$ of $n+2$ points of X satisfying the inequality $\varrho(y_i, B(x, r/2)) < r$ for $i = 1, 2, \dots, n+2$, there exist natural numbers $i, j \leq n+2$ such that $i \neq j$ and $\varrho(y_i, y_j) < r$.
- (iv) For every point $x \in X$ and every sequence $y_1, y_2, \dots, y_{n+2}$ of $n+2$ points of X there exist natural numbers $i, j \leq n+2$ such that $i \neq j$ and $\varrho(y_i, y_j) \leq \varrho(x, y_i)$.

One easily verifies that every metric ϱ which satisfies (i) satisfies also (ii), and that if on a space X there exists a metric ϱ which satisfies (ii), then $\dim X \leq n$ (see Problem 4.2.D). Let us note that a separable metrizable space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if on the space X there exists a metric ϱ which satisfies the first part of (i); it is not known whether the second part of (i) can be omitted also in the case of an arbitrary metrizable space.

An interesting problem is connected with conditions (iii) and (iv), namely it is not known whether the inequality $\dim X \leq n$ follows from the existence, on the space X , of a metric ϱ which satisfies the following condition

- (v) For every point $x \in X$, each positive number r and every sequence $y_1, y_2, \dots, y_{n+2}$ of $n+2$ points of X satisfying the inequality $\varrho(x, y_i) < r$ for $i = 1, 2, \dots, n+2$, there exist natural numbers $i, j \leq n+2$ such that $i \neq j$ and $\varrho(y_i, y_j) < r$.

One easily verifies that every metric ϱ which satisfies (iii) satisfies also (v), so that the problem stated above is equivalent to the question whether the existence on a space X of a metric ϱ which satisfies (v) means that $\dim X \leq n$. Similarly, one can readily check that condition (v) is equivalent to the following condition:

- (v') For every point $x \in X$ and every sequence $y_1, y_2, \dots, y_{n+2}$ of $n+2$ points of X there exist natural numbers $i, j, k \leq n+2$ such that $i \neq j$ and $\varrho(y_i, y_j) \leq \varrho(x, y_k)$,

and that every metric ϱ which satisfies (iv) satisfies also (v') and (v). Let us note that if on a separable metrizable space X there exists a totally bounded metric ϱ which satisfies (v), then $\dim X \leq n$ (see Problem 4.2.F); it is not known if the assumption of total boundedness can be omitted.

We now turn to the universal space theorem. As in the case of normal spaces discussed in Section 3.3, we shall deduce this theorem from a factorization theorem. We start with a simple lemma on paracompact spaces.

4.2.4. Lemma. *Every locally finite open cover $\mathcal{U}$ of a paracompact space X has a locally finite open star refinement $\mathcal{V}$ such that $|\mathcal{V}| \leq \max(|\mathcal{U}|, \aleph_0)$.*

Proof. Let $\mathcal{W}$ be an open star refinement of $\mathcal{U}$ (see [GT], Theorem 5.1.12). For every pair $\mathcal{U}_0, \mathcal{U}_1$ of finite subfamilies of the cover $\mathcal{U}$ denote by $\mathcal{W}(\mathcal{U}_0, \mathcal{U}_1)$ the family of all $W \in \mathcal{W}$ such that $\mathcal{U}_0 = \{U \in \mathcal{U} : W \subset U\}$ and $\mathcal{U}_1 = \{U \in \mathcal{U} : \text{St}(W, \mathcal{W}) \subset U\}$. From the local finiteness of $\mathcal{U}$ it follows that all the sets of the form $\bigcup \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1)$ constitute an open cover $\mathcal{V}'$ of the space X such that $|\mathcal{V}'| \leq \max(|\mathcal{U}|, \aleph_0)$. We shall show that $\mathcal{V}'$ is a star refinement of $\mathcal{U}$.

Consider a set $V = \bigcup \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1) \in \mathcal{V}'$. Let U be an arbitrary member of $\mathcal{U}_1$; clearly

$$(4) \quad \text{St}(W, \mathcal{W}) \subset U \quad \text{for every } W \in \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1).$$

For every set $V' = \bigcup \mathcal{W}(\mathcal{U}'_0, \mathcal{U}'_1) \in \mathcal{V}'$ which intersects V there exist $W \in \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1)$ and $W' \in \mathcal{W}(\mathcal{U}'_0, \mathcal{U}'_1)$ such that $W \cap W' \neq \emptyset$. From

(4) it follows that $W' \subset U$, so that $U \in \mathcal{U}'_0$, which implies that $V' \subset U$. Hence, $\text{St}(V, \mathcal{V}') \subset U$.

To complete the proof it suffices to consider a locally finite open shrinking $\mathcal{V}$ of the cover $\mathcal{V}'$. $\square$

4.2.5. Pasynkov's factorization theorem. *For every continuous mapping $f: X \rightarrow Y$ of a metrizable space X to a metrizable space Y there exist a metrizable space Z and continuous mappings $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.*

Proof. If $\dim X = \infty$ or $w(Y) < \aleph_0$, then $Z = f(X)$, $g = f$ and $h = \text{id}_Z$ satisfy the theorem. Thus one can suppose that $\dim X = n < \infty$ and $w(Y) = m \geq \aleph_0$. For $i = 1, 2, \dots$ consider a locally finite open cover $\mathcal{U}_i$ of the space Y such that $\text{mesh } \mathcal{U}_i < 1/i$ and $|\mathcal{U}_i| \leq m$. Applying Lemma 4.2.4 and Proposition 3.2.2 one can easily define by induction a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of locally finite open covers of the space X such that for $i = 1, 2, \dots$

$$(5) \quad \text{ord } \mathcal{V}_i \leq n, \quad |\mathcal{V}_i| \leq m$$

and

$$(6) \quad \mathcal{V}_{i+1} \text{ is a star refinement of } \mathcal{V}_i \wedge f^{-1}(\mathcal{U}_{i+1}).$$

We shall now consider another topology on the set X , coarser than the original one, which is defined by declaring that a set $U \subset X$ is open if for every $x \in U$ there exists a natural number i such that $\text{St}(x, \mathcal{V}_i) \subset U$. The set X with this new topology will be denoted by X' ; let us note that generally X' is not a T_0 -space. We shall show that for every $A \subset X'$ the interior of the set A in the space X' coincides with the set

$$A^* = \{x \in X: \text{there exists an } i \text{ such that } \text{St}(x, \mathcal{V}_i) \subset A\}.$$

Obviously, it suffices to verify that A^* is an open subset of X' . Consider an arbitrary point $x \in A^*$ and an i such that $\text{St}(x, \mathcal{V}_i) \subset A$. For every point $y \in \text{St}(x, \mathcal{V}_{i+1})$ we have $\text{St}(y, \mathcal{V}_{i+1}) \subset \text{St}(x, \mathcal{V}_i) \subset A$, so that $y \in A^*$. Thus $\text{St}(x, \mathcal{V}_{i+1}) \subset A^*$, which shows that A^* is an open subset of X' .

Let $\mathcal{V}_i^* = \{V^*: V \in \mathcal{V}_i\}$ for $i = 1, 2, \dots$. As for every $W \in \mathcal{V}_{i+1}$ there exists a $V \in \mathcal{V}_i$ such that $\text{St}(W, \mathcal{V}_{i+1}) \subset V$ and consequently $W \subset V^* \in \mathcal{V}_i^*$, the family $\mathcal{V}_i^*$ is an open cover of the space X' . Since A^* is the interior of A in X' , from (6) it follows that for $i = 1, 2, \dots$

(7) $\mathcal{V}_{i+1}^*$ is a star refinement of $\mathcal{V}_i^*$.

