Sigma Series in Pure Mathematics
Volume 10

Ryszard Engelking

Theory of Dimensions
Finite and Infinite

Heldermann Verlag

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Preface

Dimension theory is the branch of topology devoted to the definition and study of the notion of dimension in various classes of topological spaces. It originated in the early twenties and rapidly developed during the next fifteen years. The investigations of that period were concentrated almost exclusively on separable metric spaces; they are brilliantly recapitulated in Hurewicz and Wallman's book *Dimension Theory*, published in 1941. After the initial impetus, dimension theory was at a standstill for ten years or more. A fresh start was made at the beginning of the fifties, when it was discovered that many results obtained for separable metric spaces can be extended to larger classes of spaces, provided that the dimension is properly defined. The last reservation necessitates an explanation. It is possible to define the dimension of a topological space X in three different ways, leading to the small inductive dimension $\operatorname{ind} X$, the large inductive dimension $\operatorname{Ind} X$, and the covering dimension $\dim X$. The three dimensions coincide in the class of separable metric spaces, i.e., $\operatorname{ind} X = \operatorname{Ind} X = \dim X$ for every separable metric space X. In larger classes of spaces the dimensions ind, Ind, and dim diverge. At first, the small inductive dimension ind was chiefly used; this notion has a great intuitive appeal and leads quickly and economically to an elegant theory. The dimensions Ind and dim played an auxiliary role and often were not even explicitly defined. To attain the next stage of development of dimension theory, namely its extension to larger classes of spaces, first and foremost to the class of metrizable spaces, it was necessary to realize that in fact there are three theories of dimension and to decide which is the proper one. The adoption of such a point of view quickly led to the understanding that the dimension ind is practically of no importance outside the class of separable metric spaces and that the dimension dim prevails over the dimension Ind. The greatest achievement in dimension theory during the fifties was the discovery that $\operatorname{Ind} X = \dim X$ for every metric space X and the creation of a satisfactory dimension theory for metrizable spaces. Since that time many important results on dimension of topological spaces have been obtained; they primarily bear upon the covering dimension dim. Included among them are theorems of an entirely new type, such as the factorization theorems, with no counterpart in the classical theory, and a few quite complicated examples, which finally demarcated the range of applicability of various dimensions. Then the interest shifted towards the spaces of infinite dimension: one tried to classify such spaces according to their "degree of infinite dimensionality". This was done in several ways, but it is not excluded that more satisfactory classifications are still to be discovered.

The above outline of the history of dimension theory helps to explain the arrangement of the material in the present book. In Chapter 1, which in itself constitutes more than a third of the book, the classical dimension theory of separable metric spaces is developed. The purpose of the chapter is twofold: to present a self-contained exposition of the most important section of dimension theory and to provide the necessary geometric background for the rather abstract considerations of subsequent chapters. Chapters 2 and 3 are devoted, respectively, to the large inductive dimension and the

covering dimension of general topological spaces. Chapter 4 develops the dimension theory of metrizable spaces, and the last three chapters deal with various classifications of infinite-dimensional spaces. The interdependence of Chapters 2–7 is loose. After having read Chapter 1, the reader should be able to continue the reading according to his own interests or needs; in particular, he can read small fragments of Chapters 2 and 3 and pass to Chapters 4, 5, 6 or 7 (cf. introductions to these chapters).

Chapter 1 is quite elementary; the reader is assumed to be familiar only with fundamental notions of topology of separable metric spaces. Further chapters are more difficult and demand from the reader some acquaintance with the notions and methods of general topology. The larger classes of spaces we consider, the more remote we are from the geometric source of the concept of dimension. In general spaces we often need quite involved arguments to prove facts that can be established in a simple way for separable metric spaces. This can be easily seen by comparing the proofs of corresponding theorems in Chapter 1 and in Chapters 2 and 3. Although, with some hesitation, I decided to present the results on infinite-dimensional spaces in a rather general setting, in the introductions to Chapters 5–7, I inform the reader interested only in separable metric spaces how the proofs can be simplified in this important case.

Each section ends with historical and bibliographic notes. These are followed by problems which aim both at testing the reader's comprehension of the material and at providing additional information; the harder problems usually contain detailed hints, which, in fact, are outlines of proofs.

The mark ■ indicates the end of a proof, of a remark, or of an example. If it appears immediately after the statement of a theorem, a proposition or a corollary, it means that the statement is obviously valid.

Numbers in square brackets refer to the bibliography at the end of the book. The papers of each author are numbered separately, the number being the year of publication. In referring to my *General Topology* (Engelking [1989]), which is quite often cited in the book, the symbol [GT] is used.

This book is a revised and completed edition of my earlier book *Dimension Theory* published in 1978. To the four chapters of the latter, three new chapters devoted to infinite-dimensional spaces have been added. In 1978 the study of these spaces just started and there were more problems than results in the area. Since then a fairly complete and most interesting theory was built up which is presented here for the first time in book form, unified and rounded out. Also the material in the first four chapters is updated: there are substantial changes in Sections 1.8, 2.4, 4.2 and 4.3, as well as in Chapter 3, where a new section on the behaviour of the covering dimension under mappings and results on the dimension of Cartesian products are included. Almost one half of the text has been written for this edition, and the bibliography is now twice as large as that of 1978. All these changes seem to justify the new title which better corresponds to the present content of the book.

It is a pleasure to acknowledge my deep indebtedness to E. Pol, R. Pol, J. Trzeciak and E. Wajch for their careful reading of the book and suggesting many improvements.

Ryszard Engelking

Warsaw, May 1995

Chapter 1

Dimension theory of separable metric spaces

In the present chapter the classical dimension theory of separable metric spaces is developed. Practically all the results of this chapter were obtained in the years 1920–1940. They constitute a canon on which, in subsequent years, dimension theory for larger classes of spaces was modelled. Accordingly, in Chapters 2–4 we shall follow the pattern of Chapter 1 and constantly refer to the classical theory. This arrangement influences our exposition: the classical material is discussed in relation to modern currents in the theory; in particular, the dimension functions Ind and dim are introduced at an early stage and are discussed simultaneously with the dimension function ind.

To avoid repetitions in subsequent chapters, a few definitions and theorems are stated in a more general setting, not for separable metric spaces but for topological, Hausdorff, regular or normal spaces; this is done only where the generalization does not influence the proof. If the reader is not acquainted with the notions of general topology, he should read "metric space" instead of "topological space", "Hausdorff space", "regular space", and "normal space". Reading the chapter for the first time, he can omit Sections 1.4 and 1.12–1.14, which deal with special topics; also the final parts of Sections 1.6, 1.8 and 1.9 can be skipped.

Let us describe briefly the contents of this chapter.

Section 1.1 opens with the definition of the small inductive dimension ind; next, some simple consequences and reformulations of the definition are discussed. Sections 1.2 and 1.3 are devoted to a study of zero-dimensional spaces. We prove there several important theorems, specified in the titles of the sections, which are generalized to spaces of higher dimension in Sections 1.5, 1.7 and 1.11.

In Section 1.4 we compare the properties of zero-dimensional spaces with the properties of other highly disconnected spaces. From this comparison it follows that the class of zero-dimensional spaces in the sense of the small inductive dimension is the best candidate for the zero level in a classification of separable metric spaces according to dimension. The results of this section, with the exception of Lemma 1.4.4, are used only in some problems and further on in Chapter 6.

Section 1.5 contains the first group of basic theorems on n-dimensional spaces. As will become clear later, the theorems in this group depend on the dimension ind, whereas most of the theorems in further sections depend on the dimension dim. Besides the generalizations of five theorems proved earlier for zero-dimensional spaces, Section 1.5 contains the decomposition and addition theorems.

In Section 1.6 the large inductive dimension Ind and the covering dimension dim are introduced; they both coincide with the small inductive dimension ind in the class

of separable metric spaces. In larger classes of spaces the three dimensions diverge. This will be discussed thoroughly in the following chapters. In particular, it will become evident that the dimension ind, though excellent in the class of separable metric spaces, loses its importance outside this class.

Section 1.7 opens with the compactification theorem. The location of this theorem at such an early stage in the exposition of dimension theory permits, it seems, a clearer arrangement of the material. From the compactification theorem the coincidence of ind, Ind, and dim for separable metric spaces is deduced.

In Section 1.8 we discuss the dimensional properties of Euclidean spaces. We begin with the fundamental theorem of dimension theory which states that $\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n$; then we characterize n-dimensional subsets of R^n as sets with a non-empty interior, and we show that no closed subset of dimension $\leq n-2$ separates R^n. This last result is strengthened in Mazurkiewicz's theorem, which is established in the second half of the section where also Lebesgue's covering theorem can be found.

Section 1.9 opens with the characterization of dimension in terms of extending mappings to spheres from a closed subspace over the whole space. From this characterization the Cantor-manifold theorem is deduced. In the final part of the section we give some information on the cohomological dimension.

In Section 1.10 we characterize n-dimensional spaces in terms of mappings with arbitrarily small fibers to polyhedra of geometric dimension $\leq n$ and develop the techniques of nerves and κ-mappings which are crucial for the considerations of this and the following section.

In Section 1.11 we prove that every n-dimensional space can be embedded in R^{2n+1} and we describe two subspaces of R^{2n+1} which contain topologically all n-dimensional spaces; the second of those is a compact space.

The last three sections are of a more special character. Section 1.12 is devoted to a study of relations between the dimensions of the domain and range of a continuous mapping. In Section 1.13 we characterize compact spaces of dimension $\leq n$ as spaces homeomorphic to the limits of inverse sequences of polyhedra of geometric dimension $\leq n$, and in Section 1.14 we briefly discuss the prospects for an axiomatization of dimension theory.

Symbols from set theory and general topology that are not defined in the text can be identified with the aid of the list on pp. 393–394.

1.1. Definition of the small inductive dimension

1.1.1. DEFINITION. To every regular space X one assigns the *small inductive dimension* of X, denoted by $\operatorname{ind} X$, which is an integer larger than or equal to -1, or the "infinite number" ∞; the definition of the dimension function ind consists in the following conditions:

(MU1) $\operatorname{ind} X = -1$ *if and only if* $X = \emptyset$;

(MU2) $\operatorname{ind} X \leq n$, *where* $n = 0, 1, \ldots$, *if for every point* $x \in X$ *and each neighbourhood* $V \subset X$ *of the point* x *there exists an open set* $U \subset X$ *such that*

$$x \in U \subset V \quad \text{and} \quad \operatorname{ind} \operatorname{Fr} U \leq n-1;$$

(MU3) $\operatorname{ind} X = n$ *if* $\operatorname{ind} X \leq n$ *and* $\operatorname{ind} X > n-1$, *i.e., the inequality* $\operatorname{ind} X \leq n-1$ *does not hold*;

(MU4) $\operatorname{ind} X = \infty$ *if* $\operatorname{ind} X > n$ *for* $n = -1, 0, 1, \ldots$

The small inductive dimension ind is also called the *Menger–Urysohn dimension*.

Applying induction with respect to $\operatorname{ind} X$, one can easily verify that whenever regular spaces X and Y are homeomorphic, then $\operatorname{ind} X = \operatorname{ind} Y$, i.e., the small inductive dimension is a topological invariant.

A separable metric space X will be called *n-dimensional* if $\operatorname{ind} X = n$, *finite-dimensional* if $\operatorname{ind} X < \infty$, and *infinite-dimensional* if $\operatorname{ind} X = \infty$. However, the term *zero-dimensional* will be used in a wider sense: it will refer to every regular space X with $\operatorname{ind} X = 0$.

In order to simplify some statements, we shall assume that the formulas $n \leq \infty$ and $n + \infty = \infty + n = \infty + \infty = \infty$ hold for every integer n.

Since every subspace M of a regular space X is itself regular, if the dimension ind is defined for a space X it is also defined for every subspace M of X.

1.1.2. THE SUBSPACE THEOREM. *For every subspace M of a regular space X we have* $\operatorname{ind} M \leq \operatorname{ind} X$.

PROOF. The theorem is obvious if $\operatorname{ind} X = \infty$, so that one can suppose that $\operatorname{ind} X < \infty$. We shall apply induction with respect to $\operatorname{ind} X$. Clearly, the inequality holds if $\operatorname{ind} X = -1$.

Assume that the theorem is proved for all regular spaces whose dimension does not exceed $n-1 \geq -1$. Consider a regular space X with $\operatorname{ind} X = n$, a subspace M of X, a point $x \in M$ and a neighbourhood V of x in M. By the definition of the subspace topology, there exists an open subset V_1 of the space X satisfying the equality $V = M \cap V_1$. Since $\operatorname{ind} X \leq n$, there exists an open set $U_1 \subset X$ such that

$$x \in U_1 \subset V_1 \quad \text{and} \quad \operatorname{ind} \operatorname{Fr} U_1 \leq n-1.$$

The intersection $U = M \cap U_1$ is open in M and satisfies $x \in U \subset V$. The boundary $\operatorname{Fr}_M U$ of the set U in the space M is equal to $M \cap \overline{M \cap U_1} \cap \overline{M \setminus U_1}$, where the bar denotes the closure operation in X; thus $\operatorname{Fr}_M U$ is a subspace of the space $\operatorname{Fr} U_1$. Hence, by the inductive assumption, $\operatorname{ind} \operatorname{Fr}_M U \leq n-1$, which – together with (MU2) – yields the inequality $\operatorname{ind} M \leq n = \operatorname{ind} X$. ■

Sometimes it is more convenient to apply condition (MU2) in a slightly different form involving the notion of a partition.

1.1.3. DEFINITION. Let X be a topological space and A, B a pair of disjoint subsets of the space X; we say that a set $L \subset X$ is a *partition between A and B* if there exist open sets $U, W \subset X$ satisfying the conditions

$$(1) \qquad A \subset U, \quad B \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad X \setminus L = U \cup W.$$

Clearly, the partition L is a closed subset of X.

Since, as usual, we identify the point x and the one-point set $\{x\}$, a partition between a point and a closed set, and a partition between two points, is also defined.

The notion of a partition is related to the notion of a separator between a pair A, B of disjoint subsets of X. Let us recall that a set $T \subset X$ is a *separator between A and B*, or T *separates the space X between A and B*, if there exist two sets U_0 and W_0 open in the subspace $X \setminus T$ and such that $A \subset U_0$, $B \subset W_0$, $U_0 \cap W_0 = \emptyset$ and $X \setminus T = U_0 \cup W_0$. Obviously, a set $L \subset X$ is a partition between A and B if and only if L is a closed subset of X and L is a separator between A and B.

Separators are not to be confused with cuts, a related notion we will refer to in the notes below and in Section 1.8. Let us recall that a set $T \subset X$ is a *cut between A and B*, or T *cuts the space X between A and B*, if the sets A, B and T are pairwise disjoint and every continuum, i.e., a compact connected space $C \subset X$, intersecting both A and B also intersects T. Clearly, every separator between A and B is a cut between A and B, but the two notions do not coincide (see Problems 1.1.D and 1.8.H).

1.1.4. PROPOSITION. *A regular space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if for every point $x \in X$ and each closed set $B \subset X$ which does not contain x there exists a partition L between x and B such that* $\operatorname{ind} L \leq n-1$.

PROOF. Let X be a regular space satisfying $\operatorname{ind} X \leq n \geq 0$; consider a point $x \in X$ and a closed set $B \subset X$ such that $x \notin B$. There exist a neighbourhood $V \subset X$ of x such that $\overline{V} \subset X \setminus B$ and an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind} \operatorname{Fr} U \leq n-1$. One easily sees that the set $L = \operatorname{Fr} U$ is a partition between x and B: the sets U and $W = X \setminus \overline{U}$ satisfy conditions (1).

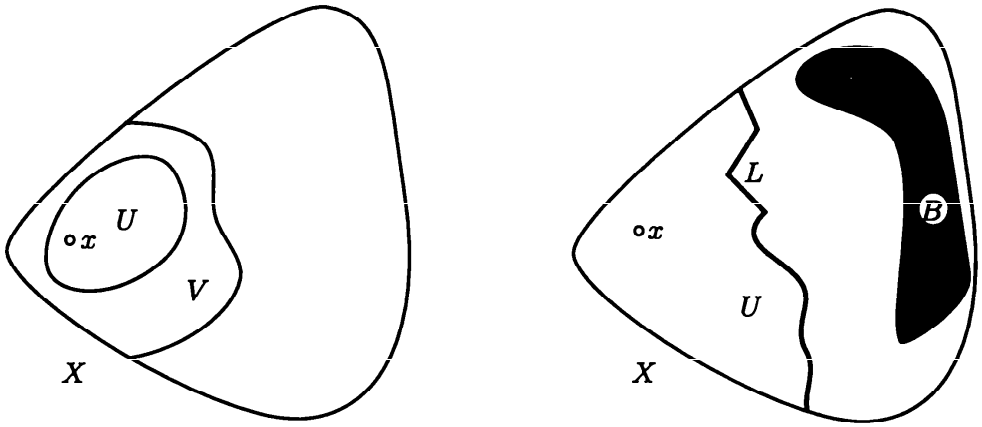


Fig. 1

Now, assume that a regular space X satisfies the condition of the proposition; consider a point $x \in X$ and a neighbourhood $V \subset X$ of x. Let L be a partition between x and $B = X \setminus V$ such that $\operatorname{ind} L \leq n-1$ and let $U, W \subset X$ be open subsets of X satisfying conditions (1). We have

$$x \in U \subset X \setminus W \subset X \setminus B = V$$

and

$$\operatorname{Fr} U \subset (X \setminus U) \cap (X \setminus W) = X \setminus (U \cup W) = L,$$

so that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ by virtue of the subspace theorem. Hence $\operatorname{ind} X \leq n$. ■

Obviously, a regular space X satisfies the inequality $\operatorname{ind} X \leq n \geq 0$ if and only if X has a base $\mathcal{B}$ such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$. In the realm of separable metric spaces this observation can be made more precise.

1.1.5. LEMMA. *If a topological space X has a countable base, then every base $\mathcal{B}$ for the space X contains a countable family $\mathcal{B}_0$ which is still a base for X.*

PROOF. Let $\mathcal{D} = \{V_i\}_{i=1}^{\infty}$ be a countable base for the space X. For each pair i, j of positive integers let

$$\mathcal{B}_{i,j} = \{U \in \mathcal{B} : V_i \subset U \subset V_j\}$$

and choose an element from $\mathcal{B}_{i,j}$ if $\mathcal{B}_{i,j} \neq \emptyset$. Clearly, the family $\mathcal{B}_0 \subset \mathcal{B}$ of all chosen elements is countable and is a base for X. ■

1.1.6. THEOREM. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if X has a countable base $\mathcal{B}$ such that* $\operatorname{ind} \operatorname{Fr} U \leq n-1$ *for every $U \in \mathcal{B}$.* ■

Historical and bibliographic notes

The dimension of simple geometric objects is one of the most intuitive mathematical notions. There is no doubt that a segment, a square and a cube have dimension 1, 2 and 3, respectively. The necessity of a precise definition of dimension became obvious only when it was established that a segment has exactly as many points as a square (G. Cantor 1878), and that a square has a continuous parametric representation on a segment, i.e., that there exist continuous functions $x(t)$ and $y(t)$ such that the points of the form $(x(t), y(t))$ fill out a square when t runs through a segment (G. Peano 1890). First and foremost, the question arose whether there exists a parametric representation of a square on a segment which is at the same time one-to-one and continuous, i.e., whether a segment and a square are homeomorphic, and – more generally – whether the n-cube I^n and the m-cube I^m are homeomorphic if $n \neq m$; clearly, a negative answer was expected. Between 1890 and 1910 a few faulty proofs of the fact that I^n and I^m are not homeomorphic if $n \neq m$ were produced and it was established that I, I^2 and I^3 are all topologically different.

The theorem that I^n and I^m are not homeomorphic if $n \neq m$ was proved by Brouwer in [1911]. It suggests itself to try to prove this theorem by defining a function d which assigns to every space a natural number, the dimension of the space, so that homeomorphic spaces have equal dimensions and $d(I^n) = n$. It was none too easy, however, to discover such functions; the search for them gave rise to dimension theory. In Brouwer's paper [1911] no such function is explicitly defined, yet an analysis of the proof shows that to distinguish between I^n and I^m for $n \neq m$ the author applies the fact that for a sufficiently small positive number ϵ it is impossible to transform the n-cube $I^n \subset R^n$ into a polyhedron $K \subset R^n$ of geometric dimension less than n by a continuous mapping $f\colon I^n \to K$ with fibres $f^{-1}(x)$ of diameter less than ϵ. As we shall show in Section 1.10, this property characterizes compact subspaces of R^n which have dimension equal to n. Another topological property of I^n was discovered by Lebesgue in [1911], viz. the fact that I^n can be covered, for every $\epsilon > 0$, by a finite family of closed sets with diameters less than ϵ such that all intersections of $n+2$ members of the family are empty, and cannot be covered by a finite family of closed sets with diameters less than 1 such that all intersections of $n+1$ members of the family are empty (see Theorem 1.8.20). Obviously, Lebesgue's observation implies that I^n and I^m are not

homeomorphic if $n \neq m$. Though the proof outlined by Lebesgue contains a gap (filled by Brouwer in [1913] and by Lebesgue in [1921]), the discovery of the new invariant was an important achievement which eventually led to the definition of the covering dimension. Lebesgue's paper [1911] contains one more important discovery. He stated (the proof was given in his paper [1921]) that for every continuous parametric representation $f(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ of the n-cube I^n with $n \geq 2$ on the closed unit interval I, some fibres of f have cardinality at least $n+1$, and that I^n has a continuous parametric representation on I with fibres of cardinality at most $n+1$.

A decisive step towards the definition of dimension was made by Poincaré in [1903], where he observed that dimension is related to the notion of separation and could be defined inductively. Poincaré called attention to the simple fact that solid bodies can be separated by surfaces, surfaces by lines, and lines by points. In [1912] he defined an n-dimensional continuum as one which can be divided into several parts by means of cuts along $(n-1)$-dimensional continua. Poincaré's definition was a sketchy one, neither a continuum nor a cut were properly defined; he did not take care, or did not have time, before his death in 1912, to give his important ideas a precise form.

The first definition of a dimension function was given by Brouwer in [1913], where he defined a topological invariant of compact metric spaces, called *Dimensionsgrad*, and proved that the *Dimensionsgrad* of the n-cube I^n is equal to n. Following Poincaré's suggestion, the definition is inductive and refers to the notion of a cut: Brouwer defined the spaces with *Dimensionsgrad* 0 as spaces which do not contain any continuum of cardinality larger than one (i.e., as punctiform spaces; cf. Section 1.4), and stated that a space X has *Dimensionsgrad* less than or equal to $n \geq 1$ if for every pair A, B of disjoint closed subsets of X there exists a closed set $T \subset X$ which cuts X between A and B and has *Dimensionsgrad* less than or equal to $n-1$. Brouwer's notion of dimension is not equivalent to what we now understand by the dimension of a compact metric space; however, the two notions coincide in the realm of locally connected complete metric spaces (the proof is based on the fact that in this class of spaces the notions of a separator and a cut coincide for closed subsets; see Engelking and Sieklucki [1992], p. 317). Brouwer did not study the new invariant closely: he only used it to give another proof that I^n and I^m are not homeomorphic if $n \neq m$.

Referring to the second part of Lebesgue's paper [1911], Mazurkiewicz showed in [1915] that for every continuous parametric representation f of the square I^2 on the interval I, some fibres of f have cardinality at least 3, and proved that every continuum $C \subset R^2$ whose interior in R^2 is empty can be represented as a continuous image of the Cantor set under a mapping with fibres of cardinality at most 2. These results led him to define the dimension of a compact metric space X as the smallest integer n with the property that the space X can be represented as the continuous image of a closed subspace of the Cantor set under a mapping f such that $|f^{-1}(x)| \leq n+1$ for every $x \in X$. It was proved later (cf. Problem 1.7.F) that this definition is equivalent to the definition of the small inductive dimension, but Mazurkiewicz's paper had no influence on the development of dimension theory.

The definition of the small inductive dimension ind was formulated by Urysohn in [1922] and by Menger in [1923]; both papers also contain Theorem 1.1.2. Menger

and Urysohn, working independently, built the framework of the dimension theory of compact metric spaces, but Urysohn was ahead of Menger by a few months and was able to establish a larger number of basic properties of the dimension. Urysohn's results are presented in a two-part paper, [1925] and [1926], published after his death in 1924, whereas Menger's results are contained in his papers [1923] and [1924] and in his book [1928]. A generalization of dimension theory to separable metric spaces is due to Tumarkin ([1925] and [1926]) and Hurewicz ([1927] and [1927b]). In [1927] Hurewicz, in a particularly successful way, made use of the inductive character of dimension and greatly simplified the proofs of some important theorems, e.g., the sum theorem and the decomposition theorem. Moreover, owing to his discovery of the compactification theorem, Hurewicz reduced, in a sense, the dimension theory of separable metric spaces to the dimension theory of compact metric spaces.

When the work of Menger and Urysohn drew the attention of mathematicians to the notion of dimension, Brouwer (in [1923], [1924], [1924a] and [1924b]) claimed that the definition of his *Dimensionsgrad* was marred by a clerical error and that it should read exactly as the definition of the large inductive dimension (see Section 1.6) and thus should yield the same notion of dimension for compact metric spaces; he also commented that even the original faulty definition of *Dimensionsgrad* could serve as a basis for an equally good, although different, dimension theory. Brouwer's arguments do not seem quite convincing. After the publication of Menger's book [1928] a heated discussion arose between Brouwer ([1928]) and Menger ([1929a], [1930], [1933]) concerning priority in defining the notion of dimension; a good account of this discussion is contained in Freudenthal's notes in the second volume of Brouwer's *Collected Papers* (Brouwer [1976]). The history of the first years of dimension theory and, in particular, an evaluation of the contributions of Menger and Urysohn can be found in Alexandroff [1951]. Johnson's two-part study ([1979] and [1981]) of the origins and early history of dimension theory is perfectly documented and most interesting.

Problems

1.1.A. Observe that a metric space X satisfies the inequality $\operatorname{ind} X \leq n \geq 0$ if and only if for every point $x \in X$ and each positive number ϵ there exists a neighbourhood $U \subset X$ of the point x such that $\delta(U) < \epsilon$ and $\operatorname{ind} \operatorname{Fr} U \leq n-1$.

1.1.B. To every regular space X and every point $x \in X$ one assigns the *dimension of X at the point x*, denoted by $\operatorname{ind}_x X$, which is an integer larger than or equal to 0, or the "infinite number" ∞; the definition consists in the following conditions: (1) $\operatorname{ind}_x X \leq n$ if for each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind} \operatorname{Fr} U \leq n-1$; (2) $\operatorname{ind}_x X = n$ if $\operatorname{ind}_x X \leq n$ and $\operatorname{ind}_x X > n-1$; (3) $\operatorname{ind}_x X = \infty$ if $\operatorname{ind}_x X > n$ for $n = 0, 1, \ldots$

(a) Note that $\operatorname{ind} X \leq n$ if and only if $\operatorname{ind}_x X \leq n$ for every $x \in X$.

(b) Formulate and prove the counterparts of 1.1.2, 1.1.4 and 1.1.6 for $\operatorname{ind}_x X$.

1.1.C. Show that whenever regular spaces X and Y are homeomorphic, then $\operatorname{ind} X = \operatorname{ind} Y$.

1.1.D. Give an example of a subspace X of the plane R^2 and of a closed set $T \subset X$ with the property that for some pair A, B of disjoint closed subsets of X the set T is a cut between A and B but is not a separator between A and B.

1.2. The separation and enlargement theorems for dimension 0

To begin, we list the results of the previous section in the case of zero-dimensional spaces.

1.2.1. PROPOSITION. *A regular space X is zero-dimensional if and only if X is non-empty and for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open-and-closed set $U \subset X$ such that $x \in U \subset V$.* ■

1.2.2. PROPOSITION. *Every non-empty subspace of a zero-dimensional space is zero-dimensional.* ■

1.2.3. PROPOSITION. *A regular space X is zero-dimensional if and only if X is non-empty and for every point $x \in X$ and each closed set $B \subset X$ which does not contain x the empty set is a partition between x and B.* ■

1.2.4. PROPOSITION. *A separable metric space X is zero-dimensional if and only if X is non-empty and has a countable base consisting of open-and-closed sets.* ■

We shall now discuss a few examples.

1.2.5. EXAMPLES. (a) The space $P \subset R$ of irrational numbers is zero-dimensional because it has a countable base consisting of open-and-closed sets, viz., the sets of the form $P \cap (a, b)$, where a and b are rational numbers.

(b) Similarly, the space $Q \subset R$ of rational numbers is zero-dimensional. More generally, if a metric space X satisfies the condition $0 < |X| < \mathbf{c}$, then $\operatorname{ind} X = 0$. Indeed, for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exist a positive number r such that $B(x, r) \subset V$ and a positive number $t < r$ such that $\rho(x, y) \neq t$ for every $y \in X$, where ρ is the metric on the space X; the set $U = B(x, t)$ satisfies the condition $x \in U \subset V$ and is open-and-closed, because $\operatorname{Fr} U \subset \{y \in X : \rho(x, y) = t\} = \emptyset$.

(c) A non-empty subspace X of the real line R is zero-dimensional if and only if it does not contain any interval. Indeed, since intervals are connected and no connected space containing at least two points is zero-dimensional, the condition is necessary by virtue of Proposition 1.2.2. The condition is also sufficient, because the sets of the form $X \cap (a, b)$, where $a, b \in R$ and $a < x < b$, constitute a base for the space X at the point x, and for each $V = X \cap (a, b)$ one can find an open-and-closed set $U \subset V$ such that $x \in U \subset V$ – it suffices to define $U = X \cap (c, d)$, where $c \in (a, x) \setminus X$ and $d \in (x, b) \setminus X$.

(d) In particular, the subspace C of the real line consisting of all numbers in the closed unit interval I that have a tryadic expansion in which the digit 1 does not occur, i.e., the set of all numbers of the form $\sum_{i=1}^{\infty} 2x_i/3^i$, where x_i is equal to 0 or 1 for $i = 1, 2, \ldots$, is zero-dimensional. Indeed, the set C does not contain any interval

because $C = \bigcap_{i=1}^{\infty} F_i$, where F_i is the subset of I consisting of all numbers having a tryadic expansion in which 1 does not occur as the j-th digit for $j \leq i$, and F_i contains no interval longer than $1/3^i$. One easily sees that the set F_1 is obtained from I by removing the "middle" interval $(1/3, 2/3)$, the set F_2 is obtained from F_1 by removing the "middle" intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ of both parts of F_1, and so on. The set F_i consists of 2^i disjoint intervals of length $1/3^i$ each.

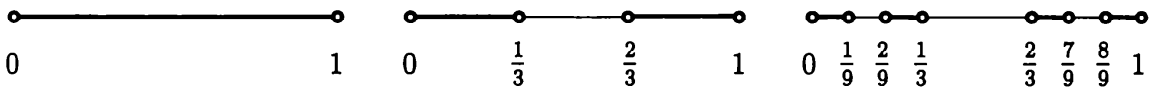


Fig. 2

The subspace C of the real line is called the *Cantor set*. Since C is a closed subset of I, it is compact.

(e) The Cartesian product $Q^n \subset R^n$ is zero-dimensional, because it is a countable space. The Cartesian product $P^n \subset R^n$ is also zero-dimensional; the proof is left to the reader (cf. Theorem 1.3.6). ■

As shown in the following theorem, zero-dimensional separable metric spaces have a separation property which is much stronger than the property described in Proposition 1.2.3.

1.2.6. THE FIRST SEPARATION THEOREM FOR DIMENSION 0. *If X is a zero-dimensional separable metric space, then for every pair A, B of disjoint closed subsets of X the empty set is a partition between A and B, i.e., there exists an open-and-closed set $U \subset X$ such that $A \subset U$ and $B \subset X \setminus U$.*

PROOF. For every $x \in X$ there exists an open-and-closed set $W_x \subset X$ such that $x \in W_x$ and

$$\text{either} \quad A \cap W_x = \emptyset \quad \text{or} \quad B \cap W_x = \emptyset. \tag{1}$$

The open cover $\{W_x\}_{x \in X}$ of the space X has a countable subcover $\{W_{x_i}\}_{i=1}^{\infty}$. The sets

$$U_i = W_{x_i} \setminus \bigcup_{j<i} W_{x_j} \subset W_{x_i}, \quad \text{where } i = 1, 2, \ldots,$$

are open and constitute a cover of X. Let us define

$$U = \bigcup \{U_i : A \cap U_i \neq \emptyset\} \quad \text{and} \quad W = \bigcup \{U_i : A \cap U_i = \emptyset\};$$

obviously $A \subset U$, and it follows from (1) that $B \subset W$. Since the sets U_i are pairwise disjoint, it follows that $W = X \setminus U$, which implies that U is open-and-closed and $B \subset X \setminus U$. ■

1.2.7. REMARK. It follows from the above proof that in Theorem 1.2.6 the assumption that X is a separable metric space can be replaced by the weaker assumption that X is a *Lindelöf space*, i.e., a regular space such that every open cover of it has a countable subcover. ■

Now we are going to prove the second separation theorem, which is still stronger than Theorem 1.2.6. Let us recall that two subsets A and B of a topological space X are

separated if $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. One easily sees that the sets A and B are separated if and only if they are disjoint and open (or – equivalently – closed) in their union $A \cup B$, i.e., if $A \cap B = \emptyset$ and if the empty set is a partition between A and B in the subspace $A \cup B$ of X. In particular, two disjoint open sets, and also two disjoint closed sets, are separated.

The second separation theorem will be deduced from two lemmas; for the purpose of future applications, the second lemma is formulated in a more general way than needed in this section.

1.2.8. LEMMA. *For every pair A, B of separated subsets of a metric space X there exist open sets $U, W \subset X$ such that*

$$A \subset U, \quad B \subset W \quad \text{and} \quad U \cap W = \emptyset. \tag{2}$$

PROOF. Let ρ be the metric on the space X, and let $f(x) = \rho(x, A)$ and $g(x) = \rho(x, B)$ denote the distance of the point $x \in X$ from A and B, respectively. Since the functions f and g are continuous, the sets

$$U = \{x \in X : f(x) - g(x) < 0\} \quad \text{and} \quad W = \{x \in X : f(x) - g(x) > 0\}$$

are open. The inclusions in (2) follow from the equalities $f^{-1}(0) = \overline{A}$ and $g^{-1}(0) = \overline{B}$; the equality $U \cap W = \emptyset$ follows directly from the definition of U and W. ■

1.2.9. LEMMA. *Let M be a subspace of a metric space X and A, B a pair of disjoint closed subsets of X. For every partition L' in the space M between $M \cap \overline{V}_1$ and $M \cap \overline{V}_2$, where V_1, V_2 are open subsets of X such that $A \subset V_1, B \subset V_2$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, there exists a partition L in the space X between A and B which satisfies the inclusion $M \cap L \subset L'$.*

If M is a closed subspace of a metric space X and A, B a pair of disjoint closed subsets of X, then for every partition L' in the space M between $M \cap A$ and $M \cap B$ there exists a partition L in the space X between A and B which satisfies the inclusion $M \cap L \subset L'$.

PROOF. Let U', W' be open subsets of M satisfying the conditions

$$M \cap \overline{V}_1 \subset U', \quad M \cap \overline{V}_2 \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad M \setminus L' = U' \cup W'.$$

Observe that

$$A \cap \overline{W'} = \emptyset = B \cap \overline{U'}. \tag{3}$$

Indeed, since $V_1 \cap W' = M \cap V_1 \cap W' \subset U' \cap W' = \emptyset$ and since the set V_1 is open, we have $V_1 \cap \overline{W'} = \emptyset$, which implies that $A \cap \overline{W'} = \emptyset$; by symmetry of assumptions, also $B \cap \overline{U'} = \emptyset$.

The sets U' and W' are disjoint and open in their union $U' \cup W'$, and thus they are separated, i.e.,

$$U' \cap \overline{W'} = \emptyset = \overline{U'} \cap W'. \tag{4}$$

It follows from (3) and (4) that the sets $A \cup U'$ and $B \cup W'$ also are separated. Hence, by Lemma 1.2.8, there exist open sets $U, W \subset X$ such that

$$A \cup U' \subset U, \quad B \cup W' \subset W \quad \text{and} \quad U \cap W = \emptyset.$$

The set $L = X \setminus (U \cup W)$ is a partition in the space X between A and B. Since

$$M \cap L = M \setminus (U \cup W) \subset M \setminus (U' \cup W') = L',$$

the first part of the lemma is established.

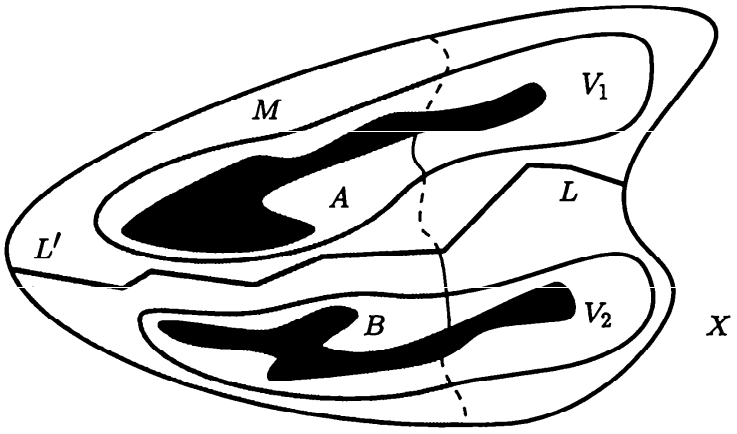


Fig. 3

To prove the second part, consider open subsets U_1, W_1 of the subspace M satisfying the conditions

$$M \cap A \subset U_1, \quad M \cap B \subset W_1, \quad U_1 \cap W_1 = \emptyset \quad \text{and} \quad M \setminus L' = U_1 \cup W_1.$$

Since $A \cap (M \setminus U_1) = \emptyset$, $B \cap (M \setminus W_1) = \emptyset$ and $A \cap B = \emptyset$, there exist open sets $V_1, V_2 \subset X$ such that

$$A \subset V_1 \subset \overline{V}_1 \subset X \setminus (M \setminus U_1), \quad B \subset V_2 \subset \overline{V}_2 \subset X \setminus (M \setminus W_1) \quad \text{and} \quad \overline{V}_1 \cap \overline{V}_2 = \emptyset.$$

Obviously, L' is a partition in the space M between $M \cap \overline{V}_1$ and $M \cap \overline{V}_2$, so that the required partition L exists by the first part of the lemma. ■

1.2.10. REMARK. Let us note that the proof of Lemma 1.2.9 is valid in any normal space X for which Lemma 1.2.8 holds; as the latter lemma holds in hereditarily normal spaces (see Theorem 2.1.1), Lemma 1.2.9 is valid in hereditarily normal spaces (cf. Problem 2.2.E(a) and Lemma 6.1.5).

Let us also note that in both situations considered in Lemma 1.2.9 one can find a partition L satisfying the equality $M \cap L = L'$. More exactly, for each pair U', W' of open sets in M witnessing that L' is a partition, one can find disjoint open sets $U, W \subset X$ such that $A \subset U, B \subset W, M \cap U = U'$ and $M \cap W = W'$. Indeed, it suffices to shrink the open sets U and W defined in the proof to the open sets $U \setminus \overline{L}'$ and $W \setminus \overline{L}'$. ■

1.2.11. THE SECOND SEPARATION THEOREM FOR DIMENSION 0. *If X is an arbitrary metric space and Z a zero-dimensional separable subspace of X, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $L \cap Z = \emptyset$.*

PROOF. Consider open sets $V_1, V_2 \subset X$ such that $A \subset V_1, B \subset V_2$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$. By virtue of Theorem 1.2.6, the empty set is a partition in the space Z between $Z \cap \overline{V}_1$ and $Z \cap \overline{V}_2$. Applying the first part of Lemma 1.2.9 we obtain the required partition L. ■

The last theorem yields a characterization of zero-dimensional subspaces in terms of neighbourhoods in the whole space (cf. the proof of Proposition 1.1.4):

1.2.12. PROPOSITION. *A separable subspace M of an arbitrary metric space X is zero-dimensional if and only if M is non-empty and for every point $x \in M$ (or – equivalently – for every point $x \in X$) and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $M \cap \operatorname{Fr} U = \emptyset$.* ■

Proposition 1.2.12 and Lemma 1.1.5 imply

1.2.13. PROPOSITION. *A subspace M of a separable metric space X is zero-dimensional if and only if M is non-empty and X has a countable base $\mathcal{B}$ such that $M \cap \operatorname{Fr} U = \emptyset$ for every $U \in \mathcal{B}$.* ■

It is natural to ask at this point whether every zero-dimensional subspace of a given space can be enlarged to a "better" zero-dimensional subspace. The example of the subspace of the real line consisting of rational numbers shows that, generally, zero-dimensional subspaces cannot be enlarged to closed zero-dimensional subspaces; however, as shown in the next theorem, they can always be enlarged to zero-dimensional G_δ-sets. Let us recall that *G_δ-sets* are defined as countable intersections of open sets, and *F_σ-sets* as their complements, i.e., countable unions of closed sets.

1.2.14. THE ENLARGEMENT THEOREM FOR DIMENSION 0. *For every zero-dimensional separable subspace Z of an arbitrary metric space X there exists a G_δ-set Z^* in X such that $Z \subset Z^*$ and the subspace Z^* of X is zero-dimensional.*

PROOF. Since every closed subset of a metric space is a G_δ-set and a G_δ-set in a subspace which itself is a G_δ-set is a G_δ-set in the space, one can assume that $\overline{Z} = X$. Thus X is separable and, by virtue of Proposition 1.2.13, has a countable base $\mathcal{B}$ such that $Z \cap \operatorname{Fr} U = \emptyset$ for every $U \in \mathcal{B}$. The union $F = \bigcup\{\operatorname{Fr} U : U \in \mathcal{B}\}$ is an F_σ-set, and its complement $Z^* = X \setminus F$ is a G_δ-set which contains Z. From Proposition 1.2.13 it follows that Z^* is zero-dimensional. ■

As the reader has undoubtedly observed, Theorem 1.2.6 states the equivalence of two properties of a separable metric space, viz., the property that the empty set is a partition between any disjoint closed sets, and the property that the empty set is a partition between every point and each closed set which does not contain it. The question arises whether the property that the empty set is a partition between any two distinct points is still the same property. As shown in the following example, the answer to this question is negative (cf. the notion of a totally disconnected space discussed in Section 1.4).

1.2.15. ERDŐS' EXAMPLE. Let us recall that *Hilbert space* H consists of all infinite sequences $\{x_i\}$ of real numbers such that the series $\sum_{i=1}^{\infty} x_i^2$ is convergent. For every point $x = \{x_i\} \in H$ its *norm* $\|x\|$ is the number $\sqrt{\sum_{i=1}^{\infty} x_i^2}$, and the distance between $x = \{x_i\}$ and $y = \{y_i\}$ is defined by the formula

$$\rho(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2},$$

i.e., it is equal to the norm of the difference $x - y$. The function ρ is a metric on H, and H is a separable metric space.

We shall show that, in the subspace H_0 of H consisting of the points $\{x_i\} \in H$ such that x_i is rational for $i = 1, 2, \ldots$, the empty set is a partition between any two distinct points, and yet H_0 is not zero-dimensional.

Let us consider a pair $x = \{x_i\}$, $y = \{y_i\}$ of distinct points of H_0. There exists a natural number i_0 such that $x_{i_0} \neq y_{i_0}$; without loss of generality one can assume that $x_{i_0} < y_{i_0}$. Take an irrational number t such that $x_{i_0} < t < y_{i_0}$ and define

$$U = \{\{z_i\} \in H_0 : z_{i_0} < t\}.$$

One can easily verify that the set U is open-and-closed in H_0. As $x \in U \subset H_0 \setminus \{y\}$, the empty set is a partition between x and y.

Now, let $x_0 \in H_0$ be the sequence whose terms are all equal to zero and let $V = B(x_0, 1) = \{x \in H_0 : \|x\| < 1\}$. We shall show that for every neighbourhood U of x_0 which is contained in V we have $\operatorname{Fr} U \neq \emptyset$.

We shall define inductively a sequence $a_1, a_2, \ldots$ of rational numbers such that

$$x_k = (a_1, a_2, \ldots, a_k, 0, 0, \ldots) \in U \quad \text{and} \quad \rho(x_k, H_0 \setminus U) \leq 1/k \tag{5}$$

for $k = 1, 2, \ldots$ Conditions (5) are satisfied for $k = 1$ if we let $a_1 = 0$. Suppose that the rational numbers $a_1, a_2, \ldots, a_{m-1}$ are already defined and conditions (5) are satisfied for $k \leq m - 1$. The sequence

$$x_i^m = (a_1, a_2, \ldots, a_{m-1}, i/m, 0, 0, \ldots)$$

is an element of H_0 for $i = 1, 2, \ldots, m$. As $x_0^m = x_{m-1} \in U$ and $x_m^m \notin U$ because $\|x_m^m\| \geq 1$, there exists an $i_0 < m$ such that $x_{i_0}^m \in U$ and $x_{i_0+1}^m \notin U$. One easily sees that conditions (5) are satisfied for $k = m$ if we let $a_m = i_0/m$. Thus the sequence $a_1, a_2, \ldots$ is defined. From the first part of (5) it follows that $\sum_{i=1}^k a_i^2 < 1$ for $k = 1, 2, \ldots$, so that $\sum_{i=1}^\infty a_i^2 \leq 1$. Hence $a = \{a_i\}$ is a point of H_0 and $a \in \overline{U}$. On the other hand, from the second part of (5) it follows that $a \in \overline{H_0 \setminus U}$, so that $a \in \operatorname{Fr} U$ and $\operatorname{Fr} U \neq \emptyset$.

Thus we have shown that there is no open-and-closed set $U \subset H_0$ such that $x_0 \in U \subset V$, and this implies that the space H_0 is not zero-dimensional.

To conclude, let us observe that the only open-and-closed bounded subset of the space H_0 is the empty set. Indeed, if there existed a non-empty open-and-closed bounded set $W \subset H_0$, then by a suitable translation and contraction of W we could obtain an open-and-closed set $U \subset H_0$ such that $x_0 \in U \subset V$. ■

Historical and bibliographic notes

Zero-dimensional spaces were defined by Sierpiński in [1921], before dimension theory was originated. Sierpiński's objective was to compare a few classes of metric spaces which are highly disconnected; a similar comparison will be made in Section 1.4. The theorems in the present section are all special cases of theorems which will be proved in Section 1.5 for an arbitrary dimension n; they were established in this more general form from the beginning. Theorem 1.2.6 was proved for compact metric spaces by Menger in [1924] and by Urysohn in [1926] and was extended to separable metric spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]; the generalization stated in Remark 1.2.7 was obtained by Vedenissoff in [1939]. Theorem 1.2.11 was proved for compact metric spaces by Menger in [1924] and extended

to separable metric spaces by Hurewicz in [1927]. Theorem 1.2.14 was obtained by Tumarkin in [1926] (announcement [1925]). The space in Example 1.2.15 was described by Erdős in [1940]; the first example of a space with similar properties was given by Sierpiński in [1921].

Problems

1.2.A. Let M be a subspace of a metric space X and let $x \in M$. Prove that $\operatorname{ind}_x M = 0$ if and only if there exists a base $\{U_i\}_{i=1}^{\infty}$ for the space X at the point x such that $M \cap \operatorname{Fr} U_i = \emptyset$ for $i = 1, 2, \ldots$

1.2.B. Show that a subspace M of a metric space X satisfies the equality $\operatorname{ind} M = 0$ if and only if M is non-empty and for every point $x \in M$ and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $M \cap \operatorname{Fr} U = \emptyset$ (cf. Proposition 1.2.12 and Problem 4.1.C).

1.2.C. Show by an example that in the second part of Lemma 1.2.9 the assumption that the subspace M is closed cannot be omitted.

1.2.D. (a) Check that every countable completely metrizable space X has isolated points.

Hint. Take a complete metric on the space X and arrange all points of X into a sequence $x_1, x_2, \ldots$, then assuming that none of the points x_i is isolated, define a decreasing sequence $G_1, G_2, \ldots$ of non-empty open sets such that $\delta(G_i) < 1/i$ and $x_i \notin \overline{G}_i$ for $i = 1, 2, \ldots$; apply the Cantor theorem (see [GT], Theorem 4.3.9).

One can as well use the Baire category theorem.

(b) Show that every non-empty completely metrizable space X of cardinality less than $\mathbf{c}$ has isolated points.

Hint. Take a complete metric on the space X and, assuming that X has no isolated points, for every finite sequence $i_1, i_2, \ldots, i_k$ of zeros and ones define inductively a non-empty open set $G_{i_1 i_2 \ldots i_k}$ such that $\delta(G_{i_1 i_2 \ldots i_k}) < 1/k$, $\overline{G}_{i_1 i_2 \ldots i_k 0} \cap \overline{G}_{i_1 i_2 \ldots i_k 1} = \emptyset$ and $\overline{G}_{i_1 i_2 \ldots i_k i} \subset G_{i_1 i_2 \ldots i_k}$ for $i = 0, 1$; to each infinite sequence $i_1, i_2, \ldots$ of zeros and ones assign the unique point in the intersection $\bigcap_{k=1}^{\infty} G_{i_1 i_2 \ldots i_k}$ and check that this is a one-to-one mapping.

Remark. The image of the set of all sequences $i_i, i_2, \ldots$ under this mapping is homeomorphic to the Cantor set (see Proposition 1.3.10 or Problem 1.3.F(a)).

1.2.E (Kuratowski [1932a]). Define a mapping f of the Cantor set C to the interval $[-1, 1]$ by letting

$$f(0) = 0 \qquad \text{and} \qquad f(x) = \frac{(-1)^{i_1}}{2} + \frac{(-1)^{i_2}}{2^2} + \ldots \quad \text{for each } x = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \in C \setminus \{0\},$$

where $i_1 < i_2 < \ldots$ is the sequence (finite or infinite) of all i's such that $x_i = 1$; let C_0 denote the set of all points $x \in C \setminus \{0\}$ for which the sequence $i_1, i_2, \ldots$ is infinite and let $C_1 = C \setminus C_0$.

(a) Observe that the set C_1 consists of the number 0 and the right end-points of the intervals $(1/3, 2/3), (1/9, 2/9), (7/9, 8/9), \ldots$ removed from I to obtain the Cantor set; note that C_1 is a countable set.

(b) Check that f is continuous at all points of C_0.

(c) Verify that for every $x = \frac{2}{3^{i_1}} + \frac{2}{3^{i_2}} + \ldots + \frac{2}{3^{i_k}} \in C_1$, the upper limit $\overline{f(x)}$ and the lower limit $\underline{f(x)}$ of the function f at the point x are equal to $f(x) + \frac{1}{2^k}$ and $f(x) - \frac{1}{2^k}$, respectively; observe that f is discontinuous at all points of C_1.

(d) For each point $x \in C_1$ define two sequences, $\{x'_n\}$ and $\{x''_n\}$, of points in C_1 such that

$$\lim x'_n = x, \quad \lim \overline{f(x'_n)} = \overline{f(x)}, \quad \lim \underline{f(x'_n)} = f(x)$$

and

$$\lim x''_n = x, \quad \lim \underline{f(x''_n)} = \underline{f(x)}, \quad \lim \overline{f(x''_n)} = f(x).$$

(e) Consider the graph $K = \{(x, f(x)) : x \in C\}$ of the function f and prove that $\operatorname{ind}_{(x,f(x))} K = 1$ for every $x \in C_1$.

Hint. Prove that the space $D = K \cup \bigcup_{x \in C_1} (\{x\} \times [\underline{f(x)}, \overline{f(x)}]) \subset R^2$ is compact. Assuming that $\operatorname{ind}_{(x_0,f(x_0))} K = 0$ for an $x_0 \in C_1$, show that there exists a partition L in the space D between $(x_0, f(x_0))$ and $(x_0, \underline{f(x_0)})$ such that $L \cap K = \emptyset$. Show that the set M consisting of all points $x \in C_1$ such that L is a partition between either $(x, f(x))$ and $(x, \underline{f(x)})$ or $(x, f(x))$ and $(x, \overline{f(x)})$ is contained in the projection of L onto C and contains an isolated point (see Problem 1.2.D); deduce a contradiction with (d).

(f) Observe that $\operatorname{ind}_{(x,f(x))} K = 0$ for every $x \in C_0$.

(g) Check that the empty set is a partition between any two distinct points of the space K.

(h) Note that the space K is completely metrizable.

Hint. Check that K is a G_δ-set in D and apply Lemma 1.3.12.

1.3. The sum, Cartesian product, universal space, compactification and embedding theorems for dimension 0

The theorems enumerated in the title of the section belong to the most important results of dimension theory. For the time being we only prove their special cases pertaining to zero-dimensional spaces. We begin with the sum theorem.

1.3.1. THE SUM THEOREM FOR DIMENSION 0. *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed zero-dimensional subspaces, then X is zero-dimensional.*

PROOF. Consider a pair A, B of disjoint closed subsets of the space X. We shall prove that there exist open sets $U, W \subset X$ such that

$$A \subset U, \quad B \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad X = U \cup W, \tag{1}$$

i.e., that the empty set is a partition between A and B.

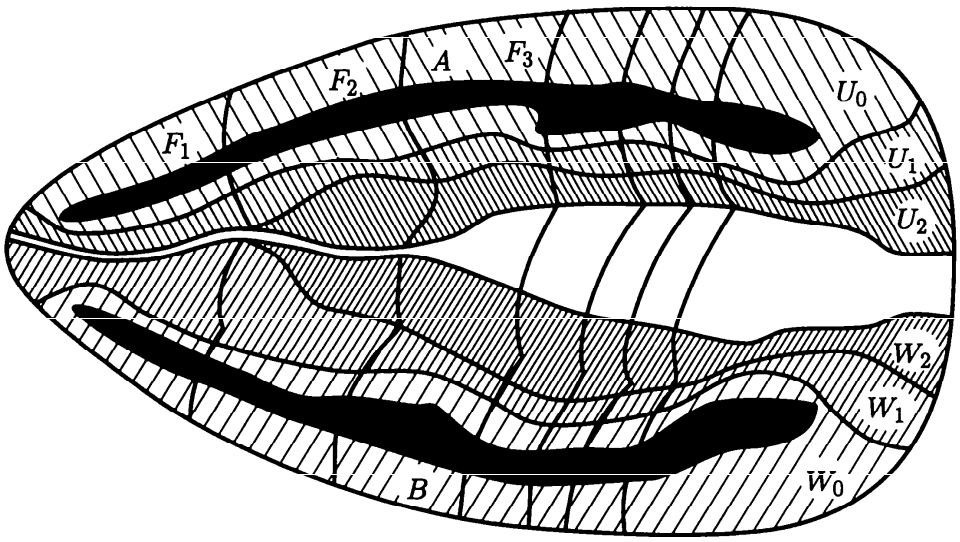


Fig. 4

Let U_0, W_0 be open subsets of X such that

$$A \subset U_0, \quad B \subset W_0 \quad \text{and} \quad \overline{U}_0 \cap \overline{W}_0 = \emptyset. \tag{2}$$

We shall define inductively two sequences $U_0, U_1, U_2, \ldots$ and $W_0, W_1, W_2, \ldots$ of open subsets of X satisfying for $i = 0, 1, 2, \ldots$ the conditions:

$$U_{i-1} \subset U_i, \quad W_{i-1} \subset W_i \quad \text{if} \quad i \geq 1 \quad \text{and} \quad \overline{U}_i \cap \overline{W}_i = \emptyset. \tag{3}$$

$$F_i \subset U_i \cup W_i, \quad \text{where} \quad F_0 = \emptyset. \tag{4}$$

Clearly, the sets U_0, W_0 defined above satisfy both conditions for $i = 0$. Assume that the sets U_i, W_i satisfying (3) and (4) are defined for all $i < k$.

The sets $\overline{U}_{k-1} \cap F_k$ and $\overline{W}_{k-1} \cap F_k$ are closed and disjoint; since the space F_k is zero-dimensional, by virtue of Theorem 1.2.6 there exists an open-and-closed subset V of F_k such that

$$\overline{U}_{k-1} \cap F_k \subset V \quad \text{and} \quad \overline{W}_{k-1} \cap F_k \subset F_k \setminus V. \tag{5}$$

Since F_k is closed in X, so are V and $F_k \setminus V$; from (5) it follows that

$$(\overline{U}_{k-1} \cup V) \cap [\overline{W}_{k-1} \cup (F_k \setminus V)] = (V \cap \overline{W}_{k-1}) \cup [\overline{U}_{k-1} \cap (F_k \setminus V)] = \emptyset,$$

so that there exist open sets $U_k, W_k \subset X$ satisfying

$$\overline{U}_{k-1} \cup V \subset U_k, \quad \overline{W}_{k-1} \cup (F_k \setminus V) \subset W_k \quad \text{and} \quad \overline{U}_k \cap \overline{W}_k = \emptyset.$$

The sets U_k, W_k satisfy (3) and (4) for $i = k$; thus, the construction of the sequences $U_0, U_1, U_2, \ldots$ and $W_0, W_1, W_2, \ldots$ is completed. It follows from (2), (3) and (4) that the unions $U = \bigcup_{i=0}^{\infty} U_i$ and $W = \bigcup_{i=0}^{\infty} W_i$ satisfy (1). ■

1.3.2. REMARK. The reader has doubtless noticed that in the proof of Theorem 1.3.1 only the normality of the space X and the fact that the empty set is a partition between each pair of disjoint closed subsets of the space F_k were applied. Thus we proved that if a normal space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces with the property that for $i = 1, 2, \ldots$ the empty set is a partition between each pair of disjoint closed subsets of the space F_i, then the empty set is a partition between any disjoint closed subsets of the space X. ■

From Theorem 1.3.1 several corollaries follow:

1.3.3. COROLLARY. *If a separable metric space X can be represented as the union of a sequence $A_1, A_2, \ldots$ of zero-dimensional subspaces, where A_i is an F_σ-set for $i = 1, 2, \ldots$, then X is zero-dimensional.* ■

1.3.4. COROLLARY. *If a separable metric space X can be represented as the union of two zero-dimensional subspaces A_1 and A_2, one of them being both an F_σ-set and a G_δ-set, then X is zero-dimensional.*

PROOF. Suppose that A_1 is both an F_σ-set and a G_δ-set. Then $X \setminus A_1 \subset A_2$ is an F_σ-set and $\operatorname{ind}(X \setminus A_1) \leq 0$ by virtue of the subspace theorem. Hence $X = A_1 \cup (X \setminus A_1)$ is zero-dimensional by Corollary 1.3.3. ■

Let us note that Corollary 1.3.4 holds in particular if one of the sets A_1, A_2 is closed, or open, or is the intersection of a closed and an open set.

1.3.5. COROLLARY. *If by adjoining a finite number of points to a zero-dimensional separable metric space one obtains a metrizable space, then the space obtained is zero-dimensional and separable.* ■

In connection with the last corollary, let us note that by adjoining countably many points to the space of irrational numbers one can obtain the real line, i.e., a space of positive dimension.

We shall now prove the Cartesian product theorem.

1.3.6. THE CARTESIAN PRODUCT THEOREM FOR DIMENSION 0. *The Cartesian product $X = \prod_{i=1}^{\infty} X_i$ of a countable family $\{X_i\}_{i=1}^{\infty}$ of regular spaces is zero-dimensional if and only if all spaces X_i are zero-dimensional.*

PROOF. If $X \neq \emptyset$, then each X_i is homeomorphic to a subspace of X, so that if X is zero-dimensional, then all spaces X_i are zero-dimensional. To prove the reverse implication, it is enough to consider for $i = 1, 2, \ldots$ a base $\mathcal{B}_i$ for the space X_i consisting of open-and-closed sets and observe that the sets of the form $U_1 \times U_2 \times \ldots \times U_k \times \prod_{i=k+1}^{\infty} X_i$, where $U_i \in \mathcal{B}_i$ for $i \leq k$ and $k = 1, 2, \ldots$, constitute a base for X and are open-and-closed in X. ■

Theorems 1.1.2 and 1.3.6 yield

1.3.7. COROLLARY. *The limit of an inverse sequence $\{X_i, \pi_j^i\}$ of zero-dimensional spaces is either zero-dimensional or empty.* ■

The sum and Cartesian product theorems allow us to increase our stock of zero-dimensional spaces.

1.3.8. EXAMPLES. (a) For every pair k, n of integers satisfying $0 \leq k \leq n \geq 1$ denote by Q_k^n the subspace of Euclidean n-space R^n consisting of all points which have exactly k rational coordinates. As observed in 1.2.5, $Q_n^n = Q^n$ is zero-dimensional; we shall prove that Q_k^n is zero-dimensional also for $k < n$.

For each choice of k distinct natural numbers $i_1, i_2, \ldots, i_k$ not larger than n and each choice of k rational numbers $r_1, r_2, \ldots, r_k$, the Cartesian product $\prod_{i=1}^{n} R_i$, where

$R_{i_j} = \{r_j\}$ for $j = 1, 2, \ldots, k$ and $R_i = R$ for $i \neq i_j$, is a closed subspace of R^n, and thus $Q_k^n \cap \prod_{i=1}^n R_i$ is a closed subspace of Q_k^n. Since the space $Q_k^n \cap \prod_{i=1}^n R_i$ is homeomorphic to the subspace of R^{n-k} consisting of all points with irrational coordinates, it follows from Example 1.2.5(a) and Theorem 1.3.6 that it is a zero-dimensional space. Theorem 1.3.1 then implies that the space Q_k^n is also zero-dimensional, because the family of all subspaces of the form $Q_k^n \cap \prod_{i=1}^n R_i$ is countable and its union is equal to the whole of Q_k^n.

(b) It also follows from Theorem 1.3.6 that $Q^{\aleph_0}$ and $P^{\aleph_0}$, the Cartesian products of $\aleph_0$ copies of the spaces of rational numbers and irrational numbers, respectively, are zero-dimensional.

(c) The reader can show, as an easy exercise, that for each integer $k \geq 0$ the subspace Q_k of the Hilbert cube $I^{\aleph_0}$ consisting of all points which have exactly k rational coordinates is zero-dimensional. ■

1.3.9. DEFINITION. We say that a topological space X is *universal* for a class $\mathcal{K}$ of topological spaces if X belongs to $\mathcal{K}$ and every space in the class $\mathcal{K}$ is embeddable in X, i.e., is homeomorphic to a subspace of X.

We are now going to prove that the Cantor set C and the space P of irrational numbers are universal spaces for the class of all zero-dimensional separable metric spaces; in the proof we shall apply the fact that both C and P can be represented as countable Cartesian products.

1.3.10. PROPOSITION. *The Cantor set C is homeomorphic to the Cartesian product $D^{\aleph_0} = \prod_{i=1}^{\infty} D_i$, where D_i, for $i = 1, 2, \ldots$, is the two-point discrete space $D = \{0, 1\}$.*

PROOF. As one readily verifies, for each point $x \in C$ the representation in the form $x = \sum_{i=1}^{\infty} 2x_i/3^i$, where x_i's are equal to 0 or 1, is unique. Hence, letting

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{2x_i}{3^i} \quad \text{for} \quad \{x_i\} \in D^{\aleph_0},$$

we define a one-to-one mapping of $D^{\aleph_0}$ onto C. Since the function $f_k\colon D^{\aleph_0} \to I$ defined by $f_k(\{x_i\}) = 2x_k/3^k$ is continuous for $k = 1, 2, \ldots$ and the sequence $f_1, f_2 + f_2, f_1 + f_2 + f_3, \ldots$ uniformly converges to f, the latter function is continuous. It follows from the compactness of $D^{\aleph_0}$ that f is a homeomorphism. ■

The proof of the counterpart of Proposition 1.3.10 for the space of irrational numbers requires some calculation to remedy the lack of compactness. Let us recall that if X is a metric space, then by a *metric on the space X* we mean any metric on the set X which is equivalent to the original metric on X, i.e., induces the same convergence as the original metric.

1.3.11. LEMMA. *Let ρ be an arbitrary metric on the space P of irrational numbers and ϵ a positive number. For every non-empty open set $U \subset P$ there exists an infinite sequence $F_1, F_2, \ldots$ of pairwise disjoint non-empty open-and-closed subsets of P such that $U = \bigcup_{i=1}^{\infty} F_i$ and the diameters with respect to ρ of all sets F_i are less than ϵ.*

PROOF. Consider an interval $(a, b) \subset R$ with rational end-points such that $(a, b) \cap P \subset U$ and divide it into $\aleph_0$ pairwise disjoint non-empty intervals $(a_1, b_1), (a_2, b_2), \ldots$

with rational end-points. Thus we have $(a,b)\cap P = \bigcup_{i=1}^{\infty} A_i$, where $A_i = (a_i,b_i)\cap P \neq \emptyset$, $a_i, b_i \in Q$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$; in addition, let $A_0 = U \setminus (a,b)$. The sets $A_0, A_1, A_2, \ldots$ are open in P, and by virtue of Proposition 1.2.4 for $i = 0,1,2,\ldots$ there exist in P open-and-closed sets $A_{i,1}, A_{i,2}, \ldots$, all of diameter less than ϵ, such that $A_i = \bigcup_{j=1}^{\infty} A_{i,j}$; letting $B_{i,j} = A_{i,j} \setminus \bigcup_{k<j} A_{i,k}$ for $j = 1,2,\ldots$, we obtain pairwise disjoint open-and-closed subsets of P whose union is equal to A_i. To complete the proof it suffices to arrange all non-empty sets $B_{i,j}$ into a simple sequence $F_1, F_2, \ldots$ ■

The next lemma is an important theorem on complete spaces. For the purpose of future applications it is stated in full generality and not merely for the subspace P of the real line.

1.3.12. LEMMA. *Every G_δ-set X in a completely metrizable space X_0 is completely metrizable.*

PROOF. Let ρ be a complete metric on the space X_0 and let $X_0 \setminus X = \bigcup_{i=1}^{\infty} F_i$, where F_i is closed in X_0 for $i = 1,2,\ldots$ Let

$$f_i(x) = \rho(x, F_i) \quad \text{for } x \in X \text{ and } i = 1,2,\ldots;$$

the functions $f_1, f_2, \ldots$ from X to the subspace $R_0 = (0,\infty)$ of the real line R are continuous. One readily sees that the formula $f(x) = (x, f_1(x), f_2(x), \ldots)$ defines a homeomorphic embedding $f: X \to \prod_{i=0}^{\infty} X_i$, where $X_i = R_0$ for $i \geq 1$. Since the Cartesian product of countably many completely metrizable spaces and a closed subspace of a completely metrizable space are completely metrizable, to complete the proof it suffices to show that $f(X)$ is a closed subset of $\prod_{i=0}^{\infty} X_i$. We shall show that every point $x = \{x_i\} \in \prod_{i=0}^{\infty} X_i \setminus f(X)$ has a neighbourhood V contained in the complement of $f(X)$.

First consider the case where $x_0 \in X$. As $x \notin f(X)$, there exists a $k > 0$ such that $x_k \neq f_k(x_0)$. Let U_1 and U_2 be disjoint neighbourhoods of x_k and $f_k(x_0)$ in R_0. The function f_k being continuous, there exists a neighbourhood $U_0 \subset X_0$ of the point x_0 such that $f_k(U_0 \cap X) \subset U_2$. One easily checks that

$$x = \{x_i\} \in V = p_0^{-1}(U_0) \cap p_k^{-1}(U_1) \subset \prod_{i=0}^{\infty} X_i \setminus f(X), \tag{6}$$

where p_i denotes the projection of $\prod_{i=0}^{\infty} X_i$ onto X_i.

Now, consider the case where $x_0 \notin X$; thus we have $x_0 \in F_k$ for some $k > 0$. Take a positive number r such that $x_k > r$ and let $U_0 = B(x_0, r)$ and $U_1 = (r,\infty)$. One easily checks that formula (6) holds also in this case. ■

1.3.13. PROPOSITION. *The space P of irrational numbers is homeomorphic to the Cartesian product $N^{\aleph_0} = \prod_{i=1}^{\infty} N_i$, where N_i, for $i = 1,2,\ldots$, is the discrete space N of natural numbers.*

PROOF. By virtue of Lemma 1.3.12, there exists a complete metric ρ on the space P. Applying Lemma 1.3.11, for every finite sequence $k_1, k_2, \ldots, k_i$ of natural numbers define an open-and-closed subset $F_{k_1 k_2 \ldots k_i}$ of P such that

$$P = \bigcup_{k=1}^{\infty} F_k \quad \text{and} \quad F_{k_1 k_2 \ldots k_i} = \bigcup_{k=1}^{\infty} F_{k_1 k_2 \ldots k_i k}, \tag{7}$$

(8) $$F_{k_1 k_2 \ldots k_i} \neq \emptyset \quad \text{and} \quad \delta(F_{k_1 k_2 \ldots k_i}) < 1/i,$$

(9) $$F_{k_1 k_2 \ldots k_i} \cap F_{m_1 m_2 \ldots m_i} = \emptyset \quad \text{whenever} \quad (k_1, k_2, \ldots, k_i) \neq (m_1, m_2, \ldots, m_i).$$

It follows from (7) and (8) that for every $\{k_i\} \in N^{\aleph_0}$ the subsets $F_{k_1}, F_{k_1 k_2}, F_{k_1 k_2 k_3}, \ldots$ of the space P form a decreasing sequence of non-empty closed sets whose diameters converge to zero; hence, by virtue of the Cantor theorem (see [GT], Theorem 4.3.9), the set $\bigcap_{i=1}^{\infty} F_{k_1 k_2 \ldots k_i}$ contains exactly one point, which we shall denote by $f(\{k_i\})$. Conditions (7)–(9) imply that by assigning $f(\{k_i\})$ to $\{k_i\} \in N^{\aleph_0}$ one defines a one-to-one mapping of $N^{\aleph_0}$ onto P. Since, as one readily verifies,

$$f\Big(\{k_1\} \times \{k_2\} \times \ldots \times \{k_j\} \times \prod_{i=j+1}^{\infty} N_i\Big) = F_{k_1 k_2 \ldots k_j},$$

the mapping f is a homeomorphism, because the sets in brackets form a base for $N^{\aleph_0}$ and the sets on the right-hand side form a base for P. ■

Propositions 1.3.10 and 1.3.13 yield the following corollary (see Problem 1.3.D(a)).

1.3.14. COROLLARY. *The Cantor set is homeomorphic to a subspace of the space of irrational numbers.* ■

Now we are ready to prove the universality of C and P.

1.3.15. THE UNIVERSAL SPACE THEOREM FOR DIMENSION 0. *The Cantor set and the space of irrational numbers are universal spaces for the class of all zero-dimensional separable metric spaces.*

PROOF. By virtue of Corollary 1.3.14, it suffices to show that for every zero-dimensional separable metric space X there exists a homeomorphic embedding $f: X \to D^{\aleph_0}$.

It follows from Proposition 1.2.4 that the space X has a countable base $\mathcal{B} = \{U_i\}_{i=1}^{\infty}$ consisting of open-and-closed sets. For $i = 1, 2, \ldots$ define a mapping $f_i: X \to D_i = D$ by letting

$$f_i(x) = \begin{cases} 1 & \text{for } x \in U_i, \\ 0 & \text{for } x \in X \setminus U_i. \end{cases}$$

Consider the mapping $f: X \to D^{\aleph_0}$ defined by the formula $f(x) = (f_1(x), f_2(x), \ldots)$. Since for every natural number k we have

$$f(U_k) = f(X) \cap \{\{x_i\} \in D^{\aleph_0} : x_k = 1\},$$

the mapping f is a homeomorphic embedding. ■

The universal space theorem implies the compactification theorem and the embedding theorem.

1.3.16. THE COMPACTIFICATION THEOREM FOR DIMENSION 0. *For every zero-dimensional separable metric space X there exists a zero-dimensional compactification $\widetilde{X}$, i.e., a zero-dimensional compact metric space $\widetilde{X}$ which contains a dense subspace homeomorphic to X.*

PROOF. Let $f: X \to C$ be a homeomorphic embedding of X in the Cantor set C. Since the Cantor set is compact, so is its closed subspace $\widetilde{X} = \overline{f(X)}$; the space $\widetilde{X}$ is zero-dimensional by virtue of the subspace theorem. ■

1.3.17. THE EMBEDDING THEOREM FOR DIMENSION 0. *Every zero-dimensional separable metric space is embeddable in the real line R.* ■

Let us conclude this section by observing that in the theory of zero-dimensional spaces the key role is played by four theorems, viz., the two separation theorems, the sum theorem and the universal space theorem. All the remaining results either are elementary or easily follow from one of the four cited theorems. As the reader will see later, the situation changes when we pass to higher dimensions.

1.3.18. REMARK. In Theorems 1.3.6, 1.3.15 and 1.3.16 countability is not essential. In the same way one proves that the Cartesian product $X = \prod_{s \in S} X_s$ of a family $\{X_s\}_{s \in S}$ of regular spaces satisfies the equality $\operatorname{ind} X = 0$ if and only if $\operatorname{ind} X_s = 0$ for every $s \in S$, and that every regular space X satisfying $\operatorname{ind} X = 0$ is embeddable in the Cantor cube $D^{\mathbf{m}}$ (i.e., the Cartesian product of $\mathbf{m}$ copies of the two-point discrete space D), which implies that X has a compactification $\widetilde{X} \subset D^{\mathbf{m}}$ such that $\operatorname{ind} \widetilde{X} = 0$; the cardinal number $\mathbf{m}$ is the cardinality of any base for X consisting of open-and-closed sets. ■

Historical and bibliographic notes

The theorems in the present section are all special cases of theorems which will be proved in Sections 1.5, 1.7 and 1.11 for an arbitrary dimension n. Theorem 1.3.1 was proved for compact spaces (and for an arbitrary dimension n) by Menger in [1924] and by Urysohn in [1926] (announcement in [1922]); it was extended to separable metric spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]. Theorem 1.3.6 was established by Kuratowski in [1933] and Theorem 1.3.15 by Sierpiński in [1921].

Problems

1.3.A. (a) Note that for a subspace X of the real line Theorem 1.3.1 is a consequence of the Baire category theorem. Deduce from Theorem 1.3.1 the Baire category theorem for the real line.

(b) Prove that if a separable metric space X can be represented as the union of a sequence $F_0, F_1, F_2, \ldots$ of closed subspaces such that $\operatorname{ind} F_i = 0$ for $i = 1, 2, \ldots$, then $\operatorname{ind}_x X = 0$ for every $x \in F_0$ such that $\operatorname{ind}_x F_0 = 0$.

(c) Give an example of a separable metric space X which can be represented as the union of two closed subspaces F_1 and F_2 in such a way that for some $x \in F_1 \cap F_2$ we have $\operatorname{ind}_x F_1 = \operatorname{ind}_x F_2 = 0$ and yet $\operatorname{ind}_x X > 0$.

1.3.B. Show that if a subspace X of a metric space X_0 is completely metrizable, then X is a G_δ-set in X_0.

Hint. One can assume that X is dense in X_0. Take a complete metric on the space X and for $i = 1, 2, \ldots$ and each $x \in X$ pick an open subset $U_{i,x}$ of X_0 such that the

intersection $X \cap U_{i,x}$ is equal to the ball $B(x, 1/i)$ in the space X. Consider the union U_i of all sets $U_{i,x}$ with $x \in X$ and apply the Cantor theorem to show that $\bigcap_{i=1}^{\infty} U_i \subset X$.

One can as well use the Lavrentieff theorem (see [GT], Theorem 4.3.21).

Remark. In the same way one shows that if a dense subspace X of a Hausdorff space X_0 is completely metrizable, then X is a G_δ-set in X_0.

1.3.C. (a) (implicitly, Sierpiński [1928]) Prove that every non-empty closed subset A of a zero-dimensional separable metric space X is a *retract* of X, i.e., that there exists a continuous mapping $r : X \to A$ such that $r(x) = x$ for every $x \in A$.

Hint. Represent the complement $X \setminus A$ as the union of a sequence $F_1, F_2, \ldots$ of pairwise disjoint open-and-closed sets such that $\lim \delta(F_i) = 0$. For $i = 1, 2, \ldots$ choose a point $x_i \in A$ such that $\rho(x_i, F_i) < \rho(A, F_i) + 1/i$ and let $r(x) = x_i$ for $x \in F_i$.

(b) Note that if a non-empty regular space X has the property that every non-empty closed set $A \subset X$ is a retract of X, then X is zero-dimensional.

Remark. The characterization of zero-dimensional spaces contained in (a) and (b) generalizes to higher dimensions; see Problem 4.1.G.

1.3.D. (a) Show that there exists a real number r such that $\{r + c : c \in C\} \subset P$, where C denotes the Cantor set and P the set of irrational numbers.

Hint. Apply the Baire category theorem.

(b) (Alexandroff [1927a] (announcement [1925]), Hausdorff [1927]) Check that letting

$$f(\{x_i\}) = \sum_{i=1}^{\infty} \frac{x_i}{2^i} \quad \text{for } \{x_i\} \in D^{\aleph_0},$$

one defines a continuous mapping of the Cantor set $D^{\aleph_0}$ onto the closed unit interval I. Observe that if $D^{\aleph_0}$ is identified with C as described in the proof of Proposition 1.3.10, then f consists in identifying the end-points of each interval removed from I to obtain the Cantor set. Verify that there exists a countable dense set $A \subset I$ such that $|f^{-1}(y)| = 2$ for every $y \in A$, and otherwise $|f^{-1}(y)| = 1$.

Define a continuous mapping of the Cantor set $D^{\aleph_0}$ onto the Hilbert cube $I^{\aleph_0}$ and – applying Problem 1.3.C(a) together with the fact that $I^{\aleph_0}$ is a universal space for the class of all separable metric spaces (see [GT], Theorem 4.2.10) – show that every non-empty compact metric space X is a continuous image of the Cantor set. Deduce that for every separable metric space X there exists a subspace of the Cantor set that can be mapped onto X by a one-to-one continuous mapping.

(c) Let us recall that a topological space X is *homogeneous* if for every pair x, y of distinct points of X there exists a homeomorphism of X onto itself which maps x to y.

Show that the Cantor set C is homogeneous.

(d) Prove that every non-empty separable completely metrizable space X is a continuous image of the space of irrational numbers.

Hint. Take a complete metric on the space X and for every finite sequence k_1, $k_2, \ldots, k_i$ of natural numbers define a non-empty closed subset $F_{k_1 k_2 \ldots k_i}$ of X such that $X = \bigcup_{k=1}^{\infty} F_k$, $F_{k_1 k_2 \ldots k_i} = \bigcup_{k=1}^{\infty} F_{k_1 k_2 \ldots k_i k}$ and $\delta(F_{k_1 k_2 \ldots k_i}) < 1/i$.

1.3.E. (a) (Mazurkiewicz [1917]) Prove that every G_δ-set which is dense, and whose complement is also dense, in a completely metrizable separable zero-dimensional space is homeomorphic to the space of irrational numbers.

Hint. Modify the proof of Proposition 1.3.13.

(b) (Alexandroff and Urysohn [1928]) Show that every completely metrizable separable zero-dimensional space which does not contain any non-empty compact open subspace is homeomorphic to the space of irrational numbers.

Hint. Apply (a).

(c) Note that the subspace of the Cantor set C consisting of all points which are not end-points of intervals removed from I to obtain the Cantor set is homeomorphic to the space of irrational numbers.

1.3.F. (a) (Brouwer [1910]) Prove that every zero-dimensional compact metric space with no isolated points is homeomorphic to the Cantor set.

Hint. Modify the construction in the proof of Proposition 1.3.13 in such a way that the sets $F_{k_1 k_2 \ldots k_i}$ will be defined for $k_1 \leq m_1, k_2 \leq m_2, \ldots, k_i \leq m_i$, where $m_1, m_2, \ldots$ is a sequence of powers of 2.

(b) (Hausdorff [1914]) Show that every non-empty completely metrizable space with no isolated points contains a subspace homeomorphic to the Cantor set.

(c) (Hausdorff [1914]) Show that every separable completely metrizable space either is countable or contains a subspace homeomorphic to the Cantor set, and thus has cardinality $\mathbf{c}$.

Hint. Note that in every separable metric space the set of all isolated points is countable.

1.3.G. (a) (Brouwer [1913a]; implicitly, Fréchet [1910]) Prove that for any two countable dense subsets A, B of the real line R there exists a homeomorphism $f\colon R \to R$ such that $f(A) = B$ (see Problem 1.8.D).

Hint. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$; define inductively a function f from A to R by letting $f(a_1) = b_1$ and taking as $f(a_i)$ an element of B with the smallest possible index such that the conditions $a_j < a_k$ and $f(a_j) < f(a_k)$ are equivalent for $j, k \leq i$. Extend f over R and prove that this extension is a homeomorphism.

(b) (Fréchet [1910]) Prove that the space Q of rational numbers is a universal space for the class of all countable metric spaces.

Hint. Observe that countable metric spaces are embeddable in the real line; for a countable $X \subset R$ apply (a) to the sets $X \cup Q$ and Q.

1.3.H. (a) Prove that if A_1, A_2, B_1 and B_2 are countable dense subsets of the real line R satisfying the condition $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, then there exists a homeomorphism $f\colon R \to R$ such that $f(A_1) = B_1$ and $f(A_2) = B_2$.

Hint. See Problem 1.3.G(a).

(b) Show that for any two countable dense subsets A, B of the space of irrational numbers P there exists a homeomorphism $f\colon P \to P$ such that $f(A) = B$.

(c) Show that for any two countable dense subsets A, B of the Cantor set C there exists a homeomorphism $f\colon C \to C$ such that $f(A) = B$.

Hint. Observe that for every countable set $A \subset C$ there exists a homeomorphism $g : C \to C$ such that $g(A)$ is disjoint from the set consisting of the end-points of all intervals removed from I to obtain the Cantor set.

(d) (Sierpiński [1920a] (announcement [1915])) Prove that every countable metric space which is dense in itself is homeomorphic to the space of rational numbers.

Hint. By virtue of (b) it suffices to show that every countable metric space X dense in itself is homeomorphic to a dense subspace of P. To that end, embed X in P, consider the closure $\overline{X} \subset P$, remove $\aleph_0$ points from $\overline{X} \setminus X$ in an appropriate way, and apply Problem 1.3.E(a).

One can as well use Problem 1.3.F(a) and apply (c).

(e) Note that the subspace of the Cantor set C consisting of the end-points of all intervals removed from I to obtain the Cantor set is homeomorphic to the space of rational numbers.

1.4. Various kinds of disconnectedness

We shall now compare the class of zero-dimensional spaces with three other classes of highly disconnected spaces. It will follow from this comparison that none of these three classes satisfies the counterparts of all the theorems proved for zero-dimensional spaces in Sections 1.2 and 1.3, so that zero-dimensional spaces form the nicest class of highly disconnected spaces. The dimension functions that one could define inductively, in the way the function ind is defined, starting at the zero-level with one of these three classes would not lead to a dimension theory as rich and harmonious as that developed in this chapter.

1.4.1. DEFINITION. A topological space X is called *totally disconnected* if for every pair x, y of distinct points of X there exists an open-and-closed set $U \subset X$ such that $x \in U$ and $y \in X \setminus U$, i.e., if the empty set is a partition between any two distinct points of the space X.

Clearly, every zero-dimensional space is totally disconnected.

Totally disconnected spaces are characterized by the property that their quasi-components are one-point sets. Let us recall that quasi-components of a topological space X are defined as minimal non-empty intersections of open-and-closed subsets of X, i.e., a non-empty set $K \subset X$ is a *quasi-component* of the space X if K can be represented as the intersection of open-and-closed sets and for every open-and-closed set $U \subset X$ such that $K \cap U \neq \emptyset$ we have $K \subset U$. The quasi-components of a space X constitute a decomposition of X into pairwise disjoint closed subsets.

1.4.2. DEFINITION. A topological space X is called *hereditarily disconnected* if X does not contain any connected subspace of cardinality larger than one.

Every totally disconnected space is hereditarily disconnected. Indeed, if X is a totally disconnected space, then for each subspace $M \subset X$ which contains at least two distinct points x, y the sets $M \cap U$ and $M \setminus U$, where U is an open-and-closed subset

of X such that $x \in U$ and $y \in X \setminus U$, form a decomposition of the space M into two non-empty disjoint open subsets, so that M is not connected.

Hereditarily disconnected spaces are characterized by the property that their components are one-point sets. Let us recall that components of a topological space X are defined as maximal non-empty connected subsets of X, i.e., a non-empty set $S \subset X$ is a *component* of the space X if S is connected and for every connected set $A \subset X$ such that $S \subset A$ we have $S = A$. The components of a space X constitute a decomposition of X into pairwise disjoint closed subsets.

1.4.3. DEFINITION. A topological space X is called *punctiform*, or *discontinuous*, if X does not contain any continuum of cardinality larger than one (let us recall that by a *continuum* we understand a connected compact space).

Clearly, every hereditarily disconnected space is punctiform and every compact punctiform space is hereditarily disconnected. As shown in Example 1.4.8 below, there exist connected punctiform spaces of cardinality larger than one.

The reader can easily verify that the above three classes of spaces are closed with respect to taking subspaces.

We shall now show that in the realm of non-empty locally compact spaces the three classes under consideration coincide with the class of zero-dimensional spaces.

1.4.4. LEMMA. *In every compact space quasi-components and components coincide.*

PROOF. To begin, we shall prove that in an arbitrary topological space X quasi-components are not smaller than components. Consider a component S of the space X. Let x be a point in S and K the quasi-component of X which contains x; we shall show that $S \subset K$. Take an open-and-closed set $U \subset X$ which contains x. Since the sets $S \cap U$ and $S \setminus U$ are open in S and disjoint and since $S \cap U \neq \emptyset$, it follows from the connectedness of S that $S \setminus U = \emptyset$, i.e., that $S \subset U$. The set K being the intersection of all open-and-closed subsets of X which contain x, we have $S \subset K$.

To complete the proof it suffices to show that the quasi-components of any compact space are connected. Let us consider the decomposition of a quasi-component K of a compact space X into two disjoint closed sets A, B and let us assume that $A \neq \emptyset$. By the normality of compact spaces there exist open sets $V, W \subset X$ such that

$$A \subset V, \quad B \subset W \quad \text{and} \quad V \cap W = \emptyset.$$

Denote by $\mathcal{U}$ a family of open-and-closed subsets of X satisfying $\bigcap \mathcal{U} = K$. Since $\bigcap \mathcal{U} \subset V \cup W$, the family $\mathcal{F} = \{U \setminus (V \cup W) : U \in \mathcal{U}\}$ of closed subsets of X has an empty intersection. It follows from the compactness of X that a finite subfamily of $\mathcal{F}$ also has an empty intersection, i.e., that there exists a finite number of sets $U_1, U_2, \ldots, U_k \in \mathcal{U}$ satisfying

$$U = U_1 \cap U_2 \cap \ldots \cap U_k \subset V \cup W.$$

The set U is open-and-closed. Since

$$\overline{V \cap U} \subset \overline{V} \cap U = \overline{V} \cap (V \cup W) \cap U = V \cap U,$$

the set $V \cap U$ also is open-and-closed. From the relation $\emptyset \neq A \subset V \cap U$ it follows that $K \subset V \cap U$, and so $B \subset K \cap W \subset V \cap U \cap W = \emptyset$, which proves that the quasi-component K is connected. ■

1.4.5. THEOREM. *Zero-dimensionality, total disconnectedness, hereditary disconnectedness and punctiformness are equivalent in the realm of non-empty locally compact spaces.*

PROOF. It suffices to prove that every non-empty locally compact punctiform space X is zero-dimensional. Consider a point $x \in X$ and a neighbourhood $V \subset X$ of x. By the local compactness of X the point x has a neighbourhood $W \subset X$ such that the closure $\overline{W}$ is compact. The subspace $M = \overline{V \cap W}$ of the space X is compact and punctiform, so that it is hereditarily disconnected. By virtue of Lemma 1.4.4, the component $\{x\}$ of the space M can be represented as the intersection of a family $\mathcal{U}$ of open-and-closed subsets of M. It follows from the compactness of M that there exists a finite number of sets $U_1, U_2, \ldots, U_k \in \mathcal{U}$ such that the intersection $U = U_1 \cap U_2 \cap \ldots \cap U_k$ is disjoint from the set $M \setminus (V \cap W)$. The set U is closed in M, and thus it is closed in X; on the other hand, U is open in $V \cap W$, so that U is an open-and-closed subset of X. As $x \in U \subset V$, the space X is zero-dimensional. ■

We shall now describe three subspaces of the plane which exhibit the difference between the classes of zero-dimensional, totally disconnected, hereditarily disconnected and punctiform spaces. They are all closely related to the totally disconnected not zero-dimensional space H_0 described in Example 1.2.15.

Our first example is a totally disconnected non zero-dimensional subspace of the plane.

1.4.6. EXAMPLE. One readily checks that by assigning to every point $\{x_i\}$ of the space H_0 described in Example 1.2.15 the same point $\{x_i\}$ in the Cartesian product $Q^{\aleph_0}$ of $\aleph_0$ copies of the space of rational numbers one defines a one-to-one continuous mapping of H_0 to $Q^{\aleph_0}$. Hence, by virtue of Theorems 1.3.6 and 1.3.15, there exists a one-to-one continuous mapping $f: H_0 \to C$ of H_0 to the Cantor set C. Since the Cantor set is homogeneous (see Problem 1.3.D(c)), one can suppose that $f(x_0) = 0$, where $x_0 \in H_0$ is the sequence whose terms are all equal to zero. Letting

$$h(x) = \left(\frac{f(x)}{\max(1, \|x\|)}, \frac{\|x\|}{1 + \|x\|} \right) \quad \text{for } x \in H_0,$$

one defines a continuous mapping $h: H_0 \to I^2$. One can prove that h is a homeomorphic embedding (see Problem 1.4.B(a)), which implies that the subspace $X = h(H_0) \subset I^2$ of the plane is totally disconnected but is not zero-dimensional; however, a direct proof of these properties of the space X is simpler.

To begin, let us observe that by letting

$$G(y_1, y_2) = y_1 \cdot \max\left(1, \frac{y_2}{1 - y_2}\right) \quad \text{for } 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 < 1$$

one defines a continuous mapping $G: I \times [0, 1) \to R$ which to the point $h(x) \in X$ assigns the point $f(x) \in C$, so that the restriction $g = G|X: X \to C$ is a one-to-one continuous mapping of the space X to the Cantor set C. From the existence of such a mapping it follows that X is a totally disconnected space. Indeed, for every pair x, y

of distinct points of X there exists an open-and-closed set $V \subset C$ such that $g(x) \in V$ and $g(y) \in C \setminus V$; the inverse image $U = g^{-1}(V)$ is an open-and-closed subset of X such that $x \in U$ and $y \in X \setminus U$.

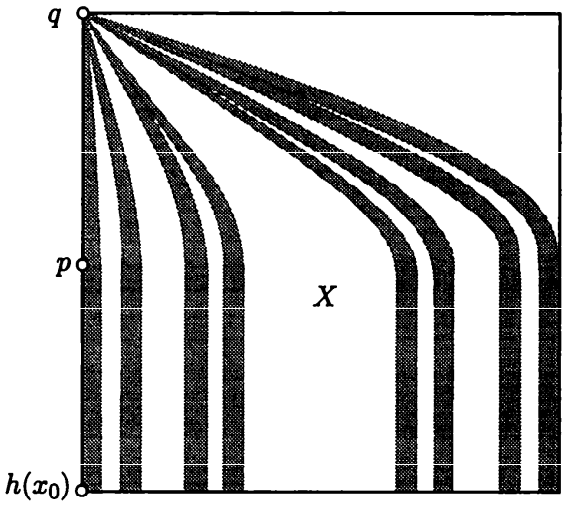


Fig. 5

Now, consider an open-and-closed set $U \subset X$ such that $U \subset (I \times [0,1/2)) \cap X$. The inverse image $h^{-1}(U) \subset H_0$ is an open-and-closed bounded subset of H_0 and thus, by virtue of the final observation in Example 1.2.15, it is empty. Hence, there is no open-and-closed set $U \subset X$ such that $h(x_0) = (0,0) \in U \subset (I \cap [0,1/2)) \cap X$, which shows that the space X is not zero-dimensional. ■

The following two examples are: a space $Y \subset I^2$ which is hereditarily disconnected but is not totally disconnected and a space $Z \subset I^2$ which is punctiform and connected; in both examples we shall use the notation introduced in Example 1.4.6.

1.4.7. EXAMPLE. We shall show that the subspace $Y = X \cup \{p\}$ of the plane, where $p = (0,1/2)$, is hereditarily disconnected but is not totally disconnected.

Consider a connected subspace A of the space Y. As $G(p) = 0 \in C$, the image $G(A)$ is a connected subspace of the Cantor set and thus contains at most one point. It follows that A is contained in a fibre of the mapping $G|Y$. Since all fibres of $G|Y$ are at most of cardinality 2, the set A either is empty or consists of exactly one point, and this implies that the space Y is totally disconnected.

Consider now an open-and-closed set $U \subset Y$ such that $p \in U$. There exists a number $a \in I \setminus C$ such that $([0,a) \times \{1/2\}) \cap Y \subset U$. One readily sees that the set

$$V = ([0,a) \times [0,1/2)) \cap Y \setminus U = ([0,a] \times [0,1/2]) \cap Y \setminus U$$

is open-and-closed in X. The inverse image $h^{-1}(V) \subset H_0$ is an open-and-closed bounded subset of H_0 and thus is empty. Hence the set V also is empty, which implies that $h(x_0) = (0,0) \in U$. Thus for the pair $x = p$, $y = h(x_0)$ of distinct points of Y there exists no open-and-closed set $U \subset X$ such that $x \in U$ and $y \in Y \setminus U$, i.e., the space Y is not totally disconnected. ■

1.4.8. EXAMPLE. We shall show that the subspace $Z = X \cup \{q\}$ of the plane, where $q = (0,1)$, is punctiform and connected.

Consider a continuum $A \subset Z$. The difference $A \setminus \{q\} \subset X$ is an F_σ-set in A, and so $A \setminus \{q\} = \bigcup_{i=1}^{\infty} A_i$, where A_i is compact for $i = 1, 2, \ldots$ For every i the restriction $g_i = g|A_i: A_i \to g_i(A_i) \subset C$ is a homeomorphism, because it is a one-to-one continuous mapping defined on a compact space. Since $\operatorname{ind} g_i(A_i) \leq 0$, we have $\operatorname{ind} A_i \leq 0$ for $i = 1, 2, \ldots$ and – by virtue of Theorems 1.1.2 and 1.3.1– $\operatorname{ind} A \leq \operatorname{ind}(\{q\} \cup \bigcup_{i=1}^{\infty} A_i) = 0$. Hence the set A either is empty or consists of exactly one point, and this implies that the space Z is punctiform.

Consider now an open-and-closed set $U \subset Z$ such that $q \in U$. One readily sees that there exists a number $a \in (0, 1)$ such that $(I \times (a, 1]) \cap Z \subset U$. The inverse image $h^{-1}(Z \setminus U) \subset H_0$ is an open-and-closed bounded subset of H_0 and thus is empty. Hence $U = Z$, i.e., the space Z is connected. ■

In the table below the basic properties of countable, zero-dimensional, totally disconnected, hereditarily disconnected, and punctiform spaces are compared; a plus means that a theorem holds in the corresponding class, a minus that it does not hold. Formal statements of the results in the table together with hints how to obtain them can be found in the problems.

Various kinds of disconnectedness

	Enlargment to a G_δ-set	Countable sum	Finite sum	Adjoining one point	Countable Cartesian product	Finite Cartesian product	Universal space	Compactification	Embedding in the real line	Embedding in a Euclidean space
Countable spaces	–	+	+	+	–	+	+	–	+	+
Zero-dimensional spaces	+	+	+	+	+	+	+	+	+	+
Totally disconnected spaces	–	–	–	–	+	+	–	–	–	–
Hereditarily disconnected spaces	–	–	–	–	+	+	–	–	–	–
Punctiform spaces	–	+	+	+	+	+	–	–	–	–

Historical and bibliographic notes

Totally disconnected spaces were introduced by Sierpiński in [1921], hereditarily disconnected spaces – by Hausdorff in [1914], and punctiform spaces – by Janiszewski in [1912]. Theorem 1.4.5 was proved (for compact metric spaces) by Menger in [1923] and by Urysohn in [1925]. The first example of a totally disconnected space which is not zero-dimensional was given by Sierpiński in [1921]; Sierpiński's space is a completely metrizable subspace of the plane. The first example of a hereditarily disconnected space which is not totally disconnected was also given by Sierpiński in [1921]; this space is also a completely metrizable subspace of the plane. Example 1.4.7 is a simplified version of an example described by Roberts in [1956]. The first example of a punctiform space which is not hereditarily disconnected was described by Sierpiński in [1920]; this space is

a connected subspace of the plane. An example of a completely metrizable punctiform and connected subspace of the plane was given by Mazurkiewicz in [1920]. A simple modification that leads from spaces described in Examples 1.4.6–1.4.8 to similar spaces which are, moreover, completely metrizable is sketched in Problem 1.4.B(b). A dimension function associated with the class of totally disconnected spaces was defined by O'Connor and Rogers in [1992].

Problems

1.4.A. (a) Observe that zero-dimensionality, total disconnectedness, hereditary disconnectedness and punctiformness are equivalent in the realm of regular spaces which can be represented as countable unions of compact subspaces.

(b) Note that zero-dimensionality, total disconnectedness, hereditary disconnectedness and punctiformness are equivalent in the realm of subspaces of the real line. Deduce that there exist totally disconnected spaces which are not embeddable in the real line.

1.4.B. (a) (Roberts [1956]) Prove that the mapping $h: H_0 \to I^2$ defined in Example 1.4.6 is a homeomorphic embedding.

Hint. Prove that in Hilbert space H a sequence of points $x^1, x^2, \ldots$, where $x^m = \{x_i^m\}$ for $m = 1, 2, \ldots$, converges to a point $x = \{x_i\}$ if and only if the sequence $\|x^m\|$ converges to $\|x\|$ and the sequence $x_i^1, x_i^2, \ldots$ converges to x_i for $i = 1, 2, \ldots$ (cf. Example 1.5.17).

(b) Give examples of completely metrizable spaces $X_1, Y_1, Z_1 \subset I^2$ such that X_1 is totally disconnected but is not zero-dimensional, Y_1 is hereditarily disconnected but is not totally disconnected, and Z_1 is punctiform and connected.

Hint. Consider the subspace H_1 of Hilbert space H consisting of the points $\{x_i\} \in H$ such that x_i is irrational for every i and suitably modify the constructions in Examples 1.4.6–1.4.8. When proving that X_1 is completely metrizable, apply Lemma 1.3.12. In the proof of complete metrizability of Y_1 and Z_1 use the fact that every completely metrizable subspace of I^2 is a G_δ-set (see Problem 1.3.B).

1.4.C. (a) (Knaster and Kuratowski [1921]) Let C be the Cantor set on the interval $[0,1] \times \{0\} \subset R^2$; denote by Q the set of the end-points of all intervals removed from $[0,1] \times \{0\}$ to obtain the Cantor set and let $P = C \setminus Q$. Join every point $c \in C$ to the point $q = (1/2, 1/2) \in R^2$ by a segment L_c and denote by F_c the set of all points $(x, y) \in L_c$, where y is rational if $c \in Q$ and y is irrational if $c \in P$. The subspace $F = \bigcup_{c \in C} F_c$ of the plane is called the *Knaster–Kuratowski fan.*

Prove that the Knaster–Kuratowski fan is connected and punctiform.

Hint. Let $r_1, r_2, \ldots$ be the sequence of all rational numbers in the interval $[0, 1/2]$ and let $P_i \subset R^2$ be the horizontal line $y = r_i$. Suppose that $F = (F \cap A) \cup (F \cap B)$, where the sets A and B are closed in R^2, $F \cap A \cap B = \emptyset$ and $q \in A$. Consider the sets $K_i = \{c \in C : A \cap B \cap L_c \cap P_i \neq \emptyset\}$, check that they are closed and that $\bigcup_{i=1}^{\infty} K_i \subset P$. Show that $F_c \cap B = \emptyset$ for every point $c \in P \setminus \bigcup_{i=1}^{\infty} K_i$, apply the Baire category theorem to prove that the set $P \setminus \bigcup_{i=1}^{\infty} K_i$ is dense in C, and deduce that $F \cap B = \emptyset$.

(b) (Knaster and Kuratowski [1921]) Prove that the space $F \setminus \{q\}$ is hereditarily disconnected but is not totally disconnected.

(c) (E. Pol [1978]) Prove that every completely metrizable space X which contains a subspace homeomorphic to $F \setminus \{q\}$ also contains a subspace homeomorphic to the closed unit interval.

Hint. Apply the Lavrentieff theorem (see [GT], Theorem 4.3.21) to reduce the problem to the case where X is a G_δ-set in the plane and $F \setminus \{q\} \subset X$. Consider the set $P \cap \bigcap_{i=1}^{\infty} p(X \cap P_i)$, where p is the projection from q onto C and P_i is defined in the hint to part (a).

(d) Observe that modifying the construction of the space F by taking all points $(x, y) \in L_c$, where y is irrational if $c \in Q$ and y is rational if $c \in P$, one obtains a zero-dimensional space.

1.4.D. Let K and D be the spaces discussed in Problem 1.2.E.

(a) Show that if $\operatorname{ind}_z K = 1$, then there exists a point $p \in D \setminus K$ such that the empty set is not a partition in the space $K' = K \cup \{p\}$ between z and p.

Hint. See the hint to Problem 1.2.E(e).

(b) Prove that the space K' defined in (a) is hereditarily disconnected but is not totally disconnected.

(c) Verify that the space K' is completely metrizable.

1.4.E. (a) Note that by adjoining a point to a totally disconnected space one can obtain a space which is not totally disconnected. Deduce that the counterpart of the sum theorem does not hold either for totally disconnected spaces or for hereditarily disconnected spaces, even in the case where the space is represented as the union of two closed subspaces. Show that the counterpart of the sum theorem holds for punctiform spaces.

(b) Check that the Cartesian product of a family of totally disconnected, hereditarily disconnected, or punctiform spaces is a space in the same class.

(c) Note that the counterpart of the compactification theorem holds for no class of spaces in the table on p. 28 other than the class of zero-dimensional spaces.

1.4.F (Hilgers [1937]). (a) Let Z be a topological space and T a subspace of Z whose cardinality is equal to $\mathbf{c}$. Prove that for every family $\mathcal{G}$ of subspaces of the Cartesian product $Z \times Z$ such that $|\mathcal{G}| \leq \mathbf{c}$ there exists a set $H \subset Z \times Z$ satisfying the following conditions:

(1) *The projection of $Z \times Z$ onto the first factor maps H in a one-to-one way onto the subspace T.*

(2) *If $H \subset G$ for a $G \in \mathcal{G}$, then G contains a set homeomorphic to Z.*

Hint. Let φ be an arbitrary function from T onto $\mathcal{G}$; define a mapping f of T to Z by letting $f(t)$ be a point $z \in Z$ such that $(t, z) \in (\{t\} \times Z) \setminus \varphi(t)$ if such points exist, and an arbitrary point $z \in Z$ otherwise. Consider the set $H = \{(t, f(t)) : t \in T\}$, i.e., the graph of the mapping f.

(b) Applying Theorem 1.5.11 and the equality $\operatorname{ind} R^n = n$ (see Theorem 1.8.3), for every natural number n define a separable metric space X such that $\operatorname{ind} X = n$ and X can be mapped by a continuous and one-to-one mapping onto a zero-dimensional

space. Observe that X is totally disconnected and deduce that there exists a totally disconnected separable metric space which cannot be embedded in a Euclidean space (cf. Example 6.2.4).

Hint. Consider the space $Z = R^n$, a subspace $T \subset Z$ homeomorphic to the Cantor set, and the family $\mathcal{G}$ of all G_δ-sets in the Cartesian product $R^n \times R^n$; then apply (a).

Remark. The first example of a totally disconnected separable metric space of an arbitrary dimension $n \geq 1$ was given by Mazurkiewicz in [1927]; Mazurkiewicz's spaces are completely metrizable (cf. Problem 6.2.A). Clearly, such spaces do not contain any compact subspace of positive dimension.

1.4.G (R. Pol [1973]). (a) Prove that every separable metric space X which topologically contains all punctiform separable metric spaces contains topologically the Hilbert cube $I^{\aleph_0}$.

Hint. One can assume that $X \subset I^{\aleph_0}$. Consider the space $Z = I^{\aleph_0}$, a subspace $T \subset Z$ homeomorphic to the Cantor set, and the family $\mathcal{G}$ consisting of all sets of the form $f^{-1}(X)$, where f is a continuous mapping defined on a G_δ-set in the Cartesian product $I^{\aleph_0} \times I^{\aleph_0}$ and taking values in $I^{\aleph_0}$; then apply Problem 1.4.F(a) and the Lavrentieff theorem (see [GT], Theorem 4.3.21).

(b) Define a totally disconnected separable metric space X with the property that every completely metrizable separable space which contains a subspace homeomorphic to X also contains a subspace homeomorphic to the Hilbert cube.

Hint. Consider the space $Z = I^{\aleph_0}$, a subspace $T \subset Z$ homeomorphic to the Cantor set and the family $\mathcal{G}$ of all G_δ-sets in the Cartesian product $I^{\aleph_0} \times I^{\aleph_0}$; then apply Problem 1.4.F(a) and the Lavrentieff theorem.

1.5. The sum, decomposition, addition, enlargement, separation and Cartesian product theorems

We begin with some observations on the dimension of subspaces. The subspace theorem established in Section 1.1 states that for every subspace M of a regular space X we have $\operatorname{ind} M \leq \operatorname{ind} X$. In this context it is natural to ask whether among the subspaces of a space X such that $\operatorname{ind} X = n$ one can find subspaces of all intermediate dimensions between 0 and $n-1$. As shown in the next theorem, the answer is positive and there even exist closed subspaces of all intermediate dimensions.

1.5.1. THEOREM. *If X is a regular space and $\operatorname{ind} X = n \geq 1$, then for $k = 0, 1, \ldots, n-1$ the space X contains a closed subspace M_k such that $\operatorname{ind} M_k = k$.*

PROOF. It is enough to show that X contains a closed subspace M such that $\operatorname{ind} M = n - 1$. As $\operatorname{ind} X > n - 1$, there exist a point $x \in X$ and a neighbourhood $V \subset X$ of the point x such that for every open set $U \subset X$ satisfying the condition $x \in U \subset V$ we have $\operatorname{ind} \operatorname{Fr} U > n - 2$. On the other hand, as $\operatorname{ind} X \leq n$, there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind} \operatorname{Fr} U \leq n - 1$. The closed subspace $M = \operatorname{Fr} U$ of the space X has the required property. ■

The situation is quite different in spaces of dimension ∞. In Example 1.8.23 we shall describe, applying the continuum hypothesis, a separable metric space of dimension ∞ whose finite-dimensional subspaces are all countable (it turns out that the existence of such a space is equivalent to the continuum hypothesis). Let us observe that spaces with the above property are rather peculiar; in particular, no such space is completely metrizable, because every uncountable completely metrizable separable space contains a subspace homeomorphic to the Cantor set (see Problem 1.3.F(c)). There also exist compact metric spaces of dimension ∞ whose finite-dimensional subspaces are all zero-dimensional, but examples of such spaces are much more complicated (see Example 5.2.23).

We now pass to the sum theorem.

1.5.2. LEMMA. *If a separable metric space X can be represented as the union of two subspaces Y and Z such that* $\operatorname{ind} Y \leq n-1$ *and* $\operatorname{ind} Z \leq 0$, *then* $\operatorname{ind} X \leq n$.

PROOF. Consider a point $x \in X$ and a neighbourhood $V \subset X$ of the point x. By virtue of Theorem 1.2.11, there exist disjoint open sets $U, W \subset X$ such that $x \in U$, $X \setminus V \subset W$ and $[X \setminus (U \cup W)] \cap Z = \emptyset$. Clearly, $x \in U \subset V$; as $\operatorname{Fr} U \subset [X \setminus (U \cup W)] \subset X \setminus Z \subset Y$, we have $\operatorname{ind} \operatorname{Fr} U \leq n-1$. Hence $\operatorname{ind} X \leq n$. ■

1.5.3. THE SUM THEOREM. *If a separable metric space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that* $\operatorname{ind} F_i \leq n$ *for* $i = 1, 2, \ldots$, *then* $\operatorname{ind} X \leq n$.

PROOF. We shall apply induction with respect to n. For $n = 0$ the theorem is already proved. Assume that it holds for dimensions less than n and consider a space $X = \bigcup_{i=1}^{\infty} F_i$, where F_i is closed and $\operatorname{ind} F_i \leq n \geq 1$ for $i = 1, 2, \ldots$ Applying Theorem 1.1.6, for $i = 1, 2, \ldots$ choose a countable base $\mathcal{B}_i$ for the space F_i such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}_i$, where Fr denotes the boundary operator in the space F_i. By the inductive assumption the subspace $Y = \bigcup \{\operatorname{Fr} U : U \in \bigcup_{i=1}^{\infty} \mathcal{B}_i\}$ of X satisfies the inequality $\operatorname{ind} Y \leq n-1$. Now, Proposition 1.2.13 implies that for $i = 1, 2, \ldots$ the subspace $Z_i = F_i \setminus Y$ of the space F_i satisfies the inequality $\operatorname{ind} Z_i \leq 0$; hence, by the sum theorem for dimension 0, the subspace $Z = \bigcup_{i=1}^{\infty} Z_i = X \setminus Y$ of X also satisfies the inequality $\operatorname{ind} Z \leq 0$, because it follows from the relation $Z_i = F_i \setminus Y = F_i \cap Z$ that all the Z_i's are closed in Z. Thus by virtue of Lemma 1.5.2 we have $\operatorname{ind} X \leq n$. ■

As in the case of zero-dimensional spaces, the sum theorem implies three corollaries:

1.5.4. COROLLARY. *If a separable metric space X can be represented as the union of a sequence $A_1, A_2, \ldots$ of subspaces such that* $\operatorname{ind} A_i \leq n$ *and A_i is an F_σ-set for* $i = 1, 2, \ldots$, *then* $\operatorname{ind} X \leq n$. ■

1.5.5. COROLLARY. *If a separable metric space X can be represented as the union of two subspaces A_1 and A_2, one of them being both an F_σ-set and a G_δ-set, such that* $\operatorname{ind} A_1 \leq n$ *and* $\operatorname{ind} A_2 \leq n$, *then* $\operatorname{ind} X \leq n$. ■

1.5.6. COROLLARY. *If by adjoining a finite number of points to a separable metric space X such that* $\operatorname{ind} X \leq n$ *one obtains a metrizable space Y, then the space Y satisfies the inequality* $\operatorname{ind} Y \leq n$ *and is separable.* ■

Let us note that in the sum theorem the countability of the family of closed subspaces is essential (see Problem 1.5.I(c)).

Let us also observe that the sum theorem plays a key role in the dimension theory of separable metric spaces. Indeed, all the remaining results in this section follow either from the sum theorem or from one of the decomposition theorems which are easy consequences of the sum theorem.

Applying the sum theorem, one readily shows that the condition in Lemma 1.5.2 characterizes separable metric spaces of dimension $\leq n$:

1.5.7. THE FIRST DECOMPOSITION THEOREM. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if X can be represented as the union of two subspaces Y and Z such that* $\operatorname{ind} Y \leq n-1$ *and* $\operatorname{ind} Z \leq 0$.

PROOF. It suffices to show the necessity of the condition. Consider a separable metric space X such that $\operatorname{ind} X \leq n \geq 0$. By Theorem 1.1.6 the space X has a countable base $\mathcal{B}$ such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$. It follows from the sum theorem that the subspace $Y = \bigcup\{\operatorname{Fr} U : U \in \mathcal{B}\}$ has dimension $\leq n-1$, and from Proposition 1.2.13 that the subspace $Z = X \setminus Y$ has dimension ≤ 0. ■

From the first decomposition theorem we obtain by an easy induction

1.5.8. THE SECOND DECOMPOSITION THEOREM. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \ldots, Z_{n+1}$ such that* $\operatorname{ind} Z_i \leq 0$ *for* $i = 1, 2, \ldots, n+1$. ■

Let us emphasize that, as will be shown in Section 1.8 (see Theorem 1.8.21), the Hilbert cube cannot be represented as a countable union of zero-dimensional subspaces, or – equivalently – of finite-dimensional subspaces. Hence, the second decomposition theorem does not extend to separable metric spaces of dimension ∞.

1.5.9. EXAMPLES. (a) For every point x in the real line R or in the circle S^1 and each neighbourhood V of the point x there exists a neighbourhood U of x such that $U \subset V$ and the boundary $\operatorname{Fr} U$ is a two-point set. Hence, $\operatorname{ind} R \leq 1$ and $\operatorname{ind} S^1 \leq 1$. Since $\operatorname{ind} I > 0$ by virtue of Example 1.2.5(c), the subspace theorem implies that $\operatorname{ind} R = \operatorname{ind} S^1 = \operatorname{ind} I = 1$.

(b) For every point x in Euclidean n-space R^n or in the n-sphere S^n and each neighbourhood V of the point x there exists a neighbourhood U of x such that $U \subset V$ and the boundary $\operatorname{Fr} U$ is homeomorphic to S^{n-1}. Hence, as shown by an inductive argument, $\operatorname{ind} R^n \leq n$, $\operatorname{ind} S^n \leq n$ and $\operatorname{ind} I^n \leq n$ for every natural number n.

The small inductive dimension of R^n, S^n and I^n is indeed equal to n, but the proof of this fact is much more difficult than the above evaluations; it will be given in Section 1.8. The equality $\operatorname{ind} R^n = n$ is of utmost importance for dimension theory. In a sense, it justifies the definition of the dimension function by showing that this definition yields a

notion conforming to geometric intuition. The fact that $\operatorname{ind} R^n = n$ is sometimes called the fundamental theorem of dimension theory.

Let us note that the inequality $\operatorname{ind} R^n \leq n$ can also be deduced from the formula

$$R^n = Q_0^n \cup Q_1^n \cup \ldots \cup Q_n^n$$

which yields an explicit decomposition of R^n into $n+1$ zero-dimensional subspaces (see Example 1.3.8(a)).

(c) For every pair k, n of integers satisfying $0 \leq k \leq n \geq 1$ define

$$N_k^n = Q_0^n \cup Q_1^n \cup \ldots \cup Q_k^n \quad \text{and} \quad L_k^n = Q_k^n \cup Q_{k+1}^n \cup \ldots \cup Q_n^n;$$

thus N_k^n is the subspace of Euclidean n-space R^n consisting of all points with at most k rational coordinates and L_k^n is the subspace of R^n consisting of all points with at least k rational coordinates. From the second decomposition theorem it follows that $\operatorname{ind} N_k^n \leq k$ and $\operatorname{ind} L_k^n \leq n-k$; we shall show in Section 1.8, applying the equality $\operatorname{ind} R^n = n$, that $\operatorname{ind} N_k^n = k$ and $\operatorname{ind} L_k^n = n-k$. The space $N_n^{2n+1} \subset R^{2n+1}$ will play a particularly important role in the sequel: it turns out to be a universal space for the class of all separable metric spaces of dimension $\leq n$ (see Theorem 1.11.5).

(d) From Example 1.3.8(c) it follows that for each integer $n \geq 0$ the subspace $N_n = Q_0 \cup Q_1 \cup \ldots \cup Q_n$ of the Hilbert cube $I^{\aleph_0}$ consisting of all points with at most n rational coordinates satisfies the inequality $\operatorname{ind} N_n \leq n$. Since N_n contains topologically the n-cube I^n, from the equality $\operatorname{ind} I^n = n$ it follows that $\operatorname{ind} N_n = n$. ■

We shall now state further consequences of the sum and decomposition theorems. Let us begin with the addition theorem, which follows immediately from the second decomposition theorem.

1.5.10. THE ADDITION THEOREM. *If a separable metric space X is represented as the union of two subspaces X_1 and X_2, then*

$$\operatorname{ind} X \leq \operatorname{ind} X_1 + \operatorname{ind} X_2 + 1. \blacksquare$$

1.5.11. THE ENLARGEMENT THEOREM. *For every separable subspace M of an arbitrary metric space X satisfying the inequality* $\operatorname{ind} M \leq n$ *there exists a G_δ-set M^* in X such that $M \subset M^*$ and* $\operatorname{ind} M^* \leq n$.

PROOF. By the second decomposition theorem $M = Z_1 \cup Z_2 \cup \ldots \cup Z_{n+1}$, where $\operatorname{ind} Z_i \leq 0$ for $i = 1, 2, \ldots, n+1$. Applying Theorem 1.2.14, enlarge each Z_i to a G_δ-set Z_i^* in X such that $\operatorname{ind} Z_i^* \leq 0$. The union $M^* = Z_1^* \cup Z_2^* \cup \ldots \cup Z_{n+1}^*$ has the required properties. ■

1.5.12. THE FIRST SEPARATION THEOREM. *If X is a separable metric space such that* $\operatorname{ind} X \leq n \geq 0$, *then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that* $\operatorname{ind} L \leq n-1$.

PROOF. By the first decomposition theorem $X = Y \cup Z$, where $\operatorname{ind} Y \leq n-1$ and $\operatorname{ind} Z \leq 0$. Applying Theorem 1.2.11, we obtain a partition L between A and B such that $L \cap Z = \emptyset$. As $L \subset X \setminus Z \subset Y$, we have $\operatorname{ind} L \leq n-1$ by the subspace theorem. ■

In a similar way, applying the first decomposition theorem to the subspace M, we obtain

1.5.13. THE SECOND SEPARATION THEOREM. *If X is an arbitrary metric space and M is a separable subspace of X such that* $\operatorname{ind} M \leq n \geq 0$, *then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that* $\operatorname{ind}(M \cap L) \leq n-1$. ■

Clearly, the first separation theorem is a special case of the second separation theorem. On the other hand, the second separation theorem easily follows (cf. the proof of Theorem 1.2.11) from the first separation theorem and Lemma 1.2.9 which is an elementary topological fact; hence, both separation theorems are in a sense equivalent.

The second separation theorem yields a characterization of the dimension of subspaces in terms of neighbourhoods in the whole space, which generalizes Proposition 1.2.12:

1.5.14. PROPOSITION. *A separable subspace M of an arbitrary metric space X satisfies the inequality* $\operatorname{ind} M \leq n \geq 0$ *if and only if for every point $x \in M$ (or - equivalently - for every point $x \in X$) and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and* $\operatorname{ind}(M \cap \operatorname{Fr} U) \leq n-1$. ■

Proposition 1.5.14 and Lemma 1.1.5 imply the following generalization of Proposition 1.2.13.

1.5.15. PROPOSITION. *A subspace M of a separable metric space X satisfies the inequality* $\operatorname{ind} M \leq n \geq 0$ *if and only if X has a countable base $\mathcal{B}$ such that* $\operatorname{ind}(M \cap \operatorname{Fr} U) \leq n-1$ *for every $U \in \mathcal{B}$.* ■

The general Cartesian product theorem reads as follows:

1.5.16. THE CARTESIAN PRODUCT THEOREM. *For every pair X, Y of separable metric spaces, not both empty, we have*

$$\operatorname{ind}(X \times Y) \leq \operatorname{ind} X + \operatorname{ind} Y.$$

PROOF. The theorem is obvious if $\operatorname{ind} X = \infty$ or $\operatorname{ind} Y = \infty$, so that we can suppose that $k(X, Y) = \operatorname{ind} X + \operatorname{ind} Y$ is finite. We shall apply induction with respect to that number. If $k(X, Y) = -1$, then either $X = \emptyset$ or $Y = \emptyset$, and our inequality holds. Assume that the inequality is proved for every pair of separable metric spaces with the sum of dimensions less than $k \geq 0$ and consider separable metric spaces X and Y such that $\operatorname{ind} X = n \geq 0$, $\operatorname{ind} Y = m \geq 0$ and $n + m = k$. Let (x, y) be a point of $X \times Y$ and $W \subset X \times Y$ a neighbourhood of (x, y). There exist neighbourhoods $U', U \subset X$ of the point x and $V, V' \subset Y$ of the point y such that $U' \times V' \subset W$, $U \subset U'$, $V \subset V'$, $\operatorname{ind} \operatorname{Fr} U \leq n-1$ and $\operatorname{ind} \operatorname{Fr} V \leq m-1$. Since

$$\operatorname{Fr}(U \times V) \subset (X \times \operatorname{Fr} V) \cup (\operatorname{Fr} U \times Y),$$

by virtue of the inductive assumption and the sum theorem we have $\operatorname{ind} \operatorname{Fr}(U \times V) \leq k-1$. Hence $\operatorname{ind}(X \times Y) \leq k$ and the proof is complete. ■

The inequality in the Cartesian product theorem cannot be replaced by equality. In the next example we shall describe a separable metric space X such that $\operatorname{ind} X = 1$, and yet $\operatorname{ind}(X \times X) = 1$, because X is homeomorphic to the square $X \times X$; let us note that also for the space K defined in Problem 1.2.E we have $\operatorname{ind} K = 1$ and

$\operatorname{ind}(K \times K) = 1$ (see the remark to Problem 1.5.H(c)). There even exist pairs of compact metric spaces such that the dimension of their Cartesian product is less than the sum of their dimensions, but they are more complicated, and in checking their properties one has to apply the methods of algebraic topology; let us observe that such spaces are necessarily of dimension ≥ 2, because for any compact metric spaces X, Y such that $\operatorname{ind} Y = 1$ we have $\operatorname{ind}(X \times Y) = \operatorname{ind} X + 1 = \operatorname{ind} X + \operatorname{ind} Y$ (see Problem 1.9.E(b)).

1.5.17. EXAMPLE. We shall show that for the space H_0 defined in Example 1.2.15 we have $\operatorname{ind} H_0 = 1 = \operatorname{ind}(H_0 \times H_0)$.

To establish the equality $\operatorname{ind} H_0 = 1$ it is enough to prove that $\operatorname{ind} H_0 \leq 1$. Since every point $x \in H_0$ can be mapped by a suitable translation to the point $x_0 \in H_0$, it suffices to show that for every natural number n the boundary $F_n = \{x \in H_0 : \|x\| = 1/n\}$ of the $1/n$-ball $U_n = \{x \in H_0 : \|x\| < 1/n\}$ about x_0 is zero-dimensional. This, however, is a consequence of Example 1.3.8(b) and the fact that the mapping $h\colon F_n \to h(F_n) \subset Q^{\aleph_0}$ defined by letting $h(\{x_i\}) = \{x_i\}$ is a homeomorphism, which, in turn, is implied by the fact that both in $Q^{\aleph_0}$ and in F_n a sequence $x^1 = \{x_i^1\}$, $x^2 = \{x_i^2\}, \ldots$ converges to $x = \{x_i\}$ if and only if the sequence $x_i^1, x_i^2, \ldots$ converges to x_i for $i = 1, 2, \ldots$ The last equivalence is well known to hold in the Cartesian product $Q^{\aleph_0}$; it does not generally hold in Hilbert space H, but it does hold in the subspace F_n, because all points of F_n have the same norm (cf. the hint to Problem 1.4.B(a)).

Now, to show that H_0 is homeomorphic to the square $H_0 \times H_0$ it suffices to note that by assigning to a point $(x, y) = (\{x_i\}, \{y_i\}) \in H_0 \times H_0$ the point $(x_1, y_1, x_2, y_2, \ldots) \in H_0$ one defines a homeomorphism of $H_0 \times H_0$ onto H_0.

To conclude, let us observe that the Cartesian products $H_0^2, H_0^3, \ldots$ are all homeomorphic to H_0, and thus are one-dimensional spaces. Since the countable Cartesian product $H_0^{\aleph_0}$ is homeomorphic to the limit of an inverse sequence consisting of finite Cartesian products H_0^i, and since the limit of an inverse sequence consisting of separable metric spaces of dimension $\leq n$ is itself of dimension $\leq n$ (see Theorem 1.13.4), the countable Cartesian product $H_0^{\aleph_0}$ is also one-dimensional. Let us add that every infinite Cartesian product of compact metric spaces of positive dimension is infinite-dimensional (see Problem 1.8.K(b)). ■

Historical and bibliographic notes

Theorem 1.5.3 was proved for compact metric spaces by Menger in [1924] and by Urysohn in [1926] (announcement in [1922]) and was extended to separable metric spaces by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz [1927] (the latter gave the simple proof reproduced here). Theorems 1.5.7, 1.5.8 and 1.5.10 were established for compact metric spaces by Urysohn in [1926] (announcement in [1922]) and were extended to separable metric spaces by Tumarkin and Hurewicz in the above quoted papers. Theorem 1.5.11 was proved by Tumarkin in [1926] (announcement in [1925]). As the reader will see in the next section, Theorem 1.5.12 states, in substance, that for every separable metric space X we have the equality $\operatorname{ind} X = \operatorname{Ind} X$. For compact metric spaces this equality was announced by Brouwer in [1924] while he was discussing relationships between his *Dimensionsgrad* and the small inductive dimension

ind (Brouwer commented that the equality was also known to Urysohn); the proof was given by Menger in [1924] and by Urysohn in [1926]. For separable metric spaces, the equality of ind and Ind was established by Tumarkin in [1926] (announcement in [1925]) and by Hurewicz in [1927]. Theorem 1.5.13 was proved by Menger in [1924] for compact metric spaces, and extended by Hurewicz in [1927] to separable metric spaces. Theorem 1.5.16 was obtained by Menger in [1928]. Example 1.5.17 was described by Erdős in [1940]; let us add that there exist for $n = 1, 2, \ldots$ separable completely metrizable spaces $K(n) \subset I^{n+1}$ such that $\operatorname{ind}[K(n)]^m = \operatorname{ind}[K(n)]^{\aleph_0} = n$ for $m = 1, 2, \ldots$ (see Problem 6.2.C). The first example of two-dimensional compact metric spaces X and Y such that the Cartesian product $X \times Y$ is three-dimensional was given by Pontrjagin in [1930]; examples of such spaces can be found in Kodama [1970]. All the first examples of that kind had rather unpleasant local structure, only in [1988] Dranišnikov defined two compact metric absolute retracts X and Y such that $\operatorname{ind} X = \operatorname{ind} Y = 4$, and yet $\operatorname{ind}(X \times Y) = 7$. It is an open problem if there exist compact metric absolute neighbourhood retracts X, Y such that $\operatorname{ind}(X \times Y) < \operatorname{ind} X + \operatorname{ind} Y < 8$; if such X and Y exist, they satisfy the inequality $\operatorname{ind} X + \operatorname{ind} Y \geq 6$ (Kodama [1957]). Let us note that there exists an interesting geometric characterization of compact metric spaces X, Y for which the "logarithmic law" for dimension of the Cartesian product fails: $\operatorname{ind}(X \times Y) < \operatorname{ind} X + \operatorname{ind} Y$ if and only if for each $\epsilon > 0$ and any two continuous mappings $f: X \to R^m$ and $g: Y \to R^m$, where $m = \operatorname{ind} X + \operatorname{ind} Y$, there exist continuous mappings $f': X \to R^m$ and $g': Y \to R^m$ such that $\rho(f(x), f'(x)) < \epsilon$ for every $x \in X$, $\rho(g(y), g'(y)) < \epsilon$ for every $y \in Y$ and $f'(X) \cap g'(Y) = \emptyset$ (see Krasinkiewicz [1989], Spież [1990] and [1990a], and Dranišnikov, Repovš and Ščepin [1991]; see also Borsuk [1936] for a condition which enforces the "logarithmic law").

Problems

1.5.A (Hurewicz [1927]). Show that if A is a subspace of a separable metric space X and $\operatorname{ind} A = m < n = \operatorname{ind} X$, then for $k = m+1, m+2, \ldots, n-1$ there exists a subspace M_k of X such that $A \subset M_k$ and $\operatorname{ind} M_k = k$; moreover, if A is closed one can assume that M_k is also closed.

1.5.B (Hurewicz [1927]). Prove that each separable metric space X with $\operatorname{ind} X = n \geq 1$ contains a closed non-empty subspace M such that $\operatorname{ind} U = n$ for every non-empty open subset U of the subspace M.

Hint. Let $M = X \setminus \bigcup\{V \subset X : V \text{ is open in } X \text{ and } \operatorname{ind} V \leq n-1\}$ and apply Corollary 1.5.5.

Remark. The assumption of separability can be omitted if one replaces ind by the covering dimension dim (see Definition 1.6.7).

1.5.C. (a) Observe that if a separable metric space X can be represented as the union of a family $\{F_s\}_{s \in S}$ of closed subspaces such that every point $x \in X$ has a neighbourhood which meets at most countably many sets F_s and $\operatorname{ind} F_s \leq n$ for every $s \in S$, then $\operatorname{ind} X \leq n$.

(b) Prove that if a separable metric space X can be represented as the union of a sequence $F_0, F_1, F_2, \ldots$ of closed subspaces such that $\operatorname{ind} F_i \leq n$ for $i = 1, 2, \ldots$, then $\operatorname{ind}_x X \leq n$ for every point $x \in F_0$ such that $\operatorname{ind}_x F_0 \leq n$.

1.5.D. Prove that a subspace M of a metric space X satisfies the inequality $\operatorname{ind} M \leq n \geq 0$ if and only if for every point $x \in M$ and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind}(M \cap \operatorname{Fr} U) \leq n-1$ (cf. Proposition 1.5.14 and Problem 4.1.C).

Hint. Apply Lemma 1.2.9.

1.5.E. Give an example of a space $X \subset R^2$ and of a zero-dimensional subspace M of X with the property that for some disjoint closed subsets A, B of X there is no partition L between A and B such that $L \cap M = \emptyset$ and $\operatorname{ind} L < \operatorname{ind} X$ (cf. Lemma 4.3.8).

Hint. Let $X = I^2 \setminus (Q \times P)$, where Q and P denote respectively the set of rational and irrational numbers in the interval I, and consider the subspace $M = P^2$ of the space X.

1.5.F (Menger [1928]). (a) Show that if $\operatorname{ind}_x X \leq 0$ and $\operatorname{ind}_y Y \leq 0$, then $\operatorname{ind}_{(x,y)}(X \times Y) \leq 0$.

(b) Applying the equality $\operatorname{ind} I^2 = 2$, give an example of two subspaces X and Y of the real line such that for some points $x \in X$ and $y \in Y$ we have $\operatorname{ind}_x X = 0$ and $\operatorname{ind}_y Y = 1$, and yet $\operatorname{ind}_{(x,y)}(X \times Y) = 2$.

1.5.G. Give an example of a completely metrizable separable space X such that $\operatorname{ind} X = 1$ and X is homeomorphic to the square $X \times X$.

1.5.H. Let X be a separable metric space such that $\operatorname{ind} X = n \geq 1$; the set $\{x \in X : \operatorname{ind}_x X = n\}$ is called the *dimensional kernel* of the space X.

(a) (Menger [1924], Urysohn [1926]) Check that the dimensional kernel is an F_σ-set.

(b) (Menger [1926]) Show that the dimensional kernel of a separable metric space X such that $\operatorname{ind} X = n \geq 1$ has dimension $\geq n - 1$.

Hint. Represent the complement of the kernel as the union of two subspaces Y and Z such that $\operatorname{ind} Y \leq n - 2$, $\operatorname{ind} Z \leq 0$ and Y is an F_σ-set in X.

(c) (Menger [1926]) Prove that the dimensional kernel of a compact metric space X such that $\operatorname{ind} X = n \geq 1$ has dimension n at each point (cf. Theorem 1.9.9).

Hint. To begin, prove that for every F_σ-set A in a compact metric space X and every family $\mathcal{B}$ of subsets of X which has the property that for each $x \in A$ and every $\epsilon > 0$ one can find a $U \in \mathcal{B}$ with $x \in U$ and $\delta(U) < \epsilon$ there exists a countable family $\{U_i\}_{i=1}^{\infty}$ of members of $\mathcal{B}$ such that

$$A \subset \bigcup_{i=1}^{\infty} U_i \subset \overline{\bigcup_{i=1}^{\infty} U_i} \subset A \cup \bigcup_{i=1}^{\infty} \overline{U}_i.$$

Now suppose that for a point x of the dimensional kernel M the inequality $\operatorname{ind}_x M \leq n - 1$ holds, and for every positive number ϵ define a neighbourhood U of the point x in the space X such that $\delta(U) < \epsilon$ and $\operatorname{ind} \operatorname{Fr} U \leq n-2$. To that end find a neighbourhood U_0 of x in the space X such that $\delta(U_0) < \epsilon/2$ and $\operatorname{ind}(M \cap \operatorname{Fr} U_0) \leq n - 2$.

Then enlarge $M \cap \operatorname{Fr} U_0$ to an $(n-2)$-dimensional G_δ-set M^* in X. Let $U = U_0$ if $\operatorname{Fr} U_0 \subset M^*$, and if $\operatorname{Fr} U_0 \setminus M^* \neq \emptyset$, define a countable family $\{U_i\}_{i=1}^\infty$ of open subsets of X such that

$$\delta(U_i) < \epsilon/2, \quad \operatorname{ind} \operatorname{Fr} U_i \leq n-2, \quad U_i \cap (\operatorname{Fr} U_0 \setminus M^*) \neq \emptyset$$

for $i = 1, 2, \ldots$, and

$$\operatorname{Fr} U_0 \setminus M^* \subset \bigcup_{i=1}^\infty U_i \subset \overline{\bigcup_{i=1}^\infty} U_i \subset \operatorname{Fr} U_0 \cup \bigcup_{i=1}^\infty \overline{U}_i,$$

and let $U = \bigcup_{i=0}^\infty U_i$; note that $\operatorname{Fr} U \subset (\operatorname{Fr} U_0 \setminus \bigcup_{i=1}^\infty U_i) \cup \bigcup_{i=1}^\infty \operatorname{Fr} U_i$.

Remark. A separable metric space X such that $\operatorname{ind} X = n \geq 1$ and the dimensional kernel of X has dimension $n-1$ is called a *weakly n-dimensional* space. Clearly, a weakly n-dimensional space contains no compact subspace of dimension n. The space K described in Problem 1.2.E is weakly one-dimensional; the first example of such a space was given by Sierpiński in [1921]. First examples of weakly n-dimensional spaces for $n = 2, 3, \ldots$ were given by Mazurkiewicz in [1929]; Mazurkiewicz's spaces are completely metrizable (for a simpler construction of weakly n-dimensional spaces for $n = 1, 2, \ldots$ see Problem 6.2.B). Tomaszewski in [1979] proved that if X is a weakly n-dimensional space and Y is a weakly m-dimensional space, then $\operatorname{ind}(X \times Y) \leq n+m-1 = \operatorname{ind} X + \operatorname{ind} Y - 1$.

(d) (Hurewicz [1927]) Prove that for each point x_0 of a compact metric space X with $\operatorname{ind}_{x_0} X \geq n \geq 1$ there exists a continuum C of cardinality larger than one such that $x_0 \in C \subset \{x \in X : \operatorname{ind}_x X \geq n\}$.

Hint. Consider the set $C = \{x \in X :$ there is no partition of dimension $\leq n-2$ between x and $x_0\}$. First note that C is closed and has cardinality larger than one; then prove that C is connected by modifying the second half of the proof of Lemma 1.4.4 (which corresponds to the case where $n = 1$).

1.5.I. (a) Applying transfinite induction, prove that for every set X of cardinality $\mathbf{c}$ and any family $\mathcal{C}$ of cardinality $\mathbf{c}$ consisting of subsets of X which all have cardinality $\mathbf{c}$, there exists a set $Z \subset X$ such that $|Z| = \mathbf{c}$ and $Z \cap C \neq \emptyset \neq (X \setminus Z) \cap C$ for every $C \in \mathcal{C}$.

(b) Deduce from (a), making use of Problem 1.3.F(c), that the Cantor set C contains a subset Z such that $|Z| = \mathbf{c}$ and every compact set contained either in Z or in $C \setminus Z$ is countable (any set Z with the last property is called a *Bernstein set*).

(c) (R. Pol [1981]) Give an example of a separable metric space of positive dimension which can be represented as the union of $\aleph_1$ closed zero-dimensional subspaces.

Hint. Modify the construction of the Knaster–Kuratowski fan (see Problem 1.4.C(a)); apply (b) to define a set $P' \subset C \setminus Q$ such that $|P'| = \aleph_1$ and for every non-empty open set $U \subset C$ the intersection $P' \cap U$ is not an F_σ-set in U.

Remark. From Martin's axiom, a statement consistent with the usual axioms of set theory, it follows that in every compact metric space the complement of any union of $\aleph_1$ closed nowhere dense sets is a dense set. Since the continuum hypothesis, i.e., the equality $\aleph_1 = \mathbf{c}$, is also consistent with the axioms, the statement: "the closed unit interval I can be represented as the union of $\aleph_1$ closed zero-dimensional subspaces" is independent of the axioms of set theory.

1.6. Definitions of the large inductive dimension and the covering dimension. Metric dimension

The first separation theorem, established in the preceding section, suggests a modification in the definition of the small inductive dimension – replacing the point x by a closed set A. In this way we are led to the notion of the large inductive dimension Ind, defined for all normal spaces. Both dimensions coincide in the realm of separable metric spaces. They diverge, however, in the class of metric spaces and also in the class of compact spaces, but the relevant examples are very difficult and will not be discussed in this book (see notes to Section 2.4). The theory of the dimension function Ind will be developed in Chapter 2, and in Chapter 4 it will be shown that in the realm of all metric spaces this theory is quite similar to the theory of the dimension function ind in separable metric spaces. In the present chapter, the large inductive dimension Ind, just as the covering dimension dim discussed later in this section, will play an auxiliary role: introducing these dimension functions leads to some simplifications in the exposition.

We pass to the formal definition of the dimension Ind.

1.6.1. DEFINITION. To every normal space X one assigns the *large inductive dimension* of X, denoted by $\operatorname{Ind} X$, which is an integer larger than or equal to -1, or the "infinite number" ∞; the definition of the dimension function Ind consists in the following conditions:

(BČ1) $\operatorname{Ind} X = -1$ *if and only if* $X = \emptyset$;

(BČ2) $\operatorname{Ind} X \leq n$, *where* $n = 0, 1, \ldots$, *if for every closed set* $A \subset X$ *and each open set* $V \subset X$ *which contains the set* A *there exists an open set* $U \subset X$ *such that*

$$A \subset U \subset V \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} U \leq n - 1;$$

(BČ3) $\operatorname{Ind} X = n$ *if* $\operatorname{Ind} X \leq n$ *and* $\operatorname{Ind} X > n - 1$;

(BČ4) $\operatorname{Ind} X = \infty$ *if* $\operatorname{Ind} X > n$ *for* $n = -1, 0, 1, \ldots$

The large inductive dimension Ind is also called the *Brouwer–Čech dimension*.

Applying induction with respect to $\operatorname{Ind} X$, one can easily verify that whenever normal spaces X and Y are homeomorphic, then $\operatorname{Ind} X = \operatorname{Ind} Y$, i.e., the large inductive dimension is a topological invariant.

Modifying slightly the proof of Proposition 1.1.4, one obtains

1.6.2. PROPOSITION. *A normal space* X *satisfies the inequality* $\operatorname{Ind} X \leq n \geq 0$ *if and only if for every pair* A, B *of disjoint closed subsets of* X *there exists a partition* L *between* A *and* B *such that* $\operatorname{Ind} L \leq n - 1$. ■

Applying induction with respect to $\operatorname{Ind} X$, one can easily prove the following theorem, which justifies the names of the small and large inductive dimensions.

1.6.3. THEOREM. *For every normal space* X *we have* $\operatorname{ind} X \leq \operatorname{Ind} X$. ■

Both dimensions coincide in the realm of separable metric spaces.

1.6.4. THEOREM. *For every separable metric space* X *we have* $\operatorname{ind} X = \operatorname{Ind} X$.

PROOF. It suffices to show that $\operatorname{Ind} X \leq \operatorname{ind} X$; obviously, one can suppose that $\operatorname{ind} X < \infty$. We shall apply induction with respect to $\operatorname{ind} X$. The inequality holds if $\operatorname{ind} X = -1$. Assume that the inequality is proved for all separable metric spaces of small inductive dimension less than $n \geq 0$ and consider a separable metric space X such that $\operatorname{ind} X = n$. Let A, B be a pair of disjoint closed subsets of X. By virtue of the first separation theorem, there exists a partition L between A and B such that $\operatorname{ind} L \leq n-1$. It follows from the inductive assumption that $\operatorname{Ind} L \leq n-1$, so that $\operatorname{Ind} X \leq n$ by Proposition 1.6.2. Hence $\operatorname{Ind} X \leq \operatorname{ind} X$ and the proof is complete. ■

Using the dimension function Ind one can reformulate Theorem 1.2.6 in the following form (cf. Remark 1.2.7):

1.6.5. THEOREM. *For every Lindelöf space X the conditions* $\operatorname{ind} X = 0$ *and* $\operatorname{Ind} X = 0$ *are equivalent.* ■

Besides the inductive dimensions ind and Ind, in dimension theory one studies another dimension function, namely the covering dimension dim defined for all normal spaces. In the following section we shall prove that the dimensions ind and dim coincide in the realm of separable metric spaces. Later on the reader will see that they diverge in the class of metric spaces and also in the class of compact spaces (see Remark 4.1.6 and Example 3.1.31). On the other hand, the dimensions Ind and dim coincide in the realm of all metric spaces (see Theorem 4.1.3) and diverge in the class of compact spaces (see Example 3.1.31). The theory of the dimension function dim will be developed in Chapter 3. The reason why we introduce the covering dimension now is that this notion comes out in a natural way in the proofs of the compactification, embedding and universal space theorems.

Let us briefly comment on the significance of ind, Ind and dim. There are three ways of defining the dimension of separable metric spaces. They are all equivalent and equally natural, but they are based on different geometric properties of spaces. In the realm of all metric spaces Ind and dim coincide and lead to a dimension theory almost as rich as the theory for separable metric spaces. The dimension ind is practically of no importance outside the class of separable metric spaces. The dimension theories of Ind and dim in the class of normal spaces diverge and are poorer than the theory for separable metric spaces; the theory of dim, however, is richer than that of Ind. Finally, in the dimension theory of separable metric spaces some theorems depend – roughly speaking – on the dimension ind, and other ones depend on the dimension dim; so far we have discussed theorems of the first group, in the subsequent sections we shall discuss those of the second group.

In the definition of the covering dimension the notion of the order of a family of sets will be applied.

1.6.6. DEFINITION. Let X be a set and $\mathcal{A}$ a family of subsets of X. By the *order* of the family $\mathcal{A}$ we mean the largest integer n such that the family $\mathcal{A}$ contains $n+1$ sets with a non-empty intersection; if no such integer exists, we say that $\mathcal{A}$ has order ∞. The order of $\mathcal{A}$ is denoted by $\operatorname{ord} \mathcal{A}$.

Thus, if the order of a family $\mathcal{A} = \{A_s\}_{s\in S}$ equals n, then for any $n+2$ distinct indices $s_1, s_2, \ldots, s_{n+2} \in S$ we have $A_{s_1} \cap A_{s_2} \cap \ldots \cap A_{s_{n+2}} = \emptyset$. In particular, a family of order -1 consists of the empty set alone, and a family of order 0 consists of pairwise disjoint sets which are not all empty.

Let us recall that a cover $\mathcal{B}$ is a *refinement* of another cover $\mathcal{A}$ of the same space, in other words $\mathcal{B}$ *refines* $\mathcal{A}$, if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $B \subset A$. Clearly, every subcover $\mathcal{A}_0$ of $\mathcal{A}$ is a refinement of $\mathcal{A}$.

1.6.7. DEFINITION. To every normal space X one assigns the *covering dimension* of X, denoted by $\dim X$, which is an integer larger than or equal to -1, or the "infinite number" ∞; the definition of the dimension function dim consists in the following conditions:

(ČL1) $\dim X \leq n$, *where* $n = -1, 0, 1, \ldots$, *if every finite open cover of the space* X *has a finite open refinement of order* $\leq n$;

(ČL2) $\dim X = n$ *if* $\dim X \leq n$ *and* $\dim X > n-1$;

(ČL3) $\dim X = \infty$ *if* $\dim X > n$ *for* $n = -1, 0, 1, \ldots$

The covering dimension dim is also called the *Čech–Lebesgue dimension*.

One readily sees that whenever normal spaces X and Y are homeomorphic, then $\dim X = \dim Y$, i.e., the covering dimension is a topological invariant. Clearly, $\dim X = -1$ if and only if $X = \emptyset$.

The next proposition contains two useful characterizations of the covering dimension; in the second one the notion of a shrinking is used. Let us recall that by a *shrinking* of the cover $\{A_s\}_{s\in S}$ of a topological space X we mean any cover $\{B_s\}_{s\in S}$ of the space X such that $B_s \subset A_s$ for every $s \in S$; a shrinking is *open* (*closed*) if all its members are open (closed) subsets of X.