Note that by letting $f'(x) = f(x)$ for $x \in X'$ we define a continuous mapping $f': X' \rightarrow Y$; indeed, it follows from (6) that $\delta(f'(V^*)) < 1/i+1$ for every $V^* \in \mathcal{V}_{i+1}^*$.

For $x, y \in X'$ define

xEy if and only if for every i there exists a $V \in \mathcal{V}_i$ such that $x, y \in V$.

One easily checks that E is an equivalence relation on the space X' . Clearly

$$[x] = \bigcap_{i=1}^{\infty} \text{St}(x, \mathcal{V}_i) \quad \text{for } x \in X',$$

where $[x]$ denotes the equivalence class that contains x . Let Z be the quotient space X'/E and $g: X \rightarrow Z$ the composition of the identity mapping $i: X \rightarrow X'$ and the natural quotient mapping $g': X' \rightarrow X'/E$. By virtue of (6), $f'(x) = f'(y)$ whenever xEy , so that by letting $h([x]) = f'(x)$ we define a mapping h of Z to Y ; from the relation $hg' = f'$ it follows that $h: Z \rightarrow Y$ is a continuous mapping. Obviously, $g(X) = Z$ and $f = hg$. Let us note that

$$(8) \quad g'^{-1}g'(A^*) = A^* \quad \text{for every } A \subset X',$$

so that

$$(9) \quad \text{the set } g'(A^*) \text{ is open in } Z \text{ for every } A \subset X'.$$

We shall show that Z is a metrizable space. To begin with, observe that Z is a T_1 -space. Indeed, if $[x] \neq [y]$ then there exists an i such that $y \notin \text{St}(x, \mathcal{V}_i)$, and then the set $g'(\mathcal{V}^*)$, where $V = \text{St}(x, \mathcal{V}_i)$, is a neighbourhood of the point $[x]$ which does not contain the point $[y]$. Let $\mathcal{W}_i = \{g'(V^*): V \in \mathcal{V}_i\}$ for $i = 1, 2, \dots$; from (9) it follows that $\mathcal{W}_i$ is an open cover of the space Z . Now, from the definition of topology on X' it follows that the sequence $\mathcal{V}_1^*, \mathcal{V}_2^*, \dots$ is a development for the space X' , so that by virtue of (8) the sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ is a development for the space Z ; moreover, (7) implies that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \dots$. Hence, the space Z is metrizable (see [GT], Corollary 5.4.10).

From the first part of (5) it follows that $\text{ord } \mathcal{W}_i \leq n$ for $i = 1, 2, \dots$, so that $\dim Z \leq n$ by virtue of Theorem 4.2.3. To complete the proof it suffices to show that $w(Z) \leq m$. The family $\mathcal{B}$ consisting of all sets $\text{St}(z, \mathcal{W}_i)$, where $z \in Z$ and $i = 1, 2, \dots$, is a base for the space Z . Since

the covers $\mathcal{V}_i$ of the space X are all locally finite, it follows that the covers $\mathcal{V}_i^*$ and $\mathcal{W}_i$ are point-finite, so that $|\mathcal{B}| \leq m$. $\square$

Let us note that in the proof of Pasynkov's factorization theorem only the paracompactness of the space X , and not its metrizability, was used; it turns out that the theorem holds under the even weaker assumption of normality of X (see Problem 4.2.G).

4.2.6. The universal space theorem. *For every integer $n \geq 0$ and every cardinal number $m \geq \aleph_0$ there exists a universal space J_m^n for the class of all metrizable spaces whose covering dimension is not larger than n and whose weight is not larger than m .*

Proof. Let $\{X_s\}_{s \in S}$ be the family of all subspaces of the Cartesian product $[J(m)]^{\aleph_0}$ of $\aleph_0$ copies of the hedgehog $J(m)$ (see [GT], Example 4.1.5) whose covering dimension is not larger than n , and let $i_s: X_s \rightarrow [J(m)]^{\aleph_0}$ be the embedding of X_s in $[J(m)]^{\aleph_0}$. Consider the sum $X = \bigoplus_{s \in S} X_s$ and the mapping $i: X \rightarrow [J(m)]^{\aleph_0}$ defined by letting $i(x) = i_s(x)$ for $x \in X_s$. Since $\dim X \leq n$, by virtue of Theorem 4.2.5 there exist a metrizable space J_m^n and continuous mappings $g: X \rightarrow J_m^n$ and $h: J_m^n \rightarrow [J(m)]^{\aleph_0}$ such that $\dim J_m^n \leq \dim X = n$, $w(J_m^n) \leq w([J(m)]^{\aleph_0}) = m$ and $f = hg$.

Consider now an arbitrary metrizable space Y such that $\dim Y \leq n$ and $w(Y) \leq m$. Since Y is embeddable in $[J(m)]^{\aleph_0}$ (see [GT], Theorem 4.4.9), there exists an $s \in S$ such that X_s is homeomorphic to Y . The composition $h_0 g_0$ of the restrictions $g_0 = g|_{X_s}: X_s \rightarrow g(X_s) \subset J_m^n$ and $h_0 = h|_{g(X_s)}: g(X_s) \rightarrow X_s \subset [J(m)]^{\aleph_0}$ is a homeomorphism, so that g_0 is also a homeomorphism. Thus J_m^n is the required universal space. $\square$

Historical and bibliographic notes

Theorem 4.2.2 was established by Morita in [1954]. The equivalence of conditions (a) and (b) in Theorem 4.2.3 was proved by Nagata in [1956a]; the equivalence of conditions (a) and (c) was proved by Nagami and Roberts in [1967]. Characterizations of dimension in terms of metrics satisfying (i) or (ii) were established by Nagata in [1963], the characterization in terms of metrics satisfying (iii) was obtained by Nagata in [1956], and the characterization in terms of metrics satisfying (iv) was obtained independently by Nagata in [1964] and by Ostrand in [1965a]. Let us add

that in [1936] Marczewski proved that a separable metric space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if on the space X there exists a metric ϱ such that for every point $x \in X$ we have $\dim \text{Fr} B(x, r) \leq n-1$ for almost all (in the sense of Lebesgue measure) positive numbers r . The question whether the inequality $\dim X \leq n$ follows from the existence on the space X of a metric ϱ which satisfies (v) was raised by de Groot in [1957]. Theorem 4.2.5, under the weaker assumption of normality of X (see Problem 4.2.G), was proved by Pasynkov in [1967a] (announcement in [1964]); the present proof is obtained by amalgamating the proofs given by Arhangel'skiĭ in [1967] and by Morita in [1975]. Theorem 4.2.5 for a separable Y and an analogous theorem with no evaluation of $w(Z)$ were given by Pasynkov in [1963]. Theorem 4.2.6 was established by Nagata in [1960a]; the present proof was given by Pasynkov in [1964]. In Nagata's original proof of Theorem 4.2.6 a universal space is explicitly defined (cf. Pears [1975]). Let us mention, in connection with Theorem 4.2.6, that in [1975] Lipscomb defined for every cardinal number $m \geq \aleph_0$ a metrizable space $L(m)$ with $\dim L(m) = 1$ and $w(L(m)) = m$ such that each metrizable space X satisfying the inequalities $\dim X \leq n$ and $w(X) \leq m$ is embeddable in the Cartesian product $[L(m)]^{n+1}$, and proved that the Cartesian product $[L(m)]^{n+1}$ contains an easily definable subspace which is a universal space for the class of all metrizable spaces whose covering dimension is not larger than n and whose weight is not larger than m . The fact that each metrizable space X with $\dim X = n$ is embeddable in the Cartesian product of $n+1$ metrizable spaces whose covering dimension is equal to 1 was discovered by Nagata in [1958]; Borsuk proved in [1975] that the two-sphere S^2 cannot be embedded in the Cartesian product of two one-dimensional spaces.

Problems

4.2.A (Morita [1954]). (a) Prove the Katětov–Morita theorem by applying only Lemmas 4.2.1 and 2.3.16 and Theorems 4.1.9, 4.1.10 and 3.1.28.

Hint. Adjoin to conditions (a)–(c) in Theorem 4.2.2 condition (d) stating that $\text{Ind} X \leq n$.

(b) Note that in Theorem 4.2.2 one can replace the σ -local finiteness of $\mathcal{B}$ by σ -discreteness.

4.2.B. (a) Show that every locally finite family $\{F_s\}_{s \in S}$ of closed subsets of a metrizable space X has a locally finite open swelling $\{U_s\}_{s \in S}$.

Hint. For $i = 1, 2, \dots$ consider the set A_i consisting of all points $x \in X$ such that the set $\{s \in S: B(x, 1/i) \cap F_s \neq \emptyset\}$ is finite, let $V_s = \bigcup_{i=1}^{\infty} [B(F_s, 1/3i) \cap \text{Int} A_i]$ and apply Problem 3.1.A.

(b) (Morita [1955], Nagami [1960]) Prove that a metrizable space X satisfies the inequality $\dim X \leq n$ if and only if there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ of locally finite closed covers of the space X such that $\text{ord } \mathcal{F}_i \leq n$ and $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ for $i = 1, 2, \dots$ and for every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that $\text{St}(x, \mathcal{F}_i) \subset U$.

Hint. Let $\mathcal{F}_i = \{F_s\}_{s \in S_i}$ for $i = 1, 2, \dots$. Apply part (a) to define a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open covers of X , where $\mathcal{W}_i = \{W_s\}_{s \in S_i}$ is a swelling of the cover $\mathcal{F}_i$ and $W_s \subset B(F_s, 1/i)$ for $s \in S_i$, such that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \dots$; check that $\mathcal{W}_1, \mathcal{W}_2, \dots$ is a strong development for the space X .

4.2.C (Nagata [1963] (announcement [1961])). Prove that a metrizable space X satisfies the inequality $\dim X \leq n$ if and only if X has a base of rank $\leq n$ (see Problem 3.2.C).

Hint (Arhangel'skii [1963]). When proving that X has a base of rank $\leq n$ use the inequality $\text{Ind} X \leq n$; apply Theorem 4.1.17.

4.2.D (Nagata [1963]). (a) Note that every metric ρ which satisfies condition (i) satisfies also condition (ii).

(b) Show that if on a space X there exists a metric ρ which satisfies condition (ii), then $\dim X \leq n$.

Hint. See the second part of the hint to Problem 4.1.A(b).

4.2.E. Note that for $n = 0$ conditions (iii), (iv) and (v) reduce to the condition that the metric ρ is non-Archimedean; check that each non-Archimedean metric ρ satisfies conditions (i) and (ii) for $n = 0$.

4.2.F (de Groot [1957]). Applying the fact that on every compact metrizable space of dimension $\leq n$ there exists a metric ρ satisfying condition (iii), show that a separable metrizable space X satisfies the inequality $\dim X \leq n$ if and only if on the space X there exists a totally bounded metric ρ which satisfies condition (v).

Hint. Note that if ϱ satisfies (v), then the metric $\tilde{\varrho}$ on the completion $\tilde{X}$ of the space X also satisfies (v). For an arbitrary finite open cover $\mathcal{U}$ of the completion $\tilde{X}$ consider a set $A \subset \tilde{X}$ maximal with respect to the property that $\varrho(y, y') \geq \delta/2$ for distinct $y, y' \in A$, where δ is a Lebesgue number for the cover $\mathcal{U}$.

4.2.G (Pasyukov [1967a] (announcement [1964])). Prove that for every continuous mapping $f: X \rightarrow Y$ of a normal space X to a metrizable space Y there exist a metrizable space Z and continuous mappings $g: X \rightarrow Z$ and $h: Z \rightarrow Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.

Hint. Generalize Lemma 4.2.4 to normal spaces; make use of the mapping of X to the metrizable space assigned to the cover $\mathcal{U}$ as described in the hint to Problem 3.2.H(b).

4.3. Dimension and mappings in metrizable spaces

In this section we shall study the behaviour of dimension of metrizable spaces under continuous mappings. We start with extending to arbitrary metrizable spaces the theorems on dimension-raising and dimension-lowering mappings established for separable metric spaces in Section 1.12. The proofs of these extensions closely follow the pattern of the proofs in Chapter 1 and differ from them only in technical details.

4.3.1. Lemma. *A metrizable space X satisfies the inequality $\text{Ind} X \leq n \geq 0$ if and only if X has a σ -locally finite network $\mathcal{N}$ such that $\text{Ind Fr } M \leq n-1$ for every $M \in \mathcal{N}$.*

Proof. By virtue of Theorem 4.1.15, it suffices to show that if a metrizable space X has a σ -locally finite network $\mathcal{N}$ such that $\text{Ind Fr } M \leq n-1$ for $M \in \mathcal{N}$, then $\text{Ind} X \leq n$. Let $Y = \bigcup \{\text{Fr } M: M \in \mathcal{N}\}$ and $Z = X \setminus Y$. Since the family $\{\text{Fr } M: M \in \mathcal{N}\}$ is σ -locally finite, it follows from Theorems 4.1.9 and 4.1.10 that $\text{Ind} Y \leq n-1$. The family $\{Z \cap M: M \in \mathcal{N}\}$ is a σ -locally finite network for the subspace Z ; as the members of this family are open-and-closed in Z , it is a base for Z , so that $\text{Ind} Z \leq 0$ by virtue of Theorem 4.1.15. The inequality $\text{Ind} X \leq n$ now follows from Theorem 4.1.16. $\square$

Let us recall that a continuous mapping $f: X \rightarrow Y$ defined on a Hausdorff space X is *perfect* if f is a closed mapping and for every $y \in Y$ the

fibre $f^{-1}(y)$ is a compact subspace of X . Obviously, every closed mapping with finite fibres defined on a Hausdorff space is perfect.