Clearly, every shrinking $\mathcal{B}$ of a cover $\mathcal{A}$ is a refinement of $\mathcal{A}$ and satisfies the inequality $\operatorname{ord}\mathcal{B} \leq \operatorname{ord}\mathcal{A}$.

1.6.8. PROPOSITION. *For every normal space* X *the following conditions are equivalent*:

(i) *The space* X *satisfies the inequality* $\dim X \leq n$.

(ii) *Every finite open cover of the space* X *has an open refinement of order* $\leq n$.

(iii) *Every finite open cover of the space* X *has an open shrinking of order* $\leq n$.

PROOF. The implications (i)⇒(ii) and (iii)⇒(i) are obvious. Consider a normal space X which satisfies (ii). Let $\{U_i\}_{i=1}^k$ be a finite open cover of the space X and $\mathcal{V}$ an open refinement of this cover such that $\operatorname{ord}\mathcal{V} \leq n$. For every $V \in \mathcal{V}$ choose an $i(V) \leq k$ such that $V \subset U_{i(V)}$ and define $V_i = \bigcup\{V : i(V) = i\}$. One readily verifies that $\{V_i\}_{i=1}^k$ is a shrinking of $\{U_i\}_{i=1}^k$ and has order $\leq n$, so that (ii)⇒(iii). ■

We shall now show that when checking the inequality $\dim X \leq n$ it suffices to consider $(n+2)$-element covers.

1.6.9. THEOREM. *A normal space* X *satisfies the inequality* $\dim X \leq n$ *if and only if every* $(n+2)$*-element open cover* $\{U_i\}_{i=1}^{n+2}$ *of the space* X *has an open shrinking* $\{W_i\}_{i=1}^{n+2}$ *of order* $\leq n$, *i.e., such that* $\bigcap_{i=1}^{n+2} W_i = \emptyset$.

PROOF. It suffices to show that every normal space X such that $\dim X > n$ has an $(n+2)$-element open cover $\{U_i\}_{i=1}^{n+2}$ with the property that each open shrinking $\{W_i\}_{i=1}^{n+2}$ of $\{U_i\}_{i=1}^{n+2}$ satisfies the condition $\bigcap_{i=1}^{n+2} W_i \neq \emptyset$.

Since $\dim X > n$, by virtue of Proposition 1.6.8 there exists an open cover $\mathcal{V} = \{V_i\}_{i=1}^{k}$ of the space X which has no open shrinking of order $\leq n$. Moreover, one can assume – replacing $\mathcal{V}$, if necessary, by a suitable shrinking – that if $\mathcal{V}' = \{V_i'\}_{i=1}^{k}$ is an open shrinking of $\{V_i\}_{i=1}^{k}$, then

$$V_{i_1}' \cap V_{i_2}' \cap \ldots \cap V_{i_m}' \neq \emptyset \quad \text{whenever } V_{i_1} \cap V_{i_2} \cap \ldots \cap V_{i_m} \neq \emptyset, \tag{1}$$

where $i_1, i_2, \ldots, i_m$ is a sequence of natural numbers less than or equal to k. Indeed, if $\mathcal{V}$ has an open shrinking $\mathcal{V}'$ which does not satisfy (1), one replaces $\mathcal{V}$ by $\mathcal{V}'$ and one continues this procedure until an open cover with the required property is obtained; as the number of intersections in (1) is finite, the process will come to an end after finitely many steps. Since $\operatorname{ord} \mathcal{V} > n$, rearranging if need be the members of $\mathcal{V}$, we have

$$\bigcap_{i=1}^{n+2} V_i \neq \emptyset. \tag{2}$$

We shall show that the $(n+2)$-element open cover $\{U_i\}_{i=1}^{n+2}$ of the space X, where $U_i = V_i$ for $i \leq n+1$ and $U_{n+2} = \bigcup_{i=n+2}^{k} V_i$, has the required property. Consider an open shrinking $\{W_i\}_{i=1}^{n+2}$ of $\{U_i\}_{i=1}^{n+2}$. The cover

$$\{W_1, W_2, \ldots, W_{n+1}, W_{n+2} \cap V_{n+2}, W_{n+2} \cap V_{n+3}, \ldots, W_{n+2} \cap V_k\}$$

of the space X is an open shrinking of $\mathcal{V}$, so that by (1) and (2) we have

$$\bigcap_{i=1}^{n+2} W_i \supset \Big(\bigcap_{i=1}^{n+1} W_i\Big) \cap (W_{n+2} \cap V_{n+2}) \neq \emptyset. \blacksquare$$

Let us note that the last theorem immediately yields

1.6.10. THEOREM. *For every normal space X the conditions* $\operatorname{Ind} X = 0$ *and* $\dim X = 0$ *are equivalent.* ■

1.6.11. COROLLARY. *If a normal space X satisfies the condition* $0 < |X| < \mathbf{c}$, *then* $\operatorname{Ind} X = 0$ *and* $\dim X = 0$.

PROOF. It suffices to show that $\operatorname{Ind} X = 0$. Consider a pair A, B of disjoint closed subsets of X; by Urysohn's lemma ([GT], Theorem 1.5.11) there exists a continuous function $f\colon X \to I$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. Obviously, for every $t \in (0,1)$ the set $f^{-1}(t)$ is a partition between A and B. Since $f^{-1}(t) \cap f^{-1}(t') = \emptyset$ whenever $t \neq t'$ and since $|X| < \mathbf{c}$, one of these partitions is empty. ■

In the realm of compact metric spaces, the covering dimension can be characterized in terms of the metric, viz., by the condition that the space has finite covers of order $\leq n$ by open sets of arbitrarily small diameter. Let us recall that the *mesh* of a family $\mathcal{A}$ of subsets of a metric space X, denoted by $\operatorname{mesh} \mathcal{A}$, is defined as the least upper bound of the diameters of all members of $\mathcal{A}$, i.e.,

$$\operatorname{mesh} \mathcal{A} = \sup\{\delta(A) : A \in \mathcal{A}\};$$

the mesh is either a non-negative real number or the "infinite number" ∞.

Let us also recall that for every open cover $\mathcal{U}$ of a compact metric space there exists a positive number ϵ such that every subset of X which has diameter less than ϵ is contained in a member of $\mathcal{U}$ (see [GT], Theorem 4.3.31); any such number is called a *Lebesgue number* of the cover $\mathcal{U}$.

1.6.12. THEOREM. *For every compact metric space X the following conditions are equivalent*:

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *For every metric ρ on the space X and for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that* $\operatorname{mesh}\mathcal{U} < \epsilon$ *and* $\operatorname{ord}\mathcal{U} \leq n$.

(iii) *There exists a metric ρ on the space X with the property that for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that* $\operatorname{mesh}\mathcal{U} < \epsilon$ *and* $\operatorname{ord}\mathcal{U} \leq n$.

PROOF. Let X be a compact metric space satisfying $\dim X \leq n$; consider a metric ρ on the space X and a positive number ϵ. By the compactness of X, the open cover $\{B(x, \epsilon/3)\}_{x \in X}$ has a finite open subcover; by applying (ČL1) to this subcover we obtain a finite open cover $\mathcal{U}$ such that $\operatorname{mesh}\mathcal{U} < \epsilon$ and $\operatorname{ord}\mathcal{U} \leq n$. Hence (i)⇒(ii).

The implication (ii)⇒(iii) being obvious, to conclude the proof it suffices to show that (iii)⇒(i). Let ρ be a metric on the space X which has the property stated in (iii). Consider a finite open cover $\{U_i\}_{i=1}^k$ of X and denote by ϵ a Lebesgue number of the cover $\{U_i\}_{i=1}^k$. The cover $\mathcal{U}$ in condition (iii) is a refinement of $\{U_i\}_{i=1}^k$, so that $\dim X \leq n$. ■

The attempts to extend the last theorem to separable metric spaces led to the notion of the metric dimension which we shall discuss presently. Before that, let us briefly comment upon the status of conditions (ii) and (iii) in separable metric spaces.

To begin, recall that if ρ is a metric on a space X and for every positive number ϵ there exists a finite cover of X with mesh less than ϵ, then we say that the metric ρ and the space (X, ρ) are *totally bounded*. Clearly, every totally bounded metric space has finite open covers and finite closed covers with arbitrarily small mesh. One easily checks that every totally bounded space is separable, and on every separable metric space there exists a totally bounded metric (see [GT], Theorem 4.3.5). When passing to separable metric spaces, we obviously have to replace condition (ii) by the following condition:

(ii′) *For every totally bounded metric ρ on the space X and for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that* $\operatorname{mesh}\mathcal{U} < \epsilon$ *and* $\operatorname{ord}\mathcal{U} \leq n$.

Now, one proves that conditions (i) and (ii′) are equivalent for every separable metric space X (see Problem 1.6.B), whereas conditions (i) and (iii) are generally not equivalent for such spaces (see Example 1.10.23).

To every metric space (X, ρ) one assigns the *metric dimension* of (X, ρ), denoted by $\mu\dim(X, \rho)$, which is an integer larger than or equal to -1, or the "infinite number" ∞; the definition follows the pattern of the definition of the covering dimension dim except that condition (ČL1) is replaced by the condition that for every positive number

ϵ there exists an open cover $\mathcal{U}$ of the space X such that $\operatorname{mesh}\mathcal{U} < \epsilon$ and $\operatorname{ord}\mathcal{U} \leq n$. Clearly, if (X, ρ) is a compact space, then $\mu\dim(X, \rho) = \dim X$.

From our comments on conditions (ii) and (iii) it follows that in the realm of separable metric spaces the metric dimension is not a topological invariant; in general, the number $\mu\dim(X, \rho)$ depends upon the metric ρ on the space X. Nevertheless, a theory of the metric dimension $\mu\dim$ can be developed, and it shows a resemblance to the theory of the covering dimension dim. Let us observe that – as the reader can readily verify – if the metrics ρ and σ on a space X are uniformly equivalent, i.e., if the identity mapping of (X, ρ) to (X, σ) and also the identity mapping of (X, σ) to (X, ρ) are uniformly continuous, then $\mu\dim(X, \rho) = \mu\dim(X, \sigma)$; this means that the metric dimension is a uniform invariant.

Let us also note that in the definition of the metric dimension one has to consider an arbitrary open cover $\mathcal{U}$, because the restriction to finite open covers would imply the restriction of the definition to totally bounded spaces. However, in the case where (X, ρ) is a totally bounded metric space, the restriction to finite open covers yields an equivalent definition (see Problem 1.6.E).

From Problem 1.7.E below it follows that for every separable metric space (X, ρ) we have $\mu\dim(X, \rho) \leq \dim X$; by virtue of an important characterization of the covering dimension, to be established in Chapter 3 (see Theorem 3.2.1), this inequality extends to all metric spaces. On the other hand, for every metric space (X, ρ) we have $\dim X \leq 2\mu\dim(X, \rho)$; a proof is sketched out in the hint to Problem 1.6.F(b).

We shall return briefly to the metric dimension in Section 1.10, where a characterization of this dimension function in terms of mappings to polyhedra will be given (see Problem 1.10.L).

Historical and bibliographic notes

As we have already observed in the notes to Section 1.1, the notion of the large inductive dimension Ind is related to Brouwer's notion of *Dimensionsgrad*. A formal definition of the dimension function Ind in the class of normal spaces was first given by Čech in [1931], a short announcement of results proved in his paper [1932] devoted to a study of the large inductive dimension. Theorem 1.6.4 is a restatement of Theorem 1.5.12; its history is described in the notes to Section 1.5. The covering dimension dim was formally introduced and discussed in Čech's paper [1933]; it is related to a property of covers of the n-cube I^n discovered by Lebesgue in [1911] (see the notes to Section 1.1). Theorem 1.6.9 was proved by Hemmingsen in [1946], and Theorem 1.6.10 by Vedenissoff in [1939].

The notion of the metric dimension was introduced by Alexandroff around 1930. As a definition he used the characterization given here in Problem 1.10.L, which is connected with his famous theorem on ϵ-translations to polyhedra (see Theorem 1.10.19). Alexandroff's question whether the metric dimension coincides with the covering dimension in the realm of separable metric spaces was solved in the negative by Sitnikov in [1953] (see Example 1.10.23). This last paper called the topologists' attention back to the notion of the metric dimension. The basic properties of the dimension $\mu\dim$ were established by Smirnov in [1956] and by Egorov in [1959]. Besides the notion of the

metric dimension discussed in this section, which is a natural geometric notion with a sound intuitive background, a few other metric dimension functions have been studied; they are all obtained by replacing topological conditions by the corresponding metrical ones in various characterizations of the dimension dim. A discussion of this topic can be found in Nagami's book [1970].

Problems

1.6.A. (a) Show that if F is a closed subspace of a normal space X and every finite open cover $\mathcal{U}$ of the space X has a finite open refinement $\mathcal{V}$ such that $\operatorname{ord}(\{V \cap F : V \in \mathcal{V}\}) \leq n$, then $\dim F \leq n$ (see Lemma 3.1.5).

(b) Give a direct proof of Theorem 1.6.10.

1.6.B (Hurewicz [1930b]). Prove that a separable metric space X satisfies the inequality $\dim X \leq n$ if and only if for every totally bounded metric ρ on the space X and for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that $\operatorname{mesh}\mathcal{U} < \epsilon$ and $\operatorname{ord}\mathcal{U} \leq n$.

Hint. For a finite open cover $\{U_i\}_{i=1}^k$ of the space X define a metric ρ on the space X with the property that every subset of X which has diameter less than 1 is contained in one of the sets U_i. To that end define continuous functions $f_1, f_2, \ldots, f_k$ from X to I such that $f_i(X \setminus U_i) \subset \{0\}$ for $i = 1, 2, \ldots, k$ and $\bigcup_{i=1}^k f_i^{-1}(1) = X$; observe that by adding to the original distance of x and y the sum $\sum_{i=1}^k |f_i(x) - f_i(y)|$ one obtains a metric on the space X.

1.6.C. Show that Theorem 1.6.12 remains valid if in conditions (ii) and (iii) the words "finite open cover" are replaced by the words "finite closed cover".

1.6.D (Hadziivanov [1976]). (a) Prove that for every positive number ϵ each separable metric space has a countable closed cover $\mathcal{F}$ such that $\operatorname{mesh}\mathcal{F} < \epsilon$ and $\operatorname{ord}\mathcal{F} \leq 1$.

Hint. Note first that the interior of the n-cube I^n in R^n has a countable cover of order 1 by closed subsets of I^n. Then prove the assertion for the n-cube I^n, and finally for the Hilbert cube $I^{\aleph_0}$.

(b) Prove that a compact metric space X is zero-dimensional if and only if for every positive number ϵ there exists a countable closed cover $\mathcal{F}$ of the space X such that $\operatorname{mesh}\mathcal{F} < \epsilon$ and $\operatorname{ord}\mathcal{F} \leq 0$.

Hint. Apply Lemma 1.10.20 and Theorem 1.4.5.

1.6.E (Egorov [1959]). Prove that a totally bounded metric space (X, ρ) satisfies the inequality $\mu\dim(X, \rho) \leq n$ if and only if for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that $\operatorname{mesh}\mathcal{U} < \epsilon$ and $\operatorname{ord}\mathcal{U} \leq n$.

Hint. Consider a finite subset $\{x_1, x_2, \ldots, x_k\}$ of the space X with the property that for every point $x \in X$ there exists an $i \leq k$ such that $\rho(x, x_i) < \epsilon/4$ and observe that every subset of X which has diameter less than $\epsilon/4$ is contained in a member of the cover $\{B(x_i, \epsilon/2)\}_{i=1}^k$.

1.6.F. (a) Prove that if for a metric space (X, ρ) we have $\mu\dim(X, \rho) = 0$, then $\dim X = 0$. Deduce that for every separable metric space (X, ρ) the conditions $\mu\dim(X, \rho) = 0$ and $\dim X = 0$ are equivalent.

Hint. To prove the first part apply Theorem 1.6.10 and modify the proof of Theorem 1.2.6.

Remark. As follows from Proposition 3.2.2, the assumption of separability is not essential.

(b) (Katětov [1958]) Prove that for every metric space (X, ρ) we have $\dim X \leq 2\mu\dim(X, \rho)$.

Hint. For $i = 1, 2, \ldots$ choose an open cover $\mathcal{U}_i$ of the space X such that $\operatorname{mesh}\mathcal{U}_i < 1/3^i$ and $\operatorname{ord}\mathcal{U}_i \leq n = \mu\dim(X, \rho)$. Consider a finite open cover $\{U_j\}_{j=1}^k$ of the space X and denote by $\mathcal{V}_i$ the family of all sets $U \in \mathcal{U}_i$ such that $\rho(U, X \setminus U_j) \geq 1/3^i$ for some $j \leq k$. Check that the sets $V_1, V_2, \ldots$, where $V_i = \bigcup \mathcal{V}_i$, form a cover of the space X and that $\overline{V}_i \subset V_{i+1}$ for $i = 1, 2, \ldots$ Consider the sets $F_i = \overline{V}_{i-1} \cup \overline{C}_i$, where $V_0 = \emptyset$ and C_i consists of all points of X which belong to $n+1$ members of $\mathcal{V}_i$, and the families $\mathcal{W}_i = \{V \setminus F_{i-1} : V \in \mathcal{V}_i\}$, where $F_0 = \emptyset$; define $\mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ and $W_i = \bigcup \mathcal{W}_i$. Show that $\overline{C}_i \subset V_i$ for $i = 1, 2, \ldots$ and deduce that $\mathcal{W}$ is a cover of the space X; observe that it is a refinement of $\{U_j\}_{j=1}^k$. Check that $W_i \cap W_m = \emptyset$ whenever $m \geq i+2$ and deduce that $\operatorname{ord}\mathcal{W} \leq 2n$.

Remark. The evaluation of $\dim X$ in Problem 1.6.F(b) cannot be improved (see Example 1.10.23 and Problem 1.10.J).

1.7. The compactification and coincidence theorems. Characterization of dimension in terms of partitions

The compactification theorem belongs to the group of theorems depending on the dimension dim, and in its proof covers are used in an essential way; accordingly, we formulate this theorem in terms of the covering dimension dim. The compactification theorem is an important step towards the proof of the coincidence theorem, which states that the dimensions ind, Ind and dim coincide in the realm of separable metric spaces.

We begin with introducing three simple operations on covers which will be used throughout the book.

If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ are covers of a topological space X, then the family of all intersections $A_1 \cap A_2 \cap \ldots \cap A_k$, where $A_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, k$, is a cover of the space X. We denote this cover by $\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \ldots \wedge \mathcal{A}_k$; obviously, it is a refinement of $\mathcal{A}_i$ for $i = 1, 2, \ldots, k$.

If $f: X \to Y$ is a continuous mapping of a topological space X to a topological space Y and $\mathcal{A}$ is a cover of the space Y, then the family of all inverse images $f^{-1}(A)$, where $A \in \mathcal{A}$, is a cover of X. We denote this cover by $f^{-1}(\mathcal{A})$.

If M is a subspace of a topological space X and $\mathcal{A}$ is a cover of X, then the family of all intersections $M \cap A$, where $A \in \mathcal{A}$, is a cover of M. We denote this cover by $\mathcal{A}|M$.

One readily sees that the above operations applied to open (closed) covers yield open (closed) covers, and applied to finite covers yield finite covers.

1.7.1. LEMMA. *If (X,ρ) is a totally bounded metric space, then for every finite sequence $f_1, f_2, \ldots, f_k$ of continuous functions from X to I and for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that* $\operatorname{mesh}\mathcal{U} < \epsilon$ *and* $|f_i(x) - f_i(y)| < \epsilon$ *for $i = 1, 2, \ldots, k$ whenever x and y belong to the same member of $\mathcal{U}$.*

PROOF. Let $\mathcal{V}$ be a finite open cover of the space X such that $\operatorname{mesh}\mathcal{V} < \epsilon$ and let $\mathcal{W}$ be a finite open cover of the interval I such that $\operatorname{mesh}\mathcal{W} < \epsilon$. The reader can readily check that the cover $\mathcal{U} = \mathcal{V} \wedge f_1^{-1}(\mathcal{W}) \wedge f_2^{-1}(\mathcal{W}) \wedge \ldots \wedge f_k^{-1}(\mathcal{W})$ has the required properties. ■

The compactification theorem says that each separable metric space (X,ρ) can be embedded in a compact metric space of the same dimension. The proof consists in replacing the original metric ρ by a suitably chosen totally bounded metric $\widetilde{\rho}$ on X, and then taking the completion of $(X,\widetilde{\rho})$. Let us recall that for every metric space X there exists a *completion*, i.e., a complete metric space $\widetilde{X}$ such that X is a dense subspace of $\widetilde{X}$; the metric on $\widetilde{X}$ extends the metric ρ from X and is also denoted by ρ. Moreover, every uniformly continuous function $f\colon X \to I$ can be extended to a continuous function $\tilde{f}\colon \widetilde{X} \to I$, and if X is totally bounded, then its completion is compact (see [GT], Theorems 4.3.19, 4.3.17 and 4.3.29).

1.7.2. THE COMPACTIFICATION THEOREM. *For every separable metric space X there exists a dimension preserving compactification, i.e., a compact metric space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequality* $\dim\widetilde{X} \leq \dim X$.

More exactly, *for every totally bounded metric ρ on the space X there exists an equivalent metric $\widetilde{\rho}$ on X such that $\rho(x,y) \leq \widetilde{\rho}(x,y)$ for $x, y \in X$ and the completion $\widetilde{X}$ of the space $(X,\widetilde{\rho})$ is a compactification of X which satisfies the inequality* $\dim\widetilde{X} \leq \dim X$.

PROOF. For $m = 1, 2, \ldots$ we shall define a finite open cover $\mathcal{U}_m = \{U_{m,k}\}_{k=1}^{k_m}$ of the space X satisfying the following conditions:

(1) $\operatorname{ord}\mathcal{U}_m \leq \dim X$;

(2) $\operatorname{mesh}\mathcal{U}_m \leq 1/2^m$;

and for $i < m$ and $k \leq k_i$,

(3) $|f_{i,k}(x) - f_{i,k}(y)| \leq 1/2^m$ whenever x and y belong to the same member of $\mathcal{U}_m$,

where

$$f_{i,k}(x) = \frac{\rho(x, X \setminus U_{i,k})}{\sum_{j=1}^{k_i} \rho(x, X \setminus U_{i,j})} \quad \text{for } x \in X.$$

To obtain $\mathcal{U}_1$ it suffices to apply the total boundedness of the space X. When the covers $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{m-1}$ are defined, we apply Lemma 1.7.1 to obtain $\mathcal{U}_m$.

Now, let us arrange all pairs (m,k) with $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, k_m$ into an infinite sequence and denote by $n(m,k)$ the place of (m,k) in it. One easily checks that

the formula

$$\widetilde{\rho}(x,y) = \rho(x,y) + \sum_{n(i,k)=1}^{\infty} \frac{1}{2^{n(i,k)}} |f_{i,k}(x) - f_{i,k}(y)| \quad \text{for } x, y \in X$$

defines a metric on the set X, and that ρ and $\widetilde{\rho}$ are equivalent.

We shall now show that the sequence $\widetilde{\operatorname{mesh}}\,\mathcal{U}_m$, where $\widetilde{\operatorname{mesh}}$ denotes the mesh with respect to $\widetilde{\rho}$, converges to 0; this will imply that the metric $\widetilde{\rho}$ is totally bounded and that the completion $\widetilde{X}$ of the space $(X, \widetilde{\rho})$ is compact.

Consider an arbitrary positive number ϵ. Let N be a natural number satisfying the inequality $1/2^N < \epsilon/3$ and $M = \max\{N, \max\{i : n(i,k) \leq N\} + 1\}$. From (2) and (3) it follows by an easy evaluation that $\widetilde{\operatorname{mesh}}\,\mathcal{U}_m < \epsilon$ whenever $m \geq M$, so that indeed $\lim \widetilde{\operatorname{mesh}}\,\mathcal{U}_m = 0$.

Since the functions $f_{m,k}: X \to I$ are uniformly continuous with respect to $\widetilde{\rho}$, they can be extended to continuous functions $\tilde{f}_{m,k}: \widetilde{X} \to I$; and since $\sum_{k=1}^{k_m} f_{m,k}(x) = 1$ for every $x \in X$, we have $\sum_{k=1}^{k_m} \tilde{f}_{m,k}(x) = 1$ for every $x \in \widetilde{X}$. Thus the family $\widetilde{\mathcal{U}}_m = \{\widetilde{U}_{m,k}\}_{k=1}^{k_m}$, where $\widetilde{U}_{m,k} = \tilde{f}_{m,k}^{-1}((0,1])$, is an open cover of the space $\widetilde{X}$ for $m = 1, 2, \ldots$ Now, by the density of X in $\widetilde{X}$ and since $X \cap \widetilde{U}_{m,k} = U_{m,k}$ for $m = 1, 2, \ldots$ and $k \leq k_m$, we have $\lim \widetilde{\operatorname{mesh}}\,\widetilde{\mathcal{U}}_m = 0$; moreover, we have $\operatorname{ord} \widetilde{\mathcal{U}}_m \leq \dim X$ for $m = 1, 2, \ldots$ by virtue of (1). Thus $\dim \widetilde{X} \leq \dim X$ by Theorem 1.6.12. ■

1.7.3. REMARK. Our proof of Theorem 1.7.2 yields a slightly more general result. Under the assumption that the covers $\mathcal{U}_m$ satisfy only (2) and (3), the compactification $\widetilde{X}$ has the following property: if for a subset A of X there exists an m_0 such that $\operatorname{ord}(\mathcal{U}_m | A) \leq n$ for $m \geq m_0$, then the closure $\bar{A}$ of A in $\widetilde{X}$ satisfies the inequality $\dim \bar{A} \leq n$. ■

A variant of the above proof of the compactification theorem is outlined in Problem 1.7.B.

Let us observe that from the equality $\operatorname{ind} \widetilde{X} = \dim \widetilde{X}$, which is a consequence of Lemmas 1.7.5 and 1.7.6, and from Theorem 1.1.2 and Lemma 1.7.5 it follows that $\dim \widetilde{X} = \dim X$ in the compactification theorem.

We now pass to the coincidence theorem.

1.7.4. LEMMA. *Let X be a metric space and M a subspace of X. For every family $\{G_i\}_{i=1}^k$ of pairwise disjoint open subsets of M there exists a family $\{H_i\}_{i=1}^k$ of pairwise disjoint open subsets of X such that $G_i \subset H_i$ for $i = 1, 2, \ldots, k$.*

PROOF. The sets

$$H_i = \{x \in X : \rho(x, G_i) < \rho(x, M \setminus G_i)\}$$

have all the required properties. ■

1.7.5. LEMMA. *For every separable metric space X we have* $\dim X \leq \operatorname{ind} X$.

PROOF. We can suppose that $\operatorname{ind} X < \infty$. If $\operatorname{ind} X = -1$, we clearly have $\dim X \leq \operatorname{ind} X$. Consider the case where $\operatorname{ind} X = 0$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of the space X. By virtue of Proposition 1.2.4, the cover $\mathcal{U}$ has a refinement $\{V_i\}_{i=1}^{\infty}$ consisting

of open-and-closed subsets of X. The pairwise disjoint sets

$$W_1 = V_1, \quad W_2 = V_2 \setminus W_1, \quad \ldots,$$
$$W_i = V_i \setminus (W_1 \cup W_2 \cup \ldots \cup W_{i-1}),$$

are open-and-closed and form a cover of the space X which refines the cover $\mathcal{U}$. From Proposition 1.6.8 it follows that $\dim X \leq 0$, so that again $\dim X \leq \operatorname{ind} X$.

Now, consider the case where $\operatorname{ind} X = n > 0$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of the space X. By virtue of the second decomposition theorem

$$X = Z_1 \cup Z_2 \cup \ldots \cup Z_{n+1}, \quad \text{where } \operatorname{ind} Z_j \leq 0 \quad \text{for } j = 1, 2, \ldots, n+1.$$

It follows from the already established special case of our lemma that $\dim Z_j \leq 0$ for $j = 1, 2, \ldots, n+1$. Thus the cover $\mathcal{U}|Z_j$ of the space Z_j has a shrinking $\{G_{j,i}\}_{i=1}^k$ consisting of pairwise disjoint open subsets of Z_j. Applying Lemma 1.7.4, we obtain a family $\{H_{j,i}\}_{i=1}^k$ of pairwise disjoint open subsets of X such that $G_{j,i} \subset H_{j,i}$ for $i = 1, 2, \ldots, k$. The sets $V_{j,i} = H_{j,i} \cap U_i$, where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n+1$, form an open cover of the space X which refines the cover $\mathcal{U}$; the order of this cover is not larger than n, because any $n+2$ sets $V_{j,i}$ include at least two with the same index j, and the latter have an empty intersection. Thus $\dim X \leq n$, i.e., $\dim X \leq \operatorname{ind} X$. ∎

Let us observe that in the case where $\operatorname{ind} X = 0$ the inequality $\dim X \leq \operatorname{ind} X$ follows from Theorems 1.6.4 and 1.6.10, but the argument given above is much simpler.

1.7.6. LEMMA. *For every compact metric space X we have* $\operatorname{ind} X \leq \dim X$.

PROOF. We can suppose that $\dim X < \infty$. We shall apply induction with respect to $\dim X$. If $\dim X = -1$, we clearly have $\operatorname{ind} X \leq \dim X$. Assume that our inequality holds for all compact metric spaces with covering dimension $\leq n-1$ and consider a compact metric space X such that $\dim X = n \geq 0$, a point $x \in X$, and a closed set B which does not contain x. It suffices to define open sets $K, M \subset X$ which, together with the set $L = X \setminus (K \cup M)$, satisfy the conditions

$$x \in K, \quad B \subset M, \quad K \cap M = \emptyset \quad \text{and} \quad \dim L \leq n-1; \tag{4}$$

indeed, the set L is then a partition between x and B, and $\operatorname{ind} L \leq n-1$ by virtue of the inductive assumption. To that end we shall define two sequences $K_0, K_1, K_2, \ldots$ and $M_0, M_1, M_2, \ldots$ of closed subsets of X satisfying for $i = 1, 2, \ldots$, together with the set $L_i = X \setminus (K_i \cup M_i)$, the following conditions:

$$x \in K_{i-1} \subset \operatorname{Int} K_i, \quad B \subset M_{i-1} \subset \operatorname{Int} M_i \quad \text{and} \quad K_i \cap M_i = \emptyset. \tag{5}$$

(6) L_i has a finite open cover $\mathcal{W}_i$ with $\operatorname{mesh} \mathcal{W}_i < 1/i$ and $\operatorname{ord} \mathcal{W}_i \leq n-1$.

Let $K_0 = \{x\}$, $M_0 = B$, and assume that the sets K_i, M_i are already defined for $i < j$ and satisfy (5) and (6) for $0 < i < j$. Since in a compact metric space the distance of two disjoint closed sets is positive, there exists a finite open cover $\mathcal{U}_j$ of the space X such that $\operatorname{mesh} \mathcal{U}_j < \min(1/j, \rho(K_{j-1}, M_{j-1}))$ and $\operatorname{ord} \mathcal{U}_j \leq n$. Let $K_j = X \setminus H_j$ and $M_j = X \setminus G_j$, where

$$G_j = \bigcup \{U \in \mathcal{U}_j : \overline{U} \cap M_{j-1} = \emptyset\}$$

and

$$H_j = \bigcup \{U \in \mathcal{U}_j : \overline{U} \cap M_{j-1} \neq \emptyset\}.$$

As the closure of no member of $\mathcal{U}_j$ meets both K_{j-1} and M_{j-1}, it follows from the definitions of G_j and H_j that

$$\overline{G}_j \cap M_{j-1} = \emptyset = \overline{H}_j \cap K_{j-1},$$

which implies that $K_{j-1} \subset X \setminus \overline{H}_j = \operatorname{Int} K_j$ and $M_{j-1} \subset X \setminus \overline{G}_j = \operatorname{Int} M_j$; moreover, $K_j \cap M_j = 0$, because $G_j \cup H_j = X$. Thus, condition (5) is satisfied for $i = j$. The family $\mathcal{W}_j = \{U \cap L_j : U \in \mathcal{U}_j \text{ and } \overline{U} \cap M_{j-1} \neq \emptyset\}$ is an open cover of the set $L_j = X \setminus (K_j \cup M_j) = G_j \cap H_j$ and $\operatorname{mesh} \mathcal{W}_j < 1/j$. Since every point $x \in L_j \subset G_j$ belongs to at least one set $U \in \mathcal{U}_j$ such that $\overline{U} \cap M_{j-1} = \emptyset$, the order of $\mathcal{W}_j$ is not larger than $n-1$. Thus condition (6) is also satisfied for $i = j$, so that the construction of the sequences $K_0, K_1, K_2, \ldots$ and $M_0, M_1, M_2, \ldots$ is completed.

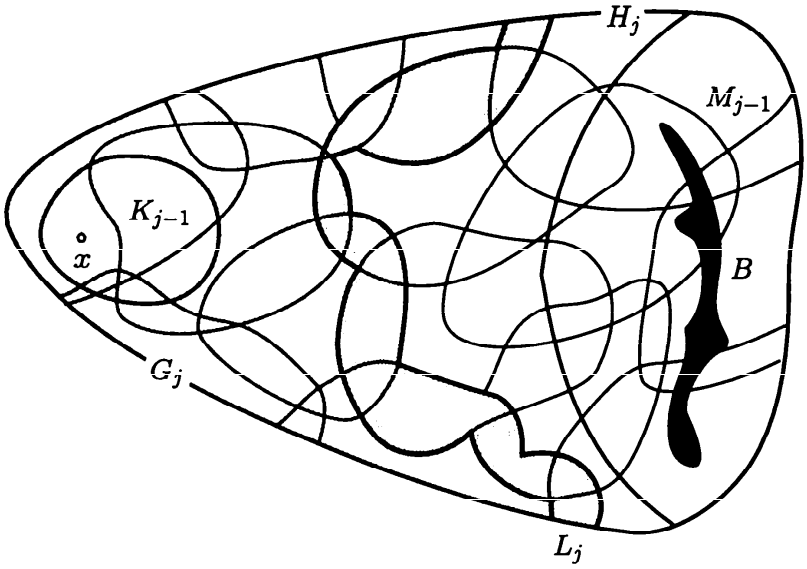


Fig. 6

The open sets $K = \bigcup_{i=0}^{\infty} K_i$ and $M = \bigcup_{i=0}^{\infty} M_i$ are disjoint and contain x and B respectively, and the set $L = X \setminus (K \cup M) = \bigcap_{i=1}^{\infty} L_i$ satisfies, by virtue of (6) and Theorem 1.6.12, the inequality $\dim L \leq n-1$; hence conditions (4) hold, and the proof of the theorem is complete. ■

1.7.7. THE COINCIDENCE THEOREM. *For every separable metric space X we have* $\operatorname{ind} X = \operatorname{Ind} X = \dim X$.

PROOF. By virtue of Theorem 1.6.4 and Lemma 1.7.5 it suffices to show that $\operatorname{ind} X \leq \dim X$. Apply the compactification theorem to obtain a compactification $\widetilde{X}$ of the space X such that $\dim \widetilde{X} \leq \dim X$. It follows from Lemma 1.7.6 that $\operatorname{ind} \widetilde{X} \leq \dim \widetilde{X}$, so that $\operatorname{ind} X \leq \dim X$ by virtue of the subspace theorem. ■

Let us now make some comments on the last theorem. To prove that for every separable metric space X we have $\operatorname{ind} X = \dim X$ one has to establish two inequalities: $\dim X \leq \operatorname{ind} X$ and $\operatorname{ind} X \leq \dim X$. The proof of the former is fairly easy, but the proof of the latter is much more difficult. Usually, we prove the inequality $\operatorname{ind} X \leq \dim X$ for compact metric spaces, which is much simpler, and then apply the compactification theorem to extend it over all separable metric spaces. In all the existing proofs of the inequality $\operatorname{ind} X \leq \dim X$ one can detect an auxiliary integer-valued invariant $d(X)$ for

which the inequalities $\operatorname{ind} X \leq d(X)$ and $d(X) \leq \dim X$ are established. In our proof the invariant $d(X)$, defined for every compact metric space X, was the smallest integer $n \geq 0$ such that the space X has finite open covers of order $\leq n$ with arbitrarily small meshes; the inequality $d(X) \leq \dim X$ was trivial, and the proof reduced to showing that $\operatorname{ind} X \leq d(X)$. In Section 1.11 we shall give incidentally another proof of the inequality $\operatorname{ind} X \leq \dim X$ for compact metric spaces. In that proof the invariant $d(X)$ will be the smallest integer $n \geq 0$ such that X is embeddable in the space $N_n^{2n+1} \subset R^{2n+1}$ defined in Example 1.5.9(c); since $\operatorname{ind} X \leq d(X)$, the proof will reduce to showing that $d(X) \leq \dim X$. One more proof of the inequality $\operatorname{ind} X \leq \dim X$ is sketched in Problem 1.7.F.

Let us also note that in Section 4.1 the proof of Lemma 1.7.6 will reappear almost verbatim as part of the proof that the inequality $\operatorname{Ind} X \leq \dim X$ holds for every metric space X.

We conclude this section with the theorem on partitions, an important characterization of dimension. At the core of this characterization lies an interesting geometric property of the n-cube I^n; viz., the property that if L_i is a partition between the pair of opposite faces $A_i = \{\{x_j\} \in I^n : x_i = 0\}$ and $B_i = \{\{x_j\} \in I^n : x_i = 1\}$ of I^n for $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^n L_i \neq \emptyset$; this property is closely related to the fixed-point property of I^n (see Theorem 1.8.1 and Problem 1.8.B(a)). The importance of the theorem on partitions consists in the fact that it provides an internal characterization of n-dimensional spaces which, in effect, is but a reformulation of a very useful external characterization of such spaces, namely the characterization in terms of mappings to the n-sphere S^n (see Theorem 1.9.3), which in turn is very close to the cohomological characterization of dimension (see the final part of Section 1.9). The theorem on partitions will be used in the proof of the fundamental equality $\operatorname{ind} R^n = n$ (see Theorem 1.8.3).

Let us begin with an auxiliary theorem which will often be used in our study of the covering dimension.

1.7.8. THEOREM. *Every finite open cover $\{U_i\}_{i=1}^k$ of a normal space X has a closed shrinking $\{F_i\}_{i=1}^k$.*

PROOF. We shall apply induction with respect to k beginning with $k = 2$. In this case, applying the definition of normality to disjoint closed subsets $A = X \setminus U_1$ and $B = X \setminus U_2$ of the space X, we obtain disjoint open sets $V_1, V_2 \subset X$ such that $A \subset V_1$ and $B \subset V_2$. The sets $F_1 = X \setminus V_1$ and $F_2 = X \setminus V_2$ form the required shrinking. Assume that the theorem is proved for every natural number k less than $m \geq 3$ and consider an m-element open cover $\{U_i\}_{i=1}^m$ of the space X. Let

$$U_i' = U_i \text{ for } i \leq m-2 \quad \text{and} \quad U_{m-1}' = U_{m-1} \cup U_m;$$

applying the theorem to the cover $\{U_i'\}_{i=1}^{m-1}$ we obtain a closed shrinking $\{F_i'\}_{i=1}^{m-1}$. The closed subspace F_{m-1}' of the space X is normal; applying the theorem again, this time to the two-element open cover $\{F_{m-1}' \cap U_{m-1}, F_{m-1}' \cap U_m\}$ of the space F_{m-1}', we obtain a closed cover $\{F_{m-1}, F_m\}$ of F_{m-1}' such that $F_{m-1} \subset U_{m-1}$ and $F_m \subset U_m$. One readily checks that the family $\{F_i\}_{i=1}^m$, where $F_i = F_i'$ for $i \leq m-2$, is the required shrinking of the cover $\{U_i\}_{i=1}^m$ of the space X. ■

1.7.9. THEOREM ON PARTITIONS. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if for every sequence* $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ *of* $n+1$ *pairs of disjoint closed subsets of* X *there exist closed sets* $L_1, L_2, \ldots, L_{n+1}$ *such that* L_i *is a partition between* A_i *and* B_i *and* $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

PROOF. If $\operatorname{ind} X \leq n \geq 0$, then – applying the second separation theorem – we can define, one by one, partitions $L_1, L_2, \ldots, L_{n+1}$ such that $\operatorname{ind}(L_1 \cap L_2 \cap \ldots \cap L_i) \leq n - i$ for $i = 1, 2, \ldots, n+1$; clearly $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

We shall now show that if X satisfies the condition in the theorem, then $\dim X \leq n$ and this, by the coincidence theorem, will complete the proof. We are going to apply Theorem 1.6.9; to that end consider an $(n+2)$-element open cover $\{U_i\}_{i=1}^{n+2}$ of the space X. By Theorem 1.7.8, the cover $\{U_i\}_{i=1}^{n+2}$ has a closed shrinking $\{B_i\}_{i=1}^{n+2}$; let $A_i = X \setminus U_i$ for $i = 1, 2, \ldots, n+1$. The sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ consists of $n+1$ pairs of disjoint closed subsets of X. Hence, there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Consider open sets $V_i, W_i \subset X$ such that

$$A_i \subset V_i, \quad B_i \subset W_i, \quad V_i \cap W_i = \emptyset \quad \text{and} \quad X \setminus L_i = V_i \cup W_i \tag{7}$$

for $i = 1, 2, \ldots, n+1$. Observe that

$$\bigcup_{i=1}^{n+1} V_i \cup \bigcup_{i=1}^{n+1} W_i = \bigcup_{i=1}^{n+1} (V_i \cup W_i) = \bigcup_{i=1}^{n+1} (X \setminus L_i) = X \setminus \bigcap_{i=1}^{n+1} L_i = X. \tag{8}$$

From (8), (7) and the inclusion $B_{n+2} \subset U_{n+2}$ it follows that

$$\bigcup_{i=1}^{n+1} W_i \cup \Big[U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i\Big] = \Big[\bigcup_{i=1}^{n+1} W_i \cup U_{n+2}\Big] \cap \Big[\bigcup_{i=1}^{n+1} W_i \cup \bigcup_{i=1}^{n+1} V_i\Big] \supset \bigcup_{i=1}^{n+2} B_i = X,$$

so that the family $\{W_i\}_{i=1}^{n+2}$, where $W_{n+2} = U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i$, is an open shrinking of the cover $\{U_i\}_{i=1}^{n+2}$. It follows from (7) that

$$\bigcap_{i=1}^{n+2} W_i = \bigcap_{i=1}^{n+1} W_i \cap \Big[U_{n+2} \cap \bigcup_{i=1}^{n+1} V_i\Big] \subset \bigcap_{i=1}^{n+1} W_i \cap \bigcup_{i=1}^{n+1} V_i = \emptyset,$$

therefore $\dim X \leq n$ by virtue of Theorem 1.6.9. ■

1.7.10. REMARK. Let us note that in the second part of the above proof only the normality of the space X was applied; hence, we have shown that if for every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of a normal space X there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$, then $\dim X \leq n$. ■

Let us call the reader's attention to the structure of the last proof. We started with the inequality $\operatorname{ind} X \leq n$, showed that this inequality implies a property of the space X, then proved that this property implies the inequality $\dim X \leq n$, and, finally, applied the coincidence theorem. Thus, we incidentally gave another proof of Lemma 1.7.5; obviously, the original proof of Lemma 1.7.5 is more transparent. The proof of Theorem 1.9.3 below has a similar structure. The conditions in both theorems characterize the

covering dimension dim in the realm of normal spaces; however, in the realm of separable metric spaces the proofs are considerably shortened by applying the coincidence theorem (cf. Theorems 3.2.6 and 3.2.10).

Historical and bibliographic notes

Theorem 1.7.2 was established by Hurewicz in [1927b] (another proof in [1930b]). This theorem permitted or facilitated the extension of many theorems of dimension theory from compact metric spaces to separable metric spaces; it was one of the most important achievements in the development of the theory. Lemma 1.7.5 was proved for metric compact spaces by Menger in [1924] and by Urysohn in [1926] (announcement in [1922]) and was extended to separable metric spaces by Hurewicz in [1927b]. Lemma 1.7.6 was obtained by Urysohn in [1926]. The equality $\operatorname{ind} X = \dim X$ in Theorem 1.7.7 was established by Hurewicz in [1927b]; the history of the equality $\operatorname{ind} X = \operatorname{Ind} X$ is summarized in the notes to Section 1.5. Theorem 1.7.9 was proved by Eilenberg and Otto in [1938].

Problems

1.7.A. Observe that the enlargement theorem readily follows from the compactification theorem and the Lavrentieff theorem ([GT], Theorem 4.3.21).

1.7.B. Let X be a separable metric space and $f_{i,j}\colon X \to I$, where $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, k_i$, the functions defined in the proof of Theorem 1.7.2. Show that by assigning to every point $x \in X$ the point $f(x)$ in the Hilbert cube $I^{\aleph_0}$ whose $n(i,j)$th coordinate is equal to $f_{i,j}(x)$, one defines a homeomorphic embedding $f\colon X \to I^{\aleph_0}$ such that $\dim \overline{f(X)} \leq \dim X$.

1.7.C. (a) (Engelking [1960], Forge [1961]) Show that for every separable metric space X and for every sequence of continuous functions $f_1, f_2, \ldots$, where $f_i\colon X \to I$ for $i = 1, 2, \ldots$, there exists a compactification $\widetilde{X}$ of the space X such that $\dim \widetilde{X} \leq \dim X$ and each f_i is extendable to a continuous function $\tilde{f}_i\colon \widetilde{X} \to I$.

(b) (Engelking [1960]; for $n = 0$, de Groot and McDowell [1960]) Prove that for every separable metric space X and for every sequence of continuous mappings $g_1, g_2, \ldots$, where $g_i\colon X \to X$ for $i = 1, 2, \ldots$, there exists a compactification $\widetilde{X}$ of the space X such that $\dim \widetilde{X} \leq \dim X$ and each g_i is extendable to a continuous mapping $\tilde{g}_i\colon \widetilde{X} \to \widetilde{X}$.

Hint. One can assume that $g_1 = \mathrm{id}_X$ and that for every pair i, j there exists a k such that $g_j g_i = g_k$. As in the proof of Theorem 1.7.2, for $m = 1, 2, \ldots$ define a finite open cover $\mathcal{U}_m$ which moreover is a refinement of $g_1^{-1}(\mathcal{U}_{m-1}) \wedge g_2^{-1}(\mathcal{U}_{m-1}) \wedge \ldots \wedge g_{m-1}^{-1}(\mathcal{U}_{m-1})$, where $\mathcal{U}_0 = \{X\}$. Consider the metric $\tilde{\rho}$ on the space X defined by letting

$$\tilde{\rho}(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho(g_i(x), g_i(y)) + \sum_{n(i,j,k)=1}^{\infty} \frac{1}{2^{n(i,j,k)}} |f_{i,j} g_k(x) - f_{i,j} g_k(y)| \quad \text{for } x, y \in X,$$

where $n(i,j,k)$ is the place of the triple (i,j,k) in some infinite sequence consisting of all triples of natural numbers.

1.7.D. Prove that for every separable subspace M of a metric space X such that $\operatorname{ind} M \leq n$ and for every family $\{U_i\}_{i=1}^k$ of open subsets of X such that $M \subset \bigcup_{i=1}^k U_i$ there exists a finite family $\mathcal{V}$ of open subsets of X satisfying the conditions $M \subset \bigcup \mathcal{V}$ and $\operatorname{ord} \mathcal{V} \leq n$ and such that each of its members is contained in a set U_i.

Hint. See the proof of Lemma 1.7.5.

1.7.E. Show that a separable metric space X satisfies the inequality $\dim X \leq n$ if and only if every open cover of the space X has an open refinement of order $\leq n$ or – equivalently – if every open cover of the space X has an open shrinking of order $\leq n$ (cf. Proposition 3.2.2).

Hint. When proving that every open cover of a separable metric space X satisfying $\dim X \leq n$ has an open refinement of order $\leq n$, first reduce the problem to countable covers, then consider the special case where $n = 0$, and in the general case apply the second decomposition theorem and Lemma 1.2.8.

1.7.F. (a) (Kuratowski [1932]) Prove that for every compact metric space X with no isolated point such that $0 \leq \dim X \leq n$ there exists a continuous mapping $f: C \to X$ of the Cantor set C onto the space X with fibres of cardinality at most $n+1$. Deduce the topological characterization of the Cantor set stated in Problem 1.3.F(a).

Hint. Observe that the subspace $\Phi = \{f \in X^C : f(C) = X\}$ of the function space X^C is non-empty and closed in X^C, and hence is completely metrizable (see Problem 1.3.D(b) and the beginning of Section 1.11). For $k = 1, 2, \ldots$ consider the subset Ψ_k of Φ consisting of all functions $f \in X^C$ which have the property that for some $n+2$ points $x_1, x_2, \ldots, x_{n+2}$ of the Cantor set C such that $|x_i - x_j| > 1/k$ whenever $i \neq j$ we have the equality $f(x_1) = f(x_2) = \ldots = f(x_{n+2})$. Prove that the sets Ψ_k are closed and nowhere dense in Φ; then apply the Baire category theorem. When proving that Ψ_k is nowhere dense, observe that every finite open cover of the space X consisting of non-empty sets has a closed shrinking of order $\leq n$ consisting of non-empty sets; then apply Problem 1.3.D(b).

(b) (Kuratowski [1932]) Prove that for every compact metric space X such that $\dim X \leq n \geq 0$ there exists a continuous mapping $f: A \to X$ of a closed subspace A of the Cantor set C onto the space X with fibres of cardinality at most $n+1$.

Hint. Applying the definition of dim, prove that $\dim(X \times C) \leq n$.

(c) (Hurewicz [1926]) Prove that if for a compact metric space X there exists a continuous mapping $f: A \to X$ of a closed subspace A of the Cantor set C onto the space X with fibres of cardinality at most $n+1$, then $\operatorname{ind} X \leq n$ (cf. Theorem 1.12.2).

Hint. Apply induction with respect to n. Assuming that (c) holds for every $n < m$, consider a continuous mapping $f: A \to X$ of a closed subspace A of the Cantor set C onto the space X with fibres of cardinality at most $m+1$, a point $x \in X$, and a neighbourhood $V \subset X$ of the point x; then take an open-and-closed set $W \subset A$ such that $f^{-1}(x) \subset W \subset f^{-1}(V)$ and show that the open set $U = X \setminus f(A \setminus W) \subset X$ satisfies the conditions $x \in U \subset V$ and $\operatorname{Fr} U \subset f(W) \cap f(A \setminus W)$.

(d) (Kuratowski [1932]) Observe that (b) and (c) imply that for every compact metric space X we have $\operatorname{ind} X \leq \dim X$.

Remark. Hurewicz proved in [1926] that a separable metric space X satisfies the inequality $\operatorname{ind} X \leq n \geq 0$ if and only if there exists a closed mapping (see the beginning of Section 1.12) $f: Z \to X$ of a zero-dimensional separable metric space Z onto the space X with fibres of cardinality at most $n+1$. This fact can be deduced from the compactification theorem, the coincidence theorem, part (b) of the present problem, and the hint to part (c); obviously, Hurewicz's original proof was direct (cf. Theorem 4.3.15).

Let us also note that a separable metric space X is a continuous image of a zero-dimensional separable metric space under a closed mapping with finite fibres if and only if X can be represented as the union of countably many finite-dimensional subspaces (cf. Theorem 5.4.14).

1.8. Dimensional properties of Euclidean spaces and the Hilbert cube. Infinite-dimensional spaces

The main result in this section is the fundamental theorem of dimension theory, which states that Euclidean n-space R^n has dimension n, i.e., that $\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n$ for $n = 1, 2, \ldots$ This theorem justifies the definitions of our three dimension functions; indeed, any dimension function assigning to R^n a number distinct from n would disagree with the intuitive notion of dimension and thus would not be acceptable.

It readily follows from the evaluation of dimensions of Euclidean n-space R^n that the dimensions ind, Ind and dim of the n-cube I^n and the n-sphere S^n are also equal to n. The spaces R^n, I^n and S^n with $n \geq 2$ are our first examples of spaces of dimension larger than one; so far we have not shown that such spaces exist.

The proof that R^n has dimension n requires a deeper insight into the structure of this space; by the nature of things, some combinatorial or algebraic arguments must appear in it. In this book, instead of producing a complete proof of the fundamental theorem of dimension theory, we shall deduce this result from the Brouwer fixed-point theorem, which states that for every continuous mapping $f: I^n \to I^n$ there exists a point $x \in I^n$ such that $f(x) = x$. The latter theorem is certainly well known to the reader; let us note that it is closely related to the "geometric versions" of the fundamental theorem of dimension theory, i.e., to Theorems 1.8.1 and 1.8.20 (see Problem 1.8.B(a)).

We begin with a theorem describing an interesting geometric property of the n-cube I^n.

1.8.1. THEOREM. *Let A_i and B_i, where $i = 1, 2, \ldots, n$, be the subsets of the n-cube I^n defined by letting*

$$A_i = \{\{x_j\} \in I^n : x_i = 0\} \quad \text{and} \quad B_i = \{\{x_j\} \in I^n : x_i = 1\},$$

i.e., the pairs of opposite faces of I^n. If L_i is a partition between A_i and B_i for $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^n L_i \neq \emptyset$.

PROOF. Assume that $\bigcap_{i=1}^n L_i = \emptyset$. It follows from Theorem 1.7.8 that the partitions L_i can be enlarged to open sets H_i with $\bigcap_{i=1}^n H_i = \emptyset$. The open sets $G_i = H_i \setminus (A_i \cup B_i) \subset$

I^n clearly satisfy

$$\bigcap_{i=1}^{n} G_i = \emptyset. \tag{1}$$

For $i = 1, 2, \ldots, n$ consider open sets $U_i, W_i \subset I^n$ such that

$$A_i \subset U_i, \quad B_i \subset W_i, \quad U_i \cap W_i = \emptyset \quad \text{and} \quad I^n \setminus L_i = U_i \cup W_i.$$

The sets

$$E_i = U_i \setminus G_i = (U_i \cup L_i) \setminus G_i \quad \text{and} \quad F_i = W_i \setminus G_i = (W_i \cup L_i) \setminus G_i$$

are closed, disjoint and satisfy

$$I^n \setminus G_i = E_i \cup F_i \quad \text{for } i = 1, 2, \ldots, n. \tag{2}$$

Take a continuous function $f_i: I^n \to I$ such that

$$f_i(E_i) \subset \{1\} \quad \text{and} \quad f_i(F_i) \subset \{0\}, \tag{3}$$

and define a continuous mapping $f: I^n \to I^n$ by letting $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$.

Consider now an arbitrary point $x \in I^n$. By (1) and (2) there exists an i such that $x \in E_i \cup F_i$. It follows from (3) that if $x \in E_i$, then $f(x) \in B_i \subset F_i$; similarly, if $x \in F_i$, then $f(x) \in A_i \subset E_i$. Since $E_i \cap F_i = \emptyset$, in both cases $f(x) \neq x$. This contradiction with the Brouwer fixed-point theorem shows that $\bigcap_{i=1}^{n} L_i \neq \emptyset$. ■

The fixed-point theorem for $I^{\aleph_0}$ (see Problem 1.8.B(b)), by the argument in the above proof, with the sets H_i defined by the equality

$$H_i = \{x \in I^{\aleph_0} : 2\rho(x, L_i) < \sup\{\rho(x, L_j) : j = 1, 2, \ldots\}\},$$

yields

1.8.2. THEOREM. *Let A_i and B_i, where $i = 1, 2, \ldots$, be the subsets of the Hilbert cube $I^{\aleph_0}$ defined by letting*

$$A_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 0\} \quad \textit{and} \quad B_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 1\},$$

i.e., the pairs of opposite faces of $I^{\aleph_0}$. If L_i is a partition between A_i and B_i for $i = 1, 2, \ldots$, then $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. ■

1.8.3. THE FUNDAMENTAL THEOREM OF DIMENSION THEORY. *For every natural number n we have*

$$\operatorname{ind} R^n = \operatorname{Ind} R^n = \dim R^n = n.$$

PROOF. By virtue of the inequality $\operatorname{ind} R^n \leq n$ established in Example 1.5.9(b) and by the coincidence theorem, it suffices to show that $\operatorname{ind} R^n \geq n$; the latter inequality follows immediately from Theorem 1.8.1 and the theorem on partitions. ■

1.8.4. COROLLARY. *For every natural number n we have*

$$\underline{\operatorname{ind}}\, I^n = \underline{\operatorname{Ind}}\, I^n = \underline{\dim}\, I^n = n = \dim S^n = \operatorname{Ind} S^n = \operatorname{ind} S^n.$$ ■

1.8.5. COROLLARY. *For the Hilbert cube $I^{\aleph_0}$ we have*

$$\operatorname{ind} I^{\aleph_0} = \operatorname{Ind} I^{\aleph_0} = \dim I^{\aleph_0} = \infty.$$ ■

1.8.6. THEOREM. *For the subspace N_k^n of Euclidean n-space R^n consisting of all points with at most k rational coordinates and the subspace L_k^n of R^n consisting of all points with at least k rational coordinates we have*

$$\operatorname{ind} N_k^n = k \quad \text{and} \quad \operatorname{ind} L_k^n = n - k.$$

PROOF. Since both equalities are obvious for $k = 0$ and for $k = n$, we can assume that $0 < k < n$. In Example 1.5.9(c) we have shown that $\operatorname{ind} N_k^n \leq k$ and $\operatorname{ind} L_k^n \leq n-k$. Since $R^n = N_k^n \cup L_{k+1}^n = N_{k-1}^n \cup L_k^n$, the reverse inequalities follow from the equality $\operatorname{ind} R^n = n$ and the addition theorem. ■

Looking closer at our proof of the fundamental theorem of dimension theory we can see that first the inequalities $\operatorname{ind} R^n \leq n$ and $\operatorname{Ind} R^n \geq n$ were established, then – by applying Theorem 1.6.4 – it was deduced that $\operatorname{ind} R^n = \operatorname{Ind} R^n = n$, and finally Lemmas 1.7.5 and 1.7.6 were applied to obtain the equality $\dim R^n = n$. The equality $\dim I^n = n$ can also be obtained in a direct way. Theorem 1.7.8 and Theorem 1.8.20 below imply that $\dim I^n \geq n$, and the reverse inequality is obtained by defining a finite open cover of I^n which has order $\leq n$ and arbitrarily small mesh. This can be done by dividing I^n into small "bricks" and then enlarging the bricks to open sets in such a way that the order of the family does not increase; Fig. 7 illustrates the procedure in the case of $n = 2$.

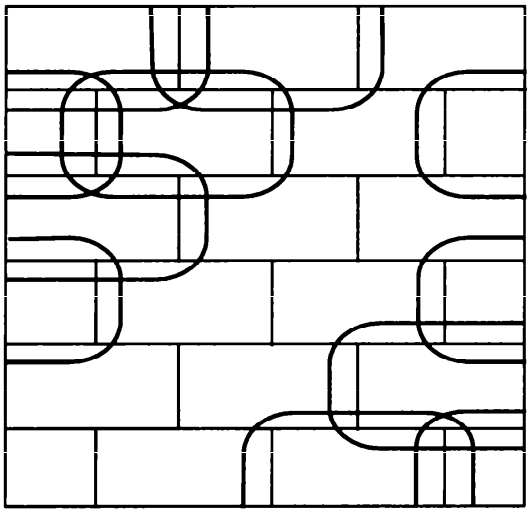


Fig. 7

Our next task is to characterize n-dimensional subspaces of Euclidean n-space R^n; we shall show that they are subsets of R^n which have a non-empty interior. We start with an auxiliary geometric result, interesting in itself, which states that if a subset C of R^n has an empty interior, then C is homeomorphic to a subset D of R^n contained together with its closure in N_{n-1}^n; in particular, each such subset is homeomorphic to a nowhere dense subset of R^n.

1.8.7. LEMMA. *If a subset C of R^n has an empty interior and is dense in R^n, then every subset D of R^n which is homeomorphic to C has an empty interior.*

PROOF. Let $f\colon D \to C$ be a homeomorphism of D onto C. Suppose that $\operatorname{Int} D \neq \emptyset$ and consider a non-empty open subset V of R^n such that the closure $\overline{V}$ of V in R^n is compact and contained in D. The image $f(V)$ is open in C and its closure in C equals

$f(\overline{V})$; the last set being compact, the closure $\overline{f(V)}$ of $f(V)$ in R^n also equals $f(\overline{V})$. Now, let U be an open subset of R^n such that $f(V) = C \cap U$. As the set C is dense in R^n, we have $\emptyset \neq U \subset \overline{U} = \overline{C \cap U} = \overline{f(V)} = f(\overline{V}) \subset C$, which contradicts the assumption that $\operatorname{Int} C = \emptyset$. ■

1.8.8. LEMMA. *Let A be a subset of R^n and a an arbitrary point of $R^n \setminus A$. For every positive number r there exists a homeomorphism $f\colon A \to f(A) \subset R^n$ such that for all $x, y \in A$ we have*

$$\rho(x, f(x)) \leq r, \quad \rho(x, y) \leq \rho(f(x), f(y)) \quad \text{and} \quad B(a, r) \cap f(A) = \emptyset.$$

PROOF. It suffices to assign to every $x \in A$ the point $f(x)$ situated on the ray starting from a and passing through x which satisfies the condition $\rho(a, f(x)) = \rho(a, x) + r$. ■

1.8.9. THEOREM. *For every subset C of R^n which has an empty interior there exists a subset D of R^n homeomorphic to C and such that $\overline{D} \subset N^n_{n-1}$.*

PROOF. The set C can easily be enlarged to a dense subset of R^n which has an empty interior, and it suffices to prove the theorem for this larger set. Hence, without loss of generality, one can assume that the set C is dense in R^n. Let us arrange all points of the complement $R^n \setminus N^n_{n-1}$, i.e., all points of R^n with rational coordinates, into a sequence $a_1, a_2, \ldots$ We shall define inductively a sequence $C_0, C_1, C_2, \ldots$ of subsets of R^n, where $C_0 = C$, and a sequence of homeomorphisms $f_0, f_1, f_2, \ldots$, where $f_i\colon C_i \to C_{i+1}$, satisfying the following conditions:

$$\rho(x, f_i(x)) \leq 1/3^{i+1} \quad \text{for every } x \in C_i. \tag{4}$$

$$\rho(x, y) \leq \rho(f_i(x), f_i(y)) \quad \text{for all } x, y \in C_i. \tag{5}$$

$$B(a_i, 1/(4 \cdot 3^i)) \cap C_{i+1} = \emptyset \quad \text{for } i \geq 1. \tag{6}$$

Conditions (4)–(6) are satisfied for $i = 0$ if we let $C_1 = C$ and $f_0 = \mathrm{id}_C$. Assume that the sets $C_0, C_1, C_2, \ldots, C_k$ and the homeomorphisms $f_0, f_1, f_2, \ldots, f_{k-1}$ are defined and satisfy (4)–(6) for $i < k \geq 1$. The set $C_k = f_{k-1}(C_{k-1})$ is homeomorphic to $C_0 = C$, so that $\operatorname{Int} C_k = \emptyset$ by virtue of Lemma 1.8.7; thus there exists a point $a \in B(a_k, 1/(4 \cdot 3^{k+1})) \setminus C_k$. One readily checks that Lemma 1.8.8 applied to the set $A = C_k$, the point a, and the number $r = 1/3^{k+1}$ yields a homeomorphism $f_k = f\colon C_k \to f(C_k)$ which together with the set $C_{k+1} = f(C_k)$ satisfies conditions (4)–(6) for $i = k$. Thus our construction is completed.

For $i = 1, 2, \ldots$ consider the composition $h_i = g_i f_{i-1} \ldots f_1 f_0\colon C \to R^n$, where g_i denotes the embedding of C_i in R^n. By virtue of (4), for $k \geq 1$ we have

$$\begin{aligned} (7) \quad & \rho(h_i(x), h_{i+k}(x)) \\ & \leq \rho(h_i(x), h_{i+1}(x)) + \rho(h_{i+1}(x), h_{i+2}(x)) + \ldots + \rho(h_{i+k-1}(x), h_{i+k}(x)) \\ & \leq 1/3^{i+1} + 1/3^{i+2} + \ldots + 1/3^{i+k} < 1/(2 \cdot 3^i) \quad \text{for every } x \in C; \end{aligned}$$

hence, for every $x \in C$ the sequence $h_1(x), h_2(x), \ldots$ is a Cauchy sequence and thus converges to a point $h(x) \in R^n$. In this way a mapping h of C to R^n is defined. Passing in (7) to the limit with respect to k, we see that the sequence of mappings $h_1, h_2, \ldots$ uniformly converges to h, so that h is continuous. Condition (5) implies that $\rho(x, y) \leq \rho(h_i(x), h_i(y))$ for all $x, y \in C$ and $i = 1, 2, \ldots$ Hence, $\rho(x, y) \leq \rho(h(x), h(y))$

for all $x, y \in C$, which implies that h is a one-to-one mapping and that the inverse mapping h^{-1} is continuous, i.e., that h is a homeomorphism of the set C onto the set $D = h(C) \subset R^n$.

Now, consider a point $a_i \in R^n \setminus N^n_{n-1}$ and an arbitrary point $z \in D$; let $h^{-1}(z) = x \in C$. It follows from (6) that

$$\rho(a_i, h_{i+1}(x)) \geq 1/(4 \cdot 3^i). \tag{8}$$

By virtue of (7) we have $\rho(h_{i+1}(x), h_{i+k}(x)) \leq 1/(2 \cdot 3^{i+1})$ for $k = 1, 2, \ldots$, so that

$$\rho(h_{i+1}(x), z) \leq 1/(2 \cdot 3^{i+1}). \tag{9}$$

Relations (8) and (9) yield the inequality $\rho(a_i, z) \geq 1/(4 \cdot 3^{i+1})$; hence $B(a_i, 1/(4 \cdot 3^{i+1})) \cap D = \emptyset$, which implies that $a_i \notin \overline{D}$. Thus the inclusion $\overline{D} \subset N^n_{n-1}$ is established. ■

Theorem 1.8.9 yields

1.8.10. THEOREM. *Every subset of R^n which has an empty interior is homeomorphic to a nowhere dense subset of R^n.* ■

1.8.11. THEOREM. *A subspace M of Euclidean n-space R^n satisfies the condition* $\operatorname{ind} M = n$ *if and only if the interior of M in R^n is non-empty.*

PROOF. If $\operatorname{Int} M \neq \emptyset$, then M contains a subspace homeomorphic to I^n, so that $\operatorname{ind} M = n$. On the other hand, if $\operatorname{Int} M = \emptyset$, then – as follows from Theorem 1.8.9 – the space M is homeomorphic to a subspace of N^n_{n-1}, so that $\operatorname{ind} M \leq n - 1$ by virtue of Theorem 1.8.6. ■

1.8.12. COROLLARY. *A subspace M of the n-cube I^n, or the n-sphere S^n, satisfies the condition* $\operatorname{ind} M = n$ *if and only if the interior of M is non-empty.* ■

The next two theorems are further consequences of Theorem 1.8.11.

1.8.13. THEOREM. *Let X be Euclidean n-space R^n, the n-cube I^n, or the n-sphere S^n. If a set $F \subset X$ is the boundary of a non-empty open subset of X which is not dense in X, then* $\operatorname{ind} F = n - 1$.

PROOF. Consider first a subspace F of S^n, and let $F = \operatorname{Fr} U$, where U is an open subset of S^n such that $U \neq \emptyset \neq S^n \setminus \overline{U}$. From Corollary 1.8.12 it follows that $\operatorname{ind} F \leq n - 1$; suppose that $\operatorname{ind} F \leq n - 2$. Let x be a point of U. Since $\overline{U} \neq S^n$, for every positive number ϵ one can readily define a homeomorphism f of the n-sphere S^n onto itself such that $f(x) = x$ and $f(\overline{U}) \subset B(x, \epsilon)$. Thus the point x has arbitrarily small neighbourhoods with at most $(n-2)$-dimensional boundaries. Since S^n is homogeneous, the same holds for any of its points. This contradiction with Theorem 1.8.3 shows that $\operatorname{ind} F = n - 1$.

As R^n is homeomorphic to the open subspace $S^n \setminus \{a\}$ of S^n and for every open set $U \subset R^n$ the boundary of U in R^n and the boundary of its counterpart in S^n differ topologically by at most one point, the validity of our theorem for subspaces of R^n follows from its validity for subspaces of S^n and Corollary 1.5.6.

The case of $X = I^n$ is left to the reader. ■

Let us recall that a set $T \subset X$ *separates* a topological space X, or is a *separator* of X, if T separates X between some pair of points, i.e., if the complement $X \setminus T$ is disconnected.

1.8.14. THEOREM. *Let X be Euclidean n-space R^n, the n-cube I^n, or the n-sphere S^n. If a closed subset L of X satisfies the inequality* $\operatorname{ind} L \leq n-2$, *then L does not separate the space X, i.e., the complement $X \setminus L$ is connected.*

PROOF. Suppose that $X \setminus L = U \cup V$, where U, V are non-empty disjoint open sets. Clearly $\operatorname{Fr} U \subset L$, so that $\operatorname{ind} \operatorname{Fr} U \leq n-2$, which contradicts Theorem 1.8.13. ■

The last theorem leads to the notion of a Cantor-manifold which will be discussed in the following section.

We shall now pass to Mazurkiewicz's theorem, which is much stronger than Theorem 1.8.14 and states that if a subset M of a connected open subspace G of R^n satisfies the inequality $\operatorname{ind} M \leq n-2$, then M does not cut G; let us recall that a set $T \subset X$ *cuts* a topological space X, or is a *cut* in X, if T cuts X between some pair of points x and y, i.e., if each continuum $C \subset X$ which contains x and y meets the set T (see Problem 1.8.H). The theorem will be preceded by two lemmas; the second lemma is in fact a special case of the theorem, to which the general situation will be reduced by a relatively simple argument.

1.8.15. LEMMA. *Let $(A_1, B_1), (A_2, B_2), \ldots, (A_n, B_n)$ be the sequence of pairs of opposite faces of the n-cube I^n with $n \geq 2$. If L_i is a partition between A_i and B_i for $i = 2, 3, \ldots, n$, then the intersection $\bigcap_{i=2}^n L_i$ contains a continuum C such that $A_1 \cap C \neq \emptyset \neq B_1 \cap C$.*

PROOF. Assume that there is no such continuum. Then for each component C of $M = \bigcap_{i=2}^n L_i$ that meets A_1 we have $C \cap B_1 = \emptyset$, and by Lemma 1.4.4 there exists in M an open-and-closed set U_C such that $C \subset U_C$ and $B_1 \cap U_C = \emptyset$. The family of all U_C's covers $M \cap A_1$ and, by the compactness of this subspace, so does some finite subfamily; the union of all members of this subfamily being open-and-closed in M and disjoint from $M \cap B_1$, we conclude that the empty set is a partition between $M \cap A_1$ and $M \cap B_1$ in M. By Lemma 1.2.9 there exists a partition L_1 in I^n between A_1 and B_1 such that $M \cap L_1 = \bigcap_{i=1}^n L_i = \emptyset$, and this contradicts Theorem 1.8.1. ■

1.8.16. LEMMA. *Let A, B be a pair of opposite faces of the n-cube I^n. If a subspace M of I^n meets every continuum $C \subset I^n$ such that $A \cap C \neq \emptyset \neq B \cap C$, then* $\operatorname{ind} M \geq n-1$.

PROOF. We can suppose that $n \geq 2$, and that $A = A_1$ and $B = B_1$, where $(A_1, B_1), (A_2, B_2), \ldots, (A_n, B_n)$ is the sequence of pairs of opposite faces of I^n. Let $V_{i,1}, V_{i,2}$, for $i = 2, 3, \ldots, n$, be open subsets of I^n such that $A_i \subset V_{i,1}$, $B_i \subset V_{i,2}$ and $\overline{V}_{i,1} \cap \overline{V}_{i,2} = \emptyset$.

Assume now that $\operatorname{ind} M \leq n-2$. By virtue of Theorem 1.7.9 there exist closed subsets $L'_2, L'_3, \ldots, L'_n$ of M such that L'_i is a partition in M between $M \cap \overline{V}_{i,1}$ and $M \cap \overline{V}_{i,2}$ and $\bigcap_{i=2}^n L'_i = \emptyset$. Applying Lemma 1.2.9 we obtain partitions L_i in I^n between A_i and B_i, where $i = 2, 3, \ldots, n$, such that $M \cap \bigcap_{i=2}^n L_i = \emptyset$, and this contradicts Lemma 1.8.15. ■

1.8.17. REMARK. By Theorem 1.8.2, Lemma 1.8.15 holds for $I^{\aleph_0}$, i.e., if (A_1, B_1), $(A_2, B_2), \ldots$ is the sequence of pairs of opposite faces of $I^{\aleph_0}$, and L_i is a partition between A_i and B_i for $i = 2, 3, \ldots$, then the intersection $\bigcap_{i=2}^{\infty} L_i$ contains a continuum C such that $A_1 \cap C \neq \emptyset \neq B_1 \cap C$. This fact and the last proof yield the infinite-dimensional counterpart of Lemma 1.8.16: If A, B is a pair of opposite faces of $I^{\aleph_0}$ and a subspace of $I^{\aleph_0}$ meets all continua in $I^{\aleph_0}$ which intersect A and B, then there exists in this subspace a sequence $(E_1, F_1), (E_2, F_2), \ldots$ of pairs of disjoint closed sets such that $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$ whenever L_i is a partition between E_i and F_i for $i = 1, 2, \ldots$ (this means that the subspace is strongly infinite-dimensional, according to Definition 6.1.1). ■

We are ready now to prove Mazurkiewicz's theorem. Let us recall that a *region* in R^n is a connected open subset of R^n.

1.8.18. MAZURKIEWICZ'S THEOREM. *If a subset M of a region $G \subset R^n$ satisfies the inequality* $\operatorname{ind} M \leq n-2$, *then M does not cut G, i.e., for every pair of points $x, y \in G \setminus M$ there exists a continuum $C \subset G \setminus M$ which contains x and y.*

PROOF. Let us start with the special case where $G = R^n$. Consider a pair of points $x, y \in R^n \setminus M$ and denote by K the closed ball in R^n whose centre coincides with the centre of the segment with end-points x and y and whose radius is equal to $\frac{1}{2}\rho(x, y)$. Let $f\colon I^n \to K$ be a continuous mapping of I^n onto K which maps a pair A, B of opposite faces of I^n to x and y, respectively, and has the property that the restriction $g = f|[I^n \setminus (A \cup B)]$ is a homeomorphism of $I^n \setminus (A \cup B)$ onto $K \setminus \{x, y\}$; such a mapping can easily be obtained by an application of Problem 1.8.A and Lemma 1.8.8. The set $M' = f^{-1}(K \cap M) \subset I^n \setminus (A \cup B)$ satisfies the inequality $\operatorname{ind} M' \leq n - 2$. Hence by Lemma 1.8.16 there exists a continuum $C' \subset I^n$ such that $A \cap C' \neq \emptyset \neq B \cap C'$ and $C' \cap M' = \emptyset$. The set $C = f(C') \subset R^n \setminus M$ is a continuum which contains x and y, so that Mazurkiewicz's theorem is proved for $G = R^n$.

Now, consider an arbitrary region $G \subset R^n$ and a pair of points $x, y \in G \setminus M$. Let $B_1, B_2, \ldots, B_k$ be a sequence of open balls in R^n such that $x \in B_1, y \in B_k, B_i \subset G$ for $i = 1, 2, \ldots, k$ and $B_i \cap B_{i+1} \neq \emptyset$ for $i \leq k - 1$. The existence of such a sequence follows from the connectedness of G, because the set of all points in G that can be joined to the point x by such a sequence is open-and-closed. Since the set M has an empty interior, for $i = 1, 2, \ldots, k - 1$ there exists a point $z_i \in B_i \cap B_{i+1} \setminus M$; let, additionally, $z_0 = x$ and $z_k = y$. By the special case of Mazurkiewicz's theorem established above, for $i = 1, 2, \ldots, k$ there exists a continuum $C_i \subset B_i \setminus M$ which contains z_{i-1} and z_i. The union $C = C_1 \cup C_2 \cup \ldots \cup C_k \subset G \setminus M$ is a continuum which contains x and y. ■

We conclude our study of dimensional properties of Euclidean spaces with Lebesgue's covering theorem describing an important geometric property of the n-cube I^n which is closely related to the equality $\dim I^n = n$. As stated in the notes to Section 1.1, this theorem played a significant role in the formation of dimension theory.

1.8.19. LEMMA. *Let A_i, B_i, where $i = 1, 2, \ldots, n$, be the pairs of opposite faces of I^n. If $I^n = L'_0 \supset L'_1 \supset \ldots \supset L'_n$ is a decreasing sequence of closed sets such that L'_i is a partition in L'_{i-1} between $L'_{i-1} \cap A_i$ and $L'_{i-1} \cap B_i$ for $i = 1, 2, \ldots, n$, then $L'_n \neq \emptyset$.*

PROOF. By virtue of the second part of Lemma 1.2.9, for $i = 2, 3, \ldots, n$ there exists a partition L_i in I^n between A_i and B_i such that

$$L'_{i-1} \cap L_i \subset L'_i \quad \text{for } i = 2, 3, \ldots, n; \quad \text{let, additionally, } L_1 = L'_1. \tag{10}$$

From (10) we obtain, one by one, the inclusions $L_1 \subset L'_1, L_1 \cap L_2 \subset L'_2, \ldots, L_1 \cap L_2 \cap \ldots \cap L_n \subset L'_n$, so that $L'_n \neq \emptyset$ by Theorem 1.8.1. ■

1.8.20. LEBESGUE'S COVERING THEOREM. *If $\mathcal{F}$ is a finite closed cover of the n-cube I^n no member of which meets two opposite faces of I^n, then* $\operatorname{ord} \mathcal{F} \geq n$.

PROOF. Let A_i, B_i, where $i = 1, 2, \ldots, n$, be the pairs of opposite faces of I^n. For $i = 1, 2, \ldots, n$ define

$$\mathcal{F}_i = \{F \in \mathcal{F} : F \cap A_i \neq \emptyset\}$$

and consider the families

$$\mathcal{K}_1 = \mathcal{F}_1, \quad \mathcal{K}_2 = \mathcal{F}_2 \setminus \mathcal{K}_1, \quad \ldots, \quad \mathcal{K}_n = \mathcal{F}_n \setminus (\mathcal{K}_1 \cup \ldots \cup \mathcal{K}_{n-1})$$

and

$$\mathcal{K}_{n+1} = \mathcal{F} \setminus (\mathcal{K}_1 \cup \mathcal{K}_2 \cup \ldots \cup \mathcal{K}_n).$$

It follows from the assumptions of the theorem that the closed sets $K_i = \bigcup \mathcal{K}_i$ satisfy the inclusions

$$A_i \subset I^n \setminus (K_{i+1} \cup \ldots \cup K_{n+1}) \quad \text{and} \quad B_i \subset I^n \setminus K_i \quad \text{for } i = 1, 2, \ldots, n. \tag{11}$$

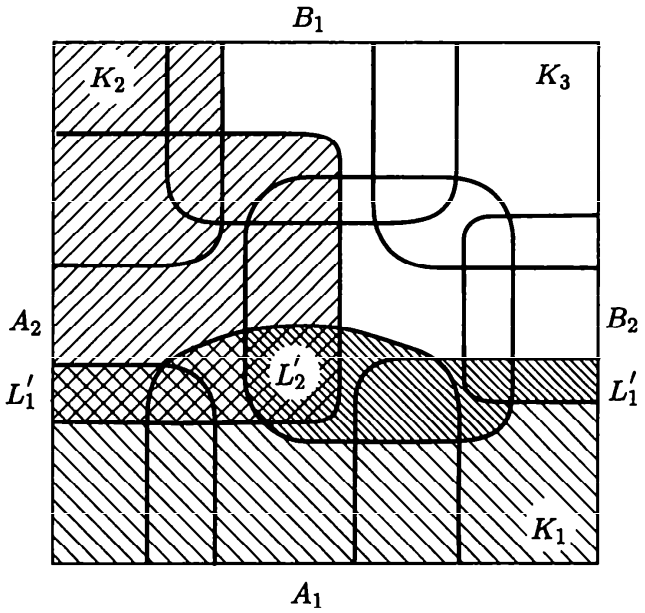


Fig. 8

The sets $L'_1, L'_2, \ldots, L'_n$, where

$$L'_i = K_1 \cap K_2 \cap \ldots \cap K_i \cap (K_{i+1} \cup K_{i+2} \cup \ldots \cup K_{n+1}),$$

are closed and form a decreasing sequence. Observe that $I^n \setminus L'_i = U_i \cup W_i$, where

$$U_i = I^n \setminus (K_{i+1} \cup K_{i+2} \cup \ldots \cup K_{n+1}) \quad \text{and} \quad W_i = I^n \setminus (K_1 \cap K_2 \cap \ldots \cap K_i).$$

Inclusions (11) yield

$$L'_{i-1} \cap A_i \subset L'_{i-1} \cap U_i \quad \text{and} \quad L'_{i-1} \cap B_i \subset L'_{i-1} \cap W_i$$

for $i = 1, 2, \ldots, n$, where $L'_0 = I^n$. One readily checks that $(L'_{i-1} \cap U_i) \cap (L'_{i-1} \cap W_i) = \emptyset$; moreover,

$$L'_{i-1} \setminus [(L'_{i-1} \cap U_i) \cup (L'_{i-1} \cap W_i)] = L'_{i-1} \setminus (U_i \cup W_i) = L'_{i-1} \cap L'_i = L'_i,$$

so that L'_i is a partition in L'_{i-1} between $L'_{i-1} \cap A_i$ and $L'_{i-1} \cap B_i$ for $i = 1, 2, \ldots, n$. By virtue of Lemma 1.8.19, we have $L'_n = K_1 \cap K_2 \cap \ldots \cap K_{n+1} \neq \emptyset$. Since every member of the cover $\mathcal{F}$ belongs to exactly one family $\mathcal{K}_i$, the last inequality implies that $\operatorname{ord} \mathcal{F} \geq n$. ■

Now we shall establish an important dimensional property of the Hilbert cube announced in Section 1.5 (cf. Problem 1.8.J).

1.8.21. THEOREM. *The Hilbert cube $I^{\aleph_0}$ cannot be represented as a countable union of finite-dimensional subspaces.*

PROOF. By virtue of the second decomposition theorem it is enough to show that $I^{\aleph_0} \neq \bigcup_{i=1}^{\infty} Z_i$ for every sequence $Z_1, Z_2, \ldots$ of zero-dimensional subspaces of $I^{\aleph_0}$. Let $(A_1, B_1), (A_2, B_2), \ldots$ be the sequence of opposite faces of $I^{\aleph_0}$. Applying Theorem 1.2.11, we can find a partition L_i between A_i and B_i such that

$$L_i \cap Z_i = \emptyset \quad \text{for } i = 1, 2, \ldots \tag{12}$$

From Theorem 1.8.2 it follows that $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$, and by (12) we have $\bigcap_{i=1}^{\infty} L_i \subset I^{\aleph_0} \setminus \bigcup_{i=1}^{\infty} Z_i$, so that $I^{\aleph_0} \neq \bigcup_{i=1}^{\infty} Z_i$. ■

In the light of Theorem 1.8.21 we see that infinite-dimensional separable metric spaces can differ by their "degree of infinite dimensionality". We intuitively feel that a space which can be represented as a countable number of finite-dimensional subspaces (i.e., a countable-dimensional space, according to Definition 5.1.1) has lower "degree of infinite dimensionality" than a space with no such representation. The problem of defining this "degree" is, however, by no means easy and can be approached in various ways, as shown in the three final chapters of this book. For the time being let us discuss a few examples.

1.8.22. EXAMPLES. (a) For $n = 1, 2, \ldots$ consider the subspace I'_n of the Hilbert cube $I^{\aleph_0}$ consisting of all points $\{x_j\}$ such that $1/(2n-1) \leq x_1 \leq 1/(2n)$ and $x_j = 0$ if $j > n$; clearly I'_n is homeomorphic to I^n. The union $S = \bigcup_{n=1}^{\infty} I'_n$ is one of the simplest infinite-dimensional spaces, it consists of disjoint open-and-closed pieces with increasing dimensions. Using the operation of the sum of topological spaces (see [GT], Section 2.2), we can say that S is the sum of the n-cubes I^n for $n = 1, 2, \ldots$, i.e., that $S = \bigoplus_{n=1}^{\infty} I^n$.

(b) The subspace K_ω of the Hilbert cube $I^{\aleph_0}$ consisting of all points with finitely many non-zero coordinates can be represented as a countable union of finite-dimensional cubes; however, its "degree of infinite dimensionality" seems higher than that of the space S from (a), because no point of K_ω has a finite-dimensional neighbourhood.

(c) The subspace N_ω of the Hilbert cube $I^{\aleph_0}$ consisting of all points with finitely many rational coordinates is of still higher "degree of infinite dimensionality". Since $N_\omega = \bigcup_{n=1}^{\infty} N_n$, where N_n's are the spaces defined in Example 1.5.9(d), it can be represented as a countable union of finite-dimensional spaces, and yet – in contrast with S

and K_ω – it cannot be represented as a countable union of closed finite-dimensional subspaces (i.e., N_ω is not strongly countable-dimensional, see Definition 5.1.4 and Problem 5.3.E). ■

Let us also mention that it is possible to extend the inductive definition of ind and Ind from natural numbers to ordinal numbers (see Definitions 7.1.1 and 7.1.11). In this way one obtains the transfinite small inductive dimension trind and the transfinite large inductive dimension trInd, which satisfy the inequality $\operatorname{trind} X \leq \operatorname{trInd} X$. It turns out, however, that there exist separable metric spaces with trind larger than any given ordinal number. For example the inequality $\operatorname{trind} K_\omega \leq \alpha$, and, *a fortiori*, the inequality $\operatorname{trInd} K_\omega \leq \alpha$, holds for no ordinal number α; so that for the space K_ω neither trind nor trInd is defined. One can also prove that trInd is not defined for the space S, although, as can easily be seen, $\operatorname{trind} S = \omega_0$. Finally, there exist compact metric spaces for which both trind and trInd are defined but are distinct ordinal numbers.

We close this section with an example of a very peculiar infinite-dimensional space announced in Section 1.5.

1.8.23. EXAMPLE. Applying the *continuum hypothesis*, i.e., the equality $\aleph_1 = \mathbf{c}$, we shall describe a space $X \subset I^{\aleph_0}$ of dimension ∞ whose finite-dimensional subspaces are all countable.

One readily checks that the family of all G_δ-sets in $I^{\aleph_0}$ has cardinality $\mathbf{c}$; let us arrange – applying the continuum hypothesis – all finite-dimensional members of this family into a transfinite sequence

$$M_1, M_2, \ldots, M_\alpha, \ldots, \quad \alpha < \omega_1,$$

where ω_1 denotes the first uncountable ordinal number, i.e., the initial number of cardinality $\aleph_1$. As all one-point subsets of $I^{\aleph_0}$ are among the sets M_α, we have

$$\bigcup_{\alpha<\omega_1} M_\alpha = I^{\aleph_0}. \tag{13}$$

From Theorem 1.8.21 it follows that $\bigcup_{\alpha<\gamma} M_\alpha \neq I^{\aleph_0}$ for every $\gamma < \omega_1$, so that by virtue of (13) there exists an uncountable set Γ of countable ordinal numbers such that $M_\gamma \setminus \bigcup_{\alpha<\gamma} M_\alpha \neq \emptyset$ for $\gamma \in \Gamma$. Let us choose a point $x_\gamma \in M_\gamma \setminus \bigcup_{\alpha<\gamma} M_\alpha$ for every $\gamma \in \Gamma$ and consider the subspace $X = \{x_\gamma : \gamma \in \Gamma\}$ of $I^{\aleph_0}$. By Theorem 1.5.11, for each finite-dimensional subspace $M \subset X \subset I^{\aleph_0}$ there exists a finite-dimensional G_δ-set $M^* \subset I^{\aleph_0}$ such that $M \subset M^*$, i.e., there exists an index $\alpha < \omega_1$ such that $M \subset M_\alpha$. Thus we have $M \subset \{x_\gamma : \gamma \leq \alpha\}$, which implies that the subspace M is countable. As the space X itself is uncountable, $\operatorname{ind} X = \infty$.

Conversely, the existence of a space X with the above properties implies the continuum hypothesis. Indeed, since every non-empty metric space of cardinality $< \mathbf{c}$ is zero-dimensional (see Example 1.2.5(b)), $|X| \geq \mathbf{c}$ and every subset of X which has cardinality $< \mathbf{c}$ is countable; considering a set $A \subset X$ such that $|A| = \aleph_1$, we conclude that $\aleph_1 = \mathbf{c}$. As the continuum hypothesis is independent of the standard axioms of set theory, the existence of a metric space X such that $\operatorname{ind} X = \infty$ and all finite-dimensional subspaces of X are countable is also independent of the standard axioms of set theory. ■

Historical and bibliographic notes

A proof of the Brouwer fixed-point theorem can be found in the appendix to Section 7.3 of [GT]. Theorem 1.8.1 is implicit in Eilenberg and Otto's paper [1938] and Theorem 1.8.2 is implicit in Hurewicz and Wallman's book [1941]. As already mentioned in the notes to Section 1.1, the equality $\dim R^n = n$ (more exactly, Theorem 1.8.20) was discovered by Lebesgue in [1911] and proved by Brouwer in [1913]; the gap in Lebesgue's original outline of proof, given in [1911], was filled in his later paper [1921]. Brouwer's paper [1913] contains also a proof of the equality $\operatorname{Ind} R^n = n$, which is reduced to the equality $\dim R^n = n$. The equality $\operatorname{ind} R^n = n$ was established – also by a reduction to the equality $\dim R^n = n$ and an application of Lebesgue's result – by Menger in [1924] and by Urysohn in [1925] (announcement in [1922]). Theorem 1.8.10 was given by Sierpiński in [1922]; Theorem 1.8.9 is obtained by a small modification of Sierpiński's proof. Theorems 1.8.11, 1.8.13 and 1.8.14 were obtained by Menger in [1924] and by Urysohn in [1925] (announcement in [1922]). Urysohn in his proof of Theorem 1.8.11 applied an earlier result of Fréchet and Brouwer (see Problem 1.8.D), and Menger showed that if a subspace M of R^n has an empty interior, then for every point $x \in R^n$ and each positive number ϵ there exists a neighbourhood $U \subset R^n$ of the point x such that $\delta(U) < \epsilon$, the boundary $\operatorname{Fr} U$ is homeomorphic to S^{n-1} and the intersection $M \cap \operatorname{Fr} U$ has an empty interior in the space $\operatorname{Fr} U$. Mazurkiewicz established Theorem 1.8.18 in [1929]; the proof given here, as well as Remark 1.8.17, is taken from Rubin, Schori and Walsh [1979]. Theorem 1.8.21 and Example 1.8.23 were given by Hurewicz in [1928] and [1932], respectively; the example was announced in [1928]. Urysohn in [1925] and [1926] and Hurewicz in [1928] were the first to hint at the possibility of a classification of infinite-dimensional spaces.

Problems

1.8.A. Let us recall that a subset A of Euclidean n-space R^n is *convex* if for each pair x, y of points of A the line segment with end-points x and y is contained in A.

Show that every convex compact subspace $A \subset R^n$ which has a non-empty interior is homeomorphic to the n-ball B^n and its boundary $\operatorname{Fr} A$ is homeomorphic to the $(n-1)$-sphere S^{n-1}. Note that, in particular, I^n and B^n, and also $\operatorname{Fr} I^n$ and S^{n-1}, are homeomorphic to each other for $n = 1, 2, \ldots$

Hint. Consider a point $x \in \operatorname{Int} A$ and prove that every ray starting from x meets $\operatorname{Fr} A$ at exactly one point.

1.8.B. (a) Show that the Brouwer fixed-point theorem follows easily both from Theorem 1.8.1 and from Theorem 1.8.20.

Hint. The Brouwer fixed-point theorem is equivalent to the statement that S^{n-1} is not a retract of B^n. Deduce from Theorem 1.8.1 and from Theorem 1.8.20 that $\operatorname{Fr} I^n$ is not a retract of I^n.

(b) Prove, using the Brouwer fixed-point theorem, that for every continuous mapping $f: I^{\aleph_0} \to I^{\aleph_0}$ there exists a point $x \in I^{\aleph_0}$ such that $f(x) = x$.

(c) Deduce Theorem 1.8.2 from Theorem 1.8.1.

(d) Show that the fixed-point theorem for $I^{\aleph_0}$ stated in (b) follows from Theorem 1.8.2.

Hint. Suppose that there exists a continuous mapping $f: I^{\aleph_0} \to I^{\aleph_0}$ such that $f(x) \neq x$ for every $x \in I^{\aleph_0}$; one can assume that $0 < (f(x))_i < 1$, where $(f(x))_i$ is the ith coordinate of $f(x)$. Consider the sets $L_i = \{x = \{x_j\} \in I^{\aleph_0} : x_i = (f(x))_i\}$.

Remark. The same argument solves the first part of (a). ■

1.8.C (Nöbeling [1932]). Prove that if the projection of a compact subspace X of R^m onto the Cartesian product of any n coordinate axes of R^m, where $0 < n < m$, has an empty interior, then the subspace X is embeddable in N^m_{n-1}. Deduce that if an F_σ-set $X \subset R^m$ satisfies the inequality $\operatorname{ind} X \geq n > 0$, then there exists a set of n coordinate axes of R^m such that the projection of X onto the Cartesian product of these coordinate axes has dimension n (the assumption that X is an F_σ-set cannot be omitted; see Example 1.10.23).

Hint. Let $H_1, H_2, \ldots$ be the sequence of all $(m-n)$-dimensional affine subspaces of R^m defined by conditions of the form $x_{i_1} = r_1$, $x_{i_2} = r_2$, $\ldots$, $x_{i_n} = r_n$, where $1 \leq i_1 < i_2 < \ldots < i_n \leq m$ and $r_1, r_2, \ldots, r_n$ are arbitrary rational numbers. Observe that for $i = 1, 2, \ldots$ the set $A_i \subset R^m$ of all points $a \in R^m$ such that $\{x + a : x \in X\} \cap H_i \neq \emptyset$ is closed and has an empty interior.

1.8.D (Brouwer [1913a]; implicitly, Fréchet [1910]). Prove that for any two countable dense subsets A, B of Euclidean n-space R^n there exists a homeomorphism $f: R^n \to R^n$ such that $f(A) = B$. Note that this result yields Theorem 1.8.11.

Hint. A set $A \subset R^n$ is in general position with respect to the coordinate axes if for every pair of distinct points $x = \{x_i\}, y = \{y_i\} \in A$ the difference $x_i - y_i$ does not vanish for $i = 1, 2, \ldots, n$. Prove first that for every countable set $A \subset R^n$ there exists a homeomorphism of R^n onto itself that maps A onto a set in general position with respect to the coordinate axes. Then show that the elements of two countably infinite sets in general position with respect to the coordinate axes can be arranged into two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, where $x_j = \{x_i^j\}$ and $y_j = \{y_i^j\}$ for $j = 1, 2, \ldots$, in such a way that the differences $x_i^j - x_i^k$ and $y_i^j - y_i^k$ have the same sign for $j, k = 1, 2, \ldots$, $j \neq k$, and $i = 1, 2, \ldots, n$ (cf. the hint to Problem 1.3.G(a)).

Remark. It was proved by Fort in [1962] that the Hilbert cube also has the above strong homogeneity property; more general results were obtained by Bessaga and Pełczyński in [1970] and by Bennett in [1972].

1.8.E. Show that every subspace X of R^n such that $\operatorname{ind} X \leq n-1$ has a compactification of dimension $\leq n-1$ which is embeddable in R^n.

1.8.F. (a) Deduce from Theorem 1.8.14 that if a subset M of a region $G \subset R^n$ satisfies the inequality $\operatorname{ind} M \leq n-2$, then M does not separate G, i.e., the complement $G \setminus M$ is connected.

Hint. Observe first that every set T which separates a topological space X between a pair of points x and y contains a partition between x and y. Then reduce the problem to the special case where $G = R^n$ (cf. the proof of Theorem 1.8.18).

(b) Give an example of a one-dimensional subset of the plane R^2 which does not separate any region $G \subset R^2$.

Remark. Sitnikov gave in [1954] an example of a two-dimensional subset of R^3 which does not separate any region $G \subset R^3$. On the other hand, every $(n-1)$-dimensional closed subset of R^n does separate a region $G \subset R^n$ (see Alexandroff [1930], Frankl and Pontrjagin [1930], and Frankl [1930]).

1.8.G. (a) (Sierpiński [1921]) Prove that if a subset M of Euclidean n-space R^n, where $n \geq 2$, is punctiform, then M does not separate R^n, i.e., the complement $R^n \setminus M$ is connected.

Hint. Apply Theorems 1.4.5 and 1.8.14; then apply the fact that any two points of a region $G \subset R^n$ can be joined by a broken line contained in G.

(b) Note that (a) holds if R^n is replaced by any region $G \subset R^n$.

1.8.H. Give an example of a subset of the plane R^2 which cuts R^2 but does not separate it.

1.8.I (Lebesgue [1921]). Prove that if L is a partition between a pair of opposite faces of the n-cube I^n and $\mathcal{F}$ is a finite closed cover of L no member of which meets two opposite faces of I^n, then $\operatorname{ord} \mathcal{F} \geq n-1$. ■

1.8.J (Smirnov [1958]). Show that every non-empty separable metric space can be represented as the union of an increasing transfinite sequence $X_1 \subset X_2 \subset \ldots \subset X_\alpha \subset \ldots$, $\alpha < \omega_1$, of zero-dimensional subspaces.

Hint. (S. Proizvolov, cited in Smirnov [1958]). Show that the interval I can be represented in this way, and apply the universality of the Hilbert cube for the class of all separable metric spaces.

1.8.K. (a) (Holsztyński [1968], Lifanov [1968a]) Prove that if X_i is a compact metric space with $\operatorname{ind} X_i > 0$ for $i = 1, 2, \ldots, n$, then in the Cartesian product $X = X_1 \times X_2 \times \ldots \times X_n$ there exists a sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_n, B_n)$ of n pairs of disjoint closed sets such that for any closed sets $L_1, L_2, \ldots, L_n$, where L_i is a partition between A_i and B_i, we have $\bigcap_{i=1}^n L_i \neq \emptyset$.

Hint. One can suppose that X_i is a continuum contained in a copy Y_i of the Hilbert cube $I^{\aleph_0}$. First for $i = 1, 2, \ldots, n$ choose a pair a_i, b_i of distinct points in X_i and show that every open subset of Y_i that contains X_i also contains an arc (i.e., a set homeomorphic to the interval I) with end-points a_i and b_i. Then consider the disjoint closed sets $A_i = p_i^{-1}(a_i)$ and $B_i = p_i^{-1}(b_i)$ in X, where $p_i \colon X \to X_i$ is the projection, and assume the existence of partitions L_i' between A_i and B_i such that $\bigcap_{i=1}^n L_i' = \emptyset$. Extend L_i' to a partition L_i in $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ between A_i and B_i in such a way that the intersection $L = \bigcap_{i=1}^n L_i$ is disjoint from X, and consider open sets $U_i, W_i \subset Y_i$ witnessing that L_i is a partition. Find open sets $V_i \subset Y_i$ such that $X \subset V_1 \times V_2 \times \ldots \times V_n \subset Y \setminus L$ and arcs $I_i \subset V_i$ with end-points a_i and b_i such that $I_1 \times \ldots \times \{a_i\} \times \ldots \times I_n \subset U_i$ and $I_1 \times \ldots \times \{b_i\} \times \ldots \times I_n \subset W_i$ (see [GT], Theorem 3.2.10). Deduce a contradiction with Theorem 1.8.1.

(b) (Hurewicz [1930a]) Prove that if X_i is a compact metric space with $\operatorname{ind} X_i > 0$ for $i = 1, 2, \ldots, n$, then $\operatorname{ind}(X_1 \times X_2 \times \ldots \times X_n) \geq n$ (see Problem 1.9.E (b)).

Remark. In (a) and (b), the assumption that the spaces X_i are metric can be omitted if one considers dim rather than ind; it suffices to replace in the hint the Hilbert cubes Y_i by appropriate Tychonoff cubes.

1.9. Characterization of dimension in terms of mappings to spheres. Cantor-manifolds. Cohomological dimension

In the preceding sections we chiefly studied the internal properties of n-dimensional spaces. In the present one, and the next two to follow, we shall study the external properties; more exactly, we shall discuss the relations of n-dimensional spaces to standard spaces such as spheres, polyhedra and Euclidean spaces.

We begin with a characterization of dimension in terms of extending mappings to spheres from a closed subspace over the whole space.

Let us recall that a continuous mapping $f: A \to Y$ defined on a subspace A of a space X is *continuously extendable* over X if there exists a continuous mapping $F: X \to Y$ such that $F|A = f$, i.e. $F(x) = f(x)$ for every $x \in A$; the mapping F is called a *continuous extension* of f. One of the most important results on extending mappings is the Tietze–Urysohn theorem, which states that every continuous function from a closed subspace A of a normal space X to the closed interval I is continuously extendable over X. Urysohn's lemma is a special case of this theorem; it states that for every pair A, B of disjoint closed subsets of a normal space X there exists a continuous function $f: X \to I$ such that $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. When X is a metric space, the function f in Urysohn's lemma can be obtained in a particularly simple way, viz., by defining

$$f(x) = \frac{\rho(x, A)}{\rho(x, A) + \rho(x, B)} \quad \text{for } x \in X.$$

The Tietze–Urysohn theorem implies that for every continuous mapping $f: A \to I^n$ of a closed subspace A of a normal space X to the n-cube I^n there exists a continuous extension $F: X \to I^n$ of f; indeed, it suffices to choose for $i = 1, 2, \ldots, n$ a continuous extension $F_i: X \to I$ of the composition $p_i f$, where $p_i: I^n \to I$ is the projection of I^n onto the ith coordinate axis, and define $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))$.

On the other hand, for every continuous mapping $f: A \to S^n$ of a closed subspace A of a normal space X to the n-sphere S^n there exists an open set $U \subset X$ containing A such that f is continuously extendable over U. Indeed, one can replace the sphere S^n by the boundary $\operatorname{Fr} I^{n+1}$ of the $(n+1)$-cube I^{n+1} in R^{n+1} which is homeomorphic to S^n (see Problem 1.8.A) and, applying the above observation, obtain a continuous mapping $F_0: X \to I^{n+1}$ such that $F_0(x) = f(x)$ for every $x \in A$; now, the mapping f is continuously extendable over the open set $U = F_0^{-1}(I^{n+1} \setminus \{a\}) \subset X$, where $a = (1/2, 1/2, \ldots, 1/2)$; the composition F of the restriction $F_0|U$ and the central projection p of $I^{n+1} \setminus \{a\}$ from the point a to $\operatorname{Fr} I^{n+1}$ is a continuous extension of f over U.

It is well known (cf. the hint to Problem 1.8.B(a)) that in general continuous mappings to the n-sphere S^n are not continuously extendable from a closed subspace to

the whole space. We shall now show that the extendability of such mappings depends exclusively on the dimension of the complement of the subspace under consideration.

1.9.1. LEMMA. *Let X be a separable metric space and A a closed subspace of X such that* $\operatorname{ind}(X \setminus A) \leq n \geq 0$. *For every pair A_1, A_2 of closed subsets of A such that $A_1 \cup A_2 = A$ there exist closed subspaces X_1, X_2 of X which satisfy the conditions*

(1) $$X = X_1 \cup X_2, \quad A_1 \subset X_1, \quad A_2 \subset X_2, \quad A_1 \cap X_2 = A_1 \cap A_2 = X_1 \cap A_2$$

and

(2) $$\operatorname{ind}[(X_1 \cap X_2) \setminus (A_1 \cap A_2)] \leq n - 1.$$

PROOF. The sets $A_1 \setminus A_2 = A_1 \setminus (A_1 \cap A_2)$ and $A_2 \setminus A_1 = A_2 \setminus (A_1 \cap A_2)$ are disjoint and closed in the subspace $X \setminus (A_1 \cap A_2)$ of the space X. By virtue of the second separation theorem, there exists a partition L in $X \setminus (A_1 \cap A_2)$ between $A_1 \setminus A_2$ and $A_2 \setminus A_1$ such that $\operatorname{ind}[L \cap (X \setminus A)] \leq n-1$; since $L \cap A = [L \cap (A_1 \setminus A_2)] \cup [(L \cap (A_2 \setminus A_1)] \cup [L \cap A_1 \cap A_2] = \emptyset$, the last inequality can be rewritten as $\operatorname{ind} L \leq n - 1$.

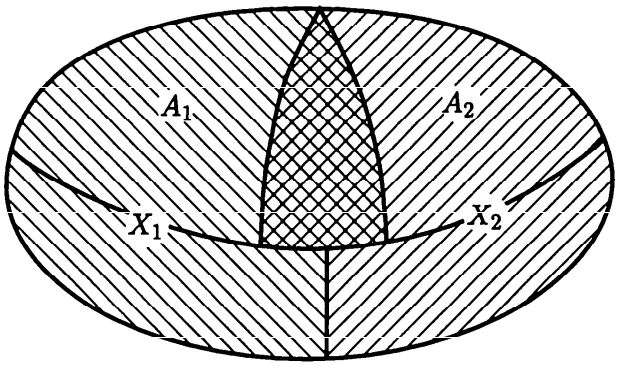


Fig. 9

Consider a pair of sets $U, V \subset X \setminus (A_1 \cap A_2)$ which are open in $X \setminus (A_1 \cap A_2)$ and satisfy the conditions

$$A_1 \setminus A_2 \subset U, \quad A_2 \setminus A_1 \subset V, \quad U \cap V = \emptyset \quad \text{and} \quad [X \setminus (A_1 \cap A_2)] \setminus L = U \cup V.$$

The sets U and V are open in X; we shall verify that their complements $X_1 = X \setminus V$ and $X_2 = X \setminus U$ satisfy (1) and (2).

First of all,

$$X_1 \cup X_2 = (X \setminus V) \cup (X \setminus U) = X \setminus (U \cap V) = X.$$

Next,

$$A_1 = (A_1 \setminus A_2) \cup (A_1 \cap A_2) \subset U \cup [X \setminus (U \cup V)] \subset X \setminus V = X_1,$$

and – similarly – $A_2 \subset X_2$. Then,

$$A_1 \cap A_2 \subset A_1 \cap X_2 = A_1 \setminus U \subset A_1 \setminus (A_1 \setminus A_2) = A_1 \cap A_2,$$

so that $A_1 \cap A_2 = A_1 \cap X_2$, and – similarly – $A_1 \cap A_2 = X_1 \cap A_2$. Finally, since $X_1 \cap X_2 = X \setminus (U \cup V) = L \cup (A_1 \cap A_2)$, we have $(X_1 \cap X_2) \setminus (A_1 \cap A_2) = L$, so that $\operatorname{ind}[(X_1 \cap X_2) \setminus (A_1 \cap A_2)] \leq n - 1$. ■

1.9.2. THEOREM. *If X is a separable metric space and A a closed subspace of X such that* $\operatorname{ind}(X \setminus A) \leq n \geq 0$, *then for every continuous mapping $f: A \to S^n$ there exists a continuous extension $F: X \to S^n$ of f over X.*

PROOF. We shall apply induction with respect to n. When $n = 0$, the mapping f takes values in the two-point space $S^0 = \{-1, 1\}$ and it follows from Lemma 1.9.1, applied to the sets $A_1 = f^{-1}(1)$ and $A_2 = f^{-1}(-1)$, that there exist closed subspaces X_1, X_2 of X such that $X = X_1 \cup X_2$, $A_1 \subset X_1$, $A_2 \subset X_2$ and $X_1 \cap X_2 = \emptyset$. The mapping $F: X \to S^0$ defined by letting

$$F(x) = \begin{cases} 1 & \text{for } x \in X_1, \\ -1 & \text{for } x \in X_2, \end{cases}$$

is then the required continuous extension of f.

Consider now an $n \geq 1$ and assume that the theorem holds for continuous mappings to the $(n-1)$-sphere S^{n-1}. Let $f: A \to S^n$ be a continuous mapping defined on a closed subspace A of a separable metric space X such that $\operatorname{ind}(X \setminus A) \leq n$. Denote by S^n_+ and S^n_- the upper and lower hemispheres of S^n. Let $A_1 = f^{-1}(S^n_+)$ and $A_2 = f^{-1}(S^n_-)$; since $S^n_+ \cap S^n_- = S^{n-1}$, the restriction $g = f|A_1 \cap A_2$ maps $A_1 \cap A_2$ to S^{n-1}. Applying Lemma 1.9.1, we obtain closed subspaces $X_1, X_2 \subset X$ which satisfy conditions (1) and (2). From (2) and the inductive assumption it follows that the mapping $g: A_1 \cap A_2 \to S^{n-1}$ has a continuous extension $G: X_1 \cap X_2 \to S^{n-1}$ over the subspace $X_1 \cap X_2$ of X. Since $A_1 \cap X_2 = A_1 \cap A_2 = X_1 \cap A_2$, the formulas

$$f_1(x) = \begin{cases} f(x) & \text{for } x \in A_1, \\ G(x) & \text{for } x \in X_1 \cap X_2, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} f(x) & \text{for } x \in A_2, \\ G(x) & \text{for } x \in X_1 \cap X_2, \end{cases}$$

define continuous mappings

$$f_1: A_1 \cup (X_1 \cap X_2) \to S^n_+ \quad \text{and} \quad f_2: A_2 \cup (X_1 \cap X_2) \to S^n_-.$$

The hemispheres S^n_+ and S^n_- being homeomorphic to I^n, it follows from the Tietze–Urysohn theorem that f_1 and f_2 are continuously extendable to mappings $F_1: X_1 \to S^n_+$ and $F_2 : X_2 \to S^n_-$. Letting

$$F(x) = \begin{cases} F_1(x) & \text{for } x \in X_1, \\ F_2(x) & \text{for } x \in X_2, \end{cases}$$

we define a continuous extension $F: X \to S^n$ of the mapping f. ■

1.9.3. THEOREM ON EXTENDING MAPPINGS TO SPHERES. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if for every closed subspace A of the space X and each continuous mapping $f: A \to S^n$ there exists a continuous extension $F: X \to S^n$ of f over X.*

PROOF. By virtue of Theorem 1.9.2, it suffices to show that extendability of mappings implies the inequality $\operatorname{ind} X \leq n$. We shall apply the theorem on partitions.

Let $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ be a sequence of $n+1$ pairs of disjoint closed subsets of X. Define $A = \bigcup_{i=1}^{n+1}(A_i \cup B_i)$ and, for $i = 1, 2, \ldots, n+1$, consider a continuous function $f_i: A \to I$ such that

$$f_i(A_i) \subset \{0\} \quad \text{and} \quad f_i(B_i) \subset \{1\}.$$

Letting $f(x) = (f_1(x), f_2(x), \ldots, f_{n+1}(x))$ for $x \in A$, we define a continuous mapping f of A to the boundary $\operatorname{Fr} I^{n+1}$ of the $(n+1)$-cube in R^{n+1}. As the space $\operatorname{Fr} I^{n+1}$ is homeomorphic to S^n, the mapping f has a continuous extension $F: X \to \operatorname{Fr} I^{n+1}$. The composition $F_i: X \to I$ of F and the projection of $\operatorname{Fr} I^{n+1}$ onto the ith coordinate axis

is an extension of f_i for $i = 1, 2, \ldots, n+1$, so that the set $L_i = F_i^{-1}(1/2)$ is a partition between A_i and B_i. Since $\bigcap_{i=1}^{n+1} L_i = \emptyset$, we have $\operatorname{ind} X \leq n$ by virtue of the theorem on partitions. ■

1.9.4. REMARK. Let us note that in the second part of the above proof only the normality of the space X was applied; hence, we have shown that if a normal space X has the property that for every closed subspace A of X and each continuous mapping $f: A \to S^n$ there exists a continuous extension $F: X \to S^n$ of f over X, then for every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. ■

The characterization of dimension in terms of mappings to spheres will now be applied to establish the Cantor-manifold theorem; another important application of this characterization will be given in Section 1.12 (see Theorem 1.12.4).

1.9.5. DEFINITION. A compact metric space X such that $\operatorname{ind} X = n \geq 1$ is an *n-dimensional Cantor-manifold* if no closed subset L of X satisfying the inequality $\operatorname{ind} L \leq n-2$ separates the space X, i.e., if for every such set the complement $X \setminus L$ is connected.

Obviously, one-dimensional Cantor-manifolds coincide with one-dimensional metric continua, and from Theorem 1.8.14 it follows that for every $n \geq 1$ the n-cube I^n and the n-sphere S^n are n-dimensional Cantor-manifolds. On the other hand, for every $n \geq 2$ the union of two copies of the n-cube I^n with exactly one point in common is an example of an n-dimensional compact metric space which is not a Cantor-manifold. Let us observe that if X is an n-dimensional Cantor-manifold, then the dimension of X at each point $x \in X$ (see Problem 1.1.B) is equal to n, but – as shown by the last example – a compact metric space of dimension n at each point need not be a Cantor-manifold.

As we have already observed (see remarks to Problems 1.4.F(b) and 1.5.H), for every $n \geq 1$ there exist separable metric spaces of dimension n which contain no compact subspace of dimension n, and, *a fortiori*, no n-dimensional Cantor-manifold. However, every compact metric space of dimension $n \geq 1$ does contain an n-dimensional Cantor-manifold. The proof of this important theorem is preceded by two lemmas.

Let us recall that continuous mappings f_0, f_1 of a topological space X to a topological space Y are *homotopic* if there exists a continuous mapping $F: X \times I \to Y$ such that $F(x, i) = f_i(x)$ for $i = 0, 1$ and $x \in X$; the mapping F is called a *homotopy* between f_0 and f_1.

1.9.6. LEMMA. *Let $f, g: X \to S^n$ be continuous mappings of a separable metric space X to the n-sphere S^n. If the set*

$$D(f, g) = \{x \in X : f(x) \neq g(x)\}$$

satisfies the inequality $\operatorname{ind} D(f, g) \leq n-1$, *then the mappings f and g are homotopic.*

PROOF. Consider the closed subset $A = (X \times \{0,1\}) \cup ((X \setminus D(f,g)) \times I)$ of the space $Y = X \times I$ and the continuous mapping $h\colon A \to S^n$ defined by letting

$$h(x,0) = f(x), \quad h(x,1) = g(x) \quad \text{for } x \in X$$

and

$$h(x,t) = f(x) = g(x) \quad \text{for } x \in X \setminus D(f,g) \text{ and } t \in I.$$

As $Y \setminus A \subset D(f,g) \times I$, we have $\operatorname{ind}(Y \setminus A) \leq n$, and by virtue of Theorem 1.9.2 there exists a continuous extension $H\colon Y \to S^n$ of the mapping $h\colon A \to S^n$; clearly, H is a homotopy between f and g. ■

The next lemma describes an important property of mappings to spheres.

1.9.7. LEMMA (THE BORSUK HOMOTOPY EXTENSION THEOREM). *Let X be a topological space such that the Cartesian product $X \times I$ is normal (in particular, a metric or a compact space), A a closed subspace of X, and $f, g\colon A \to S^n$ a pair of homotopic continuous mappings of A to the n-sphere S^n. If f is continuously extendable over X, then so is g; moreover, for every extension $F\colon X \to S^n$ of f there exists an extension $G\colon X \to S^n$ of g such that F and G are homotopic.*

PROOF. Let $h\colon A \times I \to S^n$ be a homotopy between f and g. The continuous mapping $h'\colon (X \times \{0\}) \cup (A \times I) \to S^n$ defined by letting

$$h'(x,0) = F(x) \quad \text{for } x \in X$$

and

$$h'(x,t) = h(x,t) \quad \text{for } x \in A \text{ and } t \in I$$

can be extended to a continuous mapping $H'\colon U \to S^n$ defined on an open set $U \subset X \times I$ which contains $(X \times \{0\}) \cup (A \times I)$. It follows from the compactness of I and the definition of the Cartesian product topology that every point $a \in A$ has a neighbourhood $V_a \subset X$ such that $V_a \times I \subset U$; the union $V = \bigcup_{a \in A} V_a$ is an open subset of X such that $A \times I \subset V \times I \subset U$. The space X, being homeomorphic to the closed subspace $X \times \{0\}$ of the normal space $X \times I$, is itself normal, so that by Urysohn's lemma there exists a continuous function $u\colon X \to I$ such that

$$u(X \setminus V) \subset \{0\} \quad \text{and} \quad u(A) \subset \{1\}.$$

One readily verifies that the formula

$$H(x,t) = H'(x, t \cdot u(x)) \quad \text{for } (x,t) \in X \times I$$

defines a continuous mapping $H\colon X \times I \to S^n$ such that $H(x,0) = F(x)$ for $x \in X$ and $H(x,1) = g(x)$ for $x \in A$. Hence, the mapping $G\colon X \to S^n$ defined by letting $G(x) = H(x,1)$ for $x \in X$ is a continuous extension of g, and H is a homotopy between F and G. ■

1.9.8. REMARK. The Borsuk homotopy extension theorem holds under the weaker assumption that X is a normal space, but the proof of this result is much more difficult than that of Lemma 1.9.7 (see [GT], hint to Problem 5.5.21(b)). ■

1.9.9. THE CANTOR-MANIFOLD THEOREM. *Every compact metric space X such that $\operatorname{ind} X = n \geq 1$ contains an n-dimensional Cantor-manifold.*

PROOF. By virtue of Theorem 1.9.3, there exists a closed subspace A of the space X and a continuous mapping $f\colon A \to S^{n-1}$ which cannot be continuously extended over X. Denote by $\mathcal{C}$ the family consisting of all closed sets $C \subset X$ such that the mapping f cannot be continuously extended over $A \cup C$, and define an order $\leq$ in the family $\mathcal{C}$ by letting $C_1 \leq C_2$ whenever $C_2 \subset C_1$; since $X \in \mathcal{C}$, the family $\mathcal{C}$ is non-empty.

Now, consider a subfamily $\mathcal{C}_0$ of $\mathcal{C}$ which is linearly ordered by $\leq$. We shall show that the closed set $C_0 = \bigcap \mathcal{C}_0$ is a member of $\mathcal{C}$. Assume the contrary, i.e., that the mapping f is continuously extendable over $A \cup C_0$. There exists then an open set $U \subset X$ containing $A \cup C_0$ such that f is continuously extendable over U. Since $C_0 = \bigcap \mathcal{C}_0 \subset U$, the compactness of X yields the existence of a finite number of sets $C_1, C_2, \ldots, C_k \in \mathcal{C}_0$ such that $C_1 \cap C_2 \cap \ldots \cap C_k \subset U$. The family $\mathcal{C}_0$ being linearly ordered by $\leq$, there exists an $i_0 \leq k$ such that $C_{i_0} \subset C_i$ for every $i \leq k$. Hence, $C_{i_0} \subset U$ and the function f is continuously extendable over $A \cup C_{i_0} \subset U$; but this is impossible, because C_{i_0} is a member of $\mathcal{C}$. The contradiction shows that $C_0 \in \mathcal{C}$; clearly $C \leq C_0$ for every $C \in \mathcal{C}_0$. Applying the Kuratowski–Zorn lemma, we obtain a maximal set $M \in \mathcal{C}$, i.e., a closed set $M \subset X$ such that the mapping f cannot be continuously extended over $A \cup M$, and yet for every proper closed subset M' of M it is continuously extendable over $A \cup M'$. We shall show that M is an n-dimensional Cantor-manifold.

As $\operatorname{ind} M \leq \operatorname{ind} X = n$, it suffices to show that if $M = M_1 \cup M_2$, where M_1 and M_2 are proper closed subsets of M, then $\operatorname{ind}(M_1 \cap M_2) \geq n-1$. Assume that there exist proper closed subsets M_1, M_2 of M such that $M = M_1 \cup M_2$ and $\operatorname{ind}(M_1 \cap M_2) \leq n-2$; consider the sets $A_1 = A \cup M_1$ and $A_2 = A \cup M_2$. For $i = 1, 2$ the mapping f can be extended to a continuous mapping $f_i\colon A_i \to S^{n-1}$. The restrictions $f_1|B$ and $f_2|B$ of f_1 and f_2 to the set $B = A \cup (M_1 \cap M_2)$ differ only on a subset of $M_1 \cap M_2$, so that by virtue of Lemma 1.9.6 they are homotopic. Since $f_2|B$ is continuously extendable over $A \cup M_2$, it follows from Lemma 1.9.7 that $f_1|B$ is also extendable to a continuous mapping $f_1' : A \cup M_2 \to S^{n-1}$. From the equality $(A \cup M_1) \cap (A \cup M_2) = B$ it follows that letting

$$F(x) = \begin{cases} f_1(x) & \text{for } x \in A \cup M_1, \\ f_1'(x) & \text{for } x \in A \cup M_2, \end{cases}$$

we define a continuous mapping $F\colon A \cup M \to S^{n-1}$ which extends f. This contradiction concludes the proof. ■

1.9.10. COROLLARY. *Every n-dimensional compact metric space X has an n-dimensional component.* ■

The characterization of dimension in terms of mappings to spheres leads, via Hopf's extension theorem, to the cohomological characterization of dimension. We shall describe briefly this process.

It is a well-known fact that the Čech cohomology groups with coefficients in the group of integers, based on all open covers, satisfy the Eilenberg–Steenrod axioms. In particular, for every topological space X and every closed subset A of X the cohomology sequence

$$H^0(X, A) \xrightarrow{j^*} \ldots \xrightarrow{\delta} H^n(X, A) \xrightarrow{j^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta} H^{n+1}(X, A) \xrightarrow{j^*} \ldots$$

of the pair (X, A) is an exact sequence, and for every continuous mapping $f: X \to S^n$ an element $f^*(s^n)$ of the cohomology group $H^n(X)$ is defined, the image of a fixed generator of the group $H^n(S^n)$ under the induced homomorphism $f^*: H^n(S^n) \to H^n(X)$. Hopf's extension theorem, which was mentioned above, states that

(H) *A continuous mapping $f: A \to S^n$ defined on a closed subspace A of a paracompact space X such that $\dim X \leq n+1$ is continuously extendable over X if and only if $f^*(s^n) \in i^*(H^n(X))$.*

It follows from the definition of cohomology groups that if X is a paracompact space such that $\dim X \leq n$, then $H^{n+1}(X, A) = 0$ for every closed set $A \subset X$ (cf. Proposition 3.2.2). On the other hand, if a finite-dimensional separable metric space X satisfies the inequality $\dim X > n$, then by virtue of Theorem 1.5.1 there exists a closed subspace X' of X such that $\dim X' = n+1$, and by virtue of Theorem 1.9.3 there exist a closed subspace A of X' and a continuous mapping $f : A \to S^n$ which cannot be continuously extended over X'; from Hopf's extension theorem and the elementary properties of cohomology groups it follows that $f^*(s^n) \notin i^*(H^n(X))$, so that $H^{n+1}(X, A) \neq 0$ by the exactness of the cohomology sequence of the pair (X, A). Thus we obtain the cohomological characterization of dimension, which states that a finite-dimensional separable metric space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if $H^{n+i}(X, A) = 0$ for every closed set $A \subset X$ and for $i = 1, 2, \ldots$ This characterization remains valid in the class of all paracompact spaces (cf. Theorem 3.2.10). Let us observe that for a metric space X it would be enough to consider the group $H^{n+1}(X, A)$, but for arbitrary paracompact spaces it is necessary to take into account all groups $H^{n+i}(X, A)$, because no counterpart of Theorem 1.5.1 holds in paracompact spaces for the covering dimension dim. Let us also note that the assumption of finite-dimensionality of X in the cohomological characterization of dimension is essential.

The cohomological characterization of dimension was the point of departure for cohomological dimension theory, a section of dimension theory on the border-line of general and algebraic topology, which studies the notion of the cohomological dimension with respect to an abelian group. For a fixed abelian group $G \neq 0$, to every topological space X one assigns the *cohomological dimension of X* with respect to G, denoted by $\dim_G X$, which is an integer larger than or equal to -1, or the "infinite number" ∞; the definition of the dimension function $\dim_G$ consists in the following conditions:

(CD1) $\dim_G X = -1$ *if and only if* $X = \emptyset$;

(CD2) $\dim_G X \leq n$, *where* $n = 0, 1, \ldots$, *if* $H^{n+i}(X, A; G) = 0$ *for every closed set* $A \subset X$ *and for* $i = 1, 2, \ldots$;

(CD3) $\dim_G X = n$ *if* $\dim_G X \leq n$ *and* $\dim_G X > n-1$;

(CD4) $\dim_G X = \infty$ *if* $\dim_G X > n$ *for* $n = -1, 0, 1, \ldots$

One proves that the functions $\dim_G$, although distinct for different groups G, have many properties of dimension. In particular, for every abelian group $G \neq 0$ and every natural number n we have $\dim_G R^n = \dim_G I^n = \dim_G S^n = n$. One also proves that, under suitable assumptions on the space X and the group G, cohomological dimensions satisfy the counterparts of the subspace, sum, Cartesian product and compactification

theorems. In the proofs of these theorems methods of algebraic topology and homological algebra are applied.

To conclude, let us note that the dimension of compact spaces can also be characterized in terms of Čech homology groups with coefficients in the group R_1 of real numbers modulo 1. One proves that a finite-dimensional compact space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if $H_{n+i}(X, A; R_1) = 0$ for every closed set $A \subset X$ and for $i = 1, 2, \ldots$. In the proof, the exactness of the homology sequence and the homological counterpart of Hopf's extension theorem are used; since compactness is crucial for the validity of both these results, when passing to larger classes of spaces one has to replace homology with cohomology.

Historical and bibliographic notes

Theorem 1.9.2 was established by Hurewicz in [1935a]. The same paper contains Theorem 1.9.3 for compact metric spaces. It is a restatement of an earlier result of Alexandroff (see Problem 1.9.A); Hurewicz's contribution was to find a clever formulation and a simple proof (cf. Problem 1.9.C). The extension of Theorem 1.9.3 to separable metric spaces was given by Hurewicz and Wallman in [1941]. The notion of a Cantor-manifold was introduced by Urysohn in [1925]; Theorem 1.9.9 was proved independently by Hurewicz and Menger in [1928] and by Tumarkin in [1928]. The original proofs were much more involved than the one presented here, discovered by Hurewicz in [1937] (a similar proof was given by Freudenthal in [1937]).

Homological dimension theory was originated by Alexandroff in [1932]. Complete proofs of homological and cohomological characterizations of dimension in the realm of compact metric spaces can be found in Hurewicz and Wallman's book [1941]. The important problem of coincidence of dim and $\dim_Z$, where Z is the group of integers, remained open for a long time; it was solved by Dranišnikov in [1988a], where he defined an infinite-dimensional compact metric space X with $\dim_Z X = 3$. A modern and comprehensive exposition of cohomological dimension theory can be found in Dranišnikov, Dydak and Walsh [199?], older references are Kuz'minov [1968] and Kodama [1970]. Dranišnikov [1988b] is a survey of his results, among others those mentioned above and in notes to Section 1.5.

Problems

1.9.A (Alexandroff [1932] for compact metric spaces). A continuous mapping $f: X \to B^{n+1}$ of a topological space X to the $(n+1)$-ball in R^{n+1} is *essential* if there is no continuous mapping $g: X \to B^{n+1}$ such that $g|f^{-1}(S^n) = f|f^{-1}(S^n)$ and $B^{n+1} \setminus g(X) \neq \emptyset$; essential mappings with values in sets homeomorphic to balls are defined in a similar way.

Show that a separable metric space X satisfies the inequality $\operatorname{ind} X \leq n \geq 0$ if and only if no continuous mapping $f: X \to B^{n+1}$ is essential.

1.9.B (Hurewicz and Wallman [1941]). A point $y \in f(X)$ is an *unstable value* of a continuous mapping $f: X \to Y$ of a topological space X to a metric space Y if for every

positive number ϵ there exists a continuous mapping $g\colon X \to Y$ such that $\rho(f(x), g(x)) < \epsilon$ for every $x \in X$ and $y \notin g(X)$; otherwise, $y \in f(X)$ is a *stable value* of f.

(a) Show that $y \in f(X)$ is an unstable value of a continuous mapping $f\colon X \to I^n$ if and only if for every neighbourhood U of the point y there exists a continuous mapping $g\colon X \to I^n$ such that $g(x) = f(x)$ whenever $f(x) \notin U$, $g(x) \in U$ whenever $f(x) \in U$, and $y \notin g(X)$.

(b) Show that a separable metric space X satisfies the inequality $\operatorname{ind} X \leq n \geq 0$ if and only if for every continuous mapping $f\colon X \to I^{n+1}$ all points in $f(X)$ are unstable values of f. Observe that instead of I^{n+1} one can consider S^{n+1} or R^{n+1}.

1.9.C (Hurewicz [1935a]). (a) Let Y be a complete metric space and Z a totally bounded metric space. Prove that for every open set $U \subset Y \times Z$ which is dense in $Y \times Z$ there exists a set $A \subset Y$ which is dense in Y and such that for every $a \in A$ the set $\{z \in Z : (a, z) \in U\}$ is dense in Z.

(b) Prove that if X is a compact metric space and the set $\{f \in (R^{n+1})^X : f(x) \neq (0, 0, \ldots, 0)$ for every $x \in X\}$ is dense in the function space $(R^{n+1})^X$, then $\operatorname{ind} X \leq n$ (see the beginning of Section 1.11).

Hint. Apply induction with respect to n; use the equality $(R^{n+1})^X = (R^n)^X \times R^X$ and part (a).

(c) Apply part (b) to show that if for every closed subspace A of a compact metric space X and each continuous mapping $f\colon A \to S^n$ there exists a continuous extension $F\colon X \to S^n$ of f over X, then $\operatorname{ind} X \leq n$.

1.9.D. (a) (Borsuk [1937]; for compact metric spaces, Eilenberg [1936]) Prove that for every continuous mapping $f\colon A \to S^k$ defined on a closed subspace A of a separable metric space X such that $\operatorname{ind}(X \setminus A) \leq n$, where $0 \leq k \leq n$, there exists a closed subspace B of X such that $A \cap B = \emptyset, \operatorname{ind} B \leq n - k - 1$, and the mapping f has a continuous extension $F\colon X \setminus B \to S^k$ over $X \setminus B$.

Hint. Apply induction with respect to $k + n$; modify the proof of Theorem 1.9.2.

(b) (Lisica [1979]) Let X be a separable metric space and k, n a pair of integers satisfying $0 \leq k \leq n$. Prove that if for every continuous mapping $f\colon A \to S^k$ defined on a closed subspace A of X there exists a closed subspace B of X such that $A \cap B = \emptyset$, $\operatorname{ind} B \leq n - k - 1$, and the mapping f has a continuous extension $F\colon X \setminus B \to S^k$ over $X \setminus B$, then $\operatorname{ind} X \leq n$.

Hint. Consider a sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n + 1$ pairs of disjoint closed subsets of X and for $i = 1, 2, \ldots, k+1$ take a continuous function $f_i\colon X \to I$ such that $f_i(A_i) \subset \{0\}$ and $f_i(B_i) \subset \{1\}$; let $f(x) = (f_1(x), f_2(x), \ldots, f_{k+1}(x))$ for $x \in X$ and $A = f^{-1}(\operatorname{Fr} I^{k+1})$. Extend $f|A\colon A \to \operatorname{Fr} I^{k+1}$ to a continuous mapping $F\colon X \setminus B \to S^{k+1}$, where B is a closed subspace of X disjoint from A with $\operatorname{ind} B \leq n-k-1$. Note that $L_i = B \cup (p_i F)^{-1}(1/2)$, where p_i is the projection of R^{k+1} onto the ith coordinate axis, is a partition in X between A_i and B_i for $i = 1, 2, \ldots, k+1$ and that $\bigcap_{i=1}^{k+1} L_i = B$. Consider the $n - k$ pairs $(A_{k+2} \cap B, B_{k+2} \cap B), (A_{k+3} \cap B, B_{k+3} \cap B), \ldots, (A_{n+1} \cap B, B_{n+1} \cap B)$ of disjoint closed subsets of B, for $i = k + 2, k + 3, \ldots, n + 1$ take a partition L'_i in B between $A_i \cap B$ and $B_i \cap B$ such that $\bigcap_{i=k+2}^{n+1} L'_i = \emptyset$ and extend L'_i to a partition L_i in X between A_i and B_i.

Remark. The argument sketched in the hint shows that the result in (b) holds for dim in normal spaces; to extend L_i''s one applies then Lemmas 6.1.4 and 6.1.5 rather than Lemma 1.2.9 (cf. Problem 2.2.B).

1.9.E. (a) Let $p : R \to S^1$ be the continuous mapping of the real line to the circle defined by letting $p(t) = (\cos 2\pi t, \sin 2\pi t)$ for $t \in R$. Applying the fact that for every continuous mapping $g: X \to S^1$ of a metric space X to S^1 which is homotopic to a constant mapping there exists a continuous mapping $\tilde{g}: X \to R$ such that $g = p\tilde{g}$ (see Engelking and Sieklucki [1992], p. 145), show that for every one-dimensional compact metric space Z there exists a closed set $M \subset Z$ such that the quotient space Z/M can be mapped onto S^1 by a mapping which is not homotopic to the constant mapping.

Hint. Let M be a closed subspace of the space Z with the property that there exists a continuous mapping $f: M \to S^0$ which cannot be continuously extended over Z, and let $F: Z \to I$ be a continuous mapping such that $F(x) = f(x)$ for every $x \in M$. Consider the quotient space Z/M and the continuous mapping $g: Z/M \to I/S^0 = S^1$ induced by F.

(b) (Hurewicz [1935]) Applying the fact that for every n-dimensional compact metric space Z, where $n = 1, 2, \ldots$, there exists a closed set $M \subset Z$ such that the quotient space Z/M can be mapped onto S^n by a mapping which is not homotopic to the constant mapping (see Dowker [1947], p. 235), show that for every compact metric space X and every one-dimensional separable metric space Y we have $\operatorname{ind}(X \times Y) = \operatorname{ind} X + 1 = \operatorname{ind} X + \operatorname{ind} Y$ (cf. Problem 1.8.K(b)).

Hint. Let $\operatorname{ind} X = n$; reduce the problem to the special case where there exists a continuous mapping $g: X \to S^n$ which is not homotopic to the constant mapping. Then consider a pair A, B of disjoint closed subsets of Y such that the empty set is not a partition between A and B and, assuming that $\operatorname{ind}(X \times Y) \leq n$, extend the mapping $f: X \times (A \cup B) \to S^n$ defined by letting

$$f(x,y) = (1,0,0,\ldots,0) \quad \text{for } y \in A \quad \text{and} \quad f(x,y) = g(x) \quad \text{for } y \in B$$

to a continuous mapping $F: X \times Y \to S^n$; examine the set of those $y \in Y$ for which the restriction $F|(X \times \{y\})$ is homotopic to the constant mapping.

Remark. As observed by Morita in [1953], the proof outlined in the above hint shows that for every compact space X and every paracompact space Y with $\dim Y = 1$ we have $\dim(X \times Y) = \dim X + 1 = \dim X + \dim Y$. Dowker's result applied in (b) is established with the aid of cohomology groups; a more elementary proof of the equality in (b) can be found in Karno [1991] and [1995].

(c) Note that the equality in (b) holds under the weaker assumption that X is locally compact.

1.9.F (Kuratowski and Otto [1939]). Let $\{X_s\}_{s \in S}$ be a family of n-dimensional Cantor-manifolds contained in an n-dimensional compact metric space X. Prove that if there exists an $s_0 \in S$ such that the inequality $\operatorname{ind}(X_s \cap X_{s_0}) \geq n - 1$ holds for all $s \in S$, then the subspace $\overline{\bigcup_{s \in S} X_s}$ of X is an n-dimensional Cantor-manifold.

Remark. As proved by Borsuk in [1951], the Cartesian product of two Cantor-manifolds is not necessarily a Cantor-manifold.

1.9.G. (a) (Anderson and Keisler [1967]) Let X be an open subspace of an n-dimensional Cantor-manifold. Prove that if a subset M of X meets every continuum $C \subset X$ which has cardinality larger than one, then $\operatorname{ind} M \geq n-1$ (cf. Theorem 1.8.18).

Hint. Assuming that $\operatorname{ind} M \leq n-2$, define inductively a decreasing sequence $X_1 \supset X_2 \supset \ldots \supset X_{n-1}$ of subsets of X such that X_k is an $(n-k)$-dimensional Cantor-manifold and $\operatorname{ind}(M \cap X_k) \leq n-k-2$ for $k = 1, 2, \ldots, n-1$.

(b) Give an example of a two-dimensional Cantor-manifold of which a one-point subset is a cut.

1.9.H. (a) (Alexandroff [1932]) Show that every n-dimensional Cantor-manifold contained in an n-dimensional compact metric space X can be enlarged to a maximal Cantor-manifold contained in X, i.e., to a Cantor manifold which is not a proper subset of any other Cantor manifold contained in X; maximal n-dimensional Cantor-manifolds contained in an n-dimensional compact metric space X are called *dimensional components* of X. Check that dimensional components of a one-dimensional compact metric space X coincide with those components of X which have cardinality larger than one. Note that the union of all dimensional components of a compact metric space X is not necessarily equal to X. Observe that the intersection of two dimensional components of an n-dimensional compact metric space has dimension $\leq n-2$.

(b) (Mazurkiewicz [1933]) Prove that if A is a dimensional component of an n-dimensional compact metric space, and B is the union of all the remaining dimensional components of X, then $\operatorname{ind}(A \cap B) \leq n-2$.

Hint. Define a decreasing transfinite sequence $X = F_1 \supset F_2 \supset \ldots \supset F_\alpha \supset \ldots$, $\alpha < \omega_1$, of closed sets containing A such that if $F_\alpha \setminus A \neq \emptyset$, then $F_\alpha = F_{\alpha+1} \cup \overline{(F_\alpha \setminus F_{\alpha+1})}$, where $\operatorname{ind}[F_{\alpha+1} \cap \overline{(F_\alpha \setminus F_{\alpha+1})}] \leq n-2$, and $F_\lambda = \bigcap_{\alpha<\lambda} F_\alpha$ for every limit number $\lambda < \omega_1$. Applying the fact that there exists an $\alpha_0 < \omega_1$ such that $F_\alpha = F_{\alpha_0}$ for every $\alpha \geq \alpha_0$ (see [GT], Problem 3.12.7(b)), show that $F_{\alpha_0} = A$ and $X = A \cup \bigcup_{\alpha<\alpha_0}(F_\alpha \setminus F_{\alpha+1})$.

1.9.I (Stone, cited in O'Connor and Rogers [1992]). (a) Prove that no compact metric space X with $\operatorname{ind} X = n \geq 0$ can be represented as the union of n totally disconnected subspaces.

Hint. Apply induction with respect to n. For $n \geq 1$ consider an n-dimensional Cantor manifold $Y \subset X = X_1 \cup X_2 \cup \ldots \cup X_n$ with X_i totally disconnected, assume that $|Y \cap X_1| \geq 2$, decompose $Y \cap X_1$ into two disjoint non-empty closed sets F_1, F_2, enlarge these sets to disjoint open sets $W_1, W_2 \subset Y$ and note that $\operatorname{ind}[Y \setminus (W_1 \cup W_2)] \geq n-1$.

(b) Observe that Euclidean n-space R^n cannot be represented as the union of n totally disconnected subspaces (cf. Problem 1.8.G(a)).

1.10. Characterization of dimension in terms of mappings to polyhedra

The characterization of dimension which is the object of the present section will be formulated in terms of mappings with "arbitrarily small fibres" to polyhedra of geometric dimension $\leq n$.

We begin by recalling the notions of a simplex, a simplicial complex, and a polyhedron.

Let $p = \{p_i\}$ and $q = \{q_i\}$ be points of Euclidean m-space R^m. The sum $p + q$ of the points p and q and the product λp of the point p by the real number λ are defined by the formulas

$$p + q = r = \{r_i\}, \quad \text{where } r_i = p_i + q_i,$$

and

$$\lambda p = s = \{s_i\}, \quad \text{where } s_i = \lambda p_i.$$

The point of R^m which has all coordinates equal to zero, i.e., the origin of R^m, will be denoted by the symbol 0; the distance $\rho(0, x)$ will be denoted by $\|x\|$. One can readily verify that

$$\|p - q\| = \rho(p, q), \quad \|\lambda p\| = |\lambda| \cdot \|p\| \quad \text{and} \quad \|p + q\| \leq \|p\| + \|q\|;$$

the last inequality is a reformulation of the triangle inequality in R^m.

A finite system of points $p_0, p_1, \ldots, p_k \in R^m$ is *affinely independent* if for each sequence $\lambda_0, \lambda_1, \ldots, \lambda_k$ of $k + 1$ real numbers the conditions

$$\lambda_0 p_0 + \lambda_1 p_1 + \ldots + \lambda_k p_k = 0 \quad \text{and} \quad \lambda_0 + \lambda_1 + \ldots + \lambda_k = 0$$

imply that $\lambda_i = 0$ for $i = 0, 1, \ldots, k$.

Let $p_0, p_1, \ldots, p_n$ be an affinely independent system of $n + 1$ points in R^m; the subset of R^m consisting of all points

$$p = \lambda_0 p_0 + \lambda_1 p_1 + \ldots + \lambda_n p_n, \tag{1}$$

where

$$\lambda_0 + \lambda_1 + \ldots + \lambda_n = 1 \quad \text{and} \quad \lambda_i \geq 0 \quad \text{for } i = 0, 1, \ldots, n, \tag{2}$$

is called the *n-simplex* spanned by the points $p_0, p_1, \ldots, p_n$ and is denoted by $p_0 p_1 \ldots p_n$. Clearly, the simplex $p_0 p_1 \ldots p_n$ does not depend on the ordering of the points $p_0, p_1, \ldots, p_n$, it depends on the set $\{p_0, p_1, \ldots, p_n\}$ only. One proves that the simplex $p_0 p_1 \ldots p_n$ determines the points $p_0, p_1, \ldots, p_n$ and can be characterized as the smallest convex subset of R^m which contains these points; one also proves that the diameter of the simplex $p_0 p_1 \ldots p_n$ is equal to the diameter of the set $\{p_0, p_1, \ldots, p_n\}$ (see Problem 1.10.A).

Consider an n-simplex $p_0 p_1 \ldots p_n \subset R^m$; for each choice of $k + 1$ distinct non-negative integers $i_0, i_1, \ldots, i_k$ not larger than n, where $0 \leq k \leq n$, the points $p_{i_0}, p_{i_1}, \ldots$ $\ldots, p_{i_k}$ form an affinely independent system, so that the k-simplex $p_{i_0} p_{i_1} \ldots p_{i_k}$ is well defined. Every simplex of this form is called a *k-face* of the simplex $p_0 p_1 \ldots p_n$; the 0-faces $p_0, p_1, \ldots, p_n$ are also called the *vertices* of the simplex $p_0 p_1 \ldots p_n$. One easily sees that the k-face $p_{i_0} p_{i_1} \ldots p_{i_k}$ consists of all points of the form (1) satisfying (2) and such that

$$\lambda_i = 0 \quad \text{whenever } i \neq i_j \text{ for } j = 0, 1, \ldots, k.$$

To denote the fact that $S_0 = p_{i_0} p_{i_1} \ldots p_{i_k}$ is a face of $S = p_0 p_1 \ldots p_n$ we write symbolically $S_0 \leq S$. For each n-simplex S the union of all k-faces of S with $k < n$ is called the *boundary* of the simplex S; the complement of the boundary is called the *interior* of S.

It follows from the affine independence of the set of vertices that every point p of the simplex spanned by the points $p_0, p_1, \ldots, p_n \in R^m$ can be represented in the form (1), under conditions (2), in a unique way. The coefficients $\lambda_0, \lambda_1, \ldots, \lambda_n$ in

(1) are called the *barycentric coordinates* of the point p; they will also be denoted by $\lambda_0(p), \lambda_1(p), \ldots, \lambda_n(p)$.

One readily checks that every simplex $S = p_0p_1 \ldots p_n \subset R^m$ is a compact subspace of R^m and that the barycentric coordinates $\lambda_0, \lambda_1, \ldots, \lambda_n$ are continuous functions from S to I. This implies, in particular, that any two n-simplexes are homeomorphic. Hence (see Problems 1.8.A and 1.10.A(b)), every n-simplex is homeomorphic to the n-cube I^n. It follows from Corollary 1.8.4 that for every n-simplex S we have $\operatorname{ind} S = \operatorname{Ind} S = \dim S = n$.