4.3.2. Lemma. *If $f: X \rightarrow Y$ is a perfect mapping, then for every locally finite family $\mathcal{A}$ of subsets of X the family $\{f(A): A \in \mathcal{A}\}$ is locally finite in Y .*

Proof. Since the fibres of f are compact, for every $y \in Y$ there exists an open set $U \subset X$ which contains $f^{-1}(y)$ and meets only finitely many members of $\mathcal{A}$. The mapping f being closed, the set $V = Y \setminus f(X \setminus U)$ is a neighbourhood of the point y . From the inclusion $f^{-1}(V) \subset U$ it follows that V meets only finitely many members of the family $\{f(A): A \in \mathcal{A}\}$. $\square$

4.3.3. Theorem on dimension-raising mappings. *If $f: X \rightarrow Y$ is a closed mapping of a metrizable space X onto a metrizable space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $\text{Ind } Y \leq \text{Ind } X + (k-1)$.*

Proof. We can suppose that $0 \leq \text{Ind } X < \infty$. We shall apply induction with respect to the number $n+k$, where $n = \text{Ind } X$. If $n+k = 1$, we have $k = 1$, so that f is a homeomorphism and the theorem holds. Assume that the theorem holds whenever $n+k < m \geq 2$ and consider a closed mapping $f: X \rightarrow Y$ such that $f(X) = Y$ and $n+k = m$.

Let $\mathcal{B}$ be a σ -locally finite base for X such that $\text{Ind Fr } U \leq n-1$ for every $U \in \mathcal{B}$. Consider an arbitrary $U \in \mathcal{B}$; by the closedness of f we have

$$\begin{aligned} (1) \quad \text{Fr } f(U) &= \overline{f(U)} \cap \overline{Y \setminus f(U)} \subset f(\overline{U}) \cap f(X \setminus U) \\ &= [f(U) \cup f(\text{Fr } U)] \cap f(X \setminus U) \subset f(\text{Fr } U) \cup B, \end{aligned}$$

where $B = f(U) \cap f(X \setminus U)$. Since the restriction $f|_{\text{Fr } U}: \text{Fr } U \rightarrow f(\text{Fr } U)$ is a closed mapping, it follows from the inductive assumption that

$$\text{Ind } f(\text{Fr } U) \leq (n-1) + (k-1) = n+k-2.$$

Assume that $B \neq \emptyset$. Consider the restriction $f_B: f^{-1}(B) \rightarrow B$ and the restriction $f' = f_B|_{(X \setminus U)}: (X \setminus U) \cap f^{-1}(B) \rightarrow B$; both f_B and f' are closed, and the fibres of f' all have cardinality $\leq k-1$, because $f^{-1}(y) \cap U \neq \emptyset$ for every $y \in B$. It follows from the inductive assumption that

$$\text{Ind } B \leq n + (k-1) - 1 = n+k-2.$$

As U is an F_σ -set in X , both $f(U)$ and B are F_σ -sets in Y ; applying Theorems 4.1.7 and 4.1.9, we obtain the inequality $\text{Ind}[f(\text{Fr } U) \cup B] \leq n+k-2$.

From the last inequality and from (1) it follows that $\text{Ind Fr } f(U) \leq n+k-2$ for every $U \in \mathcal{B}$; the same inequality holds if $B = \emptyset$. One readily checks that the family $\mathcal{N} = \{f(U) : U \in \mathcal{B}\}$ is a network for the space Y . In view of Lemma 4.3.2 the network $\mathcal{N}$ is σ -locally finite, so that $\text{Ind } Y \leq n+k-1 = \text{Ind } X+k-1$ by virtue of Lemma 4.3.1. $\square$

We now pass to the theorem on dimension-lowering mappings. To begin with, let us note that reproducing the proof of Lemma 1.9.6 with the application of Theorems 3.2.9, 4.1.3, 4.1.7 and 4.1.21 one obtains

4.3.4. Lemma. *Let $f, g: X \rightarrow S^n$ be continuous mappings of a metrizable space X to the n -sphere S^n . If the set*

$$D(f, g) = \{x \in X: f(x) \neq g(x)\}$$

satisfies the inequality $\text{Ind } D(f, g) \leq n-1$, then the mappings f and g are homotopic. $\square$

Now we shall establish a counterpart of Lemma 1.12.3.

4.3.5. Lemma. *If a metrizable space X has a closed cover $\mathcal{K}$ such that $\text{Ind } K \leq m \geq 0$ for each $K \in \mathcal{K}$ and a σ -locally finite open cover $\mathcal{U}$ such that for every $K \in \mathcal{K}$ and each open set $V \subset X$ that contains K there exists a $U \in \mathcal{U}$ satisfying*

$$K \subset U \subset \bar{U} \subset V \quad \text{and} \quad \text{Ind Fr } U \leq m-1,$$

then $\text{Ind } X \leq m$.

Proof. By virtue of Theorem 4.1.3 and Remarks 1.7.10 and 1.9.4 it suffices to show that for every closed subspace A of the space X and each continuous mapping $f: A \rightarrow S^m$ there exists a continuous extension $F: X \rightarrow S^m$ of f over X . To begin with, let us observe that

- (2) for every $K \in \mathcal{K}$ there exists a $U_K \in \mathcal{U}$ such that $K \subset U_K$, $\text{Ind Fr } U_K \leq m-1$ and f is continuously extendable over $A \cup \bar{U}_K$.

Indeed, it follows from Theorem 3.2.9 (cf. Problem 4.3.B(a)) that the mapping f is continuously extendable over $A \cup K$, so that there exists an open set $V \subset X$ containing $A \cup K$ such that f is continuously extendable over V ; the existence of a set U_K satisfying (2) now follows from the properties of the family $\mathcal{U}$.

The subfamily $\mathcal{U}_0$ of the family $\mathcal{U}$ which consists of all sets of the form U_κ can be represented as the union of a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of locally finite families. Let us arrange the members of $\mathcal{U}_0$ into a transfinite sequence $U_1, U_2, \dots, U_\alpha, \dots, \alpha < \xi$ placing first all members of $\mathcal{U}_1$, then all members of $\mathcal{U}_2$ etc. Clearly, for every $\alpha_0 < \xi$ the family $\{U_\alpha\}_{\alpha < \alpha_0}$ is locally finite.

We shall inductively define a transfinite sequence $F_1, F_2, \dots, F_\alpha, \dots, \alpha < \xi$ of continuous mappings, where $F_\alpha: A \cup \bigcup_{\beta \leq \alpha} \bar{U}_\beta \rightarrow S^m$, such that

$$(3) \quad F_\alpha(A \cup \bigcup_{\beta \leq \gamma} \bar{U}_\beta) = F_\gamma \quad \text{for every } \gamma \leq \alpha.$$

Let F_1 be an arbitrary continuous extension of f over $A \cup \bar{U}_1$. Assume that the mappings F_α satisfying (3) are defined for $\alpha < \alpha_0$. The set $A \cup \bigcup_{\alpha < \alpha_0} \bar{U}_\alpha$ can be represented as the union of two closed sets

$$A' = A \cup \bigcup_{\alpha < \alpha_0} \bar{U}_\alpha \quad \text{and} \quad A'' = A \cup (\bar{U}_{\alpha_0} \setminus \bigcup_{\alpha < \alpha_0} U_\alpha)$$

which, by virtue of the local finiteness of the family $\{U_\alpha\}_{\alpha < \alpha_0}$, satisfy the relation

$$(4) \quad A' \cap A'' \subset A \cup \text{Fr}(\bigcup_{\alpha < \alpha_0} U_\alpha) \subset A \cup \bigcup_{\alpha < \alpha_0} \text{Fr} U_\alpha.$$