A *simplicial complex*, or, briefly, a *complex*, is an arbitrary finite family $\mathcal{K}$ of simplexes in a Euclidean space such that if $S \in \mathcal{K}$ and $S_0 \leq S$ then $S_0 \in \mathcal{K}$, and if $S_1, S_2 \in \mathcal{K}$ then the intersection $S_1 \cap S_2$ either is empty or is a face of both S_1 and S_2; all 0-simplexes that belong to $\mathcal{K}$ are called *vertices* of the complex $\mathcal{K}$. Every subfamily $\mathcal{K}_0$ of a complex $\mathcal{K}$ which itself is a complex, i.e. which together with a simplex $S \in \mathcal{K}$ contains all faces of S, is called a *subcomplex* of $\mathcal{K}$.

Let $\mathcal{K}$ be a simplicial complex consisting of simplexes in Euclidean m-space R^m. The union $|\mathcal{K}| = \bigcup\{S : S \in \mathcal{K}\} \subset R^m$ is the *underlying polyhedron* of the complex $\mathcal{K}$; it is a compact subspace of R^m. By a *polyhedron* we mean a subspace of a Euclidean space which is the underlying polyhedron $|\mathcal{K}|$ of a simplicial complex $\mathcal{K}$; clearly, the representation of a polyhedron as the underlying polyhedron of a complex is not unique. One can prove (see Problem 1.10.B) that for every non-empty polyhedron $K = |\mathcal{K}|$, the largest integer n such that the complex $\mathcal{K}$ contains n-simplexes does not depend on the complex $\mathcal{K}$ but on the polyhedron K only. The number n is called the *geometric dimension* of the polyhedron K; the geometric dimension of an empty polyhedron is equal to -1. From the sum theorem and the fact that an n-simplex is an n-dimensional space it follows that the geometric dimension of a polyhedron coincides with its topological dimensions ind, Ind and dim. Let us note that from the same two premises it follows that the integer n referred to above depends on the polyhedron K only. However, since one of the aspects of the characterization of dimension in terms of mappings to polyhedra is that it reduces the topological notion of dimension to the elementary notion of geometric dimension, it is an important point that the correctness of the definition of geometric dimension can be checked in an elementary way. In the sequel we shall abbreviate "geometric dimension" to "dimension" and call a polyhedron whose geometric dimension equals n an n-dimensional polyhedron.

Every point p of the underlying polyhedron $|\mathcal{K}| = K$ of a simplicial complex $\mathcal{K}$ with vertices $p_0, p_1, \ldots, p_k$ can be represented in the form

$$p = \lambda_0 p_0 + \lambda_1 p_1 + \ldots + \lambda_k p_k, \tag{3}$$

where $\lambda_0 + \lambda_1 + \ldots + \lambda_k = 1$ and $\lambda_i \geq 0$ for $i = 0, 1, \ldots, k$, and if $p \in p_{i_0}p_{i_1} \ldots p_{i_l} \in \mathcal{K}$, then $\lambda_i = 0$ whenever $i \neq i_j$ for $j = 0, 1, \ldots, l$. It follows from the definition of a simplicial complex that such a representation is unique, so that the coefficients in (3) can be written as $\lambda_0(p), \lambda_1(p), \ldots, \lambda_k(p)$; by the continuity of barycentric coordinates, $\lambda_0, \lambda_1, \ldots, \lambda_k$ are continuous functions from K to I. For every vertex p_i of the complex $\mathcal{K}$ the *star of the vertex* p_i is the subset of the underlying polyhedron defined by

$$\operatorname{St}_{\mathcal{K}}(p_i) = |\mathcal{K}| \setminus \bigcup\{S \in \mathcal{K} : p_i \notin S\};$$

one readily checks that $\mathrm{St}_{\mathcal{K}}(p_i) = \{p \in |\mathcal{K}| : \lambda_i(p) > 0\}$, so that the stars of vertices of $\mathcal{K}$ are open subsets of $|\mathcal{K}|$.

We are now going to prepare tools that will be applied later in this section to prove theorems on mappings to polyhedra and in the next section to prove embedding and universal space theorems.

1.10.1. DEFINITION. A finite system of points $p_1, p_2, \ldots, p_k \in R^m$ is in *general position* if for each sequence $i_0 < i_1 < \ldots < i_l \leq k$ of $l+1$ natural numbers, where $l \leq m$, the system $p_{i_0}, p_{i_1}, \ldots, p_{i_l}$ is affinely independent.

Since no system of $m+2$ points in R^m is affinely independent, general position means the minimum of bonds between points in the system $p_1, p_2, \ldots, p_k$.

1.10.2. THEOREM. *For every finite system of points $q_1, q_2, \ldots, q_k \in R^m$ and every positive number α there exists a system of points $p_1, p_2, \ldots, p_k \in R^m$ in general position such that $\rho(p_i, q_i) < \alpha$ for $i = 1, 2, \ldots, k$. If, moreover, a system of points $r_1, r_2, \ldots, r_l \in R^m$ in general position is given, one can choose the points $p_1, p_2, \ldots, p_k$ in such a way that the whole system $r_1, r_2, \ldots, r_l, p_1, p_2, \ldots, p_k$ is in general position.*

PROOF. The points $p_1, p_2, \ldots, p_k$ will be defined by induction. Let $p_1 = q_1$; assume that for a $j \leq k$ the points $p_1, p_2, \ldots, p_{j-1}$ in general position are defined in such a way that $\rho(p_i, q_i) < \alpha$ for $i = 1, 2, \ldots, j-1$. For every system of points $p_{i_0}, p_{i_1}, \ldots, p_{i_n}$, where $i_0 < i_1 < \ldots < i_n \leq j-1$ and $n \leq m-1$, the n-dimensional affine subspace determined by these points is nowhere dense in R^m; hence the union of all such affine subspaces is also nowhere dense in R^m, and there exists a point p_j outside this union such that $\rho(p_j, q_j) < \alpha$. Clearly the system of points $p_1, p_2, \ldots, p_j$ is in general position; thus the first part of the theorem is proved. The proof of the second part is much the same, only – when defining the point p_j – one has to consider all systems of $n \leq m-1$ points in the set $\{r_1, r_2, \ldots, r_l, p_1, p_2, \ldots, p_{j-1}\}$. ■

1.10.3. DEFINITION. Let X be a topological space and $\mathcal{U} = \{U_i\}_{i=1}^k$ a finite open cover of X. By a *nerve* of the cover $\mathcal{U}$ we understand an arbitrary simplicial complex $\mathcal{N}(\mathcal{U})$ whose vertices can be arranged into a sequence $p_1, p_2, \ldots, p_k$ in such a way that

$$p_{i_0} p_{i_1} \ldots p_{i_m} \in \mathcal{N}(\mathcal{U}) \quad \text{if and only if} \quad U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_m} \neq \emptyset.$$

When discussing a nerve $\mathcal{N}(\mathcal{U})$ of a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$, we shall always assume that the vertices $p_1, p_2, \ldots, p_k$ of $\mathcal{N}(\mathcal{U})$ are arranged in such a way that the above equivalence holds. The underlying polyhedron of the nerve $\mathcal{N}(\mathcal{U})$ will be denoted by $N(\mathcal{U})$. One readily sees that if $\operatorname{ord}\mathcal{U} \leq n$, then the dimension of the polyhedron $N(\mathcal{U})$ is not larger than n. Note that every finite open cover has a nerve, although not uniquely determined (cf. Problem 1.10.C). Indeed, for a given open cover $\{U_i\}_{i=1}^k$ one can, for example, consider an arbitrary affinely independent system $p_1, p_2, \ldots, p_k$ of k points in R^{k-1} and define $\mathcal{N}(\mathcal{U})$ as the simplicial complex consisting of all faces $p_{i_0} p_{i_1} \ldots p_{i_m}$ of the simplex $p_1 p_2 \ldots p_k$ such that $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_m} \neq \emptyset$. It turns out, however, that nerves can be defined in a more economical way.

1.10.4. THEOREM. *Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of a topological space X. If $\operatorname{ord}\mathcal{U} \leq n$, then there exists a nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$ consisting of simplexes*

contained in Euclidean $(2n+1)$-space R^{2n+1}. If, moreover, an n-dimensional affine subspace $H \subset R^{2n+1}$, a system $q_1, q_2, \ldots, q_k$ of points in R^{2n+1}, and a positive number α are given, one can choose the nerve $\mathcal{N}(\mathcal{U})$ in such a way that $H \cap N(\mathcal{U}) = \emptyset$ and the vertices $p_1, p_2, \ldots, p_k$ of $\mathcal{N}(\mathcal{U})$ satisfy the inequality $\rho(p_i, q_i) < \alpha$ for $i = 1, 2, \ldots, k$.

PROOF. Let $r_1, r_2, \ldots, r_{n+1}$ be an affinely independent system of points in R^{2n+1} which spans the affine subspace H. By virtue of Theorem 1.10.2, there exists a system of points $p_1, p_2, \ldots, p_k \in R^{2n+1}$ such that $\rho(p_i, q_i) < \alpha$ for $i = 1, 2, \ldots, k$ and the system $r_1, r_2, \ldots, r_{n+1}, p_1, p_2, \ldots, p_k$ is in general position. Since $\operatorname{ord}\mathcal{U} \leq n$, for each sequence $i_0 < i_1 < \ldots < i_l \leq k$ of $l+1$ natural numbers such that $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_l} \neq \emptyset$ we have $l \leq n < 2n+1$, so that the points $p_{i_0}, p_{i_1}, \ldots, p_{i_l}$ form an affinely independent system and the simplex $p_{i_0}p_{i_1}\ldots p_{i_l} \subset R^{2n+1}$ is well defined. Let us denote by $\mathcal{N}(\mathcal{U})$ the family of all simplexes $p_{i_0}p_{i_1}\ldots p_{i_l} \subset R^{2n+1}$ obtained in this way. To complete the proof it suffices to show that $\mathcal{N}(\mathcal{U})$ is a simplicial complex and that $H \cap N(\mathcal{U}) = \emptyset$. Since each face of a simplex in $\mathcal{N}(\mathcal{U})$ is in $\mathcal{N}(\mathcal{U})$, to show that $\mathcal{N}(\mathcal{U})$ is a complex we only have to check that if $S_1 = p_{i_0}p_{i_1}\ldots p_{i_l}$ and $S_2 = p_{j_0}p_{j_1}\ldots p_{j_m}$ belong to $\mathcal{N}(\mathcal{U})$ and the intersection $S_1 \cap S_2$ is non-empty, then $S_1 \cap S_2$ is a face of both S_1 and S_2. Obviously, it is enough to prove that if $p \in S_1 \cap S_2$ and in the representation $p = \sum_{s=0}^{l} \lambda_s p_{i_s}$, where $\sum_{s=0}^{l} \lambda_s = 1$, we have $\lambda_{s_0} > 0$, then $p_{i_{s_0}}$ is a vertex of S_2. Since $p \in S_2$, it follows that $p = \sum_{t=0}^{m} \lambda'_t p_{j_t}$, where $\sum_{t=0}^{m} \lambda'_t = 1$; we have

$$\sum_{s=0}^{l} \lambda_s p_{i_s} - \sum_{t=0}^{m} \lambda'_t p_{j_t} = 0 \quad \text{and} \quad \sum_{s=0}^{l} \lambda_s - \sum_{t=0}^{m} \lambda'_t = 0.$$

As observed before, $l \leq n$; similarly, $m \leq n$. The points $p_{i_0}, p_{i_1}, \ldots, p_{i_l}, p_{j_0}, p_{j_1}, \ldots, p_{j_m}$ form an affinely independent system, because its cardinality is not larger than $l+m+2 \leq 2n+2$, and the whole system $r_1, r_2, \ldots, r_{n+1}, p_1, p_2, \ldots, p_k$ is in general position. It follows that $p_{i_{s_0}}$ occurs among the points $p_{j_0}, p_{j_1}, \ldots, p_{j_m}$, i.e., $p_{i_{s_0}}$ is a vertex of S_2.

In a similar way one shows that all simplexes in $\mathcal{N}(\mathcal{U})$ are disjoint from the affine subspace H. ■

For each finite open cover $\mathcal{U} = \{U_i\}_{i=1}^{k}$ of a metric space X and a sequence of points $p_1, p_2, \ldots, p_k$ in R^m we now define a certain natural continuous mapping of X to R^m.

1.10.5. DEFINITION. Let $\mathcal{U} = \{U_i\}_{i=1}^{k}$ be a finite open cover of a metric space X and $p_1, p_2, \ldots, p_k$ a sequence of points in Euclidean m-space R^m. The continuous mapping $\kappa: X \to R^m$ defined by letting

$$\kappa(x) = \kappa_1(x)p_1 + \kappa_2(x)p_2 + \ldots + \kappa_k(x)p_k, \tag{4}$$

where

$$\kappa_i(x) = \frac{\rho(x, X \setminus U_i)}{\sum_{j=1}^{k} \rho(x, X \setminus U_j)} \quad \text{for } i = 1, 2, \ldots, k, \tag{5}$$

is called the *κ-mapping* determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \ldots, p_k$.

Let us observe that the denominator in (5) does not vanish, because $\rho(x, X \setminus U_j) > 0$ whenever $x \in U_j$. Let us also note that $\sum_{i=1}^{k} \kappa_i(x) = 1$ for every $x \in X$.

As explained in the following theorem, all continuous mappings of a metric space to a Euclidean space can be approximated by κ-mappings.

1.10.6. THEOREM. *Let $f: X \to R^m$ be a continuous mapping of a metric space X to Euclidean m-space R^m and let δ be a positive number. If $\mathcal{U} = \{U_i\}_{i=1}^k$ is a finite open cover of the space X and $p_1, p_2, \ldots, p_k \in R^m$ is a sequence of points such that*

$$\delta(\{p_i\} \cup f(U_i)) < \delta \quad \text{for } i = 1, 2, \ldots, k, \tag{6}$$

then the κ-mapping $\kappa: X \to R^m$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \ldots, p_k$ has the property that $\rho(f(x), \kappa(x)) < \delta$ for every $x \in X$.

PROOF. Consider a point $x \in X$ and let $U_{i_0}, U_{i_1}, \ldots, U_{i_l}$ be all members of the cover $\mathcal{U}$ that contain the point x. By the definition, $\kappa_i(x) = 0$ whenever $i \neq i_j$ for $j = 0, 1, \ldots, l$. It follows from (6) that $\|f(x) - p_{i_j}\| = \rho(f(x), p_{i_j}) < \delta$ for $j = 0, 1, \ldots, l$; applying (4), we obtain

$$\rho(f(x), \kappa(x)) = \|f(x) - \kappa(x)\| = \Big\| \sum_{j=0}^{l} \kappa_{i_j}(x) f(x) - \sum_{j=0}^{l} \kappa_{i_j}(x) p_{i_j} \Big\|$$

$$\leq \sum_{j=0}^{l} \kappa_{i_j}(x) \|f(x) - p_{i_j}\| < \delta \sum_{j=0}^{l} \kappa_{i_j}(x) = \delta. \blacksquare$$

The notion of a κ-mapping determined by a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of a metric space and points $p_1, p_2, \ldots, p_k \in R^m$ proves particularly useful in the case where the points $p_1, p_2, \ldots, p_k$ are the vertices of a nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$.

1.10.7. THEOREM. *If $\mathcal{U} = \{U_i\}_{i=1}^k$ is a finite open cover of a metric space X and $\mathcal{N}(\mathcal{U})$ a nerve of $\mathcal{U}$ with vertices $p_1, p_2, \ldots, p_k \in R^m$, then the κ-mapping $\kappa: X \to R^m$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \ldots, p_k$ satisfies the conditions*

$$\kappa(X) \subset N(\mathcal{U}) \tag{7}$$

and

$$\kappa^{-1}(\mathrm{St}_{\mathcal{N}(\mathcal{U})}(p_i)) = U_i \quad \text{for } i = 1, 2, \ldots, k. \tag{8}$$

PROOF. Consider a point $x \in X$ and let $U_{i_0}, U_{i_1}, \ldots, U_{i_l}$ be all members of the cover $\mathcal{U}$ that contain the point x. Since $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_l} \neq \emptyset$, the simplex $p_{i_0} p_{i_1} \ldots p_{i_l}$ belongs to the nerve $\mathcal{N}(\mathcal{U})$. It follows from (5) and (4) that $\kappa(x) \in p_{i_0} p_{i_1} \ldots p_{i_l}$, so that (7) is satisfied.

Since the representation of points of $N(\mathcal{U})$ in the form (3) is unique, we have $\lambda_i(\kappa(x)) = \kappa_i(x)$ for $i = 1, 2, \ldots, k$. Thus

$$\kappa^{-1}(\mathrm{St}_{\mathcal{N}(\mathcal{U})}(p_i)) = \{x \in X : \kappa(x) \in \mathrm{St}_{\mathcal{N}(\mathcal{U})}(p_i)\} = \{x \in X : \lambda_i(\kappa(x)) > 0\}$$

$$= \{x \in X : \kappa_i(x) > 0\} = U_i \quad \text{for } i = 1, 2, \ldots, k,$$

i.e., (8) is also satisfied. $\blacksquare$

1.10.8. DEFINITION. Let $\mathcal{E}$ be an open cover of a topological space X and $f: X \to Y$ a continuous mapping of X to a topological space Y; we say that f is an *$\mathcal{E}$-mapping* if there exists an open cover $\mathcal{U}$ of the space Y such that the cover $f^{-1}(\mathcal{U})$ is a refinement of $\mathcal{E}$.

1.10.9. DEFINITION. Let ϵ be a positive number and $f: X \to Y$ a continuous mapping of a metric space X to a topological space Y; we say that f is an *ϵ-mapping* if $\delta(f^{-1}(y)) < \epsilon$ for every $y \in Y$.

Obviously, if $\mathcal{E}$ is an open cover of a metric space and $\operatorname{mesh}\mathcal{E} < \epsilon$, then each $\mathcal{E}$-mapping is an ϵ-mapping.

1.10.10. THEOREM. *If X is a compact metric space, then for every open cover $\mathcal{E}$ of the space X there exists a positive number ϵ such that each ϵ-mapping of X to a Hausdorff space is an $\mathcal{E}$-mapping.*

PROOF. Let ϵ be a Lebesgue number of the cover $\mathcal{E}$. Consider an ϵ-mapping $f: X \to Y$ of X to a Hausdorff space Y. For every $y \in Y$ there exists a $V_y \in \mathcal{E}$ such that $f^{-1}(y) \subset V_y$. Since every continuous mapping of a compact space to a Hausdorff space maps closed sets onto closed sets, the set $U_y = Y \setminus f(X \setminus V_y)$ is open for every $y \in Y$; one readily checks that $y \in U_y$ and $f^{-1}(U_y) \subset V_y$. Hence $\mathcal{U} = \{U_y\}_{y \in Y}$ is an open cover of the space Y such that the cover $f^{-1}(\mathcal{U})$ is a refinement of $\mathcal{E}$, i.e., f is an $\mathcal{E}$-mapping. ■

1.10.11. THEOREM. *If for every finite open cover $\mathcal{E}$ of a normal space X there exists an $\mathcal{E}$-mapping of X to a compact space Y such that $\dim Y \leq n$, then $\dim X \leq n$.*

PROOF. Consider an arbitrary finite open cover $\mathcal{E}$ of the space X and an $\mathcal{E}$-mapping $f: X \to Y$ of X to a compact space Y such that $\dim Y \leq n$. Let $\mathcal{U}$ be an open cover of the space Y such that the cover $f^{-1}(\mathcal{U})$ is a refinement of $\mathcal{E}$. As Y is compact, $\mathcal{U}$ has a finite open refinement $\mathcal{V}$ which, by virtue of the inequality $\dim Y \leq n$, has in turn a finite open refinement $\mathcal{W}$ such that $\operatorname{ord}\mathcal{W} \leq n$. Now, $f^{-1}(\mathcal{W})$ is a finite open cover of X, has order $\leq n$, and refines $\mathcal{E}$, so that $\dim X \leq n$. ■

Let us observe that if the space Y in the last theorem is a polyhedron, one can slightly modify the above proof so as to use only the geometric dimension of Y. Indeed, in this case the existence of a refinement $\mathcal{W}$ of the cover $\mathcal{V}$ such that $\operatorname{ord}\mathcal{W} \leq n$ follows from the elementary fact that every polyhedron of geometric dimension $\leq n$ has finite covers of order $\leq n$ by open sets of arbitrarily small diameter.

Theorems 1.10.10 and 1.10.11 imply

1.10.12. THEOREM. *If X is a compact metric space and for every positive number ϵ there exists an ϵ-mapping of X to a compact space Y such that $\dim Y \leq n$, then $\dim X \leq n$.* ■

We are now ready to characterize dimension in terms of mappings to polyhedra.

1.10.13. THEOREM ON $\mathcal{E}$-MAPPINGS. *A metric space X satisfies the inequality $\dim X \leq n$ if and only if for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$-mapping of X to a polyhedron of dimension $\leq n$.*

PROOF. The theorem is obvious if $\dim X = -1$. Consider a metric space X such that $0 \leq \dim X \leq n$ and a finite open cover $\mathcal{E}$ of the space X. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be an open refinement of $\mathcal{E}$ such that $\operatorname{ord}\mathcal{U} \leq n$ and let $\mathcal{N}(\mathcal{U})$ be a nerve of $\mathcal{U}$ with vertices $p_1, p_2, \ldots, p_k \in R^m$. From Theorem 1.10.7 it follows that the κ-mapping $\kappa: X \to N(\mathcal{U})$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \ldots, p_k$ is an $\mathcal{E}$-mapping, because $\{\operatorname{St}_{\mathcal{N}(\mathcal{U})}(p_i)\}_{i=1}^k$ is an open cover of the underlying polyhedron $N(\mathcal{U})$. The inequality

$\operatorname{ord}\mathcal{U} \leq n$ implies that $N(\mathcal{U})$ has dimension $\leq n$. To complete the proof it suffices to apply Theorem 1.10.11. ■

Let us note that, referring to Theorem 1.10.11 in the above proof, we used the coincidence of the geometric dimension and the topological dimension of polyhedra. The observation which follows Theorem 1.10.11 shows how this can be eliminated.

1.10.14. THEOREM ON ϵ-MAPPINGS. *A compact metric space X satisfies the inequality $\dim X \leq n$ if and only if for every positive number ϵ there exists an ϵ-mapping of X to a polyhedron of dimension $\leq n$.*

PROOF. Consider a compact metric space X such that $\dim X \leq n$ and a positive number ϵ. Let $\mathcal{E}$ be a finite open refinement of the open cover $\{B(x, \epsilon/2)\}_{x \in X}$ of the space X. By virtue of the theorem on $\mathcal{E}$-mappings, there exists an $\mathcal{E}$-mapping of X to a polyhedron of dimension $\leq n$; one readily checks that this is an ϵ-mapping. To complete the proof it suffices to apply Theorem 1.10.12. ■

Our comment on the proof of the theorem on $\mathcal{E}$-mappings applies as well to the proof of the theorem on ϵ-mappings. Let us also observe that in the theorem on ϵ-mappings the assumption of compactness is essential; indeed, there exist separable metric spaces of dimension larger than one that can be mapped to the interval I by a one-to-one continuous mapping (see Example 1.4.6 and Problem 1.4.F(b)).

The reader has of course noted that in the proofs of Theorems 1.10.13 and 1.10.14 only the existence of a nerve was applied and not the much stronger Theorem 1.10.4. The latter theorem will be applied in the next section, where it will prove to be pivotal, together with Theorem 1.10.6, in the proofs of the embedding and universal space theorems.

We shall now show that the theorems on $\mathcal{E}$-mappings and on ϵ-mappings can be somewhat strengthened, viz.; that the existence of mappings onto polyhedra can be established. In the proof we shall apply an auxiliary theorem which states that every subset A of the underlying polyhedron $|\mathcal{K}|$ of a complex $\mathcal{K}$ can be continuously "swept out" of the interiors of all simplexes in $\mathcal{K}$ which are not contained in A into their boundaries.

1.10.15. THE SWEEPING OUT THEOREM. *For every simplicial complex $\mathcal{K}$ and each subspace A of the underlying polyhedron $|\mathcal{K}|$ there exist a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ and a continuous mapping $f\colon A \to |\mathcal{K}_0|$ such that $f(A) = |\mathcal{K}_0|$ and $f(A \cap S) \subset S$ for every $S \in \mathcal{K}$.*

PROOF. Let $\boldsymbol{K}$ be the collection of all subcomplexes $\mathcal{K}'$ of $\mathcal{K}$ for which there exists a continuous mapping $f\colon A \to |\mathcal{K}'|$ such that

$$f(A \cap S) \subset S \quad \text{for every } S \in \mathcal{K}. \tag{9}$$

As the embedding $f\colon A \to |\mathcal{K}|$ of the subspace A in $|\mathcal{K}|$ satisfies (9), the complex $\mathcal{K}$ itself belongs to the collection $\boldsymbol{K}$. Hence $\boldsymbol{K}$ is non-empty and, being finite, contains a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ such that no proper subcomplex of $\mathcal{K}_0$ belongs to $\boldsymbol{K}$. Consider a continuous mapping $f\colon A \to |\mathcal{K}_0|$ which satisfies (9); we shall show that $f(A) = |\mathcal{K}_0|$.

Suppose that there exists a point $x_0 \in |\mathcal{K}_0| \setminus f(A)$. Let $S_0 = p_0p_1 \ldots p_k$ be the intersection of all simplexes in $\mathcal{K}_0$ which contain x_0. Define

$$\mathcal{S}_0 = \{S \in \mathcal{K}_0 : S_0 \leq S\}$$

and

$$\mathcal{B}_0 = \{T \in \mathcal{K}_0 : T \notin \mathcal{S}_0 \text{ and } T \leq S \text{ for an } S \in \mathcal{S}_0\};$$

clearly, both $\mathcal{K}_1 = \mathcal{K}_0 \setminus \mathcal{S}_0$ and $\mathcal{B}_0$ are subcomplexes of $\mathcal{K}_0$. The set $G = \bigcap_{i=0}^{k} \mathrm{St}_{\mathcal{K}_0}(p_i)$ is open in $|\mathcal{K}_0|$; one readily checks that

$$\overline{G} = \bigcup \mathcal{S}_0 = G \cup B_0, \quad \text{where } B_0 = |\mathcal{B}_0|.$$

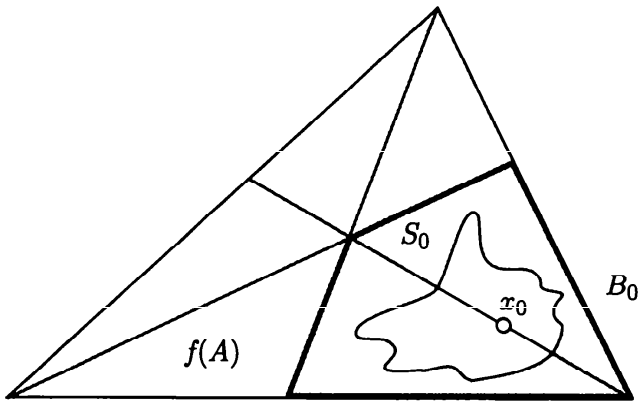


Fig. 10

Let p denote the central projection of $f(A) \cap \overline{G}$ from the point x_0 onto B_0. The restriction

$$p|f(A) \cap S : f(A) \cap S \to B_0 \cap S \tag{10}$$

is continuous for every $S \in \mathcal{S}_0$; thus $p : f(A) \cap \overline{G} \to B_0$ is a continuous mapping. Since $p(x) = x$ for $x \in f(A) \cap B_0$, the formula

$$g(x) = \begin{cases} p(x) & \text{for } x \in f(A) \cap \overline{G}, \\ x & \text{for } x \in f(A) \setminus G, \end{cases}$$

defines a continuous mapping $g : f(A) \to |\mathcal{K}_1|$; let $f_1 = gf : A \to |\mathcal{K}_1|$. It follows from (10) that $f_1(A \cap S) \subset S$ for every $S \in \mathcal{K}$; hence $\mathcal{K}_1 \in \boldsymbol{K}$, which contradicts the definition of $\mathcal{K}_0$. The contradiction shows that $f(A) = |\mathcal{K}_0|$. ■

1.10.16. THEOREM. *A metric (compact metric) space X satisfies the inequality $\dim X \leq n$ if and only if for every finite open cover $\mathcal{E}$ of the space X (for every positive number ϵ) there exists an $\mathcal{E}$-mapping (an ϵ-mapping) of X onto a polyhedron of dimension $\leq n$.*

PROOF. It suffices to show that if a metric space X satisfies the inequality $0 \leq \dim X \leq n$, then for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$-mapping of X onto a polyhedron of dimension $\leq n$. Let $\mathcal{U} = \{U_i\}_{i=1}^{k}$ be an open refinement of $\mathcal{E}$ such that $\operatorname{ord} \mathcal{U} \leq n$ and let $\kappa : X \to |\mathcal{K}|$ be the κ-mapping of X to the underlying polyhedron of a nerve $\mathcal{K} = \mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$. By virtue of the sweeping out theorem, there exists a continuous mapping f of the subspace $A = \kappa(X)$ of $|\mathcal{K}|$ onto the underlying polyhedron $|\mathcal{K}_0|$ of a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ which satisfies (9). Let us note that for every $x \in A$ and every vertex p_i of $\mathcal{K}_0$, if $f(x) \in \mathrm{St}_{\mathcal{K}_0}(p_i)$, then $x \in \mathrm{St}_{\mathcal{K}}(p_i)$, i.e.,

$$f^{-1}(\mathrm{St}_{\mathcal{K}_0}(p_i)) \subset \mathrm{St}_{\mathcal{K}}(p_i). \tag{11}$$

Indeed, if $x \notin \mathrm{St}_{\mathcal{K}}(p_i)$, then the intersection S of all simplexes in $\mathcal{K}$ that contain the point x does not contain p_i; since $f(x) \in S$ by virtue of (9), it follows that $S \in \mathcal{K}_0$, and thus $f(x) \notin \mathrm{St}_{\mathcal{K}_0}(p_i)$. Formulas (8) and (11) show that the composition $f\kappa\colon X \to |\mathcal{K}_0|$ is an $\mathcal{E}$-mapping of X onto $|\mathcal{K}_0|$. ■

The final part of this section will be devoted to subspaces of Euclidean spaces. We start by introducing the notion of an ϵ-translation.

1.10.17. DEFINITION. Let ϵ be a positive number, A, B subspaces of a metric space X, and $f\colon A \to B$ a continuous mapping of A to B; we say that f is an *ϵ-translation* if $\rho(x, f(x)) < \epsilon$ for every $x \in A$.

Obviously, each ϵ-translation is a 3ϵ-mapping and an ϵ-translation defined on a compact subspace is a 2ϵ-mapping.

For subspaces of Euclidean spaces it is much more interesting to discuss ϵ-translations to polyhedra rather than ϵ-mappings to polyhedra. Since every subspace of R^m which can be mapped to a polyhedron by an ϵ-translation is bounded, we have to restrict ourselves to bounded subspaces. It turns out that for such subspaces we have a theorem on ϵ-translations which strictly parallels the theorem on ϵ-mappings. In the proof of the former we shall apply the elementary fact that, for every finite family $\mathcal{S}$ of simplexes in R^m, the union $L = \bigcup \mathcal{S}$ is a polyhedron whose dimension is equal to the largest integer n such that $\mathcal{S}$ contains an n-simplex; more exactly, we shall assume that there exists a simplicial complex $\mathcal{L}$ such that $L = |\mathcal{L}|$ and every simplex of $\mathcal{L}$ is contained in a member of $\mathcal{S}$. A proof of this fact is outlined in Problem 1.10.I.

1.10.18. THEOREM ON ϵ-TRANSLATIONS. *If X is a bounded subspace of Euclidean m-space R^m, and X satisfies the inequality* $\dim X \leq n$, *then for every positive number ϵ there exists an ϵ-translation of X onto a polyhedron $K \subset R^m$ of dimension $\leq n$.*

PROOF. Without loss of generality one can suppose that $0 \leq n \leq m$. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of the space X such that $\mathrm{mesh}\,\mathcal{U} < \epsilon/4$ and $U_i \neq \emptyset$ for $i = 1, 2, \ldots, k$; since $\dim X \leq n$, one can assume – replacing $\mathcal{U}$ by a refinement if necessary – that $\mathrm{ord}\,\mathcal{U} \leq n$. For $i = 1, 2, \ldots, k$ choose a point $q_i \in U_i$ and apply Theorem 1.10.2 to obtain a system of points $p_1, p_2, \ldots, p_k \in R^m$ in general position such that

$$\delta(\{p_i\} \cup U_i) < \epsilon/4 \quad \text{for } i = 1, 2, \ldots, k. \tag{12}$$

Since $\mathrm{ord}\,\mathcal{U} \leq n$, for each sequence $i_0 < i_1 < \ldots < i_l \leq k$ of $l+1$ natural numbers such that $U_{i_0} \cap U_{i_1} \cap \ldots \cap U_{i_l} \neq \emptyset$ we have $l \leq n \leq m$, so that – the system $p_1, p_2, \ldots, p_k$ being in general position – the simplex $p_{i_0} p_{i_1} \ldots p_{i_l} \subset R^m$ is well defined; from (12) it follows that $\delta(p_{i_0} p_{i_1} \ldots p_{i_l}) = \delta(\{p_{i_0}, p_{i_1}, \ldots, p_{i_l}\}) < \epsilon/2$. Let $\mathcal{S}$ be the family of all simplexes obtained in this way; clearly, $\mathrm{mesh}\,\mathcal{S} < \epsilon/2$. The union $L = \bigcup \mathcal{S}$ is a polyhedron of dimension $\leq n$; moreover, one can assume that $L = |\mathcal{L}|$, where $\mathcal{L}$ is a simplicial complex with every simplex contained in a member of $\mathcal{S}$, so that we have $\mathrm{mesh}\,\mathcal{L} < \epsilon/2$. It follows from (12) and Theorem 1.10.6 applied to the embedding of X in R^m that the κ-mapping $\kappa : X \to R^m$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \ldots, p_k$ has the property that

$$\rho(x, \kappa(x)) < \epsilon/4 \quad \text{for every } x \in X. \tag{13}$$

Moreover, as one easily sees, $A = \kappa(X) \subset L$. By virtue of the sweeping out theorem, there exists a continuous mapping $g\colon A \to K$ of A to a polyhedron $K \subset L$ of dimension $\leq n$ such that

$$\rho(\kappa(x), g\kappa(x)) < \epsilon/2 \quad \text{for every } x \in X. \tag{14}$$

By (13) and (14) the composition $f = g\kappa\colon X \to K$ is the required ϵ-translation. ■

Theorems 1.10.12 and 1.10.18 yield the following

1.10.19. THEOREM. *A compact subspace X of Euclidean m-space R^m satisfies the inequality* $\dim X \leq n$ *if and only if for every positive number ϵ there exists an ϵ-translation of X onto a polyhedron $K \subset R^m$ of dimension $\leq n$.* ■

We shall now describe a two-dimensional subspace X of the cube I^3 such that for every positive number ϵ there exists an ϵ-translation of X onto a polyhedron $K \subset R^3$ of dimension ≤ 1. In this way it will be proved that in the theorem on ϵ-translations the assumption of compactness is essential. Let us at once note that the space X has finite open covers of order ≤ 1 by sets with arbitrarily small diameters (cf. Theorem 1.6.12).

We start with a lemma on decompositions of continua, which will be applied to evaluate the dimension of the space X. The lemma states an important topological fact and is known as Sierpiński's theorem; it is preceded by two technical lemmas.

1.10.20. LEMMA. *If A is a closed subspace of a continuum X such that $\emptyset \neq A \neq X$, then for every component C of the space A we have $C \cap \operatorname{Fr} A \neq \emptyset$, where $\operatorname{Fr} A$ is the boundary of A in X.*

PROOF. Assume that $C \cap \operatorname{Fr} A = \emptyset$ and consider the family $\{U_s\}_{s \in S}$ of all open-and-closed subsets of A which contain the component C; it follows from Lemma 1.4.4 that $\bigcap_{s \in S} U_s = C$. The subspace $F = \operatorname{Fr} A$ of X is compact, and the family $\{F \setminus U_s\}_{s \in S}$ is an open cover of F; thus there exists a finite number of indices $s_1, s_2, \ldots, s_k \in S$ such that $F \subset \bigcup_{i=1}^{k} (F \setminus U_{s_i}) = F \setminus \bigcap_{i=1}^{k} U_{s_i}$. The set $U = \bigcap_{i=1}^{k} U_{s_i}$ is disjoint from F, i.e., $U \subset \operatorname{Int} A$; being open-and-closed in A, the set U is open-and-closed in the continuum X. Now, $\emptyset \neq C \subset U$, so that $U = X$, and thus $\operatorname{Fr} A = \emptyset$, which contradicts the connectedness of X. ■

1.10.21. LEMMA. *If a continuum X is represented as the union of a sequence $X_1, X_2, \ldots$ of pairwise disjoint closed sets of which at least two are non-empty, then for every natural number i there exists a continuum $C \subset X$ such that $C \cap X_i = \emptyset$ and at least two sets in the sequence $C \cap X_1, C \cap X_2, \ldots$ are non-empty.*

PROOF. If $X_i = \emptyset$, we let $C = X$; hence, we can assume that $X_i \neq \emptyset$. Consider a natural number $j \neq i$ such that $X_j \neq \emptyset$ and a pair $U, V \subset X$ of disjoint open sets such that $X_i \subset U$ and $X_j \subset V$. Let x be a point in X_j and C the component of the space $\overline{V}$ which contains x. Clearly, C is a continuum, $C \cap X_i = \emptyset$ and $C \cap X_j \neq \emptyset$. Since, by virtue of Lemma 1.10.20, we have $C \cap \operatorname{Fr} \overline{V} \neq \emptyset$, and since $X_j \subset \operatorname{Int} \overline{V}$, there exists a natural number $k \neq j$ such that $C \cap X_k \neq \emptyset$. ■

1.10.22. LEMMA (SIERPIŃSKI'S THEOREM). *If a continuum X is represented as the union of a sequence $X_1, X_2, \ldots$ of pairwise disjoint closed sets, then at most one of the sets X_i is non-empty.*

PROOF. Assume that $X = \bigcup_{i=1}^{\infty} X_i$, where the sets X_i are closed, $X_i \cap X_j = \emptyset$ whenever $i \neq j$, and at least two of the X_i's are non-empty. From Lemma 1.10.21 it follows that there exists a decreasing sequence $C_1 \supset C_2 \supset \ldots$ of non-empty continua contained in X such that $C_i \cap X_i = \emptyset$ for $i = 1, 2, \ldots$ Thus $\bigcap_{i=1}^{\infty} C_i = (\bigcap_{i=1}^{\infty} C_i) \cap (\bigcup_{i=1}^{\infty} X_i) = \emptyset$, and this contradicts the compactness of X. ■

1.10.23. SITNIKOV'S EXAMPLE. For every natural number i consider the family of all planes in R^3 determined by equations of the form $x_j = z/i$, where $j = 1, 2, 3$ and z is an arbitrary integer. The planes yield a decomposition of R^3 into congruent cubes whose edges have length $1/i$ and whose interiors are pairwise disjoint; denote by $\mathcal{K}_i$ the family of all the cubes thus obtained and by A_i the union of all the edges of cubes in $\mathcal{K}_i$.

Let $B_1 = A_1$ and for $i = 2, 3, \ldots$ translate the set A_i as a rigid body to obtain a set B_i disjoint from the union $B_1 \cup B_2 \cup \ldots \cup B_{i-1}$ of all the sets previously defined. Clearly, the sets B_i are closed in R^3 and the union $B = \bigcup_{i=1}^{\infty} B_i$ is dense in R^3. We shall show that the subspace $X = I^3 \setminus B$ of the cube I^3 is two-dimensional and yet for every positive number ϵ there exists an ϵ-translation of X to a polyhedron $K \subset R^3$ of dimension ≤ 1.

Since the set $I^3 \setminus X = I^3 \cap B$ is dense in I^3, it follows from Corollary 1.8.12 that $\operatorname{ind} X \leq 2$. Assume that $\operatorname{ind} X \leq 1$ and consider arbitrary points x and y in the interior G of I^3 in R^3 such that $x \in B_2$ and $y \in B_3$. By virtue of Mazurkiewicz's theorem there exists a continuum $C \subset G \setminus X = G \cap B$ which contains x and y; the continuum C is the union of the sequence $C \cap B_1, C \cap B_2, \ldots$ of pairwise disjoint closed sets of which at least two are non-empty. The contradiction of Sierpiński's theorem shows that $\operatorname{ind} X = 2$.

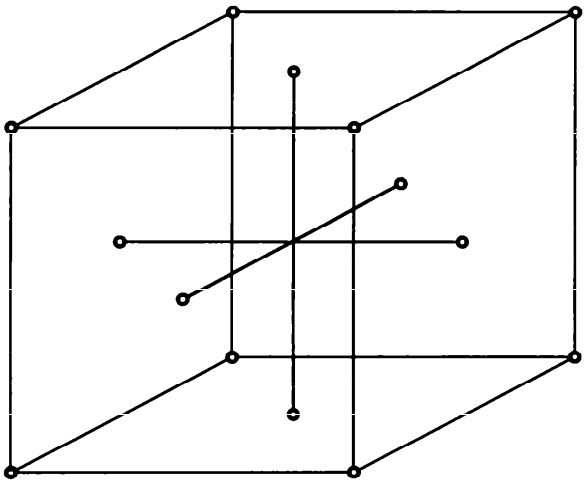


Fig. 11

Now, let ϵ be an arbitrary positive number. Consider a natural number $i > 4/\epsilon$ and translate the set $X \cup B_i$ as a rigid body to make the joints of B_i coincide with the centres of the cubes in $\mathcal{K}_i$. One can assume that the translation $g\colon X \cup B_i \to R^3$ does not shift points by more than $2/i$, i.e., that $\rho(x, g(x)) \leq 2/i$ for $x \in X$; clearly $g(X) \subset R^3 \setminus g(B_i)$. Consider now a fixed cube $T \in \mathcal{K}_i$. One readily checks that the central projection from the centre of T onto the boundary of T maps the set $T \cap g(X)$ onto a subset of the

boundary which does not contain the centres of the faces of T. On the boundary of T the projection coincides with the identity mapping, so that by performing such projections simultaneously on all the cubes in $\mathcal{K}_i$ we obtain a continuous mapping p_1 that sends $g(X)$ to the union of all faces of cubes in $\mathcal{K}_i$ in such a way that the centres of the faces do not belong to $p_1 g(X)$. Now, consider a fixed face S of T. The projection from the centre of S onto the boundary of S maps the set $S \cap p_1 g(X)$ onto a subset of A_i and coincides with the identity mapping on the boundary of S. By performing such projections simultaneously on all faces of cubes in $\mathcal{K}_i$ we obtain a continuous mapping p_2 that sends $p_1 g(X)$ to A_i. Since the points $g(x)$ and $p_2 p_1 g(x)$ lie in the same cube of $\mathcal{K}_i$, we have $\rho(g(x), p_2 p_1 g(x)) < 2/i$. Hence $\rho(x, p_2 p_1 g(x)) < 4/i < \epsilon$ for every $x \in X$ and the set $p_2 p_1 g(X)$ is contained in a one-dimensional polyhedron $K \subset A_i$; thus the required ϵ-translation $f: X \to K$ is defined by letting $f(x) = p_2 p_1 g(x)$ for every $x \in X$.

Though $\dim X = \operatorname{ind} X = 2$, the space X has, for every positive number ϵ, a finite open cover $\mathcal{U}$ such that $\operatorname{mesh} \mathcal{U} < \epsilon$ and $\operatorname{ord} \mathcal{U} \leq 1$. Indeed, if $f: X \to K$ is an $\epsilon/3$-translation of X to a polyhedron $K \subset R^3$ of dimension ≤ 1, then the family $\mathcal{U} = f^{-1}(\mathcal{V})$, where $\mathcal{V}$ is a finite open cover of K such that $\operatorname{mesh} \mathcal{V} < \epsilon/3$ and $\operatorname{ord} \mathcal{V} \leq 1$, has the required properties. ■

Historical and bibliographic notes

Nerves of covers were introduced and studied by Alexandroff in [1927], and κ-mappings – by Hurewicz in [1933] and Kuratowski in [1933a]. The discovery of these two notions was a turning point in the development of dimension theory, and even of the whole of topology; it made possible the combining of the point-set methods of general topology and the combinatorial methods of traditional algebraic topology. Theorem 1.10.13 was proved by Kuratowski in [1933a] and Theorem 1.10.14 by Alexandroff in [1928]. Theorem 1.10.15 for compact subspaces of polyhedra was obtained by Alexandroff in [1928] (cf. Problem 1.10.H); the extension to arbitrary subspaces was given by Kuratowski in [1933a]. Theorem 1.10.18 and 1.10.19 were proved by Alexandroff in [1928] (announcement in [1926]). Example 1.10.23 was described by Sitnikov in [1953].

Problems

1.10.A. (a) Check that the diameter of the simplex $p_0 p_1 \dots p_n$ is equal to the diameter of the set $\{p_0, p_1, \dots, p_n\}$ of its vertices.

(b) Prove that the simplex $p_0 p_1 \dots p_n$ is the smallest convex set which contains the points $p_0, p_1, \dots, p_n$.

(c) Show that every simplex determines its vertices, i.e., that if $p_0 p_1 \dots p_n = q_0 q_1 \dots q_n$, then $\{p_0, p_1, \dots, p_n\} = \{q_0, q_1, \dots, q_n\}$.

Hint. For every point x of a simplex S which is not a vertex of S the set $S \setminus \{x\}$ is not convex.

1.10.B. Prove that if $\mathcal{K}$ and $\mathcal{L}$ are simplicial complexes such that $|\mathcal{K}| = |\mathcal{L}|$ and $\mathcal{K}$ contains an n-simplex, then $\mathcal{L}$ also contains an n-simplex.

Hint. If $m < n$, then every m-simplex contained in an n-simplex S is nowhere dense in S.

1.10.C. Show that if the simplicial complexes $\mathcal{N}_1$ and $\mathcal{N}_2$ are nerves of the same finite open cover, then the underlying polyhedra $|\mathcal{N}_1|$ and $|\mathcal{N}_2|$ are homeomorphic.

1.10.D. Applying the fact that the one-dimensional polyhedron in Fig. 14 on p. 101 cannot be embedded in the plane (see Example 1.11.8), define a finite open cover $\mathcal{U}$ of the interval I such that $\operatorname{ord}\mathcal{U} = 1$ and yet $\mathcal{U}$ has no nerve in R^2.

1.10.E. (a) Observe that if for every finite open cover $\mathcal{E}$ of a normal space X there exists a continuous mapping $f: X \to K$ to a zero-dimensional polyhedron such that the cover $\{f^{-1}(y)\}_{y \in K}$ is a refinement of $\mathcal{E}$, then $\dim X \leq 0$.

(b) For every natural number $n \geq 2$ define a separable metric space X such that $\dim X = n$ and for every finite open cover $\mathcal{E}$ of the space X there exists a continuous mapping $f: X \to I$ such that the cover $\{f^{-1}(t)\}_{t \in I}$ is a refinement of $\mathcal{E}$.

Hint. Apply Problem 1.4.F(b).

1.10.F (Alexandroff [1932]). (a) Let $f: X \to B^{n+1}$ be an essential mapping of a compact metric space X to the $(n+1)$-ball (cf. Problem 1.9.A) and ϵ a positive number smaller than 1. Show that if a continuous mapping $g: X \to B^{n+1}$ satisfies $\rho(f(x), g(x)) < \epsilon$ for every $x \in f^{-1}(S^n)$, then the image $g(X)$ contains the ball of radius $1 - \epsilon$ concentric with B^{n+1}.

(b) Apply (a) and Theorem 1.10.6 to show that if a compact metric space X satisfies the inequality $\dim X \leq n \geq 0$, then no continuous mapping $f: X \to B^{n+1}$ is essential (see Problem 1.9.A).

(c) Deduce from the theorem on $\mathcal{E}$-mappings that if a compact metric space X satisfies the equality $\dim X = n \geq 0$, then there exists an essential mapping $f: X \to B^n$.

1.10.G (Alexandroff [1928a]). (a) Prove that a compact subspace X of Euclidean m-space R^m satisfies the inequality $\dim X \leq n$ if and only if for every polyhedron $K \subset R^m$ of dimension $m - n - 1$ and every positive number ϵ there exists an ϵ-translation $f: X \to R^m$ such that $f(X) \cap K = \emptyset$.

Hint. For a compact space $X \subset R^m$ with $\dim X \leq n$, a polyhedron $|\mathcal{K}| \subset R^m$ of dimension $m - n - 1$, and an $\epsilon > 0$, consider an open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of X satisfying $\operatorname{ord}\mathcal{U} \leq n$ and $\operatorname{mesh}\mathcal{U} < \epsilon$, and pick points $p_1, p_2, \ldots, p_k \in R^m$ such that for every simplex $S \in \mathcal{K}$ the vertices of S and $p_1, p_2, \ldots, p_k$ are in general position and that $\delta(\{p_i\} \cup U_i) < \epsilon$ for $i = 1, 2, \ldots, k$. Then apply Theorem 1.10.6 to obtain the required ϵ-translation.

To prove the reverse implication, consider a compact space $X \subset R^m$ with the property that for every polyhedron $K \subset R^m$ of dimension $m - n - 1$ and every $\epsilon > 0$ there exists an ϵ-translation $f: X \to R^m$ such that $f(X) \cap K = \emptyset$. Define a closed cover of R^m by m-dimensional cubes of diameter less than ϵ such that each point that belongs to $k \leq m+1$ cubes lies in the union of $(m-k+1)$-dimensional faces of these cubes (cf. Fig. 7 on p. 58). Let $C_1, C_2, \ldots, C_i$ be all the cubes in the cover which intersect X and let K be the union of $(m-n-1)$-dimensional faces of these cubes. Use an ϵ-translation

$f: X \to R^m$ such that $f(X) \cap K = \emptyset$ and $f(X) \subset C_1 \cup C_2 \cup \ldots \cup C_i$ to define a finite closed cover $\mathcal{F}$ of X with $\operatorname{ord} \mathcal{F} \leq n$ and $\operatorname{mesh} \mathcal{F} < 3\epsilon$. Then apply Problem 1.6.C.

Remark. Chogoshvili in [1938] claimed that a compact subspace X of R^m satisfies $\dim X \leq n$ if for every $(m-n-1)$-dimensional affine subspace $H \subset R^m$ and every $\epsilon > 0$ there exists an ϵ-translation $f: X \to R^m$ such that $f(X) \cap H = \emptyset$; his proof contains a gap and it is not known if the theorem holds. In an additional remark he claimed that the theorem was even true in a stronger version, viz., that it suffices to consider $(m-n-1)$-dimensional affine subspaces parallel to those spanned by some $m-n-1$ coordinate axes; as shown by Sternfeld in [1993], this stronger claim is false.

(b) Show that in (a) the assumption of compactness of X is essential.

(c) Applying the fact that R^3 contains the *Antoine necklace*, i.e., a subspace A homeomorphic to the Cantor set such that for a circle $S \subset R^3 \setminus A$ the embedding of S in $R^3 \setminus A$ is not homotopic to the constant mapping of S to a point of $R^3 \setminus A$ (see Engelking and Sieklucki [1992], p. 195), show that in (a) the words "every polyhedron $K \subset R^m$" cannot be replaced by "every compact subspace K of R^m".

Hint. Consider the disk X bounded by S.

1.10.H. Note that in the case where A is a compact subspace of $|\mathcal{K}|$, the sweeping out theorem can be proved in a simpler way.

Hint. Sweep out A consecutively from the interiors of all simplexes in $\mathcal{K}$ which are not contained in A starting with the simplexes of highest dimension.

1.10.I. Prove that the union and the intersection of a finite family of polyhedra contained in a Euclidean space are also polyhedra.

Hint. A bounded subset of R^m which can be represented as the intersection of a finite family of closed half-spaces in R^m is a *geometric cell*. Define the interior and the boundary of a geometric cell and the geometric dimension of a geometric cell. Prove that polyhedra can be defined as finite unions of geometric cells. To this end, by analogy to the notion of a simplicial complex, introduce the notion of a cell complex, observe that every polyhedron can be represented as the union of all cells in a cell complex, and – applying induction with respect to the maximal geometric dimension of cells – prove that every cell complex has a subdivision which is a simplicial complex (all the details are worked out in Engelking and Sieklucki [1992], Section 2.6).

1.10.J (Sitnikov [1953]). Modify the construction in Example 1.10.23 to obtain, for every natural number $n \geq 4$, an $(n-1)$-dimensional subspace X of the n-cube I^n such that for every positive number ϵ there exists an ϵ-translation of X to a polyhedron $K \subset R^n$ of dimension $\leq k$, where $k = n/2$ if n is even, and $k = (n-1)/2$ if n is odd.

1.10.K. Prove that if X is a bounded subspace of Euclidean m-space R^m and X satisfies the inequality $\dim X \leq n$, then for every positive number ϵ there exists a finite open cover $\mathcal{U}$ of the space X such that $\operatorname{mesh} \mathcal{U} < \epsilon$, $\operatorname{ord} \mathcal{U} \leq n$, and $\mathcal{U}$ has a nerve in R^m.

1.10.L (Egorov [1959]). Show that a bounded subspace X of Euclidean m-space R^m satisfies the inequality $\mu\dim(X, \rho) \leq n$, where $\mu\dim$ is the metric dimension defined in

Section 1.6 and ρ is the natural metric on R^m, if and only if for every positive number ϵ there exists an ϵ-translation of X onto a polyhedron $K \subset R^m$ of dimension $\leq n$.

Hint. Apply Problem 1.6.E.

1.11. The embedding and universal space theorems

In the present section the notion of a function space is largely applied. Let us recall that if (X, σ) is a non-empty compact metric space and (Y, ρ) an arbitrary metric space, then by letting

$$\widehat{\rho}(f, g) = \sup\{\rho(f(x), g(x)) : x \in X\} \quad \text{for } f, g \in Y^X$$

we define a metric $\widehat{\rho}$ on the set Y^X of all continuous mappings of X to Y; the metric space $(Y^X, \widehat{\rho})$ obtained in this way is a *function space*. One readily shows that if (X, σ) is a compact metric space and (Y, ρ) is a complete metric space, then the function space $(Y^X, \widehat{\rho})$ is complete. As the reader will see, this simple observation leads, via the Baire category theorem, to important applications of function spaces. We begin with three lemmas on function spaces.

1.11.1. LEMMA. *For every positive number ϵ the set of all ϵ-mappings is open in the function space Y^X.*

PROOF. Consider an ϵ-mapping $f: X \to Y$. The closed subspace

$$A = \{(x, x') \in X \times X : \sigma(x, x') \geq \epsilon\}$$

of the Cartesian product $X \times X$ is compact, so that, since for each pair $(x, x') \in A$ the inequality $\rho(f(x), f(x')) > 0$ holds, there exists a positive number δ such that

$$\rho(f(x), f(x')) \geq \delta \quad \text{for each } (x, x') \in A. \tag{1}$$

To complete the proof it suffices to observe that every continuous mapping $g: X \to Y$ such that $\widehat{\rho}(f, g) < \delta/2$ is an ϵ-mapping. Indeed, if $g(x) = g(x')$, then $\rho(f(x), f(x')) < \delta$ and this implies by virtue of (1) that $(x, x') \notin A$, i.e., $\sigma(x, x') < \epsilon$; thus $\delta(g^{-1}(y)) < \epsilon$ for every $y \in Y$. ■

1.11.2. LEMMA. *For every closed set $F \subset Y$ the set $\{f \in Y^X : f(X) \cap F = \emptyset\}$ is open in the function space Y^X.*

PROOF. Consider a continuous mapping $f: X \to Y$ such that $f(X) \cap F = \emptyset$. As $f(X)$ is a compact subspace of Y, the distance $\delta = \rho(f(X), F)$ is positive. To complete the proof it suffices to note that for every continuous mapping $g: X \to Y$ such that $\widehat{\rho}(f, g) < \delta$ the relation $g(X) \cap F = \emptyset$ holds. ■

1.11.3. LEMMA. *If X is a compact metric space such that $0 \leq \dim X \leq n$ and H is an n-dimensional affine subspace of R^{2n+1}, then for every positive number ϵ the set of all ϵ-mappings of X to R^{2n+1} whose values miss H is dense in the function space $(R^{2n+1})^X$; in particular, the set of all ϵ-mappings of X to R^{2n+1} is dense in the function space $(R^{2n+1})^X$.*

PROOF. Consider an arbitrary continuous mapping $f: X \to Y$ and a positive number δ. It follows from the compactness of X that the mapping f is uniformly continuous; thus there exists a positive number η such that $\delta(f(A)) < \delta$ whenever $\delta(A) < \eta$. By virtue of Theorem 1.6.12 there exists a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space X such that $\operatorname{mesh}\mathcal{U} < \min(\epsilon, \eta)$, $\operatorname{ord}\mathcal{U} \leq n$ and the sets U_i are non-empty; obviously, $\delta(f(U_i)) < \delta$ for $i = 1, 2, \ldots, k$. Choose a point $q_i \in f(U_i)$ for $i = 1, 2, \ldots, k$, and apply Theorem 1.10.4 to obtain a nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$, consisting of simplexes contained in R^{2n+1} and such that $H \cap N(\mathcal{U}) = \emptyset$ and the vertices $p_1, p_2, \ldots, p_k$ of $\mathcal{N}(\mathcal{U})$ satisfy the inequality $\rho(p_i, q_i) < \alpha$ for $i = 1, 2, \ldots, k$, where $\alpha = \min\{\delta - \delta(f(U_i)) : i = 1, 2, \ldots, k\}$. We have

$$\delta(\{p_i\} \cup f(U_i)) \leq \rho(p_i, q_i) + \delta(f(U_i)) < \alpha + \delta(f(U_i)) \leq \delta \quad \text{for } i = 1, 2, \ldots, k.$$

From Theorem 1.10.6 it follows that the κ-mapping $\kappa: X \to R^{2n+1}$ determined by the cover $\mathcal{U}$ and the points $p_1, p_2, \ldots, p_k$ satisfies the inequality $\widehat{\rho}(f, \kappa) < \delta$. To conclude the proof it suffices to observe that by virtue of Theorem 1.10.7 we have $\kappa(X) \subset N(\mathcal{U})$ and $\kappa^{-1}(\operatorname{St}_{\mathcal{N}(\mathcal{U})}(p_i)) = U_i$ for $i = 1, 2, \ldots, k$, so that $\kappa(X) \cap H = \emptyset$ and κ is an ϵ-mapping. ■

The above lemmas will be applied in the proof of the embedding theorem and in the proof of the first universal space theorem.

1.11.4. THE EMBEDDING THEOREM. *Every separable metric space X such that $0 \leq \dim X \leq n$ is embeddable in Euclidean $(2n+1)$-space R^{2n+1}; if, moreover, the space X is compact, then all homeomorphic embeddings of X in R^{2n+1} form a G_δ-set dense in the function space $(R^{2n+1})^X$.*

PROOF. First, consider a compact metric space X such that $0 \leq \dim X \leq n$. For $i = 1, 2, \ldots$ let Φ_i denote the subset of the function space $(R^{2n+1})^X$ consisting of all $(1/i)$-mappings; it follows from Lemmas 1.11.1 and 1.11.3 that the sets Φ_i are open and dense in $(R^{2n+1})^X$. By virtue of the Baire category theorem, the G_δ-set $\Phi = \bigcap_{i=1}^\infty \Phi_i$ is dense in the function space $(R^{2n+1})^X$. Since the space X is compact, a continuous mapping $f: X \to R^{2n+1}$ is a homeomorphic embedding if and only if it is a one-to-one mapping, i.e., if f is an $(1/i)$-mapping for $i = 1, 2, \ldots$ Thus Φ is the set of all homeomorphic embeddings of X in R^{2n+1}.

Now, consider a separable metric space X such that $0 \leq \dim X \leq n$. By virtue of the compactification theorem, there exists a compact metric space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequality $\dim \widetilde{X} \leq n$. As established above, $\widetilde{X}$ is embeddable in R^{2n+1}, and so X is also embeddable in R^{2n+1}. ■

We shall show in Example 1.11.8 below that the exponent $2n+1$ in the embedding theorem cannot be lowered.

1.11.5. THE FIRST UNIVERSAL SPACE THEOREM. *The subspace N_n^{2n+1} of Euclidean $(2n+1)$-space R^{2n+1} consisting of all points with at most n rational coordinates is a universal space for the class of all separable metric spaces whose covering dimension is not larger than n.*

PROOF. It follows from Theorem 1.8.6 and the coincidence theorem that $\dim N_n^{2n+1} = n$; hence – by virtue of the compactification theorem – it suffices to prove that every compact metric space X such that $0 \leq \dim X \leq n$ is embeddable in N_n^{2n+1}.

The complement $R^{2n+1} \setminus N_n^{2n+1} = L_{n+1}^{2n+1}$ can be represented as the union of a countable family of n-dimensional affine subspaces of R^{2n+1}, viz., of all sets defined by conditions of the form $x_{i_1} = r_1$, $x_{i_2} = r_2$, ..., $x_{i_{n+1}} = r_{n+1}$, where $1 \leq i_1 < i_2 < \ldots < i_{n+1} \leq 2n+1$ and $r_1, r_2, \ldots, r_{n+1}$ are arbitrary rational numbers. Arrange all affine subspaces in the family under consideration into a sequence $H_1, H_2, \ldots$ and for $i = 1, 2, \ldots$ denote by Φ_i the subset of the function space $(R^{2n+1})^X$ consisting of all $(1/i)$-mappings whose values miss H_i. It follows from Lemmas 1.11.1–1.11.3 that the intersection $\Phi = \bigcap_{i=1}^{\infty} \Phi_i$ is a G_δ-set dense in $(R^{2n+1})^X$, and from the compactness of X it follows that Φ consists of homeomorphic embeddings. ■

Let us observe that the last paragraph yields incidentally another proof of the inequality $\operatorname{ind} X \leq \dim X$ for compact metric spaces (cf. our comments on Theorem 1.7.7).

The space N_n^{2n+1} is called *Nöbeling's universal n-dimensional space*. Obviously, Nöbeling's universal 0-dimensional space is the space of irrational numbers; thus the last theorem extends to higher dimensions our earlier result that the space of irrational numbers is universal for the class of all zero-dimensional separable metric spaces. As the reader remembers, the Cantor set is another universal space for the same class of spaces. We shall now describe the n-dimensional counterpart of the latter universal space, i.e., *Menger's universal n-dimensional space* M_n^{2n+1}. The construction will be more general: for every pair n, m of integers satisfying $0 \leq n \leq m \geq 1$ we shall define a compact subspace M_n^m of Euclidean m-space R^m.

For $i = 0, 1, 2, \ldots$ let $\mathcal{K}_i$ be the family of 3^{mi} congruent cubes obtained by dividing the m-cube I^m by all $(m-1)$-dimensional affine subspaces of R^m determined by equations of the form $x_j = k/3^i$, where $j = 1, 2, \ldots, m$ and $0 \leq k \leq 3^i$. For every family $\mathcal{K}$ of cubes let

$$|\mathcal{K}| = \bigcup\{K : K \in \mathcal{K}\} \quad \text{and} \quad \mathcal{S}_n(\mathcal{K}) = \bigcup\{\mathcal{S}_n(K) : K \in \mathcal{K}\},$$

where $\mathcal{S}_n(K)$ is the family of all faces of K which have dimension $\leq n$; moreover, for $\mathcal{K} \subset \mathcal{K}_i$ let

$$\mathcal{K}' = \{K \in \mathcal{K}_{i+1} : K \subset |\mathcal{K}|\}.$$

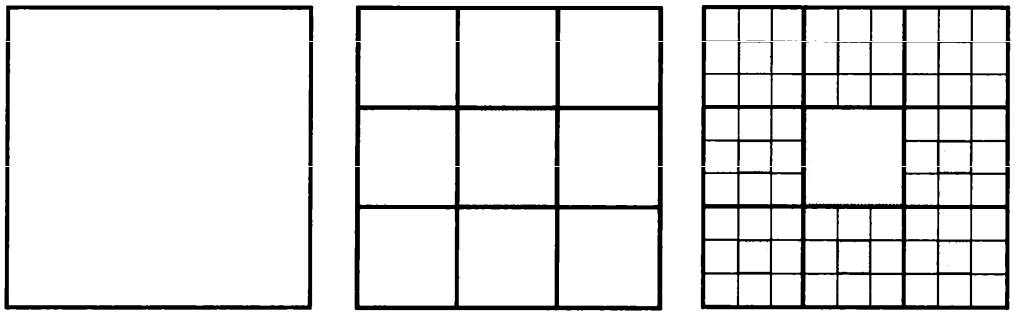
For every pair n, m of integers satisfying $0 \leq n \leq m \geq 1$ define inductively a sequence $\mathcal{F}_0, \mathcal{F}_1, \ldots$ of finite collections of cubes, where $\mathcal{F}_i \subset \mathcal{K}_i$ for $i = 0, 1, 2, \ldots$, and a decreasing sequence $F_0 \supset F_1 \supset F_2 \supset \ldots$ of closed subsets of I^m by letting $\mathcal{F}_0 = \{I^m\}$, $F_0 = |\mathcal{F}_0| = I^m$ and

$$F_i = |\mathcal{F}_i|, \quad \text{where } \mathcal{F}_i = \{K \in \mathcal{F}'_{i-1} : K \cap |\mathcal{S}_n(\mathcal{F}_{i-1})| \neq \emptyset\},$$

for $i = 1, 2, \ldots$ The intersection

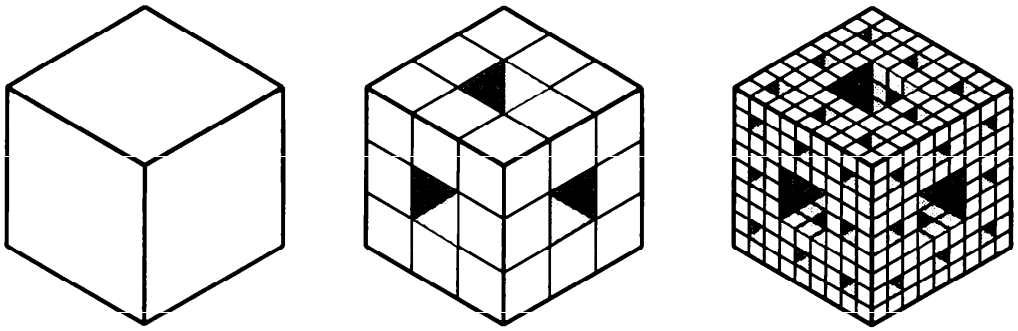
$$M_n^m = \bigcap_{i=0}^{\infty} F_i \subset I^m$$

is a compact subspace of I^m. One readily sees that the construction of the Cantor set described in Example 1.2.5(d) is a special case of the above construction, viz., the case where $m = 1$ and $n = 0$; thus the space M_0^1 is the Cantor set. One proves that the spaces M_0^m are all homeomorphic to the Cantor set (see Problem 1.11.D(a)); obviously,



F_0 F_1 F_2

Fig. 12



F_0 F_1 F_2

Fig. 13

$M_m^m = I^m$ for $m = 1, 2, \ldots$ In Figs. 12 and 13 the first three steps in constructing M_1^2 and M_1^3 are exhibited.

Let us observe that $\dim M_n^m = n$. Indeed, the inequality $\dim M_n^m \geq n$ follows directly from the inclusion $|S_n(\mathcal{F}_0)| \subset M_n^m$ and the reverse inequality is a consequence of Theorem 1.10.12, because the set F_i, and, *a fortiori*, the set M_n^m, can be translated by a $(\sqrt{m}/(2 \cdot 3^{i-1}))$-mapping to the space $|S_n(\mathcal{F}_{i-1})|$ which has dimension $\leq n$. The construction of such a mapping is left to the reader; it should be defined separately on each set of the form $F_i \cap K$, where $K \in \mathcal{F}_{i-1}$.

To prove that the space M_n^{2n+1} is a universal space for the class of all separable metric spaces of dimension $\leq n$, rather laborious computations are necessary; some of them will be left to the reader. Since the same arguments yield an interesting theorem exhibiting a relationship between the spaces N_n^m and M_n^m, we shall prove this theorem, and then deduce from it the universality of M_n^{2n+1}.

1.11.6. THEOREM. *Every compact subspace X of the space N_n^m is embeddable in the space M_n^m.*

PROOF. We shall consider on I^m the metric σ defined by letting

$$\sigma(x, y) = \max\{|x_j - y_j| : j = 1, 2, \ldots, m\}, \quad \text{where } x = \{x_j\} \text{ and } y = \{y_j\};$$

obviously, σ is equivalent to the natural metric on I^m. Let us note that if two points x and y are contained in the same cube $K \in \mathcal{K}_i$, then $\sigma(x, y) \leq 1/3^i$. For $i = 1, 2, \ldots$ denote by S_i the subset of I^m consisting of all points with at least $n+1$ coordinates of the form $k/3^i + 1/(2 \cdot 3^i)$, where $0 \leq k \leq 3^{i-1}$, and by T_i the subset of I^m consisting of all points with at least $m - n$ coordinates of the form $k/3^i$, where $0 \leq k \leq 3^i$; obviously,

$T_i = |\mathcal{S}_n(\mathcal{K}_i)|$. One readily checks that, for $i = 0, 1, 2, \ldots$,

$$\sigma(S_i, T_i) = 1/(2 \cdot 3^i) \tag{2}$$

and

$$F_{i+1} = F_i \setminus B(S_i, 1/(2 \cdot 3^{i+1})), \tag{3}$$

where $\sigma(A, B) = \inf\{\sigma(a, b) : a \in A,\ b \in B\}$ and $B(A, r)$ denotes the r-ball about A with respect to the metric σ.

The proof consists in defining by induction a sequence $f_0, f_1, f_2, \ldots$ of homeomorphisms of I^m onto itself which transform the intersection $I^m \cap N_n^m$ to N_n^m and map X consecutively to $F_0, F_1, F_2, \ldots$, and which uniformly converge to a homeomorphism that maps X to the intersection $\bigcap_{i=0}^{\infty} F_i = M_n^m$. In the inductive step one observes that the set $f_i(X)$ is disjoint from S_i, and one modifies f_i to f_{i+1} by sweeping out the set $f_i(X)$ from the ball $B(S_i, 1/(2 \cdot 3^{i+1}))$. The modification of f_i to f_{i+1} is performed separately on each coordinate axis and is described by a piecewise linear homeomorphism h_i of I onto itself.

We shall now define inductively a sequence $h_0, h_1, h_2, \ldots$ of homeomorphisms of I onto itself and a sequence $f_0, f_1, f_2, \ldots$ of homeomorphisms of I^m onto itself such that for $i = 0, 1, 2, \ldots$ the following conditions will be satisfied:

(4) The interval I can be divided into finitely many closed intervals with pairwise disjoint interiors in such a way that on each of these intervals h_i is a linear function with slope $\geq 2/3$.

$$h_i(k/(2 \cdot 3^i)) = k/(2 \cdot 3^i) \quad \text{for } 0 \leq k \leq 2 \cdot 3^i. \tag{5}$$

$$f_{i+1} = H_i f_i, \quad \text{where } H_i((x_1, x_2, \ldots, x_m)) = (h_i(x_1), h_i(x_2), \ldots, h_i(x_m)). \tag{6}$$

$$f_i(I^m \cap N_n^m) \subset N_n^m. \tag{7}$$

$$f_i(X) \subset F_i. \tag{8}$$

Without loss of generality one can assume that $X \subset I^m$. Hence, if we let $f_0 = \mathrm{id}_{I^m}$, conditions (7) and (8) are satisfied for $i = 0$. Assume that the homeomorphisms $f_0, f_1, \ldots, f_i$ and $h_0, h_1, \ldots, h_{i-1}$ with all the required properties are already defined. We shall first define a homeomorphism h_i which satisfies (4) and (5) and then show that the homeomorphism f_{i+1} defined in (6) satisfies (7) and (8) with i replaced by $i + 1$.

The relations $f_i(X) \subset f_i(I \cap N_n^m) \subset N_n^m$ and $S_i \cap N_n^m = \emptyset$ imply that $f_i(X) \cap S_i = \emptyset$; since $f_i(X)$ is a compact space, there exists a positive rational number ϵ such that

$$0 < \epsilon < \min\left(\sigma(f_i(X), S_i), \frac{1}{2 \cdot 3^i} - \frac{1}{3^{i+1}}\right).$$

Consider the division of the interval I into closed intervals with pairwise disjoint interiors determined by the points

$$0 = b_0 < a_0 < b_1 < a_1 < \ldots < a_{3^i - 1} < b_{3^i} < a_{3^i} = 1,$$

where

$$a_k = \frac{k}{3^i} + \frac{1}{2 \cdot 3^i} - \epsilon \quad \text{and} \quad b_{k+1} = \frac{k}{3^i} + \frac{1}{2 \cdot 3^i} + \epsilon$$

for $k = 0, 1, \ldots, 3^i - 1$. The reader can easily check that the functions $g'_0, g'_1, \ldots, g'_{3^i}$, where

$$g'_k(t) = \frac{1}{3^{i+1}}\left(\frac{1}{2 \cdot 3^i} - \epsilon\right)^{-1}\left(t - \frac{k}{3^i}\right) + \frac{k}{3^i} \quad \text{for } b_k \leq t \leq a_k,$$

and the functions $g''_0, g''_1, \ldots, g''_{3^i-1}$, where

$$g''_k(t) = \epsilon^{-1}\left(\frac{1}{2 \cdot 3^i} - \frac{1}{3^{i+1}}\right)\left[t - \left(\frac{k}{3^i} + \frac{1}{2 \cdot 3^i}\right)\right] + \frac{k}{3^i} + \frac{1}{2 \cdot 3^i} \quad \text{for } a_k \leq t \leq b_{k+1},$$

satisfy the equalities:

$$(9) \qquad g'_k(a_k) = g''_k(a_k) = \frac{k}{3^i} + \frac{1}{3^{i+1}} \quad \text{and} \quad g''_k(b_{k+1}) = g'_{k+1}(b_{k+1}) = \frac{k}{3^i} + \frac{1}{3^i} - \frac{1}{3^{i+1}}$$

for $k = 0, 1, \ldots, 3^i - 1$. As $g'_0(0) = 0$ and $g'_{3^i}(1) = 1$, the functions g'_k and g''_k determine a homeomorphism h_i of I onto itself which, as one easily checks, satisfies (4) and (5).

Consider now the homeomorphism f_{i+1} defined in (6). Since ϵ is a rational number, the homeomorphism h_i transforms each rational number in I into a rational number, so that by virtue of (7) we have $f_{i+1}(I^m \cap N_n^m) \subset N_n^m$.

Let x be a point in X; by virtue of (8), $f_i(x) = (t_1, t_2, \ldots, t_m) \in F_i$. Since $\sigma(f_i(X), S_i) > \epsilon$, there exist $m-n$ coordinates of the point $(t_1, t_2, \ldots, t_m)$, say $t_{j_1}, t_{j_2}, \ldots, t_{j_{m-n}}$, and $m - n$ non-negative integers $k_1, k_2, \ldots, k_{m-n} \leq 3^i$ such that

$$\left|t_{j_l} - \frac{k_l}{3^i}\right| < \frac{1}{2 \cdot 3^i} - \epsilon \quad \text{for } l = 1, 2, \ldots, m - n.$$

The last inequality means that $b_{k_l} \leq t_{j_l} \leq a_{k_l}$, which together with (9) yields

$$\frac{k_l}{3^i} - \frac{1}{3^{i+1}} \leq h_i(t_{j_l}) \leq \frac{k_l}{3^i} + \frac{1}{3^{i+1}} \quad \text{for } l = 1, 2, \ldots, m - n,$$

so that

$$(10) \qquad f_{i+1}(x) = (h_i(t_1), h_i(t_2), \ldots, h_i(t_m)) \in B(T_i, 1/3^{i+1}).$$

From (2) and (10) it follows that

$$(11) \qquad f_{i+1}(x) \notin B(S_i, 1/(2 \cdot 3^{i+1})).$$

Since, by virtue of (5), the homeomorphism H_i maps each cube in $\mathcal{K}_i$ onto itself, it follows that $f_{i+1}(x) \in F_i$; the last relation together with (11) and (3) shows that $f_{i+1}(x) \in F_{i+1}$. Hence we have $f_{i+1}(X) \subset F_{i+1}$.

It remains to check that the sequence of homeomorphisms $f_0, f_1, f_2, \ldots$ uniformly converges to a homeomorphism f of I^m onto itself; the inclusion $f(X) \subset M_n^m$ will then follow from (8). This amounts to checking that the sequence $h_0, h_1 h_0, h_2 h_1 h_0, \ldots$ of homeomorphisms of I onto itself uniformly converges to a homeomorphism of I onto itself. The last fact can be deduced from (4) and (5) by a straightforward computation, which we leave to the reader. ■

1.11.7. THE SECOND UNIVERSAL SPACE THEOREM. *The compact subspace M_n^{2n+1} of Euclidean $(2n+1)$-space R^{2n+1} is a universal space for the class of all separable metric spaces whose covering dimension is not larger than n.*

PROOF. As observed above, $\dim M_n^{2n+1} = n$; hence – by virtue of the compactification theorem – it suffices to prove that every compact metric space X such that

$\dim X \leq n$ is embeddable in M_n^{2n+1}. This is, however, an immediate consequence of Theorems 1.11.5 and 1.11.6. ■

In connection with the above universal space theorems one can ask whether there exists a universal space for the class of all subspaces of R^m which have dimension $\leq n$. It is an old hypothesis that M_n^m is such a space. For the time being we know that M_n^m is indeed a universal space for the class of all compact subspaces of R^m which have dimension $\leq n$; this is a very deep and difficult result. Hence, the hypothesis of the universality of M_n^m is now reduced to the question whether every subspace X of Euclidean m-space R^m has a dimension preserving compactification embeddable in R^m. Let us observe that in some special cases the universality of the space M_n^m can be deduced from our earlier results. Indeed, the universality of M_0^m and M_m^m is obvious, and the universality of M_n^m for $n \leq \frac{1}{2}(m-1)$ is a consequence of Theorem 1.11.7; finally, Theorems 1.8.11, 1.8.9, and 1.11.6, together with Problem 1.8.E, imply that the space M_{m-1}^m is universal for the class of all subspaces of R^m which have dimension $\leq m-1$ (a direct proof of this fact is outlined in the hint to Problem 1.11.D(c)).

The considerations of the preceding paragraph imply, in particular, that the space N_n^m is embeddable in M_n^m for $n \leq \frac{1}{2}(m-1)$, $n = m-1$, and $n = m$; the problem whether N_n^m is always embeddable in M_n^m is still open. On the other hand, the space M_n^m is embeddable in N_n^m for every pair of integers n, m satisfying $0 \leq n \leq m \geq 1$ (see Problem 1.11.E). Hence the question if N_n^m is a universal space for the class of all subspaces of R^m which have dimension $\leq n$ is a weaker version of the problem of the universality of the space M_n^m.

To conclude, let us note that from the above discussion of special cases it follows that for every pair of integers n, m satisfying $0 \leq n \leq m \leq 3$ and $m \geq 1$, the spaces M_n^m and N_n^m are universal for the class of all subspaces of R^m which have dimension $\leq n$; for every such pair n, m, the space N_n^m is embeddable in M_n^m and each n-dimensional subspace of R^m has an n-dimensional compactification embeddable in R^m.

We close this section by showing that the exponent $2n+1$ in the embedding theorem cannot be lowered. Let us recall that by an *arc* we understand a set homeomorphic to the interval I, and by a *simple closed curve* – a set homeomorphic to the circle S^1.

1.11.8. EXAMPLE. Let K_1 be the union of all 1-faces of the 4-simplex $p_0p_1p_2p_3p_4$ (see Fig. 14). Applying the θ-curve theorem (see Problem 1.11.G(b)), we shall show that the one-dimensional polyhedron K_1 cannot be embedded in R^2.

Assume that there exists a homeomorphic embedding $f\colon K_1 \to R^2$ and let $a_i = f(p_i)$ for $0 \leq i \leq 4$. The θ-curve C consisting of the arcs $L_0 = f(p_0p_1 \cup p_0p_2), L_1 = f(p_1p_3 \cup p_2p_3)$ and $L_3 = f(p_1p_2)$ separates the plane into three regions D_0, D_1, D_2 in such a way that $\operatorname{Fr} D_0 = L_0 \cup L_1, \operatorname{Fr} D_1 = L_1 \cup L_2$ and $\operatorname{Fr} D_2 = L_0 \cup L_2$. One can suppose that D_2 is the unbounded component of $R^2 \setminus C$ (otherwise, one should replace the homeomorphic embedding f by the composition of f and a suitable inversion). Since the points a_3 and a_4 can be joined in $f(K_1)$ by an arc disjoint from $L_0 \cup L_1$, we have $a_4 \notin D_4$, and thus either $a_4 \in D_0$ or $a_4 \in D_1$.

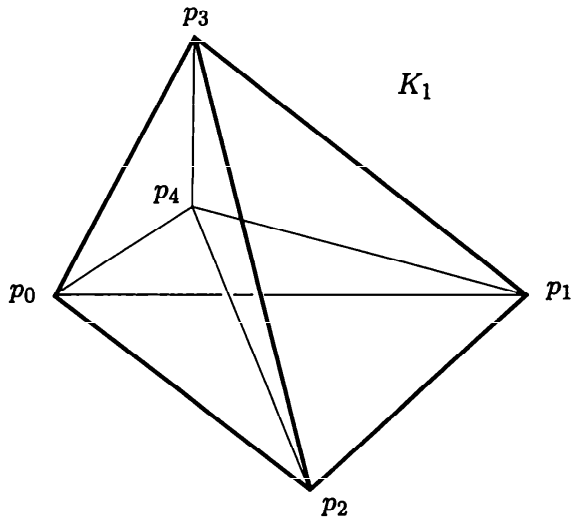
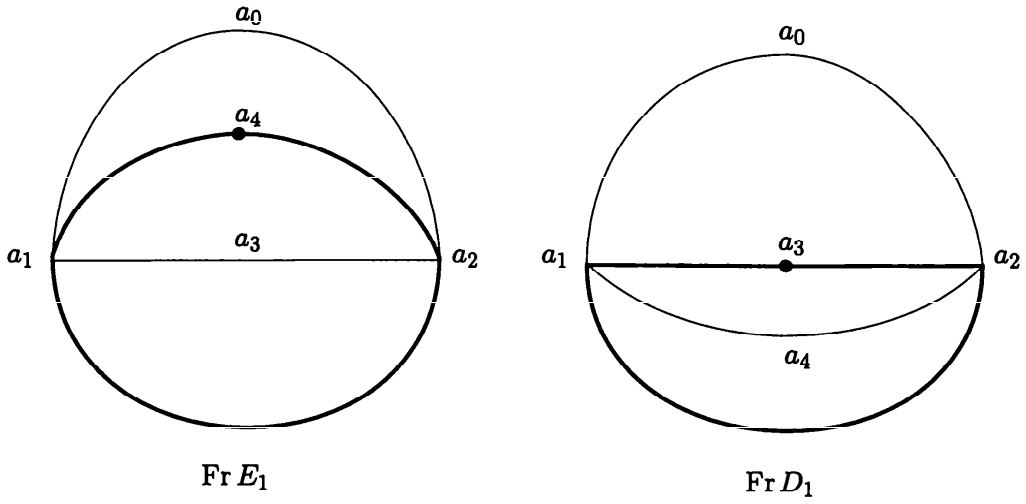


Fig. 14



Fr E_1 Fr D_1

Fig. 15

Assume that $a_4 \in D_0$. The θ-curve $L_0 \cup L \cup L_2$, where $L = f(p_1p_4 \cup p_2p_4)$, separates the plane into the regions E_0, E_1 and D_2 in such a way that $\operatorname{Fr} E_0 = L_0 \cup L$ and $\operatorname{Fr} E_1 = L \cup L_1$. Since the set $E_0 \cup \{a_4\}$ is connected, we have $E_0 \subset D_0$; thus $a_3 \in E_1$ – a contradiction, because $a_3 \in E_1$ and $a_0 \notin E_1$ can be joined in $f(K_1)$ by an arc disjoint from $\operatorname{Fr} E_1 = L \cup L_1$.

The assumption that $a_4 \in D_1$ also yields a contradiction, because $a_4 \in D_1$ and $a_0 \notin D_1$ can be joined in $f(K_1)$ by an arc disjoint from $\operatorname{Fr} D_1 = L_1 \cup L_2$. ■

One can prove that the union K_n of all n-faces of a $(2n + 2)$-simplex cannot be embedded in R^{2n} for any natural number n; a proof of this fact, based on the Borsuk–Ulam antipodal theorem, is outlined in the hint to Problem 1.11.H.

Historical and bibliographic notes

The first part of Theorem 1.11.4 was formulated, for compact metric spaces, by Menger in [1926] and was proved there for $n = 1$. In [1928] Menger again proved the theorem for $n = 1$ and hinted at the modifications in the proof that should permit to obtain the theorem in full generality. For an arbitrary n the first part of the embedding theorem was proved simultaneously by Nöbeling in [1931], Pontrjagin and Tolstowa in

[1931], and Lefschetz in [1931]; the three proofs consisted in constructing a sequence of continuous mappings uniformly converging to a homeomorphic embedding and were rather involved. The present proof was given by Hurewicz in [1933] (announcement in [1931a]); application of function spaces yielded the stronger result about the set of all homeomorphic embeddings. The consideration of function spaces and resorting to the Baire category theorem (proofs by the category method) proved very useful in the dimension theory of separable metric spaces. This idea, originated by Hurewicz in [1931a], was repeatedly exploited by Hurewicz, Kuratowski (cf. Problem 1.7.F) and their followers. Theorem 1.11.5 was established by Nöbeling in [1931]. The spaces M_n^m were introduced by Menger in [1926]. They are generalizations of the Cantor set and of *Sierpiński's universal curve*, i.e., the space M_1^2 described by Sierpiński in [1916], where it was also proved that M_1^2 is a universal space for the class of all compact subspaces of the plane which have an empty interior (in [1922] Sierpiński observed that the assumption of compactness is not essential). In [1926] Menger proved that the space M_1^3 is universal for the class of all compact metric spaces of dimension ≤ 1, observed that Sierpiński's argument yields the universality of M_{m-1}^m for the class of all compact subspaces of R^m which have dimension $\leq m-1$, announced Theorem 1.11.7 for compact spaces and put forward the hypothesis that the space M_n^m is universal for the class of all compact subspaces of R^m which have dimension $\leq n$. Theorem 1.11.6 was proved in [1963] by Bothe who thus established Theorem 1.11.7. It is, however, Lefschetz who is considered to be the author of Theorem 1.11.7. In [1931] for every pair n, m of integers satisfying $0 \leq n \leq m \geq 1$ he defined a compact subspace S_n^m of R^m which is very much like M_n^m (the difference consists in considering simplexes rather than cubes) and proved that S_n^{2n+1} is a universal space for the class of all separable metric spaces whose covering dimension is not larger than n; he also proved there that the space S_n^m is embeddable in N_n^m. From the beginning it was believed that the spaces M_n^m and S_n^m are homeomorphic, but for a long time no proof was produced. Now a homeomorphism is known to exist in some special cases, including the important case of M_n^{2n+1} and S_n^{2n+1}. The last result is a consequence of a topological characterization of M_n^{2n+1} obtained by Bestvina in [1988]; this characterization also implies that M_n^m and S_n^m are homeomorphic to M_n^{2n+1} for all $m > 2n+1$. Bestvina's result for $n = 1$ was known earlier: Anderson gave in [1958] a topological characterization of M_1^3 which implies that M_1^3 and S_1^3 are homeomorphic. Similarly, from a topological characterization of M_1^2 given by Whyburn in [1958] it follows that M_1^2 and S_1^2 are homeomorphic. Furthermore, it was proved by Cannon in [1973] that M_{m-1}^m and S_{m-1}^m are homeomorphic for $m \neq 4$. Finally, M_n^m and S_n^m are obviously homeomorphic for $n = 0$ and $n = m$. The theorem stating that M_n^m is a universal space for the class of all compact subspaces of R^m which have dimension $\leq n$ was proved by Štan'ko in [1971]; for a simpler proof and generalization to σ-compact subspaces of R^n see Edwards [1986].

Problems

1.11.A (Kuratowski [1937], Hurewicz and Wallman [1941]). (a) Check that if (X, σ) is a non-empty metric space and (Y, ρ) is a compact metric space then the function $\widehat{\rho}$ defined at the beginning of this section is a metric on the set Y^X of all continu-

ous mappings of X to Y and the function space $(Y^X, \widehat{\rho})$ is complete. Verify that for every open cover $\mathcal{E}$ of the space X the set of all $\mathcal{E}$-mappings is open in the function space Y^X.

(b) Show that for every separable metric space X there exists a sequence $\mathcal{E}_1, \mathcal{E}_2, \ldots$ of finite open covers of X such that every continuous mapping of X onto a topological space Y which is an $\mathcal{E}_i$-mapping for $i = 1, 2, \ldots$ is a homeomorphism.

Hint. Consider a countable base $\mathcal{B}$ for the space X and arrange into a sequence all covers of the form $\{W, X \setminus \overline{U}\}$, where $U, W \in \mathcal{B}$ and $\overline{U} \subset W$.

(c) Prove that for every finite open cover $\mathcal{E}$ of a separable metric space X such that $0 \leq \dim X \leq n$ the set of all $\mathcal{E}$-mappings of X to I^{2n+1} is dense in the function space $(I^{2n+1})^X$.

(d) Deduce from (a), (b) and (c) that every separable metric space X such that $0 \leq \dim X \leq n$ is embeddable in the $(2n+1)$-cube I^{2n+1}; observe that the set of all homeomorphic embeddings of X in I^{2n+1} contains a G_δ-set dense in the function space $(I^{2n+1})^X$.

Remark. As opposed to the case where X is a compact space, the set of all homeomorphic embeddings of a separable metric space X such that $0 \leq \dim X \leq n$ in I^{2n+1} is generally not a G_δ-set (see Roberts [1948]).

1.11.B (Kuratowski [1937], Hurewicz and Wallman [1941]). (a) Check that for every metric space X and every closed subset F of a compact metric space Y the set $\{f \in Y^X : f(X) \cap F = \emptyset\}$ is open in the function space Y^X.

(b) Prove that for every finite open cover $\mathcal{E}$ of a separable metric space X such that $0 \leq \dim X \leq n$ and every n-dimensional affine subspace H of R^{2n+1} the set of all $\mathcal{E}$-mappings of X to I^{2n+1} whose values miss H is dense in the function space $(I^{2n+1})^X$.

(c) Prove that for every separable metric space X such that $\dim X \leq n \geq 0$ there exists a homeomorphic embedding $f: X \to I^{2n+1}$ which satisfies the inclusion $\overline{f(X)} \subset N_n^{2n+1}$. Observe that this fact implies the compactification theorem.

1.11.C. (a) Show that for every continuous mapping $f: X \to Y$ of a separable metric space X to a separable metric space Y and for every compact metric space $\widetilde{Y}$ that contains Y there exists a compact metric space $\widetilde{X}$ that contains X as a dense subset and a continuous mapping $\tilde{f}: \widetilde{X} \to \widetilde{Y}$ such that $\tilde{f}|X = f$ (cf. Lemma 1.13.3).

Hint. Consider the completion $\widetilde{X}$ of the space X with respect to the metric $\widetilde{\rho}$ defined by letting $\widetilde{\rho}(x, y) = \rho(x, y) + \sigma(f(x), f(y))$, where ρ is a totally bounded metric on the space X and σ is an arbitrary metric on the space $\widetilde{Y}$.

(b) Let $\boldsymbol{S} = \{X_i, \pi_j^i\}$ be an inverse sequence of completely metrizable separable spaces. Prove that if $\pi_i^{i+1}(X_{i+1})$ is a dense subset of X_i for $i = 1, 2, \ldots$, then $\varprojlim \boldsymbol{S} \neq \emptyset$.

Hint. Apply (a) to define an inverse sequence $\widetilde{\boldsymbol{S}} = \{\widetilde{X}_i, \widetilde{\pi}_j^i\}$ of compact metric spaces such that X_i is a dense subset of $\widetilde{X}_i$ and $\widetilde{\pi}_i^{i+1}|X_{i+1} = \pi_i^{i+1}$ for $i = 1, 2, \ldots$ Show that for every i the inverse image $G_i = \widetilde{\pi}_i^{-1}(X_i) \subset \widetilde{X} = \varprojlim \widetilde{\boldsymbol{S}}$, where $\widetilde{\pi}_i: \widetilde{X} \to \widetilde{X}_i$ denotes the projection, is a G_δ-set dense in the space $\widetilde{X}$; use the fact that $\widetilde{\pi}_i$ maps $\widetilde{X}$ onto $\widetilde{X}_i$ (see [GT], Corollary 3.2.15).

(c) Prove that for every continuous mapping $f: X \to R^{2n+1}$ of a separable metric space X such that $\dim X \leq n$ to R^{2n+1} and for every positive number ϵ there exists a homeomorphic embedding $g: X \to R^{2n+1}$ such that $\rho(f(x), g(x)) < \epsilon$ for every $x \in X$.

Hint. Let $Z = R^{2n+1} \cup \{\omega\}$ be the one-point compactification of R^{2n+1}. Define a compact metric space $\widetilde{X}$ that contains X as a dense subset and satisfies the inequality $\dim \widetilde{X} \leq n$, and a continuous mapping $\tilde{f}: \widetilde{X} \to Z$ such that $\tilde{f}|X = f$. For $i = 1, 2, \ldots$ let $X_i = \tilde{f}^{-1}(B_i) \subset \widetilde{X}$, where $B_i = \overline{B(0, i)} \subset R^{2n+1}$; assume that $X_i \neq \emptyset$ and consider the subspace Φ_i of the function space $(R^{2n+1})^{X_i}$ consisting of all homeomorphic embeddings $g: X_i \to R^{2n+1}$ such that $\rho(g(x), \tilde{f}(x)) < \epsilon$ for every $x \in X_i$. Apply (b) to the inverse sequence $\boldsymbol{S} = \{\Phi_i, \pi_j^i\}$, where $\pi_j^i(g) = g|X_j$ for every pair i, j of natural numbers satisfying $j \leq i$.

1.11.D. (a) Show that for every natural number $m \geq 1$ the space M_0^m is homeomorphic to the Cantor set.

Hint. Apply Problem 1.3.F(a).

(b) (Sierpiński [1916] and [1922]) Prove directly that M_1^2 is a universal space for the class of all subspaces of the plane with dimension ≤ 1.

Hint. For a compact one-dimensional subspace X of the plane define a subspace of R^2 which contains X and is homeomorphic to M_1^2. To this end, consider a rectangle containing X and remove from it smaller rectangles disjoint from X in the same way as one removes squares from I^2 to obtain M_1^2 (see Fig. 16, where X has the shape of the letter a). Apply Theorem 1.8.10 to extend the result to all one-dimensional subspaces of the plane.

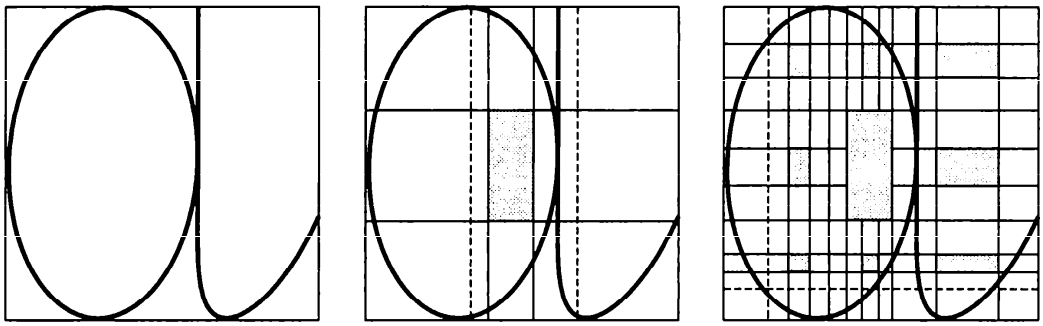


Fig. 16

(c) (Menger [1926] for compact spaces) Prove directly that the space M_{m-1}^m is a universal space for the class of all subspaces of R^m with dimension $\leq m - 1$.

Hint. Modify the construction described in the hint to part (b).

1.11.E. Prove that the space M_n^m is embeddable in N_n^m for every pair of integers m, n satisfying $0 \leq n \leq m \geq 1$.

Hint. Apply Problem 1.8.C.

1.11.F. (a) (Kuratowski [1931]) Show that if X is a non-empty compact metric space and Y is a metric space, then the set of all homeomorphic embeddings of X in Y is a G_δ-set in the function space Y^X.

(b) (Menger [1929]) Prove that every G_δ-set in R^n that contains all points with at least one rational coordinate (i.e., every G_δ-set $M \subset R^n$ such that $R^n \setminus N_0^n \subset M$) contains topologically all subspaces of R^n which have dimension $\leq n-1$.

Hint. Apply the category method. Consider a compact subspace X of R^n such that $0 \leq \dim X \leq n-1$ and the function space $(R^n)^X$. Use (a) and the fact that R^n can be divided into arbitrarily small n-cubes whose boundaries are contained in the G_δ-set under consideration.

(c) (Hayashi [1990]) Prove that every G_δ-set in R^{2n+1} that contains all points with at least $n+1$ rational coordinates (i.e., every G_δ-set $M \subset R^{2n+1}$ such that $R^{2n+1} \setminus N_n^{2n+1} \subset M$) contains topologically all separable metric spaces which have dimension $\leq n$.

Hint. Apply the category method. Consider a compact space X such that $0 \leq \dim X \leq n$ and the function space $(R^{2n+1})^X$. Use Theorem 1.11.4 and show that for every compact set $F \subset R^{2n+1}$ contained in N_n^{2n+1} the set of all continuous mappings $f: X \to R^{2n+1}$ whose values miss F is dense in $(R^{2n+1})^X$. To that end, first approximate f by a continuous mapping to an n-dimensional polyhedron $K \subset R^{2n+1}$ (see Theorem 1.10.6) and then sweep out K from all m-dimensional faces, where $m > n$, of a decomposition of R^{2n+1} into small cubes whose n-dimensional faces are disjoint from N^{2n+1}.

Remark. The result in (c) is implicitly contained in Geoghegan and Summerhill [1974] and Dijkstra [1985] where some related results can also be found.

1.11.G. Recall that the Jordan curve theorem states that every simple closed curve in the plane R^2 separates R^2 into two regions of which it is the common boundary (see Kuratowski's book [1968], p. 510). Recall also that by a *θ-curve* we understand any subset of the plane R^2 that can be represented as the union $L_0 \cup L_1 \cup L_2$ of three arcs that have only their end-points in common.

(a) Show that if $C \subset R^2$ is a simple closed curve and S is a component of $R^2 \setminus C$, then for every pair of points $x, y \in C$ and every $\varepsilon > 0$ there exist points $x', y' \in C$ and an arc L with end-points x', y' such that $\rho(x, x') < \varepsilon$, $\rho(y, y') < \varepsilon$ and $L \subset S \cup \{x', y'\}$.

(b) Show that from the Jordan curve theorem it follows that every θ-curve $L_0 \cup L_1 \cup L_2$ separates R^2 into three regions D_0, D_1, D_2 in such a way that $\operatorname{Fr} D_0 = L_0 \cup L_1$, $\operatorname{Fr} D_1 = L_1 \cup L_2$ and $\operatorname{Fr} D_2 = L_0 \cup L_2$ (this is the *θ-curve theorem*).

Hint. Denote by a and b the common end-points of the arcs L_0, L_1 and L_2, and by T the θ-curve $L_0 \cup L_1 \cup L_2$. Note that $L_0 \cup L_1$, $L_1 \cup L_2$ and $L_0 \cup L_2$ are simple closed curves, and denote by E_0 the component of $R^2 \setminus (L_0 \cup L_1)$ which contains $L_2 \setminus \{a, b\}$, and by D_0 the other component; define in a similar way the regions E_1, D_1, E_2 and D_2. Check that the connected sets D_0, D_1 and D_2 are open-and-closed in $R^2 \setminus T$, so that they are pairwise disjoint, and it remains to prove that their union equals $R^2 \setminus T$.

Suppose, on the contrary, that $E = E_0 \cap E_1 \cap E_2 \neq \emptyset$ and find an arc $L_x \subset E \cup \{x_0, x_1\}$ with end-points x_0, x_1, where $x_i \in L_i \setminus \{a, b\}$ for $i = 0, 1$. Then, applying part (a), find an arc $L_y \subset D_0 \cup \{y_0, y_1\}$ with end-points y_0, y_1, where $y_i \in L_i \setminus \{a, b, x_i\}$ for $i = 0, 1$, and consider the simple closed curve $C \subset E \cup D_0 \cup (T \setminus \{a, b\})$ consisting of L_x, L_y and the arcs with end-points x_0, y_0 and x_1, y_1, contained respectively in L_0 and L_1. Let U and V be the two components of $R^2 \setminus C$, with $a \in U$. Note that $V \cap D_0 \neq \emptyset \neq V \cap E_0$

and deduce that $V \cap (L_0 \cup L_1) \neq \emptyset$. Applying part (a), show that if $V \cap L_0 \neq \emptyset$, then $C \cap D_2 \neq \emptyset$, and if $V \cap L_1 \neq \emptyset$, then $C \cap D_1 \neq \emptyset$; check that in both cases we have a contradiction with the inclusion $C \subset E \cup D_0 \cup T$.

1.11.H (Flores [1935]). (a) Let $p_0, p_1, \ldots, p_{2n+2}$ be the vertices of a regular $(2n+2)$-simplex T^{2n+2} inscribed in the $(2n+1)$-sphere $S^{2n+1} \subset R^{2n+2}$ and let $q_i = -p_i$ for $i = 0, 1, \ldots, 2n+2$. Check that for each sequence $i_0 < i_1 < \ldots < i_{2n+1} \leq 2n+2$ of $2n+2$ non-negative integers, the system of points $p_{i_0}, p_{i_1}, \ldots, p_{i_n}, q_{i_{n+1}}, \ldots, q_{i_{2n+1}} \in R^{2n+2}$ is affinely independent and the $(2n+1)$-dimensional affine subspace of R^{2n+2} spanned by these points does not contain the origin. Denote by S_1^{2n+1} the union of all $(2n+1)$-simplexes of the form $p_{i_0} p_{i_1} \ldots p_{i_n} q_{i_{n+1}} \ldots q_{i_{2n+1}}$ and show that the central projection p of S_1^{2n+1} from the origin onto S^{2n+1} is a homeomorphism. Observe that $p(-x) = -p(x)$ for every $x \in S_1^{2n+1}$.

(b) Let $\mathcal{K}_n$ be the family of all faces of T^{2n+2} which have dimension $\leq n$ and let $K_n = |\mathcal{K}_n|$. Consider the cone $C(K_n)$ over K_n with vertex at the origin, i.e., the subset of R^{2n+2} consisting of all points of the form tx, where $x \in K_n$ and $0 \leq t \leq 1$, and the subspace S_2^{2n+1} of the Cartesian product $C(K_n) \times C(K_n)$ consisting of all pairs (x, ty) and (tx, y), where $0 \leq t \leq 1$, $x \in T_1$, $y \in T_2$ and T_1, T_2 are disjoint members of $\mathcal{K}_n$. Show that by mapping, in a linear way, every segment with end-points $(x, 0)$ and (x, y) contained in S_2^{2n+1} onto the segment with end-points x and $\frac{1}{2}(x-y)$ contained in S_1^{2n+1}, and every segment with end-points $(0, y)$ and (x, y) contained in S_2^{2n+1} onto the segment with end-points $-y$ and $\frac{1}{2}(x-y)$ contained in S_1^{2n+1}, one obtains a homeomorphism h of S_2^{2n+1} onto S_1^{2n+1}. Observe that $h(x, y) = -h(y, x)$.

(c) Applying the Borsuk–Ulam antipodal theorem, i.e., the fact that for every continuous mapping $g: S^n \to R^n$ there exists a point $x \in S^n$ such that $g(x) = g(-x)$ (see Spanier [1966], p. 266), prove that the n-dimensional polyhedron K_n defined in (b) cannot be embedded in R^{2n} for any $n \geq 1$.

Hint. Assume that there exists a homeomorphic embedding $f: K_n \to R^{2n}$. Observe that f determines a homeomorphic embedding $f_1: C(K_n) \to R^{2n+1}$, consider the mapping $f_2: S_2^{2n+1} \to R^{2n+1}$ defined by letting $f_2(x, y) = f_1(y) - f_1(x)$ and the composition $g = f_2 h^{-1} p^{-1}: S^{2n+1} \to R^{2n+1}$.

1.12. Dimension and mappings

We shall now study the relations between the dimensions of the domain and range of a continuous mapping. Let us begin with the observation that since one-to-one continuous onto mappings can arbitrarily raise or lower the dimension (see Problems 1.3.D(b) and 1.4.F(b)), we have to restrict ourselves to special classes of mappings. It turns out that for closed mappings and open mappings many interesting results can be established.

Let us recall that a continuous mapping $f: X \to Y$ is *closed* (*open*) if for every closed (open) set $A \subset X$, the image $f(A)$ is closed (open) in Y. One readily checks that if $f: X \to Y$ is a closed (an open) mapping, then for every closed (open) subset A of X the restriction $f|A: A \to Y$ is closed (open); similarly, if $f: X \to Y$ is a closed (an open) mapping, then for every subset B of Y the restriction $f_B: f^{-1}(B) \to B$ is closed (open).

Clearly, a mapping $f: X \to Y$ is closed if and only if $f(\bar{A}) = \overline{f(A)}$ for every $A \subset X$, and each continuous mapping of a compact space to a Hausdorff space is closed.

We shall first discuss closed mappings and begin with the theorem on dimension-raising mappings. In the lemma to this theorem the notion of a network appears; a family $\mathcal{N}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and each neighbourhood U of the point x there exists an $M \in \mathcal{N}$ such that $x \in M \subset U$. The definition of a network mimicks the definition of a base, the only difference being that one does not require the members of a network to be open sets. Clearly, every base for a topological space is a network for that space. The family of all one-point subsets is another example of a network.

1.12.1. LEMMA. *A separable metric space X satisfies the inequality* $\operatorname{ind} X \leq n \geq 0$ *if and only if X has a countable network $\mathcal{N}$ such that* $\operatorname{ind} \operatorname{Fr} M \leq n-1$ *for every $M \in \mathcal{N}$.*

PROOF. By virtue of Theorem 1.1.6, it suffices to show that if a separable metric space X has a network $\mathcal{N} = \{M_i\}_{i=1}^{\infty}$ such that $\operatorname{ind} \operatorname{Fr} M_i \leq n-1$ for $i = 1, 2, \ldots$, then $\operatorname{ind} X \leq n$. Let $Y = \bigcup_{i=1}^{\infty} \operatorname{Fr} M_i$, and $Z = X \setminus Y$. It follows from the sum theorem that $\operatorname{ind} Y \leq n-1$; we shall show that $\operatorname{ind} Z \leq 0$. For every point $x \in Z$ and each neighbourhood $V \subset X$ of the point x there exists an $M_i \in \mathcal{N}$ such that $x \in M_i \subset V$. Since $x \in X \setminus Y \subset X \setminus \operatorname{Fr} M_i$, we have $x \in U = \operatorname{Int} M_i \subset V$. The inclusion $\operatorname{Fr} \operatorname{Int} M_i \subset \operatorname{Fr} M_i$ implies that $Z \cap \operatorname{Fr} U = \emptyset$ and thus we have $\operatorname{ind} Z \leq 0$. The inequality $\operatorname{ind} X \leq n$ now follows from Lemma 1.5.2. ■

1.12.2. THEOREM ON DIMENSION-RAISING MAPPINGS. *If $f: X \to Y$ is a closed mapping of a separable metric space X onto a separable metric space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then* $\operatorname{ind} Y \leq \operatorname{ind} X + (k-1)$.

PROOF. We can suppose that $0 \leq \operatorname{ind} X < \infty$. We shall apply induction with respect to the number $n + k$, where $n = \operatorname{ind} X$. If $n + k = 1$, we have $k = 1$, so that f is a homeomorphism and the theorem holds. Assume that it holds whenever $n + k < m \geq 2$ and consider a closed mapping $f: X \to Y$ such that $f(X) = Y$ and $n + k = m$.

Let $\mathcal{B}$ be a countable base for X such that $\operatorname{ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$. Consider an arbitrary $U \in \mathcal{B}$; by the closedness of f we have

$$\begin{aligned} \operatorname{Fr} f(U) &= \overline{f(U)} \cap \overline{Y \setminus f(U)} \subset f(\bar{U}) \cap f(X \setminus U) \\ &= [f(U) \cup f(\operatorname{Fr} U)] \cap f(X \setminus U) \subset f(\operatorname{Fr} U) \cup B, \end{aligned} \tag{1}$$

where $B = f(U) \cap f(X \setminus U)$. Since the restriction $f | \operatorname{Fr} U: \operatorname{Fr} U \to f(\operatorname{Fr} U)$ is a closed mapping, it follows from the inductive assumption that

$$\operatorname{ind} f(\operatorname{Fr} U) \leq (n-1) + (k-1) = n + k - 2.$$

Assume that $B \neq \emptyset$. Consider the restriction $f_B: f^{-1}(B) \to B$ and the restriction $f' = f_B | (X \setminus U): (X \setminus U) \cap f^{-1}(B) \to B$; both f_B and f' are closed, and the fibres of f' have cardinality $\leq k-1$, because $f^{-1}(y) \cap U \neq \emptyset$ for every $y \in B$. It follows from the inductive assumption that

$$\operatorname{ind} B \leq n + (k-1) - 1 = n + k - 2.$$

As U is an F_σ-set in X, both $f(U)$ and B are F_σ-sets in Y; applying Corollary 1.5.4, we obtain the inequality $\operatorname{ind}[f(\operatorname{Fr} U) \cup B] \leq n + k - 2$. From the last inequality and from

(1) it follows that $\operatorname{ind}\operatorname{Fr} f(U) \leq n+k-2$ for every $U \in \mathcal{B}$; the same inequality holds if $B = \emptyset$. One readily checks that the family $\mathcal{N} = \{f(U) : U \in \mathcal{B}\}$ is a network for the space Y, so that $\operatorname{ind} Y \leq n+k-1 = \operatorname{ind} X + k - 1$ by virtue of Lemma 1.12.1. ■

We now pass to the theorem on dimension-lowering mappings. The theorem will be preceded by a lemma which, roughly speaking, shows that in condition (MU2), in the definition of the dimension function ind, points can be replaced by closed sets of small dimension.

1.12.3. LEMMA. *If a separable metric space X has a closed cover $\{A_s\}_{s\in S}$ such that $\operatorname{ind} A_s \leq m \geq 0$ for each $s \in S$ and if for every $s \in S$ and each open set $V \subset X$ that contains A_s there exists an open set $U \subset X$ such that*

$$A_s \subset U \subset \overline{U} \subset V \quad \text{and} \quad \operatorname{ind}\operatorname{Fr} U \leq m-1,$$

then $\operatorname{ind} X \leq m$.

PROOF. By virtue of Theorem 1.9.3 it suffices to show that for every closed subspace A of the space X and each continuous mapping $f\colon A \to S^m$ there exists a continuous extension $F\colon X \to S^m$ of f over X. It follows from Theorem 1.9.2 that for each $s \in S$ the mapping f is continuously extendable over $A \cup A_s$, so that there exists an open set $V_s \subset X$ containing $A \cup A_s$ such that f is continuously extendable over V_s. Consider an open set $U_s \subset X$ satisfying

$$A_s \subset U_s \subset \overline{U}_s \subset V_s \quad \text{and} \quad \operatorname{ind}\operatorname{Fr} U_s \leq m-1; \tag{2}$$

obviously, f is continuously extendable over $A \cup \overline{U}_s$. The open cover $\{U_s\}_{s\in S}$ of the space X has a countable subcover $\{U_{s_j}\}_{j=1}^{\infty}$. We shall inductively define a sequence $F_1, F_2, \ldots$ of continuous mappings, where $F_i : A \cup \bigcup_{j=1}^{i} \overline{U}_{s_j} \to S^m$, such that

$$F_i\Big|\Big(A \cup \bigcup_{j=1}^{i-1} \overline{U}_{s_j}\Big) = F_{i-1} \quad \text{for } i > 1. \tag{3}$$

Let F_1 be an arbitrary continuous extension of f over $A \cup \overline{U}_{s_1}$. Assume that the mappings F_i satisfying (3) are defined for $i < k$. The set $A \cup \bigcup_{j=1}^{k} \overline{U}_{s_j}$ can be represented as the union of two closed sets

$$A' = A \cup \bigcup_{j=1}^{k-1} \overline{U}_{s_j} \quad \text{and} \quad A'' = A \cup \Big(\overline{U}_{s_k} \setminus \bigcup_{j=1}^{k-1} U_{s_j}\Big).$$

The mapping f is extendable to a continuous mapping $f''\colon A'' \to S^m$ and

$$D = \{x \in A' \cap A'' : F_{k-1}(x) \neq f''(x)\} \subset \bigcup_{j=1}^{k-1} \overline{U}_{s_j} \cap \Big(X \setminus \bigcup_{j=1}^{k-1} U_{s_j}\Big) \subset \bigcup_{j=1}^{k-1} \operatorname{Fr} U_{s_j},$$

so that $\operatorname{ind} D \leq m-1$ by virtue of (2) and the sum theorem. It follows from Lemma 1.9.6 that the mappings $F_{k-1}|A' \cap A''$ and $f''|A' \cap A''$ are homotopic. Since the latter is continuously extendable over A'', it follows from Lemma 1.9.7 that $F_{k-1}|A' \cap A''$ is extendable to a continuous mapping $F''\colon A'' \to S^m$.

Letting

$$F_k(x) = \begin{cases} F_{k-1}(x) & \text{for } x \in A', \\ F''(x) & \text{for } x \in A'', \end{cases}$$

we define a continuous mapping F_k of $A' \cup A'' = A \cup \bigcup_{j=1}^{k} \overline{U}_{s_j}$ to S^m which satisfies (3) for $i = k$.

As $X = \bigcup_{i=1}^{\infty} U_{s_i}$, the formula

$$F(x) = F_i(x) \quad \text{for } x \in U_{s_i}$$

defines a continuous mapping $F: X \to S^m$ which is the required extension of f over X. ■

1.12.4. THEOREM ON DIMENSION-LOWERING MAPPINGS. *If $f: X \to Y$ is a closed mapping of a separable metric space X to a separable metric space Y and there exists an integer $k \geq 0$ such that $\operatorname{ind} f^{-1}(y) \leq k$ for every $y \in Y$, then $\operatorname{ind} X \leq \operatorname{ind} Y + k$.*

PROOF. We can suppose that $\operatorname{ind} Y < \infty$. We shall apply induction with respect to $n = \operatorname{ind} Y$. If $n = -1$, we have $Y = \emptyset$ and $X = \emptyset$, and so the theorem holds. Assume that it holds for closed mappings to spaces of dimension less than $n \geq 0$ and consider a closed mapping $f: X \to Y$ to a space Y such that $\operatorname{ind} Y = n$.

We shall show that the closed cover $\{f^{-1}(y)\}_{y \in Y}$ of the space X satisfies the conditions of Lemma 1.12.3 for $m = n + k$. Clearly, $\operatorname{ind} f^{-1}(y) \leq k \leq m$ for each $y \in Y$. Consider now a $y \in Y$ and an open set $V \subset X$ which contains $f^{-1}(y)$. The set $W = Y \setminus f(X \setminus V)$ is a neighbourhood of y; since $\operatorname{ind} Y = n$, there exists an open set $U' \subset Y$ such that

$$y \in U' \subset \overline{U'} \subset W \quad \text{and} \quad \operatorname{ind} \operatorname{Fr} U' \leq n - 1.$$

Applying the inductive assumption to the restriction $f_{\operatorname{Fr} U'}: f^{-1}(\operatorname{Fr} U') \to \operatorname{Fr} U'$, we obtain the inequality $\operatorname{ind} f^{-1}(\operatorname{Fr} U') \leq n + k - 1$. The open set $U = f^{-1}(U')$ satisfies the conditions

$$f^{-1}(y) \subset U \subset \overline{U} = \overline{f^{-1}(U')} \subset f^{-1}(\overline{U'}) \subset f^{-1}(W) \subset V$$

and

$$\operatorname{Fr} U = \operatorname{Fr} f^{-1}(U') = \overline{f^{-1}(U')} \setminus f^{-1}(U') \subset f^{-1}(\overline{U'}) \setminus f^{-1}(U') \subset f^{-1}(\operatorname{Fr} U'),$$

so that $\operatorname{ind} \operatorname{Fr} U \leq n+k-1$. Lemma 1.12.3 now implies that $\operatorname{ind} X \leq n+k = \operatorname{ind} Y + k$. ■

Let us note that in Theorem 1.12.4 the assumption that f is a closed mapping cannot be replaced by the assumption that f is open (see Problem 1.12.C). On the other hand, Theorem 1.12.2 holds for open mappings as well; we shall show below that even more is true: open mappings with finite – or, more generally, discrete – fibres do not change dimension. We shall also show that open mappings with countable fibres defined on locally compact spaces do not change dimension. In the proofs of both theorems the following lemma will be applied; we recall that the symbol A^{d} denotes the set of all accumulation points of the set A, i.e., the set of all points x such that $x \in \overline{A \setminus \{x\}}$.

1.12.5. LEMMA. *Let $f: X \to Y$ be an open mapping of a metric space X onto a metric space Y. For every base $\mathcal{B} = \{U_s\}_{s \in S}$ for the space X there exists a family $\{A_s\}_{s \in S}$ of subsets of X such that $A_s \subset U_s$ for each $s \in S$ and*

(i) *A_s and $f(A_s)$ are F_σ-sets in X and Y respectively,*

(ii) *$f|A_s: A_s \to f(A_s)$ is a homeomorphism,*

(iii) $X = (\bigcup_{s \in S} A_s) \cup (\bigcup_{y \in Y} [f^{-1}(y)]^{\mathrm{d}})$.

PROOF. For each $s \in S$ let

$$A_s = \{x \in U_s : U_s \cap f^{-1}f(x) = \{x\}\}.$$

Observe first that the set $f(U_s) \setminus f(A_s)$ is open in Y. Indeed, for every point $y \in f(U_s) \setminus f(A_s)$ there exist two distinct points $x_1, x_2 \in U_s$ such that $f(x_1) = f(x_2) = y$, and – as one readily checks – the open set $V = f(W_1) \cap f(W_2)$, where W_1, W_2 are disjoint open subsets of U_s which contain x_1 and x_2 respectively, contains the point y and is contained in the set $f(U_s) \setminus f(A_s)$. As $f(A_s) \subset f(U_s)$, we have

$$f(A_s) = f(U_s) \setminus [f(U_s) \setminus f(A_s)],$$

so that the set $f(A_s)$, being the intersection of an open set and a closed set, is an F_σ-set in Y. From the obvious equality

$$A_s = U_s \cap f^{-1}f(A_s) \tag{4}$$

it follows that A_s is an F_σ-set in X; thus (i) is proved.

The restriction $f|U_s: U_s \to Y$ is an open mapping, and so is its restriction $(f|U_s)_{f(A_s)}: U_s \cap f^{-1}f(A_s) \to f(A_s)$. By virtue of (4) the last mapping coincides with $f|A_s: A_s \to f(A_s)$. Thus the mapping $f|A_s: A_s \to f(A_s)$ is open and, by the definition of A_s, one-to-one, i.e., we have (ii).

To prove (iii), it suffices to observe that if $x \notin [f^{-1}f(x)]^{\mathrm{d}}$, then there exists a member U_s of the base $\mathcal{B}$ such that $U_s \cap f^{-1}f(x) = \{x\}$, and thus $x \in A_s$. ■

1.12.6. THEOREM. *If $f: X \to Y$ is an open mapping of a separable metric space X onto a separable metric space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point, then* $\operatorname{ind} Y \leq \operatorname{ind} X$.

PROOF. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base for the space X; consider a family $\{A_i\}_{i=1}^{\infty}$ of subsets of X which satisfy (i)–(iii) of Lemma 1.12.5. By the assumption on the fibres, $Y = \bigcup_{i=1}^{\infty} f(A_i)$, and since the subspaces $A_i \subset X$ and $f(A_i) \subset Y$ are homeomorphic, $\operatorname{ind} f(A_i) = \operatorname{ind} A_i \leq \operatorname{ind} X$ for $i = 1, 2, \ldots$ Hence we have $\operatorname{ind} Y \leq \operatorname{ind} X$ by virtue of Corollary 1.5.4. ■

It is not hard to see that an open mapping with an isolated point in each fibre can lower dimension; it is not so if the fibres are discrete.

1.12.7. THEOREM. *If $f: X \to Y$ is an open mapping of a separable metric space X onto a separable metric space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X, then* $\operatorname{ind} X = \operatorname{ind} Y$.

PROOF. Let $\{U_i\}_{i=1}^{\infty}$ be a countable base for the space X; consider a family $\{A_i\}_{i=1}^{\infty}$ of subsets of X which satisfy (i)–(iii) of Lemma 1.12.5. By the assumption on the fibres, $X = \bigcup_{i=1}^{\infty} A_i$, and since the subspaces $A_i \subset X$ and $f(A_i) \subset Y$ are homeomorphic, $\operatorname{ind} A_i = \operatorname{ind} f(A_i) \leq \operatorname{ind} Y$ for $i = 1, 2, \ldots$ Hence we have $\operatorname{ind} X \leq \operatorname{ind} Y$ by virtue of Corollary 1.5.4; to complete the proof it suffices to apply Theorem 1.12.6. ■

1.12.8. ALEXANDROFF'S THEOREM. *If $f: X \to Y$ is an open mapping of a separable locally compact metric space X onto a separable metric space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then* $\operatorname{ind} X = \operatorname{ind} Y$.

PROOF. Since each closed subspace of a locally compact space is locally compact, the fibres of f are locally compact. It follows from the Baire category theorem and complete metrizability of locally compact spaces (or from Problem 1.2.D(a)) that for every $y \in Y$ the fibre $f^{-1}(y)$, being countable, has an isolated point. Hence we have $\operatorname{ind} Y \leq \operatorname{ind} X$ by Theorem 1.12.6.

Let $\{F_i\}_{i=1}^{\infty}$ be a countable cover of X consisting of compact subspaces. For every i the restriction $f|F_i : F_i \to f(F_i)$ is a closed mapping with zero-dimensional fibres, so that $\operatorname{ind} F_i \leq \inf f(F_i) \leq \operatorname{ind} Y$ by virtue of the theorem on dimension-lowering mappings. Hence we have $\operatorname{ind} X \leq \operatorname{ind} Y$ by the sum theorem. ■

It is possible to define open mappings with countable fibres which arbitrarily raise or lower dimension (see Problems 1.12.E and 1.12.F, and Problem 1.12.G(c)). Open mappings with zero-dimensional fibres defined on compact spaces can also arbitrarily raise dimension, but examples of such mappings are much more complicated and will not be discussed here. Let us note, however, that the space M_1^3 defined in the last section can be mapped onto every locally connected metric continuum by an open mapping whose fibres are all homeomorphic to the Cantor set.

We conclude this section with a discussion of *open-and-closed* mappings, i.e., mappings which are both open and closed. It follows from the last paragraph that open-and-closed mappings with zero-dimensional fibres can arbitrarily raise dimension. We shall now show that open-and-closed mappings with countable fibres preserve dimension.

1.12.9. VAĬNŠTEĬN'S LEMMA. *If $f: X \to Y$ is a closed mapping of a metric space X onto a metric space Y, then for every $y \in Y$ the boundary $\operatorname{Fr} f^{-1}(y)$ of the fibre $f^{-1}(y)$ is a compact subspace of X.*

PROOF. It suffices to show that every countably infinite subset $A = \{x_1, x_2, \ldots\}$ of the boundary $\operatorname{Fr} f^{-1}(y)$ of an arbitrary fibre $f^{-1}(y)$ has an accumulation point. Let $\{V_i\}_{i=1}^{\infty}$ be a base for the space Y at the point y. For $i = 1, 2, \ldots$ choose a point $x_i' \in f^{-1}(V_i) \setminus f^{-1}(y)$ satisfying $\rho(x_i, x_i') < 1/i$; such a choice is possible, because the intersection $B(x_i, 1/i) \cap f^{-1}(V_i)$ is a neighbourhood of x_i. Consider the set $B = \{x_1', x_2', \ldots\} \subset X$. We have $y \in \overline{f(B)} \setminus f(B)$ so that $\overline{B} \neq B$, i.e., $B^{\mathrm{d}} \neq \emptyset$. Now, since $\rho(x_i, x_i') < 1/i$ for $i = 1, 2, \ldots$, $A^{\mathrm{d}} = B^{\mathrm{d}} \neq \emptyset$ and the proof is complete. ■

1.12.10. THEOREM. *If $f: X \to Y$ is an open-and-closed mapping of a separable metric space X onto a separable metric space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{ind} X = \operatorname{ind} Y$.*

PROOF. Let us note first that Theorem 1.12.4 yields the inequality $\operatorname{ind} X \leq \operatorname{ind} Y$.

Now, let $Y_0 = \{y \in Y : \operatorname{Int} f^{-1}(y) \neq \emptyset\}$, $Y_1 = Y \setminus Y_0$, and $X_1 = f^{-1}(Y_1)$. The mapping f being open, for every $y \in Y_0$ the one-point set $\{y\} = f(\operatorname{Int} f^{-1}(y))$ is open, so that the set Y_0 is open in Y and $\operatorname{ind} Y_0 \leq 0$. The restriction $f_1 = f_{Y_1}: X_1 \to Y_1$ is an open-and-closed mapping, and since $f_1^{-1}(y) = \operatorname{Fr} f^{-1}(y) \subset f^{-1}(y)$, the fibres of f_1 are countable and compact, the latter by virtue of Vaĭnšteĭn's lemma; in particular, for every $y \in Y$ the fibre $f_1^{-1}(y)$ has an isolated point (see Problem 1.2.D(a)). It then

follows from Theorem 1.12.6 that $\operatorname{ind} Y_1 \leq \operatorname{ind} X_1 \leq \operatorname{ind} X$; as $Y = Y_0 \cup Y_1$, Corollary 1.5.5 yields the inequality $\operatorname{ind} Y \leq \operatorname{ind} X$. ■

Historical and bibliographic notes

Theorem 1.12.2 was proved by Hurewicz in [1927a]. The same paper contains Theorem 1.12.4 for continuous mappings defined on compact metric spaces; the extension of this theorem to separable metric spaces was given by Hurewicz and Wallman in [1941]. Theorem 1.12.6 was first stated by Taĭmanov in [1955]; it is implicitly contained in Alexandroff's paper [1936]. Theorem 1.12.7 was proved by Hodel in [1963] (for the special case of a mapping with finite fibres it was proved earlier by Nagami in [1960]). Alexandroff established Theorem 1.12.8 in [1936]. The first example of a dimension-raising open mapping with zero-dimensional fibres defined on a compact metric space was described by Kolmogoroff in [1937]. Keldyš defined in [1954] an open mapping with zero-dimensional fibres which maps a one-dimensional compact metric space onto the square I^2; a detailed description of Keldyš' example can be found in Alexandroff and Pasynkov's book [1973]. Simpler examples of open mappings with zero-dimensional fibres which map m-dimensional compact metric spaces onto n-dimensional polyhedra for all $0 < m < n$ were constructed by Kozlovskiĭ in [1986]. The fact that M_1^3 can be mapped onto every locally connected metric continuum by an open mapping whose fibres are all homeomorphic to the Cantor set was established by Wilson in [1972]. Theorem 1.12.10 was given by Vaĭnšteĭn in [1949]. Lelek's paper [1971] contains a comprehensive discussion of the topic of the present section and an extensive bibliography. We shall return to this subject in Section 4.3, where two theorems of a more special character will be proved (see Theorems 4.3.7 and 4.3.10).

Problems

1.12.A (Hurewicz [1926]). Prove that if $f: X \to Y$ is a closed mapping of a separable metric space X onto a separable metric space Y and $|f^{-1}(y)| = k < \infty$ for every $y \in Y$, then $\operatorname{ind} X = \operatorname{ind} Y$.

Hint. Consider a countable base $\mathcal{B}$ for the space X and the family of all intersections $\bigcap_{i=1}^{k} f(\overline{U}_i)$, where $U_i \in \mathcal{B}$ for $i = 1, 2, \ldots, k$ and $\overline{U}_i \cap \overline{U}_j = \emptyset$ whenever $i \neq j$.

1.12.B (Hurewicz [1937]). Observe that, under the additional hypothesis that X is a finite-dimensional compact space, the theorem on dimension-lowering mappings is a direct consequence of the Cantor-manifold theorem.

Hint. Assume that X is a Cantor-manifold and apply induction with respect to $\operatorname{ind} Y$.

1.12.C. Give an example of an open mapping with zero-dimensional fibres which maps a one-dimensional separable metric space onto the Cantor set.

Hint. Use the Knaster–Kuratowski fan.

Remark. There even exist such mappings with countable fibres (cf. Problem 1.12.G(c)).

1.12.D. (a) Observe that in Theorem 1.12.2 the inequality $|f^{-1}(y)| \leq k$ can be replaced by the weaker inequality $|\operatorname{Fr} f^{-1}(y)| \leq k$.

Hint. Consider the restriction $f|X_1$, where X_1 is obtained by adjoining to the union $\bigcup_{y \in Y} \operatorname{Fr} f^{-1}(y)$ one point from each fibre $f^{-1}(y)$ which has an empty boundary.

(b) Observe that in Theorem 1.12.6 it suffices to assume that for every $y \in Y$ the boundary $\operatorname{Fr} f^{-1}(y)$ either has an isolated point or is empty.

1.12.E (Hausdorff [1934]). Show that for every separable metric space X there exists a subspace Z of the space P of irrational numbers that can be mapped onto X by an open mapping.

Hint. Consider a countable base $\{U_k\}_{k=1}^{\infty}$ for the space X and the subspace Z of the Cartesian product $N^{\aleph_0} = \prod_{i=1}^{\infty} N_i$, where $N_i = N$ for $i = 1, 2, \ldots$, consisting of all sequences $\{k_i\}$ such that the family $\{U_{k_i}\}_{i=1}^{\infty}$ is a base for X at some point x; assign the point x to the sequence $\{k_i\}$ (see Proposition 1.3.13).

1.12.F (Roberts [1947]). Show that for every open mapping $f: X \to Y$ of a separable metric space X onto a separable metric space Y there exists a set $X_1 \subset X$ such that the restriction $f|X_1$ is an open mapping of X_1 onto Y and has countable fibres.

Hint. Consider a countable base $\{U_i\}_{i=1}^{\infty}$ for the space X and choose one point from each non-empty intersection of the form $U_i \cap f^{-1}(y)$, where $y \in X$.

1.12.G. (a) Observe that if $f: X \to Y$ is an open mapping of a complete separable metric space X onto a separable metric space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{ind} Y \leq \operatorname{ind} X$.

(b) Prove that for every continuous mapping f of a separable metric space X onto the Cantor set C there exist a separable metric space Y containing X and an open mapping $F: Y \to C$ extending f such that $Y \setminus X$ can be represented as the union of countably many open subspaces $C_1, C_2, \ldots$ homeomorphic to C on each of which F is a homeomorphic embedding; show that if, moreover, X is complete (compact) then there exists such a space Y which is also complete (compact).

Hint. For every sequence $i_1, i_2, \ldots, i_m$ consisting of zeros and ones, let

$$C(i_1, i_2, \ldots, i_m) = \left\{ x \in C : \sum_{k=1}^{m} \frac{2i_k}{3^k} \leq x \leq \sum_{k=1}^{m} \frac{2i_k}{3^k} + \frac{1}{3^m} \right\},$$

and let $C'(i_1, i_2, \ldots, i_m)$ be a subspace of I which is homeomorphic to the Cantor set and is contained in the interval

$$\left(\sum_{k=1}^{m} \frac{2i_k}{3^k} + \frac{1}{3^{m+1}}, \sum_{k=1}^{m} \frac{2i_k}{3^k} + \frac{2}{3^{m+1}} \right)$$

removed from I in the process of constructing the Cantor set. Take a countable set $\{a_1, a_2, \ldots\}$ dense in X and define

$$Y = X' \cup \bigcup_{m=1}^{\infty} \bigcup_{i_1, \ldots, i_m = 0,1} \bigcup_{k=1}^{m} [\{a_k\} \times C'(i_1, i_2, \ldots, i_m)] \subset X \times I,$$

where $X' = \{(x, f(x)) : x \in X\} \subset X \times C$ is the graph of f homeomorphic to the space X. Then define $F: Y \to C$ by letting $F(x, f(x)) = f(x)$ for $(x, f(x)) \in X'$ and mapping

each of the sets $\{a_k\} \times C'(i_1, i_2, \ldots, i_m)$ homeomorphically onto $C(i_1, i_2, \ldots, i_m)$. Check that if X is complete, then Y is also complete, and that if X is compact, then there exists a compact $Y' \subset Y$ containing X' such that $F'|Y'$ is open.

(c) Give an example of an open mapping with countable fibres which maps a one-dimensional complete separable metric space onto the Cantor set.

(d) For every natural number $n \geq 2$ give an example of an open mapping with countable fibres which maps an n-dimensional separable metric space onto the Cantor set.

Hint. Follow the hint to part (b) and use the space in Problem 1.4.F(b).

1.12.H. (a) (Hurewicz [1933] (announcement [1931])). Prove that for every separable metric space X such that $\dim X \leq n \geq 1$ there exists a continuous mapping of X to R^n with all fibres zero-dimensional.

Hint. One can assume that X is compact. Consider the function space $(R^n)^X$ and prove that for $j = 1, 2, \ldots$ its subset Φ_j, consisting of all mappings $f: X \to R^n$ such that each fibre of f can be represented as the union of finitely many disjoint closed subsets of diameter less than $1/j$, is open and dense. To prove that Φ_j is open, show that for every $f \in \Phi_j$ there exists an $\epsilon > 0$ such that for each closed set $A \subset R^n$ with diameter less than ϵ the inverse image $f^{-1}(A)$ can be represented as the union of finitely many disjoint closed subsets of diameter less than $1/j$. To prove that Φ_j is dense, for a continuous mapping $f: X \to R^n$ and an $\epsilon > 0$ take an open finite cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of X and a system of points $p_1, p_2, \ldots, p_k \in R^n$ in general position such that $\operatorname{ord} \mathcal{U} \leq n$, $\delta(U_i) < 1/j$ and $\delta(\{p_i\} \cup f(U_i)) < \epsilon$ for $i = 1, 2, \ldots, k$. Then show that the corresponding κ-mapping $\kappa: X \to R^n$ belongs to Φ_j; consider a nerve $\mathcal{N}(\mathcal{U})$ with vertices $p_1', p_2', \ldots, p_k'$, the κ-mapping $\kappa': X \to N(\mathcal{U})$, and define a continuous mapping with finite fibres $g: N(\mathcal{U}) \to R^n$ such that $g\kappa' = \kappa$.

(b) Note that a compact metric space X satisfies the inequality $\dim X \leq n$ if and only if there exists a continuous mapping of X to R^n with all fibres zero-dimensional.

(c) Show that for the one-dimensional space $X = I \times Q$, where Q is the space of rational numbers, there is no closed mapping of X to R with all fibres zero-dimensional.

Hint. Assume that $f: X \to R$ is such a mapping. First note that $\operatorname{Int} f^{-1}(y) = \emptyset$ for each $y \in R$, and deduce from Vaĭnšteĭn's lemma that $f^{-1}(y)$ is compact. Then show that for each compact set $A \subset R$ the inverse image $f^{-1}(A)$ is compact (cf. [GT], Theorem 3.7.18) and obtain a contradiction.

1.13. Dimension and inverse sequences of polyhedra

Besides theorems on ϵ-mappings and ϵ-translations there is one more characterization of dimension in the realm of compact metric spaces through their relations to polyhedra. It has been discovered that the class of compact metric spaces which have dimension $\leq n$ coincides with the class of spaces which are homeomorphic to the limits of inverse sequences of polyhedra of dimension $\leq n$. We shall deduce this characteri-

zation from two theorems which we are now going to prove: the theorem on expansion in an inverse sequence and the theorem on the dimension of the limit of an inverse sequence.

As the subject of this section is more specific than our previous considerations, we shall assume here that the reader is familiar with the basic definitions and theorems in the theory of inverse systems (see, e.g., [GT], Section 2.5 and pp. 141–142). We shall only recall that an inverse sequence $\{X_i, \pi_j^i\}$ is an inverse system $\{X_i, \pi_j^i, N\}$ where N is the set of natural numbers directed by its natural order.

In Lemma 1.13.1 and in the proof of Theorem 1.13.2 below the symbol T^m will denote the m-simplex in R^{m+1} spanned by the points $p_1 = (1, 0, \ldots, 0)$, $p_2 = (0, 1, \ldots, 0)$, $\ldots$, $p_{m+1} = (0, 0, \ldots, 1)$. We shall consider on T^m the metric σ defined by letting

$$\sigma(x, y) = |x - y|, \quad \text{where } |z| = \sum_{i=1}^{m+1} |\lambda_i| \quad \text{for } z = (\lambda_1, \lambda_2, \ldots, \lambda_{m+1}) \in R^{m+1};$$

obviously, the metric σ is equivalent to the natural metric on T^m.

1.13.1. LEMMA. *Let a set $A_j \subset \{1, 2, \ldots, n+1\}$ be given for $j = 1, 2, \ldots, l$ and let π be the mapping of T^{l-1} to $T^n = p_1 p_2 \ldots p_{n+1}$ defined by the formula*

$$\pi((\lambda_1, \lambda_2, \ldots, \lambda_l)) = \sum_{j=1}^{l} \frac{\lambda_j}{n_j} \sum_{i \in A_j} p_i, \quad \text{where } n_j = |A_j|.$$

If $A_1 \cap A_2 \cap \ldots \cap A_l \neq \emptyset$, then $\sigma(\pi(x), \pi(y)) \leq \frac{n}{n+1}\sigma(x, y)$ for all $x, y \in T^{l-1}$.

PROOF. Consider arbitrary points $x = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $y = (\mu_1, \mu_2, \ldots, \mu_l)$ in T^{l-1}. Let $\alpha_j = \lambda_j - \mu_j$ for $j = 1, 2, \ldots, l$, $B = \{j : \alpha_j \geq 0\}$ and $C = \{j : \alpha_j < 0\}$. From the equality $\sum_{j=1}^{l} \alpha_j = 0$ it follows that $\sum_{j \in B} |\alpha_j| = \sum_{j \in C} |\alpha_j| = \frac{1}{2}\sigma(x, y)$. One easily checks that

$$\left| \sum_{j=1}^{l} \frac{\alpha_j}{n_j} \right| = \sum_{j=1}^{l} \frac{|\alpha_j|}{n_j} - 2\min(|b|, |c|), \quad \text{where } b = \sum_{j \in B} \frac{\alpha_j}{n_j} \text{ and } c = \sum_{j \in C} \frac{\alpha_j}{n_j}.$$

Since $n_j \leq n + 1$ for $j = 1, 2, \ldots, l$, we have

$$|b| = \sum_{j \in B} \frac{|\alpha_j|}{n_j} \geq \frac{1}{n+1} \sum_{j \in B} |\alpha_j| = \frac{1}{2} \frac{\sigma(x, y)}{n+1}; \quad \text{similarly} \quad |c| \geq \frac{1}{2} \frac{\sigma(x, y)}{n+1}.$$

Choosing arbitrarily an $i_0 \in A_1 \cap A_2 \cap \ldots \cap A_l$ and letting $A_j' = A_j \setminus \{i_0\}$ for $j = 1, 2, \ldots, l$ we have

$$\begin{aligned}
\sigma(\pi(x), \pi(y)) &= \left| \sum_{j=1}^{l} \frac{\alpha_j}{n_j} \sum_{i \in A_j} p_i \right| = \left| \left(\sum_{j=1}^{l} \frac{\alpha_j}{n_j} \right) p_{i_0} + \sum_{j=1}^{l} \frac{\alpha_j}{n_j} \sum_{i \in A_j'} p_i \right| \\
&\leq \left| \sum_{j=1}^{l} \frac{\alpha_j}{n_j} \right| + \sum_{j=1}^{l} \frac{|\alpha_j|(n_j - 1)}{n_j} \\
&= \sum_{j=1}^{l} \frac{|\alpha_j|}{n_j} + \sum_{j=1}^{l} \frac{|\alpha_j|(n_j - 1)}{n_j} - 2\min(|b|, |c|) = \sum_{j=1}^{l} |\alpha_j| - 2\min(|b|, |c|) \\
&= \sigma(x, y) - 2\min(|b|, |c|) \leq \sigma(x, y) - \frac{\sigma(x, y)}{n+1} = \frac{n}{n+1}\sigma(x, y). \blacksquare
\end{aligned}$$

1.13.2. THEOREM ON EXPANSION IN AN INVERSE SEQUENCE. *For every compact metric space X such that $\dim X \leq n$ there exists an inverse sequence $\{K_i, \pi_j^i\}$ of polyhedra of dimension $\leq n$ whose limit is homeomorphic to X; moreover, one can assume that each K_i is the underlying polyhedron of a nerve $\mathcal{K}_i$ of a finite open cover of the space X, and that for every $j \leq i$ the bonding mapping π_j^i is an affine transformation on each simplex in $\mathcal{K}_i$.*

PROOF. We can suppose that $\dim X \geq 0$. Consider a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of finite open covers of the space X, where $\mathcal{U}_i = \{U_{i,k}\}_{k=1}^{k_i}$ and $U_{i,k} \neq \emptyset$ for $k = 1, 2, \ldots, k_i$, such that

$$\operatorname{ord} \mathcal{U}_i \leq n \tag{1}$$

and

$$\operatorname{mesh} \mathcal{U}_{i+1} \leq \min(1/i + 1, \epsilon_i/2), \tag{2}$$

where ϵ_i is a Lebesgue number of the cover $\mathcal{U}_i$. It follows from (2) that

(3) if $U_{i+1,j_1} \cap U_{i+1,j_2} \cap \ldots \cap U_{i+1,j_l} \neq \emptyset$, then there exists a $k \leq k_i$ such that $U_{i+1,j_m} \subset U_{i,k}$ for $m = 1, 2, \ldots, l$.

For $i = 1, 2, \ldots$ let $\mathcal{K}_i = \mathcal{N}(\mathcal{U}_i)$ be a nerve of the cover $\mathcal{U}_i$ consisting of faces of the simplex $T^{k_i - 1}$. By virtue of (1) the underlying polyhedra $K_i = |\mathcal{K}_i|$ all have dimension $\leq n$.

We shall now define continuous mappings $\pi_i^{i+1} \colon K_{i+1} \to K_i$ for $i = 1, 2, \ldots$ Let $p_{i+1,j}$ be a vertex of the complex $\mathcal{K}_{i+1}$. Consider the member $U_{i+1,j}$ of the cover $\mathcal{U}_{i+1}$ which corresponds to $p_{i+1,j}$. By virtue of (3) the family

$$\mathcal{U}_{i,j} = \{U \in \mathcal{U}_i : U_{i+1,j} \subset U\}$$

is non-empty. Since $\bigcap \mathcal{U}_{i,j} \neq \emptyset$, the vertices of $\mathcal{K}_i$ which correspond to the members of $\mathcal{U}_{i,j}$ span a simplex $S_{i,j} \in \mathcal{K}_i$; we let

$$\pi_i^{i+1}(p_{i+1,j}) = b(S_{i,j}), \tag{4}$$

where $b(S)$ denotes the barycentre of S.

We shall prove that

(5) for every simplex $S \in \mathcal{K}_{i+1}$, the images of vertices of S under π_i^{i+1} are contained in a simplex $T \in \mathcal{K}_i$.

Indeed, if $S = p_{i+1,j_1} p_{i+1,j_2} \cdots p_{i+1,j_l}$, then

$$\emptyset \neq U_{i+1,j_1} \cap U_{i+1,j_2} \cap \ldots \cap U_{i+1,j_l} \subset \bigcap \mathcal{U}_{i,j_1} \cap \bigcap \mathcal{U}_{i,j_2} \cap \ldots \cap \bigcap \mathcal{U}_{i,j_l},$$

so that the vertices of $\mathcal{K}_i$ which correspond to the members of $\mathcal{U}_{i,j_1} \cup \mathcal{U}_{i,j_2} \cup \ldots \cup \mathcal{U}_{i,j_l}$ span a simplex $T \in \mathcal{K}_i$ which contains the point $\pi_i^{i+1}(p_{i+1,j_m})$ for $m = 1, 2, \ldots, l$.