Since the family $\{A\} \cup \{\bar{U}_\alpha\}_{\alpha < \alpha_0}$ is locally finite and $\{f\} \cup \{F_\alpha|_{\bar{U}_\alpha}\}_{\alpha < \alpha_0}$ is a family of compatible mappings, one can define a continuous mapping $F': A' \rightarrow S^m$ which is a common extension of all mappings F_α with $\alpha < \alpha_0$. By virtue of (2) the mapping f is extendable to a continuous mapping $f'': A'' \rightarrow S^m$, and in view of (4)

$$D = \{x \in A' \cap A'': F'(x) \neq f''(x)\} \subset \bigcup_{\alpha < \alpha_0} \text{Fr} U_\alpha,$$

so that $\text{Ind} D \leq m-1$ by virtue of (2) and Theorem 4.1.10. It follows from Lemma 4.3.4 that the mappings $F'|_{A' \cap A''}$ and $f''|_{A' \cap A''}$ are homotopic. Since the mapping $f''|_{A' \cap A''}$ is continuously extendable over A'' , it follows from Lemma 1.9.7 that the mapping $F'|_{A' \cap A''}$ is extendable to a continuous mapping $F'': A'' \rightarrow S^m$. Letting

$$F_{\alpha_0}(x) = \begin{cases} F'(x) & \text{for } x \in A', \\ F''(x) & \text{for } x \in A'', \end{cases}$$

we define a continuous mapping F_{α_0} of $A' \cup A'' = A \cup \bigcup_{\alpha \leq \alpha_0} \bar{U}_\alpha$ to S^m , which satisfies (3) for $\alpha = \alpha_0$.

As $X = \bigcup_{\alpha < \xi} U_\alpha$, the formula

$$F(x) = F_\alpha(x) \quad \text{for } x \in U_\alpha$$

defines a continuous mapping $F: X \rightarrow S^m$, which is the required extension of f over X . $\square$

4.3.6. Theorem on dimension-lowering mappings. *If $f: X \rightarrow Y$ is a closed mapping of a metrizable space X to a metrizable space Y and there exists an integer $k \geq 0$ such that $\text{Ind} f^{-1}(y) \leq k$ for every $y \in Y$, then $\text{Ind} X \leq \text{Ind} Y + k$.*

Proof. We can suppose that $\text{Ind} Y < \infty$. We shall apply induction with respect to $n = \text{Ind} Y$. If $n = -1$, we have $Y = \emptyset$ and $X = \emptyset$, and so the theorem holds. Assume that the theorem holds for closed mappings to spaces of large inductive dimension less than $n \geq 0$ and consider a closed mapping $f: X \rightarrow Y$ to a space Y such that $\text{Ind} Y = n$.

Let $\mathcal{B}$ be a σ -locally finite base for Y such that $\text{Ind Fr } U \leq n-1$ for every $U \in \mathcal{B}$. Applying the inductive assumption to the restriction $f_{\text{Fr } U}: f^{-1}(\text{Fr } U) \rightarrow \text{Fr } U$, we obtain the inequality $\text{Ind} f^{-1}(\text{Fr } U) \leq n+k-1$. Since

$$\text{Fr} f^{-1}(U) = \overline{f^{-1}(U)} \setminus f^{-1}(U) \subset f^{-1}(\overline{U}) \setminus f^{-1}(U) = f^{-1}(\text{Fr } U),$$

we have $\text{Ind Fr} f^{-1}(U) \leq n+k-1$ for every $U \in \mathcal{B}$. One readily checks that the covers $\mathcal{K} = \{f^{-1}(y)\}_{y \in Y}$ and $\mathcal{U} = f^{-1}(\mathcal{B})$ satisfy the conditions of Lemma 4.3.5 with $m = n+k$, so that $\text{Ind} X \leq n+k = \text{Ind} Y + k$. $\square$

We now turn to two more specialized theorems on the relations between the dimensions of the domain and the range of a closed mapping. In our considerations an important role will be played by the sets $C_k(f)$ and $D_k(f)$ which for each closed mapping $f: X \rightarrow Y$ of a normal space X to a normal space Y and for $k = 1, 2, \dots$ are defined by the formulas

$$C_k(f) = \{y \in Y: |f^{-1}(y)| \geq k\}$$

and

$$D_k(f) = \{y \in Y: \dim f^{-1}(y) \geq k\}.$$

4.3.7. Lemma. *For every closed mapping $f: X \rightarrow Y$ of a metrizable space X to a metrizable space Y and for $k = 1, 2, \dots$ $C_k(f)$ and $D_k(f)$ are F_σ -sets in Y .*

Proof. Consider first the sets $C_k(f)$. For $i = 1, 2, \dots$ let $\mathcal{U}_i$ be an open cover of the space X such that $\text{mesh } \mathcal{U}_i < 1/i$ and let

$U_i = \{y \in Y: \text{there exists a } \mathcal{U} \subset \mathcal{U}_i \text{ such that } |\mathcal{U}| < k \text{ and } f^{-1}(y) \subset \bigcup \mathcal{U}\}.$

From the closedness of the mapping f it follows that the sets U_i are open.

One easily checks that $Y \setminus C_k(f) = \bigcap_{i=1}^{\infty} U_i$, so that $C_k(f)$ is an F_σ -set in Y .

Now consider the sets $D_k(f)$. We start with a particular case where f is a perfect mapping. For $i = 1, 2, \dots$ let V_i be the set of all points $y \in Y$ for which there exists a finite family $\mathcal{U}$ of open subsets of X such that $\text{mesh } \mathcal{U} < 1/i$, $\text{ord } \mathcal{U} < k$ and $f^{-1}(y) \subset \bigcup \mathcal{U}$. From the closedness of the mapping f it follows that the sets V_i are open. Applying the compactness of fibres of the mapping f and Theorems 1.6.12, 3.1.3 and 3.1.2, one easily checks that $Y \setminus D_k(f) = \bigcap_{i=1}^{\infty} V_i$, so that $D_k(f)$ is an F_σ -set in Y .

For an arbitrary closed mapping $f: X \rightarrow Y$ consider the subspaces $X_0 = \bigcup_{y \in Y} \text{Int } f^{-1}(y)$ and $X_1 = X \setminus X_0 = \bigcup_{y \in Y} \text{Fr } f^{-1}(y)$ of the space X and the restrictions $f_0 = f|_{X_0}: X_0 \rightarrow Y$ and $f_1 = f|_{X_1}: X_1 \rightarrow Y$. From the countable sum theorem it follows that $D_k(f) = D_k(f_0) \cup D_k(f_1)$, because $\text{Int } f^{-1}(y)$ is for every $y \in Y$ an F_σ -set in X . Being the image of an F_σ -set in X under a closed mapping, the set

$$D_k(f_0) = f(\bigcup \{\text{Int } f^{-1}(y): \dim \text{Int } f^{-1}(y) \geq k\})$$

is an F_σ -set in Y . From Lemma 1.12.9 it follows that f_1 is a perfect mapping, so that the set $D_k(f_1)$ is an F_σ -set in Y by virtue of the particular case of our theorem established in the preceding paragraph. Thus $D_k(f)$ is an F_σ -set in Y . $\square$

4.3.8. Lemma. *If X is a metrizable space and $M_1, M_2, \dots, M_k$ is a sequence of subspaces of X such that M_i is an F_σ -set in X and $\text{Ind } M_i \leq n_i \geq 0$ for $i = 1, 2, \dots, k$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\text{Ind}(M_i \cap L) \leq n_i - 1$ for $i = 1, 2, \dots, k$.*

Proof. By the first decomposition theorem $M_i = Y_i \cup Z_i$, where $\text{Ind } Y_i \leq n_i - 1$ and $\text{Ind } Z_i \leq 0$. From the enlargement theorem it follows that there exists a G_δ -set Y_i^* in M_i such that $Y_i \subset Y_i^*$ and $\text{Ind } Y_i^* \leq n_i - 1$. The set $Z_i^* = M_i \setminus Y_i^* \subset Z_i$ is an F_σ -set in M_i and consequently an F_σ -set in X . By virtue of the countable sum theorem the set $Z = Z_1^* \cup Z_2^* \cup \dots \cup Z_k^*$ satisfies the inequality $\text{Ind } Z \leq 0$, so that from the separation theorem it follows that there exists a partition L between A and B such that $L \cap Z$

$= \emptyset$. Hence for $i = 1, 2, \dots, k$ we have $M_i \cap L \subset Y_i^*$, which implies that $\text{Ind}(M_i \cap L) \leq n_i - 1$ for $i = 1, 2, \dots, k$. $\square$

If for a continuous mapping $f: X \rightarrow Y$ of a metrizable space X to a metrizable space Y there exists an integer $n \geq 0$ such that $\text{Ind} f^{-1}(y) \leq n$ for every $y \in Y$, then we say that f is *finite-dimensional*; the smallest n with this property will be denoted by $\text{Ind} f$. In order to simplify the statements of the two theorems we are now going to prove, for every finite-dimensional mapping f we define

$$\begin{aligned} D(f) &= \max\{\text{Ind} D_i(f) + i : i = 1, 2, \dots, \text{Ind} f\} & \text{if } \text{Ind} f \geq 1 \\ \text{and} \quad D(f) &= -1 & \text{if } \text{Ind} f = 0. \end{aligned}$$

4.3.9. Vainšteĭn's first theorem. *Let $f: X \rightarrow Y$ be a finite-dimensional closed mapping of a metrizable space X to a metrizable space Y . If $\text{Ind} Y \leq n \geq 0$ and $\text{Ind} D_i(f) \leq n - i$ for $i = 1, 2, \dots, n + 1$, then $\text{Ind} X \leq n$; in other words, $\text{Ind} X \leq \max\{\text{Ind} Y, D(f)\}$.*

Proof. We shall apply induction with respect to n . If $n = 0$, we have $\text{Ind} D_1(f) = -1$, so that $\text{Ind} f = 0$ and consequently $\text{Ind} X \leq n$ by virtue of Theorem 4.3.6. Assume that the theorem holds for all natural numbers less than $n \geq 1$ and consider a closed mapping $f: X \rightarrow Y$ which satisfies the assumptions of the theorem.

From Lemma 4.3.7 it follows that $D_1(f), D_2(f), \dots, D_n(f)$ are F_σ -sets in Y . Applying Lemma 4.3.8 one can easily define (cf. the proof of Proposition 4.1.14) a σ -locally finite base $\mathcal{B}$ for the space Y such that for each $i \leq n$

$$\text{Ind}(D_i(f) \cap \text{Fr} U) \leq \text{Ind} D_i(f) - 1 \quad \text{for every } U \in \mathcal{B},$$

provided that $D_i(f) \neq \emptyset$, and that $\text{Ind} \text{Fr} U \leq n - 1$ for every $U \in \mathcal{B}$. Consider the restriction $f_{\text{Fr} U}: f^{-1}(\text{Fr} U) \rightarrow \text{Fr} U$, where $U \in \mathcal{B}$; clearly, $D_i(f_{\text{Fr} U}) \subset D_i(f) \cap \text{Fr} U$. Since $\text{Ind} \text{Fr} U \leq n - 1 \geq 0$ and $\text{Ind} D_i(f_{\text{Fr} U}) \leq (n - 1) - i$ for $i = 1, 2, \dots, n$, applying the inductive assumption to $f_{\text{Fr} U}$ we obtain the inequality $\text{Ind} f^{-1}(\text{Fr} U) \leq n - 1$. Hence, by virtue of the inclusion $\text{Fr} f^{-1}(U) \subset f^{-1}(\text{Fr} U)$ we have $\text{Ind} \text{Fr} f^{-1}(U) \leq n - 1$ for every $U \in \mathcal{B}$. Applying the equality $D_{n+1}(f) = 0$ and the Katětov-Morita theorem, one readily checks that the covers $\mathcal{K} = \{f^{-1}(y)\}_{y \in Y}$ and $\mathcal{U} = f^{-1}(\mathcal{B})$ satisfy the conditions of Lemma 4.3.5 with $m = n$, so that $\text{Ind} X \leq n$. $\square$

The reader can easily verify that Theorem 4.3.9 is a generalization of Theorem 4.3.6.

We now pass to Vainštein's second theorem. It will be preceded by two lemmas of which the second is an important particular case of the theorem.

4.3.10. Lemma. *If a metrizable space X satisfies the inequality $\text{Ind} X \leq n \geq 0$, then for every F_σ -set $C \subset X$ such that $\text{Ind} C \leq n-1$ there exists an F_σ -set $M \subset X$ satisfying the conditions*

$$\text{Ind} M \leq 0, \quad \text{Ind}(X \setminus M) \leq n-1 \quad \text{and} \quad M \cap C = \emptyset.$$

Proof. By the first decomposition theorem $X = Y \cup Z$, where $\text{Ind} Y \leq n-1$ and $\text{Ind} Z \leq 0$. Obviously, one can assume that $Y = X \setminus Z$, and by virtue of the enlargement theorem one can assume that Z is a G_δ -set in X . Thus Y is an F_σ -set and the countable sum theorem yields the inequality $\text{Ind}(Y \cup C) \leq n-1$. By virtue of the enlargement theorem there exists a G_δ -set $K \subset X$ such that $Y \cup C \subset K$ and $\text{Ind} K \leq n-1$. One easily checks that the set $M = X \setminus K$ satisfies the required conditions. $\square$

4.3.11. Lemma (Freudenthal's theorem). *Let $f: X \rightarrow Y$ be a closed mapping of a metrizable space X onto a metrizable space Y such that $\text{Ind} f^{-1}(y) \leq 0$ for every $y \in Y$. If $\text{Ind} X \leq n$ and $\text{Ind} C_2(f) \leq n-1$, then $\text{Ind} Y \leq n$; in other words, $\text{Ind} Y \leq \max\{\text{Ind} X, \text{Ind} C_2(f) + 1\}$.*

Proof. We shall apply induction with respect to n . If $n = 0$, we have $C_2(f) = \emptyset$, so that f is a homeomorphism and $\text{Ind} Y = \text{Ind} X \leq 0$. Assume that the theorem holds for all natural numbers less than $n \geq 1$ and consider a closed mapping $f: X \rightarrow Y$ which satisfies the assumptions of the theorem.

By virtue of Theorem 4.3.6 the set $C = f^{-1}(C_2(f))$ satisfies the inequality $\text{Ind} C \leq n-1$. Lemma 4.3.7 implies that C is an F_σ -set in X , so that Lemma 4.3.10 implies that there exists an F_σ -set $M \subset X$ such that $\text{Ind} M \leq 0$, $\text{Ind}(X \setminus M) \leq n-1$ and $M \cap C = \emptyset$. Let $M = \bigcup_{i=1}^{\infty} F_i$, where the sets F_i are closed in X . For $i = 1, 2, \dots$ the restriction of f to F_i is a homeomorphism of F_i onto $f(F_i)$, so that $\text{Ind} f(F_i) \leq 0$. Thus, by the countable sum theorem, $\text{Ind} f(M) \leq 0$.