It follows from (5) that the mapping π_i^{i+1} defined on the set of all vertices of $\mathcal{K}_{i+1}$ can be extended over each simplex $S = p_{i+1,j_1} p_{i+1,j_2} \cdots p_{i+1,j_l} \in \mathcal{K}_{i+1}$ by letting

$$\pi_i^{i+1}\Big(\sum_{m=1}^{l} \lambda_{j_m} p_{i+1,j_m}\Big) = \sum_{m=1}^{l} \lambda_{j_m} \pi_i^{i+1}(p_{i+1,j_m});$$

in this way a continuous mapping $\pi_i^{i+1}\colon K_{i+1} \to K_i$ is defined for $i = 1, 2, \ldots$. One easily checks that by defining, for every pair i, j of natural numbers satisfying $j \leq i$,

$$\pi_j^i = \pi_j^{j+1}\pi_{j+1}^{j+2}\ldots\pi_{i-1}^i \quad \text{if } j < i \qquad \text{and} \qquad \pi_i^i = \mathrm{id}_{K_i},$$

one obtains continuous mappings $\pi_j^i\colon K_i \to K_j$ affine on each simplex in $\mathcal{K}_i$.

Thus the inverse sequence $\boldsymbol{S} = \{K_i, \pi_j^i\}$ is defined. It remains to show that the space X is homeomorphic to the limit K of this inverse sequence.

By virtue of (3) and (5) we can apply Lemma 1.13.1 to the restrictions of π_i^{i+1} to simplexes in $\mathcal{K}_{i+1}$, so that

$$\delta(\pi_i^{i+1}(F)) \leq \frac{n}{n+1}\delta(F) \quad \text{for every } F \subset S \in \mathcal{K}_{i+1}. \tag{6}$$

Let us observe that for every choice of a point y_i in K_i for $i = 1, 2, \ldots$

(7) if $\emptyset \neq F_i = \overline{F}_i \subset \bigcup\{S \in \mathcal{K}_i : y_i \in S\}$ and $\pi_i^{i+1}(F_{i+1}) \subset F_i$ for $i = 1, 2, \ldots$, then the limit L of the inverse sequence $\{F_i, \pi_j^i|F_i\}$ is a one-point set.

Indeed, the set L is non-empty as the limit of an inverse sequence of non-empty compact spaces; since $\pi_j(L) \subset \bigcap_{i=j}^{\infty}\pi_j^i(F_i)$ for $j = 1, 2, \ldots$, where $\pi_j\colon K \to K_i$ is the projection, and since by virtue of (6) and the inequality $\delta(F_i \cap S) \leq \delta(S) \leq 2$ we have $\delta(\pi_j^i(F_i)) \leq 4(n/(n+1))^{i-j}$ whenever $j \leq i$, the $\pi_j(L)$'s are all one-point sets, which implies that so is L.

Now, consider a point $x \in X$; for every natural number i let

$$\mathcal{U}_i(x) = \{U \in \mathcal{U}_i : x \in U\}$$

and denote by $K_i(x)$ the simplex in $\mathcal{K}_i$ spanned by the vertices of $\mathcal{K}_i$ which correspond to the members of $\mathcal{U}_i(x)$. Let us note that

$$\pi_i^{i+1}(K_{i+1}(x)) \subset K_i(x) \quad \text{for } i = 1, 2, \ldots \tag{8}$$

Indeed, if $U_{i+1,j} \in \mathcal{U}_{i+1}(x)$, then $\mathcal{U}_{i,j} \subset \mathcal{U}_i(x)$, so that the images of vertices of $K_{i+1}(x)$ under π_i^{i+1} are contained in the simplex $K_i(x)$, and this implies (8). It follows from (8) that $\boldsymbol{K}(x) = \{K_i(x), \pi_j^i|K_i(x)\}$ is an inverse sequence. By virtue of (7) the limit of this inverse sequence contains exactly one point; let it be denoted by $f(x)$. As $f(x) \in K$, a mapping f of X to K is thus defined.

We shall prove that f is a continuous mapping. Obviously, it is enough to show that for every natural number j the composition $f_j = \pi_j f$ is continuous. Consider a point $x_0 \in X$, a positive number ϵ and a natural number i such that $i \geq j$ and $(n/(n+1))^{i-j} < \epsilon/2$. The set $U = \bigcap\mathcal{U}_i(x_0)$ is a neighbourhood of x_0 and for every $x \in U$ the inclusion $\mathcal{U}_i(x_0) \subset \mathcal{U}_i(x)$ holds, so that $K_i(x_0) \subset K_i(x)$ and

$$f_j(x_0) \in \pi_j^j(K_i(x_0)) \subset \pi_j^i(K_i(x)) \quad \text{for } x \in U.$$

Since, by virtue of (6),

$$\delta(\pi_j^i(K_i(x))) \leq 2(n/(n+1))^{i-j} < \epsilon,$$

and since $f_j(x) \in \pi_j^i(K_i(x))$, it follows that

$$\sigma(f_j(x_0), f_j(x)) < \epsilon \quad \text{for } x \in U,$$

i.e., the mapping f_j is continuous.

As for each pair x, x' of distinct points of X there exists – by virtue of (2) – a natural number i such that $\mathcal{U}_i(x) \cap \mathcal{U}_i(x') = \emptyset$, i.e., $K_i(x) \cap K_i(x') = \emptyset$, the mapping $f: X \to K$ is one-to-one. Since X is a compact space, to prove that f is a homeomorphism it suffices to show that $f(X) = K$.

Consider a point $y = \{y_i\} \in K$. For $i = 1, 2, \ldots$ let

$$A_i = \bigcup\{S \in \mathcal{K}_i : y_i \in S\} \subset K_i \quad \text{and} \quad B_i = f_i^{-1}(A_i) \subset X.$$

The sets B_i are non-empty. Indeed, if $S_i = p_{i,j_1} p_{i,j_2} \ldots p_{i,j_l}$ is a maximal simplex in $\mathcal{K}_i$ which contains y_i, and x_i is a point in $U_{i,j_1} \cap U_{i,j_2} \cap \ldots \cap U_{i,j_l}$, then $K_i(x_i) = S_i \subset A_i$, which implies that $x_i \in B_i$.

Since the image of each simplex $S \in \mathcal{K}_{i+1}$ under π_i^{i+1} is contained in a simplex $T \in \mathcal{K}_i$, and since $\pi_i^{i+1}(y_{i+1}) = y_i$, we have

$$\pi_i^{i+1}(A_{i+1}) \subset A_i \quad \text{for } i = 1, 2, \ldots \tag{9}$$

The last inclusion implies that

$$B_{i+1} = f_{i+1}^{-1}(A_{i+1}) \subset f_{i+1}^{-1}(\pi_i^{i+1})^{-1}(A_i) = (\pi_i^{i+1} f_{i+1})^{-1}(A_i) = f_i^{-1}(A_i) = B_i$$

for $i = 1, 2, \ldots$, so that, by the compactness of X, there exists a point $x \in \bigcap_{i=1}^{\infty} B_i$. Let $F_i = A_i \cap K_i(x)$ for $i = 1, 2, \ldots$; as $f_i(x) \in F_i$, the sets F_i are non-empty. By virtue of (9) and (8),

$$\begin{aligned} \pi_i^{i+1}(F_{i+1}) &= \pi_i^{i+1}(A_{i+1} \cap K_{i+1}(x)) \subset \pi_i^{i+1}(A_{i+1}) \cap \pi_i^{i+1}(K_{i+1}(x)) \\ &\subset A_i \cap K_i(x) = F_i \quad \text{for } i = 1, 2, \ldots, \end{aligned}$$

so that $\{F_i, \pi_j^i | F_i\}$ is an inverse sequence; by (7), its limit is a one-point set $\{z\} \subset K$. From (9) and (7) it follows that $\{A_i, \pi_j^i | A_i\}$ is an inverse sequence whose limit is the one-point set $\{y\}$. By the definition of f, the one-point set $\{f(x)\}$ is the limit of the inverse sequence $\boldsymbol{K}(x)$. Since $\{z\} \subset \{y\} \cap \{f(x)\}$, we have $f(x) = y$. ■

Let us note that another proof of Theorem 1.13.2 (see the hint to Problem 1.13.G(a)) leads to an inverse sequence $\{K_i, \pi_j^i\}$, where each K_i is the underlying polyhedron of a nerve $\mathcal{K}_i$ of a finite open cover of the space X, and for every $j \leq i$ the bonding mapping π_j^i is a *quasi-simplicial* mapping of K_i onto K_j, i.e., it is the affine extension of a simplicial mapping of a barycentric subdivision of $\mathcal{K}_i$ onto a barycentric subdivision of $\mathcal{K}_j$ (cf. Problem 1.13.C).

Let us also note that from Problem 1.10.K it follows that every compact subspace X of Euclidean m-space R^m such that $\dim X \leq n \geq 0$ can be represented as the limit of an inverse sequence of polyhedra of dimension $\leq n$ which are all contained in R^m. The converse does not hold; simple examples show that the limit of an inverse sequence of polyhedra contained in R^m need not be embeddable in R^m (see Problem 1.13.B).

We now pass to the theorem on the dimension of the limit of an inverse sequence. In the proof we shall apply the following lemma, a slight strengthening of the compactification theorem, which allows the reduction of the problem to the special case where the inverse sequence consists of compact spaces.

1.13.3. LEMMA. *For every continuous mapping $f: X \to Z$ of a separable metric space X to a compact metric space Z there exists a compact metric space $\widetilde{X}$ containing X as*

a dense subspace such that $\dim \widetilde{X} \leq \dim X$ *and* f *is extendable to a continuous mapping* $\tilde{f}\colon \widetilde{X} \to Z$.

PROOF. Let ρ_0 be a totally bounded metric on the space X and σ an arbitrary metric on the space Z. One easily checks that the formula

$$\rho(x,y) = \rho_0(x,y) + \sigma(f(x),f(y)) \quad \text{for } x,y \in X$$

defines a metric equivalent to ρ_0 and such that the mapping f is uniformly continuous with respect to ρ and σ. The metric ρ is totally bounded. Indeed, for every positive number ϵ there exists a finite cover $\mathcal{A}$ of the space (X,ρ_0) such that $\operatorname{mesh}\mathcal{A} < \epsilon/2$ and a finite cover $\mathcal{B}$ of the space (Z,σ) such that $\operatorname{mesh}\mathcal{B} < \epsilon/2$; one can readily check that the mesh of the cover $\mathcal{A} \wedge f^{-1}(\mathcal{B})$ of the space (X,ρ) is less than ϵ. It follows from the second part of the compactification theorem that on X there exists a metric $\widetilde{\rho}$ equivalent to ρ such that $\rho(x,y) \leq \widetilde{\rho}(x,y)$ for $x,y \in X$, and that the completion $\widetilde{X}$ of the space $(X,\widetilde{\rho})$ is a compact space such that $\dim \widetilde{X} \leq \dim X$. Obviously, f is uniformly continuous with respect to $\widetilde{\rho}$ and σ, which implies that it is extendable to a continuous mapping $\tilde{f}\colon \widetilde{X} \to Z$. ■

1.13.4. THEOREM ON THE DIMENSION OF THE LIMIT OF AN INVERSE SEQUENCE. *If* $\boldsymbol{S} = \{X_i, \pi_j^i\}$ *is an inverse sequence of separable metric spaces* X_i *such that* $\dim X_i \leq n$ *for* $i = 1,2,\ldots$, *then the limit* $X = \varprojlim \boldsymbol{S}$ *satisfies the inequality* $\dim X \leq n$.

PROOF. Let $\widetilde{X}_1$ be a compact metric space which contains X_1 as a dense subspace and satisfies the inequality $\dim \widetilde{X}_1 \leq n$. An inductive construction applying Lemma 1.13.3 yields for $i = 2,3,\ldots$ a compact metric space $\widetilde{X}_i$ which contains X_i as a dense subspace such that $\dim \widetilde{X}_i \leq n$ and $\pi_{i-1}^i\colon X_i \to \widetilde{X}_{i-1}$ is extendable to a continuous mapping $\widetilde{\pi}_{i-1}^i\colon \widetilde{X}_i \to \widetilde{X}_{i-1}$. Letting

$$\widetilde{\pi}_j^i = \widetilde{\pi}_j^{j+1}\widetilde{\pi}_{j+1}^{j+2}\ldots\widetilde{\pi}_{i-1}^i \quad \text{for } j < i \qquad \text{and} \qquad \widetilde{\pi}_i^i = \mathrm{id}_{\widetilde{X}_i},$$

one obtains an inverse sequence $\widetilde{\boldsymbol{S}} = \{\widetilde{X}_i, \widetilde{\pi}_j^i\}$ of compact metric spaces $\widetilde{X}_i$ such that $\dim \widetilde{X}_i \leq n$ for $i = 1,2,\ldots$ As $\varprojlim \boldsymbol{S} \subset \varprojlim \widetilde{\boldsymbol{S}}$, it suffices to prove the theorem under the additional assumption that all the X_i's are compact.

Let $\mathcal{U}$ be a finite open cover of the space X. Since the family of all sets $\pi_i^{-1}(U_i)$, where $\pi_i\colon X \to X_i$ is the projection and U_i is an open subset of X_i, is a base for X, and since X is compact, being the limit of an inverse sequence of compact spaces, the cover $\mathcal{U}$ has a finite refinement $\{\pi_{i_k}^{-1}(U_{i_k})\}_{k=1}^m$, where $i_1 \leq i_2 \leq \ldots \leq i_m$ and U_{i_k} is an open subset of X_{i_k} for $k = 1,2,\ldots,m$. The family $\{\pi_{i_m}^{-1}(V_k)\}_{k=1}^m$, where $V_k = (\pi_{i_k}^{i_m})^{-1}(U_{i_k})$, is also a refinement of $\mathcal{U}$. Consider a finite open refinement $\mathcal{V}$ of the cover $\{V_k\}_{k=1}^m$ of the space $M = \bigcup_{k=1}^m V_k \subset X_{i_m}$ such that $\operatorname{ord}\mathcal{V} \leq n$; clearly, $\pi_{i_m}^{-1}(\mathcal{V})$ is a finite open refinement of $\mathcal{U}$ and has order $\leq n$. ■

Theorems 1.13.2 and 1.13.4 yield the characterization of dimension which was announced at the beginning of the present section.

1.13.5. THEOREM ON INVERSE SEQUENCES. *A compact metric space* X *satisfies the inequality* $\dim X \leq n$ *if and only if* X *is homeomorphic to the limit of an inverse sequence of polyhedra of dimension* $\leq n$. ■

One readily sees that the theorem on ϵ-mappings is an easy consequence of the theorem on expansion in an inverse sequence (see Problem 1.13.A). The study of relations between these two theorems led to the notion of a $\boldsymbol{\Pi}$-like space. Let $\boldsymbol{\Pi}$ be a family of polyhedra; we say that a compact metric space X is *$\boldsymbol{\Pi}$-like* if for every positive number ϵ there exists an ϵ-mapping of X onto a polyhedron $K \in \boldsymbol{\Pi}$. The main theorem on $\boldsymbol{\Pi}$-like spaces states that if either $\boldsymbol{\Pi}$ is a family of connected polyhedra (more generally, if the space X below is a continuum) or $\boldsymbol{\Pi}$ is a hereditary family of polyhedra (i.e., together with the underlying polyhedron of a simplicial complex $\mathcal{K}$, it contains the underlying polyhedra of all subcomplexes of $\mathcal{K}$), then a compact metric space X is $\boldsymbol{\Pi}$-like if and only if X is homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where $K_i \in \boldsymbol{\Pi}$ for $i = 1, 2, \ldots$ and π_j^i maps X_i onto X_j for every $j \leq i$. Obviously, this is a generalization of the theorem on inverse sequences, because the family $\boldsymbol{\Pi}$ of all polyhedra of dimension $\leq n$ is a hereditary family of polyhedra and, for this family, the class of $\boldsymbol{\Pi}$-like spaces coincides with the class of all compact metric spaces whose covering dimension is not larger than n. Various families $\boldsymbol{\Pi}$ yield interesting classes of compact metric spaces. Thus, for the family $\boldsymbol{\Pi}$ consisting of the interval I alone, one obtains the class of *snake-like continua*. Clearly, each snake-like continuum is one-dimensional; one proves that snake-like continua are embeddable in the plane (see the remark to Problem 1.13.B) and that there exists a universal space for the class of all snake-like continua. In a more general setting, one can prove that if $\boldsymbol{\Pi}$ is a hereditary family of polyhedra which is, moreover, additive (i.e., together with each pair K, L of polyhedra, it contains the disjoint sum of K and L), then there exists a universal space for the class of all $\boldsymbol{\Pi}$-like spaces. Obviously, this implies the existence of a universal space for the class of all compact metric spaces whose covering dimension is not larger than n. In a less direct way, this also implies the existence of a universal snake-like continuum.

Historical and bibliographic notes

Theorem 1.13.2 was proved by Freudenthal in [1937]. Freudenthal's proof leads to an inverse sequence of polyhedra with quasi-simplicial onto mappings which, moreover, are irreducible (see Problem 1.13.F); the simpler proof given here was outlined by Isbell in [1959]. Theorem 1.13.4 was proved by Nagami in [1959]; for compact metric spaces it is implicitly contained in Freudenthal's paper [1937]. The notion of a $\boldsymbol{\Pi}$-like space was introduced by Mardešić and Segal in [1963]. In the same paper Mardešić and Segal proved that if $\boldsymbol{\Pi}$ is a family of connected polyhedra, then a compact metric space is $\boldsymbol{\Pi}$-like if and only if it is homeomorphic to the limit of an inverse sequence of polyhedra in the family $\boldsymbol{\Pi}$; Pasynkov in [1966] showed that this is likewise true under the assumption that $\boldsymbol{\Pi}$ is a hereditary family of polyhedra. It was also shown by Pasynkov in [1966] that if $\boldsymbol{\Pi}$ is a hereditary and additive family of polyhedra, then there exists a universal space for the family of all $\boldsymbol{\Pi}$-like spaces; a somewhat stronger result was obtained by McCord in [1966]. Snake-like continua were introduced by Bing in [1951]; among other things, Bing proved that each snake-like continuum is embeddable in the plane. The existence of a universal space for the class of all snake-like continua was established by Schori in [1965].

Problems

1.13.A. Let $X \neq \emptyset$ be the limit of an inverse sequence $\{X_i, \pi_j^i\}$ of compact metric spaces and let $\epsilon_i = \operatorname{mesh}(\{\pi_i^{-1}(x) : x \in X_i\})$, where $\pi_i : X \to X_i$ is the projection. Show that the sequence $\epsilon_1, \epsilon_2, \ldots$ converges to zero.

1.13.B. Define an inverse sequence $\boldsymbol{S} = \{X_i, \pi_j^i\}$, where $X_i = I$ for $i = 1, 2, \ldots$ and π_j^i maps X_i onto X_j for every $j \leq i$, such that the limit of $\boldsymbol{S}$ cannot be embedded in the real line.

Remark. As shown by Isbell in [1959], the limit of an inverse sequence of compact subspaces of R^m is embeddable in R^{2m}.

1.13.C (Isbell [1964]). Let $\boldsymbol{S} = \{|\mathcal{K}_i|, \pi_j^i\}$ be an inverse sequence of polyhedra of dimension ≥ 1, where for $j \leq i$ the bonding mapping π_j^i maps $|\mathcal{K}_i|$ onto $|\mathcal{K}_j|$ and is the affine extension of a simplicial mapping of $\mathcal{K}_i$ to $\mathcal{K}_j$. Prove that if the limit of $\boldsymbol{S}$ contains more than one point, then it contains a subspace homeomorphic to the interval I. Applying the fact that there exist one-dimensional continua with no subspace homeomorphic to the interval I (see Kuratowski [1968], p. 206), observe that there exist one-dimensional compact metric spaces which are not homeomorphic to the limit of an inverse sequence $\{|\mathcal{K}_i|, \pi_j^i\}$ of one-dimensional polyhedra, where for $j \leq i$ the bonding mapping π_j^i is the affine extension of a simplicial mapping of $\mathcal{K}_i$ to $\mathcal{K}_j$.

1.13.D. Define an inverse sequence $\boldsymbol{S} = \{X_i, \pi_j^i\}$ of one-dimensional compact metric spaces, where for $j \leq i$ the bonding mapping π_j^i maps X_i onto X_j, such that the limit of $\boldsymbol{S}$ is homeomorphic to the Cantor set.

1.13.E (Mardešić and Segal [1963]). Let $\boldsymbol{\Pi}$ be the family consisting of all polyhedra which are unions of a 1-simplex and a finite number of 0-simplexes. Observe that the Cantor set is a $\boldsymbol{\Pi}$-like space, and yet it is not homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where $K_i \in \boldsymbol{\Pi}$ for $i = 1, 2, \ldots$ and π_j^i maps X_i onto X_j for every $j \leq i$.

1.13.F (Freudenthal [1937]). Let $f: X \to |\mathcal{K}|$ be a continuous mapping of a topological space X to the underlying polyhedron of a simplicial complex $\mathcal{K}$. A continuous mapping $g: X \to |\mathcal{K}|$ is a *modification* of f if, for every $S \in \mathcal{K}$, $g(x) \in S$ whenever $f(x) \in S$; the mapping f is *irreducible* if there is no modification $g: X \to |\mathcal{K}|$ of f such that $f(X) \setminus g(X) \neq \emptyset$.

(a) Show that if F is a closed subset of X and $g': F \to |\mathcal{K}|$ is a modification of the restriction $f|F: F \to |\mathcal{K}|$ of a continuous mapping $f: X \to |\mathcal{K}|$, then there exists a modification $g: X \to |\mathcal{K}|$ of f such that $g|F = g'$.

(b) Check that if $f: X \to |\mathcal{K}|$ is an irreducible mapping, then for every subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ the restriction $f_{|\mathcal{K}_0|}: f^{-1}(|\mathcal{K}_0|) \to |\mathcal{K}_0|$ is also irreducible.

(c) Let $\mathcal{S}$ be the simplicial complex consisting of all faces of a simplex and let $\mathcal{S}_0$ be the subcomplex of $\mathcal{S}$ consisting of all proper faces of that simplex. Prove that if a continuous mapping $f: X \to |\mathcal{S}|$ has a modification $g': X \to |\mathcal{S}|$ such that no interior point p of $|\mathcal{S}|$, i.e., no point $p \in |\mathcal{S}| \setminus |\mathcal{S}_0|$, belongs to $g'(X)$, then there also exists a modification $g: X \to |\mathcal{S}|$ of f such that $g|f^{-1}(|\mathcal{S}_0|) = f|f^{-1}(|\mathcal{S}_0|)$ and $p \in |\mathcal{S}| \setminus g(X)$.

Hint. There exists an open set $U \subset X$ containing $f^{-1}(|\mathcal{S}_0|)$ and such that the line segment with end-points $f(x)$ and $g(x)$ does not contain p for any $x \in U$.

(d) Show that a continuous mapping $f: X \to |\mathcal{K}|$ is irreducible if and only if, for every $S \in \mathcal{K}$ such that some interior point of S belongs to $f(X)$, the restriction $f_S: f^{-1}(S) \to S$ is an essential mapping (see Problem 1.9.A).

(e) Prove that for every continuous mapping $f: X \to |\mathcal{K}|$ there exist an irreducible modification $g: X \to |\mathcal{K}|$ of f and a subcomplex $\mathcal{K}_0$ of $\mathcal{K}$ such that $g(X) = |\mathcal{K}_0|$.

Hint. To begin, apply (c) to observe that if for a simplex $S \in \mathcal{K}$ the restriction $f_S: f^{-1}(S) \to S$ is essential, then so is the restriction $f_T: f^{-1}(T) \to T$ for every face T of S; then modify f on all simplexes for which the corresponding restriction is not essential, beginning with the simplexes of the highest dimension.

1.13.G. (a) (Pasynkov [1966]) Prove that if $\boldsymbol{\Pi}$ is a hereditary family of polyhedra, then a compact metric space X is $\boldsymbol{\Pi}$-like if and only if X is homeomorphic to the limit of an inverse sequence $\{K_i, \pi_j^i\}$, where $K_i \in \boldsymbol{\Pi}$ for $i = 1, 2, \ldots$ and π_j^i is a quasi-simplicial mapping of K_i onto K_j for every $j \leq i$.

Hint. Apply Problem 1.13.F(e) to define a simplicial complex $\mathcal{K}_1$, a barycentric subdivision $\mathcal{P}_1$ of $\mathcal{K}_1$, and an irreducible mapping $f_1: X \to |\mathcal{P}_1|$ of X onto $|\mathcal{P}_1|$ in such a way that $|\mathcal{K}_1| \in \boldsymbol{\Pi}$ and the inverse images of stars of vertices of $\mathcal{P}_1$ under f_1 all have diameters less than $1/2$.

Assume that for each $j < i$ a simplicial complex $\mathcal{K}_j$, a barycentric subdivision $\mathcal{P}_j$ of $\mathcal{K}_j$ and an irreducible mapping $f_j: X \to |\mathcal{P}_j|$ of X onto $|\mathcal{P}_j|$ are defined in such a way that $|\mathcal{K}_j| \in \boldsymbol{\Pi}$ and the inverse images of stars of vertices of $\mathcal{P}_j$ under f_j all have diameters less than $1/2^j$; assume, moreover, that for each pair k, j of integers satisfying $k \leq j < i$ a quasi-simplicial mapping π_k^j of $|\mathcal{P}_j|$ onto $|\mathcal{P}_k|$ is defined in such a way that $\pi_k^k = \mathrm{id}_{|\mathcal{P}_k|}$, $\pi_l^k \pi_k^j = \pi_l^j$ whenever $l \leq k$, and $\rho(\pi_k^{j-1} f_{j-1}(x), \pi_k^j f_j(x)) < 1/2^j$ for $x \in X$ and $k \leq j - 1$.

Let $\mathcal{P}$ be a barycentric subdivision of $\mathcal{P}_{i-1}$ such that $\mathrm{mesh}(\{\pi_k^{i-1}(S) : S \in \mathcal{P}\}) < 1/2^i$ for every $k \leq i-1$; consider the cover $\mathcal{U}$ of the space X consisting of inverse images of stars of vertices of $\mathcal{P}$ under f_{i-1} and let $\delta_i = \min(\epsilon_i, 1/2^i)$, where ϵ_i is a Lebesgue number of the cover $\mathcal{U}$. Define a simplicial complex $\mathcal{K}_i$, a barycentric subdivision $\mathcal{P}_i$ of $\mathcal{K}_i$ and an irreducible mapping $f_i: X \to |\mathcal{P}_i|$ of X onto $|\mathcal{P}_i|$ in such a way that $|\mathcal{K}_i| \in \boldsymbol{\Pi}$ and the inverse images of stars of vertices of $\mathcal{P}_i$ all have diameters less than δ_i. Observe that by assigning to each vertex $p \in \mathcal{P}_i$ a vertex $q \in \mathcal{P}$ such that $f_i^{-1}(\mathrm{St}_{\mathcal{P}_i}(p)) \subset f_{i-1}^{-1}(\mathrm{St}_{\mathcal{P}}(q))$ one defines a simplicial mapping π_{i-1}^i of $\mathcal{P}_i$ to $\mathcal{P}$, and extend π_{i-1}^i to a quasi-simplicial mapping $\pi_{i-1}^i: |\mathcal{P}_i| \to |\mathcal{P}_{i-1}|$. Check that if $f_{i-1}(x) \in S \in \mathcal{P}$, then $\pi_{i-1}^i f_i(x) \in S$; deduce that $\pi_{i-1}^i(|\mathcal{P}_i|) = |\mathcal{P}_{i-1}|$ and that $\rho(\pi_k^{i-1} f_{i-1}(x), \pi_k^i f_i(x)) < 1/2^i$ for $x \in X$ and $k \leq i-1$, where $\pi_k^i = \pi_k^{i-1} \pi_{i-1}^i$.

Note that for every natural number k the sequence of compositions $\pi_k^{k+1} f_{k+1}$, $\pi_k^{k+2} f_{k+2}, \ldots$ uniformly converges to a continuous mapping $g_k: X \to |\mathcal{P}_k|$ and that $\pi_k^{k+1} g_{k+1} = g_k$. Check that X is homeomorphic to the limit of the inverse sequence $\{|\mathcal{K}_i|, \pi_j^i\}$.

(b) (Freudenthal [1937]) Prove that a compact metric space X satisfies the inequality $\dim X \leq n$ if and only if X is homeomorphic to the limit of an inverse sequence

$\{K_i, \pi_j^i\}$, where K_i is a polyhedron of dimension $\leq n$ for $i = 1, 2, \ldots$ and π_j^i is a quasi-simplicial mapping of K_i onto K_j for every $j \leq i$.

1.14. Axioms for dimension

Since the origin of dimension theory attempts have been made to characterize dimension functions by a few simple properties that could serve as a basis for an axiomatization of the theory. However, no satisfactory set of axioms has been proposed so far; the main drawbacks are that the axioms include properties which are either somewhat artificial or too close to the definition of dimension and that no part of dimension theory, no matter how small, can be deduced from the axioms. As the problem of axiomatization of dimension theory is of secondary importance, we shall confine ourselves to a rather sketchy discussion of this topic.

We shall consider a class $\mathcal{K}$ of topological spaces, which together with each space X contains all closed subspaces of X, and a function d defined on $\mathcal{K}$, taking values in the set of integers larger than or equal to -1 enlarged by the "infinite number" ∞, and such that $d(X) = d(Y)$ for each pair of homeomorphic spaces $X, Y \in \mathcal{K}$. By assuming that the function d satisfies some simple conditions which are known to be satisfied by the function dim we shall obtain three sets of axioms for dimension theory.

We begin with *Alexandroff's axioms*; in this case $\mathcal{K}$ is the class of all compact subspaces of Euclidean spaces and the function d satisfies the following conditions:

(A1) $d(\emptyset) = -1$, $d(\{0\}) = 0$ *and* $d(I^n) = n$ *for* $n = 1, 2, \ldots$

(A2) *If a space* $X \in \mathcal{K}$ *is represented as the union of two closed subspaces* X_1 *and* X_2, *then* $d(X) = \max(d(X_1), d(X_2))$.

(A3) *For every space* $X \in \mathcal{K}$ *there exists a positive number* ϵ *such that if there exists an* ϵ*-mapping of* X *onto a space* $Y \in \mathcal{K}$, *then* $d(X) \leq d(Y)$.

(A4) *For every space* $X \in \mathcal{K}$ *of cardinality larger than one there exists a closed set* $L \subset X$ *separating* X *and such that* $d(L) < d(X)$.

1.14.1. THEOREM. *The covering dimension* dim *is the only function* d *which satisfies conditions* (A1)–(A4) *in the class* $\mathcal{K}$ *of all compact subspaces of Euclidean spaces.*

PROOF. Clearly, the function $d = \dim$ satisfies conditions (A1)–(A4). Consider now any function d which satisfies (A1)–(A4). It follows from (A1) and (A2) that if K is a polyhedron, then $d(K) = \dim K$, so that by virtue of (A3) and the theorem on ϵ-mappings, $d(X) \leq \dim X$ for every $X \in \mathcal{K}$.

Assume that there exists a space $X \in \mathcal{K}$ such that $d(X) < \dim X$. Let $d(X) = k$ and $\dim X = n$; it follows from (A1) that $n \geq 1$. Without loss of generality one can suppose that each $Y \in \mathcal{K}$ such that $d(Y) < k$ satisfies the equality $d(Y) = \dim Y$. By virtue of Theorems 1.9.9 and 1.7.7, the space X contains an n-dimensional Cantor-manifold M. From (A2) it follows that $d(M) \leq d(X) < n = \dim M$; since $d(M) \neq \dim M$, we have $d(M) = k$. Now apply (A4) to obtain a closed set $L \subset M$ separating M and such that $d(L) < d(M)$. Thus we have $\dim L = d(L) < k < n$, so that $\dim L \leq n-2$, which contradicts the definition of a Cantor-manifold. Hence $d(X) = \dim X$ for every $X \in \mathcal{K}$. ■

It turns out that by replacing condition (A2) with

(A2′) *If a space $X \in \mathcal{K}$ is represented as the union of a sequence $X_1, X_2, \ldots$ of closed subspaces, then $d(X) = \sup\{d(X_i) : i = 1, 2, \ldots\}$*

one obtains a set of conditions which characterizes the covering dimension dim in the class $\mathcal{K}$ of all subspaces of Euclidean spaces (see Problem 1.14.C).

We now pass to *Nishiura's axioms*; in this case $\mathcal{K}$ is the class of all separable metric spaces and the function d satisfies the following conditions:

(N1) $d(\{0\}) = 0$.

(N2) *If Y is a subspace of a space $X \in \mathcal{K}$, then $d(Y) \leq d(X)$.*

(N3) *If a space $X \in \mathcal{K}$ is represented as the union of a sequence $X_1, X_2, \ldots$ of closed subspaces, then $d(X) = \sup\{d(X_i) : i = 1, 2, \ldots\}$.*

(N4) *If a space $X \in \mathcal{K}$ is represented as the union of two subspaces X_1 and X_2, then $d(X) \leq d(X_1) + d(X_2) + 1$.*

(N5) *For every space $X \in \mathcal{K}$ there exists a compactification $\widetilde{X} \in \mathcal{K}$ such that $d(\widetilde{X}) = d(X)$.*

(N6) *If a non-empty space $X \in \mathcal{K}$ satisfies the inequality $d(X) < \infty$, then for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that*

$$x \in U \subset V \quad \text{and} \quad d(\operatorname{Fr} U) \leq d(X) - 1.$$

One can prove that the covering dimension dim is the only function d which satisfies conditions (N1)–(N6) in the class $\mathcal{K}$ of all separable metric spaces (see Problem 1.14.B).

We conclude with *Menger's axioms*, chronologically the earliest set of axioms for dimension theory; in this case $\mathcal{K}$ is the class of all subspaces of Euclidean m-space R^m and the function d satisfies the following conditions:

(M1) $d(\emptyset) = -1$, $d(\{0\}) = 0$ *and* $d(R^n) = n$ *for* $n = 1, 2, \ldots, m$.

(M2) *If Y is a subspace of a space $X \in \mathcal{K}$, then $d(Y) \leq d(X)$.*

(M3) *If a space $X \in \mathcal{K}$ is represented as the union of a sequence $X_1, X_2, \ldots$ of closed subspaces, then $d(X) \leq \sup\{d(X_i) : i = 1, 2, \ldots\}$.*

(M4) *For every space $X \in \mathcal{K}$ there exists a compactification $\widetilde{X} \in \mathcal{K}$ such that $d(\widetilde{X}) = d(X)$.*

Menger put forward the hypothesis that for every natural number m the covering dimension dim is the only function d which satisfies conditions (M1)–(M4) in the class of all subspaces of Euclidean m-space R^m and showed that the hypothesis is valid for $m \leq 2$ (see Problem 1.14.D). The problem whether the hypothesis is valid for $m > 2$ is still open. Let us recall (cf. the discussion in the final part of Section 1.11) that for $m > 3$ it is not even known whether the function $d = \dim$ satisfies condition (M4). Clearly, the covering dimension dim satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of Euclidean spaces. We find, however, that dim is not the only function with this property; each cohomological dimension $\dim_G$ with respect to a finitely generated abelian group G also satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of Euclidean spaces.

Historical and bibliographic notes

Theorem 1.14.1 was proved by Alexandroff in [1932]. Ščepin announced in [1972] that by replacing condition (A2) by the stronger condition (A2′) one obtains a set of axioms that characterizes the covering dimension dim in the class of all subspaces of Euclidean spaces; the proof was published in Alexandroff and Pasynkov's book [1973], where a stronger result, also due to Ščepin, is announced, viz., that the same set of axioms characterizes dim in the class of all metric spaces whose covering dimension is finite. Nishiura proved in [1966] that the axioms (N1)–(N6) characterize the covering dimension dim in the class of all separable metric spaces. Sakai in [1968] and Aarts in [1971] modified Nishiura's axioms to obtain a set of axioms that characterizes dim in the class of all metric spaces. The fact that conditions (M1)–(M4) characterize the covering dimension dim in the class of all subspaces of the plane, and also in the class of all subspaces of the real line, was established by Menger in [1929]. The theorem that each cohomological dimension with respect to a finitely generated abelian group satisfies conditions (M1)–(M4) in the class of all subspaces of Euclidean spaces was proved by I. Švedov; the proof was first published in Kuz'minov's paper [1968]. A set of axioms characterizing the covering dimension dim in the class of all (not necessarily metric) compact spaces whose dimension dim is finite was given by Lokucievskiĭ in [1973].

Problems

1.14.A (Ščepin, cited in Alexandroff and Pasynkov [1973]; announcement Ščepin [1972]). Verify that the axioms (A1)–(A4) are independent.

1.14.B (Nishiura [1966]). (a) Prove that the covering dimension dim is the only function d which satisfies conditions (N1)–(N6) in the class $\mathcal{K}$ of all separable metric spaces.

(b) Verify that the axioms (N1)–(N6) are independent.

Hint. To verify that (N2) is independent of the remaining axioms, observe that every separable metric space which is not compact has an infinite-dimensional compactification.

1.14.C (Ščepin, cited in Alexandroff and Pasynkov [1973]; announcement Ščepin [1972]). (a) Show that for every separable metric space X such that $\dim X = n \geq 0$ there exists a separable metric space $X_* = \{x, y\} \cup \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed in X_* and homeomorphic to X, with the property that no closed set $L \subset X_*$ satisfying the inequality $\dim L \leq n-2$ separates the space X_* between x and y.

(b) Show that for every separable metric space X such that $\dim X = n \geq 0$ there exists a separable metric space $X^* = \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed in X^* and homeomorphic to X, with the property that no closed set $L \subset X^*$ satisfying the inequality $\dim L \leq n-2$ separates the space X^*.

(c) Prove that the covering dimension dim is the only function d which satisfies conditions (A1), (A2′), (A3) and (A4) in the class $\mathcal{K}$ of all subspaces of Euclidean spaces.

(d) Verify that (A2′) is independent of the axioms (A1)–(A4).

1.14.D. (a) (Menger [1929]) Check that the covering dimension dim is the only function d which satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of the real line.

Hint. Show that every compactification of the space of rational numbers contains topologically all zero-dimensional separable metric spaces.

(b) (Menger [1929]) Applying the Mazurkiewicz–Moore theorem (stating that for any pair of distinct points in a connected and locally connected completely metrizable space there exists an arc that contains these points (see Engelking and Sieklucki [1992], Theorem 6.5.17), prove that the covering dimension dim is the only function d which satisfies conditions (M1)–(M4) in the class $\mathcal{K}$ of all subspaces of the plane.

Hint. In the proof that $\dim X \leq d(X)$ for each $X \in \mathcal{K}$, the only difficulty is to show that there is no $X \in \mathcal{K}$ such that $d(X) = 0$ and $\dim X = 1$. Suppose that such an X exists; by virtue of Corollary 1.9.10 and condition (M4) one can assume that X is a continuum. Place a copy of the continuum X in a square K in such a way that the four corners of K are the only points of the boundary of K which belong to X. Then divide K into 9 congruent squares and place in the same way a copy of X in each of these smaller squares. Continue the procedure, dividing K consecutively into smaller and smaller squares, and consider the union X^* of countably many smaller and smaller copies of X placed in these squares. Prove that $d(X^*) = 0$ and that X^* is connected and locally connected. Apply the Lavrentieff theorem (see [GT], Theorem 4.3.21) and condition (M4) to obtain a G_δ-set Y^* such that $X^* \subset Y^* \subset K$ and $d(Y^*) = 0$. Find a connected and locally connected G_δ-set Y such that $X^* \subset Y \subset Y^*$, deduce from the Mazurkiewicz–Moore theorem that Y contains an arc and obtain a contradiction.

To prove that $d(X) \leq \dim X$ for each $X \in \mathcal{K}$, observe that $d(R^2 \setminus N_0^2) = 1$ and apply Problem 1.11.F(b).

(c) (Menger [1929]) Verify that the axioms (M1)–(M4) are independent in the class $\mathcal{K}$ of all subspaces of the plane.

1.14.E (Hayashi [1990]). (a) Prove that the covering dimension dim is the only function d which in the class $\mathcal{K}$ of all subspaces of Euclidean spaces satisfies conditions (M1)–(M4) and the condition

(H) *If a space $X \in \mathcal{K}$ satisfies the inequality $d(X) \leq n \geq 0$, then X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \ldots, Z_{n+1}$ such that $d(Z_i) \leq 0$ for $i = 1, 2, \ldots, n+1$.*

Hint. In the proof that $\dim X \leq d(X)$ for each $X \in \mathcal{K}$ show first (modifying the construction in the hint to Problem 1.14.D(b)) that if $d(X) = 0$, then $\dim X = 0$.

To prove that $d(X) \leq \dim X$ for each $X \in \mathcal{K}$ observe that $d(R^{2n+1} \setminus N_n^{2n+1}) = n$ and apply Problem 1.11.F(c).

(b) Verify that the axioms (M1)–(M4) and (H) are independent in the class $\mathcal{K}$ of all subspaces of Euclidean spaces.

Chapter 2

The large inductive dimension

Outside the class of separable metric spaces the dimensions ind, Ind and dim generally do not coincide. Nevertheless, a number of theorems established in Chapter 1 extend beyond this class of spaces. In larger classes they hold either for Ind, or for dim, or else for both Ind and dim. The dimension ind is practically of no importance outside the class of separable metric spaces and from now on will reappear here only occasionally. Slightly exaggerating, one could say that ind is a satisfactory dimension function only when it is equal to Ind. Thus, for general spaces we have two separate dimension theories: the theory of the large inductive dimension Ind and the theory of the covering dimension dim. They are both poorer and less harmonius than the dimension theory of separable metric spaces, yet they contain many interesting and deep theorems and shed light on classical dimension theory. It should be noted that, while the dimension dim behaves properly in the class of all normal spaces, the dimension Ind does so only in the more restricted class of strongly hereditarily normal spaces. Chapters 2 and 3 are devoted to a closer study of Ind and dim, respectively. In Chapter 4 it will be proved that Ind and dim coincide in the class of all metrizable spaces and that a dimension theory can be developed in that class which is by no means inferior to the dimension theory of separable metric spaces.

Section 2.1 contains supplementary information about hereditarily normal spaces and an investigation of the class of strongly hereditarily normal spaces.

In Section 2.2 a few theorems on the dimension Ind are proved which hold either in all normal spaces or in all hereditarily normal spaces. In the final part of the section two important examples are described, showing that neither the subspace theorem nor the sum theorem holds for Ind in the class of all normal spaces.

Section 2.3 is crucial for the present chapter; we develop in it a dimension theory for Ind in the class of strongly hereditarily normal spaces. The main results of this theory are the subspace theorem and a group of sum theorems.

The last section is devoted to a study of the relations between ind and Ind, to the Cartesian product theorems for Ind, and to a discussion of the behaviour of Ind under continuous mappings.

2.1. Hereditarily normal and strongly hereditarily normal spaces

The large inductive dimension Ind is defined for all normal spaces (see Definition 1.6.1). It turns out, however, that in such an extensive class of spaces the dimension Ind exhibits some pathological properties. As the reader will see in Section 2.2, there exist

a compact space Z and a normal subspace X of Z such that $\operatorname{Ind} X > \operatorname{Ind} Z$, as well as a compact space X with $\operatorname{Ind} X = 2$ which can be represented as the union of two closed subspaces F_1 and F_2 such that $\operatorname{Ind} F_1 = \operatorname{Ind} F_2 = 1$. Besides, since a subspace of a normal space is not necessarily normal (see [GT], Example 2.3.36 or 3.2.7), it may happen that the dimension Ind is defined for a space X, but it is not defined for some subspace of X. From all the adduced facts one gathers that to develop a satisfactory theory of Ind one has to restrict the class of spaces under consideration. As the spaces in all the examples cited above are not hereditarily normal, we might expect that no such pathological phenomena can occur in the class of hereditarily normal spaces. It has been shown, however, that the dimension Ind is not monotonic in the latter class, so that a further restriction is necessary.

A natural class of spaces where a satisfactory theory of the large inductive dimension can be developed is the class of strongly hereditarily normal spaces; it is contained in the class of all hereditarily normal spaces and constitutes a common extension of the class of perfectly normal spaces and the class of hereditarily paracompact spaces.

The present section is devoted to a study of topological properties of hereditarily normal spaces and strongly hereditarily normal spaces.

Let us recall that a space X is *hereditarily normal* if every subspace of X is normal. We begin with two simple characterizations of hereditarily normal spaces; in the second one, there reappears the notion of separated subsets of a topological space introduced in Section 1.2.

2.1.1. THEOREM. *For every T_1-space X the following conditions are equivalent:*

(i) *The space X is hereditarily normal.*

(ii) *Every open subspace of X is normal.*

(iii) *For every pair A, B of separated subsets of X there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.*

PROOF. The implication (i)⇒(ii) is obvious. We shall show that (ii)⇒(iii). Consider a pair A, B of separated subsets of a space X which satisfies (ii) and let $M = X \setminus (\overline{A} \cap \overline{B})$. Obviously, M is an open subspace of X and $A, B \subset M$. The closures of A and B in M are disjoint, so that by the normality of M there exist sets $U, V \subset M$ open in M and such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. Since M is open in X, so are U and V, and thus X satisfies (iii).

To complete the proof it remains to show that (iii)⇒(i). Consider an arbitrary subspace M of a space X which satisfies (iii) and a pair A, B of disjoint closed subsets of M. Clearly A and B are separated in X, so that there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$. The intersections $M \cap U$ and $M \cap V$ are open in M, disjoint, and contain A and B respectively; this means that X satisfies (i). ■

Let us recall that a family $\mathcal{A}$ of subsets of a topological space X is *point-finite* (*point-countable*) if for every point $x \in X$ the family $\{A \in \mathcal{A} : x \in A\}$ is finite (countable).

2.1.2. DEFINITION. A topological space X is called *strongly hereditarily normal* if X is a T_1-space and for every pair A, B of separated subsets of X there exist open sets

$U, V \subset X$ such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$, and both U and V can be represented as the unions of point-finite families of open F_σ-sets.

Obviously, every strongly hereditarily normal space is hereditarily normal; moreover, every subspace M of a strongly hereditarily normal space is strongly hereditarily normal, because any sets A, B separated in M are also separated in X.

Besides hereditary normality one considers another strengthening of normality, namely perfect normality. Let us recall that a space X is *perfectly normal* if X is a normal space and every open subset of X is an F_σ-set. Perfect normality is a hereditary property; in particular, every perfectly normal space is hereditarily normal (cf. Lemma 3.1.22 below). This fact and Theorem 2.1.1 imply

2.1.3. THEOREM. *Every perfectly normal space is strongly hereditarily normal.* ■

2.1.4. THEOREM. *Every hereditarily paracompact space is strongly hereditarily normal.*

PROOF. Since every paracompact space is normal (see [GT], Theorem 5.1.5), every hereditarily paracompact space is hereditarily normal. Thus to complete the proof it suffices to show that every open subset U of a hereditarily paracompact space X can be represented as the union of a point-finite family of open F_σ-sets.

For every $x \in U$ consider a neighbourhood U_x of the point x such that $\overline{U}_x \subset U$. The family $\mathcal{U} = \{U_x\}_{x \in U}$ is an open cover of the subspace U of X. Since the space U is paracompact, there exists a locally finite partition of unity $\{f_s\}_{s \in S}$ on U subordinated to $\mathcal{U}$ (see [GT], Theorem 5.1.9). For every $s \in S$ the set $U_s = f_s^{-1}((0,1])$ is an open F_σ-set in U. As U is open in X, so is each U_s. Furthermore, $U_s \subset U_x \subset \overline{U}_x \subset U$ for some $x \in U$, so that U_s is an F_σ-set in the closed subspace $\overline{U}_x$ of X, which implies that U_s is an F_σ-set in X. Finally, the family $\{U_s\}_{s \in S}$ is point-finite and $U = \bigcup_{s \in S} U_s$. ■

We shall now slightly generalize the last theorem. Let us recall that a topological space X is *weakly paracompact*[1] if X is a Hausdorff space and every open cover of the space X has a point-finite open refinement. Weakly paracompact spaces are not necessarily normal (see [GT], Example 5.3.4 or Exercise 5.3.B(b)).

2.1.5. THEOREM. *Every hereditarily weakly paracompact hereditarily normal space is strongly hereditarily normal.*

PROOF. It suffices to show that every open subset U of a hereditarily weakly paracompact hereditarily normal space X can be represented as the union of a point-finite family of open F_σ-sets.

For every $x \in U$ consider a neighbourhood U_x of the point x such that $\overline{U}_x \subset U$. The family $\mathcal{U} = \{U_x\}_{x \in U}$ is an open cover of the subspace U of X. Since the space U is weakly paracompact, the cover $\mathcal{U}$ has a point-finite open refinement $\mathcal{V} = \{V_s\}_{s \in S}$. The cover $\mathcal{V}$, as any point-finite open cover of a normal space, has a closed shrinking $\{F_s\}_{s \in S}$ (see Problem 3.1.A(b) or [GT], Theorem 1.5.18). By virtue of Urysohn's lemma for every $s \in S$ there exists a continuous function $f_s: U \to I$ such that $f_s(U \setminus V_s) \subset \{0\}$

[1] The terms *metacompact* and *point-paracompact* are also used.

and $f_s(F_s) \subset \{1\}$. For every $s \in S$ the set $U_s = f_s^{-1}((0,1])$ is an open F_σ-set in U. Obviously, U_s is open in X. Furthermore, $U_s \subset V_s \subset U_x \subset \overline{U}_x \subset U$ for some $x \in U$, so that U_s is an F_σ-set in X. Finally, the family $\{U_s\}_{s\in S}$ is point-finite and $U = \bigcup_{s\in S} U_s$. ■

We conclude this section with two examples: an example of a compact strongly hereditarily normal space which is neither perfectly normal nor hereditarily weakly paracompact and an example of a compact hereditarily normal space which is not strongly hereditarily normal.

2.1.6. EXAMPLE. Let W be the set of all ordinal numbers less than or equal to the first uncountable ordinal number ω_1. The set W is well-ordered by the natural order $<$. Consider on W the topology obtained by taking as a base all sets of the form

$$(1) \qquad (\alpha,\omega_1] = \{x : \alpha < x\}, \quad [0,\beta) = \{x : x < \beta\} \quad \text{or} \quad (\alpha,\beta) = \{x : \alpha < x < \beta\},$$

where $\alpha < \beta \leq \omega_1$. One easily sees that W is a Hausdorff space. We shall show that W is compact.

Let $\{U_s\}_{s\in S}$ be an open cover of the space W and let A consist of all $\alpha \in W$ such that the set $[0,\alpha] = \{x : x \leq \alpha\}$ is contained in the union of finitely many members of the cover. It suffices to show that $W \setminus A = \emptyset$.

Assume that $W \setminus A \neq \emptyset$ and denote by x_0 the smallest element of this set. Choose an $s_0 \in S$ such that $x_0 \in U_{s_0}$; since $x_0 > 0$, there exists an $x < x_0$ such that $(x, x_0] \subset U_{s_0}$. By the definition of x_0, the point x belongs to A, so that $[0,x] \subset \bigcup_{i=1}^k U_{s_i}$. It follows that $[0,x_0] \subset \bigcup_{i=0}^k U_{s_i}$, and we have a contradiction.

We shall now prove that every open subspace U of W is normal, i.e., that the space W is hereditarily normal. We shall say that a set $C \subset W$ is convex if $(\alpha,\beta) \subset C$ whenever $\alpha,\beta \in C$. One readily sees that the union of a family of convex sets is convex provided that the intersection of the family is non-empty. Hence, considering for each point $x \in U$ the union of all convex subsets of U which contain that point, we obtain a decomposition of the set U into pairwise disjoint convex sets $\{U_s\}_{s\in S}$, which will be called the convex components of the set U. The set U being open, all U_s's are open subsets of W, so that $U = \bigoplus_{s\in S} U_s$, where the symbol $\bigoplus$ denotes the sum of topological spaces (see [GT], Section 2.2). Since the set W is well-ordered by $<$, every open and convex proper subset of W is of the form (1). Thus, to prove that U is normal it suffices to show that all subspaces in (1) are normal. The subspaces of the form $(\alpha,\omega_1] = [\alpha+1,\omega_1]$, where $\alpha < \omega_1$, are normal as closed subspaces of the normal space W. The subspaces of the form $[0,\beta)$ and (α,β), where $\beta < \omega_1$, are regular second-countable spaces and thus are metrizable (see [GT], Theorem 4.2.9) and, *a fortiori*, normal. It remains to prove that the subspace $W_0 = W \setminus \{\omega_1\}$ of W is normal.

We shall show more, viz., that for every pair A, B of disjoint closed subsets of W_0 the closures $\overline{A}$ and $\overline{B}$ of A and B in the space W are disjoint. This follows from the fact that ω_1 belongs to at most one of the sets $\overline{A}$ and $\overline{B}$. Indeed, if we had $\omega_1 \in \overline{A} \cap \overline{B}$, we could define inductively two sequences, $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$, of countable ordinal numbers satisfying

$$\alpha_i < \beta_i < \alpha_{i+1}, \quad \alpha_i \in A, \quad \beta_i \in B \quad \text{for } i = 1,2,\ldots;$$

then the smallest ordinal number γ larger than all α_i's and β_i's would belong to $\overline{A} \cap \overline{B}$, which is impossible, since $\gamma < \omega_1$ because W_0 has no countable cofinal subset. Let us recall that a subset K of an ordered set X is *cofinal* in X if for every $\alpha \in X$ there exists a $\beta \in K$ such that $\alpha \leq \beta$; a set $K \subset X$ is *bounded* in X if it is not cofinal in X.

Now we shall show that W is strongly hereditarily normal. Since the space W is hereditarily normal, each pair A, B of separated subsets of X can be enlarged to a pair of disjoint open sets $U, V \subset X$. Let $U = \bigoplus_{s \in S} U_s$ and $V = \bigoplus_{t \in T} V_t$ be the decompositions of U and V into convex components. If all the convex components U_s and V_t are bounded in W, they are countable and, *a fortiori*, they are F_σ-sets. So that, in this case U and V can be represented as the unions of point-finite families of open F_σ-sets. On the other hand, if one of the convex components of U, say U_{s_0}, is cofinal in W, then the set V is bounded in W. In this case the sets A and B are contained, respectively, in disjoint open sets $U \cup \{\omega_1\}$ and V which can be represented as the unions of point-finite families of open F_σ-sets.

Every closed subset of W which is contained in W_0 is bounded in W, so that every F_σ-set in W which is contained in W_0 is also bounded in W. It follows that the open subset W_0 of W is not an F_σ-set. Thus W is not perfectly normal.

To prove that W is not hereditarily weakly paracompact it is enough to show that if $\mathcal{U} = \{U_s\}_{s \in S}$ is an open cover of the subspace W_0 of W and all U_s's are bounded in W_0, then $\mathcal{U}$ is not point-finite. Suppose that $\mathcal{U}$ is point-finite. Hence for every $x \in W_0$ the set $\mathrm{St}(x, \mathcal{U}) = \bigcup \{U_s : x \in U_s\}$, i.e., *the star of the point x with respect to the cover $\mathcal{U}$*, is bounded; thus one can define inductively an increasing sequence $\alpha_1 < \alpha_2 < \dots$ of countable ordinal numbers satisfying

$$\alpha_{i+1} \notin \mathrm{St}(\alpha_i, \mathcal{U}) \quad \text{for } i = 1, 2, \dots \tag{2}$$

The smallest ordinal number γ larger than all α_i's belongs to a member U_{s_0} of the cover $\mathcal{U}$. The set U_{s_0} is open and thus contains almost all α_i's. This contradiction with (2) shows that the cover $\mathcal{U}$ is not point-finite. ■

We now turn to an example of a compact hereditarily normal space which is not strongly hereditarily normal.

2.1.7. EXAMPLE. Let W' be a topological space homeomorphic to the space W described in Example 2.1.6 and such that $W' \cap W = \emptyset$, and let W_0' and ω_1' denote the counterparts of W_0 and ω_1 in W'. The sum $W \oplus W'$ is hereditarily normal and so is the space X obtained from $W \oplus W'$ by identifying the points ω_1 and ω_1', i.e., the quotient space determined by the decomposition of $W \oplus W'$ into the set $\{\omega_1, \omega_1'\}$ and all one-point sets $\{x\}$ with $x \in W_0 \cup W_0'$. However, the space X is not strongly hereditarily normal, because for the separated subsets $q(W_0)$ and $q(W_0')$ of X, where $q: W \oplus W' \to X$ is the natural quotient mapping, there exist no disjoint open sets U, V which can be represented as the unions of point-finite families of open F_σ-sets such that $q(W_0) \subset U$ and $q(W_0') \subset V$. Indeed, the only disjoint open sets $U, V \subset X$ that contain $q(W_0)$ and $q(W_0')$, respectively, are $U = q(W_0)$ and $V = q(W_0')$. Now, if $q(W_0)$ could be represented as the union of a point-finite family of open F_σ-sets, the subspace W_0 of W would have a point-finite open cover by sets bounded in W_0, whereas, by the last part of Example 2.1.6, no such cover exists. ■

Historical and bibliographic notes

Various restrictions of the class of normal spaces to a class where a satisfactory theory of the large inductive dimension Ind can be developed have been proposed more than once. The first step was made by Čech in [1932], who developed the theory of the large inductive dimension in the class of perfectly normal spaces. Then, Dowker in [1953] introduced the class of totally normal spaces and extended the theory to that class. Let us recall here that a topological space X is *totally normal* if X is a normal space and every open subset U of X can be represented as the union of a family of open F_σ-sets locally finite in U. Clearly, every perfectly normal space is totally normal and so is every hereditarily paracompact space; the fact that every totally normal space is hereditarily normal is by no means obvious (see Problem 2.1.C). Subsequently, Pasynkov in [1967] defined *Dowker spaces* as hereditarily normal spaces in which every open set can be represented as the union of a point-finite family of open F_σ-sets, and announced extensions of some theorems on Ind to this class of spaces. Proofs of the announced theorems, together with further results, were published by Lifanov and Pasynkov in [1970]. Finally, Nishiura in [1977] introduced the class of super normal spaces and correspondingly extended the theory of the large inductive dimension; according to Nishiura, a topological space X is called *super normal* if X is a T_1-space and for every pair A, B of separated sets in X there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, $U \cap V = \emptyset$ and both U and V can be represented as the union of a family of open F_σ-sets in X, locally finite in U and V respectively. Our class of strongly hereditarily normal spaces is obtained by amalgamating the ideas of Pasynkov and Nishiura. Let us add that in the process of extending the theory of the large inductive dimension from the class of totally normal spaces to the larger classes mentioned above, only slight modifications in Dowker's original arguments were necessary.

Problems

2.1.A. (a) A subspace A of a topological space X is *functionally open* (*functionally closed*)[1], if there exists a continuous function $f: X \to I$ such that $A = f^{-1}((0,1])$ (such that $A = f^{-1}(0)$).

Deduce from Urysohn's lemma that a subset A of a normal space is functionally open (closed) if and only if A is an open F_σ-set (a closed G_δ-set).

(b) Show that for every pair A, B of disjoint functionally closed subset of a topological space X there exists a continuous function $f: X \to I$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

2.1.B. (a) Show that a topological space X is strongly hereditarily normal if and only if X is hereditarily normal and every open domain in X can be represented as the union of a point-finite family of open F_σ-sets. Let us recall that a subset U of a topological space X is an *open domain* in X if $U = \operatorname{Int} \overline{U}$.

(b) Applying the fact that for every pair U, V of disjoint open sets in the Cantor cube $D^{\mathfrak{c}} = \prod_{t \in I} D_t$, where $D_t = D = \{0,1\}$ is the two-point discrete space for $t \in I$,

[1] The terms *cozero-set* (*zero-set*) are also used.

there exists a countable set $I_0 \subset I$ such that the projections of U and V onto $\prod_{t\in I_0} D_t$ are disjoint (see [GT], Problem 2.7.12(b)), show that every open domain in $D^{\mathbf{c}}$ is an F_σ-set. Observe, using Remark 1.3.18, that the Cantor cube $D^{\mathbf{c}}$ is not hereditarily normal.

2.1.C (Dowker [1953]). (a) Show that if a space X can be represented as the union of a locally finite family $\{F_s\}_{s\in S}$ of closed normal subspaces, then X is a normal space.

Hint. Map the sum $\bigoplus_{s\in S} F_s$ onto X and apply the fact that normality is an invariant of closed mappings.

(b) Show that if a space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed normal subspaces such that $F_i \subset \operatorname{Int} F_{i+1}$ for $i = 1, 2, \ldots$, then X is a normal space.

Hint. Note that the family $\{A_i\}_{i=1}^{\infty}$, where $A_1 = F_1$ and $A_i = F_i \setminus \operatorname{Int} F_{i-1}$ for $i > 1$, is locally finite.

(c) Show that every totally normal space is hereditarily normal and deduce that every subspace of a totally normal space is totally normal.

Hint. Apply (a), (b) and Problem 2.1.A(a).

2.1.D. (a) Prove that every paracompact totally normal space is hereditarily paracompact.

(b) Prove that every weakly paracompact Dowker space, and thus every weakly paracompact perfectly normed space, is hereditarily weakly paracompact.

2.1.E. (a) Prove that a T_1-space X is normal if and only if for every closed set $F \subset X$ and each open set $W \subset X$ that contains F there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F \subset \bigcup_{i=1}^{\infty} W_i$ and $\overline{W}_i \subset W$ for $i = 1, 2, \ldots$

Hint. Let A and B be disjoint closed subsets of a T_1-space X which has the property under consideration. Define sequences $W_1, W_2, \ldots$ and $V_1, V_2, \ldots$ of open subsets of X such that

$$A \subset \bigcup_{i=1}^{\infty} W_i, \quad B \subset \bigcup_{i=1}^{\infty} V_i \quad \text{and} \quad B \cap \overline{W}_i = \emptyset = A \cap \overline{V}_i \quad \text{for } i = 1, 2, \ldots$$

Verify that the sets $U = \bigcup_{i=1}^{\infty} G_i$ and $V = \bigcup_{i=1}^{\infty} H_i$, where $G_i = W_i \setminus \bigcup_{j\le i} \overline{V}_j$ and $H_i = V_i \setminus \bigcup_{j\le i} \overline{W}_j$, satisfy the conditions $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

(b) Prove that a T_1-space X is perfectly normal if and only if for every open set $W \subset X$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $W = \bigcup_{i=1}^{\infty} W_i$ and $\overline{W}_i \subset W$ for $i = 1, 2, \ldots$

2.2. Basic properties of the dimension Ind in normal and hereditarily normal spaces

Only a few of the theorems of Chapter 1 are valid for the dimension Ind in normal or hereditarily normal spaces. As noted above, Ind is not monotonic in hereditarily normal spaces and the sum theorem for Ind does not hold in normal spaces, so that one cannot think of developing a dimension theory for Ind in those classes of spaces.

In Section 2.3 such a theory will be developed in the more restricted class of strongly hereditarily normal spaces. In the present section we merely clear the ground for the considerations of the next one.

The definition of the large inductive dimension Ind was stated in Section 1.6; let us recall that $\operatorname{Ind} X = -1$ if and only if $X = \emptyset$, and that a normal space X satisfies the inequality $\operatorname{Ind} X \leq n \geq 0$ if and only if for every closed set $A \subset X$ and each open set $V \subset X$ which contains the set A there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\operatorname{Ind} \operatorname{Fr} U \leq n-1$. In other words, $\operatorname{Ind} X \leq n \geq 0$ if and only if for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\operatorname{Ind} L \leq n-1$.

Since normality is not a hereditary property, it may happen that the dimension Ind is defined for a space X, but it is not defined for some subspace of X. Still, normality being hereditary with respect to closed subsets, $\operatorname{Ind} M$ is defined for every closed subspace $M \subset X$. Moreover, in much the same way as Theorem 1.1.2 one obtains

2.2.1. THEOREM. *For every closed subspace M of a normal space X we have* $\operatorname{Ind} M \leq \operatorname{Ind} X$. ■

The counterpart of Theorem 1.5.1 reads as follows:

2.2.2. THEOREM. *If X is a normal space and* $\operatorname{Ind} X = n \geq 1$, *then for* $k = 0, 1, \ldots, n-1$ *the space X contains a closed subspace M_k such that* $\operatorname{Ind} M_k = k$. ■

In Example 2.2.12 we shall define a compact space Z which contains a normal subspace X such that $\operatorname{Ind} X > \operatorname{Ind} Z$. Hence, in Theorem 2.2.1 the assumption that M is a closed subspace of X cannot be replaced by the weaker assumption that $\operatorname{Ind} M$ is defined. In fact, a more spectacular pathology was discovered: there exists a hereditarily normal space X such that $\operatorname{Ind} X = \dim X = 0$, and yet X contains, for every natural number n, a subspace A_n with $\operatorname{Ind} A_n = \dim A_n = n$. The latter example, however, is too difficult to be described in this book.

We shall now show that for subspaces of a fixed hereditarily normal space X monotonicity of the dimension Ind is equivalent to its being monotonic with respect to open subspaces.

2.2.3. PROPOSITION. *For every hereditarily normal space X the following conditions are equivalent*:

(i) *For each subspace $Y \subset X$ and every subspace M of Y we have* $\operatorname{Ind} M \leq \operatorname{Ind} Y$.

(ii) *For each subspace $Y \subset X$ and every open subspace U of Y we have* $\operatorname{Ind} U \leq \operatorname{Ind} Y$.

PROOF. The implication (i)⇒(ii) is obvious. Suppose that X satisfies (ii). Condition (i) is satisfied if $\operatorname{Ind} Y = \infty$, so that it suffices to consider subspaces $Y \subset X$ with $\operatorname{Ind} Y < \infty$. We shall apply induction with respect to $\operatorname{Ind} Y$ to show that $\operatorname{Ind} M \leq \operatorname{Ind} Y$ whenever $M \subset Y$. Clearly, the inequality holds if $\operatorname{Ind} Y = -1$. Assume that the inequality is proved for all subspaces of X whose large inductive dimension does not exceed $n-1 \geq -1$ and consider a subspace $Y \subset X$ with $\operatorname{Ind} Y = n$ and an arbitrary subspace M of Y. Let A and B be disjoint closed subsets of M. As $U = Y \setminus (\overline{A} \cap \overline{B})$ is an open subspace of Y, we have $\operatorname{Ind} U \leq \operatorname{Ind} Y = n$ by virtue of (ii). The intersections $U \cap \overline{A}$

and $U \cap \overline{B}$ are disjoint closed subsets of U; therefore there exists a partition L in U between $U \cap \overline{A}$ and $U \cap \overline{B}$ such that $\operatorname{Ind} L \leq n-1$. Since $M \subset U$, $M \cap U \cap \overline{A} = A$, and $M \cap U \cap \overline{B} = B$, the set $M \cap L$ is a partition in M between A and B. By the inductive assumption $\operatorname{Ind}(M \cap L) \leq \operatorname{Ind} L \leq n-1$, so that $\operatorname{Ind} M \leq n = \operatorname{Ind} Y$. Thus X satisfies condition (i). ■

The separation and addition theorems for the dimension Ind hold in hereditarily normal spaces. From Lemma 1.2.9, Remark 1.2.10 and Theorem 2.2.1 one easily obtains the following theorem (cf. the proof of Theorem 1.2.11):

2.2.4. THE SEPARATION THEOREM FOR Ind. *If X is a hereditarily normal space and M is a subspace of X such that $\operatorname{Ind} M \leq n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\operatorname{Ind}(M \cap L) \leq n-1$.* ■

2.2.5. THE ADDITION THEOREM FOR Ind. *If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then*

$$\operatorname{Ind} X \leq \operatorname{Ind} X_1 + \operatorname{Ind} X_2 + 1.$$

PROOF. The theorem is obvious if one of the subspaces has dimension ∞, so that we can suppose that $m(X_1, X_2) = \operatorname{Ind} X_1 + \operatorname{Ind} X_2$ is finite. We shall apply induction with respect to that number. If $m(X_1, X_2) = -2$, then $X_1 = \emptyset = X_2$ and our inequality holds. Assume that the inequality is proved for every hereditarily normal space and for each pair of subspaces with the sum of large inductive dimensions less than $n \geq -1$ and consider subspaces X_1 and X_2 of a hereditarily normal space X such that $m(X_1, X_2) = n$; clearly, we can suppose that $\operatorname{Ind} X_1 \geq 0$. Let A and B be disjoint closed subsets of X. By virtue of the separation theorem there exists a partition L in X between A and B such that $\operatorname{Ind}(X_1 \cap L) \leq \operatorname{Ind} X_1 - 1$. Since $m(X_1 \cap L, X_2 \cap L) \leq \operatorname{Ind} X_1 + \operatorname{Ind} X_2 - 1 = n-1$, we have $\operatorname{Ind} L \leq n$ by the inductive assumption. This implies that $\operatorname{Ind} X \leq n+1 = \operatorname{Ind} X_1 + \operatorname{Ind} X_2 + 1$. ■

The addition theorem yields

2.2.6. COROLLARY. *If a hereditarily normal space X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \ldots, Z_{n+1}$ such that $\operatorname{Ind} Z_i \leq 0$ for $i = 1, 2, \ldots, n+1$, then $\operatorname{Ind} X \leq n$.* ■

Let us note that it is an open problem whether every hereditarily normal space X with $\operatorname{Ind} X \leq n$ can be represented as the union of $n+1$ subspaces whose large inductive dimensions do not exceed zero (this is not true in normal spaces).

Remark 1.3.2 implies that Theorem 1.3.1 can be restated as follows:

2.2.7. THEOREM. *If a normal space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that $\overline{\operatorname{Ind}} F_i \leq 0$ for $i = 1, 2, \ldots$, then $\overline{\operatorname{Ind}} X \leq 0$.* ■

Example 2.2.14 below shows that in normal spaces the dimension Ind satisfies only the sum theorem for dimension 0, i.e., Theorem 2.2.7. It is not known whether the situation improves in hereditarily normal spaces.

A very weak version of the sum theorem for the dimension Ind in normal spaces reads as follows:

2.2.8. PROPOSITION. *Let $\{X_s\}_{s\in S}$ be a family of normal spaces and let $X = \bigoplus_{s\in S} X_s$. The inequality $\operatorname{Ind} X \leq n$ holds if and only if $\operatorname{Ind} X_s \leq n$ for every $s \in S$.* ■

Some, rather specialized, results related to Theorems 2.2.5 and 2.2.7 are stated in Problem 2.2.C.

The status of the Cartesian product theorem for Ind in normal spaces is similar to that of the sum theorem. There exist compact spaces X and Y such that $\operatorname{ind} X = \operatorname{Ind} X = 1$, $\operatorname{ind} Y = \operatorname{Ind} Y = 2$, and yet $\operatorname{Ind}(X \times X) \geq \operatorname{ind}(X \times Y) \geq 4$ (cf. Theorem 2.4.10); there also exists a normal space Z, whose square $Z \times Z$ is normal, such that $\operatorname{Ind} Z = 0$ and yet $\operatorname{Ind}(Z \times Z) > 0$. The descriptions of these examples are very complicated and cannot be reproduced in this book.

It is not known if there exist hereditarily normal compact spaces X and Y such that $\operatorname{Ind}(X \times Y) > \operatorname{Ind} X + \operatorname{Ind} Y$; however, under the assumption of the continuum hypothesis there exists a locally compact perfectly normal space X with $\operatorname{Ind} X = 0$ such that the square $X \times X$ is perfectly normal, and yet $\operatorname{Ind}(X \times X) > 0$. Several situations when $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y$ will be discussed in Section 2.4.

Let us also note that there exists an inverse sequence $\mathbf{S} = \{X_i, \pi_j^i\}$ of compact spaces with $\operatorname{Ind} X_i = 1$ for $i = 1, 2, \ldots$ such that $\operatorname{Ind} \varprojlim \mathbf{S} = 2$. This example will not be discussed here (see Problem 2.2.D).

We now turn to a discussion of dimension preserving compactifications. As the reader certainly knows, among the compactifications of a completely regular space X a particular role is played by the Čech–Stone compactification βX (see [GT], Section 3.6) which can be characterized by the property that every continuous function $f\colon X \to I$ is continuously extendable over βX (we assume here that X is actually a subspace of βX). In the realm of normal spaces the Čech–Stone compactification can also be characterized by the property that disjoint closed subsets of X have disjoint closures in βX. Hence, for every closed subspace M of a normal space X the closure $\overline{M}$ of M in βX is the Čech–Stone compactification of the space M. We shall show that the Čech–Stone compactification preserves the dimension Ind.

We start with a lemma on partitions; in the lemma and in Theorem 2.2.10 the bar denotes the closure operator in the Čech–Stone compactification.

2.2.9. LEMMA. *Let X be a normal space and A, B a pair of disjoint closed subsets of βX. For every partition L' in the space X between $X \cap \overline{V}_1$ and $X \cap \overline{V}_2$, where V_1, V_2 are open subsets of βX such that $A \subset V_1$, $B \subset V_2$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, the closure $\overline{L'}$ is a partition between A and B in βX.*

PROOF. Let U', W' be open subsets of X satisfying

$$X \cap \overline{V}_1 \subset U', \quad X \cap \overline{V}_2 \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad X \setminus L' = U' \cup W'.$$

Consider the intersection $L = C \cap D$, where $C = \overline{U' \cup L'}$ and $D = \overline{W' \cup L'}$. Since $C \cup D = \overline{U' \cup W' \cup L'} = \overline{X} = \beta X$, the sets $U = \beta X \setminus D$ and $W = \beta X \setminus C$ are disjoint; clearly, U and W are open in βX, and $U \cup W = \beta X \setminus L$. As $X \cap \overline{V}_1 \subset U'$, we have $V_1 \cap (W' \cup L') = \emptyset$, so that $V_1 \cap D = \emptyset$, and thus $A \subset V_1 \subset \beta X \setminus D = U$; similarly, $B \subset W$. Hence L is a partition between A and B.

The sets $U' \cup L'$ and $W' \cup L'$ being closed in X, we have $X \cap L = X \cap (U' \cup L') \cap (W' \cup L') = L'$, so that $\overline{L'} \subset L$.

To conclude the proof it suffices to show that $L \subset \overline{L'}$. Consider a point $x \in L = C \cap D$ and an arbitrary neighbourhod G of x in βX. Let H be an open subset of βX satisfying $x \in H \subset \overline{H} \subset G$. One easily sees that

$$x \in \overline{U' \cup L'} \cap H \subset \overline{(U' \cup L') \cap H} \quad \text{and} \quad x \in \overline{W' \cup L'} \cap H \subset \overline{(W' \cup L') \cap H},$$

so that – by the characterization of βX we have just recalled – the closures of $(U' \cup L') \cap H$ and $(W' \cup L') \cap H$ in X intersect. The intersection of these closures being contained in $(U' \cup L') \cap (W' \cup L') \cap \overline{H} = L' \cap H \subset L' \cap G$, we see that every neighbourhood of x meets L', so that $x \in \overline{L'}$, and the inclusion $L \subset \overline{L'}$ is established. ■

2.2.10. THEOREM. *For every normal space X we have* $\operatorname{Ind} \beta X = \operatorname{Ind} X$.

PROOF. To begin, we shall prove that $\operatorname{Ind} X \leq \operatorname{Ind} \beta X$. The inequality being obvious if $\operatorname{Ind} \beta X = \infty$, we can suppose that $\operatorname{Ind} \beta X < \infty$. We shall apply induction with respect to $\operatorname{Ind} \beta X$. If $\operatorname{Ind} \beta X = -1$, then $\beta X = \emptyset = X$ and our inequality holds. Assume that the inequality holds for all normal spaces whose Čech–Stone compactification has dimension less than $n \geq 0$ and consider a normal space X such that $\operatorname{Ind} \beta X = n$. Let A and B be disjoint closed subsets of X. The closures $\overline{A}$ and $\overline{B}$ are also disjoint, so that there exists a partition L in βX between $\overline{A}$ and $\overline{B}$ such that $\operatorname{Ind} L \leq n - 1$. Clearly, $L_0 = L \cap X$ is a partition in X between A and B. Since $\beta L_0 = \overline{L_0} \subset L$, it follows from Theorem 2.2.1 and the inductive assumption that $\operatorname{Ind} L_0 \leq n - 1$, so that $\operatorname{Ind} X \leq n = \operatorname{Ind} \beta X$.

Now, we shall prove that $\operatorname{Ind} \beta X \leq \operatorname{Ind} X$ (cf. Problem 3.3.A(e)). As in the first part of the proof, we shall suppose that $\operatorname{Ind} X < \infty$ and apply induction with respect to $\operatorname{Ind} X$. Our equality holds if $\operatorname{Ind} X = -1$. Assume that it is proved for all normal spaces whose dimension Ind is less than $n \geq 0$ and consider a normal space X such that $\operatorname{Ind} X = n$. Let A and B be disjoint closed subsets of βX. There exist open sets $V_1, V_2 \subset \beta X$ such that

$$A \subset V_1, \quad B \subset V_2 \quad \text{and} \quad \overline{V}_1 \cap \overline{V}_2 = \emptyset.$$

The intersections $X \cap \overline{V}_1$ and $X \cap \overline{V}_2$ are closed in X and disjoint, so that there exists a partition L' in X between these sets such that $\operatorname{Ind} L' \leq n - 1$. By Lemma 2.2.9 the closure $\overline{L'}$ is a partition between A and B in βX. Since $\overline{L'} = \beta L$, by the inductive assumption we have $\operatorname{Ind} \overline{L'} \leq n - 1$, and thus $\operatorname{Ind} \beta X \leq n = \operatorname{Ind} X$. ■

2.2.11. COROLLARY. *For every normal space X and a dense normal subspace $M \subset X$ which has the property that every continuous function $f: M \to I$ is continuously extendable over X we have* $\operatorname{Ind} M = \operatorname{Ind} X$.

In other words, $\operatorname{Ind} Y = \operatorname{Ind} X$ *for every normal space X and every normal subspace Y of βX which contains X.*

PROOF. From the extendability of every continuous function $f: M \to I$ over X it follows that $\beta M = \beta X$. ■

Let us observe that the counterpart of Theorem 2.2.10 for the dimension ind does not hold (see Problem 2.2.G).

As the Čech–Stone compactification generally raises the weight of spaces (i.e., the smallest cardinality of a base), it is natural to ask if there exist compactifications preserving both the dimension Ind and the weight. One proves that for every normal space X there exists a compactification $\widetilde{X}$ such that $\operatorname{Ind}\widetilde{X} \leq \operatorname{Ind} X$ and $w(\widetilde{X}) = w(X)$, where w denotes the weight. Let us note that even more is true, viz., that for every integer $n \geq 0$ and every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a compact space $B_n(\mathbf{m})$ such that $\operatorname{Ind} B_n(\mathbf{m}) = n$, $w(B_n(\mathbf{m})) = \mathbf{m}$, and every normal space X which satisfies the conditions $\operatorname{Ind} X \leq n$ and $w(X) \leq \mathbf{m}$ is homeomorphic to a subspace of $B_n(\mathbf{m})$. In other words, there exists a compact universal space for the class of all normal spaces whose large inductive dimension is not larger than n and whose weight is not larger than $\mathbf{m}$. Both these results can be deduced from Pasynkov's factorization theorem for Ind (see the remark to Problem 2.2.F) in the same way as Theorems 3.4.2 and 3.4.3 are deduced from Theorem 3.4.1.

We conclude this section with the counterexamples announced above. The first is a compact space Z with $\operatorname{Ind} Z = 0$ which contains a normal subspace X such that $\operatorname{Ind} X > 0$.

2.2.12. DOWKER'S EXAMPLE. Let Q denote the subspace of the interval I consisting of all rational numbers in I (clearly, Q is homeomorphic to the space of rational numbers). By letting

$$xEy \quad \text{if and only if} \quad |x - y| \in Q$$

we define an equivalence relation E on the set I. Since each equivalence class of E is countable, the family of all equivalence classes has cardinality $\mathbf{c}$. Let us choose from it a subfamily of cardinality $\aleph_1$ which does not contain the equivalence class Q and let us arrange the members of this subfamily into a transfinite sequence $Q_0, Q_1, \ldots, Q_\alpha, \ldots$, $\alpha < \omega_1$.

For every $\gamma < \omega_1$ the set $P_\gamma = I \setminus \bigcup_{\alpha \geq \gamma} Q_\alpha$ satisfies the equality $\operatorname{ind} P_\gamma = 0$; indeed $Q \subset P_\gamma$, and since the sets Q_α are dense in I, the set P_γ contains no interval. Let W be the space of all ordinal numbers $\leq \omega_1$ and let $W_0 = W \setminus \{\omega_1\}$ (see Example 2.1.6). For every $\alpha < \omega_1$ the subspace $M_\alpha = [0, \alpha] = [0, \alpha + 1)$ is open-and-closed in W. Consider the Cartesian product $W \times I$ and its subspaces

$$X_\alpha = \bigcup_{\gamma \leq \alpha} (\{\gamma\} \times P_\gamma), \qquad X = \bigcup_{\alpha < \omega_1} X_\alpha \quad \text{and} \quad X^* = X \cup (\{\omega_1\} \times I).$$

As noted in Example 2.1.6, the subspaces M_α are metrizable. Being countable, they are zero-dimensional, so that $\operatorname{ind}(M_\alpha \times P_\alpha) = 0$ by virtue of Theorem 1.3.6. Since for every $\alpha < \omega_1$ the set X_α is open-and-closed in X and since $\operatorname{ind} X_\alpha = 0$ by the inclusion $X_\alpha \subset M_\alpha \times P_\alpha$, we have $\operatorname{ind} X = 0$. From Remark 1.3.18 and Theorem 1.6.5 it follows that there exists a compact space Z with $\operatorname{Ind} Z = 0$ which contains X as a subspace.

We shall now show that the space X^* is normal, which is the first step towards establishing the normality of the space X. As X^* is a subspace of $W \times I$, it is a T_1-space. Consider a pair A, B of disjoint closed subsets of X^*. The sets

$$F_1 = \{x \in I : (\omega_1, x) \in A\} \quad \text{and} \quad F_2 = \{x \in I : (\omega_1, x) \in B\}$$

are disjoint and closed in I so that there exist open sets $H_1, H_2 \subset I$ such that $F_1 \subset H_1, F_2 \subset H_2$ and $H_1 \cap H_2 = \emptyset$. For every $x \in I \setminus H_1$ there exists a neighbourhood $U_x \subset I$

of the point x and an ordinal number $\alpha(x) < \omega_1$ such that $[(W \setminus M_{\alpha(x)}) \times U_x] \cap A = \emptyset$. The set $I \setminus H_1$ being closed in I, we can choose a finite number of points $x_1, x_2, \ldots, x_k \in I \setminus H_1$ such that $I \setminus H_1 \subset \bigcup_{i=1}^{k} U_{x_i}$. For $\alpha_1 = \max(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_k)) < \omega_1$ we have $[(W \setminus M_{\alpha_1}) \times (I \setminus H_1)] \cap A = \emptyset$, i.e.,

$$A \cap (X^* \setminus X_{\alpha_1}) \subset (W \setminus M_{\alpha_1}) \times H_1. \tag{1}$$

In a similar way we find an $\alpha_2 < \omega_1$ such that

$$B \cap (X^* \setminus X_{\alpha_2}) \subset (W \setminus M_{\alpha_2}) \times H_2; \tag{2}$$

without loss of generality we can assume that $\alpha_1 \leq \alpha_2$. The sets $A \cap X_{\alpha_2}$ and $B \cap X_{\alpha_2}$ are disjoint and closed in X_{α_2}, so that there exist open sets $U_1, V_1 \subset X_{\alpha_2}$ such that

$$A \cap X_{\alpha_2} \subset U_1, \quad B \cap X_{\alpha_2} \subset V_1 \quad \text{and} \quad U_1 \cap V_1 = \emptyset; \tag{3}$$

since X_{α_2} is an open subset of X^*, the sets U_1 and V_1 are open in X^*. The sets $U = U_1 \cup \{[(W \setminus M_{\alpha_2}) \times H_1] \cap X^*\}$ and $V = V_1 \cup \{[(W \setminus M_{\alpha_2}) \times H_2] \cap X^*\}$ are open in X^*. By virtue of (1)–(3),

$$A \subset U, \quad B \subset V \quad \text{and} \quad U \cap V = \emptyset,$$

so that the space X^* is normal.

To prove that the space X is normal, it suffices to show that, for every pair A, B of disjoint closed subsets of X, their closures $\overline{A}$ and $\overline{B}$ in X^* are disjoint. Suppose that there exists a point $(\omega_1, x) \in \overline{A} \cap \overline{B}$. It follows from the definition of P_γ that there exists an $\alpha_0 < \omega_1$ such that $x \in P_\alpha$ for every $\alpha \geq \alpha_0$. We can readily define by induction two sequences $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ of countable ordinal numbers and two sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ of real numbers in I such that

$$\alpha_0 < \alpha_i < \beta_i < \alpha_{i+1}, \quad |x - x_i| < 1/i, \quad |x - y_i| < 1/i, \quad (\alpha_i, x_i) \in A \quad \text{and} \quad (\beta_i, y_i) \in B$$

for $i = 1, 2, \ldots$ Now, for the smallest ordinal number γ larger than all α_i's we have $(\gamma, x) \in A \cap B$, which is impossible. Thus $\overline{A} \cap \overline{B} = \emptyset$ and X is a normal space.

It remains to show that $\operatorname{Ind} X > 0$. Assume that $\operatorname{Ind} X = 0$ and consider the pair $W_0 \times \{0\}$, $W_0 \times \{1\}$ of disjoint closed subsets of the space X. Then there exists an open-and-closed set $U \subset X$ such that $W_0 \times \{0\} \subset U$ and $W_0 \times \{1\} \subset X \setminus U$. As established in the preceding paragraph, the closures $\overline{A}$ and $\overline{B}$ of the sets $A = U$ and $B = X \setminus U$ in X^* are disjoint. Since $X = A \cup B$ and $\overline{X} = X^*$, we have $\overline{A} \cup \overline{B} = X^*$. The sets $A_1 = \{x \in I : (\omega_1, x) \in \overline{A}\}$ and $B_1 = \{x \in I : (\omega_1, x) \in \overline{B}\}$ are disjoint and closed in I; moreover, $A_1 \cup B_1 = I$ and $A_1 \neq \emptyset \neq B_1$, because $0 \in A_1$ and $1 \in B_1$. Thus our assumption contradicts the connectedness of the interval I. Hence $\operatorname{Ind} X > 0$. ■

We now turn to the counterexample to the sum theorem for Ind in normal spaces. To begin, we shall describe two auxiliary spaces L and L_0, which are known as the long segment and the long line.

2.2.13. EXAMPLE. Let W_0 be the set of all countable ordinal numbers and let $L_0 = W_0 \times [0, 1)$. Define a linear order $<$ in the set L_0 by letting $(\alpha_1, t_1) < (\alpha_2, t_2)$ whenever $\alpha_1 < \alpha_2$ or $\alpha_1 = \alpha_2$ and $t_1 < t_2$. Furthermore, let $L = L_0 \cup \{\omega_1\}$ and extend the linear order $<$ over L by letting $x < \omega_1$ for every $x \in L_0$. By assigning to each $\alpha \in W_0$ the point $(\alpha, 0) \in L_0$ we define a one-to-one mapping of W_0 onto the set $W_0 \times \{0\} \subset L_0$; in

the sequel we shall identify α with $(\alpha, 0)$ and we shall consider W_0 as a subset of L_0. The set W of all ordinal numbers $\leq \omega_1$ is a subset of L.

Consider on L the topology obtained by taking as a base all sets of the form:

$$(x_0, \omega_1] = \{x : x_0 < x\}, \quad [0, x_1) = \{x : x < x_1\} \quad \text{or} \quad (x_0, x_1) = \{x : x_0 < x < x_1\},$$

where $x_0, x_1 \in L$ and $x_0 < x_1$. One easily sees that L is a connected compact space and that the subspace topology on $W \subset L$ coincides with the topology on W defined in Example 2.1.6. For every $x_0 \in L_0$ the closed subspace $[0, x_0] = \{x : x \leq x_0\}$ of L has a countable base, viz., the family of all sets $((\alpha_0, t_0), (\alpha_1, t_1))$, where $\alpha_0, \alpha_1 \leq x_0$ and t_0, t_1 are rational numbers in I, and thus is a compact metric space. The space L is called the *long segment*; the subspace L_0 of L is called the *long line*. ■

We shall now describe a compact space X with $\operatorname{Ind} X \geq 2$ which contains closed subspaces F_1, F_2 such that $F_1 \cup F_2 = X$ and $\operatorname{Ind} F_1 = \operatorname{Ind} F_2 = 1$.

2.2.14. LOKUCIEVSKIĬ'S EXAMPLE. Let L be the long segment and C the Cantor set. The subspace of the Cantor set consisting of the end-points of all intervals removed from I to obtain the Cantor set will be denoted by Q (cf. Problem 1.3.H(e)). Since L and C are compact, the Cartesian product $L \times C$ is a compact space. We shall now establish a property of open subsets of $L \times C$ which will prove crucial for the evaluation of $\operatorname{Ind} X$. Namely

(4) for every open set $U \subset L \times C$ such that $U \cap (\{\omega_1\} \times C) \neq \emptyset$ and $t_U = \sup\{t : (\omega_1, t) \in U\}$ belongs to $(C \setminus Q) \setminus \{1\}$ either
(i) there exists an $x' \in L_0$ such that $(x', \omega_1] \times \{t_U\} \subset \operatorname{Fr} U$
or
(ii) there exist a $t' \in (t_U, 1]$ and an L' cofinal in L_0 such that $L' \times ([t_U, t'] \cap C) \subset U$.

Consider a sequence $t_1, t_2, \ldots$ converging to t_U and such that $(\omega_1, t_i) \in U$ for $i = 1, 2, \ldots$ The set U being open, there exists a sequence $\alpha_1, \alpha_2, \ldots$ of countable ordinal numbers such that $(\alpha_i, \omega_1] \times t_i \subset U$ for $i = 1, 2, \ldots$ For the smallest ordinal number $\alpha_0 \in W_0 \subset L_0$ larger than all α_i's we have $(\alpha_0, \omega_1] \times \{t_U\} \subset \overline{U}$. If (i) does not hold, then there exists a set $L_1 \subset L_0$ cofinal in L_0 and such that $L_1 \times \{t_U\} \subset U$. For every $x \in L_1$ we can find a $t_x \in (t_U, 1]$ satisfying $\{x\} \times ([t_U, t_x] \cap C) \subset U$. Clearly, there exists a natural number k such that the set $L(k) = \{x \in L_1 : t_x - t_U > 1/k\}$ is cofinal in L_0, and thus (ii) holds with $t' = t_U + 1/k$ and $L' = L(k)$.

Let $f: C \to I$ be the continuous mapping of C onto I consisting in identifying the end-points of each interval removed from I to obtain the Cantor set (see Problem 1.3.D(b)) and let E be the equivalence relation on the space $L \times C$ corresponding to the decomposition of $L \times C$ into one-point subsets of $L_0 \times C$ and the sets $\{\omega_1\} \times f^{-1}(t)$, where $t \in I$. Since the equivalence relation E is closed, the quotient space $Y = (L \times C)/E$ is compact (see [GT], Theorem 3.2.11). The image of the set $\{\omega_1\} \times C \subset L \times C$ under the natural quotient mapping will be denoted by I and will be identified with the interval $[0, 1]$; the image of $\{\omega_1\} \times Q$ will be denoted by K.

We shall prove that $\operatorname{ind} Y = \operatorname{Ind} Y = 1$. To begin, let us observe that $\operatorname{ind} Y \leq 1$. Indeed, every neighbourhood of a point $x \in Y \setminus I$ contains a neighbourhood with boundary homeomorphic to $C \oplus C$, and every neighbourhood of a point $x \in I$ contains

a neighbourhood with boundary homeomorphic to $C \oplus B$, where B is a discrete space of cardinality ≤ 2; the last fact is a consequence of the density of K in I. Now we shall show that $\operatorname{Ind} Y \leq 1$. Consider a closed set $A \subset Y$ and an open set $V \subset Y$ which contains the set A. For every $x \in A$ there exists a neighbourhood U_x of x such that $U_x \subset V$ and $\operatorname{Fr} U_x$ is a zero-dimensional compact metric space. The subspace A of Y being compact, we can choose a finite number of points $x_1, x_2, \ldots, x_k$ such that $A \subset U = \bigcup_{i=1}^{k} U_{x_i} \subset V$. The subspace $F = \bigcup_{i=1}^{k} \operatorname{Fr} U_{x_i}$ of Y is normal, so that by virtue of Theorems 1.6.4 and 2.2.7 we have $\operatorname{Ind} F = 0$. Since $\operatorname{Fr} U \subset F$, it follows from Theorem 2.2.1 that $\operatorname{Ind} \operatorname{Fr} U \leq 0$. Hence $\operatorname{Ind} Y \leq 1$; as the space Y contains the interval I, we have $\operatorname{ind} Y = \operatorname{Ind} Y = 1$.

Let us note that (4) yields

(5) for every open set $U \subset Y$ such that $U \cap I \neq \emptyset$ and $\sup\{t : t \in U \cap I\}$ belongs to $(I \setminus K) \setminus \{1\}$ we have $\operatorname{ind} \operatorname{Fr} U \geq 1$.

Consider now two disjoint copies Y_1 and Y_2 of the space Y, and denote by I_i and K_i the counterparts of I and K in Y_i for $i = 1, 2$. Let $K_3 \subset I_2 \setminus K_2$ be a countable set dense in I_2 which does not contain the end-points of I_2. It follows from Problem 1.3.G(a) that there exists a homeomorphism $h\colon I_1 \to I_2$ such that $h(K_1) = K_3 \subset I_2 \setminus K_2$. Let S be the equivalence relation on the space $Y_1 \oplus Y_2$ corresponding to the decomposition of $Y_1 \oplus Y_2$ into one-point subsets of $(Y_1 \setminus I_1) \cup (Y_2 \setminus I_2)$ and the sets $\{x, h(x)\}$, where $x \in I_1$. Since this equivalence relation is closed, the quotient space $X = (Y_1 \oplus Y_2)/S$ is compact. Roughly speaking, the space X is obtained by pasting Y_1 and Y_2 together along I_1 and I_2 in such a way that no point of K_1 is identified with a point of K_2. For $i = 1, 2$ the image F_i of the set Y_i under the natural quotient mapping $q\colon Y_1 \oplus Y_2 \to X$ is closed in X and homeomorphic to Y, so that $\operatorname{Ind} F_1 = \operatorname{Ind} F_2 = 1$; moreover, $F_1 \cup F_2 = X$.

It remains to show that $\operatorname{Ind} X \geq 2$. Consider the mid-point x of the interval obtained by pasting I_1 and I_2, and a neighbourhood $V \subset X$ of the point x which does not contain the end-points of this interval. From (5) it follows that for every open set $U \subset X$ such that $x \in U \subset V$ we have $\operatorname{ind} \operatorname{Fr} U \geq 1$, because $\sup\{t : t \in q^{-1}(U) \cap I_i\}$ belongs to $I_i \setminus K_i$ either for $i = 1$ or for $i = 2$. Thus $\operatorname{Ind} X \geq \operatorname{ind} X \geq 2$; one can show that actually $\operatorname{Ind} X = \operatorname{ind} X = 2$ (see Problem 2.2.C(c)). ■

Historical and bibliographic notes

Theorem 2.2.1 was noted by Čech in [1932]. An example of a hereditarily normal space X with $\operatorname{Ind} X = \dim X = 0$ which for every natural number n contains a subspace A_n with $\operatorname{Ind} A_n = \dim A_n = n$ was described by E. Pol and R. Pol in [1979] (in [1977] the same authors gave an example to show that Ind and dim are not monotonic in hereditarily normal spaces); under additional set theoretic assumptions such an example was defined by Filippov in [1973]. Proposition 2.2.3 was proved by Dowker in [1953]. Theorem 2.2.5 was given by Smirnov in [1951]. An example of a normal space X with $\operatorname{Ind} X = 2$ that cannot be represented as the union of three subspaces with large inductive dimension zero was described by Zambahidze in [1982]. Theorem 2.2.7 is implicit in Čech's paper [1933] (cf. Theorems 1.6.10 and 3.1.8); it was explicitly formulated by Vedenissoff in [1939]. An example of compact spaces X and Y such that

$\operatorname{ind} X = \operatorname{Ind} X = 1$, $\operatorname{ind} Y = \operatorname{Ind} Y = 2$ and $\operatorname{Ind}(X \times Y) \geq \operatorname{ind}(X \times Y) \geq 4$ was described by Filippov in [1972] and an example of a normal space Z, whose square $Z \times Z$ is also normal, such that $\operatorname{Ind} Z = 0$ and yet $\operatorname{Ind}(Z \times Z) > 0$ was given by Wage in [1978]. In his original construction Wage applied the continuum hypothesis; Przymusiński in [1979] showed how to modify Wage's construction to avoid the continuum hypothesis. An example of a locally compact perfectly normal space X with $\operatorname{Ind} X = 0$ such that the square $X \times X$ is perfectly normal and satisfies $\operatorname{Ind}(X \times X) > 0$ was also given by Wage in [1978] (under the assumption of the continuum hypothesis). Theorem 2.2.10 was established by Vedenissoff in [1939]. The existence of compactifications preserving both the dimension Ind and the weight of normal spaces and the corresponding universal space theorem were announced by Pasynkov in [1971]; proofs were given in Alexandroff and Pasynkov's book [1973]. Example 2.2.12 was described by Dowker in [1955], and Example 2.2.14 – by Lokucievskiĭ in [1949].

Problems

2.2.A (Urysohn [1925]). (a) Prove that a subspace M of a hereditarily normal space X satisfies the inequality $\operatorname{ind} M \leq n \geq 0$ if and only if for every point $x \in M$ and each neighbourhood V of the point x in the space X there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{ind}(M \cap \operatorname{Fr} U) \leq n - 1$ (cf. Problem 4.1.C).

(b) Show that if a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then

$$\operatorname{ind} X \leq \operatorname{ind} X_1 + \operatorname{ind} X_2 + 1.$$

Remark. The sum theorem for ind does not hold even in the class of metrizable spaces (see Problem 4.1.B). Also the compactification theorem for ind does not hold: Smirnov in [1957] defined a normal space X with $\operatorname{ind} X = 1$ such that $\operatorname{ind} \widetilde{X} > 1$ for every compactification $\widetilde{X}$ of X; it is an open problem whether there exists such a space X which is metrizable (van Mill and Przymusiński defined in [1982] a perfectly normal space X with $\operatorname{ind} X = 1$ such that every Lindelöf space Y that contains X satisfies $\operatorname{ind} Y = \infty$).

(c) Show that if a regular space X is the union of two closed subspaces F_1 and F_2, where $\operatorname{ind} F_2 = n \geq 0$, then for every point $x \in X$ and each closed set $B \subset X$ with $x \notin B$ there exists a partition L between x and B which is the union of two subspaces F_1' and F_2' such that F_i' is a closed subspace of F_i for $i = 1, 2$ and $\operatorname{ind} F_2' \leq n - 1$.

Hint. See Lemma 7.2.4.

(d) (Fedorčuk [1978] for normal spaces) Show that if a regular space X is the union of two closed subspaces F_1 and F_2, then $\operatorname{ind} X \leq \operatorname{ind} F_1 + \operatorname{ind} F_2 + 1$.

2.2.B (Aarts and Nishiura [1972]; for metrizable spaces, Akasaki [1965]). Prove that for every continuous mapping $f: A \to S^k$ defined on a closed subspace A of a hereditarily normal space X such that $\operatorname{Ind}(X \setminus A) \leq n$, where $0 \leq k \leq n$, there exists a closed subspace B of X such that $A \cap B = \emptyset$, $\operatorname{Ind} B \leq n - k - 1$, and the mapping f has a continuous extension $F: X \setminus B \to S^k$ over $X \setminus B$.

Hint. See Problem 1.9.D(a).

2.2.C. (a) Show that if a normal space X is the union of two subspaces F and M, where F is closed, M is normal and $\operatorname{Ind} M = n \geq 0$, then for every pair A, B of disjoint closed subsets of X there exists a partition L in X between A and B which is the union of two subspaces F' and M' such that F' is a closed subspace of F, M' is a closed subspace of M and $\operatorname{Ind} M' \leq n - 1$.

Hint. See Lemma 7.2.5.

(b) Deduce from (a) that if a normal space X can be represented as the union of two subspaces F and M such that F is closed, $\operatorname{Ind} F = 0$, and M is normal and non-empty, then $\operatorname{Ind} X \leq \operatorname{Ind} M$.

(c) Show that if a normal space X is represented as the union of two closed subspaces F_1 and F_2, then $\operatorname{Ind} X \leq \operatorname{Ind} F_1 + \operatorname{Ind} F_2$.

(d) Show that if a normal space X is represented as the union of two normal subspaces X_1 and X_2 of which one is either closed or open, then $\operatorname{Ind} X \leq \operatorname{Ind} X_1 + \operatorname{Ind} X_2 + 1$.

(e) (Dowker [1953]) Show that if a hereditarily normal space X contains a closed subspace F such that $\operatorname{Ind} F \leq n$ and $\operatorname{Ind}(X \setminus F) \leq n$, then $\operatorname{Ind} X \leq n$ (cf. Theorem 2.3.1).

2.2.D. Apply Example 2.2.14 and Theorems 3.1.8 and 3.4.10 to define an inverse system $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ of compact spaces with $\operatorname{Ind} X_\sigma = 1$ for $\sigma \in \Sigma$ such that $\operatorname{Ind} \varprojlim \mathbf{S} = 2$ (for an inverse sequence with similar properties see Charalambous [1981]).

2.2.E. (a) Show that the second part of Lemma 1.2.9 does not hold in normal spaces (cf. Remark 1.2.10).

Hint. Consider the Cartesian product $X = L \times I$ of the long segment L and the unit interval I.

(b) Let M be a dense subspace of a normal space X and A, B a pair of disjoint closed subsets of X. Show that for every partition L' in the space M between $M \cap \overline{V}_1$ and $M \cap \overline{V}_2$, where V_1, V_2 are open subsets of X such that $A \subset V_1, B \subset V_2$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$, there exists a partition L in the space X between A and B such that $M \cap L = L'$. Note that it is not necessarily true that $\overline{L'}$ is such a partition (cf. Lemma 6.1.5).

Hint. See the proof of Lemma 2.2.9.

2.2.F. Prove that for every continuous mapping $f: X \to Y$ of a compact space X satisfying the equality $\operatorname{Ind} X = 0$ to a compact space Y there exist a compact space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\operatorname{Ind} Z = 0$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$ (cf. Theorems 1.6.10 and 3.4.1).

Hint. One can suppose that $w(Y) = \mathbf{m} \geq \aleph_0$. Consider a base $\mathcal{B}$ for the space Y with $|\mathcal{B}| = \mathbf{m}$ and the family $\mathcal{P}$ of all pairs $P = (U, V)$ of members of $\mathcal{B}$ satisfying $\overline{U} \cap \overline{V} = \emptyset$. For every $P = (U, V) \in \mathcal{P}$ take a continuous function $g_P: X \to \{0, 1\}$ such that $g_P(f^{-1}(\overline{U})) \subset \{0\}$ and $g_P(f^{-1}(V)) \subset \{1\}$, and define $g: X \to \{0, 1\}^{\mathbf{m}}$ by letting $g(x) = \{g_P(x)\}$. Let $Z = g(X)$, check that $w(Z) \leq \mathbf{m}$ and $\operatorname{Ind} Z = 0$ (see Remark 1.3.18 and Theorems 2.2.1 and 1.6.5) and show that for every $z \in Z$ the set $f(g^{-1}(z))$ consists of a single point.

Remark. This result is a special case of *Pasynkov's factorization theorem for* Ind (announced by Pasynkov in [1971]) which states that for every continuous mapping $f: X \to Y$ of a compact space X to a compact space Y there exist a compact space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\operatorname{Ind} Z \leq \operatorname{Ind} X$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$. The proof of this theorem is rather difficult (see Alexandroff and Pasynkov [1973], pp. 555–564).

2.2.G. Note that for the space X described in Example 2.2.12 we have $\operatorname{ind} \beta X \neq \operatorname{ind} X$.

2.2.H (Dowker [1955]). (a) Prove that the space X described in Example 2.2.12 satisfies the equality $\operatorname{Ind} X = 1$.

(b) Show that by adjoining one point to the space X described in Example 2.2.12 one can obtain a normal space X_1 with $\operatorname{ind} X_1 > 0$ and a normal space X_2 with $\operatorname{Ind} X_2 = 0$.

2.2.I (Smirnov [1958]). Modify the construction of the space X in Example 2.2.12 to obtain a compact space Z with $\operatorname{Ind} Z = 0$ which contains a normal subspace Y such that $\operatorname{Ind} Y = \infty$.

Hint. Replace the interval by the Hilbert cube and the sets P_γ by the Cartesian products $P_\gamma^{\aleph_0}$.

2.2.J. (a) Show that the long segment L is a strongly hereditarily normal space.

(b) Prove that for every point x_0 of the long line L_0 the subspace $[0, x_0]$ of L is homeomorphic to the interval I.

Hint. Define a countable dense subset of $[0, x_0]$ which is ordered similarly to the set of all rational numbers in I.

(c) Check that for the subspace W of the long segment L there exists no G_δ-set W^* in L such that $W \subset W^*$ and $\operatorname{Ind} W^* = \operatorname{Ind} W$.

2.3. Basic properties of the dimension Ind in strongly hereditarily normal spaces

Strongly hereditarily normal spaces constitute a relatively wide class of spaces where two of the most important theorems of dimension theory, viz., the subspace theorem and the countable sum theorem, hold for the dimension Ind. Both theorems are tightly connected and we shall prove them simultaneously. For more clarity the proof is divided into several lemmas.

Let us consider the following properties of a hereditarily normal space X:

(μ_n) *For each subspace* $Y \subset X$ *and every open subspace* U *of* Y, *if* $\operatorname{Ind} Y \leq n$, *then* $\operatorname{Ind} U \leq n$.

(σ_n) *For each subspace* $Y \subset X$ *and every sequence* $F_1, F_2, \ldots$ *of closed subspaces of* Y *such that* $Y = \bigcup_{i=1}^{\infty} F_i$, *if* $\operatorname{Ind} F_i \leq n$ *for* $i = 1, 2, \ldots$, *then* $\operatorname{Ind} Y \leq n$.

Clearly, every hereditarily normal space X has property (μ_{-1}), and thus to prove that all strongly hereditarily normal spaces have properties (μ_n) and (σ_n) for $n = -1, 0, 1, \ldots$, it suffices to show that the implications $(\mu_{n-1}) \Rightarrow (\mu_n)$ and $(\mu_n) \Rightarrow (\sigma_n)$ hold for every strongly hereditarily normal space X. The second implication holds for all hereditarily normal spaces; it will be deduced from the following version of the sum theorem.

2.3.1. THEOREM. *If a hereditarily normal space X can be represented as the union of a sequence $K_1, K_2, \ldots$ of pairwise disjoint subspaces such that $\operatorname{Ind} K_i \leq n$ and the union $\bigcup_{j \leq i} K_j$ is closed for $i = 1, 2, \ldots$, then $\operatorname{Ind} X \leq n$.*

PROOF. We shall apply induction with respect to n. For $n = -1$ the theorem is obvious. Assume that the corresponding statements hold for dimensions less than n and consider a hereditarily normal space X which satisfies the assumptions of our theorem. Let $F_i = \bigcup_{j \leq i} K_j$ for $i = 1, 2, \ldots$

Consider a pair A, B of disjoint closed subsets of X. Let U_0, V_0 be open subsets of X such that

$$A \subset U_0, \quad B \subset V_0 \quad \text{and} \quad \overline{U}_0 \cap \overline{V}_0 = \emptyset, \tag{1}$$

and let $K_0 = F_0 = L_0 = \emptyset$. We shall define inductively two sequences $U_0, U_1, \ldots$ and $V_0, V_1, \ldots$ of open subsets of X and a sequence $L_0, L_1, \ldots$ of subsets of X satisfying for $i = 0, 1, \ldots$ the following conditions:

$$L_i \subset K_i \quad \text{and} \quad \operatorname{Ind} L_i \leq n - 1. \tag{2}$$

$$\text{The set } E_i = \bigcup_{j \leq i} L_j \text{ is closed.} \tag{3}$$

$$F_i \subset U_i \cup V_i \cup E_i. \tag{4}$$

$$U_i \cup V_i \subset X \setminus E_i \quad \text{and} \quad \overline{U}_i \cap \overline{V}_i \subset E_i. \tag{5}$$

$$U_{i-1} \subset U_i \quad \text{and} \quad V_{i-1} \subset V_i \quad \text{for } i > 0. \tag{6}$$

The sets U_0, V_0 and L_0 satisfying (2)–(6) for $i = 0$ are already defined. Let us now assume that the sets U_i, V_i and L_i satisfying (2)–(6) are defined for $i < m > 0$. The sets $\overline{U}_{m-1} \cap K_m$ and $\overline{V}_{m-1} \cap K_m$ are closed in K_m and disjoint, because by virtue of (5) and (2) with $i = m - 1$ we have

$$\overline{U}_{m-1} \cap \overline{V}_{m-1} \subset E_{m-1} \subset \bigcup_{j<m} K_j$$

and the last set is disjoint from K_m. Since $\operatorname{Ind} K_m \leq n$, there exists a partition L_m in K_m between $\overline{U}_{m-1} \cap K_m$ and $\overline{V}_{m-1} \cap K_m$ which satisfies (2) with $i = m$; there also exist open subsets G, H of K_m such that

$$\overline{U}_{m-1} \cap K_m \subset G, \quad \overline{V}_{m-1} \cap K_m \subset H, \quad G \cap H = \emptyset, \quad \text{and} \quad K_m \setminus L_m = G \cup H. \tag{7}$$

From (7) it follows that $L_m \cap (U_{m-1} \cup V_{m-1}) = \emptyset$; the union $U_{m-1} \cup V_{m-1}$ being open in X, we have $\overline{L}_m \cap (U_{m-1} \cup V_{m-1}) = \emptyset$. Since L_m is closed in K_m and F_m is closed in X, the last equality and (4) with $i = m - 1$ yield the inclusion

$$\overline{L}_m \subset L_m \cup [F_{m-1} \setminus (U_{m-1} \cup V_{m-1})] \subset E_m,$$

which shows that (3) holds for $i = m$.

Since $G \cap V_{m-1} = \emptyset = H \cap U_{m-1}$, we have $\overline{G} \cap V_{m-1} = \emptyset = \overline{H} \cap U_{m-1}$, which implies that

$$\overline{G} \cap \overline{H} \subset L_m \cup [F_{m-1} \setminus (U_{m-1} \cup V_{m-1})] \subset E_m, \tag{8}$$

because $\overline{G} \cap \overline{H} \subset \overline{K}_m \subset \overline{F}_m = F_m$. The same equalities $\overline{G} \cap V_{m-1} = \emptyset = \overline{H} \cap U_{m-1}$ together with (7), the equality $U_{m-1} \cap V_{m-1} = \emptyset$ (a consequence of (5) with $i = m-1$) and (4) with $i = m-1$ yield

$$\overline{G} \cap \overline{V}_{m-1} = [\overline{G} \cap (\overline{V}_{m-1} \setminus V_{m-1})] \setminus K_m \subset F_{m-1} \cap (\overline{V}_{m-1} \setminus V_{m-1}) \subset E_{m-1} \tag{9}$$

and

$$\overline{H} \cap \overline{U}_{m-1} = [\overline{H} \cap (\overline{U}_{m-1} \setminus U_{m-1})] \setminus K_m \subset F_{m-1} \cap (\overline{U}_{m-1} \setminus U_{m-1}) \subset E_{m-1}. \tag{10}$$

Relations (8)–(10) and the second part of (5) with $i = m-1$ show that the sets $(\overline{U}_{m-1} \cup \overline{G}) \setminus E_m$ and $(\overline{V}_{m-1} \cup \overline{H}) \setminus E_m$ are disjoint. As these sets are closed in $X \setminus E_m$, from the hereditary normality of X it follows that there exist disjoint sets U_m, V_m open in $X \setminus E_m$, and consequently open in X, such that

$$(\overline{U}_{m-1} \cup \overline{G}) \setminus E_m \subset U_m, \quad (\overline{V}_{m-1} \cup \overline{H}) \setminus E_m \subset V_m,$$

and (5) is satisfied for $i = m$. The last two inclusions imply that (4) and (6) are also satisfied for $i = m$, because $L_m \cap (U_{m-1} \cup V_{m-1}) = \emptyset$ by virtue of (7). Hence the construction of the sets U_i, V_i and L_i satisfying (2)–(6) for $i = 1, 2, \ldots$ is completed.

Let $U = \bigcup_{i=1}^{\infty} U_i$, $V = \bigcup_{i=1}^{\infty} V_i$ and $L = \bigcup_{i=1}^{\infty} L_i$; it follows from (4) that $X = U \cup V \cup L$. The sets U and V are open and, by virtue of (6) and the equality $U_i \cap V_i = \emptyset$ which is a consequence of (5), disjoint. From (5), (6) and the inclusion $E_{i-1} \subset E_i$ it follows that $U \cap L = V \cap L = \emptyset$, so that $X \setminus L = U \cup V$, which together with (1) shows that L is a partition between A and B. By virtue of (2) and (3) the inductive assumption can be applied to the space L and the sequence $L_1, L_2, \ldots$; thus $\operatorname{Ind} L \leq n-1$, which shows that $\operatorname{Ind} X \leq n$. ■

2.3.2. COROLLARY. *If a hereditarily normal space X has property (μ_n), then it also has property (σ_n).*

PROOF. Consider a subspace $Y \subset X$ and a sequence $F_1, F_2, \ldots$ of closed subspaces of Y such that $Y = \bigcup_{i=1}^{\infty} F_i$ and $\operatorname{Ind} F_i \leq n$ for $i = 1, 2, \ldots$ By virtue of (μ_n) the set $K_i = F_i \setminus \bigcup_{j<i} F_j$ satisfies the inequality $\operatorname{Ind} K_i \leq n$ for $i = 1, 2, \ldots$ Applying Theorem 2.3.1 to the space Y and the sequence $K_1, K_2, \ldots$ we obtain the inequality $\operatorname{Ind} Y \leq n$. ■

We now turn to the implication $(\mu_{n-1}) \Rightarrow (\mu_n)$. To begin, we shall establish a simple property of point-finite open covers, which will be applied here and in the following chapters.

2.3.3. LEMMA. *Let $\{U_s\}_{s \in S}$ be a point-finite open cover of a topological space X. For $i = 1, 2, \ldots$ denote by K_i the set of all points of the space X which belong to exactly i members of the cover $\{U_s\}_{s \in S}$, by $\mathcal{T}_i$ the family of all subsets of S that have exactly i elements, and let $K_T = K_i \cap \bigcap_{s \in T} U_s$ for each $T \in \mathcal{T}_i$. Then*

(11) $X = \bigcup_{i=1}^{\infty} K_i$, $K_i \cap K_j = \emptyset$ *whenever* $i \neq j$, *and the union* $F_i = \bigcup_{j \leq i} K_j$ *is closed for* $i = 1, 2, \ldots$;

moreover,

(12) $K_i = \bigcup_{T \in \mathcal{T}_i} K_T$, *where the sets* K_T *are open in* K_i *and pairwise disjoint.*

PROOF. The first two equalities in (11) follow directly from the definition of the sets K_i. If $x \notin F_i$, then $x \in U_{s_1} \cap U_{s_2} \cap \ldots \cap U_{s_{i+1}} \subset X \setminus F_i$, where $s_1, s_2, \ldots, s_{i+1}$ are distinct elements of S, so that the sets F_i are closed. To establish (12) it suffices to note that $K_T \subset K_i$ for $T \in \mathcal{T}_i$ and that whenever T, T' are distinct members of $\mathcal{T}_i$, then $K_T \cap K_{T'} = \emptyset$, because the union $T \cup T'$ contains at least $i+1$ elements of S. ■

We shall now prove a lemma which, together with Corollary 2.3.2, yields the implication $(\mu_{n-1}) \Rightarrow (\mu_n)$ for every perfectly normal space X; afterwards, we shall deduce from this lemma that the implication $(\mu_{n-1}) \Rightarrow (\mu_n)$ holds for every strongly hereditarily normal space X.

2.3.4. LEMMA. *If a hereditarily normal space* X *has property* (σ_{n-1}) *and if* $\operatorname{Ind} X \leq n$, *then* $\operatorname{Ind} U \leq n$ *for every open subspace* U *of* X *which can be represented as the union of a point-finite family of open* F_σ*-sets in* X.

PROOF. Let us first consider the special case where U is an open F_σ-set in X. Then there exists a continuous function $f: X \to I$ such that $U = f^{-1}((0,1])$ (see Problem 2.1.A(a)). The sets $B_i = f^{-1}([1/i, 1])$ are closed in X and satisfy

(13) $B_i \subset \operatorname{Int} B_{i+1}$, $\operatorname{Ind} B_i \leq n$ for $i = 1, 2, \ldots$ and $U = \bigcup_{i=1}^{\infty} B_i$.

Consider an arbitrary set $A \subset U$ which is closed in U and an open set $V \subset U$ which contains the set A. For $i = 1, 2, \ldots$ let

(14) $A_i = A \cap (B_i \setminus \operatorname{Int} B_{i-1})$ and $V_i = V \cap (\operatorname{Int} B_{i+1} \setminus B_{i-2})$,

where $B_{-1} = B_0 = \emptyset$; clearly, $A_i \subset V_i \subset B_{i+1}$, A_i is closed in B_{i+1}, and V_i is open in B_{i+1}. Since $\operatorname{Ind} B_{i+1} \leq n$, there exists an open subset U_i of B_{i+1} such that

(15) $A_i \subset U_i \subset \overline{U}_i \subset V_i$ and $\operatorname{Ind} \operatorname{Fr} U_i \leq n-1$,

where the closure and boundary operators refer to the space U. Indeed, as B_{i+1} is a closed subset of U and $U_i \subset \operatorname{Int} B_{i+1}$, the closure and the boundary of U_i in B_{i+1} and in U coincide. From the inclusion $\overline{U}_i \subset U \setminus B_{i-2}$ it follows that the family $\{\overline{U}_i\}_{i=1}^{\infty}$ is locally finite in the space U, so that for the set $U_0 = \bigcup_{i=1}^{\infty} U_i$ we have $\overline{U}_0 = \bigcup_{i=1}^{\infty} \overline{U}_i$ and $\operatorname{Fr} U_0 \subset \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i$. Applying (13)–(15) and (σ_{n-1}), we obtain

$$A = A \cap \bigcup_{i=1}^{\infty} B_i \subset \bigcup_{i=1}^{\infty} A_i \subset U_0 \subset \overline{U}_0 = \bigcup_{i=1}^{\infty} \overline{U}_i \subset V \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} U_0 \leq n-1,$$

which shows that $\operatorname{Ind} U \leq n$.

Let us now pass to the general situation and consider a point-finite family $\{U_s\}_{s \in S}$ of open F_σ-sets in X such that $U = \bigcup_{s \in S} U_s$. Applying Lemma 2.3.3 to the cover $\{U_s\}_{s \in S}$

of the space U, we obtain the sets K_i which satisfy (11) and (12) with $X = U$. Since for each $T \in \mathcal{T}_i$ we have $K_T = K_i \cap \bigcap_{s \in T} U_s = F_i \cap \bigcap_{s \in T} U_s$, the sets K_T are F_σ-sets in X.

The subspace $X_i = (X \setminus U) \cup F_i$ is closed in X, so that $\operatorname{Ind} X_i \leq n$. Since the set $K_i = F_i \setminus F_{i-1} = X_i \cap (U \setminus F_{i-1})$ is open in X_i, the sets K_T are open F_σ-sets in X_i. Applying the already established special case of our theorem to the space X_i and the set K_T, we conclude that $\operatorname{Ind} K_T \leq n$ for every $T \in \mathcal{T}_i$. Thus $\operatorname{Ind} K_i \leq n$ for $i = 1, 2, \ldots$ by virtue of (12) and Proposition 2.2.8. Theorem 2.3.1 and (11) imply that $\operatorname{Ind} U \leq n$. ■

2.3.5. LEMMA. *If a strongly hereditarily normal space X has property (μ_{n-1}), then it also has property (μ_n).*

PROOF. Since strong hereditary normality and property (μ_{n-1}) are both hereditary properties, it suffices to show that if $\operatorname{Ind} X \leq n$, then $\operatorname{Ind} U \leq n$ for every open subspace U of X. Consider a pair A, B of disjoint closed subsets of the space U. Let $U' = X \setminus (\bar{A} \cap \bar{B})$, $A' = U' \cap \bar{A}$ and $B' = U' \cap \bar{B}$. Obviously, $U \subset U'$, $A \subset A'$, $B \subset B'$ and the sets A', B' are disjoint and closed in U'. It is enough to show that there exists a partition L' in U' between A' and B' such that $\operatorname{Ind} L' \leq n-1$, because then $L = U \cap L'$ will be a partition in U between A and B satisfying, by virtue of (μ_{n-1}), the inequality $\operatorname{Ind} L \leq n-1$. Since $\overline{A'} = \bar{A}$ and $\overline{B'} = \bar{B}$, we have $U' = X \setminus (\overline{A'} \cap \overline{B'})$. Thus without loss of generality we can suppose that $U = X \setminus (\bar{A} \cap \bar{B})$ and define the required partition in U.

As the sets A and B are separated in X, there exist disjoint open sets $V, W \subset X$ such that $A \subset V$, $B \subset W$ and V and W can be represented as the unions of point-finite families of open F_σ-sets in X. From the equality $\overline{V} \cap W = \emptyset = V \cap \overline{W}$ it follows that $\bar{A} \cap W = \emptyset = V \cap \bar{B}$, so that $(\bar{A} \cap \bar{B}) \cap (V \cup W) = \emptyset$, i.e., $V \cup W \subset U$. The space U being normal, there exists an open set $O \subset U$ such that $A \subset O \subset U \cap \bar{O} \subset V$. The sets A and $F = V \setminus O$ are disjoint and closed in V; since $\operatorname{Ind} V \leq n$ by virtue of Corollary 2.3.2 and Lemma 2.3.4, there exists a partition L in V between A and F such that $\operatorname{Ind} L \leq n-1$. Thus there exist open sets $G, H \subset X$ such that

$$A \subset G, \quad F \subset H, \quad G \cap H = \emptyset \quad \text{and} \quad V \setminus L = G \cup H.$$

The set $H \cup (U \setminus \bar{O})$ is open in X, disjoint from G, and contains B, because $B \subset W \subset U \setminus V \subset U \setminus \bar{O}$; moreover,

$$G \cup [H \cup (U \setminus \bar{O})] = (V \setminus L) \cup (U \setminus \bar{O}) = U \setminus L.$$

Thus L is a partition in U between A and B. ■

From Corollary 2.3.2 and Lemma 2.3.5 it follows that all strongly hereditarily normal spaces have properties (μ_n) and (σ_n) for $n = -1, 0, 1, \ldots$; this, together with Theorems 2.2.3 and 2.3.1, yields the following two results.

2.3.6. THE SUBSPACE THEOREM FOR Ind. *For every subspace M of a strongly hereditarily normal space X we have* $\operatorname{Ind} M \leq \operatorname{Ind} X$. ■

2.3.7. PROPOSITION. *If a strongly hereditarily normal space X can be represented as the union of a sequence $K_1, K_2, \ldots$ of subspaces such that* $\operatorname{Ind} K_i \leq n$ *and the union* $\bigcup_{j \leq i} K_j$ *is closed for $i = 1, 2, \ldots$, then* $\operatorname{Ind} X \leq n$. ■

The last proposition yields

2.3.8. THE COUNTABLE SUM THEOREM FOR Ind. *If a strongly hereditarily normal space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that $\operatorname{Ind} F_i \leq n$ for $i = 1, 2, \ldots$, then $\operatorname{Ind} X \leq n$.* ■

In the dimension theory of general spaces there occur also sum theorems of a different kind, where instead of countable covers one considers locally finite ones. Such theorems are not discussed in the classical dimension theory of separable metric spaces, because – as is easy to verify – every locally finite cover of a separable metric space is countable, and thus in such spaces the locally finite sum theorem is just a particular case of the countable sum theorem. Exactly as in the case of the countable sum theorem, the locally finite sum theorem for Ind in strongly hereditarily normal spaces will be deduced from a version of this theorem which holds in all hereditarily normal spaces. Let us note that in countably paracompact strongly hereditarily normal spaces (in particular, in perfectly normal spaces and in hereditarily weakly paracompact hereditarily normal spaces), the locally finite sum theorem is an easy consequence of Theorems 2.3.1, 2.3.6 and 2.3.8 (see Problem 2.3.B).

2.3.9. THEOREM. *If a hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of pairwise disjoint subspaces such that $\operatorname{Ind} K_\alpha \leq n$ and the union $\bigcup_{\beta<\alpha} K_\beta$ is closed for each $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha<\xi}$ is locally finite, then $\operatorname{Ind} X \leq n$.*

PROOF. We shall apply induction with respect to n. For $n = -1$ the theorem is obvious. Assume that the corresponding statements hold for dimensions less than n and consider a hereditarily normal space X which satisfies the assumptions of our theorem. Let $F_\alpha = \bigcup_{\beta\leq\alpha} K_\beta$ for $\alpha < \xi$.

Consider a pair A, B of disjoint closed subsets of X. We shall define inductively three transfinite sequences $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$, $V_1, V_2, \ldots, V_\alpha, \ldots$, $\alpha < \xi$, and $L_1, L_2, \ldots, L_\alpha, \ldots$, $\alpha < \xi$ of subsets of X satisfying for $\alpha < \xi$ the following conditions:

(16) $$L_\alpha \subset K_\alpha \quad \text{and} \quad \operatorname{Ind} L_\alpha \leq n - 1.$$

(17) $$U_\alpha \text{ and } V_\alpha \text{ are open in } F_\alpha \text{ and } U_\alpha \cup V_\alpha = F_\alpha \setminus E_\alpha, \text{ where } E_\alpha = \bigcup_{\beta\leq\alpha} L_\beta.$$

(18) $$A \cap F_\alpha \subset U_\alpha, \quad B \cap F_\alpha \subset V_\alpha \quad \text{and} \quad U_\alpha \cap V_\alpha = \emptyset.$$

(19) $$U_\beta = F_\beta \cap U_\alpha, \quad V_\beta = F_\beta \cap V_\alpha \quad \text{for} \quad \beta < \alpha.$$

Since $F_1 = K_1$ and $\operatorname{Ind} K_1 \leq n$, there exist sets U_1, V_1 and L_1 satisfying (16)–(19) for $\alpha = 1$. Let us assume now that the sets U_α, V_α and L_α satisfying (16)–(19) are defined for $\alpha < \gamma > 1$. Let

$$F'_\gamma = \bigcup_{\alpha<\gamma} K_\alpha, \quad U'_\gamma = \bigcup_{\alpha<\gamma} U_\alpha, \quad V'_\gamma = \bigcup_{\alpha<\gamma} V_\alpha \quad \text{and} \quad E'_\gamma = \bigcup_{\alpha<\gamma} L_\alpha;$$

note that if $\gamma = \gamma_0 + 1$, the sets defined above are equal to $F_{\gamma_0}, U_{\gamma_0}, V_{\gamma_0}$ and E_{γ_0}, respectively. From conditions (17), (18) and (19) with $\alpha < \gamma$ it follows that

(20) $$A \cap F'_\gamma \subset U'_\gamma, \quad B \cap F'_\gamma \subset V'_\gamma, \quad U'_\gamma \cap V'_\gamma = \emptyset \quad \text{and} \quad F'_\gamma \setminus E'_\gamma = U'_\gamma \cup V'_\gamma.$$

We shall show that the sets $U'_\gamma, V'_\gamma \subset F'_\gamma$ are open in F'_γ; clearly, it suffices to consider the case where γ is a limit number.

Consider an arbitrary point $x \in U'_\gamma$. Since the family $\{K_\alpha\}_{\alpha<\gamma}$ is locally finite, there exist a neighbourhood $W \subset X$ of the point x and an ordinal number $\alpha_0 < \gamma$ such that $W \cap K_\alpha = \emptyset$ whenever $\alpha_0 < \alpha < \gamma$, i.e., such that $W \cap F'_\gamma = W \cap F_{\alpha_0}$. The set U_{α_0} being open in F_{α_0}, there exists an open set $W' \subset X$ such that $U_{\alpha_0} = F_{\alpha_0} \cap W'$. From (19) it follows that $F_{\alpha_0} \cap U'_\gamma = U_{\alpha_0}$, so that

$$x \in W \cap U'_\gamma = W \cap F'_\gamma \cap U'_\gamma = W \cap F_{\alpha_0} \cap U'_\gamma = W \cap U_{\alpha_0} \subset W \cap W',$$

i.e., the set $W \cap W'$ is a neighbourhood of x in X. As

$$F'_\gamma \cap W \cap W' = W \cap F_{\alpha_0} \cap W' = W \cap U_{\alpha_0} \subset U_{\alpha_0} \subset U'_\gamma,$$

the set U'_γ is open in F'_γ. From symmetry of our assumptions it follows that the set V'_γ is also open in F'_γ. By virtue of (20) the set E'_γ is closed in F'_γ, which implies that E'_γ is closed in X, and $(A \cup B) \cap E'_\gamma = \emptyset$. The sets $A \cup U'_\gamma$ and $B \cup V'_\gamma$ are disjoint and closed in $X \setminus E'_\gamma$. From the hereditary normality of X it follows that there exist disjoint sets $G, H \subset X \setminus E'_\gamma$ open in $X \setminus E'_\gamma$, and consequently open in X, such that

$$A \cup U'_\gamma \subset G, \quad B \cup V'_\gamma \subset H \quad \text{and} \quad \overline{G} \cap \overline{H} \subset E'_\gamma. \tag{21}$$

By virtue of the last inclusion in (21) and by equality $E'_\gamma \cap K_\gamma = \emptyset$, the sets $G \cap K_\gamma$ and $H \cap K_\gamma$ have disjoint closures in K_γ. Hence there exists a partition L_γ in K_γ between $G \cap K_\gamma$ and $H \cap K_\gamma$ which satisfies (16) with $\alpha = \gamma$; there also exist open sets $G', H' \subset K_\gamma$ such that

$$G \cap K_\gamma \subset G', \quad H \cap K_\gamma \subset H', \quad G' \cap H' = \emptyset \quad \text{and} \quad K_\gamma \setminus L_\gamma = G' \cup H'. \tag{22}$$

Since the set $K_\gamma = F_\gamma \setminus F'_\gamma$ is open in F_γ, the sets G' and H' are open in F_γ. From (20), (21) and (22) it follows that the sets

$$U_\gamma = (G \cap F_\gamma) \cup G' \quad \text{and} \quad V_\gamma = (H \cap F_\gamma) \cup H'$$

satisfy conditions (17)–(19) with $\alpha = \gamma$. Hence the construction of the sets U_α, V_α and L_α satisfying (16)–(19) for $\alpha < \xi$ is completed.

It follows from (20) that the set $L = E'_\xi$ is a partition in $F'_\xi = X$ between A and B. Applying the inductive assumption to the space L and the sequence $L_1, L_2, \ldots, L_\alpha, \ldots$, $\alpha < \xi$, we obtain the inequality $\operatorname{Ind} L \leq n-1$, which shows that $\operatorname{Ind} X \leq n$. ■

Theorems 2.3.9 and 2.3.6 yield

2.3.10. THE LOCALLY FINITE SUM THEOREM FOR Ind. *If a strongly hereditarily normal space X can be represented as the union of a locally finite family $\{F_s\}_{s\in S}$ of closed subspaces such that* $\operatorname{Ind} F_s \leq n$ *for* $s \in S$, *then* $\operatorname{Ind} X \leq n$. ■

The following two theorems are common generalizations of the countable sum theorem and the locally finite sum theorem. Let us recall that a family $\mathcal{A}$ of subsets of a topological space X is *locally countable* if for every point $x \in X$ there exists a neighbourhood U of x such that the family $\{A \in \mathcal{A} : U \cap A \neq \emptyset\}$ is countable.

2.3.11. THEOREM. *If a strongly hereditarily normal space X can be represented as the union of a σ-locally finite family $\{F_s\}_{s\in S}$ of closed subspaces such that* $\operatorname{Ind} F_s \leq n$ *for* $s \in S$, *then* $\operatorname{Ind} X \leq n$.

PROOF. The family $\{F_s\}_{s\in S}$ decomposes into countably many locally finite families whose unions are closed in X and – by virtue of the locally finite sum theorem – have dimensions not larger than n each. To complete the proof it suffices to apply the countable sum theorem. ■

2.3.12. THEOREM. *If a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \ldots, F_\alpha, \ldots$, $\alpha < \xi$ of closed subspaces such that $\operatorname{Ind} F_\alpha \leq n$ and the family $\{F_\beta\}_{\beta<\alpha}$ is locally finite for each $\alpha < \xi$, and the family $\{F_\alpha\}_{\alpha<\xi}$ is locally countable, then $\operatorname{Ind} X \leq n$.*

PROOF. If the set of all ordinal numbers less than ξ contains a countable cofinal subset, then the family $\{F_\alpha\}_{\alpha<\xi}$ is σ-locally finite and the theorem follows from Theorem 2.3.11.

Assume now that the set of all ordinal numbers less than ξ contains no countable cofinal subset. To complete the proof it suffices to show that under this assumption the family $\{F_\alpha\}_{\alpha<\xi}$ is locally finite. Consider an arbitrary point $x \in X$ and a neighbourhood U of this point such that the set $\{\alpha < \xi : U \cap F_\alpha \neq \emptyset\}$ is countable. By our assumption there exists an $\alpha_0 < \xi$ such that $U \cap F_\alpha = \emptyset$ for $\alpha \geq \alpha_0$. The family $\{F_\alpha\}_{\alpha<\alpha_0}$ being locally finite, there exists a neighbourhood V of the point x such that the set $\{\alpha < \alpha_0 : V \cap F_\alpha \neq \emptyset\}$ is finite. The intersection $W = U \cap V$ is a neighbourhood of x which meets only finitely many sets F_α. ■

The next result is still another sum theorem. It will yield two further sum theorems which hold in the class of weakly paracompact strongly hereditarily normal spaces. Let us observe that in this class Theorem 2.3.13 generalizes the locally finite sum theorem (see Problem 2.3.F) and Theorem 2.3.15 generalizes both the countable sum theorem and the locally finite sum theorem.

2.3.13. THE POINT-FINITE SUM THEOREM FOR Ind. *If a strongly hereditarily normal space X can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed subspaces such that $\operatorname{Ind} F_s \leq n$ for $s \in S$, and if there exists a point-finite open cover $\{U_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then $\operatorname{Ind} X \leq n$.*

PROOF. Consider the decomposition of the space X described in Lemma 2.3.3. From the definition of the sets K_T it follows that $K_T \subset \bigcup_{s\in T} F_s$ for $T \in \mathcal{T}_i$, so that $\operatorname{Ind} K_T \leq n$ by virtue of Theorems 2.3.8 and 2.3.6. Theorem 2.2.8 and (12) imply that $\operatorname{Ind} K_i \leq n$ for $i = 1, 2, \ldots$ To complete the proof it suffices to apply Theorem 2.3.1. ■

2.3.14. THEOREM. *If a weakly paracompact strongly hereditarily normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of open subspaces such that $\operatorname{Ind} U_s \leq n$ for $s \in S$, then $\operatorname{Ind} X \leq n$.*

PROOF. The space X being weakly paracompact, one can assume that the cover $\{U_s\}_{s\in S}$ is point-finite, and thus has a closed shrinking $\{F_s\}_{s\in S}$ (see Problem 3.1.A(b) or [GT], Theorem 1.5.18). To complete the proof it suffices to apply Theorems 2.2.1 and 2.3.13. ■

2.3.15. THEOREM. *If a weakly paracompact strongly hereditarily normal space X can be represented as the union of a locally countable family $\{F_s\}_{s\in S}$ of closed subspaces such that* $\operatorname{Ind} F_s \leq n$ *for* $s \in S$, *then* $\operatorname{Ind} X \leq n$.

PROOF. For every point $x \in X$ there exist a neighbourhood U_x of x and a countable set $S(x) \subset S$ such that $U_x \cap F_s = \emptyset$ if $s \notin S(x)$. It follows that $U_x \subset \bigcup\{F_s : s \in S(x)\}$, so that by virtue of Theorems 2.3.8 and 2.3.6 we have $\operatorname{Ind} U_x \leq n$ for each $x \in X$. To complete the proof it suffices to apply Theorem 2.3.14. ■

We close this section with a characterization of the dimension Ind in strongly hereditarily normal spaces, which will be applied in Section 2.4.

2.3.16. LEMMA. *If for a pair A, B of disjoint closed subsets of a topological space X there exists a σ-locally finite open cover $\mathcal{V}$ of the space X which has the property that for every $V \in \mathcal{V}$ either $\overline{V} \cap A = \emptyset$ or $\overline{V} \cap B = \emptyset$, then there exists a partition L between A and B such that*

$$L \subset \bigcup\{\operatorname{Fr} V : V \in \mathcal{V}\}. \tag{23}$$

PROOF. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where the families $\mathcal{V}_i$ are locally finite. For $i = 1, 2, \ldots$ define $\mathcal{W}_i = \{V \in \mathcal{V}_i : \overline{V} \cap A = \emptyset\}$ and $\mathcal{U}_i = \mathcal{V}_i \setminus \mathcal{W}_i$; consider the sets

$$U_i = \bigcup \mathcal{U}_i \quad \text{and} \quad W_i = \bigcup \mathcal{W}_i.$$

The family $\mathcal{V}_i$ being locally finite,

$$\overline{U}_i \cap B = \emptyset = \overline{W}_i \cap A \quad \text{for } i = 1, 2, \ldots; \tag{24}$$

moreover,

$$A \subset \bigcup_{i=1}^{\infty} U_i \quad \text{and} \quad B \subset \bigcup_{i=1}^{\infty} W_i. \tag{25}$$

Let

$$G_i = U_i \setminus \bigcup_{j<i} \overline{W}_i \quad \text{and} \quad H_i = W_i \setminus \bigcup_{j\leq i} \overline{U}_j. \tag{26}$$

From (24)–(26) it follows that the open sets $U = \bigcup_{i=1}^{\infty} G_i$ and $W = \bigcup_{i=1}^{\infty} H_i$ satisfy the conditions $A \subset U, B \subset W$ and $U \cap W = \emptyset$. From the local finiteness of $\mathcal{V}_i$ it follows that $\operatorname{Fr} U_i \cup \operatorname{Fr} W_i \subset \bigcup\{\operatorname{Fr} V : V \in \mathcal{V}_i\}$, so that to complete the proof it suffices to show that the partition $L = X \setminus (U \cup W)$ between A and B satisfies the inclusion

$$L \subset \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i \cup \bigcup_{i=1}^{\infty} \operatorname{Fr} W_i. \tag{27}$$

Consider a point $x \in L$ and denote by F the first element of the sequence $\overline{U}_1, \overline{W}_1, \overline{U}_2, \overline{W}_2, \ldots$ that contains the point x. If $F = \overline{U}_i$, then $x \in \operatorname{Fr} U_i = \overline{U}_i \setminus U_i$, because $x \notin G_i$ and $x \notin \overline{W}_j$ for $j < i$. On the other hand, if $F = \overline{W}_i$, then $x \in \operatorname{Fr} \overline{W}_i = \overline{W}_i \setminus W_i$, because $x \notin H_i$ and $x \notin \overline{U}_j$ for $j \leq i$. Hence in both cases $x \in \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i \cup \bigcup_{i=1}^{\infty} \operatorname{Fr} W_i$. ■

2.3.17. THEOREM. *For every strongly hereditarily normal space X and each integer $n \geq 0$ the following conditions are equivalent*:

(i) *The space X satisfies the inequality* $\operatorname{Ind} X \leq n$.

(ii) *Every locally finite open cover of the space X has a locally finite open refinement $\mathcal{V}$ such that* $\operatorname{Ind}\operatorname{Fr} V \leq n-1$ *for* $V \in \mathcal{V}$.

(iii) *Every two-element open cover of the space X has a σ-locally finite open refinement $\mathcal{V}$ such that* $\operatorname{Ind}\operatorname{Fr} V \leq n-1$ *for every* $V \in \mathcal{V}$.

PROOF. Let $\{U_s\}_{s\in S}$ be a locally finite open cover of a normal space X such that $\operatorname{Ind} X \leq n \geq 0$. Consider a closed shrinking $\{F_s\}_{s\in S}$ of the cover $\{U_s\}_{s\in S}$ (see Problem 3.1.A(b)) and for every $s \in S$ choose an open set $V_s \subset X$ such that

$$F_s \subset V_s \subset U_s \quad \text{and} \quad \operatorname{Ind}\operatorname{Fr} V_s \leq n-1.$$

Clearly, the family $\mathcal{V} = \{V_s\}_{s\in S}$ is a locally finite open refinement of the cover $\{U_s\}_{s\in S}$. Hence (i)⇒(ii) for every normal space X.

The implication (ii)⇒(iii) being obvious, to complete the proof it suffices to show that (iii)⇒(i). Let X be a strongly hereditarily normal space which satisfies (iii). Consider a pair A, B of disjoint closed subsets of X. There exist open sets $U, W \subset X$ such that $A \subset U$, $B \subset W$ and $\overline{U} \cap \overline{W} = \emptyset$. The two-element open cover $\{X \setminus \overline{U}, X \setminus \overline{W}\}$ of the space X has a σ-locally finite open refinement $\mathcal{V}$ such that $\operatorname{Ind}\operatorname{Fr} V \leq n-1$ for every $V \in \mathcal{V}$. Since for every $V \in \mathcal{V}$ either $\overline{V} \cap A = \emptyset$ or $\overline{V} \cap B = \emptyset$, by virtue of Lemma 2.3.16 there exists a partition L between A and B such that $L \subset \bigcup\{\operatorname{Fr} V : V \in \mathcal{V}\}$. From Theorem 2.3.11 it follows that $\operatorname{Ind} L \leq n-1$, so that $\operatorname{Ind} X \leq n$. ■

Historical and bibliographic notes

The study of relations between properties (μ_n) and (σ_n) was originated by Dowker in [1953]. Dowker's paper contains Theorem 2.3.1, Corollary 2.3.2, Lemma 2.3.4 with "point-finite" replaced by "locally finite", as well as Theorems 2.3.6, 2.3.7 and 2.3.8 for totally normal spaces (Theorems 2.3.6 and 2.3.8 for perfectly normal spaces were established by Čech in [1932]). Lemma 2.3.3 (used implicitly by Zarelua in [1963]) and the present version of Lemma 2.3.4 were given by Lifanov and Pasynkov in [1970]; the same paper contains Theorems 2.3.6 and 2.3.8, as well as Proposition 2.3.7, for Dowker spaces (announced in Pasynkov [1967]). The last three results were extended to super normal spaces by Nishiura in [1977]. Theorem 2.3.9 was established by Lifanov and Pasynkov in [1970]. Theorem 2.3.10 was proved by Kimura in [1967] for totally normal spaces (under the additional assumption of countable paracompactness in [1963]) and was extended to super normal spaces by Nishiura in [1977]. Theorems 2.3.12, 2.3.14 and 2.3.15 were proved by Lifanov and Pasynkov in [1970] for Dowker spaces; for totally normal spaces, Theorem 2.3.14 was given by Dowker in [1955], and Theorem 2.3.15 – by Kimura in [1967] (implicitly). Theorem 2.3.17 was proved by Nagami in [1969] and [1960a], respectively for totally normal spaces and hereditarily paracompact spaces.

Problems

2.3.A (Smirnov [1951]). Let X be a normal space with the property that for each pair F_1, F_2 of closed subspaces of X with $\operatorname{Ind} F_i \leq n$ for $i = 1, 2$ we have $\operatorname{Ind}(F_1 \cup F_2) \leq n$. Prove that $\operatorname{ind} \beta X = \operatorname{Ind} X$.

Deduce that for every strongly hereditarily normal space X we have $\operatorname{ind} \beta X = \operatorname{Ind} X$.

2.3.B. Show that under the additional hypothesis that X is a countably paracompact space Theorem 2.3.10 is an easy consequence of Theorems 2.3.1, 2.3.6 and 2.3.8 (see p. 208 for the definition of countable paracompactness).

Hint. For $i = 1, 2, \ldots$ denote by K_i the set of all points of the space X which belong to exactly i members of the cover $\{F_s\}_{s \in S}$ and by $\mathcal{T}_i$ the family of all subsets of S that have exactly i elements. Note that $K_i = \bigcup_{T \in \mathcal{T}_i} K_T$, where $K_T = K_i \cap \bigcap_{s \in T} F_s$. Show that the sets $U_i = K_1 \cup K_2 \cup \ldots \cup K_i$ are open in X and that $\operatorname{Ind} U_i \leq n$ for $i = 1, 2, \ldots$ Apply the fact that the open cover $\{U_i\}_{i=1}^{\infty}$ of the space X has a closed shrinking (see Lemma 3.4.7).