Consider an arbitrary pair A, B of disjoint closed subsets of Y . From Lemma 4.3.8 with $k = 2$, $M_1 = f(M)$ and $M_2 = C_2(f)$ it follows that there exists a partition L between A and B such that $L \cap f(M) = \emptyset$ and $\text{Ind}(L \cap C_2(f)) \leq n-2$. The equality $L \cap f(M) = \emptyset$ implies that $f^{-1}(L) \subset X \setminus M$, so that $\text{Ind} f^{-1}(L) \leq n-1$. Consider the restriction $f_L: f_L^{-1}(L) \rightarrow L$;

since $C_2(f_L) \subset L \cap C_2(f)$, we have $\text{Ind } C_2(f_L) \leq n-2$, so that by the inductive assumption $\text{Ind } L \leq n-1$. Thus $\text{Ind } Y \leq n$. $\square$

4.3.12. Vainšteĭn's second theorem. *Let $f: X \rightarrow Y$ be a finite-dimensional closed mapping of a metrizable space X onto a metrizable space Y . If $\text{Ind } X \leq n \geq 1$, $\text{Ind } C_2(f) \leq n-1$ and $\text{Ind } D_i(f) \leq n-(i+1)$ for $i = 1, 2, \dots, n$, then $\text{Ind } Y \leq n$; in other words, $\text{Ind } Y \leq \max\{\text{Ind } X, \text{Ind } C_2(f) + 1, D(f) + 1\}$.*

Proof. We shall apply induction with respect to n . If $n = 1$, we have $\text{Ind } D_1(f) = -1$, so that $\text{Ind } f = 0$ and the theorem reduces to Lemma 4.3.11. Assume that the theorem holds for all natural numbers less than $n \geq 2$ and consider a closed mapping $f: X \rightarrow Y$ which satisfies the assumption of the theorem.

By virtue of Theorem 4.3.9 the set $C = f^{-1}(C_2(f))$ satisfies the inequality $\text{Ind } C \leq n-1$. Hence from Lemmas 4.3.7 and 4.3.10 it follows that there exists an F_σ -set $M \subset X$ such that $\text{Ind } M \leq 0$, $\text{Ind}(X \setminus M) \leq n-1$ and $M \cap C = \emptyset$; by the countable sum theorem $\text{Ind } f(M) \leq 0$.

Consider an arbitrary pair A, B of disjoint closed subsets of Y . From Lemma 4.3.8 with $k = n+1$, $M_i = D_i(f)$ for $i = 1, 2, \dots, n-1$, $M_n = f(M)$ and $M_{n+1} = C_2(f)$ it follows that there exists a partition L between A and B such that $L \cap f(M) = \emptyset$, $\text{Ind}(C_2(f) \cap L) \leq n-2$ and for each $i \leq n-1$

$$\text{Ind}(D_i(f) \cap L) \leq \text{Ind } D_i(f) - 1,$$

provided that $D_i(f) \neq \emptyset$. The equality $L \cap f(M) = \emptyset$ implies that $\text{Ind } f^{-1}(L) \leq n-1$. Consider the restriction $f_L: f^{-1}(L) \rightarrow L$; since $C_2(f_L) \subset C_2(f) \cap L$ and $D_i(f_L) \subset D_i(f) \cap L$ for $i = 1, 2, \dots, n-1$, we have $\text{Ind } C_2(f_L) \leq n-2$ and $\text{Ind } D_i(f_L) \leq (n-1) - (i+1)$ for $i = 1, 2, \dots, n-1$, so that by the inductive assumption $\text{Ind } L \leq n-1$. Thus $\text{Ind } Y \leq n$. $\square$

To conclude, we shall state an important theorem which permits us to generalize part of the results obtained in this section to the case of a closed mapping f of a normal space X onto a normal space Y . The proof of this theorem is too complicated to be reproduced in this book; it involves a construction similar to but much more elaborate than the construction used in the proof of Theorem 3.3.2. The symbol $\text{rd}_Y A$ which appears below denotes the *relative dimension* of a subspace A of a normal space Y with respect to Y , i.e., the smallest integer n such that $\dim Z \leq n$ for every closed subspace Z of the space Y contained in A .

Thus, one proves that

(F) *For every closed mapping $f: X \rightarrow Y$ of a normal space X onto a normal space Y with $\dim Y < \infty$ there exist compact metrizable spaces X_0, Y_0 and a closed mapping f_0 of X_0 onto Y_0 such that $\dim X_0 = \dim X$, $\dim Y_0 = \dim Y$ and $\dim C_k(f_0) \leq \text{rd}_Y C_k(f)$ for $k = 1, 2, \dots$. If, moreover, the space Y is weakly paracompact, then $\dim D_k(f_0) \leq \text{rd}_Y D_k(f)$ for $k = 1, 2, \dots$*

Let us note that the assumption of weak paracompactness in the second part of (F) cannot be omitted. Indeed, one can define a closed mapping f of a normal space X onto a normal space Y such that $\dim X = 1$, $\dim Y = 0$ and $\dim f^{-1}(y) = 0$ for every $y \in Y$.

Historical and bibliographic notes

Theorem 4.3.3 was given by Morita in [1955]; Theorem 4.3.6 was proved independently by Morita in [1956] and by Nagami in [1957]. Theorems 4.3.9 and 4.3.12 were established by Vainšteĭn in [1952] for separable X and Y . Extensions to arbitrary metrizable spaces were given by Skljarenko in [1962] and [1963]. Lemma 4.3.11 was proved by Freudenthal in [1932]. As shown by Lelek in [1971], from Vainšteĭn's theorems many further results about the behaviour of dimension under mappings can be deduced. Lelek's paper gives a comprehensive survey of the topic considered in this section and provides a good bibliography. Theorem (F) stated at the end of the section as well as the example cited after this theorem were given by Filippov in [1972a]. As we have already noted, Filippov's theorem leads to extensions of the results obtained in this section to larger classes of spaces. Some of these extensions were obtained earlier in the above-mentioned papers by Morita, Nagami and Skljarenko.

Problems

4.3.A (Suzuki [1959]). Prove that if $f: X \rightarrow Y$ is a closed mapping of a metrizable space X onto a metrizable space Y and $|f^{-1}(y)| = k < \infty$ for every $y \in Y$, then $\text{Ind} X = \text{Ind} Y$.

Hint. See the hint to Problem 1.12.A.

4.3.B. (a) Following the pattern in Section 1.9, prove that for every continuous mapping $f: A \rightarrow S^n$ defined on a closed subspace A of metri-

zable space X such that $\text{Ind}(X \setminus A) \leq n \geq 0$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X (cf. Problem 2.2.B).

(b) Deduce from (a), Theorem 4.1.3 and Remarks 1.7.10 and 1.9.4 that a metrizable space X satisfies the inequality $\text{Ind} X \leq n \geq 0$ if and only if for every closed subspace A of the space X and each continuous mapping $f: A \rightarrow S^n$ there exists a continuous extension $F: X \rightarrow S^n$ of f over X .

4.3.C (Morita [1955]). Prove that a metrizable space X satisfies the inequality $\dim X \leq n$ if and only if there exists a closed mapping $f: Z \rightarrow X$ of a subspace Z of the Baire space $B(m)$, where $m = w(X)$, onto the space X with fibres of cardinality at most $n+1$.

Hint. For a metrizable space X with $\dim X \leq n$ consider a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ of locally finite closed covers satisfying the conditions in Problem 4.2.B(b) and such that $|\mathcal{F}_i| \leq m$ for $i = 1, 2, \dots$. Define a mapping π_i^{i+1} of $\mathcal{F}_{i+1}$ to $\mathcal{F}_i$ such that $F \subset \pi_i^{i+1}(F)$ for every $F \in \mathcal{F}_{i+1}$, let $\pi_j^i = \pi_j^{j+1} \pi_{j+1}^{j+2} \dots \pi_{i-1}^i$ for $j < i$ and $\pi_i^i = \text{id}_{\mathcal{F}_i}$. Consider the inverse sequence $S = \{\mathcal{F}_i, \pi_j^i\}$, where $\mathcal{F}_i$ has the discrete topology. Note that $S = \varprojlim S$ is a subspace of $B(m)$ and consider the subspace Z of S consisting of all

sequences $\{F_i\} \in S$ such that $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$; show that the last intersection contains exactly one point of the space X and assign this point to the sequence $\{F_i\}$. Prove that the mapping $f: Z \rightarrow X$ obtained in this way has all the required properties. When proving that $f(Z) = X$ apply the fact that the limit of an inverse sequence consisting of finite discrete spaces is non-empty. When proving that f is closed apply the equality $f^{-1}(x) = Z \cap \bigcap_{i=1}^{\infty} \{F \in \mathcal{F}_i: x \in F\}$ and show that for every open set $U \subset X$ which contains $f^{-1}(x)$ there exists a neighbourhood $V \subset X$ of the point x such that $f^{-1}(V) \subset U$.

4.3.D (Nagami [1960]). Prove the Katětov–Morita theorem by applying only Theorems 1.6.11, 4.1.13 and 3.1.28 and the facts established in Problems 4.3.A and 4.3.C.

Hint. Applying 4.3.A, deduce from 4.3.C that every metrizable space X which satisfies the inequality $\dim X \leq n$ can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \dots, Z_{n+1}$ such that $\dim Z_i \leq 0$ for $i = 1, 2, \dots, n+1$.

4.3.E. (a) (Hodel [1963]) Show that if $f: X \rightarrow Y$ is an open mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X , then $\text{Ind} X \leq \text{Ind} Y$.

Hint. Apply Lemma 1.12.5.

(b) (Nagami [1960]) Show that if $f: X \rightarrow Y$ is an open mapping of a metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $\text{Ind} X = \text{Ind} Y$.

(c) (Hodel [1963]) Show that if $f: X \rightarrow Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point, then $\text{Ind} Y \leq \text{Ind} X$.

Remark. As shown by R. Pol in [1979], in the realm of metrizable spaces there exist open mappings with discrete fibres which raise dimensions (cf. Theorem 1.12.7).

(d) (Hodel [1963]) Show that if $f: X \rightarrow Y$ is an open mapping of a locally compact metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\text{Ind} X = \text{Ind} Y$.

(e) (Arhangel'skii [1966]) Show that if $f: X \rightarrow Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\text{Ind} X = \text{Ind} Y$.

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LIST OF SPECIAL SYMBOLS

Symbols defined in the text

$\text{ind}X$	3	$p+q, \lambda p$	100
$\text{ind}_x X$	9	$p_0 p_1 \dots p_n$	100
$ x $	16, 100	$ \mathcal{X} $	101
Q_k^n	22	$\text{St}_{\mathcal{X}}(p_i)$	102
N_k^n, L_k^n	44	$\mathcal{N}(\mathcal{U}), N(\mathcal{U})$	103
$\text{Ind}X$	52	$\hat{e}(f, g)$	118
$\text{ord}\mathcal{A}$	54	Y^X	118
$\dim X$	54, 222	M_n^m	122
$\text{mesh}\mathcal{A}$	56	$\text{St}(x, \mathcal{U})$	167
$\mu\dim(X, \varrho)$	57	$w(X)$	175
$\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \dots \wedge \mathcal{A}_k$	61	$\text{St}(B, \mathcal{B})$	241
$f^{-1}(\mathcal{A})$	61	$B(m)$	263
$\mathcal{A} M$	61	$C_k(f), D_k(f)$	281
$\dim_G X$	95	$D(f)$	283

Symbols from set theory and general topology

$x \in A, x \notin A$ — x belongs to A , x does not belong to A

$\emptyset$ — empty set

$A \subset B, B \supset A$ — A is contained in B

$\{x \in X: \varphi(x)\}, \{x: \varphi(x)\}$ — the set of x 's which satisfy φ

$A \cup B, \bigcup_{s \in S} A_s, \bigcup_{i=1}^{\infty} A_i, \bigcup \mathcal{A}$ — union of sets

$A \cap B, \bigcap_{s \in S} A_s, \bigcap_{i=1}^{\infty} A_i, \bigcap \mathcal{A}$ — intersection of sets

$A \setminus B$ — difference of sets

$X \times Y, \prod_{s \in S} X_s, \prod_{i=1}^{\infty} X_i$ — Cartesian product of sets and spaces

$\{x_1, x_2, \dots, x_k\}$ — set consisting of points $x_1, x_2, \dots, x_k$

$x_1, x_2, \dots, \{x_i\}$ — infinite sequence

$\{x_1, x_2, \dots\}$ — set consisting of points $x_1, x_2, \dots$

$f(A), f^{-1}(B)$ — image and inverse image of a set under a mapping

gf — composition of mappings

$|A|$ — cardinality of a set

$\aleph_0$ — cardinality of the set of natural numbers (aleph zero)

- $\mathfrak{c}$ — cardinality of the set of real numbers (continuum)
 $A_1, A_2, \dots, A_\alpha, \dots, \alpha < \xi$ — transfinite sequence of sets
 R, I — the real line and the closed unit interval
 R^n — Euclidean n -space
 I^n — unit n -cube in R^n
 S^{n-1}, B^n — unit $(n-1)$ -sphere and unit n -ball in B^n
 $I^{\aleph_0}$ — Hilbert cube
 $\varrho(x, y)$ — distance between points
 $\varrho(x, A)$ — distance from a point to a set
 $\varrho(A, B)$ — distance between sets
 $B(x, r), B(A, r)$ — balls in a metric space
 $\delta(A)$ — diameter of a set
 $f: X \rightarrow Y$ — continuous mapping
 $f|_A, f_B$ — restrictions of a mapping
 $\bar{A}, \text{Int}A, \text{Fr}A$ — closure, interior and boundary of a set
 A^d — set of accumulation points of a set
 X^m — Cartesian power of a space
 X/E — quotient space
 $\{X_i, \pi_j^i\}, \{X_\alpha, \pi_\beta^\alpha, \Sigma\}$ — inverse sequence and inverse system of spaces
 $\varprojlim S$ — limit of an inverse sequence or an inverse system
 $X \oplus Y, \bigoplus_{s \in S} X_s$ — sum of spaces
 βX — Čech–Stone compactification

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