2.3.C. (a) Show that if a strongly hereditarily normal space X can be represented as the union of two subspaces A_1 and A_2, one of them being both an F_σ-set and a G_δ-set, such that $\operatorname{Ind} A_1 \leq n$ and $\operatorname{Ind} A_2 \leq n$, then $\operatorname{Ind} X \leq n$. Deduce that in perfectly normal spaces the same holds under the assumption that one of the subspaces is the intersection of a closed and an open set.

(b) Observe that Proposition 2.3.7 follows from Theorem 2.3.8 and Problem 2.2.C(e).

(c) Deduce from Theorem 2.3.9 that if a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of subspaces such that $\operatorname{Ind} K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for each $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\operatorname{Ind} X \leq n$. Observe that the latter fact is also a consequence of Theorem 2.3.10 and Problem 2.2.C(e).

2.3.D (Lifanov and Pasynkov [1970]). Prove that if a hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of pairwise disjoint subspaces such that $\operatorname{Ind} K_\alpha \leq n$, the union $\bigcup_{\beta < \alpha} K_\beta$ is closed and the family $\{K_\beta\}_{\beta < \alpha}$ is locally finite for each $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally countable, then $\operatorname{Ind} X \leq n$.

2.3.E. (a) (Nagata [1965] for hereditarily paracompact spaces) Prove that if a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \ldots, F_\alpha, \ldots$, $\alpha < \xi$, of closed subspaces such that $\operatorname{Ind} F_\alpha \leq n$ for each $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$, of open subsets of X such that $F_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta < \alpha}$ is locally finite for each $\alpha < \xi$, then $\operatorname{Ind} X \leq n$.

Hint. Observe that if the set of all ordinal numbers less than ξ contains no countable cofinal subset, then the family $\{U_\alpha\}_{\alpha < \xi}$ is point-finite.

(b) (Nagata [1967] for totally normal spaces) Prove that if a strongly hereditarily normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots$, $K_\alpha, \ldots$, $\alpha < \xi$, of subspaces such that $\operatorname{Ind} K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for

each $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$, of open subsets of X such that $K_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta<\alpha}$ is locally finite for each $\alpha < \xi$, then $\operatorname{Ind} X \leq n$.

2.3.F (Smith and Krajewski [1971]). Prove that for every locally finite family $\{F_s\}_{s\in S}$ of closed subsets of a weakly paracompact space X there exists a point-finite family $\{U_s\}_{s\in S}$ of open sets such that $F_s \subset U_s$ for $s \in S$.

Hint. For every point $x \in X$ choose a neighbourhood U_x which meets only finitely many sets F_s. Consider a point-finite open refinement $\mathcal{V}$ of the cover $\{U_x\}_{x\in X}$ and let $U_s = \bigcup\{V \in \mathcal{V} : V \cap F_s \neq \emptyset\}$.

2.4. Relations between the dimensions ind and Ind. Cartesian product theorems for the dimension Ind. Dimension Ind and mappings

As shown in Section 1.6, for every separable metric space X we have $\operatorname{ind} X = \operatorname{Ind} X$. In the present section this equality will be extended to a somewhat larger class of spaces. However, we should make it clear at once that the class of all spaces whose small inductive and large inductive dimensions coincide is rather restricted. Indeed, there exists a completely metrizable space X, known as *Roy's space*, such that $\operatorname{ind} X = 0$ and $\operatorname{Ind} X = \dim X = 1$, as well as a first countable compact space X such that $\operatorname{ind} X = 2$ and $\operatorname{Ind} X = 3$ (cf. Theorem 2.4.3). The description of these two spaces is quite complicated and the computation of their dimensions is rather difficult, so that they will not be reproduced in this book. The only space with non-coinciding inductive dimensions discussed here in detail is the space X described in Example 2.2.12, which satisfies the equalities $\operatorname{ind} X = 0$ and $\operatorname{Ind} X = 1$.

Let us recall that a topological space X is *strongly paracompact*[1] if X is a Hausdorff space and every open cover of the space X has a star-finite open refinement (a family of sets $\mathcal{A}$ is *star-finite* if every set $A \in \mathcal{A}$ meets only finitely many members of $\mathcal{A}$). It immediately follows from the definitions that every strongly paracompact space is paracompact. One proves that every Lindelöf space is strongly paracompact (see [GT], Corollary 5.3.11).

In the proof of the next lemma we shall apply the fact that star-finite covers decompose in a natural way into countable families of sets. Let us recall that the *component* of a member A_0 of a family of sets $\mathcal{A}$ is the subfamily $\mathcal{A}_0 \subset \mathcal{A}$ consisting of all $A \in \mathcal{A}$ such that there exists a finite sequence $A_1, A_2, \ldots, A_k$ of members of $\mathcal{A}$ with $A_k = A$ and $A_i \cap A_{i-1} \neq \emptyset$ for $i = 1, 2, \ldots, k$. The components of two distinct members of $\mathcal{A}$ either coincide or are disjoint, so that

$$\mathcal{A} = \bigcup_{s\in S} \mathcal{A}_s, \quad \text{where } \mathcal{A}_s \cap \mathcal{A}_{s'} = \emptyset \text{ for } s \neq s'$$

and for every $s \in S$ the family $\mathcal{A}_s$ is the component of a member of $\mathcal{A}$. The representation of $\mathcal{A}$ as the union of the families $\mathcal{A}_s$ will be called the decomposition of $\mathcal{A}$ into components; clearly $(\bigcup \mathcal{A}_s) \cap (\bigcup \mathcal{A}_{s'}) = \emptyset$ for $s \neq s'$. Let us observe that all components

[1] The term *hypocompact* is also used.

of a star-finite family $\mathcal{A}$ are countable. Indeed, the component $\mathcal{A}_0$ of the set $A_0 \in \mathcal{A}$ can be represented as the union of countable families $\mathcal{A}_1, \mathcal{A}_2, \ldots$, where $\mathcal{A}_k$ consists of all $A \in \mathcal{A}$ such that there exists a sequence $A_1, A_2, \ldots, A_k$ of k members of $\mathcal{A}$ with $A_k = A$ and $A_i \cap A_{i-1} \neq \emptyset$ for $i = 1, 2, \ldots, k$.

2.4.1. LEMMA. *If X is a strongly paracompact space such that* $\operatorname{ind} X \leq n \geq 0$, *then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B which can be represented as the union of a sequence $L_1, L_2, \ldots$ of closed subspaces of X such that* $\operatorname{ind} L_i \leq n - 1$ *for* $i = 1, 2, \ldots$

PROOF. For every point $x \in X$ there exists a neighbourhood U_x of x such that $\operatorname{ind} \operatorname{Fr} U_x \leq n - 1$ and either $\overline{U}_x \cap A = \emptyset$ or $\overline{U}_x \cap B = \emptyset$. Consider a star-finite open refinement $\mathcal{U}$ of the cover $\{U_x\}_{x \in X}$ of the space X. Let $\mathcal{U} = \bigcup_{s \in S} \mathcal{U}_s$ be the decomposition of the cover $\mathcal{U}$ into components; as $\mathcal{U}$ is star-finite, the components are countable, i.e., $\mathcal{U}_s = \{U_{s,i}\}_{i=1}^{\infty}$. Let $X_s = \bigcup_{i=1}^{\infty} U_{s,i}$ for every $s \in S$; the sets X_s are open and pairwise disjoint, so that $X = \bigoplus_{s \in S} X_s$. For each set $U_{s,i}$ choose a point $x(s,i) \in X$ such that $U_{s,i} \subset U_{x(s,i)}$ and let $V_{s,i} = U_{x(s,i)} \cap X_s$; clearly, $X_s = \bigcup_{i=1}^{\infty} V_{s,i}$ and we have either $\overline{V}_{s,i} \cap A = \emptyset$ or $\overline{V}_{s,i} \cap B = \emptyset$. Since for $i = 1, 2, \ldots$ the family $\{V_{s,i}\}_{s \in S}$ is discrete, the family $\mathcal{V}$ consisting of all the sets $V_{s,i}$ is a σ-locally finite open cover of the space X. By virtue of Lemma 2.3.16 there exists a partition L between A and B such that $L \subset \bigcup_{i=1}^{\infty} F_i$, where $F_i = \bigcup_{s \in S} \operatorname{Fr} V_{s,i}$. Let $L_i = L \cap F_i$ for $i = 1, 2, \ldots$ To complete the proof it suffices to show that L_i is closed in X and $\operatorname{ind} L_i \leq n - 1$ for $i = 1, 2, \ldots$ Since $\operatorname{Fr} V_{s,i} \subset X_s$ for $s \in S$, we have $F_i = \bigoplus_{s \in S} \operatorname{Fr} V_{s,i}$, and F_i is a closed subset of X; thus L_i is closed in X for $i = 1, 2, \ldots$ On the other hand, $\operatorname{Fr} V_{s,i} \subset \operatorname{Fr} U_{x(s,i)} \cup \operatorname{Fr} X_s = \operatorname{Fr} U_{x(s,i)}$, so that $\operatorname{ind} \operatorname{Fr} V_{s,i} \leq n - 1$ for $s \in S$; thus $\operatorname{ind} F_i \leq n - 1$ and $\operatorname{ind} L_i \leq n - 1$ for $i = 1, 2, \ldots$ ■

From Lemma 2.4.1 we immediately obtain the following generalization of Theorem 1.6.5.

2.4.2. THEOREM. *For every strongly paracompact space X the conditions* $\operatorname{ind} X = 0$ *and* $\operatorname{Ind} X = 0$ *are equivalent.* ■

The next theorem shows that one step forward is possible.

2.4.3. THEOREM. *For every strongly paracompact space X the conditions* $\operatorname{ind} X = 1$ *and* $\operatorname{Ind} X = 1$ *are equivalent.*

PROOF. In consequence of Theorem 2.4.2 it suffices to show that for every strongly paracompact space X such that $\operatorname{ind} X = 1$ we have $\operatorname{Ind} X = 1$. Consider a pair A, B of disjoint closed subsets of the space X. By virtue of Lemma 2.4.1 there exists a partition L between A and B such that $L = \bigcup_{i=1}^{\infty} L_i$, where L_i is closed in X and $\operatorname{ind} L_i \leq 0$ for $i = 1, 2, \ldots$ As every closed subspace of a strongly paracompact space is strongly paracompact, it follows from Theorem 2.4.2 that $\operatorname{Ind} L_i \leq 0$ for $i = 1, 2, \ldots$, so that $\operatorname{Ind} L \leq 0$ by virtue of Theorem 2.2.7. Thus $\operatorname{Ind} X \leq 1$, and from the inequality $\operatorname{ind} X \leq \operatorname{Ind} X$ it follows that $\operatorname{Ind} X = 1$. ■

2.4.4. THEOREM. *For every strongly paracompact strongly hereditarily normal space X we have* $\operatorname{ind} X = \operatorname{Ind} X$.

PROOF. It suffices to show that $\operatorname{Ind} X \leq \operatorname{ind} X$; clearly, one can suppose that $\operatorname{ind} X < \infty$. We shall apply induction with respect to $\operatorname{ind} X$. The inequality holds if $\operatorname{ind} X = -1$. Assume that the inequality is proved for all strongly paracompact strongly hereditarily normal spaces of small inductive dimension less than $n \geq 0$ and consider a strongly paracompact strongly hereditarily normal space X such that $\operatorname{ind} X = n$. Let A, B be a pair of disjoint closed subsets of X. By virtue of Lemma 2.4.1 and by the inductive assumption there exists a partition L between A and B such that $L = \bigcup_{i=1}^{\infty} L_i$, where L_i is closed in X and $\operatorname{Ind} L_i \leq n-1$ for $i = 1, 2, \ldots$ By virtue of Theorem 2.3.8 we have $\operatorname{Ind} L \leq n$, so that $\operatorname{Ind} X \leq n = \operatorname{ind} X$. ■

Let us note that in the realm of Lindelöf spaces Lemma 2.4.1 is an immediate consequence of Lemma 2.3.16, so that for Lindelöf spaces Theorems 2.4.2, 2.4.3 and 2.4.4 can be obtained in a slightly simpler way.

In Problems 2.4.B and 2.4.C at the end of this section Theorems 2.4.2, 2.4.3 and 2.4.4 are extended to some classes of spaces larger than the class of strongly paracompact spaces. However, the definitions of these classes of spaces are a little less natural than the definition of strongly paracompact spaces and, to some degree, are inspired by the methods of proofs.

It can be seen from the last proof that the coincidence of ind and Ind is related to sum theorems. These theorems also play an important role in the Cartesian product theorems for inductive dimensions that we are going to discuss. In our further considerations only a weak form of sum theorems will be needed. We shall say that *the finite sum theorem for* Ind *holds* in a normal space X if for each pair F_1, F_2 of closed subspaces of X with $\operatorname{Ind} F_i \leq n$ for $i = 1, 2$ we have $\operatorname{Ind}(F_1 \cup F_2) \leq n$; if ind of closed subspaces of a regular space X has a similar property, we shall say that *the finite sum theorem for* ind *holds* in X. Clearly, the corresponding inequalities for arbitrary finite families of closed subspaces hold in both cases.

An obvious modification of the proof of Theorem 2.4.4 yields the following theorem (cf. Problem 2.3.A):

2.4.5. THEOREM. *For every compact space X in which the finite sum theorem for* Ind, *or the finite sum theorem for* ind, *holds we have* $\operatorname{ind} X = \operatorname{Ind} X$. ■

We now turn to a study of the behaviour of the dimension Ind under Cartesian multiplication. Let us begin with recalling that, as stated in Section 2.2, there exist compact spaces X and Y such that $\operatorname{Ind}(X \times Y) > \operatorname{Ind} X + \operatorname{Ind} Y$. Thus, one sees that the Cartesian product theorem for Ind, i.e., the inequality $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y$, requires rather strong assumptions on X and Y. We shall prove here two Cartesian product theorems for Ind whose proofs are relatively simple. They are special cases of more general results that will be discussed further in this section. We start with two lemmas.

2.4.6. LEMMA. *If a topological space X is represented as the union $F_1 \cup F_2 \cup \ldots \cup F_m$ of closed subspaces, then for each $A \subset X$ we have*

$$\operatorname{Fr} A \subset \operatorname{Fr}_{F_1}(A \cap F_1) \cup \operatorname{Fr}_{F_2}(A \cap F_2) \cup \ldots \cup \operatorname{Fr}_{F_m}(A \cap F_m),$$

where Fr_{F_i} denotes the boundary operator in the subspace F_i for $i \leq m$.

PROOF. Consider a point $x \in \operatorname{Fr} A = \bar{A} \cap \overline{X \setminus A}$. Clearly, there exists an $i \leq m$ such that $x \in \overline{A \cap F_i} \subset F_i$ and a $k \leq m$ such that $x \in \overline{F_k \setminus A} \subset F_k$. If $x \in A$ we have $x \in \overline{A \cap F_k} \cap \overline{F_k \setminus A} = \operatorname{Fr}_{F_k}(A \cap F_k)$; if $x \in X \setminus A$ we have $x \in \overline{A \cap F_i} \cap \overline{F_i \setminus A} = \operatorname{Fr}_{F_i}(A \cap F_i)$. ∎

2.4.7. LEMMA. *If the finite sum theorem for* Ind *holds in a normal space X, and $\mathcal{M}$ is a finite family of non-empty closed subspaces of X, then for every point $x \in X$ and each neighbourhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that $x \in U \subset V$ and $\operatorname{Ind} \operatorname{Fr}_M(U \cap M) \leq \operatorname{Ind} M - 1$ for each $M \in \mathcal{M}$, where Fr_M denotes the boundary operator in the subspace M.*

PROOF. Arrange the set $\{\operatorname{Ind} M : M \in \mathcal{M}\}$ into an increasing sequence $n_1 < n_2 < \ldots < n_k$ and let

$$A_i = \{x\} \cup \bigcup \{M \in \mathcal{M} : \operatorname{Ind} M \leq n_i\} \quad \text{for } i = 1, 2, \ldots, k.$$

Clearly

(1) $$A_1 \subset A_2 \subset \ldots \subset A_k \quad \text{and} \quad \operatorname{Ind} A_i = n_i \quad \text{for } i = 1, 2, \ldots, k.$$

Define by recursion a sequence $U_k, U_{k-1}, \ldots, U_1$ of neighbourhoods of the point x in the space X such that

(2) $$U_1 \subset U_2 \subset \ldots \subset U_k \subset V \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr}_i(U_i \cap A_i) \leq n_i - 1 \quad \text{for } i = 1, 2, \ldots, k,$$

where Fr_i denotes the boundary operator in A_i. The set $U = U_k \setminus (F_1 \cup F_2 \cup \ldots \cup F_{k-1}) \subset V$, where $F_j = A_j \setminus U_j$ for $j = 1, 2, \ldots, k-1$, is open and contains the point x. Moreover, by (1) and (2) for $i > 1$ we have

$$\begin{aligned} U \cap A_i &= U_k \cap A_i \setminus [(F_1 \cup F_2 \cup \ldots \cup F_{i-1}) \cap A_i \cup (A_i \setminus U_i)] \\ &= U_i \cap A_i \setminus [(F_1 \cup F_2 \cup \ldots \cup F_{i-1}) \cap A_i], \end{aligned}$$

so that

$$\operatorname{Fr}_i(U \cap A_i) \subset \operatorname{Fr}_i(U_i \cap A_i) \cup \operatorname{Fr}_i((F_1 \cup F_2 \cup \ldots \cup F_{i-1}) \cap A_i),$$

and $\operatorname{Ind} \operatorname{Fr}_i(U \cap A_i) \leq n_i - 1$, because $\operatorname{Fr}_i((F_1 \cup F_2 \cup \ldots \cup F_{i-1}) \cap A_i) \subset F_1 \cup F_2 \cup \ldots \cup F_{i-1} \subset A_{i-1}$ and $\operatorname{Ind} A_{i-1} \leq n_i - 1$. Since, as one easily checks, $U \cap A_1 = U_1 \cap A_1$, also $\operatorname{Ind} \operatorname{Fr}_1(U \cap A_1) \leq n_1 - 1$.

Now, for each $M \in \mathcal{M}$ and $n_i = \operatorname{Ind} M$ we have $M \subset A_i$, so that $\operatorname{Fr}_M(U \cap M) \subset \operatorname{Fr}_i(U \cap A_i)$, and thus $\operatorname{Ind} \operatorname{Fr}_M(U \cap M) \leq n_i - 1 = \operatorname{Ind} M - 1$. ∎

2.4.8. THE FIRST CARTESIAN PRODUCT THEOREM FOR Ind. *For every pair X, Y of compact spaces, not both empty, such that the finite sum theorem for* Ind *holds both in X and in Y we have*

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

PROOF. The theorem is obvious if $\operatorname{Ind} X = \infty$ or $\operatorname{Ind} Y = \infty$, so that we can suppose that $\operatorname{Ind} X + \operatorname{Ind} Y$ is finite. Let now $A_1, A_2, \ldots, A_m$ be a finite sequence of closed subspaces of X, and $B_1, B_2, \ldots, B_m$ a sequence of closed subspaces of Y; consider

closed subspaces $F_i = A_i \times B_i$ of $X \times Y$, where $i = 1, 2, \ldots, m$, and the union $F = F_1 \cup F_2 \cup \ldots \cup F_m$. We shall show by induction with respect to k that

(3) $$\text{if } \operatorname{Ind} A_i + \operatorname{Ind} B_i \leq k \text{ for } i = 1, 2, \ldots, m, \text{ then } \operatorname{Ind} F \leq k.$$

Our theorem follows readily from (3) with $k = \operatorname{Ind} X + \operatorname{Ind} Y$.

If $k = -1$, for each $i \leq m$ we have $A_i = \emptyset$ or $B_i = \emptyset$, so that $F = \emptyset$; thus (3) holds for $k = -1$. Assume that (3) is proved for each m if $k < n \geq 0$ and consider sequences $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ such that $\operatorname{Ind} A_i + \operatorname{Ind} B_i \leq n$ for $i = 1, 2, \ldots, m$. Let A be a closed subset of F and V an open subset of $X \times Y$ such that $A \subset V$. To conclude the proof it suffices to define an open subset $U \subset X \times Y$ such that

(4) $$A \subset U \subset V \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr}_F(U \cap F) \leq n - 1.$$

Lemma 2.4.7 and the compactness of A imply the existence of finitely many open subsets $U_1, U_2, \ldots, U_n$ of X and open subsets $V_1, V_2, \ldots, V_n$ of Y such that $A \subset U = \bigcup_{j=1}^{n} (U_j \times V_j) \subset V$.

(5) $$\operatorname{Ind} \operatorname{Fr}_{A_i}(U_j \cap A_i) \leq \operatorname{Ind} A_i - 1 \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr}_{B_i}(V_j \cap B_i) \leq \operatorname{Ind} B_i - 1$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

By Lemma 2.4.6 we have

$$\begin{aligned}
\operatorname{Fr}_F(U \cap F) &\subset \bigcup_{i=1}^{m} \operatorname{Fr}_{F_i}(U \cap F_i) = \bigcup_{i=1}^{m} \operatorname{Fr}_{F_i}\Big(F_i \cap \bigcup_{j=1}^{n} (U_j \times V_j)\Big) \\
&\subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \operatorname{Fr}_{F_i}(F_i \cap (U_j \times V_j)) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \operatorname{Fr}_{F_i}((A_i \cap U_j) \times (B_i \cap V_j)) \\
&\subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [(A_i \times \operatorname{Fr}_{B_i}(B_i \cap V_j)) \cup (\operatorname{Fr}_{A_i}(A_i \cap U_j) \times B_i)].
\end{aligned}$$

Denote the last union by E. Since from (5) it follows that $\operatorname{Ind} A_i + \operatorname{Ind} \operatorname{Fr}_{B_i}(B_i \cap V_j) \leq n - 1$ and $\operatorname{Ind} \operatorname{Fr}_{A_i}(A_i \cap U_j) + \operatorname{Ind} B_i \leq n - 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, by the inductive assumption we have $\operatorname{Ind} E \leq n - 1$, so that $\operatorname{Ind} \operatorname{Fr}_F(U \cap F) \leq n - 1$. Thus the set U satisfies (4). ■

2.4.9. THEOREM. *For every pair X, Y of compact strongly hereditarily normal spaces, not both empty, we have*

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y. \blacksquare$$

From Theorems 2.2.7 and 2.4.8 we obtain

2.4.10. THEOREM. *For every pair X, Y of compact spaces such that $\operatorname{Ind} X = \operatorname{Ind} Y = 1$ we have $\operatorname{Ind}(X \times Y) \leq 2$.* ■

Now we pass to the second Cartesian product theorem for Ind.

2.4.11. LEMMA. *The Cartesian product $X \times Y$ of a metrizable space X and a perfectly normal space Y is perfectly normal.*

PROOF. As $X \times Y$ is a T_1-space it suffices to show (see Problem 2.1.E(b)) that for every open set $W \subset X \times Y$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of $X \times Y$

such that $W = \bigcup_{i=1}^{\infty} W_i$ and $\overline{W_i} \subset W$ for $i = 1, 2, \ldots$ Let $\mathcal{U}$ be a base for the space X which can be represented as the union of locally finite families $\mathcal{U}_1, \mathcal{U}_2, \ldots$ (see [GT], Corollary 4.4.4). Consider the family $\mathcal{W}$ of all sets $U \times V$, where $U \in \mathcal{U}$ and V is an open subset of Y, such that $\overline{U} \times \overline{V} \subset W$; clearly $W = \bigcup \mathcal{W}$. For every $U \in \mathcal{U}$ the union $V(U) = \bigcup\{V : U \times V \in \mathcal{W}\}$ is open in Y, therefore $V(U) = \bigcup_{j=1}^{\infty} V_j(U)$, where $V_j(U)$ is open in Y and $\overline{V_j(U)} \subset V(U)$ for $j = 1, 2, \ldots$ The family $\mathcal{W}_{j,k} = \{U \times V_j(U) : U \in \mathcal{U}_k\}$ is locally finite in $X \times Y$ for $j, k = 1, 2, \ldots$, so that the set $W_{j,k} = \bigcup \mathcal{W}_{j,k}$ satisfies the inclusion $\overline{W}_{j,k} \subset W$. Since $\bigcup_{j,k=1}^{\infty} W_{j,k} = \bigcup \mathcal{W} = W$, to complete the proof it suffices to arrange all the sets $W_{j,k}$ into a sequence $W_1, W_2, \ldots$ ■

2.4.12. THE SECOND CARTESIAN PRODUCT THEOREM FOR Ind. *For every metrizable space X and every perfectly normal space Y, not both empty, we have*

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

PROOF. One can suppose that $k(X, Y) = \operatorname{Ind} X + \operatorname{Ind} Y$ is finite. We shall apply induction with respect to that number. If $k(X, Y) = -1$, then our inequality holds. Assume that the inequality is proved for every metrizable space and every perfectly normal space such that the sum of their large inductive dimensions is less than $k \geq 0$, and consider a metrizable space X and a perfectly normal space Y with $\operatorname{Ind} X = n \geq 0$, $\operatorname{Ind} Y = m \geq 0$ and $n + m = k$. Let ρ be a metric on the space X. Since every metrizable space is paracompact, the space X has for $i = 1, 2, \ldots$ a locally finite open cover $\mathcal{V}_i$ such that $\operatorname{mesh} \mathcal{V}_i < 1/i$. By virtue of Theorem 2.3.17 the cover $\mathcal{V}_i$ has a locally finite open refinement $\mathcal{U}_i$ such that $\operatorname{Ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{U}_i$. Clearly, the family $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ is a base for X.

Consider now an arbitrary two-element open cover $\{G, H\}$ of the Cartesian product $X \times Y$; let $\mathcal{W}$ be a refinement of the cover $\{G, H\}$ consisting of sets of the form $U \times V$, where $U \in \mathcal{U}$ and V is an open subset of Y. By the perfect normality of Y, for every $U \in \mathcal{U}$ the unions

$$G(U) = \bigcup\{V : U \times V \in \mathcal{W} \text{ and } U \times V \subset G\}$$

and

$$H(U) = \bigcup\{V : U \times V \in \mathcal{W} \text{ and } U \times V \subset H\}$$

can be represented in the form

$$G(U) = \bigcup_{j=1}^{\infty} G_j(U) \quad \text{and} \quad H(U) = \bigcup_{j=1}^{\infty} H_j(U),$$

where $G_j(U), H_j(U)$ are open in Y and

$$\operatorname{Ind} \operatorname{Fr} G_j(U) \leq m - 1 \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} H_j(U) \leq m - 1 \quad \text{for} \quad j = 1, 2, \ldots$$

The family

$$\mathcal{W}_{i,j} = \{U \times G_j(U) : U \in \mathcal{U}_i\} \cup \{U \times H_j(U) : U \in \mathcal{U}_i\}$$

is locally finite in $X \times Y$ for $i, j = 1, 2, \ldots$, so that the union $\bigcup_{i,j=1}^{\infty} \mathcal{W}_{i,j}$ is a σ-locally finite open refinement of the cover $\{G, H\}$.

From Theorem 2.1.3 and Lemma 2.4.11 it follows that the Cartesian product $X \times Y$ is strongly hereditarily normal. Thus, by virtue of Theorem 2.3.17, to complete the proof

it suffices to show that $\operatorname{Ind}\operatorname{Fr} W \leq k-1$ for every $W \in \mathcal{W}_{i,j}$ and $i,j = 1,2,\ldots$ The last inequality is a consequence of the inclusions

$$\operatorname{Fr}(U \times G_j(U)) \subset (X \times \operatorname{Fr} G_j(U)) \cup (\operatorname{Fr} U \times Y)$$

and

$$\operatorname{Fr}(U \times H_j(U)) \subset (X \times \operatorname{Fr} H_j(U)) \cup (\operatorname{Fr} U \times Y),$$

the inductive assumption, and Theorem 2.3.8. ■

Let us observe that the proof of Theorem 1.5.16 yields actually the following result (see Problem 2.4.H):

2.4.13. THEOREM. *For every pair X, Y of regular spaces, not both empty, such that the finite sum theorem for* ind *holds in the Cartesian product $X \times Y$ we have*

$$\operatorname{ind}(X \times Y) \leq \operatorname{ind} X + \operatorname{ind} Y. \blacksquare$$

Theorems 2.4.4 and 2.4.13 imply

2.4.14. THEOREM. *For every pair X, Y of normal spaces, not both empty, such that the Cartesian product $X \times Y$ is strongly paracompact and strongly hereditarily normal we have*

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y. \blacksquare$$

In the last two theorems the assumptions bear upon the Cartesian product $X \times Y$, rather than on the factors X and Y as in Theorems 2.4.8 and 2.4.12; that is why their proofs are simpler. Such theorems, however, are rarely applicable, because given two spaces it is not easy to decide – except in very simple cases – if the finite sum theorem for Ind holds in their product or if this product is strongly paracompact and strongly hereditarily normal.

As we announced earlier, Theorems 2.4.8 and 2.4.12 are special cases of more general results. Indeed, in Theorem 2.4.8 it suffices to assume that $X \times Y$ is normal and only the first factor is compact (or just that the projection onto the second factor is a closed mapping; cf. [GT], Theorem 3.1.16). In Theorem 2.4.12 it suffices to assume that $X \times Y$ is normal and the finite sum theorem for Ind holds in Y; moreover, metrizability of X can be weakened to the existence of a *perfect* mapping (i.e., a closed mapping with compact fibres) of X onto a metrizable space, but then we have to assume that the finite sum theorem for Ind holds in X. The proofs of these results are much more difficult.

These results can be generalized further to a somewhat technical "rectangular product" theorem for Ind which has the advantage of embracing practically all known Cartesian product theorems for Ind and of separating their "dimensional" and "topological" parts. The Cartesian product $X \times Y$ is called *rectangular* if every finite cover of $X \times Y$ by functionally open sets has a σ-locally finite refinement consisting of sets of the form $U \times V$, where U and V are functionally open in X and Y respectively, and the *rectangular product theorem for* Ind states that for every pair X, Y of normal spaces, not both empty, such that the finite sum theorem for Ind holds both in X and Y and the Cartesian product $X \times Y$ is normal and rectangular we have $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y$. Thus to prove that the last inequality is valid for a pair of spaces X, Y in which the

finite sum theorem for Ind holds, we only have to check that the product $X \times Y$ is normal and rectangular, which has nothing to do with dimension (cf. Lemmas 3.4.5 and 3.4.8). Let us add that the rectangular product theorem for Ind generalizes to Cartesian products of finitely many spaces.

We conclude this section with a discussion of the behaviour of the large inductive dimension under mappings.

2.4.15. THEOREM ON DIMENSION-RAISING MAPPINGS FOR Ind. *If $f: X \to Y$ is a closed mapping of a strongly hereditarily normal space X onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then* $\operatorname{Ind} Y \leq \operatorname{Ind} X + (k-1)$.

PROOF. We can suppose that $0 \leq \operatorname{Ind} X < \infty$. We shall apply induction with respect to the number $n+k$, where $n = \operatorname{Ind} X$. If $n+k = 1$, we have $k = 1$, so that f is a homeomorphism and the theorem holds. Assume that it holds whenever $n+k < m \geq 2$ and consider a closed mapping $f: X \to Y$ such that $f(X) = Y$ and $n+k = m$.

Let A be a closed subset of Y and $V \subset Y$ an open set which contains the set A. Consider an open set $U \subset X$ such that $f^{-1}(A) \subset U \subset f^{-1}(V)$ and $\operatorname{Ind} \operatorname{Fr} U \leq n-1$. The set $W = Y \setminus f(X \setminus U)$ is open in Y and satisfies $A \subset W \subset f(U) \subset V$, so that to conclude the proof it suffices to check that $\operatorname{Ind} \operatorname{Fr} W \leq m-2$.

Since the restriction $f|\operatorname{Fr} U: \operatorname{Fr} U \to f(\operatorname{Fr} U)$ is a closed mapping, it follows from the inductive assumption that

$$\operatorname{Ind} f(\operatorname{Fr} U) \leq (n-1) + (k-1) = n+k-2.$$

If the set $B = \operatorname{Fr} W \setminus f(\operatorname{Fr} U)$ is empty, we have $\operatorname{Fr} W \subset f(\operatorname{Fr} U)$, so that in this case $\operatorname{Ind} \operatorname{Fr} W \leq n+k-2 = m-2$. Assume that $B \neq \emptyset$ and consider the restriction $f_B: f^{-1}(B) \to B$ and the restriction $f' = f_B|(X \setminus U): (X \setminus U) \cap f^{-1}(B) \to B$; both f_B and f' are closed, and the fibres of f' have cardinality $\leq k-1$, because $\operatorname{Fr} W \subset \overline{W} \subset \overline{f(U)} = f(\overline{U}) = f(U \cup \operatorname{Fr} U) = f(U) \cup f(\operatorname{Fr} U)$, so that $B \subset f(U)$, i.e., $f^{-1}(y) \cap U \neq \emptyset$ for every $y \in B$. Moreover, f' maps $(X \setminus U) \cap f^{-1}(B)$ onto B, because $B \subset \operatorname{Fr} W = \overline{Y \setminus f(X \setminus U)} \cap \overline{f(X \setminus U)} \subset \overline{f(X \setminus U)} = f(X \setminus U)$. It follows from Theorem 2.3.6 and the inductive assumption that

$$\operatorname{Ind} B \leq n + (k-1) - 1 = n+k-2.$$

Since hereditary normality is invariant under closed mappings, the space Y is hereditarily normal and it follows from Theorem 2.3.1 (cf. Problem 2.2.C(e)) that $\operatorname{Ind}(f(\operatorname{Fr} U) \cup B) \leq n+k-2 = m-2$, so that $\operatorname{Ind} \operatorname{Fr} W \leq m-2$ also in the case when $B \neq \emptyset$. ■

2.4.16. REMARK. Let us note that Theorem 2.4.15 also holds under the assumption that X is normal and Y perfectly normal. Indeed, in this case B is an F_σ-set, i.e., $B = \bigcup_{i=1}^{\infty} F_i$ with F_i's closed, and by applying the inductive assumption to the restriction $f_i = f_{F_i}|(X \setminus U) : (X \setminus U) \cap f^{-1}(F_i) \to F_i$ we see that $\operatorname{Ind} F_i \leq m-2$; the inequality $\operatorname{Ind} \operatorname{Fr} W \leq m-2$ follows from the countable sum theorem for Ind. ■

One easily checks that if in the last theorem $\operatorname{Ind} X = 0$, then the assumption that X is a strongly hereditarily normal space can be relaxed to its being normal; it is not known, if normality is always sufficient whenever $\operatorname{Ind} X > 0$.

The companion theorem on dimension-lowering mappings does not hold for the large inductive dimension (see the remark to Problem 3.3.F). Neither does this theorem hold for continuous mappings defined on compact spaces, as shown by the projection of $X \times Y$ to X for compact spaces X, Y with $\operatorname{Ind}(X \times Y) > \operatorname{Ind} X + \operatorname{Ind} Y$. It is an open problem if there exists a closed mapping $f: X \to Y$ of a normal space X onto a weakly paracompact hereditarily normal space Y such that $\operatorname{Ind} f^{-1}(y) \leq k$ for every $y \in Y$ and yet $\operatorname{Ind} Y > \operatorname{Ind} X + k$.

Historical and bibliographic notes

The first example of a metrizable space X such that $\operatorname{ind} X = 0$ and $\operatorname{Ind} X = \dim X = 1$ was given by Roy in [1968] (an outline in [1962]). From a few later examples of this kind, the simplest is described by Kulesza in [1990a] (see also Kulesza [1990b] and Ostaszewski [1990]). Smirnov in [1951] gave the first example of a normal space with non-coinciding inductive dimensions. The first example of a compact space X such that $\operatorname{ind} X \neq \operatorname{Ind} X$ was defined by Filippov in [1969]; the example is discussed in detail in Filippov's paper [1970b] and in Pears' book [1975]. Simpler, but still quite complicated examples of such spaces were described by Filippov in [1970], Pasynkov and Lifanov in [1970], and Pasynkov in [1970]; these last spaces, moreover, are first-countable. Filippov's paper [1970a] also contains an example of a compact space X such that $\dim X = 1$, $\operatorname{ind} X = 2$ and $\operatorname{Ind} X = 3$. A perfectly normal locally compact (and hereditarily separable) space X with $\operatorname{ind} X = 0$ and $\operatorname{Ind} X > 0$ was defined by E. Pol in [1976] under the assumption of the continuum hypothesis.

Theorems 2.4.2 and 2.4.3 were proved by Vedenissoff in [1939] under the stronger assumption that X is a Lindelöf space. He also proved Theorem 2.4.4 for perfectly normal Lindelöf spaces; for strongly paracompact metrizable spaces the theorem was proved by Morita in [1950a] (see also notes to Theorem 1.5.13). Theorems 2.4.4 and 2.4.14, for strongly paracompact and totally normal X and $X \times Y$ respectively, were given by Katuta in [1966]. Theorem 2.4.8 (in a slightly more general form) was announced by Pasynkov in [1969]; its proof given here is taken from Basmanov's paper [1981]. Theorem 2.4.10 was proved by Lifanov in [1967]. Theorem 2.4.12 was obtained by Kimura in [1963]; under the additional assumption that X is a paracompact space it was proved by Nagami in [1960a]. The fact that $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y$ if $X \times Y$ is normal, the finite sum theorem for Ind holds in both X and Y, and either the projection of $X \times Y$ onto Y is closed or there exists a perfect mapping of X onto a metrizable space was announced by Filippov in [1973], the proofs were given in [1980]. Independently Pasynkov announced in [1973] that $\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y$ if $X \times Y$ is normal, X is a metrizable space and the finite sum theorem for Ind holds in Y; to be exact we should add that Pasynkov (and also Filippov in the announcement of his more general result) assumed that $X \times Y$ is moreover countably paracompact – an assumption that later proved superfluous (see [GT], Problem 5.5.18(b)). Rectangular Cartesian products were introduced by Pasynkov in [1973], and in [1975] the rectangular product theorem for Ind

was announced; the complete proof was given in [1991] (see also Pasynkov and Tsuda [1993] where only proofs of some lemmas are omitted). A notion equivalent to rectangularity in the realm of normal Cartesian products was introduced by Nagata in [1967].

Theorem 2.4.15 was proved by Morita in [1956] under the assumption that X is a perfectly normal space, and was generalized to totally normal spaces by Nagata in [1983]; Remark 2.4.16 is taken from Alexandroff and Pasynkov [1973].

Problems

2.4.A. (a) (Nagata [1957]) Prove that the Cartesian product of the open unit interval $(0,1)$ and the Baire space $B(\aleph_1)$ (see Example 4.1.23) is not strongly paracompact. Deduce that the class of all strongly paracompact metrizable spaces is not hereditary with respect to F_σ-sets and is not finitely multiplicative. Show that a metrizable space which can be represented as the union of a locally finite countable family of strongly paracompact closed subspaces is not necessarily strongly paracompact.

(b) Show that every paracompact space X such that $\operatorname{Ind} X = 0$ is strongly paracompact.

(c) Show that the Cartesian product $X \times Y$ of a strongly paracompact space X and a compact space Y is strongly paracompact.

(d) (Morita [1954]) Prove that for every $\mathbf{m} \geq \aleph_0$ the Cartesian product $N^{\aleph_0} \times B(\mathbf{m})$, where N is the discrete space of natural numbers and $B(\mathbf{m})$ – the Baire space of weight $\mathbf{m}$, is a universal space for the class of all strongly paracompact metrizable spaces whose weight is not larger than $\mathbf{m}$.

2.4.B (Zarelua [1963] (announcement [1961])). A topological space X is called *completely paracompact* if X is a regular space and for every open cover $\mathcal{U}$ of the space X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of star-finite open covers of X such that the union $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ contains a refinement of $\mathcal{U}$.

(a) Observe that every completely paracompact space is paracompact and that every strongly paracompact space is completely paracompact. Show that in the realm of connected spaces complete paracompactness is equivalent to the Lindelöf property. Prove that complete paracompactness is hereditary with respect to F_σ-sets.

(b) Prove that a metrizable space is completely paracompact if and only if it has a base which can be represented as the union of countably many star-finite covers; such spaces are called *strongly metrizable*. Deduce that the class of strongly metrizable spaces is hereditary and countably multiplicative. Show that a metrizable space which can be represented as the union of a countable family of strongly metrizable closed subspaces is not necessarily strongly metrizable.

(c) Prove that for every $\mathbf{m} \geq \aleph_0$ the Cartesian product $I^{\aleph_0} \times B(\mathbf{m})$ is a universal space for the class of all strongly metrizable spaces whose weight is not larger than $\mathbf{m}$.

(d) Give an example of a strongly metrizable space which cannot be represented as the union of a σ-locally finite family of strongly paracompact closed subspaces.

Hint. Consider the Cartesian product $R^{\aleph_0} \times B(\aleph_1)$.

(e) Check that Theorems 2.4.2, 2.4.3 and 2.4.4 remain valid for completely paracompact spaces.

2.4.C. (a) (Nagami [1969]) A topological space X is called *σ-totally paracompact* if X is a regular space and for every base $\mathcal{B}$ for the space X there exists a σ-locally finite open cover $\mathcal{V}$ of X such that for each $V \in \mathcal{V}$ one can find a $U \in \mathcal{B}$ satisfying $V \subset U$ and $\operatorname{Fr} V \subset \operatorname{Fr} U$.

Show that the class of σ-totally paracompact spaces is hereditary with respect to F_σ-sets. Check that Theorems 2.4.2, 2.4.3 and 2.4.4 remain valid for σ-totally paracompact spaces. Observe that every σ-totally paracompact space is paracompact.

(b) (Zarelua [1963] (announcement [1961])) Prove that every completely paracompact space is σ-totally paracompact.

2.4.D. (a) (Fitzpatrick and Ford [1967]) A topological space X is *order totally paracompact* if X is a regular space and for every base $\mathcal{B}$ for the space X there exists an open cover $\{V_s\}_{s\in S}$, where the set S is linearly ordered by a relation $<$, such that for every $s_0 \in S$ the family $\{V_{s_0} \cap V_s\}_{s<s_0}$ is locally finite in the space V_{s_0} and for each $s \in S$ one can find a $U \in \mathcal{B}$ satisfying $V_s \subset U$ and $\operatorname{Fr} V_s \subset \operatorname{Fr} U$.

Show that the class of order totally paracompact spaces is hereditary with respect to closed subspaces.

(b) Prove that Theorem 2.4.2 remains valid for order totally paracompact spaces.

Hint. Check that if for a pair A, B of disjoint closed subsets of a topological space X there exists an open cover $\{V_s\}_{s\in S}$ of X, where the set S is linearly ordered by a relation $<$, such that for every $s_0 \in S$ either $\overline{V}_{s_0} \cap A = \emptyset$ or $\overline{V}_{s_0} \cap B = \emptyset$ and the family $\{V_{s_0} \cap V_s\}_{s<s_0}$ is locally finite in the space V_{s_0}, then there exists a partition L between A and B such that $L \subset \bigcup_{s\in S} \operatorname{Fr} V_s$.

Remark. It is not known if Theorems 2.4.3 and 2.4.4 remain valid for order totally paracompact spaces.

(c) (Fitzpatrick and Ford [1967]) Prove that for every order totally paracompact metrizable space X we have $\operatorname{ind} X = \operatorname{Ind} X$.

Hint. Apply the hint to (b); use the fact that X has a σ-locally finite base.

(d) (Katuta [1967]) Prove that a regular space X is paracompact if and only if every open cover $\mathcal{U}$ of X has an open refinement $\{V_s\}_{s\in S}$, where the set S is linearly ordered by a relation $<$, such that for every $s \in S_0$ the family $\{V_{s_0} \cap V_s\}_{s<s_0}$ is locally finite in the space V_{s_0}.

Hint. To prove that the space X is paracompact it suffices to show that every open cover of X has a locally finite refinement (see [GT], Theorem 5.1.11).

(e) Deduce from (d) that every order totally paracompact space is paracompact and note that every σ-totally paracompact space is order totally paracompact.

2.4.E. Show that if a metrizable space X can be represented as the union of a locally countable family $\{F_s\}_{s\in S}$ of closed subspaces such that $\operatorname{ind} F_s = \operatorname{Ind} F_s$ for $s \in S$, then $\operatorname{ind} X = \operatorname{Ind} X$.

2.4.F (announcement Pasynkov [1969]). Prove that if $X_1, X_2, \ldots, X_k$ are non-empty compact spaces and in each of them the finite sum theorem for Ind holds, then $\operatorname{Ind}(X_1 \times X_2 \times \ldots \times X_k) \leq \operatorname{Ind} X_1 + \operatorname{Ind} X_2 + \ldots + \operatorname{Ind} X_k$.

Hint. Modify the proof of Theorem 2.4.8.

2.4.G (Lifanov [1968]). Prove that if $X_1, X_2, \ldots, X_k$ are compact spaces such that $\operatorname{Ind} X_i = 1$ for $i = 1, 2, \ldots, k$, then for the Cartesian product $X = X_1 \times X_2 \times \ldots \times X_k$ we have $\dim X = \operatorname{ind} X = \operatorname{Ind} X = k$.

Hint. See the remark to Problem 1.8.K and Theorem 3.1.29.

2.4.H (Basmanov [1981]; announcement, for completely regular spaces, Pasynkov [1969]). Prove that if $X_1, X_2, \ldots, X_k$ are non-empty regular spaces and in each of them the finite sum theorem for ind holds, then $\operatorname{ind}(X_1 \times X_2 \times \ldots \times X_k) \leq \operatorname{ind} X_1 + \operatorname{ind} X_2 + \ldots + \operatorname{ind} X_k$.

Hint. Modify the proof of Theorem 2.4.8.

2.4.I. (a) Note that for every regular space X and every regular space Y with $\operatorname{ind} Y = 0$ we have $\operatorname{ind}(X \times Y) = \operatorname{ind} X$.

(b) (Filippov [1980]) Let $f_i: X \to I$ be a continuous function defined on a regular space X for $i = 1, 2$. Show that the set $F = \{(x,t) \in X \times I : f_1(x) = t\} \cup \{(x,t) \in X \times I: f_2(x) = t\}$ satisfies the inequality $\operatorname{ind} F \leq \operatorname{ind} X$.

(c) (Filippov [1980] (announcement [1972]; for metrizable X, Toulmin [1954]) Show that for every completely regular space X and for $n = 1, 2, \ldots$ we have $\operatorname{ind}(X \times I^n) \leq \operatorname{ind} X + n$.

Hint. Deduce from (b) that $\operatorname{ind}(X \times R) \leq \operatorname{ind} X + 1$; one can also apply Problem 7.2.E(a).

(d) (Pasynkov, cited in Filippov [1980]) Prove that for every completely regular space X and every separable metrizable space Y, not both empty, we have $\operatorname{ind}(X \times Y) \leq \operatorname{ind} X + \operatorname{ind} Y$.

Hint. Assume first that Y is compact, take a continuous mapping $f: Y \to I^n$ such that $\operatorname{ind} f^{-1}(z) \leq 0$ for each $z \in I^n$ (see Problem 1.12.H(a)), note that the mapping $g: X \times Y \to X \times I^n$ defined by letting $g(x,y) = (x, f(x))$ is closed (see [GT], Theorem 3.7.14) and deduce that $\operatorname{ind}(X \times Y) \leq \operatorname{ind}(X \times I^n)$ (cf. Problem 2.4.K(c)).

2.4.J. (a) A continuous mapping $f: X \to Y$ is a *local homeomorphism* if for every $x \in X$ there exists a neighbourhood U of the point x such that $f|U$ is a homeomorphism of U onto an open subspace of Y.

Verify that if all fibres of an open mapping f defined on a Hausdorff space are finite and have the same cardinality then f is a local homeomorphism and is closed (cf. Lemma 6.3.12).

(b) (Nagami [1960]) Show that if $f: X \to Y$ is an open mapping of a Hausdorff space X onto a Hausdorff space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then for $j = 1, 2, \ldots$ the restriction $f_{K_j}: f^{-1}(K_j) \to K_j$, where $K_j = \{y \in Y : |f^{-1}(y)| = j\}$, is a local homeomorphism and the union $\bigcup_{j \leq i} K_j$ is closed in Y for $i = 1, 2, \ldots$

(c) (Polkowski [1983a]; for hereditarily paracompact spaces, Nagami [1960]) Prove that if $f: X \to Y$ is an open mapping of a strongly hereditarily normal space X onto a hereditarily normal hereditarily weakly paracompact space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

Hint. To begin, deduce from (a) that $f_{K_j}: f^{-1}(K_j) \to K_j$, where $K_j = \{y \in Y : |f^{-1}(y)| = j\}$, is a closed mapping, and show that $f^{-1}(K_j)$ is weakly paracompact for

$j = 1, 2, \ldots$ Then apply (b) and Theorem 2.3.13 to prove that $\operatorname{Ind} f^{-1}(K_j) = \operatorname{Ind} K_j$ for $j = 1, 2, \ldots$ To conclude the proof use Theorem 2.3.1.

2.4.K (Pasynkov [1967]). (a) (for metrizable spaces, Kolmogoroff [1941]) Prove that if $f: X \to Y$ is an open mapping defined on a locally compact space X such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then X contains an open dense subspace on which f is a local homeomorphism.

Hint. It suffices to show that if for an open mapping $f: X \to Y$ defined on an non-empty locally compact space X there is no non-empty open set $U \subset X$ such that $f|U$ is a homeomorphism of U onto $f(U)$, then there exists a point $y \in Y$ with $|f^{-1}(y)| \geq \mathbf{c}$. To that end, for $i = 1, 2, \ldots$ and all sequences $k_1, k_2, \ldots, k_i$ of zeros and ones define inductively a $V_i \subset Y$ and open non-empty subsets $U_{k_1 k_2 \ldots k_i}$ of X such that (1) $\overline{U}_{k_1 k_2 \ldots k_i}$ is compact, (2) $\overline{U}_{k_1 k_2 \ldots k_i} \cap \overline{U}_{m_1 m_2 \ldots m_i} = \emptyset$ whenever $(k_1, k_2, \ldots, k_i) \neq (m_1, m_2, \ldots, m_i)$, (3) $\overline{U}_{k_1 k_2 \ldots k_i 0} \cup \overline{U}_{k_1 k_2 \ldots k_i 1} \subset U_{k_1 k_2 \ldots k_i}$, and (4) $f(U_{k_1 k_2 \ldots k_i}) = V_i$. Then take a point $x \in \overline{U}_0 \cap \overline{U}_{00} \cap \ldots$ and consider the fibre $f^{-1}(f(x))$.

Remark. By an obvious modification of the argument in the hint one can prove that if $f: X \to Y$ is an open mapping defined on a completely metrizable space X such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then X contains an open dense subspace on which f is a local homeomorphism.

(b) Prove that if $f: X \to Y$ is an open mapping of a locally compact normal space X onto a weakly paracompact perfectly normal space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{Ind} Y \leq \operatorname{Ind} X$.

Hint. Note first that if the open set $W = \bigcup\{V \subset Y : V$ is open in Y and $\operatorname{Ind} V \leq \operatorname{Ind} X\}$ is equal to Y, then $\operatorname{Ind} Y \leq \operatorname{Ind} X$. Then assume that $Y \setminus W \neq \emptyset$ and obtain a contradiction. To that end, consider the closed subspace $X_0 = f^{-1}(Y \setminus W)$ of X and the restriction $f|X_0: X_0 \to Y \setminus W$, find a non-empty open F_σ-set U in the space X_0 such that $f|U$ is a homeomorphism of U onto $f(U)$, pick an open set $V \subset Y$ such that $V \cap f(U) \neq \emptyset, \overline{V} \subset W \cup f(U)$ and $\overline{V}$ is compact and show that $\operatorname{Ind} V \leq \operatorname{Ind} \overline{V} \leq \operatorname{Ind} X$.

(c) Prove that if $f: X \to Y$ is a closed mapping of a normal space X to a normal space Y such that $\operatorname{ind} f^{-1}(y) \leq 0$ for every $y \in Y$, then $\operatorname{ind} X \leq \operatorname{ind} Y$.

(d) Show that if $f: X \to Y$ is an open mapping of a compact perfectly normal space X onto a normal space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

Hint. Note that ind and Ind coincide on compact perfectly normal spaces (cf. Theorem 2.4.4).

Chapter 3

The covering dimension

Chapter 2 was devoted to an examination of the question which results of the classical dimension theory of separable metric spaces hold in more general classes of spaces for the large inductive dimension Ind. In the present chapter, the covering dimension dim becomes the subject of similar considerations. It will be seen that outside the class of separable metric spaces the dimension dim behaves better than Ind, i.e., that for dim a larger number of theorems of the classical theory can be extended to topological spaces and that the extensions hold under weaker assumptions.

Section 3.1 is primarily devoted to the question of monotonicity of dim and to a study of sum theorems for dim. We also discuss the relations between the covering dimension dim and the inductive dimensions ind and Ind; in particular, we prove that $\dim X \leq \operatorname{Ind} X$ for every normal space X.

Section 3.2 begins with several characterizations of the dimension dim in normal spaces. They include generalizations of three important theorems of the classical theory of dimension, namely the theorems on partitions, on extending mappings to spheres and on $\mathcal{E}$-mappings. In the final part of the section we show that $\dim(X \times Y) \leq \dim X + \dim Y$ for compact spaces X and Y; this results will be generalized in Section 3.4.

In Section 3.3 we establish the dimension-raising and dimension-lowering theorems for dim. The proofs of these important results are a bit harder than most proofs in this book.

Section 3.4 opens with Mardešić's factorization theorem from which we deduce the compactification and the universal space theorems for dim. Then we prove two Cartesian product theorems for dim and discuss their generalizations. We conclude the section by characterizing compact spaces of covering dimension $\leq n$ as limits of inverse systems of metrizable compact spaces of dimension $\leq n$.

3.1. Basic properties of the dimension dim in normal spaces. Relations between the dimensions ind, Ind and dim

The definition of the covering dimension dim was stated in Section 1.6; let us recall that a normal space X satisfies the inequality $\dim X \leq n$ if and only if every finite open cover of the space X has a finite open refinement of order $\leq n$. We shall begin this section with a characterization of the dimension dim which demonstrates that instead of finite open refinements of order $\leq n$ one can equally well consider finite closed refinements of order $\leq n$. The characterization is preceded by an auxiliary theorem on swellings of finite families of closed subsets of normal spaces which will be applied in its proof. Recall that by a *swelling* of the family $\{A_s\}_{s \in S}$ of subsets of a topological space

X we mean any family $\{B_s\}_{s\in S}$ of subsets of the space X such that $A_s \subset B_s$ for every $s \in S$ and for every finite set of indices $s_1, s_2, \ldots, s_m \in S$,

$$A_{s_1} \cap A_{s_2} \cap \ldots \cap A_{s_m} \neq \emptyset \quad \text{if and only if} \quad B_{s_1} \cap B_{s_2} \cap \ldots \cap B_{s_m} \neq \emptyset;$$

a swelling is *open* if all its members are open subsets of the space X. Clearly, every swelling $\mathcal{B}$ of a family $\mathcal{A}$ satisfies the equality $\operatorname{ord}\mathcal{B} = \operatorname{ord}\mathcal{A}$.

The following theorem is, in a sense, dual to Theorem 1.7.8, which deals with shrinkings of finite open covers of normal spaces.

3.1.1. THEOREM. *Every finite family $\{F_i\}_{i=1}^k$ of closed subsets of a normal space X has an open swelling $\{U_i\}_{i=1}^k$. If, moreover, a family $\{V_i\}_{i=1}^k$ of open subsets of X satisfying $F_i \subset V_i$ for $i = 1, 2, \ldots, k$ is given, then the swelling can be defined in such a way that $\overline{U}_i \subset V_i$ for $i = 1, 2, \ldots, k$.*

PROOF. The union E_1 of all intersections $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m}$ satisfying the equality $F_1 \cap F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m} = \emptyset$ is a closed set disjoint from F_1, so that there exists an open set U_1 such that $F_1 \subset U_1$ and $\overline{U}_1 \cap E_1 = \emptyset$. One readily sees that the family $\{\overline{U}_1, F_1, \ldots, F_k\}$ is a swelling of the family $\{F_i\}_{i=1}^k$.

Assume that for $i = 1, 2, \ldots, n-1$ an open set U_i is defined in such a way that $F_i \subset U_i$ and the family $\{\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_{n-1}, F_n, \ldots, F_k\}$ is a swelling of $\{F_i\}_{i=1}^k$. The union E_n of all intersections of members of the family $\{\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_{n-1}, F_n, \ldots, F_k\}$ which are disjoint from F_n is a closed set disjoint from F_n, so that there exists an open set U_n such that $F_n \subset U_n$ and $\overline{U}_n \cap E_n = \emptyset$. The family $\{\overline{U}_1, \overline{U}_2, \ldots, \overline{U}_n, F_{n+1}, \ldots, F_k\}$ is a swelling of $\{F_i\}_{i=1}^k$. In this way we obtain open sets $U_1, U_2, \ldots, U_k$ such that $F_i \subset U_i$ for $i = 1, 2, \ldots, k$ and the family $\{\overline{U}_i\}_{i=1}^k$ is a swelling of $\{F_i\}_{i=1}^k$. Clearly, the family $\{U_i\}_{i=1}^k$ is the required open swelling. The second part of the theorem is obvious. ■

Theorems 1.7.8 and 3.1.1 yield

3.1.2. PROPOSITION. *For every normal space X the following conditions are equivalent:*

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *Every finite open cover of the space X has a closed shrinking of order $\leq n$.*

(iii) *Every finite open cover of the space X has a finite closed refinement of order $\leq n$.* ■

Since normality is not a hereditary property, it may happen that the dimension dim is defined for a space X, but it is not defined for some subspace of X. Still, normality being hereditary with respect to closed subsets, $\dim M$ is defined for every closed subspace $M \subset X$. Moreover, the following counterpart of Theorem 2.2.1 holds.

3.1.3. THEOREM. *For every closed subspace M of a normal space X we have* $\dim M \leq \dim X$.

PROOF. The theorem is obvious if $\dim X = \infty$, so that we can assume that $\dim X = n < \infty$. Consider a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space M. For $i = 1, 2, \ldots, k$ let W_i be an open subset of X such that $U_i = M \cap W_i$. The family $\{X \setminus M\} \cup \{W_i\}_{i=1}^k$ is an open cover of the space X and, since $\dim X \leq n$, it has a finite open refinement $\mathcal{V}$ of order $\leq n$. One easily sees that the family $\mathcal{V}|M$ is a finite open cover of the space M, refines $\mathcal{U}$ and has order $\leq n$, so that $\dim M \leq n = \dim X$. ■

From Theorem 1.6.10 it follows that the compact space Z and its normal subspace X defined in Example 2.2.12 satisfy the relation $0 < \dim X > \dim Z = 0$. Hence, in Theorem 3.1.3 the assumption that M is a closed subspace of X cannot be replaced by the weaker assumption that $\dim M$ is defined. In Section 2.2 an even stronger example is cited which shows that the dimension dim is not monotonic in hereditarily normal spaces. Let us observe that the theorem on subspaces of intermediate dimensions, i.e., the counterpart of Theorems 1.5.1 and 2.2.2, does not hold for the dimension dim in normal spaces. More exactly, for every natural number $n \geq 2$ there exists a compact space X_n such that $\dim X_n = n$ and for every closed subspace $M \subset X_n$ we have either $\dim M \leq 0$ or $\dim M = n$. The description of spaces X_n and verification of their properties are too difficult to be reproduced in this book. Let us add that under the assumption of the continuum hypothesis there exist such spaces which are perfectly normal. From Theorems 1.6.10 and 2.2.2 it follows readily that spaces X_n without subspaces of intermediate dimensions dim satisfy the equality $\operatorname{Ind} X_n = \infty$.

We shall now show that for a fixed hereditarily normal space X the monotonicity of the dimension dim is equivalent to its being monotonic with respect to open subspaces.

3.1.4. PROPOSITION. *For every hereditarily normal space X the following conditions are equivalent*:

(i) *For every subspace M of X we have* $\dim M \leq \dim X$.

(ii) *For every open subspace U of X we have* $\dim U \leq \dim X$.

PROOF. The implication (i)⇒(ii) is obvious. Suppose that X satisfies (ii). Condition (i) is satisfied if $\dim X = \infty$, so that we can assume that $\dim X = n < \infty$. Consider a subspace M of X and a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space M. For $i = 1, 2, \ldots, k$ let W_i be an open subset of X such that $U_i = M \cap W_i$. Since for the open subspace $W = \bigcup_{i=1}^k W_i$ of X we have $\dim W \leq n$, the cover $\{W_i\}_{i=1}^k$ of the space W has a finite open refinement $\mathcal{V}$ of order $\leq n$. One easily sees that the family $\mathcal{V}|M$ is a finite open cover of the space M, refines $\mathcal{U}$ and has order $\leq n$, so that $\dim M \leq n = \dim X$. Thus X satisfies condition (i). ■

We shall return to the question of the monotonicity of dim later in this section and we shall show that the dimension dim is monotonic in the class of strongly hereditarily normal spaces; the proof of this important fact depends on the countable sum theorem for dim.

We now turn to a study of sum theorems for the covering dimension.

We start with two lemmas; in the second one a convenient notion of relative dimension is used. For a subspace A of a normal space X the *relative covering dimension* of A is defined by the formula

$$\operatorname{rd}_X A = \sup\{\dim F : F \subset A \text{ and } F \text{ is closed in } X\}.$$

Clearly, for every normal subspace A of a normal space X we have $\operatorname{rd}_X A \leq \dim A$, and $\operatorname{rd}_X A = \dim A$ if A is closed in X.

3.1.5. LEMMA. *For every open cover $\mathcal{U}$ of a normal space X and a closed subspace F of X which has covering dimension not larger than n and meets only finitely many members of $\mathcal{U}$ there exists an open shrinking $\mathcal{V}$ of $\mathcal{U}$ such that* $\operatorname{ord}(\mathcal{V}|F) \leq n$.

PROOF. Let $\mathcal{U} = \{U_s\}_{s\in S}$; by Proposition 1.6.8 the open cover $\{F \cap U_s\}_{s\in S}$ of the subspace F has an open shrinking $\{W_s\}_{s\in S}$ of order $\leq n$. For each $s \in S$ there exists an open subset $W'_s \subset X$ such that $W_s = F \cap W'_s$. The sets $V_s = (U_s \setminus F) \cup (U_s \cap W'_s)$, where $s \in S$, form an open shrinking $\mathcal{V}$ of the cover $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{V}|F) \leq n$. ■

3.1.6. LEMMA. *If a normal space X can be represented as the union of a sequence $K_1, K_2, \ldots$ of subspaces such that $\operatorname{rd}_X K_i \leq n$ and the union $\bigcup_{j\leq i} K_j$ is closed for $i = 1, 2, \ldots$, then $\dim X \leq n$.*

PROOF. Consider a finite open cover $\mathcal{U} = \{U_j\}_{j=1}^k$ of the space X. We shall define inductively a sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of the space X, where $\mathcal{U}_i = \{U_{i,j}\}_{j=1}^k$, satisfying the conditions:

(1) $\quad \overline{U}_{i,j} \subset U_{i-1,j}$ if $i \geq 1$ and $U_{0,j} \subset U_j$ for $j = 1, 2, \ldots, k$.

(2) $\operatorname{ord}(\{F_i \cap \overline{U}_{i,j}\}_{j=1}^k) \leq n$, where $F_i = K_1 \cup K_2 \cup \ldots \cup K_i$ if $i \geq 1$ and $F_0 = \emptyset$.

Both conditions are satisfied for $i = 0$ if we define $U_{0,j} = U_j$ for $j = 1, 2, \ldots, k$. Assume that the covers $\mathcal{U}_i$ satisfying (1) and (2) are defined for all $i < m \geq 1$. Consider the set $A = \bigcup_{T\in\mathcal{T}} \bigcap_{j\in T} \overline{U}_{m-1,j}$, where $\mathcal{T}$ is the family of all subsets of $\{1, 2, \ldots, k\}$ which have exactly $n+2$ elements. From (2) with $i = m-1$ it follows that $A \cap F_{m-1} = \emptyset$. The intersection $Z = A \cap F_m$ is a closed subspace of X and is contained in $F_m \setminus F_{m-1} \subset K_m$, so that $\dim Z \leq n$. By virtue of Lemma 3.1.5 the cover $\mathcal{U}_{m-1}$ has an open shrinking $\mathcal{V} = \{V_j\}_{j=1}^k$ such that $\operatorname{ord}(\mathcal{V}|Z) \leq n$; one easily checks that $\operatorname{ord}(\{F_m \cap V_j\}_{j=1}^k) \leq n$. By Theorem 1.7.8 and the normality of X there exists an open shrinking $\mathcal{U}_m = \{U_{m,j}\}_{j=1}^k$ of the cover $\{V_j\}_{j=1}^k$ such that $\overline{U}_{m,j} \subset V_j$ for $j = 1, 2, \ldots, k$. Clearly, the cover $\mathcal{U}_m$ satisfies conditions (1) and (2) with $i = m$. Hence the construction of the covers $\mathcal{U}_i$ satisfying (1) and (2) for $i = 0, 1, \ldots$ is completed.

For every point $x \in X$ there exists a $j(x) \leq k$ such that x belongs to infinitely many sets $U_{i,j(x)}$; it follows from (1) that $x \in \bigcap_{i=1}^\infty U_{i,j(x)}$. Applying (1) and (2) we readily see that the family $\{\bigcap_{i=1}^\infty \overline{U}_{i,j}\}_{j=1}^k$ is a closed shrinking of the cover $\{U_j\}_{j=1}^k$ and has order $\leq n$. Therefore we have $\dim X \leq n$ by virtue of Proposition 3.1.2. ■

Lemma 3.1.6 implies

3.1.7. PROPOSITION. *If a normal space X can be represented as the union of a sequence $K_1, K_2, \ldots$ of normal subspaces such that $\dim K_i \leq n$ and the union $\bigcup_{j\leq i} K_j$ is closed for $i = 1, 2, \ldots$, then $\dim X \leq n$.* ■

The last proposition yields

3.1.8. THE COUNTABLE SUM THEOREM FOR dim. *If a normal space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that $\dim F_i \leq n$ for $i = 1, 2, \ldots$, then $\dim X \leq n$.* ■

We now pass to the locally finite sum theorem for dim. The theorem will be deduced from a lemma which is formulated here in a more general form for the purpose of an application in the following section.

3.1.9. LEMMA. *Let $\mathcal{U} = \{U_s\}_{s\in S}$ be an open cover of a normal space X. If the space X has a locally finite closed cover $\mathcal{F}$ each member of which has covering dimension not larger than n and meets only finitely many members of $\mathcal{U}$, then the cover $\mathcal{U}$ has an open shrinking of order $\leq n$.*

PROOF. Let us adjoin the set $F_0 = \emptyset$ to the cover $\mathcal{F}$ and let us arrange the members of this cover into a transfinite sequence $F_0, F_1, \ldots, F_\alpha, \ldots$, $\alpha \leq \xi$, of type $\xi+1$. We shall define inductively a transfinite sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots, \mathcal{U}_\alpha, \ldots$, $\alpha \leq \xi$, of open covers of the space X, where $\mathcal{U}_\alpha = \{U_{\alpha,s}\}_{s\in S}$, satisfying the following conditions:

$$U_{\alpha,s} \subset U_{\beta,s} \quad \text{if } \alpha > \beta \geq 0 \quad \text{and} \quad U_{0,s} \subset U_s \quad \text{for } s \in S. \tag{3}$$

$$\operatorname{ord}(\{F_\alpha \cap U_{\alpha,s}\}_{s\in S}) \leq n. \tag{4}$$

$$U_{\beta,s} \setminus U_{\alpha,s} \subset \bigcup_{\beta\leq\gamma\leq\alpha} F_\gamma \quad \text{for } \beta < \alpha \text{ and } s \in S. \tag{5}$$

All conditions are satisfied for $\alpha = 0$ if we define $U_{0,s} = U_s$ for $s \in S$. Assume that the coverings $\mathcal{U}_\alpha$ satisfying (3)–(5) are defined for all $\alpha < \alpha_0 \geq 1$. To begin, we shall show that the family $\mathcal{U}'_{\alpha_0} = \{U'_{\alpha_0,s}\}_{s\in S}$, where

$$U'_{\alpha_0,s} = \bigcap_{\alpha<\alpha_0} U_{\alpha,s} \quad \text{for } s \in S,$$

is an open cover of the space X. This is clear if $\alpha_0 = \alpha_1 + 1$, because then $U'_{\alpha_0} = U_{\alpha_1}$; thus, we can assume that α_0 is a limit number.

Consider an arbitrary point $x \in X$. Since the family $\mathcal{F}$ is locally finite, there exist a neighbourhood $U \subset X$ of the point x and an ordinal number $\beta < \alpha_0$ such that $U \cap F_\gamma = \emptyset$ whenever $\beta \leq \gamma < \alpha_0$. The family $\mathcal{U}_\beta$ being a cover of X, there exists an $s \in S$ such that $x \in U_{\beta,s}$. It follows from (5) that $U \cap U_{\beta,s} \subset U_{\alpha,s}$ whenever $\beta < \alpha < \alpha_0$, so that $x \in U \cap U_{\beta,s} \subset U'_{\alpha_0,s}$. Hence $\mathcal{U}'_{\alpha_0}$ is an open cover of the space X.

By virtue of Lemma 3.1.5 the cover $\mathcal{U}'_{\alpha_0}$ has an open shrinking $\mathcal{V} = \{V_s\}_{s\in S}$ such that $\operatorname{ord}(\mathcal{V}|F_{\alpha_0}) \leq n$. One readily checks that the family $\mathcal{U}_{\alpha_0} = \{U_{\alpha_0,s}\}_{s\in S}$, where $U_{\alpha_0,s} = (U'_{\alpha_0,s} \setminus F_{\alpha_0}) \cup V_s$, is an open cover of the space X satisfying conditions (3)–(5) with $\alpha = \alpha_0$. Hence the construction of the covers $\mathcal{U}_\alpha$ satisfying (3)–(5) for $\alpha \leq \xi$ is completed.

Now, it follows from (4) that $\operatorname{ord}\mathcal{U}_\xi \leq n$; as $\mathcal{U}_\xi$ is, by virtue of (3), a shrinking of $\mathcal{U}$, the lemma is established. ■

Lemma 3.1.9 yields

3.1.10. THE LOCALLY FINITE SUM THEOREM FOR dim. *If a normal space X can be represented as the union of a locally finite family $\{F_s\}_{s\in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, then $\dim X \leq n$.* ■

The following two theorems are common generalizations of the countable sum theorem and the locally finite sum theorem. The proofs, parallel to the proofs of Theorems 2.3.11 and 2.3.12, are left to the reader.

3.1.11. THEOREM. *If a normal space X can be represented as the union of a σ-locally finite family $\{F_s\}_{s\in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, then $\dim X \leq n$.* ■

3.1.12. THEOREM. *If a normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \ldots, F_\alpha, \ldots, \alpha < \xi$, of closed subspaces such that $\dim F_\alpha \leq n$ and the family $\{F_\beta\}_{\beta<\alpha}$ is locally finite for each $\alpha < \xi$, and the family $\{F_\alpha\}_{\alpha<\xi}$ is locally countable, then $\dim X \leq n$.* ■

The next result is the point-finite sum theorem for dim. It will yield two further sum theorems which hold in the class of weakly paracompact normal spaces. Let us note that in this last class Theorem 3.1.13 generalizes the locally finite sum theorem (see Problem 2.3.F), and Theorem 3.1.15 generalizes both the countable sum theorem and the locally finite sum theorem.

3.1.13. THE POINT-FINITE SUM THEOREM FOR dim. *If a normal space X can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, and if there exists a point-finite open cover $\{U_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then $\dim X \leq n$.*

PROOF. Consider the decomposition of the space X described in Lemma 2.3.3, i.e., let

$$X = \bigcup_{i=1}^{\infty} K_i, \quad \text{where the union } \bigcup_{j\leq i} K_j \text{ is closed for } i = 1, 2, \ldots \tag{6}$$

and

$$K_i = \bigcup_{T\in\mathcal{T}_i} K_T, \quad \text{where the sets } K_T \text{ are open in } K_i \text{ and pairwise disjoint.} \tag{7}$$

Let Z be a closed subspace of the space X contained in a set K_i. It follows from (7) that for every $T \in \mathcal{T}_i$ the set $Z \cap K_T$ is closed in X. Since $Z \cap K_T \subset \bigcup_{s\in T} F_s$, by Theorems 3.1.8 and 3.1.3 we have $\dim(Z \cap K_T) \leq n$ for every $T \in \mathcal{T}_i$; and this implies that $\dim Z \leq n$. By virtue of (6), to conclude the proof it suffices to apply Lemma 3.1.6. ■

3.1.14. THEOREM. *If a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of open subspaces such that $\dim \overline{U}_s \leq n$ for $s \in S$, then $\dim X \leq n$.*

PROOF. The space X being weakly paracompact, one can assume that the cover $\{U_s\}_{s\in S}$ is point-finite and thus has a closed shrinking $\{F_s\}_{s\in S}$ (see Problem 3.1.A(b) or [GT], Theorem 1.5.18). To complete the proof it suffices to apply Theorems 3.1.3 and 3.1.13. ■

A variant of the last theorem which closely parallels Theorem 2.3.14 is given in Problem 3.1.D(a).

3.1.15. THEOREM. *If a weakly paracompact normal space X can be represented as the union of a locally countable family $\{F_s\}_{s\in S}$ of closed subspaces such that $\dim F_s \leq n$ for $s \in S$, then $\dim X \leq n$.*

PROOF. For every point $x \in X$ there exist a neighbourhood U_x of x and a countable set $S(x) \subset S$ such that $\overline{U}_x \cap F_s = \emptyset$ if $s \notin S(x)$. It follows that $\overline{U}_x \subset \bigcup\{F_s : s \in S(x)\}$,

so that by virtue of Theorems 3.1.3 and 3.1.8 we have $\dim \overline{U}_x \leq n$ for each $x \in X$. To complete the proof it suffices to apply Theorem 3.1.14. ■

We now pass to the addition theorems for dim.

3.1.16. PROPOSITION. *If a normal space X can be represented as the union of two subspaces A and B such that A is normal, $\dim A \leq n$, and $\operatorname{rd}_X B \leq m$, then $\dim X \leq n+m+1$.*

PROOF. By virtue of Lemma 3.1.6 it suffices to show that $\dim \bar{A} \leq n+m+1$. Consider a finite open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space $\bar{A}$. Since $\dim A \leq n$, there exists a family $\mathcal{V} = \{V_i\}_{i=1}^k$ of open subsets of $\bar{A}$ such that

$$A \cap V_i \subset A \cap U_i \quad \text{for } i = 1, 2, \ldots, k, \quad A \subset \bigcup_{i=1}^k V_i \quad \text{and} \quad \operatorname{ord}(\{A \cap V_i\}_{i=1}^k) \leq n;$$

without loss of generality one can assume that $V_i \subset U_i$ for $i = 1, 2, \ldots, k$. The set $Z = \bar{A} \setminus \bigcup_{i=1}^k V_i$ is closed in X and contained in B, so that $\dim Z \leq m$, which implies that there exists a closed cover $\{F_i\}_{i=1}^k$ of Z such that $F_i \subset Z \cap U_i$ for $i = 1, 2, \ldots, k$ and $\operatorname{ord}(\{F_i\}_{i=1}^k) \leq m$. By virtue of Theorem 3.1.1 the family $\{F_i\}_{i=1}^k$ of closed subsets of $\bar{A}$ has an open swelling $\mathcal{W} = \{W_i\}_{i=1}^k$ in the space $\bar{A}$ such that $W_i \subset U_i$ for $i = 1, 2, \ldots, k$. Thus

$$Z = \bar{A} \setminus \bigcup_{i=1}^k V_i \subset \bigcup_{i=1}^k W_i \quad \text{and} \quad \operatorname{ord}(\{W_i\}_{i=1}^k) \leq m.$$

The union $\mathcal{V} \cup \mathcal{W}$ covers the space $\bar{A}$ and refines $\mathcal{U}$. As $\operatorname{ord}(\mathcal{V} \cup \mathcal{W}) \leq n+m+1$, we have $\dim \bar{A} \leq n+m+1$. ■

Proposition 3.1.16 implies

3.1.17. THE ADDITION THEOREM FOR dim. *If a hereditarily normal space X is represented as the union of two subspaces X_1 and X_2, then*

$$\dim X \leq \dim X_1 + \dim X_2 + 1. \blacksquare$$

The addition theorem yields

3.1.18. COROLLARY. *If a hereditarily normal space X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \ldots, Z_{n+1}$ such that $\dim Z_i \leq 0$ for $i = 1, 2, \ldots, n+1$, then $\dim X \leq n$.* ■

Let us note that the implication in the last corollary cannot be reversed even in the class of compact perfectly normal spaces. Indeed, applying the continuum hypothesis one can define a compact perfectly normal space X such that $\dim X = 1$ and $\operatorname{Ind} X = \operatorname{ind} X = 2$; now, if the space X could be represented as the union $Z_1 \cup Z_2$, where $\dim Z_1 = \dim Z_2 = 0$, then by virtue of Theorems 1.6.10 and 2.2.5 we would have $\operatorname{Ind} X = 1$ (cf. Problem 5.1.A).

Let us return to the question of the monotonicity of dim. We are now able to prove

3.1.19. THE SUBSPACE THEOREM FOR dim. *For every subspace M of a strongly hereditarily normal space X we have $\dim M \leq \dim X$.*

PROOF. We can assume that $\dim X = n < \infty$. To begin, let us observe that from Theorems 3.1.13, 3.1.8 and 3.1.3 it follows that $\dim U \leq n$ for every open subspace U of X which can be represented as the union of a point-finite family of open F_σ-sets in X.

Now, consider an arbitrary subspace M of X and a finite open cover $\{U_i\}_{i=1}^k$ of the space M; let $\{F_i\}_{i=1}^k$ be a closed shrinking of this cover. For $i = 1, 2, \ldots, k$ the sets F_i and $M \setminus U_i$ are separated in X, so that there exists an open set $V_i \subset X$ such that $F_i \subset V_i$, $M \cap V_i \subset U_i$, and V_i can be represented as the union of a point-finite family of open F_σ-sets in X. By the above observation the set $V = \bigcup_{i=1}^k V_i$ satisfies the inequality $\dim V \leq n$; therefore there exists an open shrinking $\{W_i\}_{i=1}^k$ of the cover $\{V_i\}_{i=1}^k$ of the space V such that $\operatorname{ord}(\{W_i\}_{i=1}^k) \leq n$. Since $M = \bigcup_{i=1}^k F_i \subset \bigcup_{i=1}^k V_i = V$ and $M \cap V_i \subset U_i$, the family $\{M \cap W_i\}_{i=1}^k$ is an open shrinking of the cover $\{U_i\}_{i=1}^k$. As $\operatorname{ord}(\{M \cap W_i\}_{i=1}^k) \leq n$, we have $\dim M \leq n$. ■

The last theorem together with Theorems 2.1.3 and 2.1.5 yields the following two corollaries, which can also be deduced directly from Proposition 3.1.4 and Theorems 3.1.8 and 3.1.14.

3.1.20. COROLLARY. *For every subspace M of a perfectly normal space X we have* $\dim M \leq \dim X$. ■

3.1.21. COROLLARY. *For every subspace M of a hereditarily weakly paracompact hereditarily normal space X we have* $\dim M \leq \dim X$. ■

We shall now establish a theorem on the monotonicity of dim which slightly differs in nature from our previous theorems of this type: here, the assumptions are about the internal topological properties of the subspace M rather than the position of M in the space X.

3.1.22. LEMMA. *Every subspace M of a normal space X which is an F_σ-set in X is normal.*

PROOF. It suffices to show (see Problem 2.1.E(a) or [GT], Lemma 1.5.15) that for every closed set $F \subset M$ and each open set $W \subset M$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of M such that $F \subset \bigcup_{i=1}^\infty W_i$ and $M \cap \overline{W}_i \subset W$ for $i = 1, 2, \ldots$, where the bar denotes the closure operator in X. Let $F_1, F_2, \ldots$ be closed subsets of X such that $M = \bigcup_{i=1}^\infty F_i$ and U an open subset of X such that $W = M \cap U$. The space X being normal, for $i = 1, 2, \ldots$ there exists an open set $U_i \subset X$ such that $F \cap F_i \subset U_i \subset \overline{U}_i \subset U$. The sets $W_i = M \cap U_i$ satisfy all the required conditions. ■

3.1.23. THEOREM. *For every strongly paracompact subspace M of a normal space X we have* $\dim M \leq \dim X$.

PROOF. We can assume that $\dim X = n < \infty$. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space M. For $i = 1, 2, \ldots, k$ let W_i be an open subset of X such that $U_i = M \cap W_i$ and let $W = \bigcup_{i=1}^k W_i$. For every point $x \in M$ choose a neighbourhood V_x of x in X such that $x \in V_x \subset \overline{V}_x \subset W$. The open cover $\{M \cap V_x\}_{x \in M}$ of the space M has a star-finite open refinement $\mathcal{V}$. Consider the decomposition $\{\mathcal{V}_s\}_{s \in S}$ of the cover $\mathcal{V}$ into components; as was established in Section 2.4, the components $\mathcal{V}_s$ are countable and

the sets $V_s = \bigcup \mathcal{V}_s$ are pairwise disjoint. Let F_s be the union of the closures in X of all members of $\mathcal{V}_s$. Clearly $V_s \subset F_s \subset W$, and by virtue of Lemma 3.1.22 the subspace F_s of X is normal; from the countable sum theorem it follows that $\dim F_s \leq n$. Thus, for every $s \in S$ there exists an open cover $\{W_{s,i}\}_{i=1}^k$ of the space F_s such that $W_{s,i} \subset W_i$ for $i = 1, 2, \ldots, k$ and $\operatorname{ord}(\{W_{s,i}\}_{i=1}^k) \leq n$. One readily sees that the family $\{V_i\}_{i=1}^k$, where $V_i = M \cap \bigcup_{s \in S}(V_s \cap W_{s,i})$, is an open shrinking of the cover $\{U_i\}_{i=1}^k$ of the space M. As $\operatorname{ord}(\{V_i\}_{i=1}^k) \leq n$, we have $\dim M \leq n = \dim X$. ■

Applying Theorem 3.1.14 we can strengthen the last theorem as follows (we recall that a space X is *locally strongly paracompact* if every point of X has a neighbourhood whose closure is strongly paracompact).

3.1.24. THEOREM. *For every weakly paracompact locally strongly paracompact normal subspace M of a normal space X we have* $\dim M \leq \dim X$. ■

Let us observe that in Theorem 3.1.23 the assumption that M is strongly paracompact cannot be replaced by the weaker assumption that M is paracompact (cf. Problem 3.2.G(b)). Indeed, by virtue of Remark 1.3.18, Roy's space X (cited in Section 2.4), which satisfies $\operatorname{ind} X = 0$ and $\dim X = 1$, is embeddable in a Cantor cube $D^{\mathbf{m}}$; since $\dim D^{\mathbf{m}} = 0$ (see Theorems 1.6.5 and 1.6.10), we have $\dim X > \dim D^{\mathbf{m}}$.

We shall now show that the Čech–Stone compactification preserves the dimension dim. In Section 3.4 it will be proved that for every normal space there exist compactifications preserving both the dimension dim and the weight.

3.1.25. THEOREM. *For every normal space X we have* $\dim \beta X = \dim X$.

PROOF. To begin, we shall prove that $\dim X \leq \dim \beta X$. The inequality being obvious if $\dim \beta X = \infty$, we can suppose that $\dim \beta X = n < \infty$. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space X. By virtue of Theorem 1.7.8 there exists a closed shrinking $\{F_i\}_{i=1}^k$ of the cover $\{U_i\}_{i=1}^k$, and by Urysohn's lemma for $i = 1, 2, \ldots, k$ there exists a continuous function $f_i \colon X \to I$ such that

$$f_i(X \setminus U_i) \subset \{0\} \quad \text{and} \quad f_i(F_i) \subset \{1\}; \tag{8}$$

let $\bar{f}_i \colon \beta X \to I$ be the continuous extension of f_i over βX. By virtue of (8) the family $\{W_i\}_{i=1}^k$, where $W_i = \bar{f}_i^{-1}((0, 1])$, is an open cover of the space βX and

$$X \cap W_i \subset U_i \quad \text{for } i = 1, 2, \ldots, k. \tag{9}$$

Since $\dim \beta X \leq n$, the cover $\{W_i\}_{i=1}^k$ has an open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$. From (9) it follows that the family $\{X \cap V_i\}_{i=1}^k$ is an open shrinking of the cover $\{U_i\}_{i=1}^k$ of the space X. As $\operatorname{ord}(\{X \cap V_i\}_{i=1}^k) \leq n$, we have $\dim X \leq n$.

Now, we shall prove that $\dim \beta X \leq \dim X$ (cf. Problem 3.3.A(c)). As in the first part of the proof, we shall suppose that $\dim X = n < \infty$. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space βX. By virtue of Theorem 1.7.8 there exists an open shrinking $\{W_i\}_{i=1}^k$ of the cover $\{U_i\}_{i=1}^k$ such that $\overline{W}_i \subset U_i$ for $i = 1, 2, \ldots, k$. Since $\dim X = n$, the cover $\{X \cap W_i\}_{i=1}^k$ of the space X has an open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$ which, in turn, has a closed shrinking $\{E_i\}_{i=1}^k$. For $i = 1, 2, \ldots, k$ let $F_i = \overline{E}_i$, where the bar denotes the closure operator in βX. The family $\{F_i\}_{i=1}^k$ is a cover of

the space βX; since $F_i = \overline{E}_i \subset \overline{V}_i \subset \overline{X \cap W_i} = \overline{W}_i \subset U_i$ for $i = 1, 2, \ldots$, this is a closed shrinking of the cover $\{U_i\}_{i=1}^k$. To complete the proof it suffices to show that $\operatorname{ord}(\{F_i\}_{i=1}^k) \leq n$.

For $i = 1, 2, \ldots, k$ define a continuous function $f_i: X \to I$ such that $f_i(E_i) \subset \{0\}$ and $f_i(X \setminus V_i) \subset \{1\}$; let $\bar{f}_i: \beta X \to I$ be the continuous extension of f_i over βX. Consider an arbitrary subfamily $\{F_{i_1}, F_{i_2}, \ldots, F_{i_m}\}$ of the cover $\{F_i\}_{i=1}^k$ such that $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m} \neq \emptyset$. Let $\bar{f} = \max(\bar{f}_{i_1}, \bar{f}_{i_2}, \ldots, \bar{f}_{i_m})$. The set $U = \bar{f}^{-1}([0,1))$ is open in βX, and since $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_m} \subset U$, we have $U \neq \emptyset$, which implies that $U \cap X \neq \emptyset$. One readily checks that $U \cap X \subset V_{i_1} \cap V_{i_2} \cap \ldots \cap V_{i_m}$; thus the relation $\operatorname{ord}(\{V_i\}_{i=1}^k) \leq n$ yields the inequality $m \leq n + 1$, which shows that $\operatorname{ord}(\{F_i\}_{i=1}^k) \leq n$. ■

3.1.26. COROLLARY. *For every normal space X and a dense normal subspace $M \subset X$ which has the property that every continuous function $f: M \to I$ is continuously extendable over X we have* $\dim M = \dim X$.

In other words, $\dim Y = \dim X$ *for every normal space X and every normal subspace Y of βX which contains X.*

PROOF. From the extendability of every continuous function $f: M \to I$ over X it follows that $\beta M = \beta X$. ■

The final part of this section is devoted to a study of relations between ind, Ind and dim.

Let us begin with reminding the reader that in Section 1.7, when proving that the equality $\operatorname{ind} X = \dim X$ holds for every separable metric space X, we found out that the proof of the inequality $\dim X \leq \operatorname{ind} X$ was much easier than the proof of the reverse inequality. We shall now show that the inequality $\dim X \leq \operatorname{ind} X$ holds for every strongly paracompact space X and that the related inequality $\dim X \leq \operatorname{Ind} X$ holds for every normal space X. Both results will be deduced from a common lemma.

3.1.27. LEMMA. *If for every pair A, B of disjoint closed subsets of a normal space X there exists a partition L between A and B such that* $\dim L \leq n - 1$, *then* $\dim X \leq n$.

PROOF. Consider a finite open cover $\{U_i\}_{i=1}^k$ of the space X; let $\{F_i\}_{i=1}^k$ be a closed shrinking of $\{U_i\}_{i=1}^k$. For $i = 1, 2, \ldots, k$ there exists a partition L_i between F_i and $X \setminus U_i$ such that $\dim L_i \leq n-1$; let W_i be an open set satisfying $F_i \subset W_i \subset U_i$ and $\operatorname{Fr} W_i \subset L_i$. Since $L = \bigcup_{i=1}^k L_i$ is a normal space, $\dim L \leq n - 1$ by virtue of Theorem 3.1.8. By shrinking the open cover $\{L \cap U_i\}_{i=1}^k$ of the space L to a closed cover of order $\leq n$ and then swelling the latter according to Theorem 3.1.1, we obtain a family $\{V_i\}_{i=1}^k$ of open subsets of X such that $\overline{V}_i \subset U_i$ for $i = 1, 2, \ldots, k$,

$$L \subset V = \bigcup_{i=1}^k V_i \quad \text{and} \quad \operatorname{ord}(\{\overline{V}_i\}_{i=1}^k) \leq n - 1. \tag{10}$$

The sets $\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_k$ together with the sets $Z_1, Z_2, \ldots, Z_k$, where

$$Z_i = \overline{W}_i \setminus \Big(V \cup \bigcup_{j<i} W_j\Big),$$

constitute a closed refinement of $\{U_i\}_{i=1}^k$. To complete the proof it suffices to observe that this refinement has order $\leq n$, which, however, follows immediately from the second part of (10) and the fact that for $j < i \leq k$ we have

$$Z_j \cap Z_i \subset \overline{W}_j \cap [\overline{W}_i \setminus (V \cup W_j)] \subset (X \setminus V) \cap \operatorname{Fr} W_j = \emptyset. \blacksquare$$

From the last lemma, by applying induction with respect to $\operatorname{Ind} X$, we obtain

3.1.28. THEOREM. *For every normal space X we have* $\dim X \leq \operatorname{Ind} X$. ■

In the next chapter we shall show that for every metrizable space X we have the equality $\operatorname{Ind} X = \dim X$. From Example 3.1.31 below it follows that the equality does not hold in compact spaces. Our commentary to Corollary 3.1.18 above shows that it does not hold in perfectly normal compact spaces either.

From Lemmas 3.1.27, 2.4.1 and Theorem 3.1.8, by applying induction with respect to $\operatorname{ind} X$, we obtain the following

3.1.29. THEOREM. *For every strongly paracompact space X we have* $\dim X \leq \operatorname{ind} X$. ■

As shown by Roy's space cited above, in the last theorem the assumption that X is strongly paracompact cannot be replaced by the weaker assumption that X is paracompact (cf. Problem 3.1.F). The same example shows that the equality $\operatorname{ind} X = \dim X$ does not hold in metric spaces. That it does not hold in compact spaces either is shown in Example 3.1.31.

To conclude, let us observe that Theorems 2.4.2 and 1.6.10 yield the following

3.1.30. THEOREM. *For every strongly paracompact space X the conditions* $\operatorname{ind} X = 0$, $\operatorname{Ind} X = 0$ *and* $\dim X = 0$ *are equivalent.* ■

3.1.31. EXAMPLE. In Example 2.2.14 we described a compact space X with $\operatorname{Ind} X = \operatorname{ind} X = 2$ which contains closed subspaces F_1, F_2 such that $F_1 \cup F_2 = X$ and $\operatorname{Ind} F_1 = \operatorname{Ind} F_2 = 1$. From Theorems 3.1.28, 3.1.8 and 1.6.10 it follows that $\dim X = 1$. Thus X is a compact space such that $\dim X \neq \operatorname{Ind} X$ and $\dim X \neq \operatorname{ind} X$. ■

Historical and bibliographic notes

Proposition 3.1.2 and Theorem 3.1.3 were given by Čech in [1933]. It was proved by Fedorčuk in [1973] that for every natural number $n \geq 2$ there exists a compact space X_n such that $\dim X_n = n$ and for every closed subspace $M \subset X_n$ we have either $\dim M \leq 0$ or $\dim M = n$. Examples of such spaces which are moreover perfectly normal (and hereditarily separable) were constructed under the assumption of the continuum hypothesis by Savinov in [1976] and by Fedorčuk in [1978a]. The notion of the relative covering dimension was introduced by Alexandroff in [1947]. Proposition 3.1.4 and Lemma 3.1.6 in the case where $X = K_1 \cup K_2$ (cf. Problem 3.1.B(b)) were obtained by Dowker in [1955]. Theorem 3.1.8 was established by Čech in [1933], and Theorem 3.1.10 independently by Morita in [1950a] and Katětov in [1952]; Lemma 3.1.9

appeared in Katětov's paper [1952]. Theorems 3.1.13 and 3.1.14 were proved by Zarelua in [1963] (the latter by Dowker in [1955] and Nagami in [1955] for paracompact spaces). Theorem 3.1.15, also for paracompact spaces, was given by Nagami in [1955]. Proposition 3.1.16 was established by Zarelua in [1963a]; its particular case formulated as Theorem 3.1.17 was obtained by Smirnov in [1951]. An example of a compact perfectly normal space X such that $\dim X = 1$ and $\operatorname{ind} X = \operatorname{Ind} X = 2$ was described under the assumption of the continuum hypothesis by Filippov in [1970a] (a simpler example can be found in Fedorčuk [1978a]); the first example of such a space was constructed under the joint assumption of the continuum hypothesis and the existence of a Souslin space (see [GT], Remark to Problem 2.7.9(f)) by Lifanov and Filippov in [1970]. Theorem 3.1.19 for totally normal spaces was proved by Dowker in [1955]; it was extended to Dowker spaces by Lifanov and Pasynkov in [1970] (announcement in Pasynkov [1967]) and to super normal spaces by Nishiura in [1977]. Corollary 3.1.20 was established by Čech in [1933]. Theorem 3.1.23 was proved by Morita in [1953]; its strengthening stated as Theorem 3.1.24 was given by Lifanov and Pasynkov in [1970] (an intermediate result was obtained by Pupko in [1961]). Theorem 3.1.25 was established by Wallman in [1938]. Theorem 3.1.28 was proved by Vedenissoff in [1941] and Theorem 3.1.29 by Morita in [1950a] (the latter theorem for Lindelöf spaces was obtained independently by Morita in [1950] and by Smirnov in [1951]; for compact spaces it was proved by Alexandroff in [1941]). Example 3.1.31 was given by Lokucievskiĭ in [1949]; the first example of a compact space X such that $\dim X \neq \operatorname{ind} X$ was described by Lunc in [1949].

As observed by Katětov in [1950], the definition of the covering dimension dim can be slightly modified so as to lead to a notion of dimension which behaves relatively well in completely regular spaces. The modification consists in replacing condition (ČL1) by

(ČL1′) $\dim X \leq n$, *where* $n = -1, 0, 1, \ldots$, *if every finite functionally open cover of the space* X *has a finite functionally open refinement of order* $\leq n$.

Let us recall that a family of subsets of a topological space is *functionally open* (*closed*) if all its members are functionally open (closed) sets (see Problem 2.1.A(a)). From Theorem 3.1.1 and Proposition 3.1.2 it easily follows that the definition of dim thus modified is equivalent to the original one in the realm of normal spaces. When applied to completely regular spaces, the modified definition yields the *covering dimension for completely regular spaces*, which is also denoted by dim. Some theorems proved in this section hold in the larger class of completely regular spaces for the covering dimension thus extended (see Problems 3.1.D(b), 3.1.I, 3.1.J, and 3.2.H–3.2.K).

Occasionally, the covering dimension for completely regular spaces, or even for larger classes of spaces, was defined just by conditions (ČL1)–(ČL3) with no modifications. The dimension function obtained in this way satisfies the counterparts of a few theorems proved in this section (see Ostrand [1971]). It seems, however, that such a definition is not quite consistent with the geometrical conception of dimension. The notion obtained displays some undesirable features of a separation axiom; for example, one can easily check that every T_1-space whose dimension in that sense equals zero is normal.

Problems

3.1.A. (a) Show that for every closed subspace F of a normal space X with $\dim F \leq n$ and a family $\mathcal{U} = \{U_i\}_{i=1}^k$ of open subsets of X with $F \subset \bigcup \mathcal{U}$ there exists a family $\mathcal{V} = \{V_i\}_{i=1}^k$ of open subsets of X such that $V_i \subset U_i$ for $i = 1, 2, \ldots, k$, $F \subset \bigcup \mathcal{V}$ and $\operatorname{ord} \mathcal{V} \leq n$.

(b) (Lefschetz [1942]) Prove that for every point-finite open cover $\{U_s\}_{s \in S}$ of a normal space X there exists an open cover $\{V_s\}_{s \in S}$ of X such that $\overline{V}_s \subset U_s$ for every $s \in S$.

Hint. Consider the family $\mathcal{G}$ of all functions G from the set S to the family of open subsets of X such that $\bigcup_{s \in S} G(s) = X$ and for every $s \in S$ we have $G(s) = U_s$ or $\overline{G(s)} \subset U_s$. Order the family $\mathcal{G}$ by defining that $G_1 \leq G_2$ whenever $G_2(s) = G_1(s)$ for every $s \in S$ such that $G_1(s) \neq U_s$, and apply the Kuratowski–Zorn lemma.

(c) (Morita [1950a]) Prove that if for a family $\{F_s\}_{s \in S}$ of closed subsets of a normal space X there exists a locally finite family $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s$ for every $s \in S$, then the family $\{F_s\}_{s \in S}$ has an open swelling $\{U_s\}_{s \in S}$ such that $\overline{U}_s \subset V_s$ for $s \in S$ (cf. Problem 4.3.E(a)).

Hint. Apply transfinite induction; modify the proof of Theorem 3.1.1.

Remark. For every locally finite family $\{F_s\}_{s \in S}$ of closed subsets of a countably paracompact collectionwise normal space X there exists a locally finite family of open subsets of X such that $F_s \subset V_s$ for every $s \in S$ (see [GT], Problem 5.5.17(a)).

3.1.B. (a) Show that if a normal space X can be represented as the union of two normal subspaces A_1 and A_2, one of them being both an F_σ-set and a G_δ-set, such that $\dim A_1 \leq n$ and $\dim A_2 \leq n$, then $\dim X \leq n$. Deduce that in perfectly normal spaces the same holds under the assumption that one of the subspaces is the intersection of a closed and an open set.

(b) Observe that Lemma 3.1.6 is a consequence of its special case, viz., the case where $X = K_1 \cup K_2$, and Theorem 3.1.8.

(c) Prove that if a normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of subspaces such that $\operatorname{rd}_X K_\alpha \leq n$, the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for each $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\dim X \leq n$.

Hint. Apply transfinite induction; use Lemma 3.1.6 and Theorem 3.1.10.

(d) Prove that if a normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of normal subspaces such that $\dim K_\alpha \leq n$ and the union $\bigcup_{\beta < \alpha} K_\beta$ is closed for each $\alpha < \xi$, and the family $\{K_\alpha\}_{\alpha < \xi}$ is locally finite, then $\dim X \leq n$.

3.1.C. (a) (Nagata [1965]) Prove that if a normal space X can be represented as the union of a transfinite sequence $F_1, F_2, \ldots, F_\alpha, \ldots$, $\alpha < \xi$, of closed subspaces such that $\dim F_\alpha \leq n$ for each $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$, of open subsets of X such that $F_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta < \alpha}$ is locally finite for each $\alpha < \xi$, then $\dim X \leq n$.

Hint. See the hint to Problem 2.3.E(a).

(b) Prove that if a normal space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of subspaces such that $\operatorname{rd}_X K_\alpha \leq n$ and the union $\bigcup_{\beta<\alpha} K_\beta$ is closed for each $\alpha < \xi$, and if there exists a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$, of open subsets of X such that $K_\alpha \subset U_\alpha$ and the family $\{U_\beta\}_{\beta<\alpha}$ is locally finite for each $\alpha < \xi$, then $\dim X \leq n$.

3.1.D. (a) Show that if a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of normal open subspaces such that $\dim U_s \leq n$ for $s \in S$, then $\dim X \leq n$.

(b) Show, applying Problem 3.1.J(a) below, that the assumption of normality in (a), as well as in Proposition 3.1.7, can be omitted if one understands by dim the covering dimension for completely regular spaces as defined in the notes to this section.

3.1.E. (a) (Dowker [1955]) Note that the space X described in Example 2.2.12 satisfies the equality $\dim X = 1$.

(b) (Smirnov [1958]) Note that the space Y described in Problem 2.2.I satisfies the equality $\dim Y = \infty$.

3.1.F (Zarelua [1963] (announcement [1961])). Show that for every completely paracompact space X we have $\dim X \leq \operatorname{ind} X$ (see Problem 2.4.B).

3.1.G (Morita [1956]). Show that if $f: X \to Y$ is a closed mapping of a normal space X onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $\dim Y \leq \operatorname{Ind} X + (k-1)$ (see Theorem 3.3.7).

Hint. Modify the proof of Theorem 2.4.15 and apply Lemmas 3.1.27 and 3.1.6.

Remark. Morita also proved in [1956] that if $f: X \to Y$ is a closed mapping of a normal space X to a paracompact space Y and there exists an integer $k \geq 0$ such that $\dim f^{-1}(y) \leq k$ for every $y \in Y$, then $\dim X \leq \operatorname{Ind} Y + k$ (see Theorem 3.3.10). This result can be obtained by a modification of the method used in the proofs of Theorems 1.12.4 and 4.3.4 and an application of Remark 1.9.8 (instead of Remark 1.9.8 one can apply the notion of uniform homotopy, see Alexandroff and Pasynkov [1973] or Pears [1975]). As shown by Zarelua in [1963], the assumption that the space Y is paracompact can be relaxed to its being weakly paracompact.

3.1.H. (a) Check that the union and the intersection of finitely many functionally open (closed) sets are functionally open (closed). Show that the union (intersection) of countably many functionally open (closed) sets is functionally open (closed).

(b) Prove that every finite functionally open cover $\{U_i\}_{i=1}^k$ of a topological space X has shrinkings $\{F_i\}_{i=1}^k$ and $\{W_i\}_{i=1}^k$ which are, respectively, functionally closed and functionally open, and such that $F_i \subset \overline{W}_i \subset U_i$ for $i = 1, 2, \ldots, k$.

(c) Prove that every finite family $\{F_i\}_{i=1}^k$ of functionally closed subsets of a topological space X has a functionally open swelling $\{U_i\}_{i=1}^k$. Observe that if, moreover, a family $\{V_i\}_{i=1}^k$ of functionally open subsets of X satisfying $F_i \subset V_i$ for $i = 1, 2, \ldots, k$ is given, then the swelling can be defined in such a way that $\overline{U}_i \subset V_i$ for $i = 1, 2, \ldots, k$.

3.1.I. Show that for every completely regular space X the following conditions are equivalent (see the notes to this section):

(1) *The space X satisfies the inequality* $\dim X \leq n$.
(2) *Every finite functionally open cover of the space X has a functionally open shrinking of order* $\leq n$.
(3) *Every finite functionally open cover of the space X has a functionally closed shrinking of order* $\leq n$.
(4) *Every finite functionally open cover of the space X has a finite functionally closed refinement of order* $\leq n$.

3.1.J (Katětov [1950]). (a) Show that if a subspace M of a completely regular space X has the property that every continuous function $f: M \to I$ is continuously extendable over X, then $\dim M \leq \dim X$.

(b) Prove that for every completely regular space X we have $\dim \beta X = \dim X$.

(c) Deduce from (b) that if a completely regular space X can be represented as the union of a sequence $A_1, A_2, \ldots$ of subspaces such that $\dim A_i \leq n$ and every continuous function $f: A_i \to I$ is continuously extendable over X for $i = 1, 2, \ldots$, then $\dim X \leq n$.

Remark. Nagami in [1980] proved that for every functionally open subspace M of a completely regular space X we have $\dim M \leq \dim X$. Terasawa defined in [1977] a completely regular space X with $\dim X > 0$ which can be represented as the union of a functionally closed subspace F with $\dim F = 0$ and an open discrete subspace of cardinality $\aleph_0$. As shown by E. Pol in [1979] (announcement in [1976a]), there exists a completely regular space X with $\dim X > 0$ which can be represented as the union of two functionally closed subspaces F_1 and F_2 such that $\dim F_1 = \dim F_2 = 0$. Nagami in [1980] proved that if a completely regular space X can be represented as the union of a locally finite family $\{U_s\}_{s \in S}$ of functionally open subspaces such that $\dim U_s \leq n$ for $s \in S$, then $\dim X \leq n$. It is an open problem whether every completely regular space X which can be represented as the union of a locally finite family $\{A_s\}_{s \in S}$ of subspaces such that $\dim A_s \leq n$ and every continuous function $f: A_s \to I$ is continuously extendable over X for $s \in S$ satisfies the inequality $\dim X \leq n$.

3.2. Characterizations of the dimension dim in normal spaces

In this section several characterizations of the covering dimension in normal spaces are established. They split into two groups. The first consists of characterizations which are cognate with the definition of dim. The second is made up of generalizations of three important theorems proved in Chapter 1 for separable metric spaces, viz., of the theorems on partitions, on extending mappings to spheres, and on $\mathcal{E}$-mappings. In the final part of the section we apply one of the characterizations of the first group to obtain two relatively simple theorems on the dimension dim of Cartesian products.

3.2.1. DOWKER'S THEOREM. *For every normal space X the following conditions are equivalent*:

(i) *The space X satisfies the inequality* $\dim X \leq n$.
(ii) *Every locally finite open cover of the space X has an open shrinking of order* $\leq n$.
(iii) *Every locally finite open cover of the space X has an open refinement of order* $\leq n$.

PROOF. We shall show first that (i)⇒(ii). Consider a normal space X such that $\dim X \leq n$. Let $\mathcal{U} = \{U_s\}_{s\in S}$ be an arbitrary locally finite open cover of the space X. Denote by $\mathcal{T}$ the family of all non-empty finite subsets of S and for every $T \in \mathcal{T}$ define

$$F_T = \bigcap_{s\in T} \overline{U}_s \cap \bigcap_{s\notin T} (X \setminus U_s);$$

then $\dim F_T \leq n$ by virtue of Theorem 3.1.3. The family $\mathcal{F} = \{F_T\}_{T\in\mathcal{T}}$ is a closed cover of X each member of which meets only finitely many sets U_s. The cover $\mathcal{U}$ being locally finite, for every point $x \in X$ there exists a neighbourhood U of x and a finite set $S_0 \subset S$ such that $U \cap U_s = \emptyset$, and consequently $U \cap \overline{U}_s = \emptyset$, for $s \notin S_0$. From the definition of the sets F_T it follows that if $U \cap F_T \neq \emptyset$, then $T \subset S_0$; thus the cover $\mathcal{F}$ is locally finite. By virtue of Lemma 3.1.9 the cover $\mathcal{U}$ has an open shrinking of order $\leq n$, so that the space X satisfies condition (ii).

To complete the proof it suffices to observe that the implication (ii)⇒(iii) is obvious and the implication (iii)⇒(i) follows from Proposition 1.6.8. ■

The last theorem yields

3.2.2. PROPOSITION. *For every paracompact space X the following conditions are equivalent*:

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *Every open cover of the space X has an open shrinking of order* $\leq n$.

(iii) *Every open cover of the space X has an open refinement of order* $\leq n$. ■

Conditions (i) and (iii) in the last proposition are not equivalent in the realm of all normal spaces. Indeed, each cover of finite order is point-finite, so that every space X which satisfies (iii) is weakly paracompact, whereas there exist normal spaces which are not weakly paracompact (see Example 2.1.6). One can show that conditions (i) and (iii) are not equivalent even in the realm of all weakly paracompact spaces but the example is more difficult (see [GT], Problem 5.5.3(c)). Obviously, conditions (ii) and (iii) are equivalent for every normal space X.

As one easily sees, every family of sets which can be represented as the union of $n+1$ families of order ≤ 0, i.e., consisting of pairwise disjoint sets, has order $\leq n$. We are now going to strengthen Theorem 3.2.1 by proving that every locally finite open cover of a normal space X such that $\dim X \leq n$ has an open shrinking of this last form. This result is to a certain degree a substitute for the decomposition theorem for dim.

3.2.3. LEMMA. *For every locally finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of a normal space X such that* $\operatorname{ord}\mathcal{U} \leq n \geq 0$ *there exists an open cover $\mathcal{V}$ of the space X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n+1}$, where $\mathcal{V}_i = \{V_{i,s}\}_{s\in S}$, such that* $\operatorname{ord}\mathcal{V}_i \leq 0$ *and $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots, n+1$.*

PROOF. We shall apply induction with respect to n. The lemma is obvious if $n = 0$. Assume that the lemma is proved for all normal spaces and all locally finite open covers of order $< n \geq 1$ and consider a normal space X and a locally finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of X such that $\operatorname{ord}\mathcal{U} \leq n$. Let $\{W_s\}_{s\in S}$ and $\{F_s\}_{s\in S}$ be, respectively, an open and a closed shrinking of $\mathcal{U}$ such that $W_s \subset F_s$ for every $s \in S$ (see Problem

3.1.A(b)). Denote by $\mathcal{T}$ the family of all subsets of S that have exactly $n+1$ elements and for every $T \in \mathcal{T}$ define

$$U_T = \bigcap_{s \in T} U_s, \quad W_T = \bigcap_{s \in T} W_s \quad \text{and} \quad F_T = \bigcap_{s \in T} F_s.$$

From the local finiteness of $\mathcal{U}$ it follows that the family $\{F_T\}_{T \in \mathcal{T}}$ is locally finite. The inequality $\operatorname{ord} \mathcal{U} \leq n$ implies that $U_T \cap U_{T'} = \emptyset$ whenever $T \neq T'$.

For every $T \in \mathcal{T}$ choose arbitrarily an $s(T) \in T$ and let

$$V_{n+1,s} = \bigcup \{U_T : s(T) = s\} \quad \text{for } s \in S.$$

The family $\mathcal{V}_{n+1} = \{V_{n+1,s}\}_{s \in S}$ has order ≤ 0 and $V_{n+1,s} \subset U_s$ for $s \in S$; moreover,

$$\bigcup_{s \in S} V_{n+1,s} = \bigcup_{T \in \mathcal{T}} U_T. \tag{1}$$

Consider now two subspaces of the space X:

$$Y = X \setminus \bigcup_{T \in \mathcal{T}} W_T \quad \text{and} \quad Z = X \setminus \bigcup_{T \in \mathcal{T}} F_T;$$

clearly, Y is closed in X and Z is open; moreover, $Z \subset Y$. From the definition of the sets W_T it follows that the locally finite open cover $\{Y \cap W_s\}_{s \in S}$ of the space Y has order $\leq n-1$. Since Y is a normal space, by virtue of the inductive assumption there exists an open cover $\mathcal{V}'$ of Y which can be represented as the union of n families $\mathcal{V}'_1, \mathcal{V}'_2, \ldots, \mathcal{V}'_n$, where $\mathcal{V}'_i = \{V'_{i,s}\}_{s \in S}$, such that $\operatorname{ord} \mathcal{V}'_i \leq 0$ and $V'_{i,s} \subset Y \cap W_s \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots, n$.

For $i = 1, 2, \ldots, n$ let $\mathcal{V}_i = \{V_{i,s}\}_{s \in S}$, where $V_{i,s} = Z \cap V'_{i,s}$. One readily sees that the sets $V_{i,s}$ are open in X, that $\operatorname{ord} \mathcal{V}_i \leq 0$ for $i = 1, 2, \ldots, n$ and that $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots, n$; moreover, $Z \subset \bigcup_{i=1}^{n} \bigcup_{s \in S} V_{i,s}$. Since $F_T \subset U_T$ for $T \in \mathcal{T}$, we have

$$X \setminus Z = \bigcup_{T \in \mathcal{T}} F_T \subset \bigcup_{T \in \mathcal{T}} U_T,$$

so that by virtue of (1), the union $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$ is an open cover of the space X. ■

Theorem 3.2.1 and Lemma 3.2.3 yield

3.2.4. OSTRAND'S THEOREM. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every locally finite open cover $\mathcal{U} = \{U_s\}_{s \in S}$ of the space X there exists an open cover $\mathcal{V}$ of the space X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n+1}$, where $\mathcal{V}_i = \{V_{i,s}\}_{s \in S}$, such that $\operatorname{ord} \mathcal{V}_i \leq 0$ and $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots, n+1$.* ■

The next theorem is a strong version of the theorem on partitions.

3.2.5. MORITA'S THEOREM. *A normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every locally finite family $\{U_s\}_{s \in S}$ of open subsets of X and every family $\{F_s\}_{s \in S}$ of closed subsets of X such that $F_s \subset U_s$ for $s \in S$ there exist families $\{W_s\}_{s \in S}$ and $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s \subset \overline{V}_s \subset W_s \subset \overline{W}_s \subset U_s$ for $s \in S$ and $\operatorname{ord}(\{\overline{W}_s \setminus V_s\}_{s \in S}) \leq n-1$.*

PROOF. First we shall prove that every normal space X with $\dim X \leq n$ satisfies the condition in the theorem. Consider a locally finite family $\{U_s\}_{s \in S}$ of open subsets

of X and a family $\{F_s\}_{s\in S}$ of closed subsets of X such that $F_s \subset U_s$ for $s \in S$. Denote by $\mathcal{T}$ the family of all finite subsets of S, for each non-empty $T \in \mathcal{T}$ define

$$G_T = \bigcap_{s\in T} U_s \cap \bigcap_{s\notin T} (X \setminus F_s), \tag{2}$$

and let $G_\emptyset = X \setminus \bigcup_{s\in S} F_s$. The family $\{G_T\}_{T\in\mathcal{T}}$ is a locally finite open cover of the space X. By virtue of Dowker's theorem (cf. Remark 3.2.7) the cover $\{G_T\}_{T\in\mathcal{T}}$ has an open shrinking $\{H_T\}_{T\in\mathcal{T}}$ of order $\leq n$, and, in turn, the cover $\{H_T\}_{T\in\mathcal{T}}$ has a closed shrinking $\{A_T\}_{T\in\mathcal{T}}$. From (2) it follows that

$$\text{if } A_T \cap F_s \neq \emptyset, \text{ then } s \in T. \tag{3}$$

For every $T \in \mathcal{T}$ and each $s \in T$ we can define open sets $W_T(s)$ and $V_T(s)$ such that

$$A_T \subset V_T(s) \subset \overline{V_T(s)} \subset W_T(s) \subset \overline{W_T(s)} \subset H_T \quad \text{for } s \in T \tag{4}$$

and

$$\text{if } s, s' \in T \text{ and } s' \neq s, \text{ then either } \overline{W_T(s)} \subset V_T(s') \text{ or } \overline{W_T(s')} \subset V_T(s). \tag{5}$$

Moreover, define $W_T(s) = V_T(s) = \emptyset$ for $s \in S \setminus T$.

Let

$$W_s = \bigcup_{T\in\mathcal{T}} W_T(s) \quad \text{and} \quad V_s = \bigcup_{T\in\mathcal{T}} V_T(s) \quad \text{for } s \in S.$$

The sets W_s and V_s are open; from the local finiteness of the cover $\{H_T\}_{T\in\mathcal{T}}$ it follows that $\overline{V}_s \subset W_s$ and $\overline{W}_s \subset U_s$, because (2) implies that $\overline{W_T(s)} \subset U_s$. Now, for every point $x \in F_s$ there exists a $T \in \mathcal{T}$ such that $x \in A_T$; as $s \in T$ by virtue of (3), it follows from (4) that $x \in V_T(s) \subset V_s$, so that $F_s \subset V_s$. To complete the first part of the proof it suffices to show that $\operatorname{ord}(\{\overline{W}_s \setminus V_s\}_{s\in S}) \leq n-1$.

Consider a sequence $s_1, s_2, \ldots, s_k$ of distinct elements of the set S such that $\bigcap_{i=1}^k (\overline{W}_{s_i} \setminus V_{s_i}) \neq \emptyset$. Let x be a point in the last intersection. There exist sets $T_1, T_2, \ldots, T_k \in \mathcal{T}$ such that $s_i \in T_i$ for $i = 1, 2, \ldots, k$ and $x \in \bigcap_{i=1}^k (\overline{W_{T_i}(s_i)} \setminus V_{T_i}(s_i))$; the T_i's are distinct by (5). From (4) it follows that $x \notin \bigcup_{i=1}^k A_{T_i}$, and since $\{A_T\}_{T\in\mathcal{T}}$ is a cover of X, there exists a $T_0 \in \mathcal{T}$ such that $x \in A_{T_0} \subset H_{T_0}$; clearly $T_0 \neq T_i$ for $i = 1, 2, \ldots, k$. Thus $x \in \bigcap_{i=0}^k H_{T_i}$, and as $\operatorname{ord}(\{H_T\}_{T\in\mathcal{T}}) \leq n$ we have $k \leq n$, which shows that $\operatorname{ord}(\{\overline{W}_s \setminus V_s\}_{s\in S}) \leq n-1$.

Consider now a normal space X which satisfies the condition in the theorem. By virtue of Remark 1.7.10, to prove that $\dim X \leq n$ it suffices to show that for every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Define

$$U_i = X \setminus B_i \quad \text{and} \quad F_i = A_i \quad \text{for } i = 1, 2, \ldots, n+1.$$

Applying the condition in the theorem to $S = \{1, 2, \ldots, n+1\}$ and the sets U_i and F_i defined above, we obtain a family $\{V_i\}_{i=1}^{n+1}$ of open subsets of X such that

$$A_i \subset V_i \subset \overline{V}_i \subset X \setminus B_i \quad \text{for } i = 1, 2, \ldots, n+1 \quad \text{and} \quad \operatorname{ord}(\{\overline{V}_i \setminus V_i\}_{i=1}^{n+1}) \leq n-1. \tag{6}$$

By virtue of the first part of (6) the set $L_i = \overline{V}_i \setminus V_i$ is a partition between A_i and B_i for $i = 1, 2, \ldots, n+1$; the second part of (6) means that $\bigcap_{i=1}^{n+1} L_i = \emptyset$. ■

Theorem 3.2.5 and Remark 1.7.10 yield

3.2.6. THEOREM ON PARTITIONS. *A normal space X satisfies the inequality* $\dim X \leq n \geq 0$ *if and only if for every sequence* $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ *of* $n+1$ *pairs of disjoint closed subsets of* X *there exist closed sets* $L_1, L_2, \ldots, L_{n+1}$ *such that* L_i *is a partition between* A_i *and* B_i *and* $\bigcap_{i=1}^{n+1} L_i = \emptyset$. ■

3.2.7. REMARK. Let us note that if the family $\{U_s\}_{s\in S}$ in Morita's theorem is finite then in the proof of this theorem the cover $\{H_T\}_{T\in\mathcal{T}}$ can be obtained by applying Theorem 1.7.8 rather than Dowker's theorem. In particular, the theorem on partitions can be proved without resorting to Dowker's theorem.

Let us also note that the proof of Morita's theorem (in the case when the family $\{U_s\}_{s\in S}$ is finite) shows that if X is a normal space, $\mathcal{M}$ a family of subsets of X, and to each $M \in \mathcal{M}$ corresponds an integer $n_M \geq 0$ such that every finite open cover $\mathcal{G}$ of the space X has an open shrinking $\mathcal{H}$ with the property that $\operatorname{ord}\mathcal{H}|M \leq n_M$ for each $M \in \mathcal{M}$, then for every finite family $\{U_i\}_{i=1}^k$ of open subsets of X and every family $\{F_i\}_{i=1}^k$ of closed subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots, k$ there exist families $\{W_i\}_{i=1}^k$ and $\{V_i\}_{i=1}^k$ of open subsets of X such that $F_i \subset V_i \subset \overline{V}_i \subset W_i \subset \overline{W}_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $\operatorname{ord}(\{\overline{W}_i \setminus V_i\}_{i=1}^k | M) \leq n_M - 1$ for each $M \in \mathcal{M}$ (this more general statement will be used in Chapter 5). ■

We now turn to the theorem on extending mappings to spheres.

3.2.8. LEMMA. *Let* $f, g: X \to \operatorname{Fr} I^{n+1}$ *be continuous mappings of a topological space* X *to the boundary* $\operatorname{Fr} I^{n+1}$ *of the* $(n+1)$*-cube* I^{n+1} *in* R^{n+1}. *If for every point* $x \in X$ *the points* $f(x)$ *and* $g(x)$ *belong to the same face of* I^{n+1}, *then the mappings* f *and* g *are homotopic.*

PROOF. By assigning to every point $x \in X$ and each number $t \in I$ the point $h(x,t)$ which divides the line segment with end-points $f(x)$ and $g(x)$ in the ratio of t to $1-t$ one defines a homotopy $h: X \times I \to \operatorname{Fr} I^{n+1}$ between f and g. ■

3.2.9. THEOREM. *If* X *is a normal space and* A *is a closed subspace of* X *such that* $\operatorname{rd}_X(X \setminus A) \leq n \geq 0$, *then for every continuous mapping* $f: A \to S^n$ *there exists a continuous extension* $F: X \to S^n$ *of* f *over* X.

PROOF. There exists an open set $W \subset X$ containing A and such that f has a continuous extension $\bar{f}: W \to S^n$ over W. Consider an open set V such that $A \subset V \subset \overline{V} \subset W$ and let $Z = X \setminus V$ and $B = Z \cap \overline{V}$. To show that f is continuously extendable over X it suffices to prove that for the mapping $h = \bar{f}|B: B \to S^n$ there exists a continuous extension $H: Z \to S^n$ over Z. Indeed, the mapping $F: X \to S^n$ defined by letting

$$F(x) = \bar{f}(x) \text{ for } x \in \overline{V} \quad \text{and} \quad F(x) = H(x) \text{ for } x \in Z$$

will then be a continuous extension of f over X. Since $\dim Z \leq n$, the above observation shows that with no loss of generality one can assume that the space X satisfies the inequality $\dim X \leq n$. Moreover, one can assume that X is compact, because $\dim \beta X \leq n$ by Theorem 3.1.25 and the mapping f is continuously extendable over the closure

of the set A in βX, this last closure being the Čech–Stone compactification βA of the space A (see [GT], Corollary 3.6.8). Finally, instead of mappings to the n-sphere S^n one can consider mappings to the boundary $\operatorname{Fr} I^{n+1}$ of the $(n+1)$-cube I^{n+1} in R^{n+1}, which is homeomorphic to S^n.

Consider therefore a continuous mapping $f\colon A \to \operatorname{Fr} I^{n+1}$ defined on a closed subspace A of a compact space X such that $\dim X \leq n$. For $i = 1, 2, \ldots, n+1$ denote by $p_i\colon I^{n+1} \to I$ the projection of the $(n+1)$-cube I^{n+1} onto the ith coordinate axis and let

$$f_i = p_i f\colon A \to I, \quad A_i = f_i^{-1}(0) \quad \text{and} \quad B_i = f_i^{-1}(1);$$

obviously

$$A = \bigcup_{i=1}^{n+1} A_i \cup \bigcup_{i=1}^{n+1} B_i. \tag{7}$$

By virtue of Theorem 3.2.6 there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$. Consequently, there exist open sets $U_i, W_i \subset X$, where $i = 1, 2, \ldots, n+1$, such that

$$A_i \subset U_i, \quad B_i \subset W_i, \quad U_i \cap W_i = \emptyset \quad \text{and} \quad X \setminus L_i = U_i \cup W_i.$$

From Theorem 3.1.1 it follows that there exist open sets $V_1, V_2, \ldots, V_{n+1}$ satisfying

$$L_i \subset V_i \subset X \setminus (A_i \cup B_i) \quad \text{for } i = 1, 2, \ldots, n+1 \quad \text{and} \quad \bigcap_{i=1}^{n+1} V_i = \emptyset.$$

By virtue of Urysohn's lemma for $i = 1, 2, \ldots, n+1$ there exist continuous functions

$$g_i'\colon U_i \cup L_i \to [0, 1/2] \quad \text{and} \quad g_i''\colon W_i \cup L_i \to [1/2, 1]$$

such that

$$g_i'(U_i \setminus V_i) \subset \{0\}, \quad g_i'(L_i) \subset \{1/2\}$$

and

$$g_i''(W_i \setminus V_i) \subset \{1\}, \quad g_i''(L_i) \subset \{1/2\}.$$

Letting

$$g_i(x) = \begin{cases} g_i'(x) & \text{for } x \in U_i \cup L_i, \\ g_i''(x) & \text{for } x \in W_i \cup L_i, \end{cases}$$

we define continuous functions $g_i\colon X \to I$ such that

$$g_i(A_i) \subset \{0\}, \quad g_i(B_i) \subset \{1\} \quad \text{and} \quad g_i^{-1}(1/2) \subset V_i \quad \text{for } i = 1, 2, \ldots, n+1. \tag{8}$$

Since $\bigcap_{i=1}^{n+1} V_i = \emptyset$, the continuous mapping $g\colon X \to I^{n+1}$ defined by letting $g(x) = (g_1(x), g_2(x), \ldots, g_{n+1}(x))$ does not assume the value $a = (1/2, 1/2, \ldots, 1/2) \in I^{n+1}$. The composition of the mapping g and the central projection p of $I^{n+1} \setminus \{a\}$ from the point a onto the boundary of I^{n+1} is a continuous mapping $G\colon X \to \operatorname{Fr} I^{n+1}$. From (7) and (8) it follows that $g(A) \subset \operatorname{Fr} I^{n+1}$, so that $G|A = g|A$, i.e., the mapping G is a continuous extension of the restriction $g|A$ over the space X. Now, for every point $x \in A$ there exists an $i \leq n+1$ such that either $f_i(x) = g_i(x) = 0$ or $f_i(x) = g_i(x) = 1$, so that for every point $x \in A$ the points $f(x)$ and $g(x)$ belong to the same face of I^{n+1}. From Lemma 3.2.8 it follows that the mappings f and $g|A$ are homotopic, and Lemma 1.9.7 implies that there exists a continuous extension $F\colon X \to \operatorname{Fr} I^{n+1}$ of the mapping f over X. ■

Theorem 3.2.9, Remark 1.9.4 and Theorem 3.2.6 yield the following

3.2.10. THEOREM ON EXTENDING MAPPINGS TO SPHERES. *A normal space X satisfies the inequality* $\dim X \leq n \geq 0$ *if and only if for every closed subspace A of the space X and each continuous mapping $f\colon A \to S^n$ there exists a continuous extension $F\colon X \to S^n$ of f over X.* ■

The last characterization of the covering dimension to be established in this section uses the notion of an $\mathcal{E}$-mapping (see Definition 1.10.8).

3.2.11. THEOREM ON $\mathcal{E}$-MAPPINGS. *A normal space X satisfies the inequality* $\dim X \leq n$ *if and only if for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$-mapping of X to a polyhedron of dimension $\leq n$.*

PROOF. By virtue of Theorem 1.10.11, it suffices to show that for every finite open cover $\mathcal{E} = \{U_i\}_{i=1}^k$ of a normal space X with $0 \leq \dim X \leq n$ there exists an $\mathcal{E}$-mapping of X to a polyhedron of dimension $\leq n$.

Consider an open shrinking $\mathcal{V} = \{V_i\}_{i=1}^k$ of the cover $\mathcal{E}$ such that $\operatorname{ord}\mathcal{V} \leq n$ and a closed shrinking $\{F_i\}_{i=1}^k$ of $\mathcal{V}$. Let $\mathcal{N}(\mathcal{V})$ be a nerve of $\mathcal{V}$ with vertices $p_1, p_2, \ldots, p_k \in R^m$. By Urysohn's lemma for $i = 1, 2, \ldots, k$ there exists a continuous function $f_i\colon X \to I$ such that $f_i(X \setminus V_i) \subset \{0\}$ and $f_i(F_i) \subset \{1\}$. One readily checks (cf. the proof of Theorem 1.10.7) that the formula

$$\kappa(x) = \kappa_1(x)p_1 + \kappa_2(x)p_2 + \ldots + \kappa_k(x)p_k,$$

where

$$\kappa_i(x) = \frac{f_i(x)}{f_1(x) + f_2(x) + \ldots + f_k(x)} \quad \text{for } i = 1, 2, \ldots, k,$$

defines a continuous mapping $\kappa\colon X \to N(\mathcal{V})$ of X to the underlying polyhedron of the nerve $\mathcal{N}(\mathcal{V})$ and that κ satisfies the condition

$$\kappa^{-1}(\operatorname{St}_{N(\mathcal{V})}(p_i)) = V_i \subset U_i \quad \text{for } i = 1, 2, \ldots, k.$$

As $\{\operatorname{St}_{N(\mathcal{V})}(p_i)\}_{i=1}^k$ is an open cover of $N(\mathcal{V})$, the mapping κ is an $\mathcal{E}$-mapping. The inequality $\operatorname{ord}\mathcal{V} \leq n$ implies that $N(\mathcal{V})$ has dimension $\leq n$. ■

Applying Theorem 1.10.15 and arguing as in the proof of Theorem 1.10.16, one obtains the following strengthening of the theorem on $\mathcal{E}$-mappings.

3.2.12. THEOREM. *A normal space X satisfies the inequality* $\dim X \leq n$ *if and only if for every finite open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$-mapping of X onto a polyhedron of dimension $\leq n$.* ■

We close this section with two theorems on the covering dimension of Cartesian products that follow from Morita's theorem. A thorough discussion of Cartesian product theorems for dim is offered in Section 3.4.

3.2.13. THEOREM. *For every pair X, Y of compact spaces, not both empty, we have*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

PROOF. We can assume that $\dim X = n$ and $\dim Y = m$, where n and m are non-negative integers. Consider an arbitrary sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+m+1},$

B_{n+m+1}) of $n+m+1$ pairs of disjoint closed subsets of the Cartesian product $X \times Y$. The sets A_i being compact, for $i = 1, 2, \ldots, n+m+1$ there exist closed sets $E_{i,j} \subset X$ and $F_{i,j} \subset Y$, and open sets $U_{i,j} \subset X$ and $V_{i,j} \subset Y$, where $j = 1, 2, \ldots, k_i$, such that

$$(9) \qquad A_i \subset \bigcup_{j=1}^{k_i} (E_{i,j} \times F_{i,j}) \quad \text{and} \quad E_{i,j} \times F_{i,j} \subset U_{i,j} \times V_{i,j} \subset (X \times Y) \setminus B_i.$$

By virtue of Theorem 3.2.5 there exist open sets $G_{i,j} \subset X$ and $H_{i,j} \subset Y$ such that

$$(10) \qquad E_{i,j} \subset G_{i,j} \subset \overline{G}_{i,j} \subset U_{i,j} \quad \text{and} \quad F_{i,j} \subset H_{i,j} \subset \overline{H}_{i,j} \subset V_{i,j}$$

and

$$(11) \qquad \operatorname{ord}(\{\operatorname{Fr} G_{i,j}\}_{(i,j)\in S}) \leq n-1 \quad \text{and} \quad \operatorname{ord}(\{\operatorname{Fr} H_{i,j}\}_{(i,j)\in S}) \leq m-1,$$

where S denotes the set of all pairs (i,j) with $i = 1, 2, \ldots, n+m+1$ and $j = 1, 2, \ldots, k_i$. For $i = 1, 2, \ldots, n+m+1$ let $W_i = \bigcup_{j=1}^{k_i} (G_{i,j} \times H_{i,j})$. From (9) and (10) it follows that $A_i \subset W_i \subset \overline{W}_i \subset (X \times Y) \setminus B_i$, so that the set $L_i = \operatorname{Fr} W_i$ is a partition between A_i and B_i. Consider the family

$$\mathcal{C}_i = \{\operatorname{Fr} G_{i,j} \times \overline{H}_{i,j}\}_{j=1}^{k_i} \cup \{\overline{G}_{i,j} \times \operatorname{Fr} H_{i,j}\}_{j=1}^{k_i}$$

of subsets of $X \times Y$, denote by C_i the union $\bigcup \mathcal{C}_i$, and let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_{n+m+1}$. Since $L_i \subset \bigcup_{j=1}^{k_i} \operatorname{Fr}(G_{i,j} \times H_{i,j}) \subset C_i$, by virtue of Theorem 3.2.6 to complete the proof it suffices to show that $\bigcap_{i=1}^{n+m+1} C_i = \emptyset$. Now, the last equality follows from the inequality $\operatorname{ord} \mathcal{C} \leq n+m-1$, which in turn is a consequence of (11), because among any $n+m+1$ members of the family $\mathcal{C}$ there are either $n+1$ sets of the form $\operatorname{Fr} G_{i,j} \times \overline{H}_{i,j}$ or $m+1$ sets of the form $\overline{G}_{i,j} \times \operatorname{Fr} H_{i,j}$. ■

3.2.14. THEOREM. *For every pair X, Y of normal spaces, not both empty, such that the Cartesian product $X \times Y$ is strongly paracompact we have*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

PROOF. From Theorems 3.2.13 and 3.1.25 it follows that

$$\dim(\beta X \times \beta Y) \leq \dim \beta X + \dim \beta Y = \dim X + \dim Y.$$

To complete the proof it suffices to apply Theorem 3.1.23. ■

Historical and bibliographic notes

Theorem 3.2.1 and Proposition 3.2.2 were established by Dowker in [1947]. Lemma 3.2.3 and Theorem 3.2.4 were proved by Ostrand in [1971]; special cases of Theorem 3.2.4 were obtained earlier by Ostrand in [1965] (for metrizable spaces) and by French in [1970] (for collectionwise normal spaces). Theorem 3.2.5 was proved by Morita in [1950a], and Theorem 3.2.6 – by Hemmingsen in [1946]. Theorem 3.2.10 was obtained independently by Hemmingsen in [1946], Alexandroff in [1947], and Dowker in [1947]; for compact spaces it was proved by Alexandroff in [1940] and by Morita in [1940]. Theorem 3.2.9, which is a simple consequence of Theorem 3.2.10, was noted by Alexandroff in [1947]. As stated in the notes to Section 1.10, Theorems 3.2.11 and 3.2.12 for metric spaces were established by Kuratowski in [1933a] (who generalized a characterization of

dimension of compact metric spaces discovered by Alexandroff in [1928]); Kuratowski's proof extends without substantial changes to normal spaces. Theorem 3.2.13 was proved by Hemmingsen in [1946], and Theorem 3.2.14 – by Morita in [1953].

Problems

3.2.A. Show that a normal space X satisfies the inequality $\dim X \leq n$ if and only if every locally finite open cover of the space X has a locally finite closed refinement of order $\leq n$ or – equivalently – if every locally finite open cover of the space X has a closed shrinking of order $\leq n$.

3.2.B (Ostrand [1971]; for collectionwise normal spaces, French [1970]; for metrizable spaces, Ostrand [1965]). Prove that a normal space X satisfies the inequality $\dim X \leq n$ if and only if for every locally finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of the space X there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of discrete families of open subsets of X, where $\mathcal{V}_i = \{V_{i,s}\}_{s\in S}$, such that $V_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots$ and the union of any $n+1$ families $\mathcal{V}_i$ constitutes a cover of the space X. Show that for every locally finite open cover $\mathcal{U} = \{U_s\}_{s\in S}$ of a normal space X such that $\dim X \leq n$ besides the sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ with the above properties there also exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of discrete families of open subsets of X, where $\mathcal{W}_i = \{W_{i,s}\}_{s\in S}$, such that $\overline{V}_{i,s} \subset W_{i,s} \subset U_s$ for $s \in S$ and $i = 1, 2, \ldots$

Hint. Define the families $\mathcal{V}_i$ and $\mathcal{W}_i$ for $i \leq n+1$ by applying Theorem 3.2.4, then define inductively the families $\mathcal{V}_i$ and $\mathcal{W}_i$ for $i > n+1$. Assume that the families $\mathcal{V}_i$ and $\mathcal{W}_i$ are defined for $i \leq m-1 \geq n+1$, denote by $\mathcal{T}$ the family of all subsets of the set $\{1, 2, \ldots, m-1\}$ which have exactly n elements and, for every $T \in \mathcal{T}$, define $F_T = X \setminus \bigcup_{i\in T} V_i$, where $V_i = \bigcup_{s\in S} V_{i,s}$. Consider families $\{V_T\}_{T\in\mathcal{T}}$ and $\{W_T\}_{T\in\mathcal{T}}$ of open subsets of X such that

$$F_T \subset V_T \subset \overline{V}_T \subset W_T \text{ for } T \in \mathcal{T} \quad \text{and} \quad \overline{W}_T \cap \overline{W}_{T'} = \emptyset \quad \text{for } T \neq T',$$

for every $T \in \mathcal{T}$ choose arbitrarily an $i(T) \leq m-1$ such that $i(T) \notin T$, note that $F_T \subset V_{i(T)}$ and let

$$V_{m,s} = \bigcup_{T\in\mathcal{T}} V_T \cap V_{i(T),s} \quad \text{and} \quad W_{m,s} = \bigcup_{T\in\mathcal{T}} W_T \cap W_{i(T),s}.$$

3.2.C (Arhangel'skiĭ [1963] (announcement [1962])). A family $\mathcal{A}$ of subsets of a set X is *independent* if $A \setminus B \neq \emptyset \neq B \setminus A$ for each pair A, B of distinct members of $\mathcal{A}$. By the *rank* of a family of sets $\mathcal{A}$ we mean the largest integer n such that the family $\mathcal{A}$ contains $n+1$ sets with a non-empty intersection which form an independent family; if no such integer exists we say that the family $\mathcal{A}$ has rank ∞. Clearly, the rank of a family of sets does not exceed the order of that family.

Prove that a normal space X satisfies the inequality $\dim X \leq n$ if and only if every finite open cover of the space X has a finite open refinement of rank $\leq n$.

Hint. For an open cover $\mathcal{U} = \{U_i\}_{i=1}^k$ of the space X and its finite open refinement $\mathcal{V}$ of rank $\leq n$ define $\mathcal{V}_i = \{V \in \mathcal{V} : V \subset U_i\}$ and consider the families

$\mathcal{W}_1 = \{V \in \mathcal{V}_1 : V$ is contained in no member of $\mathcal{V}_2 \cup \mathcal{V}_3 \cup \ldots \cup \mathcal{V}_k\}$,
$\mathcal{W}_2 = \{V \in \mathcal{V}_2 : V$ is contained in no member of $\mathcal{W}_1 \cup \mathcal{V}_3 \cup \ldots \cup \mathcal{V}_k\}$,
.....
$\mathcal{W}_k = \{V \in \mathcal{V}_k : V$ is contained in no member of $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \ldots \cup \mathcal{W}_{k-1}\}$.

Check that the family $\{W_i\}_{i=1}^k$, where $W_i = \bigcup \mathcal{W}_i$, is a cover of X and has order $\leq n$.

3.2.D. (a) (Wallace [1945], Pupko [1961]) Deduce Theorems 3.1.8 and 3.1.10 from Theorem 3.2.10.

(b) (Hemmingsen [1946]) Deduce Theorem 3.2.13 from Theorem 3.2.11.

Hint. Show that every finite open cover of the Cartesian product of compact spaces X and Y has a refinement of the form $\{U \times V : U \in \mathcal{U}, V \in \mathcal{V}\}$, where $\mathcal{U}$ and $\mathcal{V}$ are finite open covers of X and Y respectively.

(c) Deduce Lemma 3.1.27 from Theorem 3.2.6 without using the sum theorem.

Hint. Consider a normal space X with $\dim X > n$ and a sequence (A_1, B_1), $(A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X such that $\bigcap_{i=1}^{n+1} L_i \neq \emptyset$ if L_i is a partition between A_i and B_i for $i = 1, 2, \ldots, n+1$. Apply Theorem 3.2.6 to show that $\dim L \geq n$ for every partition L between A_1 and B_1.

3.2.E (Alexandroff [1947]). Show that a normal space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if no continuous mapping $f: X \to B^{n+1}$ is essential (see Problem 1.9.A).

3.2.F (Alexandroff [1947]). A compact space X such that $\dim X = n \geq 1$ is an n-dimensional *Cantor-manifold* if no closed subset L of X satisfying the inequality $\dim L \leq n-2$ separates the space X (cf. Definition 1.9.5).

(a) Let $f, g: X \to S^n$ be continuous mappings of a compact space X to the n-sphere S^n. Show that if $\mathrm{rd}_X\{x \in X : f(x) \neq g(x)\} \leq n-1$, then the mappings f and g are homotopic.

(b) Prove that every compact space X such that $\dim X = n \geq 1$ contains an n-dimensional Cantor-manifold.

Hint. See the proof of Theorem 1.9.9.

3.2.G (Zolotarev [1975]). (a) Show that a normal subspace M of a normal space X satisfies the inequality $\dim M \leq n$ if and only if for every open set $U \subset X$ which contains the set M there exist a normal space Z, a normal subspace $A \subset Z$ satisfying $\dim A \leq n$ and continuous mappings $f: M \to Z$ and $g: Z \to X$ such that $f(M) \subset A \subset g^{-1}(U)$ and $gf(x) = x$ for $x \in M$.

(b) Prove that for every completely paracompact subspace M of a normal space X we have $\dim M \leq \dim X$ (see Problem 2.4.B).

Hint. One can assume that $\dim X = n < \infty$ and X is a compact space (see Theorem 3.1.25). Consider an open set $U \subset X$ which contains the set M and for every point $x \in M$ choose a neighbourhood V_x such that $x \in V_x \subset \overline{V}_x \subset U$. Let $\mathcal{V}_1, \mathcal{V}_2, \ldots$ be a sequence of star-finite open covers of the space M such that the union $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ contains a refinement $\mathcal{V}$ of the cover $\{M \cap V_x\}_{x \in M}$. For $i = 1, 2, \ldots$ consider the decomposition $\{\mathcal{V}_{s,i}\}_{s \in S_i}$ of the cover $\mathcal{V}_i$ into components, where $S_i \cap S_j = \emptyset$ whenever

$i \neq j$; for every $x \in M$ denote by $f_i(x)$ the unique $s \in S_i$ such that $x \in \bigcup \mathcal{V}_{s,i}$, and for each $s \in S_i$ let $\mathcal{V}_{s,i} \cap \mathcal{V} = \{V_{s,j}\}_{j=1}^{\infty}$. Note that the space $Z = X \times \prod_{i=1}^{\infty} S_i$, where S_i has the discrete topology, is normal, and apply Problem 2.4.A and Theorem 3.2.14 to show that $\dim Z \leq n$. Apply (a) to the continuous mapping $f: M \to Z$ defined by letting $f(x) = (x, f_1(x), f_2(x), \ldots)$ for $x \in M$, the projection $g: Z \to X$ and the set $A = \bigcup_{i,j=1}^{\infty} \bigcup_{s \in S_i} \overline{f(V_{s,j})} \subset Z$.

3.2.H. (a) Prove that every locally finite functionally open cover $\{U_s\}_{s \in S}$ of a topological space X has shrinkings $\{F_s\}_{s \in S}$ and $\{W_s\}_{s \in S}$, respectively functionally closed and functionally open, such that $F_s \subset W_s \subset \overline{W}_s \subset U_s$ for $s \in S$.

Hint. For every $s \in S$ choose a continuous function $f_s: X \to I$ such that $U_s = f_s^{-1}((0,1])$ and consider the function $f: X \to I$ defined by letting $f(x) = \sup\{f_s(x) : s \in S\}$ for $x \in X$.

(b) (Pasynkov [1965]) Prove that a completely regular space X satisfies the inequality $\dim X \leq n$ if and only if every locally finite functionally open cover of the space X has a functionally open refinement of order $\leq n$ (see notes to Section 3.1).

Hint. Let $\{U_s\}_{s \in S}$ be a locally finite functionally open cover of the space X. Consider a functionally closed shrinking $\{F_s\}_{s \in S}$ of $\{U_s\}_{s \in S}$ and continuous functions $f_s: X \to I$ such that $f_s(X \setminus U_s) \subset \{0\}$ and $f_s(F_s) \subset \{1\}$. Verify that the formula $\rho(x,y) = \sum_{s \in S} |f_s(x) - f_s(y)|$ defines a pseudometric on the set X and consider the metric space (Y, ρ) obtained by identifying each pair x, y of points in X such that $\rho(x,y) = 0$. Check that by letting $f(x) = [x]$ one defines a continuous mapping of X to Y, consider the continuous extension $F: \beta X \to \beta Y$ of f and the inverse image $T = F^{-1}(Y)$. Apply the fact that every Hausdorff space which can be mapped onto a paracompact space by a perfect mapping is itself paracompact (see [GT], Theorem 5.1.35) and use Problem 3.1.J(b) and Theorem 3.2.1.

3.2.I. (a) Observe that if M is a functionally open subspace of a space X, then a set $A \subset M$ is functionally open in X if and only if it is functionally open in M. Give an example of a functionally closed subspace M of a completely regular space X and of a set $A \subset M$ which is functionally closed in M and yet is not functionally closed in X.

(b) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every finite functionally open cover $\mathcal{U} = \{U_i\}_{i=1}^{k}$ of the space X there exists a functionally open cover $\mathcal{V}$ of the space X which can be represented as the union of $n+1$ families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n+1}$, where $\mathcal{V}_j = \{V_{j,i}\}_{i=1}^{k}$, such that $\operatorname{ord} \mathcal{V}_j \leq 0$ and $V_{j,i} \subset U_i$ for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n+1$.

3.2.J. (a) Prove that a completely regular space X satisfies the inequality $\dim X \leq n$ if and only if every $(n+2)$-element functionally open cover $\{U_i\}_{i=1}^{n+2}$ of the space X has a functionally open shrinking $\{W_i\}_{i=1}^{n+2}$ of order $\leq n$, i.e., such that $\bigcap_{i=1}^{n+2} W_i = \emptyset$.

Hint. See the proof of Theorem 1.6.9.

(b) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every finite family $\{U_i\}_{i=1}^{k}$ of functionally open subsets of X and every family $\{F_i\}_{i=1}^{k}$ of functionally closed subsets of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots, k$ there exist families $\{E_i\}_{i=1}^{k}$ and $\{V_i\}_{i=1}^{k}$ of functionally closed and functionally

open subsets of X, respectively, such that $F_i \subset V_i \subset E_i \subset U_i$ for $i = 1, 2, \ldots, k$ and $\operatorname{ord}(\{E_i \setminus V_i\}_{i=1}^k) \leq n - 1$.

(c) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint functionally closed subsets of X there exist functionally closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

Hint. Observe that if L is a functionally closed subset of a space X and $U, W \subset X$ are disjoint open sets satisfying the equality $X \setminus L = U \cup W$, then U and W are functionally open.

3.2.K. (a) (Smirnov [1956a]) Prove that a completely regular space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if for every closed subspace A of the space X and each continuous mapping $f: A \to S^n$ which is the restriction of a continuous mapping $\bar{f}: X \to B^{n+1}$ there exists a continuous extension $F: X \to S^n$ of f over X.

(b) Prove that a completely regular space X satisfies the inequality $\dim X \leq n$ if and only if for every finite functionally open cover $\mathcal{E}$ of the space X there exists an $\mathcal{E}$-mapping of X to a polyhedron of dimension $\leq n$.

3.3. Dimension dim and mappings

In this section we shall establish the theorems on dimension-raising and dimension-lowering mappings for the covering dimension. The proofs will be much more difficult than those in Section 1.12. In both of them we shall reduce the general case to the case of compact spaces by using Čech–Stone compactifications, and then we shall apply Lemma 3.3.2 to pass to the case of metrizable compact spaces, so that the results from Section 1.12 could be used. Lemma 3.3.2 will also play a crucial role in the following section.

We start with a simple lemma on normal spaces. Let us recall that a cover $\mathcal{B}$ is a *star refinement* of another cover $\mathcal{A}$ of the same space if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $\operatorname{St}(B, \mathcal{B}) \subset A$, where $\operatorname{St}(B, \mathcal{B})$ denotes the *star of the set B with respect to the cover $\mathcal{B}$*, i.e., the set $\bigcup\{B' \in \mathcal{B} : B \cap B' \neq \emptyset\}$.

3.3.1. LEMMA. *Every finite open cover of a normal space has a finite open star refinement.*

PROOF. Let $\mathcal{U} = \{U_i\}_{i=1}^k$ be a finite open cover of a normal space X. Consider a closed shrinking $\{F_i\}_{i=1}^k$ of $\{U_i\}_{i=1}^k$. By Urysohn's lemma, for $i = 1, 2, \ldots, k$ there exists a continuous function $f_i: X \to I$ such that $f_i(X \setminus U_i) \subset \{0\}$ and $f_i(F_i) \subset \{1\}$. Define $I_0 = [0, 1/2)$, $I_1 = (1/4, 3/4)$ and $I_2 = (1/2, 1]$. The family $\mathcal{V}$ of all sets

$$V_{j_1 j_2 \ldots j_k} = f_1^{-1}(I_{j_1}) \cap f_2^{-1}(I_{j_2}) \cap \ldots \cap f_k^{-1}(I_{j_k}), \tag{1}$$

where $j_i \in \{0, 1, 2\}$ for $i = 1, 2, \ldots, k$, is a finite open cover of the space X. For every non-empty $V = V_{m_1 m_2 \ldots m_k} \in \mathcal{V}$ there exists an $i \leq k$ such that $V \cap F_i \neq \emptyset$, and we clearly have $m_i = 2$. If a set of the form (1) intersects V, then j_i is equal to 1 or 2, so that $\operatorname{St}(V, \mathcal{V}) \subset U_i$. Thus $\mathcal{V}$ is a star refinement of $\mathcal{U}$. ■

3.3.2. LEMMA. *Let X be a compact space and $\mathbf{W}$ a class of finite open covers of the space X such that*

(i) *for every $\mathcal{W} \in \mathbf{W}$ there exists a $\mathcal{V} \in \mathbf{W}$ which is a star refinement of $\mathcal{W}$,*

and

(ii) *for any $\mathcal{W}_1, \mathcal{W}_2 \in \mathbf{W}$ there exists a $\mathcal{W} \in \mathbf{W}$ which is a refinement of both $\mathcal{W}_1$ and $\mathcal{W}_2$.*

By letting

$$xEy \text{ if and only if } y \in \mathrm{St}(x, \mathcal{W}) \text{ for every } \mathcal{W} \in \mathbf{W}$$

we define a closed equivalence relation E on the space X; the quotient space $Z = X/E$ is compact and satisfies the conditions $w(Z) \leq |\mathbf{W}| \cdot \aleph_0$ and $\dim Z \leq \sup\{\operatorname{ord} \mathcal{W} : \mathcal{W} \in \mathbf{W}\}$.

If, moreover, $f: X \to Y$ is a continuous mapping of X onto a compact space Y such that

(iii) *$f(x) = f(y)$ whenever xEy,*

then there exists a continuous mappings $h: Z \to Y$ satisfying $f = hg$, where $g: X \to Z$ is the quotient mapping.

PROOF. It follows directly from the definition that the relation E is reflexive and symmetric; thus to show that E is an equivalence relation it remains to check that if xEy and yEz, then xEz. To this end, by virtue of (i), it suffices to note that

(2) if $\mathcal{V}$ is a star refinement of $\mathcal{W}$, $y \in \mathrm{St}(x, \mathcal{V})$ and $z \in \mathrm{St}(y, \mathcal{V})$, then $z \in \mathrm{St}(x, \mathcal{W})$.

The relation E determines a decomposition of the space X into equivalence classes $[x]$, where

$$[x] = \bigcap_{\mathcal{W} \in \mathbf{W}} \mathrm{St}(x, \mathcal{W}) \quad \text{for } x \in X. \tag{3}$$

We shall now check that the equivalence relation E is closed. Thus, we have to show that for every open set $U \subset X$ the union of all equivalence classes that are contained in U is an open subset of X. For the purpose of a subsequent application, we shall show a little more, viz., that

(4) for every open set $U \subset X$ and each $[x] \subset U$ there exist an open set $V \subset X$ and a cover $\mathcal{W} \in \mathbf{W}$ such that $[x] \subset V \subset \bigcup_{y \in V} [y] \subset \mathrm{St}(V, \mathcal{W}) \subset U$.

To begin, let us note that if $\mathcal{V}$ is a star refinement of $\mathcal{W}$, then $\overline{\mathrm{St}(x, \mathcal{V})} \subset \mathrm{St}(x, \mathcal{W})$, so that from (3) it follows that $[x] = \bigcap_{\mathcal{W} \in \mathbf{W}} \overline{\mathrm{St}(x, \mathcal{W})}$. Now, the space X being compact, there exists a finite number of covers $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k \in \mathbf{W}$ such that $\bigcap_{i=1}^{k} \mathrm{St}(x, \mathcal{W}_i) \subset U$. By virtue of (ii) there exists a cover $\mathcal{W}_0 \in \mathbf{W}$ which refines all the covers $\mathcal{W}_i$; let $\mathcal{W} \in \mathbf{W}$ be a star refinement of $\mathcal{W}_0$. Clearly, the open set $V = \mathrm{St}(x, \mathcal{W})$ satisfies the relation $[x] \subset V \subset \bigcup_{y \in V} [y]$; the penultimate inclusion in (4) follows from (3), and the last inclusion is a consequence of the relation $\mathrm{St}(x, \mathcal{W}_0) \subset U$ and (2) with $\mathcal{V} = \mathcal{W}$ and $\mathcal{W} = \mathcal{W}_0$.

Let Z be the quotient space X/E and $g: X \to Z$ the natural quotient mapping. As E is a closed equivalence relation, the space Z is compact (see [GT], Theorem 3.2.11).

For every open set $U \subset X$ let $U^* = Z \setminus g(X \setminus U)$ and for every $\mathcal{W} \in \mathbf{W}$ let $\mathcal{W}^* = \{W^* : W \in \mathcal{W}\}$. Clearly, U^* is open in Z and $g^{-1}(U^*) \subset U$. If $\mathcal{V}$ is a star refinement of

$\mathcal{W}$, then for each $x \in X$ there exists a $W \in \mathcal{W}$ such that $[x] \subset \mathrm{St}(x, \mathcal{V}) \subset W$; thus for every $\mathcal{W} \in \mathbf{W}$ the family $\mathcal{W}^*$ is a finite open cover of the space Z.

Now we shall show that every finite open cover $\{U_i\}_{i=1}^k$ of the space Z has a refinement of the form $\mathcal{W}^*$, where $\mathcal{W} \in \mathbf{W}$. For each $x \in X$ there exists an $i \leq k$ such that $[x] \subset g^{-1}(U_i)$, and by virtue of (4) there exist a neighbourhood $V_x \subset X$ of the point x and a cover $\mathcal{W}_x \in \mathbf{W}$ such that $\mathrm{St}(V_x, \mathcal{W}_x) \subset g^{-1}(U_i)$. The open cover $\{V_x\}_{x \in X}$ of the space X has a finite subcover $\{V_{x_i}\}_{i=1}^m$. Consider a cover $\mathcal{W} \in \mathbf{W}$ which refines all the covers $\mathcal{W}_{x_i}$. For every $W \in \mathcal{W}$ there exists an $i \leq k$ such that $W \subset g^{-1}(U_i)$. Since the last inclusion implies that $W^* \subset U_i$, it follows that the cover $\mathcal{W}^*$ is a refinement of $\{U_i\}_{i=1}^k$.

From what we have shown it follows that the family $\bigcup_{\mathcal{W} \in \mathbf{W}} \mathcal{W}^*$ is a base for the space Z, so that $w(Z) \leq |\mathbf{W}| \cdot \aleph_0$. And since $\operatorname{ord} \mathcal{W}^* \leq \operatorname{ord} \mathcal{W}$, we have $\dim Z \leq \sup\{\operatorname{ord} \mathcal{W} : \mathcal{W} \in \mathbf{W}\}$.

If a continuous mapping $f: X \to Y$ satisfying (iii) is given, we define the mapping $h: Z \to Y$ by letting $h([x]) = f(x)$. From the obvious equality $hg = f$ it follows that h is continuous. ■

3.3.3. REMARK. Let us note that if a closed subset A of the space X in Lemma 3.3.2 has the property that for each $\mathcal{W} \in \mathbf{W}$ there exists a finite open cover $\mathcal{U}$ of the space X such that

(iv) *$\mathcal{U}$ is a refinement of $\mathcal{W}$, $\operatorname{ord}(\mathcal{U}|A) \leq n$, and $\mathcal{U}$ has a refinement in $\mathbf{W}$,*

then $\dim g(A) \leq n$.

Indeed, each finite open cover $\{U_i\}_{i=1}^k$ of the space Z has a refinement of the form $\mathcal{W}^* = \{Z \setminus g(X \setminus W) : W \in \mathcal{W}\}$, where $\mathcal{W} \in \mathbf{W}$; for this $\mathcal{W}$ consider a $\mathcal{U}$ satisfying (iv). Clearly $\mathcal{U}$ has a star refinement in $\mathbf{W}$, and this implies that $\mathcal{U}^* = \{Z \setminus g(X \setminus U) : U \in \mathcal{U}\}$ is a cover of Z; since $\operatorname{ord} \mathcal{U}^*|g(A) \leq n$ and $\mathcal{U}^*$ is a refinement of $\{U_i\}_{i=1}^k$, we have $\dim g(A) \leq n$ (see Problem 1.6.A(a)). ■

We shall now prove two simple lemmas on Čech–Stone compactifications which will be followed by Lemma 3.3.6 allowing the reduction of the theorem on dimension-raising mappings to the case of compact spaces. Let us recall that for every continuous mapping $f: X \to Y$ of a completely regular space X to a completely regular space Y there exists a unique continuous mapping $F: \beta X \to \beta Y$ such that $F(x) = f(x)$ for every $x \in X$, the extension of f over βX and βY. In Lemmas 3.3.4–3.3.6, 3.3.8 and 3.3.9 the bar denotes the closure operator in Čech–Stone compactifications.

3.3.4. LEMMA. *If $\{U_i\}_{i=1}^m$ is an open cover of a normal space X, then for each point $x \in \beta X$ there exist a neighbourhood $W \subset \beta X$ of x and an $i \leq m$ such that $\overline{W} \cap X \subset U_i$.*

PROOF. First consider the case when $m = 2$. Suppose that for every neighbourhood W of a point $x \in \beta X$ we have $(\overline{W} \cap X) \setminus U_i = \overline{W} \cap (X \setminus U_i) \neq \emptyset$ for $i = 1, 2$. Thus $x \in \overline{X \setminus U_1} \cap \overline{X \setminus U_2}$, and the closed subsets $X \setminus U_1$ and $X \setminus U_2$ of X have a non-empty intersection, i.e. $X \setminus (U_1 \cup U_2) \neq \emptyset$, a contradiction.

The general case is obtained by induction using the fact that $\overline{W}$ is the Čech–Stone compactification of $\overline{W} \cap X$. ■

3.3.5. LEMMA. *If $f: X \to Y$ is a closed mapping of a completely regular space X to a completely regular space Y, then for its extension $F: \beta X \to \beta Y$ and every point $y \in Y$ we have $F^{-1}(y) = \overline{f^{-1}(y)}$.*

If, moreover, Y is a normal space, then for every point $y \in \beta Y$ and each neighbourhood $W \subset \beta Y$ of y we have $F^{-1}(y) \subset \overline{f^{-1}(\overline{W} \cap Y)}$.

PROOF. For every $y \in Y$ we obviously have $\overline{f^{-1}(y)} \subset F^{-1}(y)$; let us prove the reverse inclusion. Consider a point $x \in F^{-1}(y)$ and an arbitrary neighbourhood $U \subset \beta X$ of x. We have

$$y \in F(U) \subset F(\overline{U \cap X}) = \overline{F(U \cap X)} = \overline{f(U \cap X)} \subset \overline{f(\overline{U} \cap X)} \tag{5}$$

and, since f is closed, $f(\overline{U} \cap X) = \overline{f(\overline{U} \cap X)} \cap Y$, so that $y \in f(\overline{U} \cap X)$. Thus $\overline{U} \cap f^{-1}(y) \neq \emptyset$ for every neighbourhood U of the point x, i.e., $x \in \overline{f^{-1}(y)}$. The first part of our lemma is proved.

Assume now that Y is normal; let $y \in \beta Y$ and $W \subset \beta Y$ be a neighbourhood of y. Consider a point $x \in F^{-1}(y)$ and an arbitrary neighbourhood $U \subset \beta X$ of x. By (5) we have $y \in \overline{f(\overline{U} \cap X)} \cap \overline{W} \cap Y$, so that the closed subsets $f(\overline{U} \cap X)$ and $\overline{W} \cap Y$ of the space Y have a non-empty intersection. Thus $\overline{U} \cap f^{-1}(\overline{W} \cap Y) \neq \emptyset$ for every neighbourhood U of x, i.e., $x \in \overline{f^{-1}(\overline{W} \cap Y)}$. ■

3.3.6. LEMMA. *Let $f: X \to Y$ be a closed mapping of a completely regular space X to a normal space Y and $F: \beta X \to \beta Y$ its extension. If there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $|F^{-1}(y)| \leq k$ for every $y \in \beta Y$.*

PROOF. Suppose that for some $y_0 \in \beta Y$ we have $|F^{-1}(y_0)| > k$. There exists then a finite open cover $\mathcal{W}$ of βX such that $F^{-1}(y_0)$ is not contained in any set of the form $\overline{\bigcup \mathcal{V}}$, where $\mathcal{V} \subset \mathcal{W}$ and $|\mathcal{V}| \leq k$. On the other hand, since $|F^{-1}(y)| \leq k$ for every $y \in Y$ by the first part of Lemma 3.3.5, the family

$$\left\{ Y \setminus F\left(\beta X \setminus \bigcup \mathcal{V}\right) : \mathcal{V} \subset \mathcal{W} \text{ and } |\mathcal{V}| \leq k \right\}$$

is a finite open cover of Y. Thus by Lemma 3.3.4 there exists a neighbourhood $W \subset \beta Y$ of the point y_0 and a family $\mathcal{V}_0 \subset \mathcal{W}$ with $|\mathcal{V}_0| \leq k$ such that

$$\overline{W} \cap Y \subset Y \setminus F\left(\beta X \setminus \bigcup \mathcal{V}_0\right), \quad \text{i.e.,} \quad F^{-1}(\overline{W} \cap Y) \subset \bigcup \mathcal{V}_0.$$

By the second part of Lemma 3.3.5 we have

$$F^{-1}(y_0) \subset \overline{f^{-1}(\overline{W} \cap Y)} \subset \overline{F^{-1}(\overline{W} \cap Y)} \subset \overline{\bigcup \mathcal{V}_0},$$

a contradiction. ■

3.3.7. THEOREM ON DIMENSION-RAISING MAPPINGS FOR dim. *If $f: X \to Y$ is a closed mapping of a normal space X onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $\dim Y \leq \dim X + (k-1)$.*

PROOF. Since $\dim Y = \dim \beta Y$ and $\dim X = \dim \beta X$ by Theorem 3.1.25, it suffices to show that $\dim \beta Y \leq \dim \beta X + (k-1)$. By Lemma 3.3.6 the extension $F: \beta X \to \beta Y$ of the mapping f has the property that $|F^{-1}(y)| \leq k$ for every point $y \in \beta Y$, so that it suffices to prove our theorem under the additional assumption that X and Y are compact.

We shall define inductively finite open covers $\mathcal{W}_0, \mathcal{W}_1, \ldots$ and $\mathcal{W}'_0, \mathcal{W}'_1, \ldots$ of X and Y respectively such that, for $i = 1, 2, \ldots$,

(6) $\mathcal{W}_i$ is a star refinement of $\mathcal{W}_{i-1}$,
(7) $\operatorname{ord} \mathcal{W}_i \leq \dim X$,
(8) $\mathcal{W}'_i$ is a star refinement of $\mathcal{W}'_{i-1}$,
(9) $\mathcal{W}'_i$ has no open refinement of order $< \min\{i, \dim Y\}$,
(10) $\mathcal{W}_i$ is a refinement of $f^{-1}(\mathcal{W}'_{i-1})$,
(11) $\mathcal{W}'_i$ is a refinement of $\{Y \setminus f(X \setminus \bigcup \mathcal{V}) : \mathcal{V} \subset \mathcal{W}_i \text{ and } |\mathcal{V}| \leq k\}$.

Let $\mathcal{W}_0 = \{X\}$ and $\mathcal{W}'_0 = \{Y\}$; assume now that the covers $\mathcal{W}_j$ and $\mathcal{W}'_j$ are defined for $j < i \geq 1$. Applying Lemma 3.3.1 we obtain a finite open cover $\mathcal{W}_i$ of the space X which is a star refinement of $\mathcal{W}_{i-1} \wedge f^{-1}(\mathcal{W}'_{i-1})$ and satisfies (7). Since $|f^{-1}(y)| \leq k$ for every $y \in Y$, the family $\mathcal{V}_i = \{Y \setminus f(X \setminus \bigcup \mathcal{V}) : \mathcal{V} \subset \mathcal{W}_i \text{ and } |\mathcal{V}| \leq k\}$ is a finite open cover of Y. Applying again Lemma 3.3.1 we obtain a finite open cover $\mathcal{V}'_i$ of Y which is a star refinement of $\mathcal{W}'_{i-1} \wedge \mathcal{V}_i$. Among all finite open refinements of $\mathcal{V}'_i$ there must be one which has no open refinement of order $< \min\{i, \dim Y\}$, and this will be our $\mathcal{W}'_i$. The covers $\mathcal{W}_i$ and $\mathcal{W}'_i$ satisfy conditions (6)–(11), and thus the covers $\mathcal{W}_0, \mathcal{W}_1, \ldots$ and $\mathcal{W}'_0, \mathcal{W}'_1, \ldots$ are defined.

Applying Lemma 3.3.2 to the space Y and the class $\mathbf{W} = \{\mathcal{W}'_i\}_{i=0}^{\infty}$ we define a closed equivalence relation E' on Y such that the quotient space T is compact and $w(T) \leq \aleph_0$. Thus the space T is metrizable (see [GT], Theorem 4.2.8). For $i = 1, 2, \ldots$ the family $\mathcal{W}'^*_i = \{T \setminus g_Y(Y \setminus U) : U \in \mathcal{W}'_i\}$, where $g_Y : Y \to T$ is the quotient mapping, is a finite open cover of T and $g_Y^{-1}(\mathcal{W}'^*_i)$ is a shrinking of $\mathcal{W}'_i$. Thus by (9) the cover $\mathcal{W}'^*_i$ has no open refinement of order $< \min\{i, \dim Y\}$, so that $\dim Y \leq \dim T$.

Applying Lemma 3.3.2 to the space X and the class $\mathbf{W} = \{\mathcal{W}_i\}_{i=1}^{\infty}$ we define a closed equivalence relation E on X such that the quotient space Z is compact, $w(Z) \leq \aleph_0$, i.e., Z is metrizable, and $\dim Z \leq \dim X$ by virtue of (7).

Since from (10) it follows that $g_Y f(x) = g_Y f(y)$ whenever xEy, by the second part of Lemma 3.3.2 there exists a continuous mapping $h : Z \to T$ such that $g_Y f = h g_X$, where $g_X : X \to Z$ is the quotient mapping. Since $f(X) = Y$, we have $h(Z) = T$. We shall show that

$$|h^{-1}(t)| \leq k \quad \text{for every } t \in T. \tag{12}$$

Suppose that for a $t \in T$ the fibre $h^{-1}(t)$ contains $k+1$ distinct points $z_0, z_1, \ldots, z_k$ and choose $x_0, x_1, \ldots, x_k \in X$ with $g_X(x_j) = z_j$. Since $g_X(x_j) \neq g_X(x_k)$ whenever $j \neq k$, for each pair x_j, x_k with $j \neq k$ there exists an integer $i(j,k)$ such that no member of $\mathcal{W}_{i(j,k)}$ contains both x_j and x_k. By (6) there exists a cover $\mathcal{W}_i$ no member of which contains x_j and x_k with $j \neq k$. Pick a point $y \in g_Y^{-1}(t)$; we have $g_Y^{-1}(t) \subset \operatorname{St}(y, \mathcal{W}'_{i+1}) \subset W$ for some $W \in \mathcal{W}'_i$. By (11) there exists a family $\mathcal{V} \subset \mathcal{W}_i$ such that $|\mathcal{V}| \leq k$ and $W \subset Y \setminus f(X \setminus \bigcup \mathcal{V})$; thus we have

$$\{x_0, x_1, \ldots, x_k\} \subset g_X^{-1} h^{-1}(t) = f^{-1} g_Y^{-1}(t) \subset f^{-1}(W) \subset \bigcup \mathcal{V},$$

a contradiction which shows that (12) holds.

From (12) and Theorems 1.7.7 and 1.12.2 it follows that $\dim T \leq \dim Z + (k-1)$. Thus

$$\dim Y \leq \dim T \leq \dim Z + (k-1) \leq \dim X + (k-1). \blacksquare$$

We now pass to the theorem on dimension-lowering mappings (cf. Problem 3.3.B). Its proof is similar to that of Theorem 3.3.7: first, in Lemma 3.3.9 we shall reduce the general situation to the case of compact spaces, and then, applying the construction from Lemma 3.3.2, we shall make a further reduction – to the case of metrizable compact spaces. We start with a simple property of Čech–Stone compactifications (cf. Problem 3.3.A).

3.3.8. LEMMA. *If a closed subset A of a normal space X is contained in the union $U_1 \cup U_2 \cup \ldots \cup U_m$ of open subsets of X, then in βX we have the inclusion $\bar{A} \subset V_1 \cup V_2 \cup \ldots \cup V_m$, where $V_i = \beta X \setminus \overline{X \setminus U_i}$.*

PROOF. Consider a point $x \in \bar{A}$. Since $\bar{A}$ is disjoint from $\overline{X \setminus (U_1 \cup U_2 \cup \ldots \cup U_m)}$ there exists a neighbourhood $U \subset \beta X$ of the point x such that $U \cap X \subset U_1 \cup U_2 \cup \ldots \cup U_m$. By Lemma 3.3.4, applied to the cover $\{\overline{U} \cap U_i\}_{i=1}^m$ of the space $\overline{U} \cap X$ and to the Čech–Stone compactification $\overline{U}$ of that space, there exist a neighbourhood $W \subset \overline{U}$ of the point x and an $i \leq m$ such that $W \cap (\overline{U} \cap X) \subset U_i$. Thus $(W \cap U) \cap X \subset U_i$, and $x \notin \overline{X \setminus U_i}$, i.e., $x \in V_i$. ■

3.3.9. LEMMA. *Let $f: X \to Y$ be a closed mapping of a normal space X to a weakly paracompact normal space Y with $\dim Y < \infty$ and $F: \beta X \to \beta Y$ its extension. If there exists an integer $k \geq 0$ such that $\dim f^{-1}(y) \leq k$ for every $y \in Y$, then $\dim F^{-1}(y) \leq k$ for every $y \in \beta Y$.*

PROOF. Suppose that for a point $y_0 \in \beta Y$ we have $\dim F^{-1}(y_0) > k$. There clearly exists a finite open cover $\{W_j\}_{j=1}^n$ of βX such that for every family $\{V_j\}_{j=1}^n$ of open subsets of βX satisfying $V_j \subset \overline{W}_j$ for $j = 1, 2, \ldots, n$ and $F^{-1}(y_0) \subset \bigcup_{j=1}^n V_j$ we have $\operatorname{ord}(\{V_j\}_{j=1}^n) > k$.

We start with defining an open cover $\{G_i\}_{i=0}^\infty$ of Y and families $\mathcal{W}_0, \mathcal{W}_1, \ldots$ of open subsets of X, where $\mathcal{W}_i = \{W_{i,j}\}_{j=1}^n$, satisfying for $i = 0, 1, \ldots$ the following conditions:

$$f^{-1}(G_i) \subset \bigcup \mathcal{W}_i \quad \text{and} \quad \operatorname{ord} \mathcal{W}_i \leq k. \tag{13}$$

$$W_{i,j} \subset W_j \quad \text{for } j = 1, 2, \ldots, n. \tag{14}$$

By the first part of Lemma 3.3.5 and by Theorem 3.1.25 for every point $y \in Y$ we have $\dim F^{-1}(y) \leq k$, so that there exists a family $\mathcal{U}(y) = \{U_j(y)\}_{j=1}^n$ of open subsets of βX such that $F^{-1}(y) \subset \bigcup \mathcal{U}(y)$, $\operatorname{ord} \mathcal{U}(y) \leq k$ and $U_j(y) \subset W_j$ for $j = 1, 2, \ldots, n$. The set $U(y) = Y \setminus F(\beta X \setminus \bigcup \mathcal{U}(y))$ is a neighbourhood of y in Y and $F^{-1}(U(y)) \subset \bigcup \mathcal{U}(y)$. Since Y is weakly paracompact, the cover $\{U(y)\}_{y \in Y}$ has a point-finite refinement $\{U_s\}_{s \in S}$. For $i = 1, 2, \ldots$ denote by K_i the set of all points of Y which belong to exactly i members of the cover $\{U_s\}_{s \in S}$, by $\mathcal{T}_i$ the family of all subsets of S that have exactly i elements, and let $K_T = K_i \cap \bigcap_{s \in T} U_s$ for each $T \in \mathcal{T}_i$. By virtue of Lemma 2.3.3 the union

$$F_i = K_1 \cup K_2 \cup \ldots \cup K_i \text{ is closed in } Y \text{ for } i = 1, 2, \ldots$$

and

$$K_i = \bigcup_{T \in \mathcal{T}_i} K_T, \text{ where the sets } K_T \text{ are open in } K_i \text{ and pairwise disjoint.}$$

The sets G_i we are going to define will also satisfy for $i = 0, 1, \ldots$ the inclusion

$$F_i \subset G_0 \cup G_1 \cup \ldots \cup G_i, \tag{15}$$

where $F_0 = \emptyset$. Let $G_0 = \emptyset$ and $W_{0,j} = \emptyset$ for $j = 1, 2, \ldots, n$. Assume now that for an $i \geq 1$ open sets $G_0, G_1, \ldots, G_{i-1} \subset Y$ and families $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_{i-1}$ with the required properties are already defined. The subspace $B = F_i \setminus (G_0 \cup G_1 \cup \ldots \cup G_{i-1}) \subset K_i$ of the space Y is closed, and $\{B \cap K_T\}_{T \in \mathcal{T}_i}$ is an open cover of B by disjoint sets. For each $T \in \mathcal{T}_i$ choose an $s_T \in T$ and a $y_T \in Y$ such that $U_{s_T} \subset U(y_T)$; we have

$$f^{-1}(B \cap K_T) \subset f^{-1}(U_{s_T}) \subset F^{-1}(U(y_T)) \subset \bigcup \mathcal{U}(y_T).$$

The family $\mathcal{H} = \{H_j\}_{j=1}^n$, where

$$H_j = \bigcup \{f^{-1}(B \cap K_T) \cap U_j(y_T) : T \in \mathcal{T}_i\} \subset W_j,$$

is an open cover of the closed subspace $f^{-1}(B)$ of X, and $\operatorname{ord} \mathcal{H} \leq k$. Let us shrink $\mathcal{H}$ to a closed cover $\mathcal{F}$ of $f^{-1}(B)$ and then take an open swelling $\mathcal{W}_i = \{W_{i,j}\}_{j=1}^n$ of $\mathcal{F}$ such that $W_{i,j} \subset W_j$ for $j = 1, 2, \ldots, n$. Since $f^{-1}(B) \subset \bigcup \mathcal{W}_i$, and since f is closed, there exists an open subset G_i of Y such that $B \subset G_i$ and $f^{-1}(G_i) \subset \bigcup \mathcal{W}_i$. The set G_i and the family $\mathcal{W}_i$ satisfy conditions (13)–(15). Since $Y = \bigcup_{i=0}^{\infty} F_i$, it follows from (15) that $\{G_i\}_{i=0}^{\infty}$ is a cover of Y. Thus the cover $\{G_i\}_{i=0}^{\infty}$ and the families $\mathcal{W}_0, \mathcal{W}_1, \ldots$ satisfying (13) and (14) are defined.

The space Y, being weakly paracompact, is countably paracompact (see [GT], Theorem 5.2.6), so that we can assume without loss of generality that the cover $\{G_i\}_{i=0}^{\infty}$ is locally finite. Since $\dim Y = m < \infty$, by Theorem 3.2.4 there exists an open cover $\bigcup_{l=1}^{m+1} \mathcal{V}_l$ of Y, where $\mathcal{V}_l = \{V_{l,i}\}_{i=0}^{\infty}$, such that $\operatorname{ord} \mathcal{V}_l \leq 0$ and $V_{l,i} \subset G_i$ for $i = 0, 1, \ldots$ and $l = 1, 2, \ldots, m+1$.

By virtue of Lemma 3.3.4 there exists a neighbourhood $W \subset \beta Y$ of the point y_0 and an $l \leq m+1$ such that $\overline{W} \cap Y \subset \bigcup \mathcal{V}_l$. We have

$$f^{-1}(\overline{W} \cap Y) \subset f^{-1}\Big(\bigcup_{i=0}^{\infty} V_{l,i}\Big) = \bigcup_{i=0}^{\infty} f^{-1}(V_{l,i}),$$

so that the family $\{f^{-1}(\overline{W} \cap Y) \cap f^{-1}(V_{l,i})\}_{i=0}^{\infty}$, being a cover of $f^{-1}(\overline{W} \cap Y)$ by disjoint open sets, consists of closed subsets of X and is discrete in X. Let $\{M_i\}_{i=0}^{\infty}$ be a family of pairwise disjoint open subsets of X such that

$$f^{-1}(\overline{W} \cap Y) \cap f^{-1}(V_{l,i}) \subset M_i \quad \text{for } i = 0, 1, \ldots$$

(see Lemma 5.5.20). As $f^{-1}(V_{l,i}) \subset f^{-1}(G_i)$, it follows from (13) that the family $\mathcal{W} = \{W_{i,j} \cap M_i : i = 0, 1, \ldots \text{ and } j = 1, 2, \ldots, n\}$ satisfies

$$f^{-1}(\overline{W} \cap Y) \subset \bigcup \mathcal{W} \quad \text{and} \quad \operatorname{ord} \mathcal{W} \leq k.$$

Let $U_j = \bigcup_{i=0}^{\infty} W_{i,j} \cap M_i \subset W_j$ and $V_j = \beta X \setminus \overline{X \setminus U_j}$ for $j = 1, 2, \ldots, n$. Since $f^{-1}(\overline{W} \cap Y) \subset \bigcup_{j=1}^{n} U_j$, we have

$$F^{-1}(y_0) \subset \overline{f^{-1}(\overline{W} \cap Y)} \subset \bigcup_{j=1}^{n} V_j$$

by virtue of Lemmas 3.3.5 and 3.3.8. One easily checks that $V_j \cap X = U_j$, so that $V_j \subset \overline{V}_j = \overline{U}_j \subset \overline{W}_j$ and $\operatorname{ord}(\{V_j\}_{j=1}^n) \leq k$. The contradiction concludes our proof. ■

3.3.10. THEOREM ON DIMENSION-LOWERING MAPPINGS FOR dim. *If $f: X \to Y$ is a closed mapping of a normal space X to a weakly paracompact normal space Y and there exists an integer $k \geq 0$ such that $\dim f^{-1}(y) \leq k$ for every $y \in Y$, then $\dim X \leq \dim Y + k$.*

PROOF. We can suppose that $\dim Y < \infty$. By virtue of Theorem 3.1.25 and Lemma 3.3.9 it suffices to prove our theorem under the additional assumption that X and Y are compact.

We shall define inductively finite open covers $\mathcal{W}_0, \mathcal{W}_1, \ldots$ of X, finite open covers $\mathcal{V}_0, \mathcal{V}_1, \ldots$ and $\mathcal{W}'_0, \mathcal{W}'_1, \ldots$ of Y and for each $V \in \mathcal{V}_i$ a finite open cover $\mathcal{U}_i(V)$ of X such that, for $i = 1, 2, \ldots$,

(16) $\mathcal{W}_i$ is a star refinement of $\mathcal{W}_{i-1}$ and of each $\mathcal{U}_{i-1}(V)$ with $V \in \mathcal{V}_{i-1}$,
(17) $\mathcal{W}_i$ has no open refinement of order $< \min\{i, \dim X\}$,
(18) $\mathcal{W}'_i$ is a star refinement of $\mathcal{W}'_{i-1}$ and of $\mathcal{V}_i$,
(19) $\operatorname{ord} \mathcal{W}'_i \leq \dim Y$,
(20) $\mathcal{W}_i$ is a refinement of $f^{-1}(\mathcal{W}'_{i-1})$,
(21) $\mathcal{U}_i(V)$ is a refinement of $\mathcal{W}_i$ and $\operatorname{ord}(\mathcal{U}_i(V)|f^{-1}(V)) \leq k$ for each $V \in \mathcal{V}_i$.

Let $\mathcal{W}_0 = \{X\}$, $\mathcal{V}_0 = \mathcal{W}'_0 = \{Y\}$ and $\mathcal{U}_0(Y) = \{X\}$; assume now that the covers $\mathcal{W}_j$, $\mathcal{V}_j$, $\mathcal{W}'_j$ and $\mathcal{U}_j(V)$ for $V \in \mathcal{V}_j$ are defined for $j < i \geq 1$. Applying Lemma 3.3.1 we obtain a finite open cover $\mathcal{U}$ of X which is a star refinement of $\mathcal{W}_{i-1} \wedge f^{-1}(\mathcal{W}'_{i-1})$ and of each $\mathcal{U}_{i-1}(V)$ with $V \in \mathcal{V}_{i-1}$. Among all finite open refinements of $\mathcal{U}$ there must be one which has no open refinement of order $< \min\{i, \dim X\}$, and this will be our $\mathcal{W}_i$. Conditions (16), (17) and (20) are satisfied.

For every $y \in Y$ we have $\dim f^{-1}(y) \leq k$ and thus there exists a family $\mathcal{W}_i(y)$ of open subsets of X each member of which is contained in a member of $\mathcal{W}_i$ such that $f^{-1}(y) \subset \bigcup \mathcal{W}_i(y)$ and $\operatorname{ord} \mathcal{W}_i(y) \leq k$ (see Problem 3.1.A(a)). Since f is closed, there exists a neighbourhood $V(y)$ of the point y such that $f^{-1}(\overline{V(y)}) \subset \bigcup \mathcal{W}_i(y)$. The space Y being compact, the open cover $\{V(y)\}_{y \in Y}$ of Y contains a finite subcover $\mathcal{V}_i$; for each $V = V(y) \in \mathcal{V}_i$ we let $\mathcal{U}_i(V) = \mathcal{W}_i(y) \cup \mathcal{W}_i|(X \setminus f^{-1}(\overline{V(y)}))$. Condition (21) is satisfied.

Finally, we take for $\mathcal{W}'_i$ any finite open cover of Y which is a star refinement of $\mathcal{W}'_{i-1} \wedge \mathcal{V}_i$ and has order $\leq \dim Y$. Conditions (18) and (19) are satisfied.

Thus the covers $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{V}_0, \mathcal{V}_1, \ldots$, and $\mathcal{W}'_0, \mathcal{W}'_1, \ldots$, as well as the covers $\mathcal{U}_i(V)$ with $V \in \mathcal{V}_i$ are defined.

Applying Lemma 3.3.2 to the space Y and the class $\mathbf{W} = \{\mathcal{W}'_i\}_{i=0}^{\infty}$ we define a closed equivalence relation E' on Y such that the quotient space T is compact, $w(T) \leq \aleph_0$, i.e., T is metrizable, and $\dim T \leq \dim Y$.

Applying Lemma 3.3.2 to the space X and the class $\mathbf{W} = \{\mathcal{W}_i\}_{i=0}^{\infty}$ we define a closed equivalence relation E on X such that the quotient space Z is compact, $w(Z) \leq \aleph_0$, i.e., Z is metrizable, and $\dim X \leq \dim Z$ by virtue of (17).

Let $g_Y: Y \to T$ and $g_X: X \to Z$ be the quotient mappings. Since from (20) it follows that $g_Y f(x) = g_Y f(y)$ whenever xEy, by the second part of Lemma 3.3.2 there exists a continuous mapping $h: Z \to T$ such that $g_Y f = h g_X$. We shall show that

$$\dim h^{-1}(t) \leq k \quad \text{for every } t \in T. \tag{22}$$

Since $h^{-1}(t) = g_X g_X^{-1} h^{-1}(t) = g_X f^{-1} g_Y^{-1}(t)$, by virtue of Remark 3.3.3 it suffices to show that each $\mathcal{W}_i$ has a finite open refinement $\mathcal{U}$ such that $\operatorname{ord}(\mathcal{U}|f^{-1}g_Y^{-1}(t)) \leq k$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{U}$. Pick a point $y \in g_Y^{-1}(t)$; we have $g_Y^{-1}(t) \subset \operatorname{St}(y, \mathcal{W}'_{i+1}) \subset W$ for some $W \in \mathcal{W}'_i$. By (18), W is contained in a $V \in \mathcal{V}_i$, and we have $f^{-1}g_Y^{-1}(t) \subset f^{-1}(V)$. Thus by (21) and (16) the cover $\mathcal{U} = \mathcal{U}_i(V)$ has all the required properties, and (22) holds.

From (22) and Theorems 1.7.7 and 1.12.4 it follows that $\dim Z \leq \dim T + k$. Thus

$$\dim X \leq \dim Z \leq \dim T + k \leq \dim Y + k. \blacksquare$$

Let us note that in Theorem 3.3.10 the assumption that Y is weakly paracompact cannot be omitted (see Problem 3.3.F).

Let us also note that the techniques developed in this section permit us to define, for every closed mapping $f: X \to Y$ of a normal space X to a weakly paracompact normal space Y with $\dim Y < \infty$, metrizable compact spaces $\overline{X}, \overline{Y}$ and a continuous mapping $\bar{f}: \overline{X} \to \overline{Y}$ such that the dimensional properties of the sets $\{\overline{y} \in \overline{Y} : |\bar{f}^{-1}(\overline{y})| \geq k\}$ and $\{\overline{y} \in \overline{Y} : \dim \bar{f}^{-1}(\overline{y}) \geq k\}$, where $k = 0, 1, \ldots$, exactly reflect the dimensional properties of the analogous sets defined for the mapping f (see Problem 3.3.E). In consequence, many theorems describing the behaviour of the covering dimension under closed mappings of normal spaces to weakly paracompact normal spaces can be deduced from their special cases where the spaces are metrizable and compact (cf. Problem 4.3.F).

Historical and bibliographic notes

As we already know from notes to Section 1.12, the theorems on dimension-raising and dimension-lowering mappings were established by Hurewicz for separable metrizable spaces. They were extended to arbitrary metric spaces by Morita and Nagami (see notes to Section 4.3). Theorem 3.3.7 was proved by Zarelua in [1969] (announcement in [1967]), and Theorem 3.3.10 by Skljarenko in [1962] under the assumption that X is a paracompact space; in both proofs, methods of algebraic topology were applied. In [1965a] Pasynkov deduced from Skljarenko's result that Theorem 3.3.10 holds under the assumption that Y is a paracompact space. In its present form, and by set theoretic methods, Theorem 3.3.10 was proved independently by Filippov in [1972a] and Pasynkov in [1981] (announcement in [1972]); they both established Lemma 3.3.9. Lemma 3.3.6 was proved by Zarelua in [1969] (announcement in [1967]). The proofs of Theorems 3.3.7 and 3.3.10 presented here are due to Filippov; they were sketched in [1972a] and presented in detail in [1978]. Lemma 3.3.2 is implicit in Arhangel'skiĭ's paper [1967].

Problems

3.3.A. For every open subset U of a completely regular space X let $\operatorname{Ex} U = \beta X \setminus \overline{X \setminus U}$, where the bar denotes the closure operator in βX.

(a) Check that $\operatorname{Ex} U$ is the largest open subset of βX whose intersection with X is equal to U.

(b) Note that for every finite open cover $\{U_i\}_{i=1}^m$ of a normal space X the family $\{\operatorname{Ex} U_i\}_{i=1}^m$ is a cover of βX (cf. Lemma 3.3.8).

(c) Apply part (b) to prove that $\dim \beta X \leq \dim X$ for every normal space X (cf. Theorem 3.1.25).

(d) Show that for every open subset U of a normal space X we have $\operatorname{Fr} \operatorname{Ex} U = \overline{\operatorname{Fr} U}$, where $\operatorname{Fr} \operatorname{Ex} U$ is the boundary of $\operatorname{Ex} U$ in βX and $\operatorname{Fr} U$ is the boundary of U in X.

(e) Apply part (d) to prove that $\operatorname{Ind} \beta X \leq \operatorname{Ind} X$ for every normal space X (cf. Theorem 2.2.10).

3.3.B (Morita [1956]). Prove directly that if $f: X \to Y$ is a closed mapping of a normal space X to a paracompact space Y such that $\dim f^{-1}(y) \leq 0$ for every $y \in Y$, then $\dim X \leq \dim Y$.

3.3.C (Filippov [1972a] and [1978]). (a) Verify that the interior of a closed set is an open domain and check that every finite open cover of a normal space has a shrinking consisting of open domains (recall that a subset U of a topological space is called an open domain if $\operatorname{Int} \overline{U} = U$).

(b) Prove that if $f: X \to Y$ is a closed mapping of a normal space X to a normal space Y and $F: \beta X \to \beta Y$ is its extension, then for every finite cover $\mathcal{W}$ of βX consisting of open domains and for $k = 0, 1, \ldots, |\mathcal{W}|$ the set

$$C_k(\mathcal{W}) = \left\{ y \in \beta Y : \text{there is no } \mathcal{V} \subset \mathcal{W} \text{ such that } F^{-1}(y) \subset \bigcup \mathcal{V} \text{ and } |\mathcal{V}| \leq k \right\}$$

satisfies the equality $C_k(\mathcal{W}) = \overline{C_k(\mathcal{W}) \cap Y}$.

Hint. Since $\beta Y \setminus C_k(\mathcal{W})$ is open, $\overline{C_k(\mathcal{W}) \cap Y} \subset C_k(\mathcal{W})$. Consider a point $y_0 \in C_k(\mathcal{W})$ and assume that it has a neighbourhood $U \subset \beta Y$ satisfying $U \cap (C_k(\mathcal{W}) \cap Y) = \emptyset$. Then follow the proof of Lemma 3.3.6, applying Lemma 3.3.8 at the end, to obtain a contradiction.

(c) Prove that for every closed mapping $f: X \to Y$ of a normal space X to a normal space Y there exist metrizable compact spaces $\overline{X}, \overline{Y}$ and a continuous mapping $\bar{f}: \overline{X} \to \overline{Y}$ such that $\dim \overline{X} = \dim X$, $\dim \overline{Y} = \dim Y$ and

$$\dim\{\overline{y} \in \overline{Y} : |\bar{f}^{-1}(\overline{y})| \geq k\} \leq \operatorname{rd}_Y \{y \in Y : |f^{-1}(y)| \geq k\}$$

for $k = 1, 2, \ldots$

Hint. Extend f to $F: \beta X \to \beta Y$ and for $i = 0, 1, \ldots$ define a finite cover $\mathcal{W}_i$ of βX consisting of open domains and a finite open cover $\mathcal{W}'_i$ of βY such that, for $i > 1$,

(1) $\mathcal{W}_i$ is a star refinement of $\mathcal{W}_{i-1}$,
(2) $\operatorname{ord} \mathcal{W}_i \leq \dim X$ and $\mathcal{W}_i$ has no open refinement of order $< \min\{i, \dim X\}$,
(3) $\mathcal{W}'_i$ is a star refinement of $\mathcal{W}'_{i-1}$,
(4) $\operatorname{ord} \mathcal{W}'_i \leq \dim Y$ and $\mathcal{W}'_i$ has no open refinement of order $< \min\{i, \dim Y\}$,
(5) $\mathcal{W}_i$ is a refinement of $F^{-1}(\mathcal{W}'_{i-1})$,
(6) for each $j \leq i$ and $k = 1, 2, \ldots, |\mathcal{W}_j|$ the cover $\mathcal{W}'_i$ is a refinement of a finite open cover $\mathcal{T}_{i,k,j}$ of βY which is a refinement of $\mathcal{W}'_{i-1}$ and satisfies the inequality $\operatorname{ord}(\mathcal{T}_{i,k,j} | C_{k,j}) \leq \dim C_{k,j}$, where $C_{k,j}$ is the set $C_{k-1}(\mathcal{W}_j)$ defined in (b),
(7) for each $j \leq i$ and $k = 1, 2, \ldots, |\mathcal{W}_j|$, and each $\mathcal{V} \subset \mathcal{W}_j$ with $|\mathcal{V}| < k$ the cover $\mathcal{W}'_i$ is a star refinement of $\{U(\mathcal{V}), \beta Y \setminus E(\mathcal{V})\}$, where $U(\mathcal{V}) = \beta Y \setminus F(\beta X \setminus \bigcup \mathcal{V})$ and $E(\mathcal{V}) \subset U(\mathcal{V})$ is a closed subset of βX such that $\bigcup\{E(\mathcal{V}) : \mathcal{V} \subset \mathcal{W}_j \text{ and } |\mathcal{V}| < k\} = \beta Y \setminus \operatorname{St}(C_{k,j}, \mathcal{W}'_{i-1})$.

Applying Lemma 3.3.2 define the spaces $\overline{X}, \overline{Y}$ and the mapping $\bar{f}\colon \overline{X} \to \overline{Y}$. Check that $C_{k,j} \cap Y \subset \{y \in Y : |f^{-1}(y)| \geq k\}$ and deduce from part (b) that $\dim C_{k,j} \leq \mathrm{rd}_Y\{y \in Y : |f^{-1}(y)| \geq k\}$. Then apply Remark 3.3.3 and (6) to show that $\dim g(C_{k,j}) \leq \dim C_{k,j}$, where $g\colon \beta Y \to \overline{Y}$ is the quotient mapping, and note that the union $C_k = \bigcup_{j=0}^{\infty} g(C_{k,j})$ satisfies the inequality $\dim C_k \leq \mathrm{rd}_Y\{y \in Y : |f^{-1}(y)| \geq k\}$. To conclude the proof show using (7) that $\{\overline{y} \in \overline{Y} : |\bar{f}^{-1}(\overline{y})| \geq k\} \subset C_k$.

(d) Note that Theorem 3.3.7 follows from the result in part (c).

3.3.D (Filippov [1972a] and [1978]). (a) Prove that if $f\colon X \to Y$ is a closed mapping of a normal space X to a weakly paracompact normal space Y with $\dim Y < \infty$ and $F\colon \beta X \to \beta Y$ is its extension, then for every finite cover $\mathcal{W}$ of βX consisting of open domains and for $k = -1, 0, \ldots, |\mathcal{W}| - 1$ the set

$$D_k(\mathcal{W}) = \Big\{y \in \beta Y : \text{there is no family } \mathcal{V} \text{ of open subsets of } \beta X \text{ each member of which is contained in a member of } \mathcal{W} \text{ such that } F^{-1}(y) \subset \bigcup \mathcal{V} \text{ and } \operatorname{ord}\mathcal{V} \leq k\Big\}$$

satisfies the equality $D_k(\mathcal{W}) = \overline{D_k(\mathcal{W}) \cap Y}$.

Hint. To prove that $D_k(\mathcal{W}) \subset \overline{D_k(\mathcal{W}) \cap Y}$ consider a point $y_0 \in D_k(\mathcal{W})$ and assume that it has a neighbourhood $U \subset \beta Y$ satisfying $\overline{U} \cap (D_k(\mathcal{W}) \cap Y) = \emptyset$. Then follow the proof of Lemma 3.3.9 substituting $\overline{U} \cap Y$ for Y.

(b) Prove that for every closed mapping $f\colon X \to Y$ of a normal space X to a weakly paracompact normal space Y with $\dim Y < \infty$ there exist metrizable compact spaces $\overline{X}, \overline{Y}$ and a continuous mapping $\bar{f}\colon \overline{X} \to \overline{Y}$ such that $\dim \overline{X} = \dim X$, $\dim \overline{Y} = \dim Y$ and

$$\dim\{\overline{y} \in \overline{Y} : \dim \bar{f}^{-1}(\overline{y}) \geq k\} \leq \mathrm{rd}_Y\{y \in Y : \dim f^{-1}(y) \geq k\} \quad \text{for } k = 0, 1, \ldots$$

Hint. Extend f to $F\colon \beta X \to \beta Y$ and define, for $i = 0, 1, \ldots$, (i) a finite cover $\mathcal{W}_i$ of βX consisting of open domains, (ii) finite families $\mathcal{V}_{i,k,j}$ of open subsets of βY, where $j \leq i$ and $k = 0, 1, \ldots, |\mathcal{W}_j|$, satisfying $\beta Y \setminus \mathrm{St}(D_{k,j}, \mathcal{W}'_{i-1}) \subset \bigcup \mathcal{V}_{i,k,j}$, where $D_{k,j}$ is the set $D_{k-1}(\mathcal{W}_j)$ defined in (a), (iii) open sets $U(V)$ and closed sets $E(V)$, where $V \in \mathcal{V}_{i,k,j}$, satisfying $E(V) \subset U(V) \subset \overline{U(V)} \subset V$ and $\bigcup\{E(V) : V \in \mathcal{V}_{i,k,j}\} = \beta Y \setminus \mathrm{St}(D_{k,j}, \mathcal{W}'_{i-1})$, (iv) finite open covers $\mathcal{U}_{i,k,j}(V)$ of βX, where $V \in \mathcal{V}_{i,k,j}$, and (v) a finite open cover $\mathcal{W}'_i$ of βY in such a way that, for $i > 1$,

(1) $\mathcal{W}_i$ is a star refinement of $\mathcal{W}_{i-1}$ and of each $\mathcal{U}_{i-1,k,j}(V)$ with $V \in \mathcal{V}_{i-1,k,j}$,
(2) $\operatorname{ord}\mathcal{W}_i \leq \dim X$ and $\mathcal{W}_i$ has no refinement of order $< \min\{i, \dim X\}$,
(3) $\mathcal{W}'_i$ is a star refinement of $\mathcal{W}'_{i-1}$ and of each $\mathcal{V}_{i,k,j} \cup \{\mathrm{St}(D_{k,j}, \mathcal{W}'_{i-1})\}$,
(4) $\operatorname{ord}\mathcal{W}'_i \leq \dim Y$ and $\mathcal{W}'_i$ has no open refinement of order $< \min\{i, \dim Y\}$,
(5) $\mathcal{W}_i$ is a refinement of $F^{-1}(\mathcal{W}'_{i-1})$,
(6) for each $j \leq i$ and $k = 0, 1, \ldots, |\mathcal{W}_j|$, and each $V \in \mathcal{V}_{i,k,j}$ the cover $\mathcal{U}_{i,k,j}(V)$ is a refinement of $\mathcal{W}_j$ and $\operatorname{ord}(\mathcal{U}_{i,k,j}(V)|F^{-1}(\overline{U(V)})) \leq k - 1$,
(7) for each $j \leq i$ and $k = 0, 1, \ldots, |\mathcal{W}_j|$ the cover $\mathcal{W}'_i$ is a refinement of a finite open cover $\mathcal{T}_{i,k,j}$ of βY which is a refinement of $\mathcal{W}'_{i-1}$ and satisfies the inequality $\operatorname{ord}(\mathcal{T}_{i,k,j}|D_{k,j}) \leq \dim D_{k,j}$,
(8) for each $j \leq i$ and $k = 0, 1, \ldots, |\mathcal{W}_j|$, and each $V \in \mathcal{V}_{i,k,j}$, the cover $\mathcal{W}'_i$ is a refinement of $\{U(V), \beta Y \setminus E(V)\}$.

Applying Lemma 3.3.2 define the spaces $\overline{X}, \overline{Y}$ and the mapping $\bar{f}: \overline{X} \to \overline{Y}$. Check that $D_{k,j} \cap Y \subset \{y \in Y : \dim f^{-1}(y) \geq k\}$ and deduce from part (a) that $\dim D_{k,j} \leq \mathrm{rd}_Y\{y \in Y : \dim f^{-1}(y) \geq k\}$. Then apply Remark 3.3.3 and (7) to show that $\dim g(D_{k,j}) \leq \dim D_{k,j}$, where $g: \beta Y \to \overline{Y}$ is the quotient mapping, and note that the union $D_k = \bigcup_{j=0}^{\infty} g(D_{k,j})$ satisfies the inequality $\dim D_k \leq \mathrm{rd}_Y\{y \in Y : \dim f^{-1}(y) \geq k\}$. To conclude the proof show using (6) and (8) that $\{\overline{y} \in \overline{Y} : \dim \bar{f}^{-1}(\overline{y}) \geq k\} \subset D_k$.

(c) Note that Theorem 3.3.10 follows from the result in part (b).

3.3.E (Filippov [1972a] and [1978]). Prove that for every closed mapping $f: X \to Y$ of a normal space X to a weakly paracompact normal space Y with $\dim Y < \infty$ there exist metrizable compact spaces $\overline{X}, \overline{Y}$ and a continuous mapping $\bar{f}: \overline{X} \to \overline{Y}$ such that $\dim \overline{X} = \dim X$, $\dim \overline{Y} = \dim Y$,

$$\dim\{\overline{y} \in \overline{Y} : |\bar{f}^{-1}(\overline{y}) \geq k\} \leq \mathrm{rd}_Y\{y \in Y : |f^{-1}(y)| \geq k\}$$

and

$$\dim\{\overline{y} \in \overline{Y} : \dim \bar{f}^{-1}(\overline{y}) \geq k\} \leq \mathrm{rd}_Y\{y \in Y : \dim f^{-1}(y) \geq k\}$$

for $k = 0, 1, \ldots$

Hint. Combine the constructions outlined in the hints to Problems 3.3.C(c) and 3.3.D(b).

3.3.F (Filippov [1972a] and [1978]). Define a closed mapping $f_0: X_0 \to W_0$ of a normal space X_0 to a normal space W_0 such that the covering dimension of W_0 and of all fibres of f_0 equals 0, and yet $\dim X_0 > 0$ (cf. Theorem 3.3.10).

Hint. Follow steps (a)–(e) below.

(a) Let $Y = \{(x,0) : 0 < x \leq 1\} \cup \{(x,1) : 0 \leq x < 1\}$ be the two arrows space (see [GT], Problem 3.10.C), and let $Q_0, Q_1, \ldots, Q_\alpha, \ldots$, $\alpha < \omega_1$, be the transfinite sequence of subsets of I defined in Example 2.2.12. In the Cartesian product $W \times Y$, where W is the space of all ordinal numbers $\alpha \leq \omega_1$ (see Example 2.1.6), define an equivalence relation E by letting $(\alpha,(x,i))E(\alpha',(x',i'))$ if and only if $\alpha = \alpha'$, $x = x'$, and either $x \notin \bigcup_{\beta \geq \alpha} Q_\beta$ or $i = i'$. Check that the quotient space $X = (W \times Y)/E$ is compact.

Hint. It suffices to check that X is a Hausdorff space.

(b) Define a continuous mapping $F: X \to W \times I$ by letting $F(q(\alpha,(x,i))) = (\alpha,x)$, where $q: W \times Y \to X$ is the quotient mapping, consider the mapping $f = pF: X \to W$, where $p: W \times I \to W$ is the projection, and check that the restriction $f_0: X_0 \to W_0$ of f to $X_0 = f^{-1}(W_0)$, where $W_0 = W \setminus \{\omega_1\}$, is a perfect mapping such that $\dim f_0^{-1}(\alpha) = 0$ for every $\alpha \in W_0$.

(c) Prove that for every open cover $\mathcal{U}$ of X_0 there exists an $\alpha \in W_0$ such that for each $\beta \geq \alpha$ and each $x \in I$ one can find a $U \in \mathcal{U}$ containing $q((\beta,(x,0)))$ and $q((\beta,(x,1)))$.

(d) Prove that the space X_0 is normal.

Hint. Apply (c) and normality of $W_0 \times I$ (see [GT], Theorem 5.2.8).

(e) Prove that $\dim X_0 > 0$.

Hint. Apply (c) to show that there is no open-and-closed $U \subset X_0$ such that $F^{-1}(W_0 \times \{0\}) \subset U$ and $U \cap F^{-1}(W_0 \times \{1\}) = \emptyset$.

Remark. Fedorčuk in [1976] gave an example (constructed under a set theoretical assumption independent of the axioms of set theory) of perfectly normal, locally compact and countably compact spaces X and Y such that $\dim X = \operatorname{Ind} X = 1$, $\dim Y = 0$, and yet there exists a perfect mapping f of X to Y satisfying $\dim f^{-1}(y) \leq 0$ for each $y \in Y$.

3.3.G (Pears [1975]; for paracompact spaces, Nagami [1960]). Prove that if $f: X \to Y$ is an open mapping of a weakly paracompact normal space X onto a normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then $\dim X = \dim Y$.

Hint. To begin, observe that the space Y is weakly paracompact. Then apply Problem 2.4.J(b) and Theorem 3.1.14 to show that for $K_j = \{y \in Y : |f^{-1}(y)| = j\}$ we have $\operatorname{rd}_Y K_j \leq \dim X$ and $\operatorname{rd}_X f^{-1}(K_j) \leq \dim Y$. To conclude the proof use Lemma 3.1.6.

3.3.H (Pasynkov [1967]). (a) Prove that if $f: X \to Y$ is an open mapping of a locally compact normal space X onto a weakly paracompact normal space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\dim Y \leq \dim X$.

Hint. Consider the open set $W = \bigcup\{V \subset Y : V \text{ is open in } Y \text{ and } \dim \overline{V} \leq \dim X\} \subset Y$ and follow the hint to Problem 2.4.K(b); in the proof that $\dim \overline{V} \leq \dim X$ apply Lemma 3.1.6 for two subspaces.

(b) Show that if $f: X \to Y$ is an open-and-closed mapping of a locally compact normal space X onto a paracompact space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\dim X = \dim Y$.

Hint. See Problem 3.3.B.

3.4. The compactification, universal space and Cartesian product theorems for the dimension dim. Dimension dim and inverse systems of compact spaces

In the proofs of the compactification and the universal space theorems we shall apply the following important theorem on the factorization of continuous mappings.

3.4.1. MARDEŠIĆ'S FACTORIZATION THEOREM. *For every continuous mapping $f: X \to Y$ of a compact space X to a compact space Y there exist a compact space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.*

PROOF. If $\dim X = \infty$ or $w(Y) < \aleph_0$, then $Z = f(X)$, $g = f$ and $h = \mathrm{id}_Z$ satisfy the assertion of the theorem. Thus one can suppose that $\dim X = n < \infty$ and $w(Y) = \mathbf{m} \geq \aleph_0$. We shall define inductively a sequence $\mathbf{W}_0, \mathbf{W}_1, \ldots$ of classes of finite open covers of the space X which all have order $\leq n$. Consider a base $\mathcal{B}$ for the space Y such that $|\mathcal{B}| = \mathbf{m}$ and denote by $\mathbf{W}_0$ the class of all finite covers of the space X by members of the family $f^{-1}(\mathcal{B})$ that have order $\leq n$. Clearly, for any $x, y \in X$,

(1) if $f(x) \neq f(y)$, then there exists a $\mathcal{W} \in \mathbf{W}_0$ such that $y \notin \operatorname{St}(x, \mathcal{W})$.

Assume now that the classes $\mathbf{W}_i$ are defined for all $i < k \geq 1$. By virtue of Lemma 3.3.1 and the inequality $\dim X \leq n$, for each pair of covers $\mathcal{W}, \mathcal{W}' \in \mathbf{W}_{k-1}$ we can choose a finite open star refinement of $\mathcal{W} \wedge \mathcal{W}'$ which has order $\leq n$; let $\mathbf{W}_k$ be the class of all covers of the space X thus obtained. In this way the sequence $\mathbf{W}_0, \mathbf{W}_1, \ldots$ is defined. Since $|\mathbf{W}_0| \leq \mathbf{m}$, we have $|\mathbf{W}_i| \leq \mathbf{m}$ for $i = 1, 2, \ldots$

The class $\mathbf{W} = \bigcup_{i=0}^{\infty} \mathbf{W}_i$ satisfies conditions (i) and (ii) in Lemma 3.3.2. Since $|\mathbf{W}| \leq \mathbf{m}$, the quotient space $Z = X/E$ obtained from X according to that lemma satisfies $w(Z) \leq \mathbf{m}$, and since $\operatorname{ord} \mathcal{W} \leq n$ for each $\mathcal{W} \in \mathbf{W}$, we have $\dim Z \leq n$. From (1) it follows that condition (iii) in Lemma 3.3.2 is also satisfied, so that there exists a continuous mapping $h: Z \to Y$ satisfying $f = hg$, where $g: X \to Z$ is the quotient mapping. ■

3.4.2. THE COMPACTIFICATION THEOREM FOR dim. *For every normal space X there exists a compactification preserving both the dimension* dim *and weight, i.e., a compact space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequalities $\dim \widetilde{X} \leq \dim X$ and $w(\widetilde{X}) \leq w(X)$.*

PROOF. We can suppose that $\dim X = n < \infty$ and $w(X) = \mathbf{m} \geq \aleph_0$. Consider a homeomorphic embedding $i: X \to I^{\mathbf{m}}$ of the space X in the Tychonoff cube $I^{\mathbf{m}}$ of weight $\mathbf{m}$; let $f: \beta X \to I^{\mathbf{m}}$ be the extension of i over βX. By virtue of Theorem 3.4.1 there exist a compact space $\widetilde{X}$ and continuous mappings $g: \beta X \to \widetilde{X}$ and $h: \widetilde{X} \to I^{\mathbf{m}}$ such that $\dim \widetilde{X} \leq \dim \beta X = \dim X, w(\widetilde{X}) \leq w(I^{\mathbf{m}}) = \mathbf{m}$ and $f = hg$. The composition $h_0 g_0$ of the restrictions $g_0 = g|X: X \to g(X) \subset \widetilde{X}$ and $h_0 = h|g(X): g(X) \to i(X) \subset I^m$ is a homeomorphism, so that g_0 is also a homeomorphism. Thus $\widetilde{X}$ is the required compactification of the space X. ■

3.4.3. THE UNIVERSAL SPACE THEOREM FOR dim. *For every integer $n \geq 0$ and every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a compact universal space $P_n(\mathbf{m})$ for the class of all normal spaces whose covering dimension is not larger than n and whose weight is not larger than $\mathbf{m}$.*

PROOF. Let $\{X_s\}_{s \in S}$ be the family of all normal subspaces of the Tychonoff cube $I^{\mathbf{m}}$ whose covering dimension is not larger than n, and let $i_s: X_s \to I^{\mathbf{m}}$ be the embedding of X_s in $I^{\mathbf{m}}$. Consider the sum $X = \bigoplus_{s \in S} X_s$ and the mapping $i: X \to I^{\mathbf{m}}$ defined by letting $i(x) = i_s(x)$ for $x \in X_s$; let $f: \beta X \to I^{\mathbf{m}}$ be the extension of i over βX. By virtue of Theorem 3.4.1 there exist a compact space $P_n(\mathbf{m})$ and continuous mappings $g: \beta X \to P_n(\mathbf{m})$ and $h: P_n(\mathbf{m}) \to I^{\mathbf{m}}$ such that $\dim P_n(\mathbf{m}) \leq \dim \beta X = \dim X = n$, $w(P_n(\mathbf{m})) \leq w(I^{\mathbf{m}}) = \mathbf{m}$ and $f = hg$.

Consider now an arbitrary normal space Y such that $\dim Y \leq n$ and $w(Y) \leq \mathbf{m}$. Since Y is embeddable in $I^{\mathbf{m}}$, there exists an $s \in S$ such that X_s is homeomorphic to Y. The composition $h_0 g_0$ of the restrictions $g_0 = g|X_s: X_s \to g(X_s) \subset P_n(\mathbf{m})$ and $h_0 = h|g(X_s): g(X_s) \to X_s \subset I^{\mathbf{m}}$ is a homeomorphism, so that g_0 is also a homeomorphism. Thus $P_n(\mathbf{m})$ is the required universal space. ■

Let us note that – in contrast to the case of separable metric spaces (see Theorems 1.11.5 and 1.11.7) – here we have only proved the existence of a universal space, but no

concrete, geometrical description of it was given. Let us also note that Theorem 3.4.2 is an immediate consequence of Theorem 3.4.3.

We now pass to a study of the behaviour of the covering dimension under Cartesian multiplication. The two Cartesian product theorems for dim that we are going to prove will be deduced from the following theorem which is an immediate consequence of Theorem 3.3.10, because the Cartesian product $X \times Y$ of a compact space X and a paracompact space Y is normal, and the projection of $X \times Y$ onto Y – parallel to the compact space X – is a closed mapping (see [GT], Theorems 5.1.36 and 3.1.16).

3.4.4. THEOREM. *For every compact space X and every paracompact space Y, not both empty, we have*

$$\dim(X \times Y) \leq \dim X + \dim Y. \blacksquare$$

3.4.5. LEMMA. *If the Cartesian product $X \times Y$ of a compact space X and a space Y is normal, then for every finite open cover $\{G_i\}_{i=1}^k$ of $X \times Y$ there exist a metric space Z, an open cover $\{V_s\}_{s\in S}$ of Z, a continuous mapping $g\colon Y \to Z$, and a collection $\{\mathcal{U}_s\}_{s\in S}$ of finite open covers of X such that the open cover $\{U \times g^{-1}(V_s) : U \in \mathcal{U}_s \text{ and } s \in S\}$ of the product $X \times Y$ is a refinement of $\{G_i\}_{i=1}^k$.*

PROOF. Let $\{F_i\}_{i=1}^k$ be a closed shrinking of the cover $\{G_i\}_{i=1}^k$, and for $i = 1, 2, \ldots, k$ let $f_i\colon X \times Y \to I$ be a continuous function such that

$$f_i(F_i) \subset \{0\} \quad \text{and} \quad f_i^{-1}([0,1)) \subset G_i. \tag{2}$$

By letting

$$\rho(y, y') = \sup\{|f_i(x, y) - f_i(x, y')| : x \in X \text{ and } i = 1, 2, \ldots, k\} \quad \text{for } y, y' \in Y,$$

we define a pseudometric ρ on Y. Consider the metric space Z obtained from Y by identifying each pair y, y' of points such that $\rho(y, y') = 0$; let the metric on Z be also denoted by ρ. For each $y \in Y$ denote by $[y]$ the corresponding point of Z; clearly $\rho([y], [y']) = \rho(y, y')$.

We shall show that the mapping $g\colon Y \to Z$ which assigns $[y] \in Z$ to $y \in Y$ is continuous. Take an arbitrary point $y \in Y$ and an $\epsilon > 0$, and consider the compact subspace $X \times \{y\}$ of $X \times Y$. For each $x \in X$ there exist neighbourhoods $U(x) \subset X$ and $V(x) \subset Y$ of x and y respectively such that for $x' \in U(x)$ and $y', y'' \in V(x)$ we have

$$|f_i(x', y') - f_i(x', y'')| < \epsilon/2 \quad \text{for } i = 1, 2, \ldots, k.$$

The open cover $\{U(x)\}_{x\in X}$ of the space X contains a finite subcover $\{U(x_j)\}_{j=1}^m$; the intersection $V = \bigcap_{j=1}^m V(x_j)$ is a neighbourhood of y, and for each $x \in X$ and each $y' \in V$ we have

$$|f_i(x, y) - f_i(x, y')| < \epsilon/2 \quad \text{for } i = 1, 2, \ldots, k,$$

because there exists a $j \leq m$ such that $x \in U(x_j)$ and $y, y' \in V(x_j)$. Thus for each $y' \in V$ we have $\rho([y], [y']) \leq \epsilon/2 < \epsilon$, i.e. the function g is continuous.

Let $\{V_s\}_{s\in S}$ be the family of all 1/8-balls in the space Z, and for each $s \in S$ let y_s be an arbitrary point in $g^{-1}(V_s)$. For each $y \in g^{-1}(V_s)$ we have $\rho(y_s, y) < 1/4$, so that

$$|f_i(x, y_s) - f_i(x, y)| < 1/4 \quad \text{for } x \in X,\ y \in g^{-1}(V_s) \text{ and } i = 1, 2, \ldots, k. \tag{3}$$

By the compactness of $X \times \{y_s\}$ there exists a finite open cover $\mathcal{U}_s$ of the space X such that

$$|f_i(x, y_s) - f_i(x', y_s)| < 1/2 \quad \text{for } x, x' \in U \in \mathcal{U}_s \text{ and } i = 1, 2, \ldots, k. \tag{4}$$

To conclude the proof it suffices to show that for every $s \in S$ and every $U \in \mathcal{U}_s$ the set $U \times g^{-1}(V_s)$ is contained in some G_i. Take a point $(x_0, y_0) \in U \times g^{-1}(V_s)$ and an $i \leq k$ such that $(x_0, y_0) \in F_i$. By virtue of (3) and (4) for any $(x, y) \in U \times g^{-1}(V_s)$ we have

$$\begin{aligned} |f_i(x, y) - f_i(x_0, y_0)| \leq |f_i(x, y) - f_i(x, y_s)| + |f_i(x, y_s) - f_i(x_0, y_s)| \\ + |f_i(x_0, y_s) - f_i(x_0, y_0)| < 1/4 + 1/2 + 1/4 = 1. \end{aligned}$$

By the first part of (2) we have $f_i(x_0, y_0) = 0$, so that $U \times g^{-1}(V_s) \subset f_i^{-1}([0, 1))$, and by the second part of (2) we have $U \times g^{-1}(V_s) \subset G_i$. ■

3.4.6. THE FIRST CARTESIAN PRODUCT THEOREM FOR dim. *For every compact space X and every normal space Y, not both empty, such that the Cartesian product $X \times Y$ is normal we have*

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

PROOF. Consider an arbitrary finite open cover $\{G_i\}_{i=1}^k$ of the product $X \times Y$. By Lemma 3.4.5 there exist a metric space Z, an open cover $\{V_s\}_{s \in S}$ of Z, a continuous mapping $g: Y \to Z$, and a collection $\{\mathcal{U}_s\}_{s \in S}$ of finite open covers of X such that the open cover $\{U \times g^{-1}(V_s) : U \in \mathcal{U}_s \text{ and } s \in S\}$ of the product $X \times Y$ is a refinement of $\{G_i\}_{i=1}^k$.

Let $G: \beta Y \to \beta Z$ be the continuous extension of g, and $T = G^{-1}(Z) \subset \beta Y$. Since the inverse image of a paracompact space under a perfect mapping is paracompact (see [GT], Theorem 5.1.35), the space T is paracompact, and by virtue of Theorem 3.4.4 we have $\dim(X \times T) \leq \dim X + \dim T$. From Corollary 3.1.26 it follows that $\dim T = \dim Y$, so that by Proposition 3.2.2 the open cover $\{U \times G^{-1}(V_s) : U \in \mathcal{U}_s \text{ and } s \in S\}$ of the paracompact space $X \times T$ has an open refinement $\mathcal{W}$ of order $\leq \dim X + \dim Y$. The family $\mathcal{W}|(X \times Y)$ is a refinement of $\{G_i\}_{i=1}^k$ and has order $\leq \operatorname{ord} \mathcal{W}$, so that $\dim(X \times Y) \leq \dim X + \dim Y$. ■

In the proof of the second Cartesian product theorem for dim we shall apply only a special case of Theorem 3.4.4, viz., the fact that for every compact space X and every metrizable space Y, not both empty, we have $\dim(X \times Y) \leq \dim X + \dim Y$. This latter fact can be obtained by applying Theorem 3.3.7 rather than the more difficult Theorem 3.3.10 from which we deduced Theorem 3.4.4 (see Problem 3.4.E(b)).

We start with two lemmas. Let us recall that a topological space X is *countably paracompact* if X is a Hausdorff space and every countable open cover of X has a locally finite open refinement.

3.4.7. LEMMA. *Every σ-locally finite open cover $\{W_s\}_{s \in S}$ of a countably paracompact normal space X has an open locally finite shrinking $\{H_s\}_{s \in S}$ such that $\overline{H}_s \subset W_s$ for every $s \in S$.*

PROOF. Let $S = \bigcup_{i=1}^{\infty} S_i$, where $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and the family $\{W_s\}_{s \in S_i}$ is locally finite for $i = 1, 2, \ldots$ By the countable paracompactness of X the open cover

$\{U_i\}_{i=1}^{\infty}$ of X, where $U_i = \bigcup_{s\in S_i} W_s$, has an open locally finite shrinking $\{V_i\}_{i=1}^{\infty}$. One easily checks that the sets $G_s = W_s \cap V_i$, where $s \in S_i$, form an open locally finite cover of X. Applying the normality of X we obtain an open cover $\{H_s\}_{s\in S}$ of X such that $\overline{H}_s \subset G_s \subset W_s$ for every $s \in S$ (see Problem 3.1.A(b)). ■

3.4.8. LEMMA. *If the Cartesian product $X \times Y$ of a metrizable space X and a space Y is normal and countably paracompact, then for every finite open cover $\{G_i\}_{i=1}^{k}$ of $X \times Y$ and every σ-locally finite base $\{U_s\}_{s\in S}$ for X there exist a closed cover $\{F_{s,i} : s \in S, i = 1, 2, \ldots, k\}$ of Y with an open enlargement $\{V_{s,i} : s \in S, i = 1, 2, \ldots, k\}$ such that the family $\{U_s \times F_{s,i} : s \in S, i = 1, 2, \ldots, k\}$ covers $X \times Y$ and $U_s \times V_{s,i} \subset G_i$ for every $s \in S$ and $i = 1, 2, \ldots, k$.*

PROOF. We can assume that all the U_s's are non-empty. For every $s' \in S$ and $i = 1, 2, \ldots, k$ let

$$V_{s',i} = \bigcup\{V \subset Y : V \text{ is open and } U_{s'} \times V \subset G_i\}. \tag{5}$$

One easily checks that the family $\{U_{s'} \times V_{s',i} : s' \in S, i = 1, 2, \ldots, k\}$ is a σ-locally finite open cover of $X \times Y$, so that by Lemma 3.4.7 it has an open locally finite shrinking $\{H_{s',i} : s' \in S, i = 1, 2, \ldots, k\}$ such that $\overline{H}_{s',i} \subset U_{s'} \times V_{s',i}$. For all $s', s \in S$ and $i = 1, 2, \ldots, k$ let

$$V_{s',s,i} = \bigcup\{V \subset Y : V \text{ is open and } U_s \times V \subset H_{s',i}\}.$$

One easily checks that

$$\bigcup\{U_s \times V_{s',s,i} : s \in S\} = H_{s',i}, \tag{6}$$

so that

$$\bigcup\{V_{s',s,i} : s', s \in S \text{ and } i = 1, 2, \ldots, k\} = Y. \tag{7}$$

From (6) the inclusion $U_s \times \overline{V}_{s',s,i} \subset \overline{H}_{s',i}$ follows, and since $U_s \neq \emptyset$ it implies that for every $s \in S$ and $i = 1, 2, \ldots, k$ the family $\{\overline{V}_{s',s,i} : s' \in S\}$ is locally finite so that its union $F_{s,i}$ is closed in Y. It follows from (7) that the family $\{F_{s,i} : s \in S, i = 1, 2, \ldots, k\}$ is a cover of Y. Now, for all $s', s \in S$ and $i = 1, 2, \ldots, k$ we have

$$U_s \times \overline{V}_{s',s,i} \subset \overline{H}_{s',i} \subset U_{s'} \times V_{s',i};$$

thus if $\overline{V}_{s',s,i}$ is non-empty, then

$$U_s \subset U_{s'} \quad \text{and} \quad \overline{V}_{s',s,i} \subset V_{s',i}.$$

The first of the above inclusions together with (5) implies that $V_{s',i} \subset V_{s,i}$, so that by the second one we have $\overline{V}_{s',s,i} \subset V_{s,i}$, and thus $F_{s,i} \subset V_{s,i}$. To conclude the proof it suffices to observe that $U_s \times V_{s,i} \subset G_i$ by virtue of (5). ■

In the proof of the next theorem we shall use a rather deep result asserting that if a Cartesian product $X \times Y$ of a non-discrete metrizable space X and a normal space Y is normal, then $X \times Y$ is countably paracompact (see [GT], Problem 5.5.18(b); readers discouraged by the hint there can add countable paracompactness of $X \times Y$ to the assumptions of the theorem).

3.4.9. THE SECOND CARTESIAN PRODUCT THEOREM FOR dim. *For every metrizable space X and every normal space Y, not both empty, such that the Cartesian product*

$X \times Y$ is normal we have

$$\dim(X \times Y) \leq \dim X + \dim Y.$$

PROOF. If X is discrete, the theorem is obvious; thus – by the above quoted result – we can assume that $X \times Y$ is countably paracompact.

Consider an arbitrary finite open cover $\{G_i\}_{i=1}^k$ of the product $X \times Y$ and a σ-locally finite base $\{U_s\}_{s\in S}$ for the space X; let $\{F_{s,i} : s \in S,\ i = 1, 2, \ldots, k\}$ and $\{V_{s,i} : s \in S,\ i = 1, 2, \ldots, k\}$ be covers of Y with the properties listed in Lemma 3.4.8. For each $s \in S$ and $i = 1, 2, \ldots, k$ take a continuous function $g_{s,i}\colon Y \to I$ such that

$$g_{s,i}(F_{s,i}) \subset \{0\} \quad \text{and} \quad g_{s,i}^{-1}([0,1)) \subset V_{s,i},$$

extend it to a continuous function $G_{s,i}\colon \beta Y \to I$ and let $W_{s,i} = G^{-1}([0,1))$. The family $\{U_s \times W_{s,i} : s \in S,\ i = 1, 2, \ldots, k\}$ consists of open F_σ-sets in $X \times \beta Y$ and is σ-locally finite, so that its union T is an F_σ-set in $X \times \beta Y$. The Cartesian product $X \times \beta Y$ is paracompact, and so is its subspace T (see [GT], Theorems 5.1.36 and 5.1.28). By Theorems 3.1.2, 3.1.8 and 3.4.4 we have

$$\dim T \leq \dim(X \times \beta Y) \leq \dim X + \dim Y,$$

so that by Proposition 3.2.2 the open cover $\{U_s \times W_{s,i} : s \in S,\ i = 1, 2, \ldots, k\}$ of T has an open refinement $\mathcal{W}$ of order $\leq \dim X + \dim Y$. Since $(U_s \times W_{s,i}) \cap (X \times Y) \subset U_s \times V_{s,i}$, the family $\mathcal{W}|(X \times Y)$ is a refinement of $\{G_i\}_{i=1}^k$; its order being $\leq \operatorname{ord} \mathcal{W}$, we have $\dim(X \times Y) \leq \dim X + \dim Y$. ■

As in the case of the large inductive dimension Ind (see Section 2.4), one can prove the *rectangular product theorem for* dim which states that for every pair X, Y of normal spaces, not both empty, such that the Cartesian product $X \times Y$ is normal and rectangular we have $\dim(X \times Y) \leq \dim X + \dim Y$ (as a matter of fact, the theorem holds for completely regular spaces and their covering dimension defined in notes to Section 3.1). The rectangular product theorem generalizes Theorems 3.4.6 and 3.4.9: one easily sees that from Lemmas 3.4.5 and 3.4.8 it follows that the Cartesian products in these theorems are rectangular. It also generalizes many other Cartesian product theorems for dim (cf. Problem 3.4.C).

We now turn to a study of inverse systems of compact spaces from the dimensional standpoint. To begin, let us recall that Theorem 1.13.2 established in Chapter 1 states that for every compact metric space X such that $\dim X \leq n$ there exists an inverse sequence $\{K_i, \pi_j^i\}$ of polyhedra of dimension $\leq n$ whose limit is homeomorphic to X. In Example 3.4.13 below we show that although for the compact space X described in Example 2.2.14 we have $\dim X = 1$, it cannot be represented as the limit of an inverse system of polyhedra of dimension ≤ 1. Hence, Theorem 1.13.2 in its original form does not extend to arbitrary compact spaces; yet we have the following

3.4.10. THEOREM ON EXPANSION IN AN INVERSE SYSTEM. *For every compact space X such that $\dim X \leq n$ there exists an inverse system $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$, where $|\Sigma| \leq w(X)$, of metrizable compact spaces of dimension $\leq n$ whose limit is homeomorphic to X.*

PROOF. We can suppose that $w(X) = \mathbf{m} \geq \aleph_0$. Let $h: X \to I^{\mathbf{m}} = \prod_{s \in S} I_s$, where $I_s = I$ for $s \in S$ and $|S| = \mathbf{m}$, be a homeomorphic embedding of the space X in the Tychonoff cube $I^{\mathbf{m}}$ of weight $\mathbf{m}$. For $i = 1, 2, \ldots$ denote by Σ_i the family of all subsets of S that have exactly i elements; the union $\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i$ is directed by inclusion, i.e., by the relation $\leq$ defined by letting $\rho \leq \sigma$ if $\rho \subset \sigma$, and has cardinality $\mathbf{m} = w(X)$.

Applying induction with respect to i, we shall now define for each $\sigma \in \Sigma_i$ a metrizable compact space X_σ such that $\dim X_\sigma \leq n$ if $i > 1$, and continuous mappings $\pi_\rho^\sigma: X_\sigma \to X_\rho$, where $\rho \leq \sigma$, such that

$$\pi_\tau^\rho \pi_\rho^\sigma = \pi_\tau^\sigma \quad \text{whenever} \quad \tau \leq \rho \leq \sigma \quad \text{and} \quad \pi_\sigma^\sigma = \mathrm{id}_{X_\sigma}; \tag{8}$$

at the same time we shall define continuous mappings $g_\sigma: X \to X_\sigma$ satisfying

$$g_\sigma(X) = X_\sigma \quad \text{and} \quad \pi_\rho^\sigma g_\sigma = g_\rho \quad \text{whenever} \quad \rho \leq \sigma. \tag{9}$$

For $i = 1$ all the required conditions are satisfied if for each $\sigma = \{s\} \in \Sigma_1$ we let $X_\sigma = p_s h(X) \subset I_s$, where $p_s: \prod_{s \in S} I_s \to I_s$ is the projection, $\pi_\sigma^\sigma = \mathrm{id}_{X_\sigma}$ and $g_\sigma = p_s h$. Assume that the spaces X_σ and the mappings π_ρ^σ and g_σ satisfying (8) and (9) are defined for all $\sigma \in \bigcup_{i=1}^{k-1} \Sigma_i$, where $k > 1$, and consider a set $\sigma \in \Sigma_k$. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be all $(k-1)$-element subsets of σ and let $f_\sigma: X \to X_{\sigma_1} \times X_{\sigma_2} \times \ldots \times X_{\sigma_k}$ be the continuous mapping defined by the formula $f_\sigma(x) = (g_{\sigma_1}(x), g_{\sigma_2}(x), \ldots, g_{\sigma_k}(x))$. By virtue of Theorem 3.4.1 there exist a compact space X_σ and continuous mappings $g_\sigma: X \to X_\sigma$ and $h_\sigma: X_\sigma \to X_{\sigma_1} \times X_{\sigma_2} \times \ldots \times X_{\sigma_k}$ such that $\dim X_\sigma \leq n, w(X_\sigma) \leq \aleph_0, g_\sigma(X) = X_\sigma$ and $f_\sigma = h_\sigma g_\sigma$. The last equality means that

$$\pi_{\sigma_i}^\sigma g_\sigma = g_{\sigma_i} \quad \text{for } i = 1, 2, \ldots, k, \tag{10}$$

where $\pi_{\sigma_i}^\sigma: X_\sigma \to X_{\sigma_i}$ is the composition of h_σ and the projection of $X_{\sigma_1} \times X_{\sigma_2} \times \ldots \times X_{\sigma_i}$ onto X_{σ_i}. As the space X_σ is compact and has countable weight, it is a metrizable space. For each $\rho \in \bigcup_{i=1}^{k-1} \Sigma_i$ satisfying $\rho \leq \sigma$ there exists at least one $i \leq k$ such that $\rho \leq \sigma_i \leq \sigma$. Let us observe that the composition $\pi_\rho^{\sigma_i} \pi_{\sigma_i}^\sigma$ does not depend on the choice of a particular σ_i satisfying $\rho \leq \sigma_i$. Indeed, if for some $j \leq k$ we also have $\rho \leq \sigma_j$, then

$$\pi_\rho^{\sigma_i} \pi_{\sigma_i}^\sigma g_\sigma = \pi_\rho^{\sigma_i} g_{\sigma_i} = g_\rho = \pi_\rho^{\sigma_j} g_{\sigma_j} = \pi_\rho^{\sigma_j} \pi_{\sigma_j}^\sigma g_\sigma,$$

which implies that

$$\pi_\rho^{\sigma_i} \pi_{\sigma_i}^\sigma = \pi_\rho^{\sigma_j} \pi_{\sigma_j}^\sigma, \tag{11}$$

because $g_\sigma(X) = X_\sigma$. Accordingly, for each $\rho \in \bigcup_{i=1}^{k-1} \Sigma_i$ satisfying $\rho \leq \sigma$ we define $\pi_\rho^\sigma = \pi_\rho^{\sigma_i} \pi_{\sigma_i}^\sigma$ where $\rho \leq \sigma_i \leq \sigma$. Moreover, we let $\pi_\sigma^\sigma = \mathrm{id}_{X_\sigma}$. From (10), (11) and the inductive assumption it follows that the space X_σ and the mappings π_ρ^σ and g_σ satisfy (8) and (9) for $\sigma \in \Sigma_k$. Thus, we have defined metrizable compact spaces X_σ such that $\dim X_\sigma \leq n$ if $i > 1$, and continuous mappings π_ρ^σ and g_σ satisfying (8) and (9) for $\sigma \in \Sigma$ and $\rho \leq \sigma$.

It follows from (8) that $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ is an inverse system; we shall show that X is homeomorphic to the limit $\varprojlim \mathbf{S}$. By the second equality in (9), for every $x \in X$ the point $\{g_\sigma(x)\} \in \prod_{\sigma \in \Sigma} X_\sigma$ is a thread of $\mathbf{S}$; by assigning this thread to x we define a mapping $g: X \to \varprojlim \mathbf{S}$. Since $\pi_\sigma g = g_\sigma$ for every $\sigma \in \Sigma$, where $\pi_\sigma: \varprojlim \mathbf{S} \to X_\sigma$ is the projection, the mapping g is continuous. For each $\sigma = \{s\} \in \Sigma_1$ we have $\pi_\sigma g = g_\sigma = p_s h$;

therefore, h being one-to-one, so is g. Finally, as $g_\sigma(X) = X_\sigma$ for every $\sigma \in \Sigma$, the mapping g maps X onto $\varprojlim \mathbf{S}$ (see [GT], Corollary 3.2.16). Thus g is a homeomorphism of X onto $\varprojlim \mathbf{S}$.

If $n \geq 1$, the system $\mathbf{S}$ satisfies all the required conditions; if $n = 0$, it has to be replaced by the system $\{X_\sigma, \pi^\sigma_\rho, \Sigma \setminus \Sigma_1\}$ (cf. Problem 3.4.F). ■

Let us observe that from (9) and (10) it follows that the bonding mappings π^σ_ρ in Theorem 3.4.10 map X_σ onto X_ρ.

We shall now prove the following

3.4.11. THEOREM ON THE DIMENSION OF THE LIMIT OF AN INVERSE SYSTEM. *If* $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ *is an inverse system of compact spaces* X_σ *such that* $\dim X_\sigma \leq n$ *for* $\sigma \in \Sigma$, *then the limit* $X = \varprojlim \mathbf{S}$ *satisfies the inequality* $\dim X \leq n$.

PROOF. Consider a finite open cover $\mathcal{U}$ of X. The space X being compact (see [GT], Theorem 3.2.13), the cover $\mathcal{U}$ has a finite refinement of the form $\{\pi^{-1}_{\sigma_i}(U_i)\}_{i=1}^k$, where $\pi_{\sigma_i}\colon X \to X_{\sigma_i}$ is the projection and U_i is an open subset of X_{σ_i} for $i = 1, 2, \ldots, k$. Let σ be an arbitrary element of Σ such that $\sigma_i \leq \sigma$ for $i = 1, 2, \ldots, k$ and let $W_i = (\pi^\sigma_{\sigma_i})^{-1}(U_i)$. One readily sees that the family $\{\pi^{-1}_\sigma(W_i)\}_{i=1}^k$ is an open refinement of the cover $\mathcal{U}$. Since $\pi_\sigma(X)$ is a closed subspace of X_σ, we have $\dim \pi_\sigma(X) \leq n$ and the open cover $\{\pi_\sigma(X) \cap W_i\}_{i=1}^k$ of $\pi_\sigma(X)$ has an open shrinking $\{V_i\}_{i=1}^k$ of order $\leq n$. The family $\{\pi^{-1}_\sigma(V_i)\}_{i=1}^k$ is an open cover of the space X which refines $\mathcal{U}$ and has order $\leq n$. Thus $\dim X \leq n$. ■

The assumption of compactness of X_σ's in the last theorem is essential: there exist inverse sequences $\mathbf{S} = \{X_i, \pi^i_j\}$ of Lindelöf spaces such that $\dim X_i = 0$ for $i = 1, 2, \ldots$, and yet the limit $X = \varprojlim \mathbf{S}$ is normal and $\dim X > 0$. Such examples are, however, too complicated to be discussed here. On the other hand, one can show that Theorem 3.4.11 holds under the weaker assumption that X_σ's are paracompact and the bonding mappings π^σ_ρ are perfect.

Theorems 3.4.10 and 3.4.11 yield the following

3.4.12. THEOREM ON INVERSE SYSTEMS. *A compact space* X *satisfies the inequality* $\dim X \leq n$ *if and only if* X *is homeomorphic to the limit of an inverse system of metrizable compact spaces of dimension* $\leq n$. ■

We conclude this section with the above-mentioned example of a compact space X which satisfies the equality $\dim X = 1$ and yet it is not homeomorphic to the limit of an inverse system of polyhedra of dimension ≤ 1.

3.4.13. EXAMPLE. The space X described in Example 2.2.14 is compact and satisfies the relation $\dim X = 1 < \operatorname{ind} X$ (see Example 3.1.31). Thus, to show that X has the required property it suffices to prove that for every inverse system $\mathbf{S} = \{K_\sigma, \pi^\sigma_\rho, \Sigma\}$ of polyhedra of dimension ≤ 1 the limit $K = \varprojlim \mathbf{S}$ satisfies the inequality $\operatorname{ind} K \leq 1$.

Consider a point $x \in K$ and a neighbourhood $V \subset K$ of the point x. There exists a $\sigma_0 \in \Sigma$ and a neighbourhood U_{σ_0} of the point $\pi_{\sigma_0}(x)$ in the space K_{σ_0}, where

$\pi_{\sigma_0}\colon K \to K_{\sigma_0}$ is the projection, such that the set $U = \pi_{\sigma_0}^{-1}(U_{\sigma_0})$ satisfies $x \in U \subset V$. Define $\Sigma_0 = \{\sigma \in \Sigma : \sigma_0 \leq \sigma\}$ and let $U_\sigma = (\pi_{\sigma_0}^\sigma)^{-1}(U_{\sigma_0})$ and $F_\sigma = \operatorname{Fr} U_\sigma$ for $\sigma \in \Sigma_0$. Since for each $\sigma, \rho \in \Sigma_0$ satisfying $\rho \leq \sigma$ we have

$$\begin{aligned}\pi_\rho^\sigma(F_\sigma) &= \pi_\rho^\sigma(\overline{U}_\sigma \cap \overline{K_\sigma \setminus U_\sigma}) \subset \overline{\pi_\rho^\sigma(U_\sigma)} \cap \overline{\pi_\rho^\sigma(K_\sigma \setminus U_\sigma)} \\ &= \overline{\pi_\rho^\sigma(\pi_{\sigma_0}^\sigma)^{-1}(U_{\sigma_0})} \cap \overline{\pi_\rho^\sigma(\pi_{\sigma_0}^\sigma)^{-1}(K_{\sigma_0} \setminus U_{\sigma_0})} \\ &\subset \overline{(\pi_{\sigma_0}^\rho)^{-1}(U_{\sigma_0})} \cap \overline{(\pi_{\sigma_0}^\rho)^{-1}(K_{\sigma_0} \setminus U_{\sigma_0})} = \overline{U}_\rho \cap \overline{K_\rho \setminus U_\rho} = F_\rho,\end{aligned}$$

the family $\mathbf{S}_0 = \{F_\sigma, \widetilde{\pi}_\rho^\sigma, \Sigma_0\}$, where $\widetilde{\pi}_\rho^\sigma\colon F_\sigma \to F_\rho$ is defined by letting $\widetilde{\pi}_\rho^\sigma(x) = \pi_\rho^\sigma(x)$, is an inverse system of compact spaces. Now, for each $\sigma \in \Sigma_0$ the set F_σ is nowhere dense in K_σ, so that for each 1-simplex $S \in \mathcal{K}_\sigma$, where $|\mathcal{K}_\sigma| = K_\sigma$, we have $\dim(S \cap F_\sigma) \leq 0$, and thus $\dim F_\sigma \leq 0$. Hence $\dim \varprojlim \mathbf{S}_0 \leq 0$ by virtue of Theorem 3.4.11. One readily checks that $\pi_\sigma(\operatorname{Fr} U) \subset F_\sigma$ for every $\sigma \in \Sigma_0$; thus $\operatorname{Fr} U \subset \varprojlim \mathbf{S}_0$ and by virtue of Theorem 3.1.30 we have $\operatorname{ind} \operatorname{Fr} U \leq 0$. Hence we have proved that $\operatorname{ind} K \leq 1$. ■

Let us note in connection with the last example that there exists a compact space X with a similar property which satisfies the equality $\operatorname{ind} X = \operatorname{Ind} X = \dim X = 1$.

Let us also note that every compact space is homeomorphic to the limit of an inverse system of polyhedra (see Problem 3.4.I) and that one can define a compact space X such that $\operatorname{ind} X = \operatorname{Ind} X = \dim X = 1$ which is not homeomorphic to the limit of an inverse system of polyhedra (or, more generally, of locally connected metrizable compact spaces) with onto bonding mappings.

Historical and bibliographic notes

Theorem 3.4.1 was established by Mardešić in [1960]; the present proof was given by Arhangel'skiĭ in [1967]. Theorem 3.4.2 was proved by Skljarenko in [1958]. Theorem 3.4.3 was established independently by Pasynkov in [1964] and by Zarelua in [1964].

Theorem 3.4.4 was proved by Morita in [1953]. Lemma 3.4.5 is due to Terasawa (see T. Chiba and K. Chiba [1972]). Theorem 3.4.6 was established by Filippov (announcement in [1973], proof in [1980]) and by Morita in [1973]. Lemma 3.4.8 is taken from Kodama [1969], and its proof from Engelking [1973]. Theorem 3.4.9 was announced by Filippov in [1973] and by Pasynkov in [1973]; the former gave his proof in [1980], the latter – in [1981]. The rectangular product theorem for dim was announced by Pasynkov in [1973]; the proof was given in his paper [1981]. A good discussion of Cartesian product theorems for dim can be found in Przymusiński [1980]. Theorem 3.4.10 was given by Mardešić in [1960]; Theorem 3.4.11 is implicit in Freudenthal's paper [1937].

An inverse sequence of Lindelöf spaces with dim equal to zero whose limit is a normal space with dim larger than zero was defined by Charalambous in [1980]. Theorem 3.4.11 under the weaker assumptions that X_σ's are paracompact and the bonding mappings π_ρ^σ are perfect was announced by Pasynkov in [1973]; a proof can be found in Katuta [1982] and in Yajima [1984]. Example 3.4.13 was given independently by Pasynkov in [1958] and by Mardešić in [1960]. Both examples cited at the end of this section can be found in Pasynkov's paper [1962].

Problems

3.4.A. (a) Show that if a closed subset F of a normal space X satisfies the inequality $\dim F \leq n$, then every finite open cover $\mathcal{U}$ of the space X has a finite open star refinement $\mathcal{V}$ such that $\operatorname{ord}(\mathcal{V}|F) \leq n$ (cf. Lemma 3.3.1).

(b) (Zarelua [1964a]) Prove that for every continuous mapping $f: X \to Y$ of a compact space X to a compact space Y and for any family $\{F_s\}_{s\in S}$ of closed subsets of X, where $|S| \leq w(Y)$, there exist a compact space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\dim g(F_s) \leq \dim F_s$ for $s \in S$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.

Hint (Arhangel'skiĭ [1967]). Modify the proof of Theorem 3.4.1; use (a) for each member of the family $\{F_s\}_{s\in S}$.

(c) (Zarelua [1964a]; for $|S| = \aleph_0$, Skljarenko [1958]) Prove that for every normal space X and any family $\{F_s\}_{s\in S}$ of closed subsets of X, where $|S| \leq w(X)$, there exists a compactification preserving the dimension of all F_s's and the weight of X, i.e., a compact space $\widetilde{X}$ which contains X as a dense subspace such that $w(\widetilde{X}) = w(X)$ and for each $s \in S$ the closure $\overline{F}_s$ of F_s in $\widetilde{X}$ satisfies the inequality $\dim \overline{F}_s \leq \dim F_s$.

Hint. See the proof of Theorem 3.4.2.

3.4.B. Prove directly that for every compact space X with $\dim X = 0$ and every paracompact space Y we have $\dim(X \times Y) \leq \dim X + \dim Y$ (cf. Problem 3.3.B).

3.4.C (Filippov [1980] (announcement [1973])). Check that the assumption of compactness of the space X in Theorem 3.4.6 can be weakened to the assumption that the projection of $X \times Y$ onto Y is a closed mapping.

Remark. As proved by Filippov in [1980] (announcement in [1973]) and by Pasynkov in [1981] (announcement in [1973]), the assumption of metrizability of X in Theorem 3.4.9 can be weakened to the existence of a perfect mapping of X onto a metrizable space.

3.4.D (Morita [1973]). Prove that for every locally compact paracompact space X and every normal space Y, not both empty, such that the Cartesian product $X \times Y$ is normal we have $\dim(X \times Y) \leq \dim X + \dim Y$.

3.4.E. (a) Prove directly that for every compact space X and every metrizable space Y with $\dim Y = 0$ we have $\dim(X \times Y) \leq \dim X$.

Hint. Note that the Cartesian product $X \times Y$ is strongly paracompact and apply Theorems 3.2.13 and 3.1.23.

(b) Deduce from (a) and Theorems 4.3.15 and 3.3.7 that for every compact space X and every metrizable space Y, not both empty, we have $\dim(X \times Y) \leq \dim X + \dim Y$.

3.4.F. Observe that Theorem 3.4.10 for $n = 0$ easily follows from the fact that every normal space X such that $\dim X = 0$ is embeddable in a Cantor cube (see Remark 1.3.18 and Theorem 1.6.10).

3.4.G (Pasynkov [1962]). Let $\mathbf{S} = \{X_\sigma, \pi^\sigma_\rho, \Sigma\}$ be an inverse system of compact spaces and let $X = \varprojlim \mathbf{S}$. Prove that $\dim X \leq n$ if and only if for each $\rho \in \Sigma$ and every finite

open cover $\{U_i\}_{i=1}^k$ of the space X_ρ there exists a $\sigma \in \Sigma$ satisfying $\rho \leq \sigma$ and such that the cover $\{(\pi_\rho^\sigma)^{-1}(U_i)\}_{i=1}^k$ of the space X_σ has a finite open refinement $\mathcal{V}$ satisfying the inequality $\operatorname{ord}(\mathcal{V}|\pi_\sigma(X)) \leq n$, where $\pi_\sigma: X \to X_\sigma$ is the projection.

3.4.H (Pasynkov [1958]). Prove that for every inverse system $\mathbf{S} = \{K_\sigma, \pi_\rho^\sigma, \Sigma\}$ of polyhedra of dimension ≤ 1 the limit $K = \varprojlim \mathbf{S}$ satisfies the inequality $\operatorname{Ind} K \leq 1$.

Remark. It is not known if 1 can be replaced by an arbitrary natural number.

3.4.I (Eilenberg and Steenrod [1952]). Prove that for every compact space X there exists an inverse system $\mathbf{S} = \{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ of polyhedra whose limit is homeomorphic to X.

Hint. Embed the space X in a Tychonoff cube.

Chapter 4

Dimension theory of metrizable spaces

In the realm of metrizable spaces the dimensions Ind and dim coincide. Thus for metrizable spaces both the theorems established for the dimension Ind and the theorems established for the dimension dim are valid. It will appear in the course of this chapter that the dimension theory of metrizable spaces is by no means inferior to the classical dimension theory of separable metric spaces developed in the first chapter of this book.

The present chapter can be read almost directly after Chapter 1. The results of Chapter 2 are not used here, except for Lemma 2.3.16 which belongs to general topology rather than to dimension theory. From Chapter 3 we use only the beginning of Section 3.1 up to Theorem 3.1.10 as well as Lemma 3.1.27 and Theorems 3.1.28, 3.1.29, 3.2.2 and 3.2.5; the last theorem is not used until Section 4.2. In a few instances when other theorems from Chapters 2 and 3 are quoted, we indicate how to avoid their application by some additional arguments.

In Section 4.1 the most important properties of dimension in metrizable spaces are established. We start with the Katětov–Morita theorem on the coincidence of Ind and dim and then prove the counterparts of the theorems obtained for separable metric spaces in Section 1.5.

Section 4.2 begins with two characterizations of the covering dimension in metrizable spaces, one stated in terms of special bases and the other in terms of sequences of covers. Then we discuss briefly some characterizations of dim formulated in terms of special metrics. In the second part of the section, we first prove, by applying an appropriate factorization theorem, the existence of a universal space for the class of all metrizable spaces whose dimension is not larger than n and whose weight is not larger than $\mathbf{m}$, and then we construct an example of such a universal space.

Section 4.3 resumes the considerations of Section 1.12. We generalize in it the theorems on dimension-raising and dimension-lowering mappings established in Chapter 1 and prove two more specialized theorems on the relations between the dimensions of the domain and range of a closed mapping. In the second part of the section we characterize metrizable spaces with covering dimension less than or equal to n in terms of closed mappings with fibers of cardinality $\leq n+1$ defined on spaces with covering dimension zero, and in terms of special sequences of closed covers.

Let us add that the theorems on partitions and on extending mappings to spheres established in Sections 1.7 and 1.9 extend to all normal, and, *a fortiori*, to all metrizable spaces; the proofs were given in Section 3.2 (cf. Problems 4.1.E and 4.3.B).

4.1. Basic properties of dimension in metrizable spaces

We start with one of the most important results in dimension theory, viz., with the theorem on the coincidence of the dimensions Ind and dim in metrizable spaces. In the proof we shall apply two characterizations ot the dimension dim in the class of metrizable spaces which are established in Proposition 4.1.2 below (cf. Problem 4.1.A(b)).

4.1.1. LEMMA. *Let X be a normal space. If there exists a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of the space X such that* $\operatorname{ord}\mathcal{W}_i \leq n$ *and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$, and for every point $x \in X$ the family $\{\operatorname{St}(W, \mathcal{W}_i) : x \in W \in \mathcal{W}_i,\ i = 1, 2, \ldots\}$ is a base for X at the point x, then* $\dim X \leq n$.

PROOF. For $i = 1, 2, \ldots$ take a mapping f_i^{i+1} of $\mathcal{W}_{i+1}$ to $\mathcal{W}_i$ such that $W \subset f_i^{i+1}(W)$ for each $W \in \mathcal{W}_{i+1}$; let $f_i^k = f_i^{i+1} f_{i+1}^{i+2} \ldots f_{k-1}^k$ for $i < k$ and $f_i^i = \mathrm{id}_{\mathcal{W}_i}$ for $i = 1, 2, \ldots$ Obviously

$$W \subset f_i^k(W) \quad \text{for each } W \in \mathcal{W}_k \text{ and } i \leq k. \tag{1}$$

Consider a finite open cover $\{H_j\}_{j=1}^l$ of the space X. The sets $X_1, X_2, \ldots$, where

$$X_k = \bigcup\{W \in \mathcal{W}_k : \operatorname{St}(W, \mathcal{W}_k) \subset H_j \text{ for some } j \leq l\}, \tag{2}$$

form an open cover of X. For $k = 1, 2, \ldots$ define the subfamilies

$$\mathcal{U}_k = \{U \in \mathcal{W}_k : U \cap X_k \neq \emptyset\} \quad \text{and} \quad \mathcal{V}_k = \Big\{V \in \mathcal{U}_k : V \cap \Big(\bigcup_{j<k} X_j\Big) = \emptyset\Big\}$$

of the cover $\mathcal{W}_k$. For every $U \in \mathcal{U}_k$ we have $f_k^k(U) \cap X_k = U \cap X_k \neq \emptyset$; denote by $i(U)$ the smallest integer $i \leq k$ such that $f_i^k(U) \cap X_i \neq \emptyset$. Since $\mathcal{V}_1 = \mathcal{U}_1$ and $f_{i(U)}^k(U) \subset f_{i(U)-1}^k(U) \subset \ldots \subset f_1^k(U)$ whenever $i(U) > 1$, we have

$$f_{i(U)}^k(U) \in \mathcal{V}_{i(U)}. \tag{3}$$

For every $V \in \mathcal{V}_i$ consider the open set

$$V^* = \bigcup_{k=i}^{\infty} \bigcup\{U \cap X_k : U \in \mathcal{U}_k,\ f_i^k(U) = V \text{ and } i(U) = i\}; \tag{4}$$

as $V \cap X_i \neq \emptyset$, by virtue of (2) there exist a $W \in \mathcal{W}_i$ such that $V \cap W \neq \emptyset$ and a $j(V) \leq l$ satisfying $V \subset \operatorname{St}(W, \mathcal{W}_i) \subset H_{j(V)}$. From (1) it follows that $V^* \subset V$, so that $V^* \subset H_{j(V)}$. Since $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ whenever $i \neq j$, for every $V \in \mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ the set V^* and the integer $j(V)$ are well defined.

To complete the proof it suffices to show that the family $\{V_j\}_{j=1}^l$, where $V_j = \bigcup\{V^* : V \in \mathcal{V} \text{ and } j(V) = j\} \subset H_j$, is a cover of the space X and has order $\leq n$, or – what is the same – that the family $\mathcal{V}^* = \{V^* : V \in \mathcal{V}\}$ is a cover of X and $\operatorname{ord}\mathcal{V}^* \leq n$.

Let x be an arbitrary point of X. Consider an integer k such that

$$x \in X_k \setminus \bigcup_{j<k} X_j \tag{5}$$

and a set $U \in \mathcal{W}_k$ which contains the point x; since $U \cap X_k \neq \emptyset$, we have $U \in \mathcal{U}_k$. It follows from (3) and (4) that $x \in U \cap X_k \subset (f_{i(U)}^k(U))^* \in \mathcal{V}^*$, so that $\mathcal{V}^*$ is a cover of X.

It remains to show that $\operatorname{ord}\mathcal{V}^* \leq n$. Consider a non-empty intersection $V_1^* \cap V_2^* \cap \ldots \cap V_h^*$, where $V_i \in \mathcal{V}_{m_i}$ and $V_i \neq V_j$ whenever $i \neq j$; let $x \in V_1^* \cap V_2^* \cap \ldots \cap V_h^*$. From the definition of $\mathcal{V}_{m_i}$ it follows that for the integer k satisfying (5) we have $m_i \leq k$ for $i = 1, 2, \ldots, h$. By (4) there exist sets $U_i \in \mathcal{U}_{k_i}$ such that $f_{m_i}^{k_i}(U_i) = V_i$, $i(U_i) = m_i$ and $x \in U_i \cap X_{k_i}$. Since $x \in X_{k_i}$ it follows from (5) that $k \leq k_i$. The sets $W_1, W_2, \ldots, W_h$, where $W_i = f_k^{k_i}(U_i) \in \mathcal{W}_k$, all contain the point x, so that – as $\operatorname{ord}\mathcal{W}_k \leq n$ – it suffices to show that $W_i \neq W_j$ whenever $i \neq j$. Let us note that the sets W_i belong to $\mathcal{U}_k$ and that $i(W_i) = i(U_i) = m_i$; hence $W_i \neq W_j$ whenever $m_i \neq m_j$. When $i \neq j$ and $m_i = m_j$, we also have $W_i \neq W_j$, because then

$$f_{m_i}^{k}(W_i) = f_{m_i}^{k_i}(U_i) = V_i \neq V_j = f_{m_j}^{k_j}(U_j) = f_{m_j}^{k}(W_j). \blacksquare$$

4.1.2. PROPOSITION. *For every metrizable space X the following conditions are equivalent*:

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *For every metric ρ on the space X there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite open covers of the space X such that for $i = 1, 2, \ldots$ we have* $\operatorname{mesh}\mathcal{U}_i < 1/i$, $\operatorname{ord}\mathcal{U}_i \leq n$, *and for each $U \in \mathcal{U}_{i+1}$ the set $\overline{U}$ is contained in some $V \in \mathcal{U}_i$.*

(iii) *There exist a metric ρ on the space X and a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of the space X such that for $i = 1, 2, \ldots$ we have* $\operatorname{mesh}\mathcal{W}_i < 1/i$, $\operatorname{ord}\mathcal{W}_i \leq n$, *and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$.*

PROOF. The implication (ii)⇒(iii) is obvious and the implication (iii)⇒(i) follows from Lemma 4.1.1, so that it suffices to prove that (i)⇒(ii). Consider a metrizable space X such that $\dim X \leq n$ and an arbitrary metric ρ on the space X. We shall define inductively a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X. Assume that $k = 1$ or that $k > 1$ and the covers $\mathcal{U}_i$ are defined for all $i < k$. For every point $x \in X$ there exists a neighbourhood U_x of x such that $\delta(U_x) < 1/(k+1)$ and the set $\overline{U}_x$ is contained in a member of $\mathcal{U}_{k-1}$ if $k > 1$. Since every metrizable space is paracompact (see [GT], Theorem 5.1.3), it follows from Proposition 3.2.2 that the open cover $\{U_x\}_{x \in X}$ of the space X has a locally finite open refinement $\mathcal{U}_k$ such that $\operatorname{ord}\mathcal{U}_k \leq n$. The sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ thus obtained satisfies all the conditions in (ii), and the implication (i)⇒(ii) is established. ■

4.1.3. THE KATĚTOV–MORITA THEOREM. *For every metrizable space X we have* $\operatorname{Ind} X = \dim X$.

PROOF. In consequence of Theorem 3.1.28 it suffices to show that $\operatorname{Ind} X \leq \dim X$. We can suppose that $\dim X < \infty$. We shall apply induction with respect to $\dim X$. If $\dim X = -1$, we clearly have $\operatorname{Ind} X \leq \dim X$. Assume that our inequality holds for all metrizable spaces with covering dimension $\leq n-1$, and consider a metrizable space X with $\dim X = n \geq 0$ and a pair A, B of disjoint closed subsets of the space X. It suffices to define open sets $K, M \subset X$ which, together with the set $L = X \setminus (K \cup M)$, satisfy the conditions

$$A \subset K, \quad B \subset M, \quad K \cap M = \emptyset \quad \text{and} \quad \dim L \leq n-1;$$

indeed, the set L is then a partition between A and B, and $\operatorname{Ind} L \leq n-1$ by virtue of the inductive assumption.

Let σ be an arbitrary metric on the space X and let $f: X \to I$ be a continuous function satisfying $f(A) \subset \{0\}$ and $f(B) \subset \{1\}$. One readily checks that the formula $\rho(x,y) = \sigma(x,y) + |f(x) - f(y)|$ defines another metric ρ on the space X. From now on we shall consider on X only the metric ρ. By virtue of Proposition 4.1.2 there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite open covers of X such that for $i = 1, 2, \ldots$ we have $\operatorname{mesh}\mathcal{U}_i < 1/i$, $\operatorname{ord}\mathcal{U}_i \leq n$, and for each $U \in \mathcal{U}_{i+1}$ the set $\overline{U}$ is contained in some $V \in \mathcal{U}_i$.

Let $K_0 = A, M_0 = B$, and for $i \geq 1$ let $K_i = X \setminus H_i$ and $M_i = X \setminus G_i$, where

$$G_i = \bigcup\{U \in \mathcal{U}_i : \overline{U} \cap M_{i-1} = \emptyset\}$$

and

$$H_i = \bigcup\{U \in \mathcal{U}_i : \overline{U} \cap M_{i-1} \neq \emptyset\};$$

in this way two sequences, $K_0, K_1, K_2, \ldots$ and $M_0, M_1, M_2, \ldots$, of subsets of the space X are defined.

Let us observe that

(6) if $U \in \mathcal{U}_i$ and $\overline{U} \cap M_{i-1} \neq \emptyset$, then $\overline{U} \cap K_{i-1} = \emptyset$.

The validity of (6) for $i = 1$ follows from the definition of ρ, because no set of diameter less than 1 meets both A and B. If $U \in \mathcal{U}_i$ where $i > 1$ and $\overline{U} \cap M_{i-1} \neq \emptyset$, then for any $V \in \mathcal{U}_{i-1}$ that contains the set $\overline{U}$ we have $V \cap M_{i-1} \neq \emptyset$, so that V is not contained in G_{i-1}; this implies that $V \subset H_{i-1}$, which gives the equality $\overline{U} \cap K_{i-1} = \emptyset$.

From the local finiteness of $\mathcal{U}_i$, the definitions of G_i and H_i, and (6) it follows that $\overline{G}_i \cap M_{i-1} = \emptyset = \overline{H}_i \cap K_{i-1}$ for $i = 1, 2, \ldots$, which implies that $K_{i-1} \subset X \setminus \overline{H}_i = \operatorname{Int} K_i$ and $M_{i-1} \subset X \setminus \overline{G}_i = \operatorname{Int} M_i$; moreover, as $G_i \cup H_i = X$, we have $K_i \cap M_i = \emptyset$. Hence, the sets $K = \bigcup_{i=0}^{\infty} K_i$ and $M = \bigcup_{i=0}^{\infty} M_i$ are open, disjoint and contain respectively A and B.

Let $L_i = X \setminus (K_i \cup M_i) = G_i \cap H_i$ for $i = 1, 2, \ldots$; clearly $L = X \setminus (K \cup M) = \bigcap_{i=1}^{\infty} L_i$. The family $\mathcal{W}_i = \{U \cap L : U \in \mathcal{U}_i \text{ and } \overline{U} \cap M_{i-1} \neq \emptyset\}$ is, for $i = 1, 2, \ldots$, an open cover of the space $L \subset H_i$ and $\operatorname{ord}\mathcal{W}_i \leq n-1$, because each point $x \in L \subset L_i \subset G_i$ belongs to at least one $U \in \mathcal{U}_i$ satisfying $\overline{U} \cap M_{i-1} = \emptyset$. If $U \in \mathcal{U}_{i+1}$ and $\overline{U} \cap M_i \neq \emptyset$, then for any $V \in \mathcal{U}_i$ that contains $\overline{U}$ we have $V \cap M_i \neq \emptyset$, so that V is not contained in G_i, which implies that $\overline{V} \cap M_{i-1} \neq \emptyset$, i.e., that $V \cap L \in \mathcal{W}_i$. Thus $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$. Since clearly $\operatorname{mesh}\mathcal{W}_i \leq \operatorname{mesh}\mathcal{U}_i$, we have $\dim L \leq n - 1$ by virtue of Proposition 4.1.2. ■

From the coincidence of the dimensions Ind and dim in metrizable spaces it follows that some results in the dimension theory of metrizable spaces, such as the subspace and sum theorems, are particular cases of both a theorem on Ind and a theorem on dim. However, the proofs of these particular cases are usually much simpler than the proofs of the corresponding general theorems. Moreover, what is more important, the number of theorems in the classical dimension theory which can be extended to metrizable spaces is larger than that of the theorems hitherto generalized in Chapters 2 and 3.

We are now going to list the counterparts of the theorems established for separable metric spaces in Section 1.5; we shall always indicate the theorems in Chapters 2 and 3 of which those counterparts are particular cases and, whenever possible, supply a simpler proof. The theorems will be formulated in terms of Ind; obviously, they could as well be formulated in terms of dim.

Let us begin, however, with a brief discussion of the status of the dimension ind in arbitrary metrizable spaces.

Theorems 2.4.4 and 4.1.3 (or 1.6.3, 3.1.29 and 4.1.3) yield

4.1.4. THEOREM. *For every strongly paracompact metrizable space X we have* $\operatorname{ind} X = \operatorname{Ind} X = \dim X$. ■

As a special case we recover the following important fact, stated above as Theorem 1.7.7.

4.1.5. THEOREM. *For every separable metrizable space X we have* $\operatorname{ind} X = \operatorname{Ind} X = \dim X$. ■

Let us note that Theorem 4.1.5 can be also deduced directly from Theorem 4.1.3 and Lemma 1.7.5.

4.1.6. REMARK. Let us repeat here that there exists a completely metrizable space X, known as Roy's space, such that $\operatorname{ind} X = 0$ and yet $\operatorname{Ind} X = \dim X = 1$ (see Section 2.4). The definition of that space and the computation of its dimensions ind and Ind is too difficult to be included in this book. ■

As we demonstrated in Chapter 2, the dimension ind exhibits pathological properties – and thus is practically of no importance – outside the class of separable metric spaces; suffice it to say that in metrizable spaces even the finite sum theorem for ind does not hold (see Problem 4.1.B). Therefore the dimension ind will not be discussed further in this chapter. It should be stressed, however, that the historical role of the small inductive dimension can hardly be overestimated. The dimension function ind was the first formal setting of the concept of dimension and a good base for the dimension theory of separable metric spaces. Besides, the dimension ind has a great intuitive appeal and yields quickly and economically the classical part of dimension theory.

We shall now list the basic properties of dimension in metrizable spaces.

First of all, let us observe that, since every subspace of a space X which satisfies condition (iii) in Proposition 4.1.2 also satisfies this condition, from Proposition 4.1.2 and Theorem 4.1.3 we obtain the following theorem (which is a particular case of Theorems 2.3.6 and 3.1.19).

4.1.7. THE SUBSPACE THEOREM. *For every subspace M of a metrizable space X we have* $\operatorname{Ind} M \leq \operatorname{Ind} X$. ■

The following simple result, whose proof repeats that of Theorem 1.5.1, is a particular case of Theorem 2.2.2.

4.1.8. THEOREM. *If X is a metrizable space and* $\operatorname{Ind} X = n \geq 1$, *then for* $k = 0, 1, \ldots, n-1$ *the space X contains a closed subspace M such that* $\operatorname{Ind} M_k = k$. ■

From Theorems 2.3.8 and 2.3.10 (or 3.1.8, 3.1.10 and 4.1.3) we obtain the countable sum theorem and the locally finite sum theorem. The reader can easily check that in the realm of metrizable spaces the locally finite sum theorem follows from the countable

sum theorem and the fact that every open cover of a metrizable space has an open σ-discrete refinement (see [GT], Theorem 4.4.1).

4.1.9. THE COUNTABLE SUM THEOREM. *If a metrizable space X can be represented as the union of a sequence $F_1, F_2, \ldots$ of closed subspaces such that* $\operatorname{Ind} F_i \leq n$ *for* $i = 1, 2, \ldots$, *then* $\operatorname{Ind} X \leq n$. ■

4.1.10. THE LOCALLY FINITE SUM THEOREM. *If a metrizable space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed subspaces such that* $\operatorname{Ind} F_s \leq n$ *for* $s \in S$, *then* $\operatorname{Ind} X \leq n$. ■

The next theorem is a common generalization of the last two theorems; it follows easily from these theorems and paracompactness of metrizable spaces (cf. Theorems 2.3.15 and 3.1.15).

4.1.11. THEOREM. *If a metrizable space X can be represented as the union of a locally countable family $\{F_s\}_{s \in S}$ of closed subspaces such that* $\operatorname{Ind} F_s \leq n$ *for* $s \in S$, *then* $\operatorname{Ind} X \leq n$. ■

Another common generalization of the countable sum theorem and the locally finite sum theorem is the following theorem, which is stronger than similar results in Chapters 2 and 3.

4.1.12. THEOREM. *If a metrizable space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of subspaces such that* $\operatorname{Ind} K_\alpha \leq n$ *and the union $\bigcup_{\beta<\alpha} K_\beta$ is closed for each $\alpha < \xi$, then* $\operatorname{Ind} X \leq n$.

PROOF. Let ρ be a metric on the space X. For each $\alpha < \xi$ define $F_\alpha = \bigcup_{\beta<\alpha} K_\beta$ and consider the sets

$$F_{i,\alpha} = K_\alpha \setminus B(F_\alpha, 1/i) = F_{\alpha+1} \setminus B(F_\alpha, 1/i) \quad \text{for } i = 1, 2, \ldots,$$

where $B(A, r)$ denotes the open r-ball about A with respect to the metric ρ. The sets $F_{i,\alpha}$ are closed and $\operatorname{Ind} F_{i,\alpha} \leq n$ for $\alpha < \xi$ and $i = 1, 2, \ldots$ Since $F_{i,\alpha} \cap B(F_{i,\beta}, 1/i) = \emptyset$ whenever $\beta < \alpha$, the family $\{F_{i,\alpha}\}_{\alpha<\xi}$ is discrete, so that the set $F_i = \bigcup_{\alpha<\xi} F_{i,\alpha}$ is closed and satisfies the inequality $\operatorname{Ind} F_i \leq n$ for $i = 1, 2, \ldots$

It remains to show that $X = \bigcup_{i=1}^{\infty} F_i$. Consider an arbitrary point $x \in X$; let α be the smallest ordinal number less than ξ such that $x \in K_\alpha$. Since $x \notin F_\alpha$, there exists an integer i such that $F_\alpha \cap B(x, 1/i) = \emptyset$, which implies that $x \notin B(F_\alpha, 1/i)$. Thus $x \in F_{i,\alpha} \subset F_i$. ■

Let us note, in connection with the last theorem, that there exist (non-separable) connected metrizable spaces which can be represented as the unions of increasing transfinite sequences $F_1 \subset F_2 \subset \ldots \subset F_\alpha \subset \ldots$, $\alpha < \omega_1$, of closed subspaces such that $\operatorname{Ind} F_\alpha = 0$ for each $\alpha < \omega_1$ (see also Problem 1.5.I(c)).

From Lemma 1.2.9 and Remark 1.2.10 one easily obtains (cf. the proof of Theorem 1.2.11) the following result, which is a particular case of Theorem 2.2.4.

4.1.13. THE SEPARATION THEOREM. *If X is a metrizable space and M is a subspace of X such that* $\operatorname{Ind} M \leq n \geq 0$, *then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that* $\operatorname{Ind}(M \cap L) \leq n - 1$. ■

We shall now characterize the dimension of subspaces of metrizable spaces in terms of σ-locally finite bases for the space (cf. Proposition 1.5.15); the characterization will be applied in the proofs of the decomposition, enlargement and Cartesian product theorems.

4.1.14. PROPOSITION. *A subspace M of a metrizable space X satisfies the inequality* $\operatorname{Ind} M \leq n \geq 0$ *if and only if X has a σ-locally finite base $\mathcal{B}$ such that* $\operatorname{Ind}(M \cap \operatorname{Fr} U) \leq n-1$ *for every $U \in \mathcal{B}$.*

PROOF. Consider a subspace M of a metrizable space X which satisfies the inequality $\operatorname{Ind} M \leq n \geq 0$; let ρ be an arbitrary metric on the space X. The space X being paracompact, for $i = 1, 2, \ldots$ there exists a locally finite open cover $\mathcal{V}_i = \{V_s\}_{s \in S_i}$ of X such that $\operatorname{mesh} \mathcal{V}_i \leq 1/i$; let $\{F_s\}_{s \in S_i}$ be a closed shrinking of the cover $\mathcal{V}_i$ (see Problem 3.1.A(b) or [GT], Theorem 1.5.18). By virtue of Theorem 4.1.13 for every $s \in S_i$ there exists a partition L_s between F_s and $X \setminus V_s$ such that $\operatorname{Ind}(L_s \cap M) \leq n-1$. Consequently, there exist open sets $U_s, W_s \subset X$ such that

$$F_s \subset U_s, \quad X \setminus V_s \subset W_s, \quad U_s \cap W_s = \emptyset \quad \text{and} \quad X \setminus L_s = U_s \cup W_s.$$

As $\operatorname{Fr} U_s \subset L_s$, we have $\operatorname{Ind}(M \cap \operatorname{Fr} U_s) \leq n-1$, and the inclusion $U_s \subset X \setminus W_s \subset V_s$, implies that $\delta(U_s) \leq 1/i$ for $s \in S_i$. The family $\mathcal{B}_i = \{U_s\}_{s \in S_i}$ is a locally finite open cover of X such that $\operatorname{mesh} \mathcal{B}_i \leq 1/i$ for $i = 1, 2, \ldots$, so that the union $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a σ-locally finite base for the space X, and $\operatorname{Ind}(M \cap \operatorname{Fr} U) \leq n-1$ for every $U \in \mathcal{B}$.

Conversely, if M is a subspace of a metrizable space X and X has a σ-locally finite base $\mathcal{B}$ such that $\operatorname{Ind}(M \cap \operatorname{Fr} U) \leq n-1$ for every $U \in \mathcal{B}$, then the family $\{M \cap U : U \in \mathcal{B}\}$ is a σ-locally finite base for the subspace M whose members have boundaries of large inductive dimension $\leq n-1$; by virtue of Lemma 2.3.16 and Theorems 4.1.9, 4.1.10 and 4.1.7, this implies that $\operatorname{Ind} M \leq n$. ■

The next theorem is a particular case of Proposition 4.1.14 (cf. Theorem 1.1.6).

4.1.15. THEOREM. *A metrizable space X satisfies the inequality* $\operatorname{Ind} X \leq n \geq 0$ *if and only if X has a σ-locally finite base $\mathcal{B}$ such that* $\operatorname{Ind} \operatorname{Fr} U \leq n-1$ *for every $U \in \mathcal{B}$.* ■

Let us note that in Proposition 4.1.14 and Theorem 4.1.15 the σ-local finiteness can be replaced by σ-discreteness (cf. [GT], Theorem 4.4.1).

4.1.16. THE FIRST DECOMPOSITION THEOREM. *A metrizable space X satisfies the inequality* $\operatorname{Ind} X \leq n \geq 0$ *if and only if X can be represented as the union of two subspaces Y and Z such that* $\operatorname{Ind} Y \leq n-1$ *and* $\operatorname{Ind} Z \leq 0$.

PROOF. Consider a metrizable space X such that $\operatorname{Ind} X \leq n \geq 0$. By virtue of Theorem 4.1.15, the space X has a σ-locally finite base $\mathcal{B}$ such that $\operatorname{Ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$. From Theorems 4.1.9 and 4.1.10 it follows that the subspace $Y = \bigcup\{\operatorname{Fr} U : U \in \mathcal{B}\}$ satisfies the inequality $\operatorname{Ind} Y \leq n-1$, and Proposition 4.1.14 shows that the subspace $Z = X \setminus Y$ satisfies the inequality $\operatorname{Ind} Z \leq 0$.

Conversly, if X is a metrizable space and $X = Y \cup Z$, where $\operatorname{Ind} Y \leq n-1$ and $\operatorname{Ind} Z \leq 0$, then $\operatorname{Ind} X \leq n$ by virtue of Theorems 4.1.13 and 4.1.7. ■

From the first decomposition theorem we obtain by induction

4.1.17. THE SECOND DECOMPOSITION THEOREM. *A metrizable space X satisfies the inequality* $\operatorname{Ind} X \leq n \geq 0$ *if and only if X can be represented as the union of $n+1$ subspaces $Z_1, Z_2, \ldots, Z_{n+1}$ such that* $\operatorname{Ind} Z_i \leq 0$ *for* $i = 1, 2, \ldots, n+1$. ■

Theorem 4.1.17 immediately yields the addition theorem, which is a particular case of Theorems 2.2.5 and 3.1.17.

4.1.18. THE ADDITION THEOREM. *If a metrizable space X is represented as the union of two subspaces X_1 and X_2, then*

$$\operatorname{Ind} X \leq \operatorname{Ind} X_1 + \operatorname{Ind} X_2 + 1. \blacksquare$$

We now turn to the enlargement theorem.

4.1.19. THE ENLARGEMENT THEOREM. *For every subspace M of a metrizable space X satisfying the inequality* $\operatorname{Ind} M \leq n$ *there exists a G_δ-set M^* in X such that $M \subset M^*$ and* $\operatorname{Ind} M^* \leq n$.

PROOF. Consider first the particular case of a subspace Z of X such that $\operatorname{Ind} Z \leq 0$. By virtue of Proposition 4.1.14 the space X has a σ-locally finite base $\mathcal{B}$ such that $Z \cap \operatorname{Fr} U = \emptyset$ for every $U \in \mathcal{B}$. The union $F = \bigcup\{\operatorname{Fr} U : U \in \mathcal{B}\}$ is an F_σ-set, and its complement $Z^* = X \setminus F$ is a G_δ-set which contains Z. From Proposition 4.1.14 it follows that $\operatorname{Ind} Z^* \leq 0$.

To complete the proof it suffices to use Theorem 4.1.17 and apply the particular case established above (cf. the proof of Theorem 1.5.11). ■

Since every compact metrizable space is separable, no non-separable metrizable space has a metrizable compactification. The next theorem is a substitute for the compactification theorem in the realm of metrizable spaces; it is a consequence of Theorem 4.1.19, Lemma 1.3.12, and the fact that each metrizable space is homeomorphic to a subspace of a completely metrizable space (see [GT], Corollary 4.3.15).

4.1.20. THE COMPLETION THEOREM. *For every metrizable space X there exists a completely metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the equalities* $\operatorname{Ind} \widetilde{X} = \operatorname{Ind} X$ *and* $w(\widetilde{X}) = w(X)$. ■

We now pass to the Cartesian product theorem. It is a particular case of Theorems 2.4.12 and 3.4.9; the proof of this particular case is much simpler than the proofs given in Sections 2.4 and 3.4.

4.1.21. THE CARTESIAN PRODUCT THEOREM. *For every pair X, Y of metrizable spaces, not bot empty, we have*

$$\operatorname{Ind}(X \times Y) \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

PROOF. The theorem is obvious if $\operatorname{Ind} X = \infty$ or $\operatorname{Ind} Y = \infty$, so that we can suppose that $k(X, Y) = \operatorname{Ind} X + \operatorname{Ind} Y$ is finite. We shall apply induction with respect to that number. If $k(X, Y) = -1$, then either $X = \emptyset$ or $Y = \emptyset$, and our inequality holds. Assume that the inequality is proved for every pair of metrizable spaces with the sum of large inductive dimensions less than $k \geq 0$ and consider metrizable spaces X and Y such that $\operatorname{Ind} X = n \geq 0$, $\operatorname{Ind} Y = m \geq 0$ and $n + m = k$. By virtue of Theorem

4.1.15 the space X has a base $\mathcal{C} = \bigcup_{i=1}^{\infty} \mathcal{C}_i$, where the families $\mathcal{C}_i$ are locally finite and $\operatorname{Ind}\operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{C}$. Similarly, the space Y has a base $\mathcal{D} = \bigcup_{j=1}^{\infty} \mathcal{D}_j$, where the families $\mathcal{D}_j$ are locally finite and $\operatorname{Ind}\operatorname{Fr} V \leq m-1$ for every $V \in \mathcal{D}$. For $i, j = 1, 2, \ldots$ the family

$$\mathcal{B}_{i,j} = \{U \times V : U \in \mathcal{C}_i \text{ and } V \in \mathcal{D}_j\}$$

consists of open subsets of the Cartesian product $X \times Y$ and is locally finite. Since

$$\operatorname{Fr}(U \times V) \subset (X \times \operatorname{Fr} V) \cup (\operatorname{Fr} U \times Y),$$

by virtue of the inductive assumption and Theorem 4.1.9 we have $\operatorname{Ind}\operatorname{Fr}(U \times V) \leq k-1$. Clearly, the family $\{B_{i,j}\}_{i,j=1}^{\infty}$ is a base for the Cartesian product $X \times Y$. From Theorem 4.1.15 it follows that $\operatorname{Ind}(X \times Y) \leq k$, and the proof is complete. ■

We shall now prove the theorem on dimension of the limit of an inverse sequence of metrizable spaces. Because of the character of the proof given below, the theorem is formulated in terms of the dimension dim.

4.1.22. THEOREM ON THE DIMENSION OF THE LIMIT OF AN INVERSE SEQUENCE. *If* $\mathbf{S} = \{X_i, \pi_j^i\}$ *is an inverse sequence of metrizable spaces* X_i *such that* $\dim X_i \leq n$ *for* $i = 1, 2, \ldots$, *then the limit* $X = \varprojlim \mathbf{S}$ *satisfies the inequality* $\dim X \leq n$.

PROOF. For $i = 1, 2, \ldots$ consider a metric ρ_i bounded by 1 on the space X_i. On the Cartesian product $\prod_{i=1}^{\infty} X_i$ and on its subspace X we shall consider the metric ρ defined by letting

$$\rho(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_i(x_i, y_i) \quad \text{for } \{x_i\}, \{y_i\} \in \prod_{i=1}^{\infty} X_i. \tag{7}$$

For $i, k = 1, 2, \ldots$ let $\mathcal{U}_{i,k}$ be an open cover of the space X_i such that $\operatorname{mesh}\mathcal{U}_{i,k} \leq 1/(2k)$. We shall define inductively a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of covers satisfying the conditions:

(8) $\mathcal{U}_i$ is an open cover of the space X_i and $\operatorname{ord}\mathcal{U}_i \leq n$.

(9) For each $U \in \mathcal{U}_i$, where $i > 1$, there exists a $V \in \mathcal{U}_{i-1}$ such that $U \subset (\pi_{i-1}^i)^{-1}(V)$.

(10) For every $j \leq i$, $\operatorname{mesh}(\{\pi_j^i(U) : U \in \mathcal{U}_i\}) \leq 1/(2i)$.

Conditions (8)–(10) are satisfied for $i = 1$ by an arbitrary open refinement $\mathcal{U}_1$ of the cover $\mathcal{U}_{1,1}$ of the space X_1 such that $\operatorname{ord}\mathcal{U}_1 \leq n$; such a refinement exists by virtue of Proposition 3.2.2. Assume that the covers $\mathcal{U}_i$ satisfying (8) (10) are defined for all $i < k > 1$. Let $\mathcal{U}_k$ be an arbitrary open refinement of the cover

$$[(\pi_{k-1}^k)^{-1}(\mathcal{U}_{k-1})] \wedge [(\pi_1^k)^{-1}(\mathcal{U}_{1,k})] \wedge [(\pi_2^k)^{-1}(\mathcal{U}_{2,k})] \wedge \ldots \wedge [(\pi_k^k)^{-1}(\mathcal{U}_{k,k})]$$

of the space X_k such that $\operatorname{ord}\mathcal{U}_k \leq n$. From the definition it easily follows that $\mathcal{U}_k$ satisfies (8)–(10) with $i = k$; thus the construction of the families $\mathcal{U}_i$ is complete.

It follows from (8) that for $i = 1, 2, \ldots$ the family $\mathcal{W}_i = \pi_i^{-1}(\mathcal{U}_i)$, where $\pi_i \colon X \to X_i$ denotes the projection, is an open cover of the space X and $\operatorname{ord}\mathcal{W}_i \leq n$. For every $W \in \mathcal{W}_{i+1}$ there exists a $U \in \mathcal{U}_{i+1}$ such that $W = \pi_{i+1}^{-1}(U)$; by virtue of (9) one can find a $V \in \mathcal{U}_i$ such that $U \subset (\pi_i^{i+1})^{-1}(V)$. Thus $W \subset \pi_{i+1}^{-1}(\pi_i^{i+1})^{-1}(V) = (\pi_i^{i+1}\pi_{i+1})^{-1}(V) = \pi_i^{-1}(V) \in \mathcal{W}_i$, which shows that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$. From (10) it follows that $\delta(W) \leq 1/(2i) + 1/2^i \leq 1/i$ for every $W \in \mathcal{W}_i$, so that $\dim X \leq n$ by virtue of Proposition 4.1.2. ■

We conclude this section by proving that for every cardinal number $\mathbf{m} \geq \aleph_0$ the Baire space $B(\mathbf{m})$ defined below is a universal space for the class of all metrizable spaces whose large inductive dimension is not larger than 0 and whose weight is not larger than $\mathbf{m}$. In the following section this result will be extended to higher dimensions; as the reader will see, the proof of the general theorem is much more difficult than that of the particular case discussed here.

We start with the definition of the Baire space.

4.1.23. EXAMPLE. For $i = 1, 2, \ldots$ let $X_i = D(\mathbf{m})$ be the discrete space of cardinality $\mathbf{m} \geq \aleph_0$ with the metric ρ_i defined by

$$\rho_i(x, y) = 1 \quad \text{if } x \neq y \quad \text{and} \quad \rho_i(x, x) = 0.$$

The Cartesian product $\prod_{i=1}^{\infty} X_i = [D(\mathbf{m})]^{\aleph_0}$ is a metrizable space; it is well known that formula (7) defines a metric ρ on that space. One can readily verify that by letting

$$\sigma(\{x_i\}, \{y_i\}) = \begin{cases} 1/k & \text{if } x_k \neq y_k \text{ and } x_i = y_i \text{ for } i < k, \\ 0 & \text{if } x_i = y_i \text{ for } i = 1, 2, \ldots, \end{cases} \tag{11}$$

one defines another metric on the set $\prod_{i=1}^{\infty} X_i$. The metrics ρ and σ are equivalent. Indeed, a sequence $\{x_i^1\}, \{x_i^2\}, \ldots$ in the Cartesian product $\prod_{i=1}^{\infty} X_i$ converges to a point $\{x_i\}$ if and only if for every i there exists a $k(i)$ such that $x_i^j = x_i$ whenever $j \geq k(i)$, and the same condition is necessary and sufficient for the convergence of $\{x_i^1\}, \{x_i^2\}, \ldots$ to $\{x_i\}$ with respect to σ.

The Cartesian product $[D(\mathbf{m})]^{\aleph_0}$ with the metric σ defined in (11) is called the *Baire space of weight* $\mathbf{m}$ and is denoted by $B(\mathbf{m})$. The reader can easily check that the weight of the space $B(\mathbf{m})$ is indeed equal to $\mathbf{m}$. Let us note that by virtue of Proposition 1.3.13 the Baire space $B(\aleph_0)$ is homeomorphic to the space of irrational numbers.

We shall show that $\operatorname{Ind} B(\mathbf{m}) = 0$. Consider a pair $x = \{x_i\}$, $y = \{y_i\}$ of points of $B(\mathbf{m})$ and a real number r satisfying $0 < r \leq 1$. If the intersection $B(x, r) \cap B(y, r)$ is non-empty, then there exists a point $z = \{z_i\} \in B(\mathbf{m})$ such that $x_1 = z_1 = y_1$, $x_2 = z_2 = y_2, \ldots, x_k = z_k = y_k$, where k is the integer satisfying $1/(k+1) < r \leq 1/k$; thus we have $B(x, r) = B(y, r)$. Hence in $B(\mathbf{m})$ two r-balls either are disjoint or coincide. In particular, for $i = 1, 2, \ldots$ the family $\mathcal{B}_i = \{B(x, 1/i) : x \in B(\mathbf{m})\}$ is an open cover of $B(\mathbf{m})$ which consists of pairwise disjoint sets. The union $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ is a σ-locally finite base for $B(\mathbf{m})$ which consists of open-and-closed sets, so that $\operatorname{Ind} B(\mathbf{m}) \leq 0$ by virtue of Theorem 4.1.15. ■

4.1.24. THEOREM. *For every cardinal number* $\mathbf{m} \geq \aleph_0$ *the Baire space* $B(\mathbf{m})$ *is a universal space for the class of all metrizable spaces whose large inductive dimension is not larger than* 0 *and whose weight is not larger than* $\mathbf{m}$.

PROOF. By virtue of the last example it suffices to show that every metrizable space X such that $\operatorname{Ind} X = 0$ and $w(X) = \mathbf{m}$ is embeddable in X.

Let ρ be an arbitrary metric on the space X and $\mathcal{B}$ an arbitrary base for X such that $|\mathcal{B}| = \mathbf{m}$. From Theorem 1.6.10 (or 4.1.3) and Proposition 3.2.2 it follows that the cover $\{U \in \mathcal{B} : \delta(U) < 1/i\}$ of the space X has an open shrinking $\mathcal{B}_i$ consisting of pairwise disjoint sets. Adjoining to $\mathcal{B}_i$ an appropriate number of copies of the empty

set if necessary, we can assume that $\mathcal{B}_i = \{U_{i,s}\}_{s \in X_i}$, where $X_i = D(\mathbf{m})$ is the discrete space of cardinality $\mathbf{m}$ used in Example 4.1.23 to define $B(\mathbf{m})$.

By assigning to each point $x \in X$ the element $s \in X_i$ such that $x \in U_{i,s}$ we define a continuous mapping $f_i\colon X \to X_i$. For every $x \in X$ and every closed set $F \subset X$ which does not contain x there exists a natural number i such that $\rho(x, F) > 1/i$. The set $U_{i,s}$ that contains the point x is disjoint from F, so that $f_i(x) = s \notin f_i(F) = \overline{f_i(F)}$. Thus the family $\{f_i\}_{i=1}^{\infty}$ separates points and closed sets, which implies that the mapping $F\colon X \to \prod_{i=1}^{\infty} X_i = B(\mathbf{m})$ defined by letting $F(x) = (f_1(x), f_2(x), \ldots)$ is a homeomorphic embedding (see [GT], Theorem 2.3.20). ■

Since the Cartesian product of $\aleph_0$ copies of $B(\mathbf{m})$ is homeomorphic to $B(\mathbf{m})$, Theorem 4.1.24 yields the following

4.1.25. THEOREM. *The Cartesian product $X = \prod_{i=1}^{\infty} X_i$ of a countable family $\{X_i\}_{i=1}^{\infty}$ of metrizable spaces satisfies the equality* $\operatorname{Ind} X = 0$ *if and only if* $\operatorname{Ind} X_i = 0$ *for* $i = 1, 2, \ldots$ ■

Historical and bibliographic notes

Lemma 4.1.1 was established by Nagami and Roberts in [1967]. The equivalence of conditions (i) and (ii) in Proposition 4.1.2 was proved by Dowker and Hurewicz in [1956], the equivalence of conditions (i) and (iii) – by Vopěnka in [1959]. Theorem 4.1.3 was established independently by Katětov in [1952] (announcement in [1951]) and by Morita in [1954]. Theorem 4.1.4 was given by Morita in [1950a]. References concerning Roy's space cited in Remark 4.1.6 can be found in the notes to Section 2.4. Theorems 4.1.7–4.1.11, 4.1.13 and 4.1.18 are special cases of the corresponding theorems established in Chapters 2 and 3. Theorem 4.1.12 was given by Nagami in [1957]; the example mentioned in connection with that theorem can be found in R. Pol's paper [1975]. Theorems 4.1.14 and 4.1.15 were proved by Morita in [1954]. All the remaining theorems in this section, except for Theorem 4.1.22, which was proved by Nagami in [1959], were established independently by Katětov in [1952] and by Morita in [1954].

Problems

4.1.A (Engelking [1973], Przymusiński [1974]). Let X be a metrizable space and ρ a metric on the space X; let $\operatorname{ds}(X, \rho) \leq n$ denote that the space X has a sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ with the properties stated in condition (ii) of Proposition 4.1.2.

(a) Show by modifying the proof of Theorem 4.1.3 that if $\operatorname{ds}(X, \rho) \leq n \geq 0$ then for every pair A, B of closed subsets of X satisfying $\rho(A, B) > 0$ there exists an open set $U \subset X$ such that $A \subset U \subset X \setminus B$ and $\operatorname{ds}(\operatorname{Fr} U, \rho) \leq n - 1$.

(b) Apply (a) to prove the Katětov–Morita theorem without using Lemma 4.1.1.

Hint. Observe first that if $\dim X \leq n$, then $\operatorname{ds}(X, \rho) \leq n$ for every metric ρ on the space X; then apply (a) to prove by induction that the inequality $\operatorname{ds}(X, \rho) \leq n \geq 0$ implies that X has a σ-locally finite base $\mathcal{B}$ such that $\dim \operatorname{Fr} U \leq n - 1$ for every $U \in \mathcal{B}$, and show that the existence of such a base implies the inequality $\operatorname{Ind} X \leq n$.

One can also apply (a) to prove by induction that if $\operatorname{ds}(X,\rho) \leq n \geq 0$ then, for every closed set $A \subset X$ and each open set $V \subset X$ that contains the set A, there exists an open set $U \subset X$ such that $A \subset U \subset V$ and $\operatorname{Ind}\operatorname{Fr} U \leq n-1$. To this end, for $i = 1, 2, \ldots$ define $A_i = B(A, 1/i)$ and $A'_i = B(X \setminus V, 1/i)$, consider open sets $W_i, W'_i \subset X$ such that $A_{i+1} \subset W_i \subset A_i$, $\operatorname{ds}(\operatorname{Fr} W_i, \rho) \leq n-1$ and $A'_{i+1} \subset W'_i \subset A'_i$, $\operatorname{ds}(\operatorname{Fr} W'_i, \rho) \leq n-1$, and let $U = \bigcup_{i=1}^{\infty}(W_i \setminus \overline{W'_i})$.

4.1.B (van Douwen [1973], Przymusiński [1974]). Applying the existence of a metrizable space X with the properties described in Remark 4.1.6, define a metrizable space Y with $\operatorname{ind} Y = 1$ which can be represented as the union of two closed subspaces Y_1 and Y_2 such that $\operatorname{ind} Y_1 = \operatorname{ind} Y_2 = 0$ and which contains a point a such that $\operatorname{ind}(Y \setminus \{a\}) = 0$.

Hint. Consider a pair A, B of disjoint closed subsets of X which cannot be enlarged to disjoint open-and-closed sets, and replace the set A by a point a in such a way as to obtain a metrizable space in which B is still closed (cf. Lemma 5.3.8).

Remark. It is an open problem if the enlargement theorem for ind holds in all metrizable spaces (see Problem 7.2.G).

4.1.C. Let Y be the space considered in Problem 4.1.B and let $M = Y \setminus \{a\}$. Show that, though $\operatorname{ind} M = 0$, there exists a neighbourhood V of the point a in Y such that for every open set $U \subset Y$ satisfying $a \in U \subset V$ we have $M \cap \operatorname{Fr} U \neq \emptyset$ (cf. Propositions 1.2.12 and 1.5.4 and Problem 2.2.A(a)).

4.1.D (R. Pol [1981]). Prove that if a metrizable space X can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed subspaces such that $\operatorname{Ind} F_s \leq n$ for $s \in S$ and if there exists a point-countable open cover $\{U_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then $\operatorname{Ind} X \leq n$.

Hint (Hansell [1974]). Prove that for every point-countable family $\{U_s\}_{s\in S}$ of open subsets of a metrizable space X there exist open sets $U_{s,n}$, where $s \in S$ and $n = 1, 2, \ldots$, such that the family $\{U_{s,n}\}_{s\in S}$ is locally finite for $n = 1, 2, \ldots$ and $U_s = \bigcup_{n=1}^{\infty} U_{s,n}$ for each $s \in S$. To that end, take a base $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ for the space X, where the families $\mathcal{B}_i$ are locally finite, for every non-empty $U \in \mathcal{B}$ consider a one-to-one mapping j_U of the set $\{s \in S : U \subset U_s\}$ to the set of natural numbers, and let $U_{s,i,j} = \bigcup\{U \in \mathcal{B}_i : U \subset U_s$ and $j_U(s) = j\}$.

4.1.E. Deduce from Theorems 4.1.3 and 4.1.13 and Remark 1.7.10 that a metrizable space X satisfies the inequality $\operatorname{Ind} X \leq n \geq 0$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \ldots, (A_{n+1}, B_{n+1})$ of $n+1$ pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots, L_{n+1}$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{n+1} L_i = \emptyset$.

4.1.F (Engelking [1973]). Show that a metrizable space X satisfies the inequality $\operatorname{Ind} X \leq n \geq 0$ if and only if X has a σ-locally finite network $\mathcal{N}$ such that $\operatorname{Ind}\operatorname{Fr} M \leq n-1$ for every $M \in \mathcal{N}$.

4.1.G (Levšenko [1969]; for $n = 0$, Levšenko and Smirnov [1966]; implicitly, for separable spaces, Popruženko [1931]). (a) Prove that for every non-empty closed subset A of a

metrizable space X satisfying the inequality $\operatorname{Ind} X \leq n \geq 0$ there exists a closed subset B of the subspace $X \setminus A$ of X such that $\operatorname{Ind} B \leq n-1$ and A is a retract of $X \setminus B$.

Hint. Consider a metric ρ on the space X and define a sequence $U_1, U_2, \ldots$ of open subsets of X such that $A \subset U_i \subset B(A, 1/i)$, $\overline{U}_i \subset U_{i-1}$, where $U_0 = X$, and $\operatorname{Ind} \operatorname{Fr} U_i \leq n-1$ for $i = 1, 2, \ldots$; for $i = 1, 2, \ldots$ consider a locally finite open cover $\mathcal{U}_i$ of the subspace $\overline{U}_{i-1} \setminus U_i$ of X such that $\operatorname{mesh} \mathcal{U}_i \leq 1/i$ and $\operatorname{Ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{U}_i$. Define $B = \bigcup_{i=1}^{\infty} \operatorname{Fr} U_i \cup \bigcup \{\operatorname{Fr} U : U \in \bigcup_{i=1}^{\infty} \mathcal{U}_i\}$ and see the hint to Problem 1.3.C(a).

(b) Show that if a metrizable space X has the property that for every non-empty closed set $A \subset X$ there exists a closed subset B of the subspace $X \setminus A$ of X such that $\operatorname{Ind} B \leq n-1$ and A is a retract of $X \setminus B$, then $\operatorname{Ind} X \leq n$.

4.1.H (Hausdorff [1934], de Groot [1956]). A metric ρ on a set X is called *non-Archimedean* if $\rho(x, z) \leq \max[\rho(x, y), \rho(y, z)]$ for all $x, y, z \in X$.

Let X be a metrizable space; show that on the space X there exists a non-Archimedean metric if and only if $\operatorname{Ind} X \leq 0$.

Hint. There exists a non-Archimedean metric on the Baire space $B(\mathbf{m})$.

4.2. Characterizations of dimension in metrizable spaces. The universal space theorems

In the first part of this section we shall establish four characterizations of the dimension dim in metrizable spaces formulated in terms of the existence of special bases and sequences of covers, and we shall survey characterizations of dim in terms of special metrics.

We start with characterizations in terms of σ-locally finite bases with special properties. First we shall prove a lemma related to Theorem 3.2.5.

4.2.1. LEMMA. *If a normal space X satisfies the inequality $\dim X \leq n \geq 0$, then for every σ-locally finite family $\{U_s\}_{s \in S}$ of open subsets of X and every family $\{F_s\}_{s \in S}$ of closed subsets of X such that $F_s \subset U_s$ for $s \in S$ there exists a family $\{V_s\}_{s \in S}$ of open subsets of X such that $F_s \subset V_s \subset \overline{V}_s \subset U_s$ for $s \in S$ and $\operatorname{ord}(\{\operatorname{Fr} V_s\}_{s \in S}) \leq n-1$.*

PROOF. Let $S = \bigcup_{i=1}^{\infty} S_i$, where $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and the family $\{U_s\}_{s \in S_i}$ is locally finite for $i = 1, 2, \ldots$ Applying Theorem 3.2.5, one can easily define by induction open sets $V_{s,i}, W_{s,i} \subset X$, where $s \in T_i = S_1 \cup S_2 \cup \ldots \cup S_i$ and $i = 1, 2, \ldots$, satisfying the following conditions:

(1) $$F_s \subset V_{s,i} \subset \overline{V}_{s,i} \subset W_{s,i} \subset \overline{W}_{s,i} \subset U_s \quad \text{for } s \in S_i.$$

(2) $$\overline{V}_{s,i-1} \subset V_{s,i} \subset \overline{V}_{s,i} \subset W_{s,i} \subset \overline{W}_{s,i} \subset W_{s,i-1} \quad \text{for } s \in T_{i-1} \text{ and } i > 1.$$

(3) $$\operatorname{ord}(\{\overline{W}_{s,i} \setminus V_{s,i}\}_{s \in T_i} \leq n-1.$$

For every $s \in S$ consider the open set

$$V_s = \bigcup_{j=i}^{\infty} V_{s,j}, \quad \text{where } s \in S_i.$$

Conditions (1) and (2) imply that $F_s \subset V_s \subset \overline{V}_s \subset U_s$ for $s \in S$ and that $\operatorname{Fr} V_s = \overline{V}_s \setminus V_s \subset \overline{W}_{s,j} \setminus V_{s,j}$ for $j \geq i$, where $s \in S_i$. The last inclusion together with (3) yields the inequality $\operatorname{ord}(\{\operatorname{Fr} V_s\}_{s\in S}) \leq n-1$. ■

4.2.2. THEOREM. *For every metrizable space X and each integer $n \geq 0$ the following conditions are equivalent:*

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *The space X has a σ-locally finite base $\mathcal{B}$ such that* $\operatorname{ord}(\{\operatorname{Fr} U : U \in \mathcal{B}\}) \leq n-1$.

(iii) *The space X has a σ-locally finite base $\mathcal{B}$ such that* $\dim \operatorname{Fr} U \leq n-1$ *for every* $U \in \mathcal{B}$.

PROOF. We shall show first that (i)⇒(ii). Consider a metrizable space X such that $\dim X \leq n$; let ρ be a metric on the space X. The space X being paracompact, for $i = 1, 2, \ldots$ there exists a locally finite open cover $\mathcal{U}_i = \{U_s\}_{s\in S_i}$ of X such that $\operatorname{mesh}\mathcal{U}_i \leq 1/i$; obviously, one can assume that $S_i \cap S_j = \emptyset$ whenever $i \neq j$. Let $\{F_s\}_{s\in S_i}$ be a closed shrinking of the cover $\mathcal{U}_i$ and let $S = \bigcup_{i=1}^{\infty} S_i$. Applying Lemma 4.2.1 to the families $\{U_s\}_{s\in S}$ and $\{F_s\}_{s\in S}$ we obtain a family $\mathcal{B} = \{V_s\}_{s\in S}$ which has the properties stated in (ii).

The implication (iii)⇒(i) follows from Lemmas 2.3.16 and 3.1.27 and the sum theorems for dim.

To prove that (ii)⇒(iii) we shall apply induction with respect to n. Conditions (ii) and (iii) are equivalent if $n = 0$, because then they both mean that all members of $\mathcal{B}$ are open-and-closed. Assume that the implication (ii)⇒(iii) is proved for all metrizable spaces and every $n < m$, and consider a metrizable space X which has a σ-locally finite base $\mathcal{B}$ such that $\operatorname{ord}(\{\operatorname{Fr} U : U \in \mathcal{B}\}) \leq m$. For every $U_0 \in \mathcal{B}$ the family $\mathcal{B}_0 = \{X_0 \cap U : U \in \mathcal{B}\}$, where $X_0 = \operatorname{Fr} U_0$, is a σ-locally finite base for the space X_0. Since the family of boundaries of members of $\mathcal{B}_0$ in the space X_0 has order $\leq m-1$, it follows from the inductive assumption and the implication (iii)⇒(i) that $\dim X_0 \leq m$. Thus the space X satisfies (iii) with $n = m+1$ and the proof that (ii)⇒(iii) is complete. ■

Let us note that the equivalence of conditions (i) and (iii) in Theorem 4.2.2 follows immediately from Theorems 4.1.3 and 4.1.15. In our proof, however, we applied Lemma 3.1.27, rather than those two theorems, in order to prepare the ground for another proof of the Katětov–Morita theorem (cf. Problem 4.2.A(a)). Let us also note that the σ-local finiteness of $\mathcal{B}$ in conditions (i) and (ii) can be replaced by its σ-discreteness (cf. Problem 4.2.A(b)).

We shall now introduce two topological notions which are applied in the next theorem. A sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of covers of a topological space X is a called a *development* for the space X if all covers $\mathcal{W}_i$ are open and for every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that $\operatorname{St}(x, \mathcal{W}_i) \subset U$. One easily observes that a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of a topological space X is a development for X if and only if for every point $x \in X$ each family $\{W_i\}_{i=1}^{\infty}$ of open subsets of X such that $x \in W_i \in \mathcal{W}_i$ for $i = 1, 2, \ldots$ is a base for X at the point x. A sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of covers of a topological space X is called a *strong development* for the space X if all covers $\mathcal{W}_i$ are open and for every point $x \in X$ and

each neighbourhood U of the point x there exist a neighbourhood V of x and a natural number i such that $\mathrm{St}(V, \mathcal{W}_i) \subset U$. Clearly, every strong development is a development.

4.2.3. THEOREM. *For every metrizable space X the following conditions are equivalent:*

(i) *The space X satisfies the inequality* $\dim X \leq n$.

(ii) *There exists a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ for the space X such that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ and* $\operatorname{ord} \mathcal{W}_i \leq n$ *for* $i = 1, 2, \ldots$

(iii) *There exists a strong development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ for the space X such that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ and* $\operatorname{ord} \mathcal{W}_i \leq n$ *for* $i = 1, 2, \ldots$

PROOF. We begin with the implication (i)⇒(ii). Consider a metrizable space X such that $\dim X \leq n$ and an arbitrary metric ρ on the space X. We shall define inductively a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of X which has the properties stated in (ii). Assume that $k = 1$ or that $k > 1$ and the covers $\mathcal{W}_i$ are defined for all $i < k$. The space X being paracompact, the open cover $\mathcal{W}_{k-1} \wedge \{B(x, 1/k)\}_{x \in X}$ of X, where $\mathcal{W}_0 = \{X\}$, has an open star refinement (see [GT], Theorem 5.1.12) which by virtue of Proposition 3.2.2 has an open refinement $\mathcal{W}_k$ such that $\operatorname{ord} \mathcal{W}_k \leq n$. The sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ thus obtained has the required properties, so that (i)⇒(ii).

The implication (ii)⇒(iii) follows from the fact that if $\mathrm{St}(x, \mathcal{W}_i) \subset U$ and $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$, then $\mathrm{St}(V, \mathcal{W}_{i+1}) \subset U$ for any $V \in \mathcal{W}_{i+1}$ such that $x \in V$.

Finally, the implication (iii)⇒(i) follows from Lemma 4.1.1, because – as one readily sees by applying the definition twice – if $\mathcal{W}_1, \mathcal{W}_2, \ldots$ is a strong development for a space X and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$, then the family $\{\mathrm{St}(W, \mathcal{W}_i) : x \in W \in \mathcal{W}_i,\ i = 1, 2, \ldots\}$ is a base for X at the point x. ■

Let us note that by virtue of the Nagata–Smirnov theorem ([GT], Theorem 4.4.7), Theorem 4.2.2 holds for every regular space X if condition (i) is replaced by:

(i′) *The space X is metrizable and satisfies the inequality* $\dim X \leq n$.

Similarly, by virtue of appropriate metrization theorems (see [GT], Corollary 5.4.10 and Theorem 5.4.2), Theorem 4.2.3 holds for every T_0-space X if condition (i) is replaced by condition (i′).

As follows from Problem 4.1.H, the class of all metrizable spaces X such that $\dim X \leq 0$ can be characterized in terms of the existence of special metrics. In this connection it is natural to ask whether the class of all metrizable spaces X such that $\dim X \leq n$ can be characterized in a similar manner. Investigations in this direction led to a group of interesting theorems, which are surveyed below; the proofs of those theorems are too complicated to be reproduced in this book. Thus, a metrizable space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if there exists a metric ρ on the space X which satisfies any one of the following conditions:

(μ_1) For every point $x \in X$ and each positive number r we have $\dim \mathrm{Fr}\, B(x, r) \leq n - 1$; moreover, $\overline{\bigcup_{x \in X_0} B(x, r)} = \bigcup_{x \in X_0} \overline{B(x, r)}$ for every $X_0 \subset X$.

(μ_2) For every closed set $A \subset X$ and each positive number r we have $\dim \mathrm{Fr}\, B(A, r) \leq n - 1$.

(μ_3) For every point $x \in X$, each positive number r and every sequence $y_1, y_2, \ldots, y_{n+2}$ of $n+2$ points of X satisfying the inequality $\rho(y_i, B(x, r/2)) < r$ for $i = 1, 2, \ldots, n+2$, there exist natural numbers $i, j \leq n+2$ such that $i \neq j$ and $\rho(y_i, y_j) < r$.

(μ_4) For every point $x \in X$ and every sequence $y_1, y_2, \ldots, y_{n+2}$ of $n+2$ points of X there exist natural numbers $i, j \leq n+2$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_i)$.

One easily verifies that every metric ρ which satisfies (μ_1) also satisfies (μ_2), and that if on a space X there exists a metric ρ which satisfies (μ_2), then $\dim X \leq n$ (see Problem 4.2.C). Let us note that a separable metrizable space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if there exists a metric ρ on the space X which satisfies the first part of (μ_1); it is not known whether the second part of (μ_1) can be omitted also in the case of an arbitrary metrizable space.

An interesting open problem connected with conditions (μ_3) and (μ_4) is whether the inequality $\dim X \leq n$ follows from the existence of a metric ρ on the space X which satisfies the following condition:

(μ_5) For every point $x \in X$, each positive number r and every sequence $y_1, y_2, \ldots, y_{n+2}$ of $n+2$ points of X satisfying the inequality $\rho(x, y_i) < r$ for $i = 1, 2, \ldots, n+2$, there exist natural numbers $i, j \leq n+2$ such that $i \neq j$ and $\rho(y_i, y_j) < r$.

One easily verifies that every metric ρ which satisfies (μ_3) satisfies also (μ_5), so that the problem stated above is equivalent to the question whether the existence of a metric ρ on a space X which satisfies (μ_5) means exactly that $\dim X \leq n$. One can also readily check that condition (μ_5) is equivalent to the following condition:

(μ_5') For every point $x \in X$ and every sequence $y_1, y_2, \ldots, y_{n+2}$ of $n+2$ points of X there exist natural numbers $i, j, k \leq n+2$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$,

and that every metric ρ which satisfies (μ_4) also satisfies (μ_5') and (μ_5). Let us note that if on a separable metrizable space X there exists a totally bounded metric ρ which satisfies (μ_5), then $\dim X \leq n$ (see Problem 4.2.E); it is not known if the assumption of total boundedness can be omitted.

We now turn to the universal space theorems. The first of these, which just grants the existence of a universal space, will be deduced from a factorization theorem. The proof of the second one, asserting that a specific space is universal, will be much longer.

We start with a simple lemma on paracompact spaces.

4.2.4. LEMMA. *Every locally finite open cover $\mathcal{U}$ of a paracompact space X has a locally finite open star refinement $\mathcal{V}$ such that $|\mathcal{V}| \leq \max(|\mathcal{U}|, \aleph_0)$.*

PROOF. Let $\mathcal{W}$ be an open star refinement of $\mathcal{U}$ (see [GT], Theorem 5.1.12). For every pair $\mathcal{U}_0, \mathcal{U}_1$ of finite subfamilies of the cover $\mathcal{U}$ denote by $\mathcal{W}(\mathcal{U}_0, \mathcal{U}_1)$ the family of all $W \in \mathcal{W}$ such that $\mathcal{U}_0 = \{U \in \mathcal{U} : \overline{W} \subset U\}$ and $\mathcal{U}_1 = \{U \in \mathcal{U} : \mathrm{St}(\overline{W}, \mathcal{W}) \subset U\}$. From the local finiteness of $\mathcal{U}$ it follows that the sets of the form $\bigcup \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1)$ constitute an open cover $\mathcal{V}'$ of the space X such that $|\mathcal{V}'| \leq \max(|\mathcal{U}|, \aleph_0)$. We shall show that $\mathcal{V}'$ is a star refinement of $\mathcal{U}$.

Consider a set $V = \bigcup \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1) \in \mathcal{V}'$. Let U be an arbitrary member of $\mathcal{U}_1$; clearly

$$\mathrm{St}(W, \mathcal{W}) \subset U \quad \text{for every } W \in \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1). \tag{4}$$

For every set $V' = \bigcup \mathcal{W}(\mathcal{U}_0', \mathcal{U}_1') \in \mathcal{V}'$ which intersects V there exist $W \in \mathcal{W}(\mathcal{U}_0, \mathcal{U}_1)$ and $W' \in \mathcal{W}(\mathcal{U}_0', \mathcal{U}_1')$ such that $W \cap W' \neq \emptyset$. From (4) it follows that $W' \subset U$, so that $U \in \mathcal{U}_0'$, which implies that $V' \subset U$. Hence, $\mathrm{St}(V, \mathcal{V}') \subset U$.

To complete the proof it suffices to consider a locally finite open shrinking $\mathcal{V}$ of the cover $\mathcal{V}'$. ■

4.2.5. PASYNKOV'S FACTORIZATION THEOREM. *For every continuous mapping $f: X \to Y$ of a metrizable space X to a metrizable space Y there exist a metrizable space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that* $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$, $g(X) = Z$ *and* $f = hg$.

PROOF. If $\dim X = \infty$ or $w(Y) < \aleph_0$, then $Z = f(X)$, $g = f$ and $h = \mathrm{id}_Z$ satisfy the assertion of the theorem. Thus one can suppose that $\dim X = n < \infty$ and $w(Y) = \mathbf{m} \geq \aleph_0$. For $i = 1, 2, \ldots$ consider a locally finite open cover $\mathcal{U}_i$ of the space Y such that $\mathrm{mesh}\,\mathcal{U}_i < 1/i$ and $|\mathcal{U}_i| \leq \mathbf{m}$. Applying Lemma 4.2.4 and Proposition 3.2.2 one can easily define by induction a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of locally finite open covers of X such that for $i = 1, 2, \ldots$,

$$\mathrm{ord}\,\mathcal{V}_i \leq n, \quad |\mathcal{V}_i| \leq \mathbf{m} \tag{5}$$

and

$$\mathcal{V}_{i+1} \text{ is a star refinement of } \mathcal{V}_i \wedge f^{-1}(\mathcal{U}_{i+1}). \tag{6}$$

We shall now consider another topology on the set X, coarser than the original one, which is defined by declaring that a set $U \subset X$ is open if for every $x \in U$ there exists a natural number i such that $\mathrm{St}(x, \mathcal{V}_i) \subset U$. The set X with this new topology will be denoted by X'; let us note that generally X' is not a T_0-space. We shall show that for every $A \subset X'$ the interior of A in the space X' coincides with the set

$$A^* = \{x \in X : \text{there exists an } i \text{ such that } \mathrm{St}(x, \mathcal{V}_i) \subset A\}.$$

Obviously, it suffices to verify that A^* is an open subset of X'. Consider an arbitrary point $x \in A^*$ and an i such that $\mathrm{St}(x, \mathcal{V}_i) \subset A$. For every point $y \in \mathrm{St}(x, \mathcal{V}_{i+1})$ we have $\mathrm{St}(y, \mathcal{V}_{i+1}) \subset \mathrm{St}(x, \mathcal{V}_i) \subset A$, so that $y \in A^*$. Thus $\mathrm{St}(x, \mathcal{V}_{i+1}) \subset A^*$, which shows that A^* is an open subset of X'.

Let $\mathcal{V}_i^* = \{V^* : V \in \mathcal{V}_i\}$ for $i = 1, 2, \ldots$ As for every $W \in \mathcal{V}_{i+1}$ there exists a $V \in \mathcal{V}_i$ such that $\mathrm{St}(W, \mathcal{V}_{i+1}) \subset V$ and consequently $W \subset V^* \in \mathcal{V}_i^*$, the family $\mathcal{V}_i^*$ is an open cover of the space X'. Since A^* is the interior of A in X', from (6) it follows that for $i = 1, 2, \ldots$,

$$\mathcal{V}_{i+1}^* \text{ is a star refinement of } \mathcal{V}_i^*. \tag{7}$$

Note that by letting $f'(x) = f(x)$ for $x \in X'$ we define a continuous mapping $f': X' \to Y$; indeed, it follows from (6) that $\delta(f'(V^*)) < 1/(i+1)$ for every $V^* \in \mathcal{V}_{i+1}^*$.

For $x, y \in X'$ define

$$xEy \quad \text{if and only if} \quad y \in \mathrm{St}(x, \mathcal{V}_i) \text{ for every } i.$$

One easily checks that E is an equivalence relation on the space X'. Thus E determines a decomposition of X' into equivalence classes $[x]$, where

$$[x] = \bigcap_{i=1}^{\infty} \mathrm{St}(x, \mathcal{V}_i) \quad \text{for } x \in X'.$$

Let Z be the quotient space X'/E and $g: X \to Z$ the composition of the identity mapping $i: X \to X'$ and the natural quotient mapping $g': X' \to X'/E$. By virtue of (6), $f'(x) = f'(y)$ whenever xEy, so that by letting $h([x]) = f'(x)$ we define a mapping $h: Z \to Y$; from the relation $hg' = f'$ it follows that h is continuous. Obviously, $g(X) = Z$ and $f = hg$. Let us note that

$$g'^{-1}g'(A^*) = A^* \quad \text{for every } A \subset X', \tag{8}$$

so that

$$\text{the set } g'(A^*) \text{ is open in } Z \text{ for every } A \subset X'. \tag{9}$$

We shall show that Z is a metrizable space. To begin, observe that Z is a T_1-space. Indeed, if $[x] \neq [y]$ then there exists an i such that $y \notin \mathrm{St}(x, \mathcal{V}_i)$, and then the set $g'(V^*)$, where $V = \mathrm{St}(x, \mathcal{V}_i)$, is a neighbourhood of the point $[x]$ which does not contain the point $[y]$. Let $\mathcal{W}_i = \{g'(V^*) : V \in \mathcal{V}_i\}$ for $i = 1, 2, \ldots$; from (9) it follows that $\mathcal{W}_i$ is an open cover of the space Z. Now, from the definition of the topology on X' it follows that the sequence $\mathcal{V}_1^*, \mathcal{V}_2^*, \ldots$ is a development for the space X', so that by virtue of (8) the sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ is a development for the space Z; moreover, (7) implies that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$ Hence, the space Z is metrizable (see [GT], Corollary 5.4.10).

From the first part of (5) it follows that $\operatorname{ord} \mathcal{W}_i \leq n$ for $i = 1, 2, \ldots$, so that $\dim Z \leq n$ by virtue of Theorem 4.2.3. To complete the proof it suffices to observe that since the family $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$ is a base for the space Z and $|\mathcal{B}| \leq \mathbf{m}$, we have $w(Z) \leq w(Y)$. ■

Let us note that in the proof of Pasynkov's factorization theorem only the paracompactness of the space X, and not its metrizability, was used; it turns out that the theorem holds under the even weaker assumption of normality of X (see Problem 4.2.F).

We shall now define for each $\mathbf{m} \geq \aleph_0$ the hedgehog $J(\mathbf{m})$, a space that is going to be used in the proof of the first universal space theorem.

4.2.6. EXAMPLE. Consider a set S of cardinality $\mathbf{m}$ and let $I_s = I \times \{s\}$ for each $s \in S$, where I is the closed unit interval $[0, 1]$. In the union $\bigcup_{s \in S} I_s$ identify the set $\{0\} \times S$ to a single point 0, thus obtaining the set $J(\mathbf{m}) = \bigcup_{s \in S} I_s$ with $\bigcap_{s \in S} I_s = \{0\}$, and for $x, y \in J(\mathbf{m})$ let

$$\rho(x, y) = \begin{cases} |x - y| & \text{if } x, y \in I_s \text{ for some } s \in S, \\ x + y & \text{if } x \in I_{s_1},\ y \in I_{s_2} \text{ and } s_1 \neq s_2. \end{cases}$$

One easily checks that ρ is a metric; the set $J(\mathbf{m})$ with this metric is called the *hedgehog of spininess* $\mathbf{m}$. The importance of hedgehogs comes from the fact that the Cartesian product $[J(\mathbf{m})]^{\aleph_0}$ of $\aleph_0$ copies of the hedgehog $J(\mathbf{m})$ is universal for the class of all metrizable spaces of weight $\mathbf{m} \geq \aleph_0$ (see [GT], Theorem 4.4.9). ■

4.2.7. THE FIRST UNIVERSAL SPACE THEOREM. *For every integer $n \geq 0$ and every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a universal space $J_n(\mathbf{m})$ for the class of all metrizable spaces whose covering dimension is not larger than n and whose weight is not larger than $\mathbf{m}$.*

PROOF. Let $\{X_s\}_{s \in S}$ be the family of all subspaces of the Cartesian product $[J(\mathbf{m})]^{\aleph_0}$ of $\aleph_0$ copies of the hedgehog $J(\mathbf{m})$ whose covering dimension is not larger than n, and let $i_s: X_s \to [J(\mathbf{m})]^{\aleph_0}$ be the embedding of X_s in $[J(\mathbf{m})]^{\aleph_0}$. Consider the sum $X = \bigoplus_{s \in S} X_s$ and the mapping $i: X \to [J(\mathbf{m})]^{\aleph_0}$ defined by letting $i(x) = i_s(x)$ for $x \in X_s$. Since $\dim X \leq n$, by virtue of Theorem 4.2.5 there exist a metrizable space $J_n(\mathbf{m})$ and continuous mappings $g: X \to J_n(\mathbf{m})$ and $h: J_n(\mathbf{m}) \to [J(\mathbf{m})]^{\aleph_0}$ such that $\dim J_n(\mathbf{m}) \leq \dim X = n$, $w(J_n(\mathbf{m})) \leq w([J(\mathbf{m})]^{\aleph_0}) = \mathbf{m}$ and $f = hg$.

Consider now an arbitrary metrizable space Y such that $\dim Y \leq n$ and $w(Y) \leq \mathbf{m}$. Since Y is embeddable in $[J(\mathbf{m})]^{\aleph_0}$, there exists an $s \in S$ such that X_s is homeomorphic to Y. The composition $h_0 g_0$ of the restrictions $g_0 = g|X_s: X_s \to g(X_s) \subset J_n(\mathbf{m})$ and $h_0 = h|g(X_s): g(X_s) \to X_s \subset [J(\mathbf{m})]^{\aleph_0}$ is a homeomorphism, so that g_0 is also a homeomorphism. Thus $J_n(\mathbf{m})$ is the required universal space. ■

The last theorem merely asserts that there exists a universal space, but yields no information what it looks like. We are now going to define directly a universal space $N_n(\mathbf{m})$ for the class of all metrizable spaces whose dimension is not larger than n and whose weight is not larger than $\mathbf{m}$, but the proof of universality of $N_n(\mathbf{m})$ will be rather complicated.

As the reader will see, in this context it is more natural to use the large inductive dimension; thus our second universal space theorem will be formulated in terms of Ind rather than dim. We start with the definition of the space $N_n(\mathbf{m})$.

4.2.8. EXAMPLE. For every integer $n \geq 0$ and every cardinal number $\mathbf{m} \geq \aleph_0$ we define the space $N_n(\mathbf{m})$ as the subspace of the Cartesian product $[J(\mathbf{m})]^{\aleph_0}$ consisting of points with at most n rational coordinates distinct from the point 0, where by rational points of the hedgehog $J(\mathbf{m})$ we mean those whose distance from 0 is a rational number.

From the equality $w(J(\mathbf{m})) = \mathbf{m}$ it easily follows that $w(N_n(\mathbf{m})) \leq \mathbf{m}$. We shall now show that $\operatorname{Ind} N_n(\mathbf{m}) \leq n$. By virtue of Theorem 4.1.17 it suffices to prove that for $k = 0, 1, \ldots$ the subspace $Q_k(\mathbf{m})$ of $[J(\mathbf{m})]^{\aleph_0}$ consisting of points which have exactly k rational coordinates distinct from 0 satisfies the equality $\operatorname{Ind} Q_k(\mathbf{m}) = 0$.

To begin, let us note that for each $r \in I$ the subspace $D_r = \{x \in J(\mathbf{m}) : \rho(x, 0) = r\} \subset J(\mathbf{m})$ is closed and discrete. Let us also note that – as easily follows from the countable sum theorem – the subspace $P(\mathbf{m}) = \{x \in J(\mathbf{m}) : x = 0$ or $\rho(x, 0)$ is irrational$\} \subset J(\mathbf{m})$ satisfies the equality $\operatorname{Ind} P(\mathbf{m}) = 0$. Now, for each choice of k distinct natural numbers $i_1, i_2, \ldots, i_k$ and each choice of k rational numbers $r_1, r_2, \ldots, r_k$ in the interval $(0, 1]$, the Cartesian product $\prod_{i=1}^{\infty} J_i$, where $J_{i_j} = D_{r_j}$ for $j = 1, 2, \ldots, k$ and $J_i = J(\mathbf{m})$ for $i \neq i_j$, is a closed subspace of $[J(\mathbf{m})]^{\aleph_0}$, and thus $Q_k(\mathbf{m}) \cap \prod_{i=1}^{\infty} J_i$ is a closed subspace of $Q_k(\mathbf{m})$. Since the space $Q_k(\mathbf{m}) \cap \prod_{i=1}^{\infty} J_i$ is homeomorphic to the Cartesian product $D_{r_1} \times D_{r_2} \times \ldots \times D_{r_k} \times [P(\mathbf{m})]^{\aleph_0}$, we have $\operatorname{Ind}(Q_k(\mathbf{m}) \cap \prod_{i=1}^{\infty} J_i) = 0$ by virtue of Theorem 4.1.25. Hence, by the countable sum theorem, we have $\operatorname{Ind} Q_k(\mathbf{m}) = 0$,

because the family of all subspaces of the form $Q_k(\mathbf{m}) \cap \prod_{i=1}^{\infty} J_i$ is countable and its union is equal to the whole of $Q_k(\mathbf{m})$. ■

The proof of universality of $N_n(\mathbf{m})$ will be preceded by three lemmas which, for the purpose of future application, are formulated in a way slightly more general than needed here. In particular, for our present purpose one can assume that the space X in Lemmas 4.2.9 and 4.2.10 is metrizable.

4.2.9. LEMMA. *Let X be a perfectly normal space, and $X_0, X_1, \ldots$ a sequence of subspaces of X such that $\operatorname{Ind} X_n \leq 0$ for $n = 0, 1, \ldots$ Then for every sequence $L_0 = \emptyset$, $L_1, \ldots, L_i$ of closed subspaces of X such that $\operatorname{ord}(\{L_j\}_{j=0}^{i} | X_n) \leq n-1$ for $n = 0, 1, \ldots$, and for every pair A, B of disjoint closed subsets of X there exists a partition L_{i+1} between A and B such that $\operatorname{ord}(\{L_j\}_{j=0}^{i+1} | X_n) \leq n-1$ for $n = 0, 1, \ldots$*

PROOF. To begin, note that for $n = 0, 1, \ldots,$

(10) no point of X_n belongs to more than n sets $L_0, L_1, \ldots, L_i$.

Now, let $C_0 = X_0$ and for $n = 1, 2, \ldots, i$ let

$$C_n = \bigcup \{L_{j_1} \cap L_{j_2} \cap \ldots \cap L_{j_n} \cap X_n : j_k \leq i \text{ for } k = 1, 2, \ldots, n \text{ and } j_k \neq j_{k'} \text{ whenever } k \neq k'\}.$$

Consider the unions

$$D_n = C_0 \cup C_1 \cup \ldots \cup C_n \quad \text{for } n = 0, 1, \ldots, i;$$

obviously, $D_0 \subset D_1 \subset \ldots \subset D_i$. Observe that

(11) D_n is open in D_i for $n = 0, 1, \ldots, i-1$.

Indeed, for the closed set

$$E = \bigcup_{m=n+1}^{i} \bigcup \{L_{j_1} \cap L_{j_2} \cap \ldots \cap L_{j_m} : j_k \leq i \text{ for } k = 1, 2, \ldots, m \text{ and } j_k \neq j_{k'} \text{ whenever } k \neq k'\}$$

we have $C_{n+1} \cup C_{n+2} \cup \ldots \cup C_i \subset E$ and – by virtue of (10) – $E \cap (X_0 \cup X_1 \cup \ldots \cup X_n) = \emptyset$, so that $D_n = D_i \cap (X \setminus E)$. From (10) it follows that the sets $C_0, C_1, \ldots, C_i$ are pairwise disjoint, so that $C_n = D_n \setminus D_{n-1}$ for $n = 0, 1, \ldots, i$, where $D_{-1} = \emptyset$, and this together with (11) implies that $C_0, C_1, \ldots, C_i$ are F_σ-sets in D_i. As C_n is a closed subset of X_n, we have $\operatorname{Ind} C_n \leq 0$ for $n = 0, 1, \ldots, i$, and thus $\operatorname{Ind} D_i \leq 0$. By Theorem 2.2.4 there exists a partition $\bar{L}_{i+1}$ between A and B such that $\bar{L}_{i+1} \cap D_i = \emptyset$, and – as one easily checks – this partition has the required property. ■

4.2.10. LEMMA. *Let X be a perfectly normal space, and $X_0, X_1, \ldots$ a sequence of subspaces of X such that $\operatorname{Ind} X_n \leq 0$ for $n = 0, 1, \ldots$ Then for every sequence (A_1, B_1), $(A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X and any closed interval $J = [a, b] \subset R$ there exist continuous functions $f_1, f_2, \ldots$ from X to J satisfying the following conditions:*

(i) *$f_m(A_m) \subset \{a\}$ and $f_m(B_m) \subset \{b\}$ for $m = 1, 2, \ldots$*

(ii) *$|\{m : f_m(x)$ is a rational number and $a \neq f_m(x) \neq b\}| \leq n$ for $x \in X_n$.*

PROOF. To begin, let us arrange all rational numbers in the interval (a,b) into a sequence $r_1, r_2, \ldots$, where $r_i \neq r_k$ whenever $i \neq k$, and for $j = 1, 2, \ldots$ define

$$\delta_j = \min\{|r_i - r_k|, r_i - a, b - r_i : i,k \leq j\},\ a_j = r_j - \delta_j/2^{j+1} \quad \text{and} \quad b_j = r_j + \delta_j/2^{j+1};$$

clearly, $\delta_1 \geq \delta_2 \geq \ldots$ and $a < a_j < b_j < b$ for $j = 1, 2, \ldots$

For $m = 1, 2, \ldots$ and $j = 0, 1, \ldots$ we shall now define continuous functions $f_{m,j}\colon X \to J$ such that for each m the sequence $f_{m,0}, f_{m,1}, \ldots$ is uniformly convergent to a function $f_m\colon X \to J$, and the functions $f_1, f_2, \ldots$ satisfy conditions (i) and (ii).

For $m = 1, 2, \ldots$ let $f_{m,0}\colon X \to J$ be an arbitrary continuous function satisfying

$$f_{m,0}(A_m) \subset \{a\} \quad \text{and} \quad f_{m,0}(B_m) \subset \{b\}. \tag{12}$$

The functions $f_{m,j}$ with $j \geq 1$ will be defined by induction following the ordering of all pairs (m,j), where $m, j = 1, 2, \ldots$, in the infinite sequence $(1,1), (1,2), (2,1), (1,3), \ldots$ Simultaneously we shall define open subsets $U_{m,j}$ of X which, together with the functions $f_{m,j}$, will satisfy for $m, j = 1, 2, \ldots$ the conditions:

(13) $f_{m,j}^{-1}(r_j) = \operatorname{Fr} U_{m,j}$.

(14) $\operatorname{ord}(\{X_n \cap \operatorname{Fr} U_{m',j'} : (m',j') \leq (m,j)\}) \leq n-1$ for $n = 0, 1, \ldots$

(15) If $f_{m,j}(x) \neq f_{m,j-1}(x)$, then $f_{m,j-1}(x), f_{m,j}(x) \in (a_j, b_j)$.

Assume that the functions $f_{m,j}$ and the sets $U_{m,j}$ satisfying (13)–(14) are defined for $(m,j) < (k,l)$ or else that $(k,l) = (1,1)$; we shall define $f_{k,l}$ and $U_{k,l}$.

Since the function $f_{k,l-1}$ is already defined, and since the sets

$$A = f_{k,l-1}^{-1}([a, a_l]) \quad \text{and} \quad B = f_{k,l-1}^{-1}([b_l, b])$$

are closed and disjoint, by Lemma 4.2.9 there exists an open subset $U_{k,l}$ of X such that $A \subset U_{k,l}$, $B \cap \overline{U}_{k,l} = \emptyset$ and (14) is satisfied for $(m,j) = (k,l)$. From the perfect normality of X it follows that there exists a continuous function $f_{k,l}\colon X \to J$ such that

$$f_{k,l}(x) = f_{k,l-1}(x) \text{ for } x \in A \cup B, \quad f_{k,l}(X \setminus (A \cup B)) \subset (a_l, b_l) \quad \text{and} \quad f_{k,l}^{-1}(r_l) = \operatorname{Fr} U_{k,l}.$$

Clearly, (13) and (15) are satisfied for $(m,j) = (k,l)$, so that the construction of functions $f_{m,j}$ and open sets $U_{m,j}$ satisfying (13)–(15) is complete.

Consider now the sequence $f_{m,0}, f_{m,1}, \ldots$ for an arbitrary m. It follows from (15) that $|f_{m,k-1}(x) - f_{m,k}(x)| \leq b_k - a_k = \delta_k/2^k$ for every $x \in X$ and $k = 1, 2, \ldots$ Thus for every $x \in X$, $j = 0, 1, \ldots$ and $i = 1, 2, \ldots$ we have

$$\begin{aligned} |f_{m,j}(x) - f_{m,j+i}(x)| &\leq |f_{m,j}(x) - f_{m,j+1}(x)| + |f_{m,j+1}(x) - f_{m,j+2}(x)| + \ldots \\ &\quad + |f_{m,j+i-1}(x) - f_{m,j+i}(x)| \\ &\leq \frac{\delta_{j+1}}{2^{j+1}} + \frac{\delta_{j+2}}{2^{j+2}} + \ldots + \frac{\delta_{j+i}}{2^{j+i}} \leq \frac{\delta_j}{2^j}\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^i}\right) < \frac{\delta_j}{2^j}, \end{aligned}$$

which implies that for every $x \in X$ the sequence $f_{m,0}(x), f_{m,1}(x), \ldots$ is a Cauchy sequence and thus converges to a point $f_m(x) \in J$. In this way a function f_m from X to J is defined. Since

$$|f_{m,j}(x) - f_m(x)| \leq \delta_j/2^j \quad \text{for } j = 1, 2, \ldots \text{ and } x \in X, \tag{16}$$

the sequence $f_{m,0}, f_{m,1}, \ldots$ uniformly converges to f_m, and f_m is a continuous function.

From (12) and (15) it readily follows that f_m satisfies condition (i). We shall show that condition (ii) is also satisfied.

Let x be a point in X_n; it follows from (14) that the cardinality of the set $M = \{m : x \in \operatorname{Fr} U_{m,j} \text{ for some } j\}$ is at most n. Thus, to establish (ii) it suffices to show that

(17) $$\text{if } m \notin M, \text{ then } f_m(x) \neq r_j \text{ for } j = 1, 2, \ldots$$

Consider an $m \notin M$ and an arbitrary j. Since $x \notin \operatorname{Fr} U_{m,j}$, we have $f_{m,j}(x) \neq r_j$ by virtue of (13). Let $K = \{k > j : f_{m,k}(x) \neq f_{m,j}(x)\}$. If $K = \emptyset$, then $f_m(x) = f_{m,j}(x) \neq r_j$; thus we can assume that $K \neq \emptyset$ and consider the smallest element k of the set K. Since $f_{m,k-1}(x) = f_{m,j}(x) \neq f_{m,k}(x)$, it follows from (15) that $f_{m,k}(x) \in (a_k, b_k)$, so that $|r_k - f_{m,k}(x)| \leq \delta_k/2^{k+1} \leq \delta_k/4$ and $|f_{m,k}(x) - r_j| \geq |r_j - r_k| - |r_k - f_{m,k}(x)| \geq \delta_k - \delta_k/4 = (3/4)\delta_k$.

On the other hand, it follows from (16) that $|f_{m,k}(x) - f_m(x)| \leq \delta_k/2^k \leq \delta_k/2$, so that

$$|f_m(x) - r_j| \geq |f_{m,k}(x) - r_j| - |f_{m,k}(x) - f_m(x)| \geq (3/4)\delta_k - \delta_k/2 = \delta_k/4 > 0,$$

and thus (17) is proved. ■

4.2.11. LEMMA. *For every metrizable space X of weight $\mathbf{m} \geq \aleph_0$ and for every sequence $X_0, X_1, \ldots$ of subspaces of X, where $\operatorname{Ind} X_n \leq 0$ for $n = 0, 1, \ldots$, there exists a homeomorphic embedding $h\colon X \to [J(\mathbf{m})]^{\aleph_0}$ such that $h(X_n) \subset N_n(\mathbf{m})$ for $n = 0, 1, \ldots$*

PROOF. Consider a base $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ for the space X, where $\mathcal{B}_i$ is a discrete family for $i = 1, 2, \ldots$ (see [GT], Theorem 4.4.8). With no loss of generality we can assume that $\mathcal{B}_i = \{U_{i,s}\}_{s \in S}$, where S is the set of cardinality $\mathbf{m}$ used in Example 4.2.6 to define the hedgehog $J(\mathbf{m})$. For each natural number i and each $s \in S$ represent the set $U_{i,s}$ as the union of closed sets $F_{i,j,s}$, where $j = 1, 2, \ldots$, and for $i, j = 1, 2, \ldots$ let

$$A_{i,j} = X \setminus \bigcup_{s \in S} U_{i,s} \quad \text{and} \quad B_{i,j} = \bigcup_{s \in S} F_{i,j,s};$$

the sets $A_{i,j}$ and $B_{i,j}$ are closed and disjoint. Since all pairs $(A_{i,j}, B_{i,j})$ can be arranged in a sequence, by Lemma 4.2.10 there exist continuous functions $f_{i,j}\colon X \to [0, \pi/4]$ such that for $i, j = 1, 2, \ldots$, we have $f_{i,j}(A_{i,j}) \subset \{0\}$ and $f_{i,j}(B_{i,j}) \subset \{\pi/4\}$, and

(18) $$|\{(i,j) : f_{i,j}(x) \text{ is a non-zero rational number}\}| \leq n \text{ for } x \in X_n.$$

Now, for $i, j = 1, 2, \ldots$ consider the mapping $h_{i,j}$ of X to $J(\mathbf{m})$ defined by letting

$$h_{i,j}(x) = \begin{cases} 0 & \text{for } x \in A_{i,j}, \\ (f_{i,j}(x), s) & \text{for } x \in \overline{U}_{i,s}; \end{cases}$$

since the family $\{A_{i,j}\} \cup \{\overline{U}_{i,s}\}_{s \in S}$ is locally finite, $\overline{U}_{i,s} \cap \overline{U}_{i,s'} = \emptyset$ whenever $s \neq s'$ and $f_{i,j}(x) = 0$ for $x \in A_{i,j} \cap \overline{U}_{i,s}$, the mapping $h_{i,j}$ is correctly defined and continuous. It is easily checked that the family of mappings $\{h_{i,j}\}_{i,j=1}^{\infty}$ separates points and closed sets, so that the mapping $h\colon X \to \prod_{i,j=1}^{\infty} J_{i,j}$, where $J_{i,j} = J(\mathbf{m})$ for $i, j = 1, 2, \ldots$ and $h(x) = \{h_{i,j}(x)\}$, is a homeomorphic embedding. Moreover, it follows from (18) that $h(X_n) \subset N_n(\mathbf{m})$ for $n = 0, 1, \ldots$ ■

4.2.12. REMARK. Let us observe that if in the space X from Lemma 4.2.11 a point x_0 is distinguished, then there exists a homeomorphic embedding $h\colon X \to [J(\mathbf{m})]^{\aleph_0}$ which, moreover, satisfies the condition $h(x_0) = (0, 0, \ldots)$. This can be obtained by making the following changes in the proof. First, we replace the base $\mathcal{B}$ by the base $\mathcal{B}' = \mathcal{B}_0 \cup \bigcup_{i=1}^{\infty} \mathcal{B}'_i$,

where $\mathcal{B}'_i = \{U'_{i,s}\}_{s\in S}$ with $U'_{i,s} = U_{i,s} \setminus \{x_0\}$ for $i = 1, 2, \ldots$ and $\mathcal{B}_0 = \{U_{0,j}\}_{j=1}^{\infty}$ is a base at the point x_0. Next, to the pairs $(A_{i,j}, B_{i,j})$, defined using the sets $U'_{i,s}$ rather than the sets $U_{i,s}$, we adjoin the pairs $(A_{0,j}, B_{0,j})$, where $A_{0,j} = \{x_0\}$ and $B_{0,j} = X \setminus U_{0,j}$ for $j = 1, 2, \ldots$ Finally, besides the functions $h_{i,j}$ we consider the functions $h_{0,j}\colon X \to J(\mathbf{m})$ defined by choosing an arbitrary $s_0 \in S$ and letting $h_{0,j}(x) = (f_{0,j}(x), s_0)$ for each $x \in X$ and $j = 1, 2, \ldots$ ■

4.2.13. THE SECOND UNIVERSAL SPACE THEOREM. *For every integer $n \geq 0$ and every cardinal number $\mathbf{m} \geq \aleph_0$ the subspace $N_n(\mathbf{m})$ of the Cartesian product $[J(\mathbf{m})]^{\aleph_0}$ consisting of points with at most n rational coordinates distinct from the point 0 is a universal space for the class of all metrizable spaces whose large inductive dimension is not larger than n and whose weight is not larger than $\mathbf{m}$.*

PROOF. We have shown in Example 4.2.8 that $\dim N_n(\mathbf{m}) \leq n$ and $w(N_n(\mathbf{m})) \leq \mathbf{m}$.

Consider now an arbitrary metrizable space X such that $\operatorname{Ind} X \leq n$ and $w(X) \leq \mathbf{m}$. By virtue of Theorem 4.1.17 there exist subspaces $X_0, X_1, \ldots, X_n$ of the space X such that $\operatorname{Ind} X_i \leq 0$ for $i = 0, 1, \ldots, n$ and $X = \bigcup_{i=0}^{n} X_i$. Letting $X_i = \emptyset$ for $i > n$ and applying Lemma 4.2.11 we obtain a homeomorphic embedding $h\colon X \to [J(\mathbf{m})]^{\aleph_0}$ such that $h(X) = h(\bigcup_{i=0}^{n} X_i) = \bigcup_{i=0}^{n} h(X_i) \subset \bigcup_{i=0}^{n} N_i(\mathbf{m}) = N_n(\mathbf{m})$. ■

Historical and bibliographic notes

Theorem 4.2.2 was established by Morita in [1954]. The equivalence of conditions (i) and (ii) in Theorem 4.2.3 was proved by Nagata in [1956a]; the equivalence of conditions (i) and (iii) was proved by Nagami and Roberts in [1967]. The characterizations of dimension in terms of metrics satisfying (μ_1) or (μ_2) were established by Nagata in [1963] (a simpler proof in Nagata [1983]), the characterization in terms of metrics satisfying (μ_3) was obtained by Nagata in [1956] (simpler proofs in Buzási [1979] and in Assouad [1982]), and the characterization in terms of metrics satisfying (μ_4) was obtained independently by Nagata in [1964] and by Ostrand in [1965a]. Let us add that in [1937] Marczewski proved that a separable metric space X satisfies the inequality $\dim X \leq n \geq 0$ if and only if there exists a metric ρ on the space X such that for every point $x \in X$ we have $\dim \operatorname{Fr} B(x, r) \leq n - 1$ for almost all positive numbers r (in the sense of Lebesgue measure). The question whether the inequality $\dim X \leq n$ follows from the existence of a metric ρ on the space X which satisfies (μ_5) was raised by de Groot in [1957].

Theorem 4.2.5, under the weaker assumption of normality of X (see Problem 4.2.F), was proved by Pasynkov in [1967a] (announcement in [1964]); the present proof is obtained by amalgamating the proofs given by Arhangel'skiĭ in [1967] and by Morita in [1975]. Theorem 4.2.5 for separable Y and an analogous theorem with no evaluation of $w(Z)$ were given by Pasynkov in [1963]. Theorem 4.2.7 was established by Nagata in [1960a] (where a concrete universal space is defined); our proof is taken from Pasynkov [1964]. Lemma 4.2.9 can be found in Nagata [1958a]. Theorem 4.2.13 was established by Nagata in [1963a]; the proof presented here, which simplifies the original one, is taken from E. Pol's paper [1987] (where it is proved that for each metrizable space

X with $\operatorname{Ind} X \leq n$ and $w(X) \leq \mathbf{m}$ the set of all homeomorphic embeddings of X in $N_n(\mathbf{m})$ is dense in the corresponding function space; see E. Pol [1988] for further discussion).

Let us mention, in connection with universal space theorems, that in [1975] Lipscomb defined, for every cardinal number $\mathbf{m} \geq \aleph_0$, a metrizable space $L(\mathbf{m})$ with $\dim L(\mathbf{m}) = 1$ and $w(L(\mathbf{m})) = \mathbf{m}$ such that each metrizable space X satisfying the inequalities $\dim X \leq n$ and $w(X) \leq \mathbf{m}$ is embeddable in the Cartesian product $[L(\mathbf{m})]^{n+1}$ (a simpler construction in Olszewski [1991a]); he also proved that $[L(\mathbf{m})]^{n+1}$ contains an easily defined subspace which is a universal space for the class of all metrizable spaces whose covering dimension is not larger than n and whose weight is not larger than $\mathbf{m}$. The fact that each metrizable space X with $\dim X = n$ is embeddable in the Cartesian product of $n+1$ metrizable spaces whose covering dimension is equal to 1 was discovered by Nagata in [1958]; Borsuk proved in [1975] that the two-sphere S^2 cannot be embedded in the Cartesian product of two one-dimensional spaces (simpler proofs were given by van Mill and R. Pol in [1995]).

Problems

4.2.A (Morita [1954]). (a) Prove the Katětov–Morita theorem by applying only Lemmas 4.2.1 and 2.3.16 and Theorems 4.1.9, 4.1.10 and 3.1.28.

Hint. Adjoin to conditions (i)–(iii) in Theorem 4.2.2 the condition

(iv) *the space X satisfies the inequality* $\operatorname{Ind} X \leq n$.

(b) Show that in Theorem 4.2.2 one can replace the σ-local finiteness of $\mathcal{B}$ by its σ-discreteness.

4.2.B (Nagata [1963] (announcement [1961])). Prove that a metrizable space X satisfies the inequality $\dim X \leq n$ if and only if X has a base of rank $\leq n$ (see Problem 3.2.C).

Hint (Arhangel'skiĭ 1963]). When proving that X has a base of rank $\leq n$ use the inequality $\operatorname{Ind} X \leq n$; apply Theorem 4.1.17.

4.2.C (Nagata [1963]). (a) Note that every metric ρ which satisfies condition (μ_1) also satisfies condition (μ_2).

(b) Show that if on a space X there exists a metric ρ which satisfies condition (μ_2), then $\dim X \leq n$.

Hint. See the second part of the hint to Problem 4.1.A(b).

4.2.D. Note that for $n = 0$ conditions (μ_3), (μ_4) and (μ_5) reduce to the condition that the metric ρ is non-Archimedean; check that each non-Archimedean metric ρ satisfies conditions (μ_1) and (μ_2) for $n = 0$.

4.2.E (de Groot [1957]). Applying the fact that on every compact metrizable space of dimension $\leq n$ there exists a metric ρ satisfying condition (μ_3), show that a separable metrizable space X satisfies the inequality $\dim X \leq n$ if and only if there exists a totally bounded metric ρ on the space X which satisfies condition (μ_5).

Hint. Note that if ρ satisfies (μ_5), then the metric ρ on the completion $\widetilde{X}$ of the space (X, ρ) also satisfies (μ_5). For an arbitrary finite open cover $\mathcal{U}$ of $\widetilde{X}$ consider a set $A \subset \widetilde{X}$ maximal with respect to the property that $\rho(y, y') \geq \delta/2$ for distinct $y, y' \in A$, where δ is a Lebesgue number of the cover $\mathcal{U}$.

4.2.F (Pasynkov [1967a] (announcement [1964])). Prove that for every continuous mapping $f: X \to Y$ of a normal space X to a metrizable space Y there exist a metrizable space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\dim Z \leq \dim X$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.

Hint. Generalize Lemma 4.2.4 to normal spaces making use of the mapping of X to the metrizable space assigned to the cover $\mathcal{U}$ as described in the hint to Problem 3.2.H(b).

4.2.G (Arhangel'skiĭ [1967]). (a) Prove that for every continuous mapping $f: X \to Y$ of a metrizable space X to a metrizable space Y and for any sequence $F_1, F_2, \ldots$ of closed subsets of X there exist a metrizable space Z and continuous mappings $g: X \to Z$ and $h: Z \to Y$ such that $\dim \overline{g(F_i)} \leq \dim F_i$ for $i = 1, 2, \ldots$, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$.

Hint. Modify the proof of Theorem 4.2.5 by defining locally finite open covers $\mathcal{V}_1, \mathcal{V}_2, \ldots$ of the space X which satisfy the condition $\operatorname{ord}(\mathcal{V}_i | F_j) \leq \dim F_j$ for $j = 1, 2, \ldots, i$. Observe that from the inequality $\operatorname{ord}(\mathcal{W}_i | g(F_j)) \leq \dim F_j$ it follows that $\operatorname{ord}(\mathcal{W}_i | \overline{g(F_j)}) \leq \dim F_j$.

(b) Show that the result in part (a) holds under the assumption that X is a normal space.

Hint. See Problem 4.2.F.

4.3. Dimension and mappings in metrizable spaces

In this section we shall study the behaviour of dimension of metrizable spaces under continuous mappings. Observe first that Theorem 2.4.15 (cf. Remark 2.4.16), as well as Theorem 3.3.7, yields the following result which can also be obtained by a slight modification in the proof of Theorem 1.12.2 with an application of Problem 4.1.7 and Lemma 5.4.1.

4.3.1. THEOREM ON DIMENSION-RAISING MAPPINGS. *If $f: X \to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then* $\operatorname{Ind} Y \leq \operatorname{Ind} X + (k-1)$. ■

From Theorem 3.3.10 it follows that the theorem on dimension-lowering mappings also holds in the class of metrizable spaces. Since the proof of the former theorem is long and complicated, we shall provide here another proof of the latter, one which follows the pattern of the proof given in the case of separable metric spaces (see Theorem 1.12.4). To begin, let us note that reproducing the proof of Lemma 1.9.6 with an application of Theorems 3.2.9, 4.1.3, 4.1.7 and 4.1.21 one obtains

4.3.2. LEMMA. *Let $f, g: X \to S^n$ be continuous mappings of a metrizable space X to the n-sphere S^n. If the set*

$$D(f,g) = \{x \in X : f(x) \neq g(x)\}$$

satisfies the inequality $\operatorname{Ind} D(f,g) \leq n-1$, *then the mappings f and g are homotopic.* ■

Now we shall establish a counterpart of Lemma 1.12.3.

4.3.3. LEMMA. *If a metrizable space X has a closed cover $\mathcal{K}$ such that* $\operatorname{Ind} K \leq m \geq 0$ *for each $K \in \mathcal{K}$ and a σ-locally finite open cover $\mathcal{U}$ such that for every $K \in \mathcal{K}$ and each open set $V \subset X$ that contains K there exists a $U \in \mathcal{U}$ satisfying*

$$K \subset U \subset \overline{U} \subset V \quad \text{and} \quad \operatorname{Ind} \operatorname{Fr} U \leq m-1,$$

then $\operatorname{Ind} X \leq m$.

PROOF. By virtue of Theorem 4.1.3 and Remarks 1.7.10 and 1.9.4 it suffices to show that for every closed subspace A of the space X and each continuous mapping $f: A \to S^m$ there exists a continuous extension $F: X \to S^m$ of f over X. To begin, let us observe that

(1) for every $K \in \mathcal{K}$ there exists a $U_K \in \mathcal{U}$ such that $K \subset U_K$, $\operatorname{Ind} \operatorname{Fr} U_K \leq m-1$ and f is continuously extendable over $A \cup \overline{U}_K$.

Indeed, it follows from Theorem 3.2.9 (cf. Problem 4.3.B(a)) that the mapping f is continuously extendable over $A \cup K$, so that there exists an open set $V \subset X$ containing $A \cup K$ such that f is continuously extendable over V; the existence of a set U_K satisfying (1) now follows from the properties of the family $\mathcal{U}$.

The subfamily $\mathcal{U}_0$ of the family $\mathcal{U}$ which consists of all sets of the form U_K can be represented as the union of a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of locally finite families. Let us arrange the members of $\mathcal{U}_0$ into a transfinite sequence $U_1, U_2, \ldots, U_\alpha, \ldots$, $\alpha < \xi$, placing first all members of $\mathcal{U}_1$, then all members of $\mathcal{U}_2$ etc. Clearly, for every $\alpha_0 < \xi$ the family $\{U_\alpha\}_{\alpha<\alpha_0}$ is locally finite.

We shall inductively define a transfinite sequence $F_1, F_2, \ldots, F_\alpha, \ldots$, $\alpha < \xi$, of continuous mappings, where $F_\alpha: A \cup \bigcup_{\beta\leq\alpha} \overline{U}_\beta \to S^m$, such that

$$F_\alpha \Big| \Big(A \cup \bigcup_{\beta\leq\gamma} \overline{U}_\beta \Big) = F_\gamma \quad \text{for every } \gamma \leq \alpha. \tag{2}$$

Let F_1 be an arbitrary continuous extension of f over $A \cup \overline{U}_1$. Assume that the mappings F_α satisfying (2) are defined for $\alpha < \alpha_0$. The set $A \cup \bigcup_{\alpha\leq\alpha_0} \overline{U}_\alpha$ can be represented as the union of two closed sets

$$A' = A \cup \bigcup_{\alpha<\alpha_0} \overline{U}_\alpha \quad \text{and} \quad A'' = A \cup \Big(\overline{U}_{\alpha_0} \setminus \bigcup_{\alpha<\alpha_0} U_\alpha \Big)$$

which, by virtue of the local finiteness of the family $\{U_\alpha\}_{\alpha<\alpha_0}$, satisfy the relation

$$A' \cap A'' \subset A \cup \operatorname{Fr} \Big(\bigcup_{\alpha<\alpha_0} U_\alpha \Big) \subset A \cup \bigcup_{\alpha<\alpha_0} \operatorname{Fr} U_\alpha. \tag{3}$$

Since the family $\{A\} \cup \{\overline{U}_\alpha\}_{\alpha<\alpha_0}$ is locally finite and $\{f\} \cup \{F_\alpha | \overline{U}_\alpha\}_{\alpha<\alpha_0}$ is a family of compatible mappings, one can define a continuous mapping $F': A' \to S^m$ which is a

common extension of all mappings F_α with $\alpha < \alpha_0$. By virtue of (1) the mapping f is extendable to a continuous mapping $f'': A'' \to S^m$, and by (3) we have

$$D = \{x \in A' \cap A'' : F'(x) \neq f''(x)\} \subset \bigcup_{\alpha<\alpha_0} \operatorname{Fr} U_\alpha,$$

so that $\operatorname{Ind} D \leq m-1$ by virtue of (1) and Theorem 4.1.10. It follows from Lemma 4.3.2 that the mappings $F'|A' \cap A''$ and $f''|A' \cap A''$ are homotopic. Since the latter is continuously extendable over A'', it follows from Lemma 1.9.7 that $F'|A' \cap A''$ is extendable to a continuous mapping $F'': A'' \to S^m$. Letting

$$F_{\alpha_0}(x) = \begin{cases} F'(x) & \text{for } x \in A', \\ F''(x) & \text{for } x \in A'', \end{cases}$$

we define a continuous mapping F_{α_0} of $A' \cup A'' = A \cup \bigcup_{\alpha\leq\alpha_0} \overline{U}_\alpha$ to S^m which satisfies (2) for $\alpha = \alpha_0$.

As $X = \bigcup_{\alpha<\xi} U_\alpha$, the formula

$$F(x) = F_\alpha(x) \quad \text{for } x \in U_\alpha$$

defines a continuous mapping $F: X \to S^m$ which is the required extension of f over X. ■

4.3.4. THEOREM ON DIMENSION-LOWERING MAPPINGS. *If $f: X \to Y$ is a closed mapping of a metrizable space X to a metrizable space Y and there exists an integer $k \geq 0$ such that $\operatorname{Ind} f^{-1}(y) \leq k$ for every $y \in Y$, then $\operatorname{Ind} X \leq \operatorname{Ind} Y + k$.*

PROOF. We can suppose that $\operatorname{Ind} Y < \infty$. We shall apply induction with respect to $n = \operatorname{Ind} Y$. If $n = -1$, we have $Y = \emptyset$ and $X = \emptyset$, and so the theorem holds. Assume that it holds for closed mappings to spaces of large inductive dimension less than $n \geq 0$ and consider a closed mapping $f: X \to Y$ to a space Y such that $\operatorname{Ind} Y = n$.

Let $\mathcal{B}$ be a σ-locally finite base for Y such that $\operatorname{Ind} \operatorname{Fr} U \leq n-1$ for every $U \in \mathcal{B}$. Applying the inductive assumption to the restriction $f_{\operatorname{Fr} U}: f^{-1}(\operatorname{Fr} U) \to \operatorname{Fr} U$, we obtain the inequality $\operatorname{Ind} f^{-1}(\operatorname{Fr} U) \leq n+k-1$. Since

$$\operatorname{Fr} f^{-1}(U) = \overline{f^{-1}(U)} \setminus f^{-1}(U) \subset f^{-1}(\overline{U}) \setminus f^{-1}(U) = f^{-1}(\operatorname{Fr} U),$$

we have $\operatorname{Ind} \operatorname{Fr} f^{-1}(U) \leq n+k-1$ for every $U \in \mathcal{B}$. One readily checks that the covers $\mathcal{K} = \{f^{-1}(y)\}_{y\in Y}$ and $\mathcal{U} = f^{-1}(\mathcal{B})$ satisfy the conditions of Lemma 4.3.3 with $m = n+k$, so that $\operatorname{Ind} X \leq n+k = \operatorname{Ind} Y + k$. ■

We now turn to two more specialized theorems on the relations between the dimensions of the domain and range of a closed mapping. In our considerations an important role will be played by the sets $C_k(f)$ and $D_k(f)$ defined by the formulas

$$C_k(f) = \{y \in Y : |f^{-1}(y)| \geq k\}$$

and

$$D_k(f) = \{y \in Y : \dim f^{-1}(y) \geq k\}$$

for each closed mapping $f: X \to Y$ of a normal space X to a normal space Y and for $k = 1, 2, \ldots$

Let us recall that a continuous mapping defined on a Hausdorff space is perfect if it is closed and has compact fibres.

4.3.5. LEMMA. *For every closed mapping $f: X \to Y$ of a metrizable space X to a metrizable space Y and for $k = 1, 2, \ldots$, $C_k(f)$ and $D_k(f)$ are F_σ-sets in Y.*

PROOF. Consider first the sets $C_k(f)$. For $i = 1, 2, \ldots$ let $\mathcal{U}_i$ be an open cover of the space X such that $\operatorname{mesh}\mathcal{U}_i < 1/i$ and let

$$U_i = \Big\{y \in Y : \text{there exists a } \mathcal{U} \subset \mathcal{U}_i \text{ such that } |\mathcal{U}| < k \text{ and } f^{-1}(y) \subset \bigcup \mathcal{U}\Big\}.$$

Since the mapping f is closed, the sets U_i are open. One easily checks that $Y \setminus C_k(f) = \bigcap_{i=1}^{\infty} U_i$, so that $C_k(f)$ is an F_σ-set in Y.

Now consider the sets $D_k(f)$. We start with a particular case where f is a perfect mapping. For $i = 1, 2, \ldots$ let V_i be the set of all points $y \in Y$ for which there exists a finite family $\mathcal{U}$ of open subsets of X such that $\operatorname{mesh}\mathcal{U} < 1/i$, $\operatorname{ord}\mathcal{U} < k$ and $f^{-1}(y) \subset \bigcup \mathcal{U}$. Since f is closed, the sets V_i are open. Applying compactness of fibres of f and Theorems 1.6.12, 3.1.2 and 3.1.1, one easily checks that $Y \setminus D_k(f) = \bigcap_{i=1}^{\infty} V_i$, so that $D_k(f)$ is an F_σ-set in Y.

For an arbitrary closed mapping $f: X \to Y$ consider the subspaces $X_0 = \bigcup_{y \in Y} \operatorname{Int} f^{-1}(y)$ and $X_1 = X \setminus X_0 = \bigcup_{y \in Y} \operatorname{Fr} f^{-1}(y)$ of the space X and the restrictions $f_0 = f|X_0 : X_0 \to Y$ and $f_1 = f|X_1 : X_1 \to Y$. From the countable sum theorem it follows that $D_k(f) = D_k(f_0) \cup D_k(f_1)$, because $\operatorname{Int} f^{-1}(y)$ is an F_σ-set in X for every $y \in Y$. Being the image of an F_σ-set in X under a closed mapping, the set

$$D_k(f_0) = f\Big(\bigcup\{\operatorname{Int} f^{-1}(y) : \dim \operatorname{Int} f^{-1}(y) \geq k\}\Big)$$

is an F_σ-set in Y. From Lemma 1.12.9 it follows that f_1 is a perfect mapping, so that $D_k(f_1)$ is an F_σ-set in Y by the particular case of our theorem established in the preceding paragraph. Thus $D_k(f)$ is an F_σ-set in Y. ■

4.3.6. LEMMA. *If X is a metrizable space and $M_1, M_2, \ldots, M_k$ is a sequence of subspaces of X such that M_i is an F_σ-set in X and $\operatorname{Ind} M_i \leq n_i \geq 0$ for $i = 1, 2, \ldots, k$, then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\operatorname{Ind}(M_i \cap L) \leq n_i - 1$ for $i = 1, 2, \ldots, k$.*

PROOF. By the first decomposition theorem, $M_i = Y_i \cup Z_i$, where $\operatorname{Ind} Y_i \leq n_i - 1$ and $\operatorname{Ind} Z_i \leq 0$. From the enlargement theorem it follows that there exists a G_δ-set Y_i^* in M_i such that $Y_i \subset Y_i^*$ and $\operatorname{Ind} Y_i^* \leq n_i - 1$. The set $Z_i^* = M_i \setminus Y_i^* \subset Z_i$ is an F_σ-set in M_i and consequently an F_σ-set in X. By virtue of the countable sum theorem the set $Z = Z_1^* \cup Z_2^* \cup \ldots \cup Z_k^*$ satisfies the inequality $\operatorname{Ind} Z \leq 0$, so that from the separation theorem it follows that there exists a partition L between A and B such that $L \cap Z = \emptyset$. Hence for $i = 1, 2, \ldots, k$ we have $M_i \cap L \subset Y_i^*$, which implies that $\operatorname{Ind}(M_i \cap L) \leq n_i - 1$ for $i = 1, 2, \ldots, k$. ■

If for a continuous mapping $f: X \to Y$ of a metrizable space X to a metrizable space Y there exists an integer $n \geq 0$ such that $\operatorname{Ind} f^{-1}(y) \leq n$ for every $y \in Y$, then we say that f is *finite-dimensional*; the smallest n with this property will be denoted by $\operatorname{Ind} f$. In order to simplify the statements of the two theorems we are now going to prove, for every finite-dimensional mapping f we define

$$I_f = \max\{\operatorname{Ind} D_i(f) + i : i = 1, 2, \ldots, \operatorname{Ind} f\} \quad \text{if } \operatorname{Ind} f \geq 1$$

and

$$I_f = -1 \quad \text{if } \operatorname{Ind} f = 0.$$

4.3.7. VAĬNŠTEĬN'S FIRST THEOREM. *Let $f: X \to Y$ be a finite-dimensional closed mapping of a metrizable space X to a metrizable space Y. If $\operatorname{Ind} Y \leq n \geq 0$ and $\operatorname{Ind} D_i(f) \leq n - i$ for $i = 1, 2, \ldots, n+1$, then $\operatorname{Ind} X \leq n$.*

In other words, $\operatorname{Ind} X \leq \max\{\operatorname{Ind} Y, I_f\}$ *for every finite-dimensional closed mapping f of a metrizable space X to a metrizable space Y.*

PROOF. We shall apply induction with respect to n. If $n = 0$, we have $\operatorname{Ind} D_1(f) = -1$, so that $\operatorname{Ind} f = 0$ and consequently $\operatorname{Ind} X \leq n$ by virtue of Theorem 4.3.4. Assume that the theorem holds for all natural numbers less than $n \geq 1$ and consider a closed mapping $f: X \to Y$ which satisfies the assumptions of the theorem.

From Lemma 4.3.5 it follows that $D_1(f), D_2(f), \ldots, D_n(f)$ are F_σ-sets in Y. Applying Lemma 4.3.6 one can easily define (cf. the proof of Proposition 4.1.14) a σ-locally finite base $\mathcal{B}$ for the space Y such that for each $i \leq n$,

$$\operatorname{Ind}(D_i(f) \cap \operatorname{Fr} U) \leq \operatorname{Ind} D_i(f) - 1 \quad \text{for every } U \in \mathcal{B},$$

provided that $D_i(f) \neq \emptyset$, and that $\operatorname{Ind} \operatorname{Fr} U \leq n - 1$ for every $U \in \mathcal{B}$. Consider the restriction $f_{\operatorname{Fr} U}: f^{-1}(\operatorname{Fr} U) \to \operatorname{Fr} U$, where $U \in \mathcal{B}$; clearly, $D_i(f_{\operatorname{Fr} U}) \subset D_i(f) \cap \operatorname{Fr} U$. Since $\operatorname{Ind} \operatorname{Fr} U \leq n - 1 \geq 0$ and $\operatorname{Ind} D_i(f_{\operatorname{Fr} U}) \leq (n-1) - i$ for $i = 1, 2, \ldots, n$, applying the inductive assumption to $f_{\operatorname{Fr} U}$ we obtain the inequality $\operatorname{Ind} f^{-1}(\operatorname{Fr} U) \leq n-1$. Hence, by virtue of the inclusion $\operatorname{Fr} f^{-1}(U) \subset f^{-1}(\operatorname{Fr} U)$ we have $\operatorname{Ind} \operatorname{Fr} f^{-1}(U) \leq n - 1$ for every $U \in \mathcal{B}$. Applying the equality $D_{n+1}(f) = 0$ and the Katětov–Morita theorem, one readily checks that the covers $\mathcal{K} = \{f^{-1}(y)\}_{y \in Y}$ and $\mathcal{U} = f^{-1}(\mathcal{B})$ satisfy the conditions of Lemma 4.3.3 with $m = n$, so that $\operatorname{Ind} X \leq n$. ■

The reader can easily verify that Theorem 4.3.7 is a generalization of Theorem 4.3.4.

We now pass to Vaĭnšteĭn's second theorem. It will be preceded by two lemmas of which the second is an important particular case of the theorem.

4.3.8. LEMMA. *If a metrizable space X satisfies the inequality $\operatorname{Ind} X \leq n \geq 0$, then for every F_σ-set $C \subset X$ such that $\operatorname{Ind} C \leq n - 1$ there exists an F_σ-set $M \subset X$ satisfying the conditions*

$$\operatorname{Ind} M \leq 0, \quad \operatorname{Ind}(X \setminus M) \leq n - 1 \quad \text{and} \quad M \cap C = \emptyset.$$

PROOF. By the first decomposition theorem $X = Y \cup Z$, where $\operatorname{Ind} Y \leq n - 1$ and $\operatorname{Ind} Z \leq 0$. Obviously, one can assume that $Y = X \setminus Z$, and by virtue of the enlargement theorem one can assume that Z is a G_δ-set in X. Thus Y is an F_σ-set and the countable sum theorem yields the inequality $\operatorname{Ind}(Y \cup C) \leq n - 1$. By virtue of the enlargement theorem there exists a G_δ-set $K \subset X$ such that $Y \cup C \subset K$ and $\operatorname{Ind} K \leq n - 1$. One easily checks that the set $M = X \setminus K$ satisfies the required conditions. ■

4.3.9. LEMMA (FREUDENTHAL'S THEOREM). *Let $f: X \to Y$ be a closed mapping of a metrizable space X onto a metrizable space Y such that $\operatorname{Ind} f^{-1}(y) \leq 0$ for every $y \in Y$. If $\operatorname{Ind} X \leq n$ and $\operatorname{Ind} C_2(f) \leq n - 1$, then $\operatorname{Ind} Y \leq n$.*

In other words, $\operatorname{Ind} Y \leq \max\{\operatorname{Ind} X, \operatorname{Ind} C_2(f)+1\}$ *for every closed mapping* f *of a metrizable space* X *onto a metrizable space* Y *such that* $\operatorname{Ind} f^{-1}(y) \leq 0$ *for every* $y \in Y$.

PROOF. We shall apply induction with respect to n. If $n = 0$, we have $C_2(f) = \emptyset$, so that f is a homeomorphism and $\operatorname{Ind} Y = \operatorname{Ind} X \leq 0$. Assume that the theorem holds for all natural numbers less than $n \geq 1$ and consider a closed mapping $f: X \to Y$ which satisfies the assumptions of the theorem.

By virtue of Theorem 4.3.4 the set $C = f^{-1}(C_2(f))$ satisfies the inequality $\operatorname{Ind} C \leq n-1$. Lemma 4.3.5 implies that C is an F_σ-set in X, so that Lemma 4.3.8 implies that there exists an F_σ-set $M \subset X$ such that $\operatorname{Ind} M \leq 0$, $\operatorname{Ind}(X \setminus M) \leq n-1$ and $M \cap C = \emptyset$. Let $M = \bigcup_{i=1}^{\infty} F_i$, where the sets F_i are closed in X. For $i = 1, 2, \ldots$ the restriction of f to F_i is a homeomorphism of F_i onto $f(F_i)$, so that $\operatorname{Ind} f(F_i) \leq 0$. Thus, by the countable sum theorem, $\operatorname{Ind} f(M) \leq 0$.

Consider an arbitrary pair A, B of disjoint closed subsets of Y. From Lemma 4.3.6 with $k = 2$, $M_1 = f(M)$ and $M_2 = C_2(f)$ it follows that there exists a partition L between A and B such that $f(M) \cap L = \emptyset$ and $\operatorname{Ind}(C_2(f) \cap L) \leq n-2$. The equality $f(M) \cap L = \emptyset$ implies that $f^{-1}(L) \subset X \setminus M$, so that $\operatorname{Ind} f^{-1}(L) \leq n-1$. Consider the restriction $f_L: f^{-1}(L) \to L$; since $C_2(f_L) \subset C_2(f) \cap L$, we have $\operatorname{Ind} C_2(f_L) \leq n-2$, so that by the inductive assumption $\operatorname{Ind} L \leq n-1$. Thus $\operatorname{Ind} Y \leq n$. ■

4.3.10. VAĬNŠTEĬN'S SECOND THEOREM. *Let* $f: X \to Y$ *be a finite-dimensional closed mapping of a metrizable space* X *onto a metrizable space* Y. *If* $\operatorname{Ind} X \leq n \geq 1$, $\operatorname{Ind} C_2(f) \leq n-1$ *and* $\operatorname{Ind} D_i(f) \leq n-(i+1)$ *for* $i = 1, 2, \ldots, n$, *then* $\operatorname{Ind} Y \leq n$.

In other words, $\operatorname{Ind} Y \leq \max\{\operatorname{Ind} X, \operatorname{Ind} C_2(f)+1, I_f+1\}$ *for every finite-dimensional closed mapping* f *of a metrizable space* X *onto a metrizable space* Y.

PROOF. We shall apply induction with respect to n. If $n = 1$, we have $\operatorname{Ind} D_1(f) = -1$, so that $\operatorname{Ind} f = 0$ and the theorem reduces to Lemma 4.3.9. Assume that the theorem holds for all natural numbers less than $n \geq 2$ and consider a closed mapping $f: X \to Y$ which satisfies the assumption of the theorem.

By virtue of Theorem 4.3.7 the set $C = f^{-1}(C_2(f))$ satisfies the inequality $\operatorname{Ind} C \leq n-1$. Hence from Lemmas 4.3.5 and 4.3.8 it follows that there exists an F_σ-set $M \subset X$ such that $\operatorname{Ind} M \leq 0$, $\operatorname{Ind}(X \setminus M) \leq n-1$ and $M \cap C = \emptyset$; by the countable sum theorem $\operatorname{Ind} f(M) \leq 0$.

Consider an arbitrary pair A, B of disjoint closed subsets of Y. From Lemma 4.3.6 with $k = n+1$, $M_i = D_i(f)$ for $i = 1, 2, \ldots, n-1$, $M_n = f(M)$ and $M_{n+1} = C_2(f)$ it follows that there exists a partition L between A and B such that $f(M) \cap L = \emptyset$, $\operatorname{Ind}(C_2(f) \cap L) \leq n-2$ and for each $i \leq n-1$

$$\operatorname{Ind}(D_i(f) \cap L) \leq \operatorname{Ind} D_i(f) - 1,$$

provided that $D_i(f) \neq \emptyset$. The equality $f(M) \cap L = \emptyset$ implies that $\operatorname{Ind} f^{-1}(L) \leq n-1$. Consider the restriction $f_L: f^{-1}(L) \to L$; since $C_2(f_L) \subset C_2(f) \cap L$ and $D_i(f_L) \subset D_i(f) \cap L$ for $i = 1, 2, \ldots, n-1$, we have $\operatorname{Ind} C_2(f_L) \leq n-2$ and $\operatorname{Ind} D_i(f_L) \leq (n-1)-(i+1)$ for $i = 1, 2, \ldots, n-1$, so that by the inductive assumption $\operatorname{Ind} L \leq n-1$. Thus $\operatorname{Ind} Y \leq n$. ■

In the final part of this section we shall use the covering dimension which better corresponds to the spirit of three characterizations of dimension of metrizable spaces that we are now going to prove.

We start with three lemmas.

4.3.11. LEMMA. *If X is a metrizable space and $\dim X \leq n$ then there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X satisfying the conditions:*

(i) *For every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that $\operatorname{St}(x, \mathcal{F}_i) \subset U$.*

(ii) $\operatorname{ord} \mathcal{F}_i \leq n$ *for* $i = 1, 2, \ldots$

(iii) $\mathcal{F}_{i+1}$ *is a refinement of* $\mathcal{F}_i$ *for* $i = 1, 2, \ldots$

PROOF. It suffices to define a development $\mathcal{W}_1, \mathcal{W}_2, \ldots$ for the space X such that $\operatorname{ord}(\{\overline{W} : W \in \mathcal{W}_i\}) \leq n$ and $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$, and then let $\mathcal{F}_i = \{\overline{W} : W \in \mathcal{W}_i\}$. In the standard inductive definition of such a development (see the proof of the implication (i)⇒(ii) in Proposition 4.1.2 or in Theorem 4.2.3) one has to use the fact that every locally finite open cover $\{U_s\}_{s \in S}$ of a normal space has an open shrinking $\{V_s\}_{s \in S}$ such that $\overline{V}_s \subset U_s$ for every $s \in S$ (see Problem 3.1.A(b)). ■

4.3.12. LEMMA. *If X is a metrizable space and $\dim X \leq n$ then there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X, where $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$, satisfying conditions* (i), (ii) *and*

(iii′) $F_{t_1 t_2 \ldots t_i} = \bigcup_{t \in T} F_{t_1 t_2 \ldots t_i t}$ *for all* $t_1, t_2, \ldots, t_i \in T$ *and* $i = 1, 2, \ldots$

PROOF. Denote by $\mathcal{E}_1, \mathcal{E}_2, \ldots$ a sequence of locally finite closed covers of the space X satisfying the conditions in Lemma 4.3.11. For $i = 1, 2, \ldots$ let f_i be a mapping of $\mathcal{E}_{i+1}$ to $\mathcal{E}_i$ such that $F \subset f_i(F)$ for all $F \in \mathcal{E}_{i+1}$. Now, for each $F \in \mathcal{E}_i$, where $i = 1, 2, \ldots$, consider the set

$$F^* = \bigcup \Big\{ \bigcap_{j=i}^{\infty} F_j : F_i = F,\ F_j \in \mathcal{E}_j \text{ and } f_j(F_{j+1}) = F_j \text{ for } j = i, i+1, \ldots \Big\}$$

and for $i = 1, 2, \ldots$ let $\mathcal{F}_i^* = \{F^* : F \in \mathcal{E}_i\}$. Since $F^* \subset F$ for $F \in \mathcal{E}_i$ and $i = 1, 2, \ldots$, the families $\mathcal{F}_1^*, \mathcal{F}_2^*, \ldots$ are locally finite and satisfy (i) and (ii) with $\mathcal{F}_i$ replaced by $\mathcal{F}_i^*$. To prove that $\mathcal{F}_i^*$ is a closed cover of X we shall show that

(4) if $x \notin F^* \in \mathcal{F}_i^*$, then there exists a $j > i$ such that for each $E \in \mathcal{E}_j$ satisfying $f_i f_{i+1} \ldots f_{j-1}(E) = F$ we have $x \notin E$.

Assume the contrary; then for each $j > i$ there are $F_j, F_{j-1}, \ldots, F_i$ such that

(5) $F_i = F$, $x \in F_j \in \mathcal{E}_j$ and $f_k(F_{k+1}) = F_k \in \mathcal{E}_k$ for $k = i, i+1, \ldots, j-1$.

For each $j \geq i$ consider the finite family $\mathcal{S}_j = \{F \in \mathcal{E}_j : x \in F\}$ endowed with the discrete topology; the Cartesian product $\mathcal{S} = \prod_{j=i}^{\infty} \mathcal{S}_j$ is compact, and for each $j > i$ the subspace

$$A_j = \bigcup (\{F_i\} \times \{F_{i+1}\} \times \ldots \times \{F_j\} \times \mathcal{S}_{j+1} \times \mathcal{S}_{j+2} \times \ldots),$$

where the union is taken over all $F_i, F_{i+1}, \ldots, F_j$ satisfying (5), is closed in $\mathcal{S}$ and non-empty. As obviously $A_{i+1} \supset A_{i+2} \supset \ldots$, by the compactness of $\mathcal{S}$ there exists

$(F_i^0, F_{i+1}^0, \ldots) \in \bigcap_{j=i+1}^{\infty} A_j$. Clearly, $x \in \bigcap_{j=i}^{\infty} F_j^0$, $F_i^0 = F$, $F_j^0 \in \mathcal{E}_j$ and $f_j(F_{j+1}^0) = F_j^0$ for $j = i, i+1, \ldots$, so that $x \in F^*$. Thus (4) is established.

From (4) it follows that the members of $\mathcal{F}_i^*$ are closed; indeed, if $x \notin F^* \in \mathcal{F}_i^*$, then for any j satisfying (4) the set

$$X \setminus \bigcup \{E \in \mathcal{E}_j : x \notin E\}$$

is a neighbourhood of x disjoint from F^*. It also follows from (4) that $\mathcal{F}_i^*$ is a cover of X; indeed, if $x \notin F_k^*$ for $k = 1, 2, \ldots, m$, where $F_1, F_2, \ldots, F_m$ are all the members of $\mathcal{E}_i$ that contain x, then by (4) there exists a $j > i$ such that $x \notin E$ for each $E \in \mathcal{E}_j$ satisfying $f_i f_{i+1} \ldots f_{j-1}(E) = F_k$ with $k \in \{1, 2, \ldots, m\}$, but this contradicts the fact that $\mathcal{E}_j$ is a cover of X.

Since obviously for each $F \in \mathcal{E}_i$, where $i = 1, 2, \ldots$, we have

$$F^* = \bigcup \{E^* : E \in \mathcal{E}_{i+1} \text{ and } f_i(E) = F\},$$

to obtain the required families $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$ it suffices to let $T = \bigcup_{i=1}^{\infty} \mathcal{E}_i$ and for each sequence $t_1, t_2, \ldots, t_i \in T$ such that $t_i = F \in \mathcal{E}_i$, $t_{i-1} = f_{i-1}(t_i), \ldots, t_1 = f_1(t_2)$ define $F_{t_1 t_2 \ldots t_i} = F^*$, and for all the remaining sequences $t_1, t_2, \ldots, t_i$ let $F_{t_1 t_2 \ldots t_i} = \emptyset$. ■

4.3.13. REMARK. Let us observe that in the proof of Lemma 4.3.12 we used only the existence of a sequence satisfying the conditions in Lemma 4.3.11, and not the fact that $\dim X \leq n$.

4.3.14. LEMMA. *If for a metrizable space X there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers, where $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$, satisfying conditions* (i),

(ii′) *For each $x \in X$ there exists an integer n_x such that for $i = 1, 2, \ldots$ the point x belongs to at most n_x members of $\mathcal{F}_i$,*

and (iii′), *then the space X is the image of a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping f such that $|f^{-1}(x)| \leq n_x$ for each $x \in X$.*

PROOF. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$, where $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$, be such a sequence; from the local finiteness of $\mathcal{F}_1$ it follows that $|T| \leq w(X)$. In the Cartesian product $T^{\aleph_0}$, where T is endowed with the discrete topology, i.e., in the Baire space $B(|T|)$, consider the subspace

$$Z = \Big\{(t_1, t_2, \ldots) : \bigcap_{i=1}^{\infty} F_{t_1 t_2 \ldots t_i} \neq \emptyset\Big\};$$

by virtue of (i) for each $z = (t_1, t_2, \ldots) \in Z$ the intersection $\bigcap_{i=1}^{\infty} F_{t_1 t_2 \ldots t_i}$ consists of exactly one point of X, and assigning this point to z defines a mapping of Z to X. From (i) it follows that f is continuous, from (ii′) that $|f^{-1}(x)| \leq n_x$ for each $x \in X$, and from (iii′) that $f(Z) = X$.

We shall now prove that the mapping f is closed. Consider a closed subset A of Z and an arbitrary point $x \in X \setminus f(A)$. It suffices to show that x has a neighbourhood disjoint from $f(A)$. Let $f^{-1}(x) = \{z_1, z_2, \ldots, z_n\}$, where $z_k = (t_1^k, t_2^k, \ldots)$ for $k = 1, 2, \ldots, n$. Since the set A is closed and $z_k \notin A$ for $k = 1, 2, \ldots, n$, there exists an integer i with the

property that if the first i coordinates of a point $z \in Z$ are equal to the corresponding coordinates of one of the z_k's, then $z \notin A$. The open set

$$U = X \setminus \bigcup\{F_{t_1 t_2 \ldots t_i} : (t_1, t_2, \ldots, t_i) \neq (t_1^k, t_2^k, \ldots, t_i^k) \text{ for } k = 1, 2, \ldots, n\}$$

obviously is a neighbourhood of x, and if we have $f(z) \in U$ for some $z \in Z$, then the first i coordinates of z have to be equal to the corresponding coordinates of one of the z_k's, so that $z \notin A$; hence $U \cap f(A) = \emptyset$. ■

Lemmas 4.3.12 and 4.3.14 together with Theorems 4.3.1 and 3.1.28 yield

4.3.15. THEOREM. *A metrizable space X satisfies the inequality* $\dim X \leq n$ *if and only if X is the image of a metrizable space Z with* $\dim Z \leq 0$ *and* $w(Z) \leq w(X)$ *under a closed mapping with fibers of cardinality at most* $n + 1$. ■

Let us observe that Lemmas 4.3.12 and 4.3.14 together with Theorems 4.3.1 and 1.6.10 yield the inequality $\operatorname{Ind} X \leq \dim X$ for each metrizable space X. Let us also note that the space Z in Theorem 4.3.15 satisfies the equality $w(Z) = w(X)$; indeed, closed mappings with finite fibers cannot raise the weight (see [GT], Theorem 3.7.19). Finally, let us note that the mapping f can be obtained directly, although in a more elaborate way, from a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ satisfying the conditions in Lemma 4.3.11 (see Problem 4.3.D).

The existence of a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of closed locally finite covers satisfying the conditions in Lemma 4.3.11 or 4.3.12 characterizes the covering dimension. Indeed, from Lemmas 4.3.11, 4.3.12 and 4.3.14, Remark 4.3.13, and Theorem 4.3.1 we obtain the following two theorems (see Problem 4.3.E(b)).

4.3.16. THEOREM. *A metrizable space X satisfies the inequality* $\dim X \leq n$ *if and only if there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X such that* $\operatorname{ord} \mathcal{F}_i \leq n$ *and $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ for $i = 1, 2, \ldots$, and for every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that* $\operatorname{St}(x, \mathcal{F}_i) \subset U$. ■

4.3.17. THEOREM. *A metrizable space X satisfies the inequality* $\dim X \leq n$ *if and only if there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X, where $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$, such that* $\operatorname{ord} \mathcal{F}_i \leq n$ *for $i = 1, 2, \ldots$, $F_{t_1 t_2 \ldots t_i} = \bigcup_{t \in T} F_{t_1 t_2 \ldots t_i t}$ for all $t_1, t_2, \ldots, t_i \in T$ and $i = 1, 2, \ldots$, and for every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that* $\operatorname{St}(x, \mathcal{F}_i) \subset U$. ■

Let us note that by virtue of a metrization theorem (see [GT], Exercise 5.4.D(a)) in Theorems 4.3.16 and 4.3.17 the words "A metrizable space X satisfies the inequality $\dim X \leq n$" can be replaced by "A T_0-space X is metrizable and satisfies the inequality $\dim X \leq n$".

Historical and bibliographic notes

Theorem 4.3.1 was given by Morita in [1955]; Theorem 4.3.4 was proved independently by Morita in [1956] and by Nagami in [1957]. Theorems 4.3.7 and 4.3.10

were established by Vaĭnšteĭn in [1952] for separable X and Y. Extensions to arbitrary metrizable spaces were given by Skljarenko in [1962] and [1964] (announcement in [1963]). Lemma 4.3.9 was proved by Freudenthal in [1932]. As shown by Lelek in [1971], from Vaĭnšteĭn's theorems many further results about the behaviour of dimension under mappings can be deduced. Lelek's paper gives a comprehensive survey of the topic considered in this section and provides a good bibliography. Theorems 4.3.15 and 4.3.17 were established by Morita in [1955], and Theorem 4.3.16 by Nagami in [1960] (Theorem 4.3.15 for separable metric spaces was proved by Hurewicz in [1926]).

Problems

4.3.A (Suzuki [1959], Nagami [1960]). Prove that if $f: X \to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y such that all fibers of f are finite and have the same cardinality, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

Hint. See the hint to Problem 1.12.A.

4.3.B. (a) Following the pattern of Section 1.9, prove that for every continuous mapping $f: A \to S^n$ defined on a closed subspace A of a metrizable space X such that $\operatorname{Ind}(X \setminus A) \leq n \geq 0$ there exists a continuous extension $F: X \to S^n$ of f over X (cf. Problem 2.2.B).

(b) Deduce from (a), Theorem 4.1.3 and Remarks 1.7.10 and 1.9.4 that a metrizable space X satisfies the inequality $\operatorname{Ind} X \leq n \geq 0$ if and only if for every closed subspace A of the space X and each continuous mapping $f: A \to S^n$ there exists a continuous extension $F: X \to S^n$ of f over X.

4.3.C. (a) (Hodel [1963]) Show that if $f: X \to Y$ is an open mapping of a metrizable space X to a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X, then $\operatorname{Ind} X \leq \operatorname{Ind} Y$.

Hint. Apply Lemma 1.12.5.

(b) (Junnila [1978], R. Pol [1981]) Prove that for every non-empty metrizable space X there exists a metrizable space Z with $\operatorname{Ind} Z = 0$ that can be mapped onto X by an open mapping with discrete fibers.

Hint. For $i = 1, 2, \ldots$ let X_i be the set X with the discrete topology and let Z be the subspace of the Cartesian product $\prod_{i=1}^{\infty} X_i = [D(\mathbf{m})]^{\aleph_0}$, where $\mathbf{m} = |X|$, consisting of all points $\{x_i\}$ such that (1) there exists an integer j with the property that $x_i = x_j$ for $i \geq j$, (2) if $x_i = x_j$ for $i > j$, then $x_j = x_{j+1} = \ldots = x_{i-1} = x_i$, (3) $x_j \in \bigcap_{i=1}^{j-1} B(x_i, 1/i)$ for $j > 1$, where the balls $B(x_i, 1/i)$ are taken with respect to an arbitrary metric on the space X with its original topology. To each point $\{x_i\} \in Z$ assign the point $x \in X$ which is the common value of infinitely many x_i's. Note that the subspace of Z consisting of all points with the same integer j in (1) is discrete, and deduce that $\operatorname{Ind} Z = 0$.

(c) (R. Pol [1981]) Prove that if $f: X \to Y$ is an open mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is separable and has an isolated point, then $\operatorname{Ind} Y \leq \operatorname{Ind} X$.

Deduce that if $f: X \to Y$ is an open mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a countable discrete space, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

Hint. Consider a σ-locally finite base $\mathcal{B} = \{U_s\}_{s\in S}$ for the space X, note that the family $\{f(U_s)\}_{s\in S}$ is point-countable, and apply Lemma 1.12.5 and the result in the hint to Problem 4.1.D.

(d) (Hodel [1963]) Show that if $f: X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point, then $\operatorname{Ind} Y \leq \operatorname{Ind} X$. Note that the last inequality cannot be replaced by equality.

Hint. Apply Lemma 1.12.5 and Theorem 4.1.12.

(e) (Arhangel'skiĭ [1966a]; for separable spaces, Taĭmanov [1955]) Show that if $f: X \to Y$ is an open mapping of a completely metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{Ind} Y \leq \operatorname{Ind} X$.

Hint. Apply (c) and Problem 1.3.F(b) or the remark to Problem 2.4.K(a) and the hint to Problem 2.4.K(b).

(f) (Hodel [1963]) Show that if $f: X \to Y$ is an open mapping of a locally compact metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

(g) (Arhangel'skiĭ [1966a]) Show that if $f: X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then $\operatorname{Ind} X = \operatorname{Ind} Y$.

4.3.D (Nagami [1960]). Show that for every metrizable space X satisfying the inequality $\dim X \leq n$ a closed mapping $f: Z \to X$ with fibers of cardinality at most $n+1$ from a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ onto the space X can be obtained directly from a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ satisfying the conditions in Lemma 4.3.11.

Hint. Define a mapping π_i^{i+1} of $\mathcal{F}_{i+1}$ to $\mathcal{F}_i$ such that $F \subset \pi_i^{i+1}(F)$ for every $F \in \mathcal{F}_{i+1}$, let $\pi_j^i = \pi_j^{j+1}\pi_{j+1}^{j+2}\ldots\pi_{i-1}^i$ for $j < i$ and $\pi_i^i = \mathrm{id}_{\mathcal{F}_i}$. Consider the inverse sequence $\boldsymbol{S} = \{\mathcal{F}_i, \pi_j^i\}$, where $\mathcal{F}_i$ has the discrete topology. Note that $S = \varprojlim \boldsymbol{S}$ is a subspace of $B(w(X))$ and consider the subspace Z of S consisting of all sequences $\{F_i\} \in S$ such that $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$; show that the last intersection contains exactly one point of the space X and assign this point to the sequence $\{F_i\}$. Prove that the mapping $f: Z \to X$ obtained in this way has all the required properties. When proving that $f(Z) = X$ apply the fact that the limit of an inverse sequence of finite discrete spaces is non-empty. When proving that f is closed apply the equality $f^{-1}(x) = Z \cap \prod_{i=1}^{\infty}\{F \in \mathcal{F}_i : x \in F\}$.

4.3.E. (a) Show that every locally finite family $\{F_s\}_{s\in S}$ of closed subsets of a metrizable space X has an open swelling $\{U_s\}_{s\in S}$.

Hint. For every $x \in X$ take an $\epsilon_x > 0$ such that if $F_s \cap B(x, 2\epsilon_x) \neq \emptyset$, then $x \in F_s$, and let $U_s = \bigcup\{B(x, \epsilon_x) : x \in F_s\}$.

(b) Prove without using Lemma 4.3.14 that if for a metrizable space X there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X such that $\operatorname{ord} \mathcal{F}_i \leq n$ and $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ for $i = 1, 2, \ldots$, and for every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that $\operatorname{St}(x, \mathcal{F}_i) \subset U$, then $\dim X \leq n$.

Hint. Let $\mathcal{F}_i = \{F_s\}_{s\in S_i}$ for $i = 1, 2, \ldots$ Apply part (a) to define a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open covers of X, where $\mathcal{W}_i = \{W_s\}_{s\in S_i}$ is a swelling of the cover $\mathcal{F}_i$ and

$W_s \subset B(F_s, 1/i)$ for $s \in S_i$, such that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ for $i = 1, 2, \ldots$; check that $\mathcal{W}_1, \mathcal{W}_2, \ldots$ is a strong development for the space X and apply Theorem 4.2.3.

4.3.F. If for a continuous mapping $f: X \to Y$ of a normal space X to a normal space Y there exists an integer $n \geq 0$ such that $\dim f^{-1}(y) \leq n$ for every $y \in Y$, then we say that f is *finite-dimensional*; the smallest n with this property will be denoted by $\dim f$. For every finite-dimensional mapping $f: X \to Y$ we define

$$d_f = \max\{\mathrm{rd}_Y D_i(f) + i : i = 1, 2, \ldots, \dim f\} \quad \text{if } \dim f \geq 1$$

and

$$d_f = -1 \quad \text{if } \dim f = 0.$$

(a) (Filippov [1972a] and [1978]; for paracompact spaces, Skljarenko [1962]) Prove that for every finite-dimensional closed mapping $f: X \to Y$ of a normal space X to a weakly paracompact normal space Y we have $\dim X \leq \max\{\dim Y, d_f\}$.

Hint. See Problem 3.3.E.

(b) (Filippov [1972a] and [1978]; for paracompact spaces, implicitly, Skljarenko [1964] (announcement [1963])) Prove that for every finite-dimensional closed mapping $f: X \to Y$ of a normal space X onto a weakly paracompact normal space Y with $\dim Y < \infty$ we have $\dim Y \leq \max\{\dim X, \mathrm{rd}_Y C_2(f) + 1, d_f + 1\}$.

Hint. See Problem 3.3.E.

Chapter 5

Countable-dimensional spaces

This is the first of three chapters devoted to infinite-dimensional spaces. Here we measure the "degree of infinite dimensionality" by the possibility of representing spaces as unions of spaces with finite covering dimension. The three classes of spaces under discussion are – in ascending order – locally finite-dimensional spaces (unions of open finite-dimensional subspaces), strongly countable-dimensional spaces (countable unions of closed finite-dimensional subspaces), and countable-dimensional spaces (countable unions of finite-dimensional subspaces). In the following two chapters we shall present other methods of stratification of the class of infinite-dimensional spaces.

We base the theory of countable-dimensional, strongly countable-dimensional and locally finite-dimensional spaces on the covering dimension dim and develop it in the class of normal spaces. In a similar way one could define countable dimensionality, strong countable dimensionality and locally finite dimensionality using the large inductive dimension Ind, but the results, although identical in the crucial case of metrizable spaces, would in general be less satisfactory.

In the present chapter the knowledge of the material from Chapter 2 – with the exception of Section 2.1 and Lemma 2.3.3 – is not assumed. The reader interested only in separable metric spaces (or in metrizable spaces) should be able to follow our exposition after having read Sections 1.1–1.8 and 1.12 (or Chapter 4), but would occasionally have to resort to some results placed elsewhere in the book. Such a reader is advised to skip Lemma 5.2.9, provide a direct proof of Theorem 5.2.10 for metrizable spaces (see Problem 5.2.E), and then skip Theorems 5.2.11–5.2.13 as well as Theorems 5.2.16–5.2.19.

Section 5.1 opens with the definitions of countable-dimensional and strongly countable-dimensional spaces and their simple reformulations. Then we give an example of a countable-dimensional compact metrizable space that is not strongly countable-dimensional and two characterizations of strong countable dimensionality in terms of open covers. The section continues with parallel characterizations of countable-dimensional and strongly countable-dimensional metrizable spaces in terms of partitions and special bases, and concludes with characterizations of strongly countable-dimensional metrizable spaces in terms of developments and strong developments.

In Section 5.2 we first discuss properties of subspaces of countable-dimensional and strongly countable-dimensional spaces, and then the sum theorems and the Cartesian product theorems. The final part is devoted to an example of an uncountable-dimensional compact metrizable space whose non-empty countable-dimensional subspaces are all zero-dimensional.

Section 5.3 is devoted to the compactification and the universal space theorems for countable-dimensional and strongly countable-dimensional spaces.

In Section 5.4 we prove counterparts of theorems on dimension-raising and dimension-lowering closed mappings for countable-dimensional and strongly countable-dimensional spaces. Then we discuss the behaviour of these classes of spaces under open mappings, and we conclude with a characterization of countable-dimensional and strongly countable-dimensional metrizable spaces in terms of closed mappings defined on spaces with covering dimension zero.

Section 5.5 deals with locally finite-dimensional spaces. At the beginning we prove in turn the subspace theorem, the locally finite sum theorem, and the Cartesian product theorem. Then we give several characterizations of locally finite-dimensional spaces, and we discuss S-like and almost S-like subspaces. The section continues with characterizations of locally finite-dimensional metrizable spaces in terms of partitions, special bases, and closed mappings defined on spaces with covering dimension zero. At the end of the section two universal space theorems are established.

5.1. Definitions and characterizations of countable-dimensional and strongly countable-dimensional spaces

In the final part of Section 1.8 we hinted at the possibility of distinguishing between various infinite-dimensional spaces according to their "degree of infinite-dimensionality". We defined there three subspaces of the Hilbert cube: S, K_ω, and N_ω whose "degrees of infinite-dimensionality", intuitively, go increasing. In this section we shall make these considerations precise by introducing and studying countable-dimensional and strongly countable-dimensional spaces. In this chapter we shall consider only the covering dimension. We shall call each normal space X satisfying the inequality $\dim X < \infty$ *finite-dimensional*; if $\dim X = \infty$, we shall say that X is *infinite-dimensional*.

5.1.1. DEFINITION. A normal space X is *countable-dimensional* if X can be represented as a countable union of normal finite-dimensional subspaces. Normal spaces which are not countable-dimensional are called *uncountable-dimensional*.

Thus spaces S, K_ω and N_ω are countable-dimensional, and so are all finite-dimensional spaces; the Hilbert cube $I^{\aleph_0}$ is uncountable-dimensional (see Theorem 1.8.21).

From Theorem 3.1.17 we obtain the following

5.1.2. PROPOSITION. *A hereditarily normal space X is countable-dimensional if and only if X can be represented as the union of an increasing sequence of subspaces $X_1 \subset X_2 \subset \ldots$ such that $\dim X_n \leq n$ for $n = 1, 2, \ldots$* ■

Theorems 4.1.3 and 4.1.17 yield

5.1.3. PROPOSITION. *A metrizable space X is countable-dimensional if and only if X can be represented as the union of a sequence of subspaces $X_1, X_2, \ldots$ such that $\dim X_i \leq 0$ for $i = 1, 2, \ldots$* ■

Let us note that – as will be shown later – one can assume that the subspaces in the last proposition form an increasing sequence (see Corollary 5.1.14 and Problem 5.1.H; cf. Problem 5.1.B(a)).

Let us also note that under the assumption of the continuum hypothesis there exists a perfectly normal compact space X with $\dim X = 1$ that cannot be represented as countable union of subspaces whose covering dimension is equal to zero (cf. Problem 5.1.A). No example of a hereditarily normal countable-dimensional space X which cannot be represented as the union of a sequence $X_1, X_2, \ldots$ of subspaces such that $\dim X_i \leq 0$ for $i = 1, 2, \ldots$ has been constructed yet without additional set-theoretic assumptions.

5.1.4. DEFINITION. A normal space X is *strongly countable-dimensional* if X can be represented as a countable union of closed finite-dimensional subspaces.

The spaces S and K_ω, and all finite-dimensional spaces are strongly countable-dimensional. The space N_ω is not strongly countable-dimensional (see Theorem 5.3.7 and Problem 5.3.E).

5.1.5. PROPOSITION. *A normal space X is strongly countable-dimensional if and only if X can be represented as the union of an increasing sequence of closed subspaces $X_1 \subset X_2 \subset \ldots$ such that $\dim X_n \leq n$ for $n = 1, 2, \ldots$* ■

5.1.6. PROPOSITION. *Every strongly countable-dimensional normal space is countable-dimensional.* ■

We shall now define a countable-dimensional compact metric space that is not strongly countable-dimensional.

5.1.7. EXAMPLE. The Cantor set $C \subset R$ is defined as the intersection $\bigcap_{i=1}^{\infty} F_i$, where F_i consists of 2^i disjoint closed intervals of length $1/3^i$ each (see Example 1.2.5(d)). Let us arrange into a sequence $r_1, r_2, \ldots$, where $r_i \neq r_j$ for $i \neq j$, the end-points of all intervals removed from I to obtain the Cantor set. One easily sees that $\{r_1, r_2, \ldots\}$ is a dense subset of C. For $i = 1, 2, \ldots$ in the space $X = C \times C$ (which is homeomorphic to the Cantor set by Proposition 1.3.10) consider the 2^i disjoint intersections of the Cantor set $\{r_i\} \times C$ with the 2^i disjoint intervals the set $\{r_i\} \times F_i$ consists of, and arrange all such intersections into a sequence $C_1, C_2, \ldots$ Obviously, the sets C_n are all homeomorphic to the Cantor set and are pairwise disjoint; moreover, $\lim \delta(C_n) = 0$ and the union $A = \bigcup_{n=1}^{\infty} C_n$ is dense in X.

For $n = 1, 2, \ldots$ take a continuous mapping f_n with finite fibres that maps C_n onto the n-cube I^n (the existence of such a mapping follows from Problem 1.3.D(b) or 1.7.F(a)) and consider the equivalence relation E on the space X that corresponds to the decomposition of X into one-point subsets of $X \setminus A$ and the fibres $f_n^{-1}(x)$, where $x \in I^n$ and $n = 1, 2, \ldots$ It is not hard to check (cf. Problem 5.1.C) that the relation E is closed, so that the quotient space $Y = X/E$ is a metrizable compact space (see [GT], Theorem 4.2.13). Let $q: X \to Y$ be the natural quotient mapping and $B = Y \setminus q(A)$. Since $q_B: q^{-1}(B) \to B$ is a closed one-to-one mapping, the sets $q^{-1}(B) = X \setminus A$ and B are homeomorphic, so that $Y = B \cup \bigcup_{n=1}^{\infty} q(C_n)$ is a countable-dimensional space, each $q(C_n)$ being homeomorphic to I^n.

Now, for every non-empty open subset U of Y the inverse image $q^{-1}(U)$ contains infinitely many terms of the sequence $C_1, C_2, \ldots$, so that U contains subspaces of ar-

bitrarily high dimension, and thus $\dim U = \infty$. This, together with the Baire category theorem, implies that Y is not a strongly countable-dimensional space. ■

We shall now give a covering characterization of strongly countable-dimensional spaces.

5.1.8. LEMMA. *If a normal space X is represented as the union of an increasing sequence of closed subspaces $F_1 \subset F_2 \subset \ldots$ such that $\dim F_n \leq n$ for $n = 1, 2, \ldots$, then every finite open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|F_n \leq (2 + 3 + \ldots + n + (n+1)) - 1$.*

PROOF. For each n the finite open cover $\mathcal{U}|F_n$ of the subspace F_n has a closed shrinking $\mathcal{F}_n$ such that $\operatorname{ord} \mathcal{F}_n \leq n$, and by Theorem 3.1.1 there exists a family $\mathcal{V}_n$ of open subsets of X such that $F_n \subset \bigcup \mathcal{V}_n$, $\operatorname{ord} \mathcal{V}_n \leq n$ and every member of $\mathcal{V}_n$ is contained in a member of $\mathcal{U}$. One easily checks that the union $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n|(X \setminus F_{n-1})$, where $F_0 = \emptyset$, is an open refinement of $\mathcal{U}$ and has the required property. ■

5.1.9. THEOREM. *A normal space X is strongly countable-dimensional if and only if X can be represented as the union of a sequence of closed subsets $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$*

PROOF. Every normal space X that can be thus represented is strongly countable-dimensional, because $\dim X_n \leq n$ for $n = 1, 2, \ldots$ (cf. Problem 1.6.A(a)).

If X is a strongly countable-dimensional normal space, then $X = \bigcup_{n=1}^{\infty} F_n$, where $F_1 \subset F_2 \subset \ldots$ are closed and $\dim F_n \leq n$ for $n = 1, 2, \ldots$ By Lemma 5.1.8 the sequence $F_1, F_1, F_1, F_2, \ldots$, where F_n is repeated $n + 2$ times, has the required property. ■

By a small modification in the proof of the last theorem one obtains a similar characterization of paracompact strongly countable-dimensional spaces; we shall use it in the proof of Theorem 5.1.15 (cf. Problem 5.1.G):

5.1.10. THEOREM. *A paracompact space X is strongly countable-dimensional if and only if X can be represented as the union of a sequence of closed subspaces $X_1, X_2, \ldots$ such that every open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$*

PROOF. The modification consists in a different construction of the families $\mathcal{V}_n$ in Lemma 5.1.8, the covers $\mathcal{U}|F_n$ being no longer finite. By paracompactness of X one can assume that the open cover $\mathcal{U}$ is locally finite, so that $\mathcal{U}|F_n$, as a locally finite open cover of the subspace F_n, has a closed shrinking $\mathcal{F}_n = \{F_s\}_{s \in S}$ such that $\operatorname{ord} \mathcal{F}_n \leq n$ (see Theorem 3.2.1 and Problem 3.1.A(b)). As $\mathcal{F}_n$ is a locally finite family of closed subsets of X, every $x \in X$ has a neighbourhood H_x such that

$$\text{if } x \notin F_s \text{ for an } s \in S, \text{ then } \overline{H}_x \cap F_s = \emptyset. \tag{1}$$

Take an open locally finite cover $\mathcal{W}$ of the space X such that $\{\operatorname{St}(z, \mathcal{W})\}_{z \in X}$ is a refinement of $\{H_x\}_{x \in X}$ (see [GT], Theorem 5.1.12) and for every $s \in S$ consider the open set

$$V_s = X \setminus \bigcup \{\overline{W} : W \in \mathcal{W} \text{ and } \overline{W} \cap F_s = \emptyset\};$$

clearly $F_s \subset V_s$ for $s \in S$. Now, if $z \in V_{s_1} \cap V_{s_2} \cap \ldots \cap V_{s_k}$, there exist $W_1, W_2, \ldots, W_k \in \mathcal{W}$ such that $\overline{W}_i \cap F_{s_i} \neq \emptyset$ and $z \in W_i$ for $i = 1, 2, \ldots, k$; since $\mathrm{St}(z, \mathcal{W}) \subset H_x$ for some $x \in X$, and thus $\overline{H}_x \cap F_{s_i} \neq \emptyset$, it follows from (1) that $x \in F_{s_1} \cap F_{s_2} \cap \ldots \cap F_{s_k}$, so that $\operatorname{ord}\{V_s\}_{s \in S} \leq \operatorname{ord} \mathcal{F}_n \leq n$. Intersecting each set V_s with a member of $\mathcal{U}$ that contains the set F_s, we obtain a family $\mathcal{V}_n$ with the required properties. ■

Let us note that since for every open cover $\mathcal{V}$ of a topological space X and a subset M of X we have $\operatorname{ord} \mathcal{V}|M = \operatorname{ord} \mathcal{V}|\overline{M}$, in the last two theorems the assumption that the subspaces $X_1 \subset X_2 \subset \ldots$ are closed can be omitted.

Let us also note that in both theorems one can assume that $\mathcal{V}$ is a shrinking of $\mathcal{U}$.

Now we pass to characterizations of countable-dimensional and strongly countable-dimensional spaces in terms of partitions and special bases.

First of all we shall define a notion analogous to point-finiteness. A family $\mathcal{A}$ of subsets of a topological space X is *c-point-finite* if X can be represented as the union of a sequence of closed subsets $X_1, X_2, \ldots$ such that $\operatorname{ord} \mathcal{A}|X_k$ is finite for $k = 1, 2, \ldots$ One easily sees that by dropping the assumption that the sets X_i are closed one obtains the definition of point-finiteness, so that every c-point-finite family is point-finite (see Problem 5.1.F). On the other hand, from Theorem 5.2.3 in [GT] (cf. Lemma 3.4.7) it follows that in a countably paracompact normal space every locally finite family of sets is c-point-finite; it is not hard to provide examples showing that the last implication cannot be reversed.

Before formulating our characterizations we shall prove two lemmas that – for the purpose of further application – are stated in a way slightly more general than needed here. The proof of the first one is similar to the proof of Lemma 4.2.1.

5.1.11. LEMMA. *Let X be a normal space, $\mathcal{M}$ a family of subsets of X, and let an integer $n_M \geq 0$ correspond to each member M of the family $\mathcal{M}$. If every finite open cover $\mathcal{U}$ of X has an open shrinking $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|M \leq n_M$ for each $M \in \mathcal{M}$, then for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $\operatorname{ord}(\{L_i\}_{i=1}^{\infty}|M) \leq n_M - 1$ for each $M \in \mathcal{M}$.*

PROOF. Applying the second part of Remark 3.2.7 one can easily define by induction open sets $V_{j,1}, V_{j,2}, \ldots, V_{j,j}$ and $W_{j,1}, W_{j,2}, \ldots, W_{j,j}$ for $j = 1, 2, \ldots$ satisfying the following conditions:

$$A_j \subset V_{j,j} \subset \overline{V}_{j,j} \subset W_{j,j} \subset \overline{W}_{j,j} \subset X \setminus B_j, \tag{2}$$

$$\overline{V}_{j-1,i} \subset V_{j,i} \subset \overline{V}_{j,i} \subset W_{j,i} \subset \overline{W}_{j,i} \subset W_{j-1,i} \quad \text{for } i < j, \tag{3}$$

$$\operatorname{ord}(\{(\overline{W}_{j,i} \setminus V_{j,i}) \cap M\}_{i=1}^{j}) \leq n_M - 1 \quad \text{for each } M \in \mathcal{M}. \tag{4}$$

Conditions (2) and (3) imply that the open sets $V_i = \bigcup_{j=i}^{\infty} V_{j,i}$ satisfy $A_i \subset V_i \subset \overline{V}_i \subset X \setminus B_i$ for $i = 1, 2, \ldots$, so that the set $L_i = \operatorname{Fr} V_i$ is a partition between A_i and B_i for $i = 1, 2, \ldots$ Since $L_i = \overline{V}_i \setminus V_i \subset \overline{W}_{j,i} \setminus V_{j,i}$ for $j \geq i$, it follows from (4) that $\operatorname{ord}(\{L_i\}_{i=1}^{\infty}|M) \leq n_M - 1$ for each $M \in \mathcal{M}$. ■

5.1.12. LEMMA. *Let X be a metrizable space, $\mathcal{M}$ a family of subsets of X, and let an integer $n_M \geq -1$ correspond to each member M of the family $\mathcal{M}$. If for every*

sequence $(A_1, B_1), (A_2, B_2), \ldots$ *of pairs of disjoint closed subsets of* X *there exist closed sets* $L_1, L_2, \ldots$ *such that* L_i *is a partition between* A_i *and* B_i *and* $\operatorname{ord}(\{L_i\}_{i=1}^{\infty}|M) \leq n_M$ *for each* $M \in \mathcal{M}$, *then the space* X *has a* σ*-discrete base* $\mathcal{B}$ *such that* $\operatorname{ord}(\{\operatorname{Fr} U : U \in \mathcal{B}\}|M) \leq n_M$ *for each* $M \in \mathcal{M}$.

PROOF. Consider a σ-discrete base $\{V_s\}_{s\in S}$ for X, where $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ whenever $i \neq j$, and the family $\{V_s\}_{s\in S_i}$ is discrete for $i = 1, 2, \ldots$ For every $s \in S$ let $V_s = \bigcup_{j=1}^{\infty} A_{s,j}$, where the sets $A_{s,j}$ are closed, and for $i, j = 1, 2, \ldots$ consider the disjoint closed sets

$$A_{i,j} = \bigcup_{s\in S_i} A_{s,j} \quad \text{and} \quad B_{i,j} = X \setminus W_i, \quad \text{where } W_i = \bigcup_{s\in S_i} V_s.$$

There exist open sets $U_{i,j}$ such that $A_{i,j} \subset U_{i,j} \subset W_i$ for $i, j = 1, 2, \ldots$ and $\operatorname{ord}(\{\operatorname{Fr} U_{i,j}\}_{i,j=1}^{\infty}|M) \leq n_M$ for each $M \in \mathcal{M}$. The family $\mathcal{B}_{i,j} = \{U_{s,j}\}_{s\in S_i}$, where $U_{s,j} = V_s \cap U_{i,j}$, is discrete and $\mathcal{B} = \bigcup_{i,j=1}^{\infty} \mathcal{B}_{i,j}$ is a base for X. The family $\{\operatorname{Fr} U_{s,j}\}_{s\in S_i}$ is also discrete, and – since $\bigcup \mathcal{B}_{i,j} = U_{i,j}$ – we have $\bigcup_{s\in S_i} \operatorname{Fr} U_{s,j} = \operatorname{Fr} U_{i,j}$, so that if a point belongs to $\operatorname{Fr} U_{i,j}$, then it belongs to exactly one set $\operatorname{Fr} U_{s,j}$ with $s \in S_i$. Hence $\operatorname{ord}(\{\operatorname{Fr} U : U \in \mathcal{B}\}|M) \leq n_M$ for each $M \in \mathcal{M}$. ■

5.1.13. THEOREM. *For every metrizable space* X *the following conditions are equivalent*:

(i) *The space* X *is countable-dimensional* (*strongly countable-dimensional*).

(ii) *For every sequence* $(A_1, B_1), (A_2, B_2), \ldots$ *of pairs of disjoint closed subsets of* X *there exist closed sets* $L_1, L_2, \ldots$ *such that* L_i *is a partition between* A_i *and* B_i *and the family* $\{L_i\}_{i=1}^{\infty}$ *is point-finite* (*c-point-finite*).

(iii) *The space* X *has a* σ*-discrete base* $\mathcal{B}$ *such that the family* $\{\operatorname{Fr} U : U \in \mathcal{B}\}$ *is point-finite* (*c-point-finite*).

PROOF. We shall show first that (i)⇒(ii). In the case of a strongly countable-dimensional space X the implication follows directly from Theorem 5.1.9 and Lemma 5.1.11. Consider now a countable-dimensional space X. By virtue of Proposition 5.1.3, X can be represented as the union of a sequence of subspaces $X_1, X_2, \ldots$ such that $\dim X_n \leq 0$ for $n = 1, 2, \ldots$ Now, if a sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X is given, then letting $X_0 = L_0 = \emptyset$ and applying Lemma 4.2.9 and Theorem 1.6.10 we recursively obtain closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $\operatorname{ord}(\{L_j\}_{j=1}^{i}|X_n) \leq n - 1$ for $i, n = 1, 2, \ldots$ Since $X = \bigcup_{n=1}^{\infty} X_n$, the last inequality implies that the family $\{L_i\}_{i=1}^{\infty}$ is point-finite.

The implication (ii)⇒(iii) follows from Lemma 5.1.12.

To conclude the proof it remains to show that (iii)⇒(i). Consider a σ-discrete base $\mathcal{B}$ for the space X such that the family $\mathcal{F} = \{\operatorname{Fr} U : U \in \mathcal{B}\}$ is point-finite (*c*-point-finite) and represent X as the union of subsets (closed subsets) $X_1, X_2, \ldots$ such that $\operatorname{ord} \mathcal{F}|X_k = n_k < \infty$ for $k = 1, 2, \ldots$ Since $\operatorname{Fr}_{X_k}(X_k \cap U)$, the boundary of $X_k \cap U$ in the subspace X_k, is contained in $X_k \cap \operatorname{Fr} U$, the family $\{X_k \cap U : U \in \mathcal{B}\}$ is a σ-discrete base for X_k such that $\operatorname{ord}(\{\operatorname{Fr}_{X_k}(X_k \cap U) : U \in \mathcal{B}\}) \leq n_k$. Thus $\dim X_k < \infty$ by Theorem 4.2.2, and the space X is countable-dimensional (strongly countable-dimensional). ■

Let us note that the σ-discreteness of $\mathcal{B}$ in condition (iii) of the last theorem can be replaced by its σ-local finiteness.

5.1.14. COROLLARY. *A metrizable space X is countable-dimensional if and only if X can be represented as the union of an increasing sequence $X_1 \subset X_2 \subset \dots$ of subspaces such that* $\dim X_i \leq 0$ *for* $i = 1, 2, \dots$

PROOF. We have to show that a countable-dimensional metrizable space X can be represented as the union of a sequence of subspaces $X_1 \subset X_2 \subset \dots$ such that $\dim X_i \leq 0$ for $i = 1, 2, \dots$ From Theorem 5.1.13 it follows that X has a base $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$, where $\mathcal{B}_i$ is discrete, such that the family $\{\operatorname{Fr} U : U \in \mathcal{B}\}$ is point-finite. One easily sees that the subspaces

$$X_i = X \setminus \bigcup \Big\{ \operatorname{Fr} U : U \in \bigcup_{j=i}^{\infty} \mathcal{B}_j \Big\}, \quad i = 1, 2, \dots,$$

form an increasing sequence and that their union is equal to X. To conclude the proof it suffices to observe that the inequality $\dim X_i \leq 0$ follows from the fact that the family $\bigcup_{j=i}^{\infty} \mathcal{B}_j$ enlarged by all one-point open subsets of X is a base for the space X. ■

We conclude this section with a characterization of strongly countable-dimensional spaces in terms of developments.

5.1.15. THEOREM. *For every metrizable space X the following conditions are equivalent:*

(i) *The space X is strongly countable-dimensional.*

(ii) *There exists a development $\mathcal{W}_1, \mathcal{W}_2, \dots$ for the space X and a closed cover $X_1, X_2, \dots$ of X such that $\mathcal{W}_{i+1}$ is a star refinement of $\mathcal{W}_i$ and* $\operatorname{ord} \mathcal{W}_i | X_n \leq n$ *for* $i = 1, 2, \dots$ *and* $n = 1, 2, \dots$

(iii) *There exists a strong development $\mathcal{W}_1, \mathcal{W}_2, \dots$ for the space X and a closed cover $X_1, X_2, \dots$ of X such that $\mathcal{W}_{i+1}$ is a refinement of $\mathcal{W}_i$ and* $\operatorname{ord} \mathcal{W}_i | X_n \leq n$ *for* $i = 1, 2, \dots$ *and* $n = 1, 2, \dots$

PROOF. To show that (i)⇒(ii) we apply Theorem 5.1.10 and represent X as the union of a sequence of closed subspaces $X_1, X_2, \dots$ such that every open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V} | X_n \leq n$ for $n = 1, 2, \dots$ Then we define inductively the sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ as in the proof of the corresponding implication in Theorem 4.2.3.

The implication (ii)⇒(iii) is obvious, and the implication (iii)⇒(i) follows from Theorem 4.2.3. ■

Clearly in conditions (ii) and (iii) of the last theorem the assumption that the subspaces $X_1, X_2, \dots$ are closed can be omitted.

Historical and bibliographic notes

Countable-dimensional spaces were introduced by Hurewicz in [1928] (in the realm of separable metrizable spaces); Urysohn in [1926] conjectured that the Hilbert cube cannot be represented as a countable union of zero-dimensional spaces. A perfectly normal compact space X with $\dim X = 1$ that cannot be represented as a countable union of subspaces whose dim is equal to zero was defined (under the assumption of the continuum hypothesis) by Odincov in [1986]; such spaces of higher dimension were known earlier (see Problem 5.1.A).

In [1928] Hurewicz announced the existence of a compact metrizable countable-dimensional space that cannot be represented as a union of countably many closed finite-dimensional subspaces (but he did not consider the class of strongly countable-dimensional spaces). Hurewicz never published his example which was rediscovered by Smirnov in [1959]; this paper and Nagata's [1960] were the first where the class of strongly countable-dimensional spaces has been studied. Our Example 5.1.7 is, in principle, Smirnov's original example (simplified by Alexandroff and Pasynkov in [1973] and by Engelking and E. Pol in [1983]).

Theorems 5.1.9 and 5.1.10 (in a different formulation) can be found in Arhangel'skiĭ [1963] (the first was announced in [1962]). The equivalence of conditions (i) and (iii) in Theorem 5.1.13 for countable-dimensional spaces was established by Nagata in [1958a], and the equivalence of conditions (i) and (ii) for countable-dimensional spaces by Nagami and Roberts in [1965] and Nagata in [1965]; in the case of strongly countable-dimensional spaces Theorem 5.1.13 was proved in Engelking [1992]. Corollary 5.1.14 was obtained by Nagami in [1965] (for separable metrizable spaces by Tumarkin in [1960]). The equivalence of conditions (i) and (ii) in Theorem 5.1.15 was established by Nagata in [1960] (for separable metrizable spaces by Skljarenko in [1959]); and the equivalence of conditions (i) and (iii) by Nagami and Roberts in [1965].

Problems

5.1.A (Fedorčuk [1978]). Prove that if a hereditarily normal compact space X with $\dim X > 0$ can be represented as the union of a sequence of subspaces $X_1, X_2, \ldots$ satisfying $\dim X_i \leq 0$ for $i = 1, 2, \ldots$, then X contains a closed subspace F such that $\dim F = 1$.

Deduce that the compact perfectly normal spaces, mentioned in Section 3.1, whose dim equals n, and whose non-empty closed subspaces all have dim equal to 0 or n, are not countable-dimensional (recall that these spaces are defined under the assumption of the continuum hypothesis for $n \geq 2$).

Hint. Assume that $\dim F > 1$ for each closed set $F \subset X$ such that $\dim F \geq 1$. Note that $\operatorname{Ind} X > 1$ and use Theorem 2.2.4 to define a sequence $F_1 \supset F_2 \supset \ldots$ of closed sets such that $\dim F_i > 1$ and $F_i \cap X_i = \emptyset$ for $i = 1, 2, \ldots$

5.1.B. (a) (Smirnov, cited in Arhangel'skiĭ [1963]) Show that every metrizable space X can be represented as the union of an increasing transfinite sequence of subspaces $X_1 \subset X_2 \subset \ldots \subset X_\alpha \subset \ldots$, $\alpha < \omega_1$, such that $\dim X_\alpha \leq 0$ for each $\alpha < \omega_1$.

Hint. See the hint to Problem 1.8.J. Then apply the fact that every metrizable space of weight $\mathbf{m}$ is embeddable in the Cartesian product $[J(\mathbf{m})]^{\aleph_0}$ (see [GT], Theorem 4.4.9).

(b) (Arhangel'skiĭ [1963] (announcement [1962])) Show that every metrizable space X can be represented as the union of a family $\{X_s\}_{s \in S}$ of closed subspaces such that $|S| \leq \mathbf{c}$ and $\dim X_s \leq 0$ for each $s \in S$.

Hint. See the hint to part (a).

Remark. Note that the statement: "the Hilbert cube $I^{\aleph_0}$ can be represented as the union of $\aleph_1$ closed finite-dimensional subspaces" is independent of the axioms of set theory (see the remark to Problem 1.5.I(c)).

5.1.C. Let X be a compact metric space, $A_1, A_2, \ldots$ a sequence of pairwise disjoint closed subsets of X such that $\lim \delta(A_i) = 0$, and E_i a closed equivalence relation on A_i for $i = 1, 2, \ldots$ Show that the equivalence relation on the space X that corresponds to the decomposition of X into one-point subsets of $X \setminus \bigcup_{i=1}^{\infty} A_i$ and the equivalence classes of all relations E_i is closed.

5.1.D. Prove that for every point x of the countable-dimensional space Y that is not strongly countable-dimensional, defined in Example 5.1.7, and for each neighbourhood $V \subset Y$ of the point x there exists an open set $U \subset Y$ such that $x \in U \subset V$ and $\operatorname{ind} \operatorname{Fr} U < \infty$.

5.1.E. (a) (Levšenko [1965], Urbański [1995]) Show that if X_1, X_2 are connected and locally connected completely metrizable spaces whose non-empty open subsets all contain topologically the n-cubes I^n for $n = 1, 2, \ldots$, then the Cartesian product $X_1 \times X_2$ cannot be separated by a finite-dimensional closed subset.

Hint. Assume that there exists a closed subset L of $X = X_1 \times X_2$ such that $\dim L = n$ and $X \setminus L = U \cup W$, where U and W are disjoint non-empty open subsets of X. Take points $(x_1, x_2) \in U$ and $(y_1, y_2) \in W$ for which there exist subspaces $I_1 \subset X_1$ and $I_2 \subset X_2$, both homeomorphic to I^{n+2}, such that $x_1 \in I_1$ and $y_2 \in I_2$, and – applying the Mazurkiewicz–Moore theorem (Engelking and Sieklucki [1992], Theorem 6.5.17) – find subspaces $A_1 \subset X_1$ and $A_2 \subset X_2$, both homeomorphic to the closed unit interval, such that $x_1, y_1 \in A_1$ and $x_2, y_2 \in A_2$. Check that the set $[(I_1 \times A_2) \cup (I_1 \times I_2) \cup (A_1 \times I_2)] \setminus L$ is connected and deduce a contradiction.

(b) (Urbański [1995]) Show that if X is a locally connected completely metrizable space whose non-empty open subsets all contain topologically the n-cubes I^n for $n = 1, 2, \ldots$, then its square X^2 cannot be separated by a strongly countable-dimensional closed subset.

Hint. Assume that there exists a closed subset L of X^2 such that $L = \bigcup_{n=1}^{\infty} L_n$ with L_n's closed and finite-dimensional, and $X^2 \setminus L = U \cup W$, where U and W are disjoint non-empty open subsets of X^2; without loss of generality one can suppose that $\operatorname{Fr} U = L = \operatorname{Fr} W$. By the Baire category theorem there exists an n such that the interior of L_n in L is non-empty; thus there exist points $x_1, x_2 \in X$ and their respective connected neighbourhoods V_1, V_2 such that $(x_1, x_2) \in (V_1 \times V_2) \cap L \subset L_n$. Observe that the subset $(V_1 \times V_2) \cap L$ separates the space $V_1 \times V_2$ and obtain a contradiction with (a).

(c) (Urbański [1995]) Give an example of a countable-dimensional compact metrizable space that cannot be represented as the union of a strongly countable-dimensional space and a zero-dimensional space.

Hint. Observe that every metrizable space which is the union of a strongly countable-dimensional subspace and a zero-dimensional subspace, and contains at least two points, can be separated by a strongly countable-dimensional closed subset. Then modify the space in Example 5.1.7, and apply (b).

5.1.F. Give an example of a point-finite family of finite subsets of the closed unit interval I which is not c-point-finite.

5.1.G (Engelking [1992]). (a) Show that a normal space X is strongly countable-dimensional if and only if X can be represented as the union of a sequence of closed subsets $X_1, X_2, \ldots$ such that every locally finite open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$

Hint. Apply Problem 3.1.A(c) to show that in Lemma 5.1.8 one can assume that the cover $\mathcal{U}$ is locally finite.

(b) Show that a metrizable (normal) space X is countable-dimensional (strongly countable-dimensional) if and only if X can be represented as the union of a sequence of subsets (closed subsets) $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space X has a finite closed refinement $\mathcal{F}$ with the property that $\operatorname{ord} \mathcal{F}|X_n \leq n$ for $n = 1, 2, \ldots$

Hint. To obtain the "if" part in the countable-dimensional case, it suffices to show that if a subspace Y of a metrizable space X has the property that every finite open cover $\mathcal{U}$ of X has a finite closed refinement $\mathcal{F}$ such that $\operatorname{ord} \mathcal{F}|Y \leq n$, then $\dim Y \leq 2n + 1$.

Remark. An interesting characterization of strongly countable-dimensional compact metrizable spaces was given by E. Pol in [1988a].

5.1.H (Engelking and Olszewski [1991]). (a) Prove that if a perfectly normal space X can be represented as the union of a sequence $Z_1, Z_2, \ldots$ of G_δ-sets such that $\dim Z_i \leq 0$ for $i = 1, 2, \ldots$, then X can also be represented as the union of an increasing sequence $X_1 \subset X_2 \subset \ldots$ of G_δ-sets such that $\dim X_i \leq 0$ for $i = 1, 2, \ldots$

Hint. Start with showing by induction with respect to k that the conclusion holds if $Z_i = \emptyset$ for $i \geq k$. Passing to the general case, represent $Y_j = Z_1 \cup Z_2 \cup \ldots \cup Z_j$ as the union $\bigcup_{k=1}^{\infty} Y_{j,k}$, where $Y_{j,1} \subset Y_{j,2} \subset \ldots$ are G_δ-sets in X and $\dim Y_{j,k} \leq 0$ for $k = 1, 2, \ldots$ Let $Y_0 = \emptyset$ and note that the difference $Y_j \setminus Y_{j-1}$ can be represented as the union $\bigcup_{m=1}^{\infty} A_{j,m}$, where $A_{j,m}$ is the intersection of a closed and an open subset of Y_j. Using Problem 3.1.B(a), check that the sets $X_i = \bigcup\{Y_{j,k} \cap A_{j,m} : j + k + m = i + 2\}$ have the required properties.

(b) Note that (a) generalizes Corollary 5.1.14.

5.2. Basic properties of countable-dimensional and strongly countable-dimensional spaces

In this section we discuss the behaviour of the countable-dimensional and strongly countable-dimensional spaces under the operations of taking a subspace, the union of a family of subspaces, and the Cartesian product. We start with four simple subspace theorems.

Theorem 3.1.3 yields the following two theorems:

5.2.1. THEOREM. *Every closed subspace M of a countable-dimensional space X is countable-dimensional.* ■

5.2.2. THEOREM. *Every closed subspace M of a strongly countable-dimensional space X is strongly countable-dimensional.* ■

The next two theorems follow from Theorem 3.1.19.

5.2.3. THE SUBSPACE THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *Every subspace M of a countable-dimensional strongly hereditarily normal space X is countable-dimensional.* ■

5.2.4. THE SUBSPACE THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *Every subspace M of a strongly countable-dimensional strongly hereditarily normal space is strongly countable-dimensional.* ■

There exist examples which show that "strongly hereditarily normal" cannot be weakened to "hereditarily normal" in Theorems 5.2.3 and 5.2.4. More exactly, there exists (under the assumption of the continuum hypothesis) a hereditarily normal compact space X with $\dim X = 0$ that contains an open uncountable-dimensional subspace, and there exists a hereditarily normal Lindelöf space X with $\dim X = 0$ that contains a subspace which is not strongly countable-dimensional. Let us add that we know no example of a countable-dimensional hereditarily normal space constructed without any additional set-theoretic assumptions which contains an uncountable-dimensional subspace. Let us also mention, in connection with Proposition 3.1.4, that it is not known whether in a hereditarily normal space X all subspaces are countable-dimensional (strongly countable-dimensional) if and only if all open subspaces of X are countable-dimensional (strongly countable-dimensional).

Let us now briefly discuss the question of existence of finite-dimensional subspaces in countable-dimensional and strongly countable-dimensional spaces.

5.2.5. THEOREM. *If X is a countable-dimensional completely metrizable space with $\dim X = \infty$, then for $k = 0, 1, \ldots$ the space X contains a closed subspace M_k such that $\dim M_k = k$.*

PROOF. By Proposition 5.1.3 the space X can be represented as the union $\bigcup_{i=1}^{\infty} X_i$, where $\dim X_i = 0$ for $i = 1, 2, \ldots$ Let us assume that X does not contain any closed set of dimension $k \geq 1$; it follows from Theorem 4.1.8 that

(1) if L is a closed subspace of X and $\dim L < \infty$, then $\dim L < k$.

Lemma 3.1.27 (or Theorem 4.1.3) yields the existence of a pair A, B of disjoint closed subsets of X such that for every partition L between A and B we have $\dim L = \infty$. Applying Theorems 1.6.10 and 2.2.4 (or 4.1.13) we find a partition L_1 between A and B such that $L_1 \cap X_1 = \emptyset$. Let us fix a complete metric on the space X and represent L_1 as the union of a locally finite family of closed subsets of diameter less than 1; at least one of these sets, say F_1, is infinite-dimensional, since otherwise by (1) and the locally finite sum theorem we would have $\dim L_1 < k$.

Applying induction we obtain, by a similar argument, a decreasing sequence $F_1 \supset F_2 \supset \ldots$ of infinite-dimensional closed subsets of X such that $F_i \cap X_i = \emptyset$ and $\delta(F_i) < 1/i$

for $i = 1, 2, \ldots$ By the completeness of X we have $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$, but

$$\bigcap_{i=1}^{\infty} F_i \subset \bigcap_{i=1}^{\infty} (X \setminus X_i) = X \setminus \bigcup_{i=1}^{\infty} X_i = \emptyset;$$

the contradiction shows that X contains a closed subset M_k with $\dim M_k = k$. ■

It is not known if in the last theorem the assumption of complete metrizability of X can be weakened to metrizability of X. Indeed, it is an open problem if each countable-dimensional separable metrizable space X with $\dim X = \infty$ contains a one-dimensional subspace. There exist, however, uncountable-dimensional compact metrizable spaces which contain no finite-dimensional, or even countable-dimensional, subspaces of dimension > 0 (see Example 5.2.23).

From Theorems 4.1.3, 4.1.8 and 4.1.9 we obtain

5.2.6. THEOREM. *If X is a strongly countable-dimensional metrizable space with* $\dim X = \infty$, *then for $k = 0, 1, 2, \ldots$ the space X contains a closed finite-dimensional subspace M_k such that* $\dim M_k = k$. ■

We also have the following obvious

5.2.7. PROPOSITION. *If X is a strongly countable-dimensional normal space with* $\dim X = \infty$, *then for $k = 0, 1, 2, \ldots$ the space X contains a closed subspace M_k such that* $\dim M_k \geq k$. ■

It might happen, however, that for some k there is no closed subspace of dimension k in X (see Problem 5.2.C).

We now pass to sum theorems for countable-dimensional and strongly countable-dimensional spaces.

5.2.8. THE COUNTABLE SUM THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *If a normal space X can be represented as a countable union of countable-dimensional spaces, then X is a countable-dimensional space.* ■

The locally finite sum theorem for countable-dimensional spaces is not as obvious as the countable sum theorem; its proof will be preceded by a lemma.

5.2.9. LEMMA. *If a hereditarily normal space X can be represented as the union of a family of subspaces $\{X_s\}_{s \in S}$ such that* $\dim X_s \leq n$ *for $s \in S$ and each point $x \in X$ has a neighbourhood which meets at most i sets X_s, then* $\dim X \leq i(n+1) - 1$.

PROOF. We shall apply induction with respect to i. If $i = 1$ the family $\{X_s\}_{s \in S}$ is discrete and the sets X_s are open, so that $X = \bigoplus_{s \in S} X_s$ is the sum of the X_s's (see [GT], Section 2.2), and thus $\dim X \leq n = 1(n+1) - 1$.

Assume that the lemma holds for $i = m$, and consider the case of $i = m + 1$. For each $s_0 \in S$ we have $\overline{X}_{s_0} = X_{s_0} \cup X'_{s_0}$, where

$$X'_{s_0} = \bigcup \{\overline{X}_{s_0} \cap X_s : s \in S \setminus \{s_0\}\};$$

since – by virtue of the inductive assumption – $\dim X'_{s_0} \leq m(n+1) - 1$, Theorem 3.1.17

implies that $\dim \overline{X}_{s_0} \leq n+[m(n+1)-1]+1 = (m+1)(n+1)-1$. The family $\{\overline{X}_s\}_{s\in S}$ being locally finite, we have $\dim X \leq (m+1)(n+1)-1$ by Theorem 3.1.10, and thus the proof is complete. ■

In the next theorem the notion of countable paracompactness appears; let us recall that a normal space is countably paracompact if and only if for every countable open cover $\{U_i\}_{i=1}^{\infty}$ of the space X there exists a closed cover $\{F_i\}_{i=1}^{\infty}$ of X such that $F_i \subset U_i$ for $i = 1, 2, \ldots$ or – equivalently – if and only if for every sequence $F_1, F_2, \ldots$ of closed subsets of the space X satisfying $\bigcap_{i=1}^{\infty} F_i = \emptyset$ there exists a sequence $W_1, W_2, \ldots$ of open subsets of X such that $F_i \subset W_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} W_i = \emptyset$ (see [GT], Theorem 5.2.3; cf. Lemma 3.4.7).

5.2.10. THE LOCALLY FINITE SUM THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *If a space X which is either strongly hereditarily normal or hereditarily normal and countably paracompact can be represented as the union of a locally finite family $\{X_s\}_{s\in S}$ of countable-dimensional subspaces, then X is a countable-dimensional space.*

PROOF. For each $s \in S$ we have $X_s = \bigcup_{n=1}^{\infty} X_{s,n}$, where $\dim X_{s,n} \leq n$; define $X_n = \bigcup_{s\in S} X_{s,n}$ for $n = 1, 2, \ldots$ Let U_i consist of all points $x \in X$ that have a neighbourhood which meets at most i sets X_s. The family $\{U_i\}_{i=1}^{\infty}$ is an open cover of X.

If X is strongly hereditarily normal we have $\dim X_n \cap U_i \leq i(n+1)-1$ by Theorem 3.1.19 and Lemma 5.2.9, so that the space $X = \bigcup_{n,i=1}^{\infty}(X_n \cap U_i)$ is countable-dimensional.

If X is hereditarily normal and countably paracompact we can shrink the cover $\{U_i\}_{i=1}^{\infty}$ to a closed cover $\{F_i\}_{i=1}^{\infty}$, and we have $\dim X_n \cap F_i \leq i(n+1)-1$ by Theorem 3.1.3 and Lemma 5.2.9, so that also in this case $X = \bigcup_{n,i=1}^{\infty}(X_n \cap F_i)$ is countable-dimensional. ■

It is not known if the last theorem holds under the assumption that the space X is hereditarily normal, even under the additional assumption that X_s's are closed.

5.2.11. THE POINT-FINITE SUM THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *If a normal space X can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed countable-dimensional subspaces, and if there exists a point-finite open cover $\{U_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then X is a countable-dimensional space.*

PROOF. By virtue of Urysohn's lemma we can assume that U_s's are F_σ-sets. According to Lemma 2.3.3 the set K_i consisting of all points of X which belong to exactly i members of the cover $\{U_s\}_{s\in S}$ can be represented as the union $\bigcup_{T\in\mathcal{T}_i} K_T$, where $\mathcal{T}_i$ is the family of all subsets of S that have exactly i elements, and the sets $K_T = K_i \cap \bigcap_{s\in T} U_s$ are open in K_i and pairwise disjoint. Since $K_T = \bigcap_{s\in T} U_s \cap \bigcap_{s\notin T}(X \setminus U_s)$ is an F_σ-set in X, by normality of the subspace $\bigcup_{s\in T} F_s$ and by Lemma 3.1.22 and Theorems 5.2.1 and 5.2.8, the subspace $A_T = K_T \cap \bigcup_{s\in T} F_s$ of X is normal and countable-dimensional. Since the union $A_i = \bigcup_{T\in\mathcal{T}_i} A_T$, being the sum $\bigoplus_{T\in\mathcal{T}_i} A_T$, is normal and countable-dimensional, it suffices to show that $X = \bigcup_{i=1}^{\infty} A_i$.

Consider a point $x \in X$ and let $T = \{s \in S : x \in U_s\}$. There exists an $s \in S$ such that $x \in F_s$, and since $s \in T$, we have $x \in A_T$. ■

5.2.12. THEOREM. *If a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of open countable-dimensional subspaces, then X is a countable-dimensional space.*

PROOF. The space X being weakly paracompact, the cover $\{U_s\}_{s\in S}$ has a point-finite open shrinking $\{V_s\}_{s\in S}$, and – X being normal – the latter has a closed shrinking $\{F_s\}_{s\in S}$ (see Problem 3.1.A(b)). By Theorem 5.2.1 the subspaces F_s are countable-dimensional, and from Theorem 5.2.11 it follows that so is X. ■

5.2.13. THEOREM. *If a weakly paracompact hereditarily normal space X can be represented as the union of a locally countable family $\{X_s\}_{s\in S}$ of countable-dimensional subspaces, then X is a countable-dimensional space.*

PROOF. For every point $x \in X$ there exist an open F_σ-set U_x containing x and a countable set $S(x) \subset S$ such that $U_x \cap X_s = \emptyset$ if $s \notin S(x)$. From the last relation it follows that $U_x \subset \bigcup\{X_s : s \in S(x)\}$, so that by virtue of Theorems 5.2.8 and 5.2.1 the subspace U_x is countable-dimensional for each $x \in X$. To complete the proof it suffices to apply Theorem 5.2.12. ■

It is not known if the last theorem holds in weakly paracompact normal spaces; it is easily seen, though, that it does if we assume that the sets X_s are closed.

Let us note that the assumption of weak paracompactness in Theorems 5.2.12 and 5.2.13 is essential. Indeed, there exists a perfectly normal uncountable-dimensional space which can be represented as the union of its open countable-dimensional subspaces (for a normal space with these properties see Problem 5.5.A); there also exists, under the assumption of the continuum hypothesis, a perfectly normal uncountable-dimensional space which is locally countable.

We now pass to strongly countable-dimensional spaces.

5.2.14. THE COUNTABLE SUM THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *If a normal space X can be represented as a countable union of closed strongly countable-dimensional subspaces, then X is a strongly countable-dimensional space.* ■

As shown by Example 5.1.7 the assumption that the strongly countable-dimensional subspaces are closed cannot be omitted in the last theorem (cf. Problems 5.2.F(b) and (c)).

5.2.15. THE LOCALLY FINITE SUM THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *If a normal space X can be represented as the union of a locally finite family $\{F_s\}_{s\in S}$ of closed strongly countable-dimensional subspaces, then X is a strongly countable-dimensional space.*

PROOF. For each $s \in S$ we have $F_s = \bigcup_{n=1}^{\infty} F_{s,n}$, where the sets $F_{s,n}$ are closed and $\dim F_{s,n} \leq n$. Since the family $\{F_{s,n}\}_{s\in S}$ is locally finite, the set $X_n = \bigcup_{s\in S} F_{s,n}$ is closed in X, and thus normal. By Theorem 3.1.10 we have $\dim X_n \leq n$ for $n = 1, 2, \ldots$, so that X is strongly countable-dimensional. ■

5.2.16. THE POINT-FINITE SUM THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *If a normal space X can be represented as the union of a point-finite family $\{U_s\}_{s\in S}$ of open strongly countable-dimensional subspaces, then X is a strongly countable-dimensional space.*

PROOF. The open point-finite cover $\{U_s\}_{s\in S}$ of the space X has a shrinking $\{V_s\}_{s\in S}$ consisting of open F_σ-sets (see Problem 3.1.A(b)). From Theorems 5.2.2 and 5.2.14 and Lemma 3.1.22 it follows that the subspaces V_s are strongly countable-dimensional, so that for each $s \in S$ we have $V_s = \bigcup_{n=1}^{\infty} F_{s,n}$, where the sets $F_{s,n}$ are closed, $F_{s,1} \subset F_{s,2} \subset \ldots$, and $\dim F_{s,n} \leq n$. One easily checks that the closed sets $X_n = X \setminus \bigcup_{s\in S}(V_s \setminus F_{s,n})$, where $n = 1, 2, \ldots$, form a cover of the space X. Since $\dim X_n \cap F_{s,n} \leq n$ by virtue of Theorem 3.1.3, and since $X_n \cap F_{s,n} \subset X_n \cap V_s$ and the family $\{X_n \cap V_s\}_{s\in S}$ is a point-finite open cover of X_n, from Theorem 3.1.13 it follows that $\dim X_n \leq n$, so that X is strongly countable-dimensional. ■

The reader should note that the point-finite sum theorems for countable-dimensional and strongly countable-dimensional spaces are not strictly parallel (see Problem 5.2.F(e)).

5.2.17. THEOREM. *If a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of open strongly countable-dimensional subspaces, then X is a strongly countable-dimensional space.*

PROOF. The space X being weakly paracompact and normal, the cover $\{U_s\}_{s\in S}$ has a point-finite open shrinking $\{V_s\}_{s\in S}$ which consists of open F_σ-sets. From Theorems 5.2.2 and 5.2.14 and Lemma 3.1.22 it follows that the subspaces V_s are strongly countable-dimensional, so that X is strongly countable-dimensional by virtue of Theorem 5.2.16. ■

5.2.18. THEOREM. *If a weakly paracompact normal space X can be represented as the union of a locally countable family $\{F_s\}_{s\in S}$ of closed strongly countable-dimensional subspaces, then X is a strongly countable-dimensional space.*

PROOF. For every point $x \in X$ there exist an open F_σ-set U_x containing x and a countable set $S(x) \subset S$ such that $U_x \cap F_s = \emptyset$ if $s \notin S(x)$. From the last relation it follows that $U_x \subset \bigcup\{F_s : s \in S(x)\}$, so that by virtue of Lemma 3.1.22 and Theorems 5.2.14 and 5.2.2 the subspace U_x is strongly countable-dimensional for each $x \in X$. To complete the proof it suffices to apply Theorem 5.2.17. ■

Let us note that the assumption of weak paracompactness in Theorem 5.2.17 is essential (see Problem 5.5.A); there even exists a perfectly normal space which can be represented as the union of its open subspaces with covering dimension 0 and which is not strongly countable-dimensional (it is, however, countable-dimensional). The second of the two examples mentioned after the proof of Theorem 5.2.13 shows that the assumption of weak paracompactness is also essential in Theorem 5.2.18 (let us recall that this example requires the continuum hypothesis).

The four theorems on Cartesian products of countable-dimensional and strongly countable-dimensional spaces that we are now going to state are direct consequences of Theorems 3.4.4, 4.1.21 and 4.1.3, 3.4.6, and 3.4.9 respectively.

5.2.19. THEOREM. *The Cartesian product $X \times Y$ of a strongly countable-dimensional compact space X and a countable-dimensional hereditarily paracompact space Y is countable-dimensional.* ■

It is an open problem if the Cartesian product $X \times Y$ of a countable-dimensional compact space X and a countable-dimensional compact metrizable space Y is countable-dimensional.

5.2.20. THEOREM. *The Cartesian product $X \times Y$ of two countable-dimensional metrizable spaces X and Y is countable-dimensional.* ■

5.2.21. THEOREM. *If the Cartesian product $X \times Y$ of a strongly countable-dimensional compact space X and a strongly countable-dimensional space Y is normal, then it is strongly countable-dimensional.* ■

5.2.22. THEOREM. *If the Cartesian product $X \times Y$ of a strongly countable-dimensional metrizable space X and a strongly countable-dimensional space Y is normal, then it is strongly countable-dimensional.* ■

Let us note that under the assumption of the continuum hypothesis there exists a locally compact perfectly normal space X with $\dim X = 0$ such that the square $X \times X$ is perfectly normal and uncountable-dimensional. No example of a countable-dimensional space X such that $X \times X$ is normal and uncountable-dimensional has been constructed yet without additional set-theoretic assumptions.

We shall conclude this section with a rather surprising example of an uncountable-dimensional compact metric space X whose non-empty countable-dimensional subspaces are all zero-dimensional (for related examples see Problem 6.1.G).

5.2.23. EXAMPLE. For $i = 1, 2, \ldots$ denote by p_i the projection of the Hilbert cube $I^{\aleph_0} = \prod_{j=1}^{\infty} I_j$ onto I_i, and consider the pair of the opposite faces $A_i = p_i^{-1}(0), B_i = p_i^{-1}(1)$, as well as the pair of the larger disjoint closed sets

$$A_i^* = p_i^{-1}([0, 1/3]), \quad B_i^* = p_i^{-1}([2/3, 1]).$$

First we shall show that

(2) for each infinite subset N_0 of the set N of all positive integers, each $i_0 \in N$, and each set $K \subset I_{i_0}$ homeomorphic to the Cantor set there exist partitions L_i between A_i and B_i in $I^{\aleph_0}$ for $i \in N_0$ with the property that each space $Y \subset \bigcap\{L_i : i \in N_0\}$ satisfying $K \subset p_{i_0}(Y)$ is uncountable-dimensional.

Clearly, if (2) holds for a set N_0, then it also holds for each larger subset of N. Thus it suffices to prove (2) under the additional assumption that the complement of N_0 is infinite and $i_0 \notin N_0$. Without loss of generality we can assume that $N_0 = \{2, 4, 6, \ldots\}$ and $i_0 = 1$. In the function space of all continuous mappings of $I^{\aleph_0}$ into itself (see Section 1.11) consider the subspace Z consisting of all $g\colon I^{\aleph_0} \to I^{\aleph_0}$ such that

(3) $$g(A_{2i}^*) \subset A_i \quad \text{and} \quad g(B_{2i}^*) \subset B_i \quad \text{for } i = 1, 2, \ldots$$

Since the ambient function space is metrizable and separable (see [GT], Corollary 4.2.18), so is the subspace Z, and thus there exists a continuous mapping f of a subspace

A of $K \subset I_1$ onto the space Z (see Problem 1.3.D(b)). Let $M = A \times \prod_{j=2}^{\infty} I_j$; one easily checks that by letting

$$F(\{x_j\}) = [f(x_1)](\{x_j\}) \quad \text{for each } \{x_j\} \in M$$

we define a continuous mapping $F\colon M \to I^{\aleph_0}$. From (3) it follows that

$$A^*_{2i} \cap M \subset F^{-1}(A_i) \quad \text{and} \quad B^*_{2i} \cap M \subset F^{-1}(B_i) \quad \text{for } i = 1, 2, \ldots$$

Thus the set $L'_{2i} = F^{-1}(p_i^{-1}(1/2))$ is a partition between $A^*_{2i} \cap M$ and $B^*_{2i} \cap M$ in M. For the open subsets $V_{2i,1} = \operatorname{int} A^*_{2i}$ and $V_{2i,2} = \operatorname{int} B^*_{2i}$ of $I^{\aleph_0}$ we have

$$A_{2i} \subset V_{2i,1}, \quad B_{2i} \subset V_{2i,2} \quad \text{and} \quad \overline{V}_{2i,1} \cap \overline{V}_{2i,2} = \emptyset,$$

so that by Lemma 1.2.9 for $i = 1, 2, \ldots$ there exists a partition L_{2i} between A_{2i} and B_{2i} in $I^{\aleph_0}$ such that $M \cap L_{2i} \subset L'_{2i}$. We shall show that the partitions satisfy our requirements.

Assume that a countable-dimensional space $Y \subset \bigcap_{i=1}^{\infty} L_{2i}$ satisfies $K \subset p_1(Y)$. By Proposition 5.1.3 and Theorem 1.2.11 there exist partitions L^*_{2i} between A^*_{2i} and B^*_{2i} in $I^{\aleph_0}$ such that $Y \cap \bigcap_{i=1}^{\infty} L^*_{2i} = \emptyset$ (cf. the proof of Theorem 1.8.21). Now, for $i = 1, 2, \ldots$ consider a continuous function $g_i\colon I^{\aleph_0} \to I_i$ such that

$$L^*_{2i} = g_i^{-1}(1/2), \quad g_i(A^*_{2i}) = \{0\} \quad \text{and} \quad g_i(B^*_{2i}) = \{1\}, \tag{4}$$

and define $g\colon I^{\aleph_0} \to I^{\aleph_0}$ by letting

$$g(x) = (g_1(x), g_2(x), \ldots) \quad \text{for each } x \in I^{\aleph_0}.$$

From (4) it readily follows that $g \in Z$, and thus there exists an $x_1^0 \in A \subset K \subset I_1$ such that $g = f(x_1^0)$. Since $x_1^0 \in p_1(Y)$, there exists a point $x = \{x_j\} \in Y$ with $x_1 = x_1^0$, and we have

$$x \in p_1^{-1}(A) \cap Y \subset M \cap \bigcap_{i=1}^{\infty} L_{2i} \subset \bigcap_{i=1}^{\infty} L'_{2i} = \bigcap_{i=1}^{\infty} F^{-1}(p_i^{-1}(1/2)),$$

so that $F(x) \in \bigcap_{i=1}^{\infty} p_i^{-1}(1/2)$. But $F(x) = [f(x_1^0)](x) = g(x) = (g_1(x), g_2(x), \ldots)$, and thus $g_i(x) = 1/2$ for $i = 1, 2, \ldots$, i.e., $x \in \bigcap_{i=1}^{\infty} g_i^{-1}(1/2) = \bigcap_{i=1}^{\infty} L^*_{2i}$. Since $x \in Y$ and $Y \cap \bigcap_{i=1}^{\infty} L^*_{2i} = \emptyset$, we have a contradiction. Thus each space $Y \subset \bigcap_{i=1}^{\infty} L_{2i}$ satisfying $K \subset p_1(Y)$ is uncountable-dimensional, and (2) is proved.

We shall also need the following:

(5) In the unit interval I there exists a sequence $K_1, K_2, \ldots$ of pairwise disjoint subspaces homeomorphic to the Cantor set such that each non-empty open subinterval of I contains a set from this sequence.

Indeed, it suffices to arrange all subintervals of I with rational end-points into a sequence $(a_1, b_1), (a_2, b_2), \ldots$ and define $K_1, K_2, \ldots$ inductively taking for K_i a subspace of (a_k, b_k) homeomorphic to the Cantor set and disjoint from the union $K_1 \cup K_2 \cup \ldots \cup K_{i-1}$.

Now we are ready to define our space X.

Applying (5), for $j = 1, 2, \ldots$ choose in I_j a family $\{K_{j,k}\}_{k=1}^{\infty}$ of pairwise disjoint subspaces homeomorphic to the Cantor set such that each non-empty open subset of I_j contains a $K_{j,k}$. Then, decompose the set $N \setminus \{1\}$ into pairwise disjoint infinite sets

$N_{j,k}$, where $j, k = 1, 2, \ldots$ By virtue of (2) there exist partitions L_i between A_i and B_i in $I^{\aleph_0}$ for $i \in N_{j,k}$ with the property that

(6) each space $Y \subset \bigcap\{L_i : i \in N_{j,k}\}$ satisfying $K_{j,k} \subset p_j(Y)$ is uncountable-dimensional.

We shall show that the compact space $X = \bigcap_{i=2}^{\infty} L_i = \bigcap_{j,k=1}^{\infty} \bigcap\{L_i : i \in N_{j,k}\}$ has the required properties. Indeed, if Y is a non-empty subspace of X then either there exists a j such that the projection $p_j(Y)$ has a non-empty interior, and thus contains a $K_{j,k}$, or else all the projections $p_j(Y)$ are zero-dimensional. In the first case Y is uncountable-dimensional by virtue of (6), in the second case Y is zero-dimensional as a subspace of the zero-dimensional space $\prod_{j=1}^{\infty} p_j(Y)$. It remains to show that X itself is not zero-dimensional. Suppose it were; then by Theorem 1.2.11 we could find a partition L_1 between A_1 and B_1 in $I^{\aleph_0}$ such that $X \cap L_1 = \bigcap_{i=1}^{\infty} L_i = \emptyset$, but this is impossible by Theorem 1.8.2. ■

Historical and bibliographic notes

Examples of a hereditarily normal compact space X with $\dim X = 0$ that contains an open uncountable-dimensional subspace and of a hereditarily normal Lindelöf space X with $\dim X = 0$ that contains a subspace which is not strongly countable-dimensional were constructed by Gruenhage and E. Pol in [1983] (the former under the assumption of the continuum hypothesis). Theorem 5.2.5 was proved by Hurewicz in [1928] for compact metrizable spaces; de Groot and Nagata observed in [1969] that it extends to completely metrizable spaces. Theorem 5.2.10 for metrizable spaces was proved by Wenner in [1969], Theorem 5.2.11 is a variant of a theorem given by Olszewski in [1992]. Theorems 5.2.12 and 5.2.13 were given by Olszewski in [1992]; for metrizable spaces the former was obtained by Arhangel'skiĭ in [1963], and the latter by Wenner in [1969]. Examples of a perfectly normal uncountable-dimensional space which can be represented as the union of its open countable-dimensional subspaces and of a perfectly normal uncountable-dimensional space which is locally countable were constructed by Gruenhage and E. Pol in [1983] (the latter under the assumption of the continuum hypothesis). Theorems 5.2.15, 5.2.17 and 5.2.18 for metrizable spaces were given by Wenner in [1972]; Theorem 5.2.16 was proved by Arhangel'skiĭ in [1963]. An example of a perfectly normal space which can be represented as the union of its open subspaces with covering dimension 0 and which is not strongly countable-dimensional (being countable-dimensional) was described by Gruenhage and E. Pol in [1983]. An example of a locally compact perfectly normal space X with $\dim X = 0$ such that the square $X \times X$ is perfectly normal and uncountable-dimensional was constructed (under the assumption of the continuum hypothesis) in Engelking and E. Pol [1983] and in Tsuda [1984].

Example 5.2.23 was given by R. Pol in [1986] as a finale of a long series of achievements: In [1965] Henderson defined an infinite-dimensional compact metrizable space with no closed subspaces of finite positive dimension, his example was very difficult; simpler examples of such spaces were given by Bing in [1966], by Henderson himself in [1967], and by Zarelua in [1972]. Rubin, Schori and Walsh gave in [1979]

a still simpler example that significantly clarified the matter. Then Walsh in [1979] (announcement in [1978]) defined an infinite-dimensional compact metrizable space with no subspaces of finite positive dimension. Finally R. Pol gave his, relatively simple, example of such a space. Another fairly simple example was given by Levin in [1995].

Problems

5.2.A. (a) Check that every strongly metrizable subspace of a countable-dimensional normal space is countable-dimensional.

Hint. See Problems 2.4.B and 3.2.G.

(b) Check that every completely paracompact subspace of a strongly countable-dimensional normal space is strongly countable-dimensional.

5.2.B (Tumarkin [1961], de Groot and Nagata [1969]). Show that if a metrizable space X contains for $k = 0, 1, \ldots$ a (closed) subspace M_k such that $\dim M_k = k$, then X contains a (closed) countable-dimensional subspace M with $\dim M = \infty$. Deduce that a completely metrizable space X contains a countable-dimensional closed subspace M with $\dim M = \infty$ if and only if, for $k = 0, 1, \ldots$, the space X contains a closed subspace M_k with $\dim M_k = k$.

5.2.C. Assuming the existence, for $n = 2, 3, \ldots$, of a compact (perfectly normal compact) space X_n with $\dim X_n = n$ which contains no closed subspace M such that $0 < \dim M < n$, define a strongly countable-dimensional compact (perfectly normal compact) space X with $\dim X = \infty$ which contains no closed subspace M such that $0 < \dim M < \infty$ (for the existence of X_n's see notes to Section 3.1).

Remark. No example of a strongly countable-dimensional hereditarily normal compact space X with $\dim X = \infty$ which for some $k > 0$ contains no closed subspace M such that $\dim M = k$ has been constructed yet without additional set-theoretic assumptions.

5.2.D (Hattori [1983b]). Show that if a hereditarily normal compact space X with $\dim X = \infty$ can be represented as the union of a sequence of subspaces $X_1, X_2, \ldots$ satisfying $\dim X_i = 0$ for $i = 1, 2, \ldots$, then for $k = 1, 2, \ldots$, X contains a closed subspace M_k such that $\dim M_k = k$.

Remark. It is not known if it suffices to assume that X is a countable-dimensional hereditarily normal compact space with $\dim X = \infty$.

5.2.E. Show directly that the locally finite sum theorem for countable-dimensional spaces holds for metrizable spaces.

Hint. Apply the fact that every open cover of a metrizable space has an open σ-discrete refinement (see [GT], Theorem 4.4.1).

5.2.F. (a) Show that if a hereditarily normal space X can be represented as the union of a family $\{X_s\}_{s \in S}$ of countable-dimensional subspaces and if there exists an integer i such that each point $x \in X$ has a neighbourhood which meets at most i sets X_s, then X is a countable-dimensional space.

(b) Show that if a normal space X can be represented as the union of two strongly countable-dimensional subspaces, one of them being both an F_σ-set and a G_δ-set, then X is a strongly countable-dimensional space. Deduce that in perfectly normal spaces the same holds under the assumption that one of the subspaces is the intersection of a closed and an open set.

(c) Show that if a hereditarily normal countably paracompact space X can be represented as the union of a locally finite family $\{X_s\}_{s\in S}$ of subspaces and if there exists an integer n such that $\dim X_s \leq n$ for each $s \in S$, then X is a strongly countable-dimensional space. Note that the same holds in hereditarily normal weakly paracompact spaces under the assumption that all X_s's are finite-dimensional.

(d) (Engelking and E. Pol [1983]) Give an example of a hereditarily paracompact space X which can be represented as the union of a closed strongly countable-dimensional subspace and an open discrete subspace, and yet is not strongly countable-dimensional.

Hint. Consider the space $X = I^{\aleph_0}$ with the topology generated by the base consisting of all sets of the form $U \cup A$, where U is an open subset of $I^{\aleph_0}$ and $A \cap K_\omega = \emptyset$ (see Example 1.8.22(b)). Use the fact that no non-empty open subset of K_ω is finite-dimensional.

(e) Give an example of a hereditarily paracompact space X that is not strongly countable-dimensional, and yet can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed strongly countable-dimensional subspaces that can be enlarged to open subspaces U_s such that the cover $\{U_s\}_{s\in S}$ is point-finite (cf. Theorems 5.2.11 and 5.2.16).

5.2.G. (Wenner [1969] and [1972]). Show that if a metrizable space X can be represented as the union of a transfinite sequence $K_1, K_2, \ldots, K_\alpha, \ldots$, $\alpha < \xi$, of countable-dimensional (strongly countable-dimensional) subspaces such that the union $\bigcup_{\beta<\alpha} K_\beta$ is closed for each $\alpha < \xi$, then X is a countable-dimensional (strongly countable-dimensional) space.

5.3. The compactification and universal space theorems for countable-dimensional and strongly countable-dimensional spaces

We shall start with compactification theorems for countable-dimensional and strongly countable-dimensional spaces. In contrast with the compactification theorem in Section 1.7, in both of them we have to restrict ourselves to completely metrizable spaces. Indeed, as we shall show in Example 5.3.6, there exists a strongly countable-dimensional separable metric space with no countable-dimensional metrizable compactification (cf. Problem 5.3.B(c)).

5.3.1. LEMMA. *For every completely metrizable separable space X there exist a compact metrizable space Y and a homeomorphic embedding $f\colon X \to Y$ such that the complement $Y \setminus f(X)$ is a countable union of finite-dimensional compact spaces.*

PROOF. One can assume that X is a subspace of a compact metric space Z and that the metric ρ on the space Z is bounded by 1. Being completely metrizable, the

space X can be represented as the intersection of a decreasing sequence $U_1 \supset U_2 \supset \dots$ of open subsets of Z (see Problem 1.3.B), and since ρ is totally bounded, for each i the set U_i is the union of finitely many open sets $U_{i,1}, U_{i,2}, \dots, U_{i,k_i}$ of diameter less than $1/i$. Let

$$f_{i,k}(z) = \rho(z, Z \setminus U_{i,k}) \quad \text{for } i = 1, 2, \dots \text{ and } k = 1, 2, \dots, k_i,$$

arrange all the functions $f_{i,k}\colon Z \to I$ into a sequence $f_1, f_2, \dots$, and define $F\colon Z \to I^{\aleph_0}$ by the formula $F(z) = (f_1(z), f_2(z), \dots)$. The subspace K_ω of $I^{\aleph_0}$ consisting of points with finitely many non-zero coordinates has the property that $F^{-1}(I^{\aleph_0} \setminus K_\omega) \subset X$; indeed, each point $z \in F^{-1}(I^{\aleph_0} \setminus K_\omega)$ belongs to infinitely many sets $U_{i,k}$, and thus to infinitely many U_i's. Now, the family of restrictions $\{f_i|X\}_{i=1}^{\infty}$ separates points and closed sets, so that the restriction $F|X\colon X \to I^{\aleph_0}$ is a homeomorphic embedding (see [GT], Theorem 2.3.20). Let Y be the closure of $F(X)$ in $I^{\aleph_0}$; clearly, $Y \subset F(Z)$ is a compact metric space and the mapping $f\colon X \to Y$ defined by letting $f(x) = F(x)$ for $x \in X$ is a homeomorphic embedding. Thus $f(X)$, being homeomorphic to X and completely metrizable, is a G_δ-set in Y, so that $Y \setminus f(X)$ is a countable union of compact subspaces of Y. To conclude the proof, it suffices to observe that since $F^{-1}(I^{\aleph_0} \setminus K_\omega) \subset X$, we have

$$Y \setminus f(X) \subset F(Z) \setminus F(X) \subset F(Z \setminus X) \subset F(Z \setminus F^{-1}(I^{\aleph_0} \setminus K_\omega)) \subset I^{\aleph_0} \setminus (I^{\aleph_0} \setminus K_\omega) = K_\omega,$$

and to apply the strong countable dimensionality of K_ω. ■

The lemma yields immediately

5.3.2. THE COMPACTIFICATION THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *For every countable-dimensional completely metrizable separable space X there exists a countable-dimensional metrizable compactification, i.e., a countable-dimensional compact metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X.* ■

The compactification theorem for strongly countable-dimensional spaces will be deduced from two lemmas that we are now going to prove. The first lemma is a generalization of the compactification theorem from Section 1.7.

5.3.3. LEMMA. *For every separable metrizable space X and any sequence $F_1, F_2, \dots$ of closed subsets of X there exists a metrizable compactification preserving the dimension of all F_i's, i.e., a compact metrizable space $\widetilde{X}$ which contains X as a dense subspace such that for $i = 1, 2, \dots$ the closure $\overline{F}_i$ of F_i in $\widetilde{X}$ satisfies the equality* $\dim \overline{F}_i = \dim F_i$.

PROOF. Since X is a separable metrizable space, there exists a totally bounded metric ρ on the space X. We can also assume that $\dim F_i = n_i < \infty$ for $i = 1, 2, \dots$ By virtue of Remark 1.7.3, it suffices to define a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of finite open covers of X, where $\mathcal{U}_m = \{U_{m,k}\}_{k=1}^{k_m}$, satisfying for $m = 1, 2, \dots$ the following conditions:

(1) $\operatorname{ord}(\mathcal{U}_m|F_i) \le n_i$ for $i < m$;

(2) $\operatorname{mesh}\mathcal{U}_m \le 1/2^m$;

and for $i < m$ and $k \le k_i$,

(3) $|f_{i,k}(x) - f_{i,k}(y)| \le 1/2^m$ whenever x and y belong to the same member of $\mathcal{U}_m$,

where

$$f_{i,k}(x) = \frac{\rho(x, X \setminus U_{i,k})}{\sum_{j=1}^{k_i} \rho(x, X \setminus U_{i,j})} \quad \text{for } x \in X.$$

To obtain an open cover $\mathcal{U}_1 = \{U_{1,k}\}_{k=1}^{k_1}$ such that $\operatorname{mesh}\mathcal{U}_1 \leq 1/2$ it suffices to apply the total boundedness of (X, ρ). Assume that the covers $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_{m-1}$ are already defined.

Applying Lemma 1.7.1 we obtain an open cover $\mathcal{V}_1 = \{V_{1,k}\}_{k=1}^{k_m}$ of the space X which satisfies conditions (2) and (3) with $\mathcal{U}_m$ replaced by $\mathcal{V}_1$. We shall now inductively define open covers $\mathcal{V}_2, \mathcal{V}_3, \ldots, \mathcal{V}_m$ of the space X, where $\mathcal{V}_i = \{V_{i,k}\}_{k=1}^{k_m}$, such that

$$V_{i,k} \subset V_{i-1,k} \quad \text{for } i = 2, 3, \ldots \text{ and } k = 1, 2, \ldots, k_m, \tag{4}$$

and

$$\operatorname{ord}(\mathcal{V}_i | F_{i-1}) \leq n_{i-1} \quad \text{for } i = 2, 3, \ldots, m. \tag{5}$$

Assume that the covers $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_j$ are defined and satisfy (4) and (5) for $i = 2, 3, \ldots, j$. The existence of an open cover $\mathcal{V}_{j+1} = \{V_{j+1,k}\}_{k=1}^{k_m}$ satisfying (4) and (5) for $i = j + 1$ follows from Lemma 3.1.5, so that the sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ is defined. Now we let $\mathcal{U}_m = \mathcal{V}_m$.

From (4) and the construction of $\mathcal{V}_1$ it follows that $\mathcal{U}_m$ satisfies conditions (2) and (3), and from (4) and (5) it follows that $\mathcal{U}_m$ satisfies (1). Thus the sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is defined, and the proof is complete. ■

5.3.4. LEMMA. *For every pair A, B of disjoint G_δ-sets in a metrizable space Z there exists a subspace T of Z which is both an F_σ-set and a G_δ-set such that $A \subset T \subset Z \setminus B$.*

PROOF. Let $A = \bigcap_{i=1}^{\infty} U_i$ and $B = \bigcap_{i=1}^{\infty} V_i$, where $U_1 = Z \supset U_2 \supset \ldots, V_1 = Z \supset V_2 \supset \ldots$, and the sets U_i and V_i are open. The union $T = \bigcup_{i=1}^{\infty}(U_i \setminus V_i)$ is an F_σ-set which satisfies the inclusion $A \subset T \subset Z \setminus B$, and the equality $Z \setminus T = \bigcup_{i=1}^{\infty}(V_i \setminus U_{i+1})$ implies that T is also a G_δ-set. ■

5.3.5. THE COMPACTIFICATION THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *For every strongly countable-dimensional completely metrizable separable space X there exists a strongly countable-dimensional metrizable compactification, i.e., a strongly countable-dimensional compact metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X.*

PROOF. Consider a strongly countable-dimensional completely metrizable separable space X. Let $X = \bigcup_{i=1}^{\infty} F_i$, where F_i is a closed subset of X and $\dim F_i < \infty$ for $i = 1, 2, \ldots$ By Lemma 5.3.3 there exists a compact metrizable space Z that contains X as a dense subset such that the closure $\overline{F}_i$ of F_i in Z is finite-dimensional for $i = 1, 2, \ldots$ The sets $A = X$ and $B = Z \setminus \bigcup_{i=1}^{\infty} \overline{F}_i$ are disjoint G_δ-sets in Z (see Problem 1.3.B), so that by Lemma 5.3.4 there exists a subspace T of Z which is both an F_σ-set and a G_δ-set such that $X \subset T \subset \bigcup_{i=1}^{\infty} \overline{F}_i$. The space T, being an F_σ-set contained in $\bigcup_{i=1}^{\infty} \overline{F}_i$, is the union of countably many compact finite-dimensional spaces. Now, from Lemma 1.3.12 it follows that T is completely metrizable, so that by Lemma 5.3.1 the space T has a strongly countable-dimensional metrizable compactification. Since X is a dense subset of T, every compactification of T is a compactification of X. ■

We conclude our study of countable-dimensional and strongly countable-dimensional compactifications with the example mentioned at the beginning of this section. Let us recall that the Lavrentieff theorem which will be used in the example states

that every homeomorphism $f: A \to C$, where A and C are subspaces of completely metrizable spaces X and Y respectively, is extendable to a homeomorphism $F: B \to D$, where $A \subset B \subset X$, $C \subset D \subset Y$ and B and D are G_σ-sets in X and Y respectively (see [GT], Theorem 4.3.21).

5.3.6. EXAMPLE. We shall show that the strongly countable-dimensional space K_ω defined in Example 1.8.22(b) has no countable-dimensional metrizable compactification (see Problem 5.3.C). By virtue of the Lavrentieff theorem it suffices to prove that every G_δ-set $G \subset I^{\aleph_0}$ which contains K_ω also contains a subspace of $I^{\aleph_0}$ that is homeomorphic to the whole of $I^{\aleph_0}$ (cf. Theorem 1.8.21).

Let $I^{\aleph_0} = \prod_{i=1}^{\infty} I_i$, where $I_i = I$ for $i = 1, 2, \ldots$, and let $G = \bigcap_{k=1}^{\infty} U_k$, where U_k is an open subset of $I^{\aleph_0}$ for $k = 1, 2, \ldots$ We shall define inductively an increasing sequence $n_1, n_2, \ldots$ of natural numbers and a sequence $a_1, a_2, \ldots$ of real numbers in the interval $(0, 1]$ such that

$$\prod_{i=1}^{n_k} [0, a_i] \times \prod_{i=n_k+1}^{\infty} I_i \subset U_k \tag{6}$$

for $k = 1, 2, \ldots$

To begin, observe that since

$$\{0\} \times \{0\} \times \ldots \subset K_\omega \subset U_1,$$

there exist a natural number n_1 and numbers $a_1, a_2, \ldots, a_{n_1}$ in the interval $(0, 1]$ such that

$$\prod_{i=1}^{n_1} [0, a_i] \times \prod_{i=n_1+1}^{\infty} I_i \subset U_1.$$

Assume that the natural numbers $n_1 < n_2 < \ldots < n_m$ and numbers $a_1, a_2, \ldots, a_{n_m}$ in the interval $(0, 1]$ are already defined and (6) is satisfied for $k \leq m$. Since the set

$$\prod_{i=1}^{n_m} [0, a_i] \times \{0\} \times \{0\} \times \ldots \subset K_\omega \subset U_{m+1}$$

is compact, by the Wallace theorem ([GT], Theorem 3.2.10) there exist a natural number $n_{m+1} > n_m$ and numbers $a_{n_m+1}, a_{n_m+2}, \ldots, a_{n_{m+1}}$ in the interval $(0, 1]$ such that (6) is satisfied for $k = m + 1$. Thus the sequences $n_1, n_2, \ldots$ and $a_1, a_2, \ldots$ are defined. Now, from (6) it follows that the Cartesian product $\prod_{i=1}^{\infty} [0, a_i]$, homeomorphic to $I^{\aleph_0}$, is contained in $\bigcap_{k=1}^{\infty} U_k = G$. ■

Let us note that the Čech–Stone compactification βS of the strongly countable-dimensional space $S = \bigoplus_{n=1}^{\infty} I^n$ is not countable-dimensional (see Problems 5.3.D(c) and 5.5.I).

Now we pass to the universal space theorems (cf. Problem 5.3.I(b)).

5.3.7. THE FIRST UNIVERSAL SPACE THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *The subspace N_ω of the Hilbert cube $I^{\aleph_0}$ consisting of points with finitely many rational coordinates is a universal space for the class of all countable-dimensional separable metrizable spaces.*

PROOF. We have observed in Example 1.8.22(c) that the space $N_\omega = \bigcup_{n=0}^{\infty} N_n$ is countable-dimensional.

Consider now a countable-dimensional separable metrizable space X. By virtue of Proposition 5.1.3 and Theorem 1.6.10 there exist subspaces $X_0, X_1, \ldots$ of X such that $\operatorname{Ind} X_n \leq 0$ for $n = 0, 1, \ldots$ and $X = \bigcup_{n=0}^{\infty} X_n$. Choose a metric on the space X, and let $\{U_m\}_{m=1}^{\infty}$ be a countable base for X consisting of bounded sets. For $m = 1, 2, \ldots$ define

$$A_m = X \setminus B(U_m, \delta(U_m)) \quad \text{and} \quad B_m = \overline{U}_m.$$

By Lemma 4.2.10 there exist continuous functions $f_m: X \to [\pi/5, \pi/4]$ such that for $m = 1, 2, \ldots$ we have $f_m(A_m) \subset \{\pi/5\}$, $f_m(B_m) \subset \{\pi/4\}$, and

$$|\{m : f_m(x) \text{ is a rational number}\}| \leq n \quad \text{for } x \in X_n. \tag{7}$$

One easily checks that the family of mappings $\{f_m\}_{m=1}^{\infty}$ separates points and closed sets, so that the mapping $f: X \to I^{\aleph_0}$, where $f(x) = (f_1(x), f_2(x), \ldots)$, is a homeomorphic embedding. Moreover, it follows from (7) that $f(X_n) \subset N_n$ for $n = 0, 1, \ldots$, so that $f(X) \subset N_\omega$. ■

The universal space theorem for strongly countable-dimensional separable metrizable spaces is preceded by three lemmas the first two of which will be applied again in Section 5.5. It is for the purpose of that application that we formulate Lemma 5.3.8 in a more general way; here we shall need it only for a compact metrizable space X, and in this case it can also be proved by considering the decomposition of X into A and one-point subsets of $X \setminus A$, and taking as Y the corresponding quotient space.

5.3.8. LEMMA. *For each metrizable space X and a closed non-empty subset A of X there exist a metrizable space Y and a continuous mapping $g: X \to Y$ such that $A = g^{-1}(a)$ for a point $a \in Y$, and the restriction $g|X \setminus A$ is a homeomorphism of $X \setminus A$ onto $Y \setminus \{a\}$.*

PROOF. Let $Y = (X \setminus A) \cup \{a\}$, where $a \notin X$, and let ρ be a metric on the space X. It is readily checked that by letting

$$\sigma(x, y) = \min\{\rho(x, y), \rho(x, A) + \rho(y, A)\} \quad \text{if } x \neq a \neq y,$$

and $\sigma(x, a) = \sigma(a, x) = \rho(x, A)$, one defines a metric on Y. The mapping g defined by

$$g(x) = \begin{cases} x & \text{for } x \in X \setminus A, \\ a & \text{for } x \in A, \end{cases}$$

is continuous, because $\sigma(g(x), g(y)) \leq \rho(x, y)$ for all $x, y \in X$. To show that $g|X \setminus A$ is a homeomorphism it suffices to note that for each $x_0 \in X \setminus A$ the open ball $\{x \in X : \rho(x_0, x) < r/2\} \subset X \setminus A$, where $r = \rho(x, A) > 0$, is isometrically mapped by g onto the open ball $\{x \in Y : \sigma(x_0, x) < r/2\} \subset Y \setminus \{a\}$. ■

5.3.9. LEMMA. *For every separable metrizable space X and any non-empty closed subsets X_0, X_1 of X satisfying $X_0 \subset X_1$ and $\dim(X_1 \setminus X_0) \leq n$ there exists a continuous mapping $f: X \to I^{2n+2}$ such that $X_0 = f^{-1}((0, 0, \ldots, 0))$, $X_1 = f^{-1}(\operatorname{Fr} I^{2n+2})$, where $\operatorname{Fr} I^{2n+2}$ is the boundary of the cube I^{2n+2} in R^{2n+2}, and the restriction $f|X_1 \setminus X_0$ is a homeomorphic embedding of $X_1 \setminus X_0$ in $\operatorname{Fr} I^{2n+2}$.*

PROOF. By Lemma 5.3.8 there exist a metrizable space Y and a continuous mapping $g\colon X_1 \to Y$ such that $X_0 = g^{-1}(a)$ for a point $a \in Y$, and the restriction $g|X_1 \setminus X_0$ is a homeomorphism of $X_1 \setminus X_0$ onto $Y \setminus \{a\}$. From Theorem 1.7.7 and Corollary 1.5.6 it follows that the space Y is separable and satisfies the inequality $\dim Y \leq n$. Theorem 1.11.4 implies that there exists a homeomorphic embedding $h\colon Y \to (0,1)^{2n+1}$, and by the Tietze–Urysohn extension theorem the mapping $hg\colon X_1 \to (0,1)^{2n+1}$ can be extended to a continuous mapping $H\colon X \to (0,1)^{2n+1}$. Now, take a continuous function $k\colon X \to [0,1)$ such that $k^{-1}(0) = X_1$ and for each $x \in X$ let $f_0(x) = (H(x), k(x)) \in I^{2n+2}$. Obviously, the mapping $f_0\colon X \to I^{2n+2}$ is continuous,

$$f_0^{-1}((h(a),0)) = X_0, \quad f_0^{-1}(\operatorname{Fr} I^{2n+2}) = X_1,$$

and the restriction $f_0|X_1 \setminus X_0$ is a homeomorphic embedding. There clearly exists a homeomorphism f_1 of I^{2n+2} onto itself such that the composition $f = f_1 f_0$ satisfies all the required conditions. ■

5.3.10. LEMMA. *For every compact metrizable space Z and any increasing sequence $C_1 \subset C_2 \subset \ldots$ of closed subspaces of Z with $\dim C_n \leq n$ for $n = 1, 2, \ldots$ there exists a continuous mapping $f\colon Z \to I^{\aleph_0}$ such that the restriction of f to the subspace $Y = \bigcup_{n=1}^{\infty} C_n$ is a homeomorphic embedding and $f(Y) \subset K_\omega$.*

PROOF. We can assume that $C_1 \neq \emptyset$; let C_0 be an arbitrary one-point subset $\{x_0\} \subset C_1$. From Lemma 5.3.9 it follows that for $n = 1, 2, \ldots$ there exists a continuous mapping $f_n\colon Z \to I^{2n+2}$ such that $C_{n-1} = f_n^{-1}((0,0,\ldots,0))$, $C_n = f_n^{-1}(\operatorname{Fr} I^{2n+2})$, and

(8) $\quad f_n|C_n \setminus C_{n-1}$ is a homeomorphic embedding of $C_n \setminus C_{n-1}$ in $\operatorname{Fr} I^{2n+2}$.

We shall prove that the continuous mapping $f\colon Z \to I^{\aleph_0}$ defined by letting $f(x) = (f_1(x), f_2(x), \ldots)$ for $x \in Z$ has the required properties. Observe first that for each $x \in Y$ there exists an $m \geq 1$ such that $x \in C_{n-1}$ for $n \geq m$; hence $f_n(x) = (0,0,\ldots,0)$ for $n \geq m$, and $f(x) \in K_\omega$. Thus $f(Y) \subset K_\omega$. To prove that the restriction $f|Y$ is a homeomorphic embedding, it suffices to show that the family $\{f_n|Y\}_{n=1}^{\infty}$ separates points and closed sets. To that end, consider a point $x \in Y$ and a closed set $F \subset Z$ such that $x \notin F$. Assume first that $x \neq x_0$. There exists an n such that $x \in C_n \setminus C_{n-1}$; by (8) we have $f_n(x) \notin f_n(F \cap C_n)$ and since $f_n(x) \in \operatorname{Fr} I^{2n+2}$ while $f_n(F \setminus C_n) \subset I^{2n+2} \setminus \operatorname{Fr} I^{2n+2}$, we also have $f_n(x) \notin f_n(F \setminus C_n)$. Thus $f_n(x) \notin f_n(F)$ and the compactness of F implies that $f_n(x) \notin \overline{f_n(F)}$. If $x = x_0$, we have $f_1(x) = (0,0,0,0) \notin f_1(F)$, and again the compactness of F implies that $f_1(x) \notin \overline{f_1(F_1)}$. Hence the family $\{f_n|Y\}_{n=1}^{\infty}$ separates points and closed sets. ■

5.3.11. THE FIRST UNIVERSAL SPACE THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *The subspace K_ω of the Hilbert cube $I^{\aleph_0}$ consisting of points with finitely many non-zero coordinates is a universal space for the class of all strongly countable-dimensional separable metrizable spaces.*

PROOF. We have observed in Example 1.8.22(b) that the space K_ω is strongly countable-dimensional.

Consider now a strongly countable-dimensional separable metrizable space X. By Proposition 5.1.5 the space X is the union of an increasing sequence of closed subspaces

$F_1 \subset F_2 \subset \ldots$ such that $\dim F_n \leq n$ for $n = 1, 2, \ldots$ From Lemma 5.3.3 it follows that there exists a compact metrizable space Z, containing X as a dense subset, such that for $n = 1, 2, \ldots$ the closure C_n of F_n in Z satisfies the inequality $\dim C_n \leq n$. To conclude the proof it suffices to apply Lemma 5.3.10. ■

Let us note in connection with Theorems 5.3.7 and 5.3.11 that there is no universal space either for the class of all countable-dimensional compact metrizable spaces or for the class of all strongly countable-dimensional compact metrizable spaces (see Problems 7.1.I and 6.1.I(d)).

We close this section with universal space theorems for countable-dimensional and strongly countable-dimensional metrizable spaces of weight $\mathbf{m} \geq \aleph_0$.

Let us recall that by rational points of the hedgehog $J(\mathbf{m})$ we mean those whose distance from the point 0 is a rational number.

5.3.12. THE SECOND UNIVERSAL SPACE THEOREM FOR COUNTABLE-DIMENSIONAL SPACES. *For every cardinal number $\mathbf{m} \geq \aleph_0$ the subspace $N(\mathbf{m})$ of the Cartesian product $[J(\mathbf{m})]^{\aleph_0}$ consisting of points with finitely many rational coordinates distinct from the point 0 is a universal space for the class of all countable-dimensional metrizable spaces whose weight is not larger than $\mathbf{m}$.*

PROOF. It follows from Example 4.2.8 and Theorem 4.1.3 that the space $N(\mathbf{m}) = \bigcup_{n=0}^{\infty} N_n(\mathbf{m})$ is countable-dimensional.

Consider now an arbitrary countable-dimensional metrizable space X such that $w(X) \leq \mathbf{m}$. By virtue of Proposition 5.1.3 and Theorem 1.6.10 there exist subspaces $X_0, X_1, \ldots$ of the space X such that $\operatorname{Ind} X_n \leq 0$ for $n = 0, 1, \ldots$ and $X = \bigcup_{n=0}^{\infty} X_n$. Applying Lemma 4.2.11 we obtain a homeomorphic embedding $h\colon X \to [J(\mathbf{m})]^{\aleph_0}$ such that $h(X) = h(\bigcup_{n=0}^{\infty} X_n) = \bigcup_{n=0}^{\infty} h(X_n) \subset \bigcup_{n=0}^{\infty} N_n(\mathbf{m}) = N(\mathbf{m})$. ■

The universal space for strongly countable-dimensional metrizable spaces of weight $\mathbf{m} \geq \aleph_0$ will be a subspace of the Cartesian product $[J(\mathbf{m})]^{\aleph_0} \times I^{\aleph_0}$ (cf. Problem 5.3.F). For each cardinal number $\mathbf{m} \geq \aleph_0$ and each integer $n \geq 0$ we define first the subspace $K_n(\mathbf{m})$ of $[J(\mathbf{m})]^{\aleph_0} \times I^{\aleph_0}$ as the set of points $(x_0, x_1, x_2, \ldots)$ with $x_0 \in [J(\mathbf{m})]^{\aleph_0}$ and $x_i \in I$ for $i = 1, 2, \ldots$ such that $x_0 \in N_n(\mathbf{m})$, $x_i \neq 0$ for $i \leq n$, and $x_i = 0$ for $i > n$; then we let $K(\mathbf{m}) = \bigcup_{n=0}^{\infty} K_n(\mathbf{m})$.

5.3.13. THE SECOND UNIVERSAL SPACE THEOREM FOR STRONGLY COUNTABLE-DIMENSIONAL SPACES. *For every cardinal number $\mathbf{m} \geq \aleph_0$ the subspace $K(\mathbf{m})$ of the Cartesian product $[J(\mathbf{m})]^{\aleph_0} \times I^{\aleph_0}$ is a universal space for the class of all strongly countable-dimensional metrizable spaces whose weight is not larger than $\mathbf{m}$.*

PROOF. To begin, let us observe that the space $K_n(\mathbf{m})$ is homeomorphic to the Cartesian product $N_n(\mathbf{m}) \times (0, 1]^n$, so that $\operatorname{Ind} K_n(\mathbf{m}) \leq 2n$ by Theorem 4.1.21. Furthermore, for each m the union $\bigcup_{n=0}^{m} K_n(\mathbf{m})$ is closed in $K(\mathbf{m})$; indeed, if $(x_0, x_1, x_2, \ldots) \in K(\mathbf{m}) \setminus \bigcup_{n=0}^{m} K_i(\mathbf{m})$, then $x_{m+1} > 0$, and the point $(x_0, x_1, x_2, \ldots)$ has a neighbourhood disjoint from $\bigcup_{n=0}^{m} K_n(\mathbf{m})$. Thus, by Theorems 4.1.3 and 4.1.18, the space $K(\mathbf{m})$ is strongly countable-dimensional.

Consider now an arbitrary strongly countable-dimensional metrizable space X such that $w(X) \leq \mathbf{m}$. By Proposition 5.1.5 the space X is the union of an increasing

sequence of closed subspaces $F_1 \subset F_2 \subset \ldots$ such that $\dim F_i \leq i-1$ for $i = 1, 2, \ldots$ From Theorems 4.1.3 and 4.1.17 it follows that for $i = 1, 2, \ldots$ the difference $F_i \setminus F_{i-1}$, where $F_0 = \emptyset$, is the union of i sets of large inductive dimension ≤ 0; arrange all these sets in a sequence $X_0, X_1, \ldots$ taking first the set $X_0 = F_1$, second the two sets that yield $F_2 \setminus F_1$, third the three sets that yield $F_3 \setminus F_2$, and so on. By Lemma 4.2.11 there exists a homeomorphic embedding $h: X \to [J(\mathbf{m})]^{\aleph_0}$ such that $h(X_n) \subset N_n(\mathbf{m})$ for $n = 0, 1, \ldots$ Now for $i = 1, 2, \ldots$ choose a continuous function $f_i: X \to I$ satisfying the condition $f_i^{-1}(0) = F_i$, and for each $x \in X$ let

$$f(x) = (h(x), f_1(x), f_1(x), f_2(x), \ldots),$$

where the number $f_i(x)$ is repeated $i+1$ times. Clearly, $f: X \to [J(\mathbf{m})]^{\aleph_0} \times I^{\aleph_0}$ is a homeomorphic embedding.

To conclude the proof, we have to show that $f(X) \subset K(\mathbf{m})$. Consider an arbitrary point $x \in X$. There exists an $i \geq 1$ such that $x \in F_i \setminus F_{i-1}$ and an $n \geq 0$ such that $x \in X_n \subset F_i \setminus F_{i-1}$; obviously $0+1+\ldots+(i-1) \leq n \leq (1+2+\ldots+i)-1 = m$. Now, since $x \notin F_{i-1}$, the mth coordinate of the point $(f_1(x), f_1(x), f_2(x), \ldots)$ is positive, and so are all the preceding ones; on the other hand, since $x \in F_i$, the $(m+1)$th coordinate is equal to 0, and so are all the succeeding ones. Since $h(x) \in h(X_n) \subset N_n(\mathbf{m}) \subset N_m(\mathbf{m})$, we have $f(x) \in K_m(\mathbf{m})$. Thus $f(X) \subset K(\mathbf{m})$. ■

Historical and bibliographic notes

Theorem 5.3.2 was announced by Hurewicz in [1928] (it was rediscovered and proved by Lelek in [1965]) and Theorem 5.3.5 was obtained by Schurle in [1969]; their proofs presented here were given by Engelking and R. Pol in [1988]. Lemma 5.3.3 was established by Hurewicz in [1927b]. Example 5.3.6 is taken from Skljarenko [1959a]; Hurewicz in [1928] announced that the subspace of the Hilbert space consisting of points with finitely many non-zero coordinates has no countable-dimensional metrizable compactification.

Theorem 5.3.7 was established by Nagata in [1960], and Theorem 5.3.11 by Smirnov in [1959] and by Nagata in [1960]. Theorem 5.3.12 was proved by Nagata in [1963a], our proof is taken from E. Pol's paper [1987]. The first example of a universal space for the class of strongly countable-dimensional spaces whose weight is not larger than $\mathbf{m} \geq \aleph_0$ was given by Nagata in [1960a]; the universality of the space $K(\mathbf{m})$, i.e., Theorem 5.3.13, was proved by Olszewski in [1990].

Problems

5.3.A. Note that a metrizable separable space X has a countable-dimensional (strongly countable-dimensional) metrizable compactification if and only if X is embeddable in a countable-dimensional (strongly countable-dimensional) completely metrizable space.

5.3.B. (a) (Engelking and R. Pol [1988]) Prove that if a Tychonoff space X has the following property:

(π) X is the intersection of a decreasing sequence $U_1 \supset U_2 \supset \ldots$ of open subsets of a compact space Z and for $i = 1, 2, \ldots$ the set U_i can be represented as the union

of a point-finite family $\{U_{i,s}\}_{s\in S}$ of functionally open subsets of Z in such a way that for every pair x_1, x_2 of distinct points of X there exist an $i \geq 1$ and an $s \in S$ such that the set $U_{i,s}$ contains exactly one of the points x_1, x_2,

then there exists a compact space Y of weight $w(X)$ and a homeomorphic embedding $f: X \to Y$ such that the complement $Y \setminus f(X)$ is a countable union of finite-dimensional compact spaces.

Hint. Modify the proof of Lemma 5.3.1 and apply the remark to Problem 1.3.B.

(b) Prove that every completely metrizable space of weight $\mathbf{m} \geq \aleph_0$ is homeomorphic to a closed subspace of $[J(\mathbf{m})]^{\aleph_0}$.

Hint. To begin, show that the real line is homeomorphic to a closed subspace of $[J(\mathbf{m})]^2$. Then apply the universality of $[J(\mathbf{m})]^{\aleph_0}$ and the fact that every G_δ-set in a metrizable space X is homeomorphic to a closed subspace of the Cartesian product $X \times R^{\aleph_0}$ (see the proof of Lemma 1.3.12).

(c) (Engelking and R. Pol [1988]) Prove that for every countable-dimensional completely metrizable space X of weight $\leq \mathbf{m} \geq \aleph_0$ there exists a countable-dimensional compactification of weight $\leq \mathbf{m}$.

Hint. First observe that the property (π) in (a) is preserved by countable Cartesian multiplication and by taking closed subspaces. Then note that by (b) it suffices to prove that $J(\mathbf{m})$ has the property (π); to that end, take the one-point compactification ωS of the discrete space S of cardinality $\mathbf{m}$ and consider the space Z obtained from the Cartesian product $I \times \omega S$ by identifying the set $\{0\} \times \omega S$ to a point.

Remark. The counterpart of (c) for strongly countable-dimensional spaces is also true, it was announced by Kozlovskiĭ in [1972] and proved by Olszewski in [1991].

5.3.C (Skljarenko [1959]). Show that the space K_ω has no strongly countable-dimensional compactification.

Hint. Apply the Baire category theorem for compact spaces (cf. [GT], Theorem 3.9.3).

Remark. It is an open problem if K_ω has a countable-dimensional compactification (or if there exists a countable-dimensional metrizable separable space with no countable-dimensional compactification). One can prove though that βK_ω is not countable-dimensional (see Problem 5.5.I) and that K_ω has no hereditarily normal countable-dimensional compactification (see Problem 6.1.D(a) and Theorem 6.1.10).

5.3.D. (a) (Čoban and Attia [1990]; implicitly, van Douwen [1979]) Show that if a compact subspace K of a normal space X has the property that each closed subspace $F \subset X$ disjoint from K is finite-dimensional, then also each closed subspace E of βX disjoint from K is finite-dimensional. Deduce that if a normal space X contains a compact subspace K such that $\dim F < \infty$ for each closed subspace $F \subset X$ disjoint from K, then each point $x \in \beta X \setminus X$ has a neighbourhood W with $\dim \overline{W} < \infty$.

(b) Prove that if a dense subspace X of a compact space Y is metrizable, then $Y \setminus X$ is a Lindelöf space.

Hint. Show that each set $A \subset X$ that is closed in Y is a G_δ-set in Y. To that end, choose a metric on the space X and consider a sequence $U_1, U_2, \ldots$ of open subsets of Y such that $X \cap U_i = B(A, 1/i)$ for $i = 1, 2, \ldots$

(c) (van Douwen [1979]). Prove that if a metrizable space X contains a countable-dimensional (strongly countable-dimensional) compact subspace K such that $\dim F < \infty$ for each closed subspace $F \subset X$ disjoint from K, then βX is countable-dimensional (strongly countable-dimensional).

Remark. The reverse implication is also true (see Problem 5.5.I).

5.3.E (Nagata [1960]). Show without using Theorem 5.3.7 that the space N_ω is not strongly countable-dimensional.

Hint. First note that the Cartesian product $P^{\aleph_0}$ of $\aleph_0$ copies of the irrational numbers is completely metrizable and contained in N_ω. Then show that for every finite-dimensional closed subset F of N_ω the intersection $F \cap P^{\aleph_0}$ has an empty interior in $P^{\aleph_0}$.

5.3.F. Prove that for $\mathbf{m} > \aleph_0$ the Baire space $B(\mathbf{m})$ cannot be embedded in the subspace of $[J(\mathbf{m})]^{\aleph_0}$ consisting of points with finitely many coordinates distinct from the point 0.

Hint. First show that $B(\mathbf{m})$ cannot be embedded in $[J(\mathbf{m})]^n$ for $n = 1, 2, \ldots$, and then apply the Baire category theorem.

5.3.G. Modifying the argument in the proof of Theorem 5.3.13, find in $I^{\aleph_0}$ a universal space for the class of all strongly countable-dimensional separable metrizable spaces.

5.3.H (Arhangel'skiĭ [1967]). Deduce from Problem 4.2.G(a) that for every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a universal space for the class of all strongly countable-dimensional metrizable spaces whose weight is not larger than $\mathbf{m}$.

Hint. See the proof of Theorem 3.4.3.

5.3.I. (a) (implicitly, Zarelua [1964a]) Prove that if the Čech–Stone compactification βX of a normal space X is strongly countable-dimensional, then X has a strongly countable-dimensional compactification of weight $w(X)$ (cf. Problem 5.5.I).

Hint. Apply Problem 3.4.A(b).

(b) (Pasynkov [1972]) Prove that for every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a universal space for the class of all strongly countable-dimensional normal spaces whose weight is not larger than $\mathbf{m}$.

Hint. Apply Problem 3.4.A(b) and Lemma 3.1.22.

Remark. It is not known if for $\mathbf{m} > \aleph_0$ there exists a universal space for the class of all countable-dimensional normal spaces whose weight is not larger than $\mathbf{m}$, or even if there exists a normal countable-dimensional space of weight $\mathbf{m}$ that contains topologically all countable-dimensional compact spaces whose weight is not larger than $\mathbf{m}$.

5.4. Countable dimensionality and mappings

In this section we shall study the behaviour of countable-dimensional and strongly countable-dimensional spaces under mappings. First we shall discuss closed mappings, and we shall start with two invariance theorems, counterparts of the theorem on dimension-raising mappings for countable-dimensional and strongly countable-dimensional spaces. We begin with two lemmas.

5.4.1. LEMMA. *If $f: X \to Y$ is a perfect mapping, then for every locally finite family $\mathcal{A}$ of subsets of X the family $\{f(A) : A \in \mathcal{A}\}$ is locally finite in Y.*

PROOF. Since the fibres of f are compact, for every $y \in Y$ there exists an open set $U \subset X$ which contains $f^{-1}(y)$ and meets only finitely many members of $\mathcal{A}$. The mapping f being closed, the set $V = Y \setminus f(X \setminus U)$ is a neighbourhood of the point y. From the inclusion $f^{-1}(V) \subset U$ it follows that V meets only finitely many members of the family $\{f(A) : A \in \mathcal{A}\}$. ■

5.4.2. LEMMA. *Let $f: X \to Y$ be a closed mapping of a regular space X onto a topological space Y. For every base $\mathcal{B} = \{U_s\}_{s \in S}$ for the space X there exists a family $\{A_s\}_{s \in S}$ of subsets of X such that $A_s \subset U_s$ for each $s \in S$ and*

(i) *$f|A_s: A_s \to f(A_s)$ is a homeomorphism,*

(ii) *$X = (\bigcup_{s \in S} A_s) \cup (\bigcup_{y \in Y} [f^{-1}(y)]^{\mathrm{d}})$.*

PROOF. For each $s \in S$ let

$$A_s = \{x \in U_s : \overline{U}_s \cap f^{-1}f(x) = \{x\}\}.$$

The restriction $f|\overline{U}_s: \overline{U}_s \to Y$ is a closed mapping, and so is its restriction $(f|\overline{U}_s)_{f(A_s)}$: $\overline{U}_s \cap f^{-1}f(A_s) \to f(A_s)$. Since we clearly have $A_s = \overline{U}_s \cap f^{-1}f(A_s)$, the last mapping coincides with $f|A_s: A_s \to f(A_s)$. Thus the mapping $f|A_s: A_s \to f(A_s)$ is closed and, by the definition of A_s, one-to-one, i.e., we have (i).

To prove (ii) it suffices to observe that if $x \notin [f^{-1}f(x)]^{\mathrm{d}}$, then there exists a member U_s of the base $\mathcal{B}$ such that $\overline{U}_s \cap f^{-1}f(x) = \{x\}$, and thus $x \in A_s$. ■

5.4.3. THEOREM. *If $f: X \to Y$ is a closed mapping of a countable-dimensional metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point or is of cardinality less than $\mathbf{c}$, then Y is countable-dimensional.*

PROOF. For every $y \in Y$ let

$$A(y) = \begin{cases} \operatorname{Fr} f^{-1}(y) \cup \{x\}, \text{ where } x \text{ is an isolated point of } f^{-1}(y), & \text{if } |\operatorname{Fr} f^{-1}(y)| \geq \mathbf{c}, \\ \operatorname{Fr} f^{-1}(y) \quad \text{if } 0 < |\operatorname{Fr} f^{-1}(y)| < \mathbf{c}, & \\ \{x\}, \text{ where } x \text{ is an arbitrary point of } f^{-1}(y), & \text{if } \operatorname{Fr} f^{-1}(y) = \emptyset. \end{cases}$$

The set $A = \bigcup_{y \in Y} A(y)$ is closed in X, because its complement is the union of open sets $\operatorname{Int} f^{-1}(y)$ with at most one point removed from each of them. The restriction $f|A: A \to Y$ is a closed mapping of A onto Y, and from Lemma 1.12.9 it follows that its fibres are compact, i.e., that $f|A$ is perfect. Moreover, every fibre of $f|A$ has an isolated point or is of cardinality less than $\mathbf{c}$. Hence, by virtue of Theorem 5.2.1, it suffices to prove our theorem under the additional assumption that f is a perfect mapping.

Observe first that since each non-empty compact metric space of cardinality less than $\mathbf{c}$ has isolated points (see Problem 1.2.D(b)), under our additional assumption for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point. Let $\mathcal{B} = \{U_s\}_{s \in S}$ be a base for the space X such that $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ whenever $i \neq j$, and the family $\{U_s\}_{s \in S_i}$ is locally finite for $i = 1, 2, \ldots$ Consider a family $\{A_s\}_{s \in S}$ of subsets of X which satisfy (i) and (ii) in Lemma 5.4.2. Since each fibre has an isolated point, $Y = \bigcup_{s \in S} f(A_s)$, and since the subspaces $A_s \subset X$ and $f(A_s) \subset Y$ are homeomorphic, $f(A_s)$ is countable-dimensional for each $s \in S$. To conclude the proof it suffices to apply Lemma 5.4.1 and Theorems 5.2.10 and 5.2.8. ■

In the analogue of Theorem 5.4.3 for strongly countable-dimensional spaces stronger assumptions on the fibres are necessary:

5.4.4. THEOREM. *If $f: X \to Y$ is a closed mapping of a strongly countable-dimensional normal space X onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then Y is strongly countable-dimensional.*

PROOF. If $X = \bigcup_{i=1}^{\infty} X_i$, where X_i's are closed and $\dim X_i < \infty$ for $i = 1, 2, \ldots$, then $Y = \bigcup_{i=1}^{\infty} Y_i$, where $Y_i = f(X_i)$ is closed and – by Theorem 3.3.7 – $\dim Y_i \leq \dim X_i + (k-1) < \infty$ for $i = 1, 2, \ldots$ ■

Observe that the last theorem does not hold under the weaker assumption that all fibres are finite. Indeed, the space Y described in Example 5.1.7, which is not strongly countable-dimensional, was obtained as the image of the Cantor set C under a closed mapping with finite fibres (cf. Problem 5.4.A(d)).

Let us also observe that in the case when X and Y are metrizable Theorem 5.4.4 can be deduced from Theorems 4.3.1 and 4.1.3 rather than from Theorem 3.3.7.

Now we pass to counterparts of the theorem on dimension-lowering mappings for countable-dimensional and strongly countable-dimensional spaces.

5.4.5. PROPOSITION. *If $f: X \to Y$ is a closed mapping of a normal space X to a countable-dimensional hereditarily normal hereditarily weakly paracompact space Y such that $\dim f^{-1}(y) < \infty$ for every $y \in Y$, then X is countable-dimensional.*

PROOF. Let $Y = \bigcup_{n=1}^{\infty} Y_n$, where $\dim Y_n \leq n$ for $n = 1, 2, \ldots$, and let $Y_{n,k} = \{y \in Y_n : \dim f^{-1}(y) \leq k\}$. Since the mapping $f_{Y_{n,k}}: f^{-1}(Y_{n,k}) \to Y_{n,k}$ is closed, from Theorem 3.3.10 it follows that $\dim f^{-1}(Y_{n,k}) \leq \dim Y_{n,k} + k \leq \dim Y_n + k \leq n + k$. Thus the space X, being the union of the subspaces $f^{-1}(Y_{n,k})$ for $n, k = 1, 2, \ldots$, is countable-dimensional. ■

Theorem 5.4.3, Proposition 5.4.5 and Corollary 1.6.11 yield

5.4.6. COROLLARY. *If $f: X \to Y$ is a closed mapping of a metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then X is countable-dimensional if and only if Y is countable-dimensional.* ■

Let us note that in the last proposition the assumption of finite dimensionality of fibres cannot be relaxed to strong countable-dimensionality, even in the case of open-and-closed mappings. Indeed, one can define a compact metrizable space which is not countable-dimensional, and yet can be mapped onto the Cantor set by an open mapping with strongly countable-dimensional fibres (see Problem 6.2.D(b)).

In the analogue of Proposition 5.4.5 for strongly countable-dimensional spaces stronger assumptions on the fibres are necessary: from Theorem 3.3.10 we obtain

5.4.7. PROPOSITION. *If $f: X \to Y$ is a closed mapping of a normal space X to a strongly countable-dimensional weakly paracompact normal space Y and there exists an integer $k \geq 0$ such that $\dim f^{-1}(y) \leq k$ for every $y \in Y$, then X is strongly countable-dimensional.* ■

Observe that the last proposition does not hold under the weaker assumption that all fibres have finite covering dimension, even in the case of open-and-closed mappings.

5.4.8. EXAMPLE. We shall define an open-and-closed mapping $f: Y \to C$ of the space Y described in Example 5.1.7, which is not strongly countable-dimensional, onto the Cantor set C such that $\dim f^{-1}(c) < \infty$ for every $c \in C$.

The space Y is the quotient space X/E, where $X = C \times C$ and E is a closed equivalence relation on X. Observe that for every point $y \in Y$ the inverse image $q^{-1}(y)$ under the quotient mapping $q: X \to Y$ is contained in the set $\{c\} \times C$ for some $c \in C$; by assigning this c to the point y we define a mapping f of Y onto C. One easily sees that the composition fq coincides with the projection $p: X \to C$ of the Cartesian product $C \times C$ onto the first factor. From the equality $fq = p$ it follows that $f: X \to C$ is continuous, and from the fact that p is an open mapping it follows that f is open. Indeed, for any set $U \subset Y$ we have

$$f(U) = fqq^{-1}(U) = pq^{-1}(U),$$

so that if U is open, then so is $f(U)$. The space Y being compact, f is an open-and-closed mapping, and for each $c \in C$ the fibre $f^{-1}(c) = qp^{-1}(c)$ is finite-dimensional. ■

Let us also observe that in the case when X and Y are metrizable Propositions 5.4.5 and 5.4.7 can be deduced from Theorems 4.3.4 and 4.1.3 rather than from Theorem 3.3.10.

Let us now pass to a discussion of open mappings. We shall show that under rather weak assumptions on X and Y both countable dimensionality and strong countable dimensionality are preserved by open mappings with finite fibres. Let us start with a simple lemma (cf. Problem 3.3.G).

5.4.9. LEMMA. *If $f: X \to Y$ is an open mapping of a normal space X with $\dim X = n$ onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then $\dim Y \leq (n+1)k - 1$.*

PROOF. Let $\{U_i\}_{i=1}^m$ be a finite open cover of the space Y. The open cover $\{f^{-1}(U_i)\}_{i=1}^m$ of the space X has an open shrinking $\{V_i\}_{i=1}^m$ of order $\leq n$. The family $\{f(V_i)\}_{i=1}^m$ is an open shrinking of $\{U_i\}_{i=1}^m$, and for every $y \in Y$ we have

$$y \in f(V_i) \quad \text{if and only if} \quad f^{-1}(y) \cap V_i \neq \emptyset,$$

so that $\operatorname{ord}(\{f(V_i)\}_{i=1}^m) \leq (n+1)k - 1$. Thus $\dim Y \leq (n+1)k - 1$. ■

5.4.10. THEOREM. *If $f: X \to Y$ is an open mapping of a countable-dimensional strongly hereditarily normal space X onto a hereditarily normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then Y is countable-dimensional.*

PROOF. By Proposition 5.1.2 the space X can be represented as the union of an increasing sequence of subspaces $X_1 \subset X_2 \subset \ldots$ such that $\dim X_n \leq n$ for $n = 1, 2, \ldots$ Let $Y_n = Y \setminus f(X \setminus X_n)$ and $Y_{n,k} = Y_n \cap \{y \in Y : |f^{-1}(y)| \leq k\}$ for $n, k = 1, 2, \ldots$ Since the fibres of f are finite, we have $\bigcup_{n=1}^{\infty} Y_n = Y$ and $\bigcup_{k=1}^{\infty} Y_{n,k} = Y_n$, so that $Y = \bigcup_{n,k=1}^{\infty} Y_{n,k}$. But $\dim Y_{n,k} < \infty$ by Lemma 5.4.9; indeed, $\dim f^{-1}(Y_{n,k}) \leq n$ by Theorem 3.1.19 because $f^{-1}(Y_{n,k}) \subset f^{-1}(Y_n) \subset X_n$, and $f|f^{-1}(Y_{n,k}): f^{-1}(Y_{n,k}) \to Y_{n,k}$

is an open mapping whose fibres contain at most k points. Hence Y is a countable-dimensional space. ■

By a similar argument – applying Proposition 5.1.5 to represent X as the union of an increasing sequence of closed subspaces $X_1 \subset X_2 \subset \dots$ such that $\dim X_n \leq n$ for $n = 1, 2, \dots$ and observing that for $k = 1, 2, \dots$ the set $\{y \in Y : |f^{-1}(y)| \leq k\}$ is closed in Y, so that the sets $Y_{n,k}$ are closed – we obtain

5.4.11. THEOREM. *If $f: X \to Y$ is an open mapping of a strongly countable-dimensional normal space X onto a normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then Y is strongly countable-dimensional.* ■

As follows from Problems 1.12.E and 1.12.F, the Hilbert cube can be obtained as the image of a subspace of the space of irrational numbers under an open mapping with countable fibres, so that in the last two theorems the assumption that fibres are finite cannot be relaxed to their being countable (cf. Problem 5.4.D(a)).

It turns out (see Problem 5.4.E) that if a space X can be mapped by an open mapping with finite fibres onto a countable-dimensional (strongly countable-dimensional) space Y, then – under rather weak assumptions on X and Y – the space X is countable-dimensional (strongly countable-dimensional).

We close this section with characterizations of countable-dimensional and strongly countable-dimensional spaces in terms of closed mappings defined on spaces with covering dimension zero.

To state the characterization of strongly countable-dimensional spaces we have to introduce a new notion; we shall say that a mapping $f: X \to Y$ is of *c-finite order* if the space Y can be represented as the union of closed subspaces $Y_1, Y_2, \dots$ with the property that for $k = 1, 2, \dots$ there exists an integer $n_k \geq 1$ such that $|f^{-1}(y)| \leq n_k$ for every $y \in Y_k$. One easily sees that by dropping the assumption that the sets Y_k are closed one obtains the condition that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$; to formulate and prove our two characterizations in a parallel way, we shall call any $f: X \to Y$ that satisfies the latter condition a mapping of *finite order*. Thus every mapping of c-finite order is of finite order.

5.4.12. LEMMA. *For every σ-discrete base $\mathcal{B} = \{U_s\}_{s \in S}$ for a metrizable space X, where $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ whenever $i \neq j$, and the family $\{U_s\}_{s \in S_i}$ is discrete for $i = 1, 2, \dots$, there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \dots$ of locally finite closed covers of the space X, where $\mathcal{F}_i = \{F_{t_1 t_2 \dots t_i} : t_1, t_2, \dots, t_i \in T\}$, satisfying the conditions:*

(i) *For every point $x \in X$ and each neighbourhood U of the point x there exists a natural number i such that $\operatorname{St}(x, \mathcal{F}_i) \subset U$.*

(ii) *For $i = 1, 2, \dots$ each point $x \in X$ such that $|\{s \in \bigcup_{j=1}^{i} S_j : x \in \operatorname{Fr} U_s\}| \leq n$ belongs to at most 2^n members of $\mathcal{F}_i$.*

(iii) *$F_{t_1 t_2 \dots t_i} = \bigcup_{t \in T} F_{t_1 t_2 \dots t_i t}$ for all $t_1, t_2, \dots, t_i \in T$ and $i = 1, 2, \dots$*

PROOF. Let $T = S \cup \{t_0\}$, where $t_0 \notin S$, and for $i = 1, 2, \dots$ consider the closed cover $\mathcal{E}_i = \{E_{i,t}\}_{t \in T}$ of the space X, where

$$E_{i,s} = \overline{U}_s \quad \text{for } s \in S_i, \quad E_{i,s} = \emptyset \quad \text{for } s \in S \setminus S_i \quad \text{and} \quad E_{i,t_0} = X \setminus \bigcup_{s \in S_i} U_s;$$

the cover $\mathcal{E}_i$ is locally finite and

(1) each point $x \in X$ belongs to at most 2 members of $\mathcal{E}_i$.

The sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ will now be defined inductively. Let $\mathcal{F}_1 = \{F_t : t \in T\}$, where $F_t = E_{1,t}$, and for $i > 1$ – assuming that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{i-1}$ are already defined – let $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_{i-1} t_i} : t_1, t_2, \ldots, t_{i-1}, t_i \in T\}$, where

$$F_{t_1 t_2 \ldots t_{i-1} t_i} = F_{t_1 t_2 \ldots t_{i-1}} \cap E_{i,t_i}. \tag{2}$$

From the definition it follows directly that $\mathcal{F}_1, \mathcal{F}_2, \ldots$ are locally finite closed covers of X and that condition (iii) is satisfied.

Consider a point $x \in X$ and an arbitrary neighbourhood U of x. There exists an $s \in S$ such that $x \in U_s \subset \overline{U}_s \subset U$ and a natural number i such that $s \in S_i$. Since $\overline{U}_s$ is the only member of $\mathcal{E}_i$ which contains the point x, it follows from (2) that $\operatorname{St}(x, \mathcal{F}_i) \subset \overline{U}_s \subset U$. Thus condition (i) is satisfied.

It remains to prove that condition (ii) is satisfied. We shall proceed by induction with respect to i. From (1) and the definition of $\mathcal{F}_1$ it is clear that each point $x \in X$ belongs to at most 2 members of $\mathcal{F}_1$. Assume that

$$\text{if } \left|\left\{s \in \bigcup_{j=1}^{i-1} S_j : x \in \operatorname{Fr} U_s\right\}\right| \leq m, \text{ then } |\{F \in \mathcal{F}_{i-1} : x \in F\}| \leq 2^m \tag{3}$$

and consider an $x \in X$ such that $|\{s \in \bigcup_{j=1}^{i} S_j : x \in \operatorname{Fr} U_s\}| \leq n$. If $x \notin \operatorname{Fr} U_s$ for all $s \in S_i$, then x belongs to exactly one member of $\mathcal{E}_i$, and since $|\{F \in \mathcal{F}_{i-1} : x \in F\}| \leq 2^n$ by (3), it follows from (2) that x belongs to at most 2^n members of $\mathcal{F}_i$. If $x \in \operatorname{Fr} U_s$ for some $s \in S_i$, then $|\{s \in \bigcup_{j=1}^{i-1} S_j : x \in \operatorname{Fr} U_s\}| \leq n-1$, so that $|\{F \in \mathcal{F}_{i-1} : x \in F\}| \leq 2^{n-1}$ by (3), and from (1) and (2) it follows that x belongs to at most 2^n members of $\mathcal{F}_i$. Thus (ii) is established. ■

5.4.13. LEMMA. *If $\mathcal{B}$ is a σ-discrete base for a metrizable space X and X can be represented as the union of a sequence of subsets $X_1, X_2, \ldots$ with the property that $\operatorname{ord}(\{\operatorname{Fr} U : U \in \mathcal{B}\}|X_k) = n_k < \infty$ for $k = 1, 2, \ldots$, then X is the image of a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping f such that $|f^{-1}(x)| \leq 2^{n_k+1}$ for every $x \in X_k$ and $k = 1, 2, \ldots$*

PROOF. Let $\mathcal{B} = \{U_s\}_{s \in S}$, where $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ whenever $i \neq j$, and $\{U_s\}_{s \in S_i}$ is discrete for $i = 1, 2, \ldots$ Thus for $i = 1, 2, \ldots$ and for each point $x \in X_k$ we have $|\{s \in \bigcup_{j=1}^{i} S_j : x \in \operatorname{Fr} U_s\}| \leq n_k + 1$, so that by Lemma 5.4.12 there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X, where $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$, satisfying conditions (i) and (iii) of the lemma and such that each point $x \in X_k$ belongs to at most 2^{n_k+1} members of $\mathcal{F}_i$ for $i = 1, 2, \ldots$

From Lemma 4.3.14 it now follows that the space X is the image of a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping f such that $|f^{-1}(x)| \leq 2^{n_k+1}$ for every $x \in X_k$ and $k = 1, 2, \ldots$ ■

5.4.14. THEOREM. *A metrizable space X is countable-dimensional (strongly countable-dimensional) if and only if X is the image of a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping of finite order (c-finite order).*

PROOF. From Theorem 5.1.13 and Lemma 5.4.13 it follows that each countable-dimensional (strongly countable-dimensional) metrizable space X is the image of a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping of finite order (c-finite order).

The reverse implication follows from Theorem 5.4.3 (Theorems 5.4.4 and 5.2.14), or else from Theorems 4.1.3 and 4.3.1. ■

Historical and bibliographic notes

Theorem 5.4.3 under the assumption that the fibres have cardinality less than $\mathbf{c}$ was proved by Arhangel'skiĭ in [1966] (for compact metrizable spaces by Skljarenko in [1959]), and under the assumption that every fibre has an isolated point – by Nagami in [1967]. Theorem 5.4.4 and Propositions 5.4.5 and 5.4.7 are easy consequences of theorems on dimension-raising and dimension-lowering mappings. Theorems 5.4.10 and 5.4.11 were given by Arhangel'skiĭ in [1963], our proofs are taken from Polkowski [1983a]. Theorem 5.4.14 for countable-dimensional spaces was established by Nagata in [1958a], its version for strongly countable-dimensional spaces was given in Engelking [1992].

Problems

5.4.A. (a) (Arhangel'skiĭ [1966], Nagami [1967]) Prove that if $f\colon X \to Y$ is a closed mapping of a countable-dimensional metrizable space X onto an uncountable-dimensional metrizable space Y, then the subspace $\{y \in Y : |f^{-1}(y)| \geq \mathbf{c}\}$ of Y is uncountable-dimensional.

(b) (Nagami [1967]) Observe that in Theorem 5.4.3 the assumption on the fibres can be replaced by the assumption that for every $y \in Y$ the set $\operatorname{Fr} f^{-1}(y)$ has an isolated point or is of cardinality less than $\mathbf{c}$.

(c) (Arhangel'skiĭ [1966]) Prove that Theorem 5.4.3 holds under the assumption that Y is a normal space.

Hint (Lašnev [1965]). Prove that if $f\colon X \to Y$ is a closed mapping of a metrizable space X onto a space Y, then $Y = Y_0 \cup \bigcup_{i=1}^{\infty} Y_i$, where $f^{-1}(y)$ is compact for each $y \in Y_0$ and Y_i is a discrete closed subspace of Y for $i = 1, 2, \ldots$ To that end, let $Y_i = \{y \in Y$: these is no $y' \neq y$ such that $f^{-1}(y) \subset B(f^{-1}(y'), 1/i)\}$; to show that $f^{-1}(y)$ is compact for each $y \in Y \setminus \bigcup_{i=1}^{\infty} Y_i$ choose a sequence $y_1, y_2, \ldots$, where $f^{-1}(y) \subset \overline{B(f^{-1}(y_i), 1/i)}$ for $i = 1, 2, \ldots$ and $y_i \neq y_j$ whenever $i \neq j$, observe that $f^{-1}(y) \subset \overline{\bigcup_{i=1}^{\infty} f^{-1}(y_i)}$ and apply Lemma 1.12.9 to the restriction of f to the subspace $f^{-1}(\{y, y_1, y_2, \ldots\})$ of the space X.

(d) (Arhangel'skiĭ [1967]) Prove that if $f\colon X \to Y$ is a continuous mapping of a compact strongly countable-dimensional space X onto a metrizable space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then Y is countable-dimensional.

Hint. Apply Problem 3.4.A(b).

Remark. It is not known if the result in (d) holds under the assumption that Y is a normal space. It is not known either if in the class of compact spaces the countable dimensionality is preserved by continuous mappings with fibres of cardinality less than $\mathbf{c}$.

5.4.B. (a) (Arhangel'skiĭ [1966]) Show that if $f: X \to Y$ is a closed mapping of a separable metrizable space X onto a topological space Y and for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X, then the image under f of each countable-dimensional subspace of X is countable-dimensional.

(b) Observe that if Y in (a) is a metrizable space, then the assumption of separability of X can be dropped.

5.4.C. (a) Show that if $f: X \to Y$ is an open mapping of a metrizable space X to a countable-dimensional (strongly countable-dimensional) metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X, then X is countable-dimensional (strongly countable-dimensional).

Note that there exists an open mapping with discrete fibres that maps a metrizable space X with $\dim X = 0$ onto the Hilbert cube (see Problem 4.3.C(b)).

Hint. Apply Problem 4.3.C(a) or Lemma 1.12.5.

(b) (Polkowski [1983a]) Show that if $f: X \to Y$ is an open mapping of a countable-dimensional (strongly countable-dimensional) metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is separable and has an isolated point, then Y is countable-dimensional (strongly countable-dimensional).

Deduce that if $f: X \to Y$ is an open mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a countable discrete space, then X is countable-dimensional (strongly countable-dimensional) if and only if Y is countable-dimensional (strongly countable-dimensional).

Hint. See the hint to Problem 4.3.C(c).

(c) (Wenner [1985]; implicitly, Polkowski [1983a]) Show that if $f: X \to Y$ is an open-and-closed mapping of a strongly countable-dimensional metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point, then Y is strongly countable-dimensional.

Observe that, even in the class of separable spaces, the assumption that f is open-and-closed cannot be weakened to the assumption that f is closed (cf. part (b)).

Hint. Note that the restriction $f_{Y \setminus Y_0}$, where Y_0 is the set of all isolated points of Y, is perfect and open, and apply Lemma 1.12.5.

(d) Note that the mapping of the space $I^{\aleph_0} \oplus \{0\}$ onto the one-point space $\{0\}$ is open-and-closed, has an isolated point in its unique fiber and maps an uncountable-dimensional compact metrizable space onto a strongly countable-dimensional metrizable space.

5.4.D. (a) (Arhangel'skiĭ [1966]) Show that if $f: X \to Y$ is an open-and-closed mapping of a strongly countable-dimensional metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then Y is strongly countable-dimensional.

Observe that the assumption that f is open-and-closed cannot be weakened to the assumption that f is either open or closed, even in the class of separable spaces.

Note that if $f: X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then X is strongly countable-dimensional if and only if Y is strongly countable-dimensional.

Hint. See the hint to Problem 5.4.C(c).

(b) Give an example of an open mapping with countable fibres which maps an uncountable-dimensional metrizable separable space onto the Cantor set (cf. Problem 6.2.D(a)).

Hint. Note that every countable-dimensional subspace of the Hilbert cube is contained in a countable-dimensional $G_{\delta\sigma}$-set (i.e. a countable union of G_δ-sets) and apply Problem 1.4.F(a) to define a a mapping f of the Cantor set to the Hilbert cube with an uncountable-dimensional graph.

5.4.E. (a) Show that if $f: X \to Y$ is an open mapping of a hereditarily normal space X to a countable-dimensional strongly hereditarily normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then X is countable-dimensional (cf. Problem 5.4.D(b)).

Hint. Apply Problems 2.4.J(b) and 2.4.J(a).

(b) Show that if $f: X \to Y$ is an open mapping of a weakly paracompact normal space X to a strongly countable-dimensional normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then X is strongly countable-dimensional.

Hint. Apply Problem 3.3.G.

5.4.F. Note that if $f: X \to Y$ is a closed mapping of c-finite order of a metrizable space X onto a metrizable space Y, then X is strongly countable-dimensional if and only if Y is strongly countable-dimensional.

5.4.G. (a) (Nagata [1958a]) Prove that a metrizable space X is countable-dimensional if and only if for some representation of X as the union of a sequence of subspaces $X_1, X_2, \ldots$ there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally-finite closed covers of the space X, where $\mathcal{F}_i = \{F_{t_1 t_2 \ldots t_i} : t_1, t_2, \ldots, t_i \in T\}$, such that for $k = 1, 2, \ldots$ there exists an integer n_k with the property that $\operatorname{ord} \mathcal{F}_i | X_k \leq n_k$ for $i = 1, 2, \ldots$, $F_{t_1 t_2 \ldots t_i} = \bigcup_{t \in T} F_{t_1 t_2 \ldots t_i t}$ for all $t_1, t_2, \ldots, t_i \in T$ and $i = 1, 2, \ldots$, and for every point x and each neighbourhood U of the point x there exists a natural number i such that $\operatorname{St}(x, \mathcal{F}_i) \subset U$.

(b) Prove that a metrizable space X is countable-dimensional if and only if for some representation of X as the union of a sequence of subspaces $X_1, X_2, \ldots$ there exists a sequence $\mathcal{F}_1, \mathcal{F}_2, \ldots$ of locally finite closed covers of the space X such that for $k = 1, 2, \ldots$ there exists an integer n_k with the property that $\operatorname{ord} \mathcal{F}_i | X_k \leq n_k$ for $i = 1, 2, \ldots$, $\mathcal{F}_{i+1}$ is a refinement of $\mathcal{F}_i$ for $i = 1, 2, \ldots$, and for every point x and each neighbourhood U of the point x there exists a natural number i such that $\operatorname{St}(x, \mathcal{F}_i) \subset U$.

Hint. See Lemma 4.3.14.

(c) Check that a metrizable space X is strongly countable-dimensional if and only if X satisfies the condition in (a), or the condition in (b), for some representation of X as the union of a sequence of closed subspaces $X_1, X_2, \ldots$

5.5. Locally finite-dimensional spaces

This section is devoted to locally finite-dimensional spaces, a class of spaces that is the smallest natural extension of the class of finite-dimensional spaces.

5.5.1. DEFINITION. A normal space X is *locally finite-dimensional* if for every point $x \in X$ there exists a normal open subspace U of X such that $x \in U$ and $\dim U < \infty$.

Clearly, every finite-dimensional space is locally finite-dimensional. The simplest example of an infinite-dimensional locally finite-dimensional space is the sum of the n-cubes I^n for $n = 1, 2, \ldots$, i.e., the space $S = \bigoplus_{n=1}^{\infty} I^n$ considered in Example 1.8.22(a).

5.5.2. PROPOSITION. *A normal space X is locally finite-dimensional if and only if every point $x \in X$ has a neighbourhood U such that* $\dim \overline{U} < \infty$.

PROOF. If X is locally finite-dimensional, then every point $x \in X$ belongs to an open normal subspace W of X with $\dim W < \infty$, and any neighbourhood U of the point x such that $\overline{U} \subset W$ satisfies $\dim \overline{U} < \infty$ by virtue of Theorem 3.1.3.

If U is a neighbourhood of a point $x \in X$ such that $\dim \overline{U} < \infty$, then for any continuous function $f: X \to I$ with $f(x) = 0$ and $f(X \setminus U) \subset \{1\}$ the open subspace $W = f^{-1}([0, 1)) \subset X$ contains x, is normal by Lemma 3.1.22, and satisfies $\dim W < \infty$ by Theorems 3.1.3 and 3.1.8. ■

5.5.3. PROPOSITION. *A hereditarily normal space X is locally finite-dimensional if and only if every point $x \in X$ has a neighbourhood U such that* $\dim U < \infty$. ■

The subspace, sum, and Cartesian product theorems for locally finite-dimensional spaces are easy consequences of the corresponding theorems for n-dimensional spaces.

Theorem 3.1.3 and Proposition 5.5.2 yield

5.5.4. THEOREM. *Every closed subspace M of a locally finite-dimensional space X is locally finite-dimensional, and every open subspace M of a hereditarily normal locally finite-dimensional space X is locally finite-dimensional.* ■

Theorem 3.1.19 yields

5.5.5. THE SUBSPACE THEOREM FOR LOCALLY FINITE-DIMENSIONAL SPACES. *Every subspace M of a strongly hereditarily normal locally finite-dimensional space X is locally finite-dimensional.* ■

As shown by the example of the one-point compactification of the space S, a space that can be represented as a countable union of closed finite-dimensional subspaces, or the union of two locally finite-dimensional subspaces, one of them being closed, need not be locally finite-dimensional. We have, however,

5.5.6. THE LOCALLY FINITE SUM THEOREM FOR LOCALLY FINITE-DIMENSIONAL SPACES. *If a normal space X can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of closed locally finite-dimensional subspaces, then X is a locally finite-dimensional space.*

PROOF. Consider an arbitrary point $x \in X$ and let $S_0 = \{s_1, s_2, \ldots, s_k\}$ be the set of all $s \in S$ such that $x \in F_s$. By Proposition 5.5.2 for $i = 1, 2, \ldots, k$ there exists a neighbourhood U_i of the point x in the space X such that $\dim \overline{U_i \cap F_{s_i}} < \infty$. Since x does not belong to the closed set $\bigcup\{F_s : s \notin S_0\}$, there exists a neighbourhood U of x in X such that $U \subset \bigcap_{i=1}^{k} U_i$ and $U \cap \bigcup\{F_s : s \notin S_0\} = \emptyset$. Clearly, $U \subset \bigcup_{i=1}^{k} U_i \cap F_{s_i}$, so that $\overline{U} \subset \bigcup_{i=1}^{k} \overline{U_i \cap F_{s_i}}$, and $\dim U < \infty$ by Theorem 3.1.8. ■

The definition of locally finite-dimensional spaces directly implies

5.5.7. THEOREM. *If a normal space X can be represented as the union of a family of open locally finite-dimensional subspaces, then X is a locally finite-dimensional space.* ■

Finally, from Proposition 5.5.2 and Theorems 3.4.6 and 3.4.9 we obtain

5.5.8. THE CARTESIAN PRODUCT THEOREM FOR LOCALLY FINITE-DIMENSIONAL SPACES. *If the Cartesian product $X \times Y$ of locally finite-dimensional spaces X and Y, where X is either compact or metrizable, is normal, then it is locally finite-dimensional.* ■

We shall now prove two simple theorems on relations between the classes of locally finite-dimensional, strongly countable-dimensional and finite-dimensional spaces. First, we introduce some notation.

For each normal space X and $n = 0, 1, \ldots$ we let

$$D_n(X) = \bigcup\{U \subset X : U \text{ is open and } \dim \overline{U} \leq n\}$$

and

$$K(X) = X \setminus \bigcup_{n=0}^{\infty} D_n(X);$$

clearly, $D_0(X) \subset D_1(X) \subset \ldots$, the sets $D_n(X)$ are open, $K(X)$ is closed, and X is locally finite-dimensional if and only if $\bigcup_{n=0}^{\infty} D_n(X) = X$, i.e., $K(X) = \emptyset$.

5.5.9. LEMMA. *Every locally finite-dimensional weakly paracompact normal space X can be represented as the union of a sequence of closed subspaces $F_0, F_1, \ldots$ such that $F_n \subset D_n(X)$ and $\dim F_n \leq n$ for $n = 0, 1, \ldots$*

PROOF. The space X being weakly paracompact, its open cover $\{D_n(X)\}_{n=0}^{\infty}$ has a point-finite open shrinking $\{U_n\}_{n=0}^{\infty}$, and – X being normal – the latter has a closed shrinking $\{F_n\}_{n=0}^{\infty}$. The subspaces F_n are weakly paracompact and normal, so that by Theorems 3.1.3 and 3.1.14 we have $\dim F_n \leq n$ for $n = 0, 1, 2 \ldots$ ■

From the lemma we immediately obtain

5.5.10. THEOREM. *Every locally finite-dimensional weakly paracompact normal space is strongly countable-dimensional.* ■

Let us note that the assumption of weak paracompactness cannot be omitted in the last theorem (see Problem 5.5.A).

Proposition 5.5.2 and Theorem 3.1.8 imply

5.5.11. PROPOSITION. *Every locally finite-dimensional compact space is finite-dimensional.* ■

We shall now establish three characterizations of weakly paracompact normal locally finite-dimensional spaces.

The first of these is a counterpart of Theorem 5.1.9.

5.5.12. THEOREM. *A weakly paracompact normal space X is locally finite-dimensional if and only if X can be represented as the union of a sequence of open subsets $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$*

PROOF. We shall first show that every normal space X that can be thus represented is locally finite-dimensional. Consider an arbitrary point $x \in X$ and an integer n such that $x \in X_n$. There exists a neighbourhood U of x with $\overline{U} \subset X_n$. One easily checks that $\dim \overline{U} \leq n$ (cf. Problem 1.6.A(a)), so that X is locally finite-dimensional by Proposition 5.5.2.

Consider now a weakly paracompact normal locally finite-dimensional space X. By Lemma 5.5.9 the space X can be represented as the union of a sequence of closed subspaces $E_0, E_1, \ldots$ such that $E_n \subset D_n(X)$. Applying Urysohn's lemma we find closed subsets $F_1, F_2, \ldots$ of X such that $E_n \subset \operatorname{Int} F_n \subset F_n \subset D_n(X)$; the last inclusion, together with Theorems 3.1.3 and 3.1.14, implies that $\dim F_n \leq n$ for $n = 1, 2, \ldots$, and since $D_1(X) \subset D_2(X) \subset \ldots$, we can assume without loss of generality that $F_1 \subset F_2 \subset \ldots$ By Lemma 5.1.8 the sequence $\operatorname{Int} F_1, \operatorname{Int} F_1, \operatorname{Int} F_1, \operatorname{Int} F_2, \ldots$, where the set $\operatorname{Int} F_n$ is repeated $n + 2$ times, has the required property. ■

5.5.13. THEOREM. *A weakly paracompact normal space X is locally finite-dimensional if and only if X has an open cover $\{U_n\}_{n=1}^{\infty}$ by normal subspaces such that $\dim U_n \leq n$ for $n = 1, 2, \ldots$ and $U_m \cap U_n = \emptyset$ whenever $|m - n| > 1$.*

PROOF. It suffices to prove that each weakly paracompact locally finite-dimensional space X has an open cover $\{U_n\}_{n=1}^{\infty}$ with the required properties. Consider a closed cover $\{F_n\}_{n=0}^{\infty}$ of the space X such that $F_n \subset D_n(X)$ for $n = 0, 1, \ldots$ We shall define by induction a sequence $V_0, V_1, \ldots$ of open F_σ-sets and a sequence $E_0, E_1, \ldots$ of closed G_δ-sets such that

$$F_n \subset V_n \subset E_n \subset D_n(X) \tag{1}$$

and

$$E_n \subset V_{n+1} \quad \text{for } n = 0, 1, \ldots \tag{2}$$

The sets V_0 and E_0 are obtained by applying Urysohn's lemma to the pair F_0, $X \setminus D_0(X)$ of disjoint closed subsets of X and taking the inverse images of the intervals $[0, 1/2)$ and $[0, 1/2]$ under the corresponding function. If the sets $V_0, V_1, \ldots, V_n$ and $E_0, E_1, \ldots, E_n$ satisfying the required conditions are already defined, we apply Urysohn's lemma to the pair $E_n \cup F_{n+1}$, $X \setminus D_{n+1}(X)$ and define V_{n+1} and E_{n+1} as the inverse images of the intervals $[0, 1/2)$ and $[0, 1/2]$. Thus the sequences $V_0, V_1, \ldots$ and $E_0, E_1, \ldots$ are defined.

The open F_σ-sets

$$U_1 = V_1, \quad U_2 = V_2 \setminus E_0, \quad \ldots, \quad U_n = V_n \setminus E_{n-2},$$

cover the space X. Indeed, it follows from (2) that for each $x \in X$ we have $x \in U_{n_x}$, where n_x is the smallest $n \geq 1$ such that $x \in V_n$. By Lemma 3.1.22 the subspaces U_n are normal, and by Theorem 3.1.8 we have $\dim U_n \leq n$ for $n = 1, 2, \ldots$ because $\dim F \leq n$ for every closed set F contained in $U_n \subset D_n(X)$. Finally, it follows from (1) that $U_m \cap U_n = \emptyset$ whenever $|m - n| > 1$. ■

In our third characterization of local finite dimensionality we shall refer to a class of spaces which are somewhat similar to the space $S = \bigoplus_{n=1}^{\infty} I^n$.

5.5.14. DEFINITION. A normal space X is *S-like* if $X = \bigoplus_{n=1}^{\infty} X_n$, where $n \leq \dim X_n < \infty$ for $n = 1, 2, \ldots$

Obviously, S is an S-like space, and every S-like space is locally finite-dimensional.

5.5.15. PROPOSITION. *If for a normal space* $X = \bigoplus_{n=1}^{\infty} X_n$ *we have* $\dim X_n < \infty$ *for* $n = 1, 2, \ldots$ *and* $\sup\{\dim X_n : n = 1, 2, \ldots\} = \infty$, *then* X *is an* S*-like space.* ■

The next theorem shows that weakly paracompact normal locally finite-dimensional spaces are closely related to S-like spaces (cf. Problem 5.5.C).

5.5.16. THEOREM. *A weakly paracompact normal space* X *is locally finite-dimensional if and only if* X *can be represented as the union of two closed subspaces each of which is either* S*-like or finite-dimensional.*

PROOF. The sufficiency of the condition in the theorem follows from Theorem 5.5.6. We shall prove its necessity.

By Theorem 5.5.13 the space X has an open cover $\{U_n\}_{n=1}^{\infty}$ by normal subspaces such that $\dim U_n \leq n$ for $n = 1, 2, \ldots$ and $U_m \cap U_n = \emptyset$ whenever $|m - n| > 1$. Let

$$W_1 = \bigcup_{n=0}^{\infty} U_{2n+1} \quad \text{and} \quad W_2 = \bigcup_{n=1}^{\infty} U_{2n};$$

since $U_m \cap U_n = \emptyset$ if n and m are distinct and both odd or both even,

$$W_1 = \bigoplus_{n=0}^{\infty} U_{2n+1} \quad \text{and} \quad W_2 = \bigoplus_{n=1}^{\infty} U_{2n}. \tag{3}$$

The space X being normal, there exist closed subsets $X_1, X_2 \subset X$ such that $X_1 \subset W_1, X_2 \subset W_2$ and $X_1 \cup X_2 = X$. From (3) it follows that

$$X_1 = \bigoplus_{n=0}^{\infty} X_{1,n} \quad \text{and} \quad X_2 = \bigoplus_{n=1}^{\infty} X_{2,n},$$

where $X_{1,n} = X_1 \cap U_{2n+1}$ and $X_{2,n} = X_2 \cap U_{2n}$. Now, the subspace X_i for $i = 1, 2$ is either S-like or finite-dimensional depending on whether $\sup\{\dim X_{i,n} : n = 1, 2, \ldots\}$ is infinite or finite. ■

5.5.17. COROLLARY. *If a weakly paracompact normal locally finite-dimensional space* X *contains no closed* S*-like subspace, then* X *is finite-dimensional.* ■

In connection with the last corollary it is interesting to characterize spaces which contain no closed S-like subspaces. Here we come across a difficulty: containing no closed S-like subspaces may be related either to the structure of the space under consideration or to its not containing any closed subspaces X_n with $n \leq \dim X_n < \infty$ (see Problem 5.5.D(b)). To exclude the latter possibility, we shall slightly modify the definition of S-likeness.

5.5.18. DEFINITION. A normal space X is *almost S-like* if $X = \bigoplus_{n=1}^{\infty} X_n$, where $\dim X_n \geq n$ for $n = 1, 2, \ldots$

Theorems 5.2.5 and 5.2.6 yield

5.5.19. PROPOSITION. *Every almost S-like space which is either completely metrizable and countable-dimensional or metrizable and strongly countable-dimensional contains a closed S-like subspace.* ■

The characterization of spaces which contain no closed almost S-like subspaces will be preceded by a lemma.

5.5.20. LEMMA. *For every countable discrete family $\{F_n\}_{n=1}^{\infty}$ of closed subsets of a normal space X there exists a family $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that $F_n \subset U_n$ for $n = 1, 2, \ldots$ and the family $\{\overline{U}_n\}_{n=1}^{\infty}$ is discrete.*

PROOF. Since the function $f: \bigcup_{n=1}^{\infty} F_n \to R$ defined by letting $f(x) = n$ for $x \in F_n$ is continuous, by virtue of the Tietze–Urysohn theorem it can be extended to a continuous function $F: X \to R$. One readily checks that the sets $U_n = F^{-1}((n - 1/3, n + 1/3))$, where $n = 1, 2, \ldots$, satisfy the required conditions. ■

5.5.21. THEOREM. *A weakly paracompact normal space X contains no closed almost S-like subspace if and only if X satisfies the following condition:*

(κ) *There exists a compact subspace $K \subset X$ such that $\dim F < \infty$ for each closed subspace $F \subset X$ disjoint from K.*

PROOF. Assume that the space X satisfies condition (κ) and consider an almost S-like subspace A, where $A = \bigoplus_{n=1}^{\infty} A_n$ with $\dim A_n \geq n$ for $n = 1, 2, \ldots$ We shall show that A is not closed. Suppose the contrary; then the sets $\bigoplus_{n=m}^{\infty} A_n$ are closed and infinite-dimensional for $m = 1, 2, \ldots$, so that the subspace K meets all of them. Thus there exists a sequence of integers $n_1 < n_2 < \ldots$ and a sequence of distinct points $x_1, x_2, \ldots$ such that $x_i \in A_{n_i} \cap K$ for $i = 1, 2, \ldots$ The compactness of K implies that the set $\{x_1, x_2, \ldots\}$ has an accumulation point x_0, and A being closed we have $x_0 \in A_{n_0}$ for some n_0. But this is impossible; hence no almost S-like subspace of X is closed.

Assume now that the space X contains no closed almost S-like subspace. We shall prove that condition (κ) is satisfied for $K = K(X) = X \setminus \bigcup_{n=1}^{\infty} D_n(X)$. Since the space X is weakly paracompact, so is its closed subspace K, and to prove that K is compact it suffices to show that it is countably compact (see [GT], Theorem 5.3.2), i.e., to show that every countably infinite set $\{x_1, x_2, \ldots\} \subset K$ has an accumulation point in the subspace K (see [GT], Theorem 3.10.3); we can obviously assume that all points x_n are distinct. Suppose now that the set $\{x_1, x_2, \ldots\}$ has no accumulation point. As K is closed in X this means that the family $\{\{x_n\}\}_{n=1}^{\infty}$ is discrete, and by Lemma 5.5.20 there exists a family $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that $x_n \in U_n$ for $n = 1, 2, \ldots$ and the family $\{\overline{U}_n\}_{n=1}^{\infty}$ is discrete. As $x_n \notin D_n(X)$, we have $\dim \overline{U}_n > n$, so that $\bigcup_{n=1}^{\infty} \overline{U}_n$ is a closed almost S-like subspace of X. The contradiction shows that K is compact. The fact that each closed subspace $F \subset X$ disjoint from K is finite-dimensional follows from Corollary 5.5.17. ■

The remaining part of this section will be devoted to locally finite-dimensional metrizable spaces. We begin with a characterization theorem, a counterpart of Theorems 5.1.13 and 5.4.14. To state the theorem, we need two notions that we are now going to define.

A family $\mathcal{A}$ of subsets of a topological space X is *o-point-finite* if X can be represented as the union of a sequence of open subsets $X_1, X_2, \ldots$ such that $\operatorname{ord} \mathcal{A}|X_k$ is finite for $k = 1, 2, \ldots$ One easily sees that every locally finite family is o-point-finite, and that in a countably paracompact normal space every o-point-finite family is c-point-finite. It is not hard to provide examples showing that neither of the above implications can be reversed.

The second notion we need pertains to mappings. We shall say that a mapping $f: X \to Y$ is of *o-finite-order* if the space Y can be represented as the union of open subspaces $Y_1, Y_2, \ldots$ with the property that for $k = 1, 2, \ldots$ there exists an integer $n_k \geq 1$ such that $|f^{-1}(y)| \leq n_k$ for every $y \in Y_k$. One easily sees that if Y is a countably paracompact normal space, then every mapping $f: X \to Y$ of o-finite order is of c-finite order.

Now we are ready to state our characterization theorem. Its proof will readily follow from lemmas established in Sections 5.1 and 5.4 (some of the implications can be proved in a simpler way (see Problem 5.5.E)).

5.5.22. THEOREM. *For every metrizable space X the following conditions are equivalent:*

(i) *The space X is locally finite-dimensional.*

(ii) *For every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and the family $\{L_i\}_{i=1}^{\infty}$ is o-point-finite.*

(iii) *The space X has a σ-discrete base $\mathcal{B}$ such that the family $\{\operatorname{Fr} U : U \in \mathcal{B}\}$ is o-point-finite.*

(iv) *The space X is the image of a metrizable space Z with $\dim Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping of o-finite order.*

PROOF. The implication (i)⇒(ii) follows from Theorem 5.5.12 and Lemma 5.1.11 (or from Proposition 5.5.2 and Lemmas 3.1.5 and 5.1.11); the implication (ii)⇒(iii) is a consequence of Lemma 5.1.12; the implication (iii)⇒(iv) follows from Lemma 5.4.13, and the implication (iv)⇒(i) from Theorems 4.1.3 and 4.3.1. ■

Let us note that the σ-discreteness of $\mathcal{B}$ in condition (iii) of the last theorem can be replaced by its σ-local finiteness.

Now we pass to the universal space theorems. We shall start with defining a subspace W_ω of the Hilbert cube.

Let W_n be the subspace of the Hilbert cube $I^{\aleph_0}$ consisting of all points $\{x_j\}$ such that $0 \leq x_j \leq 1/n$ for $j = 1, 2, \ldots, n$ and $x_j = 0$ for $j > n$; the required space is defined by letting $W_\omega = \bigcup_{n=1}^{\infty} W_n \setminus \{(0, 0, \ldots)\}$.

5.5.23. THE FIRST UNIVERSAL SPACE THEOREM FOR LOCALLY FINITE-DIMENSIONAL SPACES. *The subspace W_ω of the Hilbert cube $I^{\aleph_0}$ is a universal space for the class of all locally finite-dimensional separable metrizable spaces.*

PROOF. To begin, let us observe that the space W_ω is locally finite-dimensional. Indeed, for each point $x = \{x_j\} \in W_\omega$ there exists a j such that $x_j > 0$, and thus $x_j > 1/n$ for a positive integer n; the set U consisting of all points in W_ω which have the jth

coordinate larger than $1/n$ is a neighbourhood of x, and since $U \subset W_1 \cup W_2 \cup \ldots \cup W_{n-1}$, we have $\dim U < \infty$. Thus the space W_ω is locally finite-dimensional.

Consider now a locally finite-dimensional separable metrizable space X. By Theorem 5.5.13 the space X has an open cover $\{U_n\}_{n=1}^\infty$ such that $\dim U_n \leq n$ for $n = 1, 2, \ldots$ and $U_m \cap U_n = \emptyset$ whenever $|m - n| > 1$. From Lemma 5.3.9 applied to $X_0 = X \setminus U_n$ and $X_1 = X$ it follows that for $n = 1, 2, \ldots$ there exists a continuous mapping $f_n \colon X \to [0, a_n]^{2(n+1)}$, where $a_n = 1/((n+2)(n+5))$, such that

$$f_n^{-1}((0, 0, \ldots, 0)) = X \setminus U_n \tag{4}$$

and the restriction $f|U_n$ is a homeomorphic embedding. The cubes $[0, a_1]^4, [0, a_2]^6, \ldots$ are subsets of the products $I_1 \times I_2 \times I_3 \times I_4$, $I_5 \times I_6 \times \ldots \times I_{10}, \ldots$, where $I_j = I$ for $j = 1, 2, \ldots$, so that by letting

$$f(x) = (f_1(x), f_2(x), \ldots) \quad \text{for each } x \in X,$$

we define a continuous mapping $f \colon X \to \prod_{j=1}^\infty I_j = I^{\aleph_0}$. Since, as one easily checks, the family $\{f_n\}_{n=1}^\infty$ separates points and closed sets, f is a homeomorphic embedding.

To conclude the proof, it remains to show that $f(X) \subset W_\omega$. Consider an arbitrary point $x \in X$ and an integer n such that $x \in U_n$. First, note that by (4) we have $f_n(x) \neq (0, 0, \ldots, 0)$, so that $f(x) \neq (0, 0, \ldots)$. Now, observe that $f_m(x) = (0, 0, \ldots, 0)$ if $m \notin \{n-1, n, n+1\}$, so that the indices of all non-zero coordinates of $f(x) = \{y_j\} \in \prod_{j=1}^\infty I_j$ are larger than $0 + 4 + 6 + \ldots + 2(n-1) + 2$, but not larger than $4 + 6 + \ldots + 2(n+2)$. Since $a_{n+1} \leq a_n \leq a_{n-1}$, all these coordinates belong to $[0, a_{n-1}]$; as $a_{n-1} = 1/((n+1)(n+4))$ and $(n+1)(n+4) = 4 + 6 + \ldots + 2(n+2)$, we have $f(x) = \{y_j\} \in W_{(n+1)(n+4)} \setminus \{(0, 0, \ldots)\}$. Thus $f(X) \subset W_\omega$. ■

To state the second universal space theorem for locally finite-dimensional spaces we shall define the space $W(\mathbf{m})$ for each cardinal number $\mathbf{m} \geq \aleph_0$. Let $W_n(\mathbf{m})$ be the subspace of the Cartesian product $\prod_{j=1}^\infty N_j(\mathbf{m})$ – where $N_j(\mathbf{m}) \subset [J(\mathbf{m})]^{\aleph_0}$ is the space of weight $\leq \mathbf{m}$ with $\dim N_j(\mathbf{m}) \leq j$ defined in Example 4.2.8 – consisting of all points $\{x_j\}$ such that $\rho(x_j, \mathbf{0}) \leq 1/n$ for $j = 1, 2, \ldots, n$ and $x_j = \mathbf{0}$ for $j > n$, where ρ is a metric on $[J(\mathbf{m})]^{\aleph_0}$ and $\mathbf{0} = (0, 0, \ldots) \in [J(\mathbf{m})]^{\aleph_0}$; the required space is defined by letting $W(\mathbf{m}) = \bigcup_{n=1}^\infty W_n(\mathbf{m}) \setminus \{(\mathbf{0}, \mathbf{0}, \ldots)\}$.

5.5.24. THE SECOND UNIVERSAL SPACE THEOREM FOR LOCALLY FINITE-DIMENSIONAL SPACES. *For every cardinal number $\mathbf{m} \geq \aleph_0$ the subspace $W(\mathbf{m})$ of the Cartesian product $\prod_{j=1}^\infty N_j(\mathbf{m})$ is a universal space for the class of all locally finite-dimensional metrizable spaces whose weight is not larger than $\mathbf{m}$.*

PROOF. To prove that the space $W(\mathbf{m})$ is locally finite-dimensional it suffices to repeat the argument in the first part of the proof of Theorem 5.5.23.

Consider now an arbitrary locally finite-dimensional metrizable space X such that $w(X) \leq \mathbf{m}$. By Theorem 5.5.13 the space X has an open cover $\{U_n\}_{n=1}^\infty$ such that $\dim U_n \leq n$ for $n = 1, 2, \ldots$ and $U_m \cap U_n = \emptyset$ whenever $|m - n| > 1$; without loss of generality we can assume that no U_n is equal to X. From Lemma 5.3.8 it follows that for each n there exists a metrizable space Y_n and a continuous mapping $g_n \colon X \to Y_n$ such that for some point $a_n \in Y_n$ we have $g_n^{-1}(a_n) = X \setminus U_n$ and the restriction $g_n|U_n$ is a homeomorphism of U_n onto $Y \setminus \{a_n\}$. One readily sees that $w(Y_n) \leq \mathbf{m}$ and that

$\dim Y_n \leq n$, so that by Theorems 4.1.3 and 4.1.17, Lemma 4.2.11 and Remark 4.2.12 there exists a homeomorphic embedding $h_n\colon Y_n \to N_n(\mathbf{m})$ such that

$$h_n(Y_n) \subset \{x \in N_n(\mathbf{m}) : \rho(x, \mathbf{0}) \leq 1/(n+2)\} \tag{5}$$

and $h_n(a_n) = \mathbf{0}$. The family $\{f_j\}_{j=1}^{\infty}$, where $f_j = h_j g_j\colon X \to N_j(\mathbf{m})$, separates points and closed sets, so that by letting

$$f(x) = (f_1(x), f_2(x), \ldots) \quad \text{for each } x \in X,$$

we define a homeomorphic embedding of X to $\prod_{j=1}^{\infty} N_j(\mathbf{m})$; to conclude the proof, it remains to show that $f(X) \subset W(\mathbf{m})$. Consider an arbitrary point $x \in X$ and an integer n such that $x \in U_n$. First, note that $f_n(x) \neq \mathbf{0}$, so that $f(x) \neq (\mathbf{0}, \mathbf{0}, \ldots)$. Now, observe that $f_m(x) = \mathbf{0}$ if $m \notin \{n-1, n, n+1\}$, and that by (5) we have $\rho(f_{n-1}(x), \mathbf{0}) \leq 1/(n+1), \rho(f_n(x), \mathbf{0}) \leq 1/(n+2)$ and $\rho(f_{n+1}(x), \mathbf{0}) \leq 1/(n+3)$, which implies that $f(x) \in W_{n+1}(\mathbf{m}) \setminus (\mathbf{0}, \mathbf{0}, \ldots)$. Thus $f(X) \subset W(\mathbf{m})$. ■

Let us note that for every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a completely regular space $X(\mathbf{m})$ each point of which has a neighbourhood U with $\overline{U}$ compact and $\dim \overline{U} < \infty$ such that every locally finite-dimensional space whose weight is not larger than $\mathbf{m}$ is embeddable in $X(\mathbf{m})$; it is not known if there exists such a space which is normal.

Historical and bibliographic notes

The notion of a locally finite-dimensional space, and its elementary properties, have been part of the dimension-theoretic folklore since a long time. Theorems 5.5.6 and 5.5.10 were proved by Wenner in [1972] for metrizable spaces. Theorem 5.5.12 can be found in Engelking [1992]. Theorem 5.5.13 was given by Olszewski in [1990] (for metrizable spaces). Theorem 5.5.21 is implicit in Skljarenko [1959a]. Theorem 5.5.22 was proved by Engelking in [1992] (related results were obtained by Walker and Wenner in [1981]; see Engelking [1992], Remark 4.4). Theorem 5.5.23 was established by Wenner in [1973], and Theorem 5.5.24 by Olszewski in [1990]. The existence of a universal space for the class of all locally finite-dimensional metrizable spaces whose weight is not larger than $\mathbf{m} \geq \aleph_0$ was established by Luxemburg in [1979]; the first concrete example of such a space was given by Bobkov in [1982]. The existence of spaces $X(\mathbf{m})$ mentioned at the end of the section was established by Charalambous in [1989] and by Zaharov in [1989].

Problems

5.5.A. Modify Dowker's example to produce an example of a normal space X such that $X = D_0(X)$ and yet X is not strongly countable-dimensional.

Hint. The modification is described in the hint to Problem 2.2.I.

Remark. This space is even uncountable-dimensional (see Engelking and E. Pol [1983]).

5.5.B (Engelking [1992]). (a) Show that a weakly paracompact normal space X is locally finite-dimensional if and only if X can be represented as the union of a sequence of open

subsets $X_1, X_2, \ldots$ such that every locally finite open cover $\mathcal{U}$ of the space X has an open refinement $\mathcal{V}$ with the property that $\operatorname{ord} \mathcal{V}|X_n \leq n$ for $n = 1, 2, \ldots$

Hint. See the hint to Problem 5.1.G(a).

(b) Note that a weakly paracompact normal space X is locally finite-dimensional if and only if X can be represented as the union of a sequence of open subsets $X_1, X_2, \ldots$ such that every finite open cover $\mathcal{U}$ of the space X has a finite closed refinement $\mathcal{F}$ with the property that $\operatorname{ord} \mathcal{F}|X_n \leq n$ for $n = 1, 2, \ldots$

5.5.C. Prove that every locally finite-dimensional metrizable space X with $\dim X = \infty$ can be represented as the union of two closed S-like subspaces.

5.5.D. (a) Show that if a normal space X contains no closed S-like subspace, then for each closed subspace $F \subset X$ disjoint from $K(X)$ there exists an integer $n \geq 0$ such that $F \subset D_n(X)$.

(b) Use the space in Example 5.2.23 to produce an example of a separable metrizable space X that contains no closed S-like subspace and yet does not satisfy condition (κ) in Theorem 5.5.21.

5.5.E. (a) Deduce the implication (i)⇒(iv) in Theorem 5.5.22 from Theorem 4.3.15.

(b) Deduce the implication (i)⇒(iii) in Theorem 5.5.22 from Theorems 4.2.2 and 5.5.13.

(c) Deduce the implication (iii)⇒(i) in Theorem 5.5.22 from Theorem 4.2.2.

5.5.F. (a) (Wenner [1973]) Show that every locally finite-dimensional separable metrizable space is embeddable in a locally finite-dimensional locally compact separable metrizable space.

(b) Prove that every locally finite-dimensional metrizable space is embeddable in a locally finite-dimensional completely metrizable space.

Hint. Apply Theorems 5.5.16 and 4.1.19 and Lemma 1.3.12.

5.5.G (Olszewski [1990]). Prove that for every cardinal number $\mathbf{m} \geq \aleph_0$ the subspace $U(\mathbf{m})$ of the Cartesian product $\prod_{j=1}^{\infty} N_j(\mathbf{m})$ consisting of all points $\{x_j\}$ such that the set $\{j : x_j \neq (0, 0, \ldots)\}$ consists of one or two consecutive integers is a universal space for the class of all locally finite-dimensional metrizable spaces whose weight is not larger than $\mathbf{m}$.

5.5.H. (a) (Baković [1973] for metrizable spaces) Show that if $f: X \to Y$ is a closed mapping of a normal locally finite-dimensional space X onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then Y is locally finite-dimensional.

Observe that this is not true under the weaker assumption that all fibres are finite.

(b) (Baković [1973] for metrizable spaces) Show that if $f: X \to Y$ is a closed mapping of a normal space X to a locally finite-dimensional weakly paracompact normal space Y and there exists an integer $k \geq 1$ such that $\dim f^{-1}(y) \leq k$ for every $y \in Y$, then X is locally finite-dimensional.

Observe that this is not true under the weaker assumption that all fibres have finite covering dimension, even in the case of open-and-closed mappings.

(c) (Wenner [1985] for metrizable spaces) Show that if $f: X \to Y$ is an open mapping of a weakly paracompact normal space X onto a normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then X is locally finite-dimensional if and only if Y is locally finite-dimensional.

Hint. Apply Problem 3.3.G.

(d) (Wenner [1985]) Show that if $f: X \to Y$ is an open mapping of a metrizable space X to a metrizable locally finite-dimensional space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X, then X is locally finite-dimensional.

Note that there exists an open mapping with discrete fibres that maps a metrizable space X with $\dim X = 0$ onto the Hilbert cube (see Problem 4.3.C(b)).

Hint. Apply Problem 4.3.C(a) or Lemma 1.12.5.

(e) (Wenner [1985]) Show that if $f: X \to Y$ is an open-and-closed mapping of a locally finite-dimensional metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point, then Y is locally finite-dimensional.

Hint. Note that the restriction $f_{Y \setminus Y_0}$, where Y_0 is the set of all isolated points of Y, is perfect and open, and apply Problem 4.3.C(d).

(f) Show that in part (e) the assumption that f is open-and-closed cannot be weakened to the assumption that f is open.

Hint. Take as Y the one-point compactification of the space S and as X the space $S \oplus [(\bigoplus_{n=1}^{\infty} Z_n) \cup \{p\}]$, where Z_n is a subspace of the space P of irrational numbers that can be mapped onto I^n by an open mapping (see Problem 1.12.E), $p \notin P$, and the sets $(\bigoplus_{n=m}^{\infty} Z_n) \cup \{p\}$, where $m = 1, 2, \ldots$, form a base at the point p.

(g) (Wenner [1985]) Show that if $f: X \to Y$ is an open-and-closed mapping of a locally finite-dimensional metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then Y is locally finite-dimensional.

Observe that the assumption that f is open-and-closed cannot be weakened to the assumption that f is either open or closed, even in the class of separable spaces.

Note that if $f: X \to Y$ is an open-and-closed mapping of a metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then X is locally finite-dimensional if and only if Y is locally finite-dimensional.

Hint. See the hint to part (e).

5.5.I. (a) (van Douwen [1979]; for strongly metrizable spaces, E. Pol [1979a]) Prove that the Čech–Stone compactification of a metrizable space X is countable-dimensional (strongly countable-dimensional) if and only if X contains a countable-dimensional (strongly countable-dimensional) compact subspace K such that $\dim F < \infty$ for each closed subspace $F \subset X$ disjoint from K.

Hint (E. Pol, cited in Engelking and E. Pol [1983]). By Problem 5.3.D(c) and Theorem 5.5.21 it suffices to show that if βX is countable-dimensional, then X is not an almost S-like space. Assume that βX is countable-dimensional and X is an almost S-like space. Let $\beta X = \bigcup_{i=1}^{\infty} X_i$, where $\dim X_i < \infty$ for $i = 1, 2, \ldots$ and $X = \bigoplus_{n=1}^{\infty} Z_n$, where $\dim Z_n \geq n$ for $n = 1, 2, \ldots$ First, define by induction a sequence $K_1, K_2, \ldots$ of closed G_δ-sets in $A = \beta X \setminus \bigcup_{n=1}^{\infty} \overline{Z}_n$ such that $K_1 = A$, $\operatorname{Int}_A K_m \neq \emptyset$, $K_{m+1} \subset \operatorname{Int}_A K_m$, and if $X_m \cap \operatorname{Int}_A K_m$ is not dense in $\operatorname{Int}_A K_m$, then $K_{m+1} \cap X_m = \emptyset$ (where Int_A is the interior operator in the subspace A). Apply Problem 2.2.E(b) to show that for every

sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of $K = \bigcap_{m=1}^{\infty} K_m$ there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition in K between A_i and B_i and $\bigcap_{i=1}^{\infty} L_i = \emptyset$. Then find in X a closed subspace $X' = \bigoplus_{n=1}^{\infty} Z'_n$, where $\dim Z'_n \geq n$ for $n = 1, 2, \ldots$, such that $\bigcap_{i=1}^{\infty} \overline{\bigcup_{n=i}^{\infty} Z'_n} \subset K$, and in each Z'_n consider a sequence $(A_{n,1}, B_{n,1}), (A_{n,2}, B_{n,2}), \ldots, (A_{n,n}, B_{n,n})$ of pairs of disjoint closed subsets such that any partitions in Z'_n between these sets intersect. Finally, let $A_i^* = \bigcup_{n=i}^{\infty} A_{n,i}$, $B_i^* = \bigcup_{n=i}^{\infty} B_{n,i}$, $A_i = K \cap \overline{A_i^*}$, and $B_i = K \cap \overline{B_i^*}$ for $i = 1, 2, \ldots$, note that $A_i \cap B_i = \emptyset$, consider partitions L_i in K between A_i and B_i such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$, enlarge them to partitions which are G_δ-sets in K and have an empty intersection, and apply Lemma 6.1.5 to obtain a contradiction.

(b) Show that if the Čech–Stone compactification βX of a metrizable space X is countable-dimensional, then X has a countable-dimensional compactification of weight $w(X)$ (cf. Problem 5.3.I(a)).

Hint. Apply (a) and Problem 3.4.A(c).

Remark. It is not known if (b) holds for normal spaces.

Chapter 6

Weakly infinite-dimensional spaces

In the present chapter the "degree of infinite dimensionality" is measured by the possibility of separating disjoint closed sets, in countably many pairs, by partitions that have an empty intersection.

The results of Chapters 2 and 3 are not used here, except for Lemma 2.3.3 and Theorems 3.1.1 and 3.2.4. The reader interested only in separable metric spaces (or in metrizable spaces) is advised to skip Lemmas 6.1.4 and 6.1.5, Theorem 6.1.7 with Remark 6.1.8, as well as Theorems 6.1.11–6.1.13 and Lemma 6.1.14, to provide a direct proof of Theorems 6.1.10 and 6.1.15 for metrizable spaces (see Problems 6.1.B(c) and (d)), and then to skip again Proposition 6.1.16.

Section 6.1 opens with the definition of weakly infinite-dimensional spaces and its easy consequences. Then we prove two countable sum theorems and deduce from them that (under rather mild assumptions) weak infinite dimensionality is hereditary with respect to F_σ-sets and that every countable-dimensional space is weakly infinite-dimensional. We then proceed to the point-finite sum theorem, locally finite sum theorem, and compactification theorem. The section concludes with an example of a compact metrizable weakly infinite-dimensional space which is not countable-dimensional.

In Section 6.2 we give an example of a totally disconnected strongly infinite-dimensional space; such a space was necessary to carry out the construction in the conclusion of the preceding section.

In Section 6.3 we prove two counterparts of theorems on dimension-raising mappings and a counterpart of the theorem on dimension-lowering mappings. From the latter we deduce a Cartesian product theorem. In this context the notion of a C-space appears. We close the section by a simple theorem on the invariance of weakly infinite-dimensional spaces under open mappings.

6.1. Definition and basic properties of weakly infinite-dimensional spaces

As in the preceding chapter, we shall only consider the covering dimension here, and by a finite-dimensional space we shall understand a normal space X satisfying the inequality $\dim X < \infty$.

The following definition is suggested by the characterization of the covering dimension in terms of partitions (Theorem 3.2.6).

6.1.1. DEFINITION. A normal space X is *weakly infinite-dimensional*[1)] if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed

[1)] The terms *A-weakly infinite-dimensional* and *Alexandroff weakly infinite-dimensional* are also used.

sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{\infty} L_i = \emptyset$. Normal spaces which are not weakly infinite-dimensional are called *strongly infinite-dimensional.*

From Theorem 1.8.2 it follows that the Hilbert cube $I^{\aleph_0}$ is strongly infinite-dimensional. On the other hand, Theorem 3.2.6 yields

6.1.2. PROPOSITION. *Every finite-dimensional normal space is weakly infinite-dimensional.* ■

As one easily checks, the infinite-dimensional space $S = \bigoplus_{n=1}^{\infty} I^n$ considered in Example 1.8.22(a) is weakly infinite-dimensional; the spaces K_ω and N_ω from Examples 1.8.22(b) and (c) are also weakly infinite-dimensional (see Theorem 6.1.10).

Directly from the definition we obtain

6.1.3. PROPOSITION. *Every closed subspace of a weakly infinite-dimensional space is weakly infinite-dimensional.* ■

It will be shown in Example 6.1.21 that weak infinite dimensionality is not a hereditary property, even in the realm of separable metrizable spaces (cf. Theorem 6.1.9 and Problem 6.1.B(a) and (b)). It also turns out that weak infinite dimensionality is not a multiplicative property in the realm of separable metrizable spaces (see Problem 6.1.H); it is an open problem, though, whether the Cartesian product of two weakly infinite-dimensional compact metrizable spaces is weakly infinite-dimensional (cf. Theorem 6.3.11).

We are now going to prove two countable sum theorems for weakly infinite-dimensional spaces. The following two lemmas will be used in the proof of the second one.

6.1.4. LEMMA. *If for a sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of a countably paracompact normal space X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $\bigcap_{i=1}^{\infty} L_i = \emptyset$, then there also exist closed G_δ-sets with the same properties.*

If the sequence under consideration is finite, it suffices to assume that X is normal.

PROOF. Since X is normal and countably paracompact, there exist open subsets $G_1, G_2, \ldots$ of X such that $L_i \subset G_i$ for $i = 1, 2, \ldots$ and $\bigcap_{i=1}^{\infty} G_i = \emptyset$ (see [GT], Theorem 5.2.3; cf. Lemma 3.4.7). Applying Urysohn's lemma, one can enlarge L_i to a closed G_δ-set contained in $G_i \setminus (A_i \cup B_i)$, and the sets obtained in this way for $i = 1, 2, \ldots$ have the required properties. If the sequence under consideration is finite and the space X is normal, the existence of G_i's follows from Theorem 3.1.1. ■

6.1.5. LEMMA. *If M is a closed subspace of a normal space X and A, B a pair of disjoint closed subsets of X, then for every partition L' in the space M between $M \cap A$ and $M \cap B$ which is a G_δ-set in M there exists a partition L in the space X between A and B which is a G_δ-set in X and satisfies the equality $M \cap L = L'$.*

More exactly, for each pair U', W' of open sets in M satisfying the conditions

$$M \cap A \subset U', \quad M \cap B \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad M \setminus L' = U' \cup W'$$

there exist disjoint open F_σ-sets $U, W \subset X$ such that

$$A \subset U, \quad B \subset W, \quad M \cap U = U' \quad \text{and} \quad M \cap W = W'.$$

PROOF. One easily sees that L' is a G_δ-set in $M \cup (A \cup B)$, so that L' is a closed G_δ-set in the closed subspaces $X_1 = (M \setminus W') \cup A$ and $X_2 = (M \setminus U') \cup B$ of the space X. Since X_1 and X_2 are normal and $L' \cap A = \emptyset = L' \cap B$, by virtue of Urysohn's lemma (cf. Problem 2.1.A) there exist continuous functions $f_1 \colon X_1 \to [-1, 0]$ and $f_2 \colon X_2 \to [0, 1]$ such that

$$f_1(A) \subset \{-1\}, \quad f_2(B) \subset \{1\} \quad \text{and} \quad f_1^{-1}(0) = L' = f_2^{-1}(0).$$

As $X_1 \cap X_2 = L'$, by letting

$$f(x) = \begin{cases} f_1(x) & \text{for } x \in X_1, \\ f_2(x) & \text{for } x \in X_2, \end{cases}$$

we define a continuous function $f \colon X_1 \cup X_2 \to [-1, 1]$, which can be extended to a continuous function $F \colon X \to [-1, 1]$ by the Tietze–Urysohn theorem. One readily checks that the sets $L = f^{-1}(0)$, $U = F^{-1}([-1, 0))$, and $W = F^{-1}((0, 1])$ have the required properties. ■

6.1.6. THE FIRST COUNTABLE SUM THEOREM FOR WEAKLY INFINITE-DIMENSIONAL SPACES. *If a hereditarily normal space X can be represented as the union of a sequence of weakly infinite-dimensional subspaces $X_1, X_2, \ldots$, then X is a weakly infinite-dimensional space.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairs of disjoint closed subsets of X. For $i = 1, 2, \ldots$ choose open sets $V_{i,1}, V_{i,2} \subset X$ such that $A_i \subset V_{i,1}$, $B_i \subset V_{i,2}$ and $\overline{V}_{i,1} \cap \overline{V}_{i,2} = \emptyset$. Represent the set N of all positive integers as the union $\bigcup_{j=1}^{\infty} N_j$ of infinite sets, where $N_j \cap N_k = \emptyset$ whenever $j \neq k$. Since X_j is weakly infinite-dimensional, there exists a family $\{L'_i : i \in N_j\}$ of closed subsets of X_j such that L'_i is a partition in X_j between $X_j \cap \overline{V}_{i,1}$ and $X_j \cap \overline{V}_{i,2}$, and $\bigcap\{L'_i : i \in N_j\} = \emptyset$. By virtue of Lemma 1.2.9 and Remark 1.2.10, for $j = 1, 2, \ldots$ and each $i \in N_j$ there exists a partition L_i in the space X between A_i and B_i which satisfies the inclusion $X_j \cap L_i \subset L'_i$.

Since $X_j \cap \bigcap\{L_i : i \in N_j\} = \bigcap\{X_j \cap L_i : i \in N_j\} \subset \bigcap\{L'_i : i \in N_j\} = \emptyset$ for $j = 1, 2, \ldots$, the equality $X = \bigcup_{j=1}^{\infty} X_j$ implies that $\bigcap_{i=1}^{\infty} L_i = \bigcap_{j=1}^{\infty} \bigcap\{L_i : i \in N_j\} = \emptyset$, and this shows that X is a weakly infinite-dimensional space. ■

6.1.7. THE SECOND COUNTABLE SUM THEOREM FOR WEAKLY INFINITE-DIMENSIONAL SPACES. *If a countably paracompact normal space X can be represented as the union of a sequence of closed weakly infinite-dimensional subspaces $F_1, F_2, \ldots$, then X is a weakly infinite-dimensional space.*

PROOF. We proceed as in the proof of the previous theorem making the following changes. First, for $j = 1, 2, \ldots$ we take a family $\{L''_i : i \in N_j\}$ such that L''_i is a partition in F_j between $F_j \cap A_i$ and $F_j \cap B_i$, and $\bigcap\{L''_i : i \in N_j\} = \emptyset$. Next, using Lemma 6.1.4, we replace the partitions L''_i with partitions L'_i which are G_δ-sets in F_j. Finally, we use Lemma 6.1.5 to extend each L'_i to a partition L_i in the space X between A_i and B_i in such a way that $F_j \cap L_i \subset L'_i$ for $i \in N_j$. ■

6.1.8. REMARK. In fact, we have proved a little more, viz. that each subspace of a countably paracompact normal space X which can be represented as the union of countably many weakly infinite-dimensional closed subspaces of X is weakly infinite-dimensional. However, since the property of being a countably paracompact normal space is hereditary with respect to F_σ-sets (see [GT], Problem 5.5.14(a)), this more general result follows from our theorem. ■

Let us note that it is not known if Theorem 6.1.6, or even Theorem 6.1.7, holds for normal spaces.

Proposition 6.1.3 and Theorem 6.1.6 (Remark 6.1.8) imply

6.1.9. THEOREM. *Every subspace M of a hereditarily normal (normal and countably paracompact) weakly infinite-dimensional space X which is an F_σ-set in X is weakly infinite-dimensional.* ■

The sum theorems for weakly infinite-dimensional spaces yield the following two important theorems.

6.1.10. THEOREM. *Every countable-dimensional hereditarily normal space is weakly infinite-dimensional.* ■

6.1.11. THEOREM. *Every strongly countable-dimensional countably paracompact normal space is weakly infinite-dimensional.* ■

As shown in Example 6.1.21 below, there exist compact metrizable weakly infinite-dimensional spaces which are not countable-dimensional.

The question whether Theorem 6.1.10 holds for normal spaces, or compact spaces, is still open. One does not even know if Theorem 6.1.11 holds for normal spaces.

We shall now prove two simple sum theorems, and then pass to the more difficult locally finite sum theorem for weakly infinite-dimensional spaces.

6.1.12. THE POINT-FINITE SUM THEOREM FOR WEAKLY INFINITE-DIMENSIONAL SPACES. *If a hereditarily normal space X can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed weakly infinite-dimensional subspaces, and if there exists a point-finite open cover $\{U_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then X is a weakly infinite-dimensional space.*

PROOF. Let $\mathcal{T}_i$ be the family of all subsets of S that have exactly i elements and let K_i consist of all points of X which belong to exactly i members of the cover $\{U_s\}_{s\in S}$. As in the proof of Theorem 5.2.11, we can assume that U_s's are F_σ-sets and represent K_i as the sum $\bigoplus_{T\in\mathcal{T}_i} K_T$, where $K_T = K_i \cap \bigcap_{s\in T} U_s$ is an F_σ-set in X. For each $T \in \mathcal{T}_i$ the subspace $\bigcup_{s\in T} F_s$ is weakly infinite-dimensional by virtue of Theorem 6.1.6; thus, by Theorem 6.1.9, the subspace $A_T = K_T \cap \bigcup_{s\in T} F_s$ is also weakly infinite-dimensional, and so is the union $A_i = \bigcup_{T\in\mathcal{T}_i} A_T = \bigoplus_{T\in\mathcal{T}_i} A_T$. To conclude the proof it suffices to check that $X = \bigcup_{i=1}^{\infty} A_i$ and apply again Theorem 6.1.6. ■

Let us note that Theorem 6.1.12 also holds under the assumption that X is a countably paracompact normal space (see Problem 6.1.C(c)).

6.1.13. THEOREM. *If a weakly paracompact hereditarily normal space X can be represented as the union of a locally countable family $\{X_s\}_{s\in S}$ of weakly infinite-dimensional subspaces, then X is a weakly infinite-dimensional space.*

PROOF. For every point $x \in X$ there exist an open F_σ-set V_x containing x and a countable set $S(x) \subset S$ such that $V_x \cap X_s = \emptyset$ if $s \notin S(x)$. It follows that $V_x \subset \bigcup\{X_s : s \in S(x)\}$, so that by Theorems 6.1.6 and 6.1.9 the subspace V_x is weakly infinite-dimensional. Since X is weakly paracompact, the open cover $\{V_x\}_{x\in X}$ has a point-finite open shrinking $\{U_x\}_{x\in X}$, which in turn has a closed shrinking $\{F_x\}_{x\in X}$ (see Problem 3.1.A(b)). To conclude the proof it suffices to apply Theorem 6.1.12. ■

In the proof of the locally finite sum theorem for weakly infinite-dimensional spaces it is necessary to consider, together with a partition L in a space X, two open sets $U, W \subset X$ witnessing that L is a partition, in accordance with Definition 1.1.3. Since $L = X\setminus(U\cup W)$, we need not even mention the partition L when U and W are specified. The following somewhat technical lemma is formulated in such a way.

6.1.14. LEMMA. *If M is a closed subspace of a hereditarily normal (normal and countably paracompact) space X and each closed subspace of X contained in $X \setminus M$ is weakly infinite-dimensional, then for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X and every sequence $(U_1', W_1'), (U_2', W_2') \ldots$ of pairs of open sets (open F_σ-sets) in M satisfying the conditions*

$$M \cap A_i \subset U_i', \quad M \cap B_i \subset W_i', \quad U_i' \cap W_i' = \emptyset \quad \text{for } i = 1, 2, \ldots$$

and $M = \bigcup_{i=1}^{\infty}(U_i' \cup W_i')$ there exists a sequence $(U_1, W_1), (U_2, W_2), \ldots$ of pairs of open sets (open F_σ-sets) in X such that $M \cap U_i = U_i'$, $M \cap W_i = W_i'$,

$$A_i \subset U_i, \quad B_i \subset W_i, \quad U_i \cap W_i = \emptyset \quad \text{for } i = 1, 2, \ldots$$

and $X = \bigcup_{i=1}^{\infty}(U_i \cup W_i)$.

PROOF. By virtue of Lemma 1.2.9 and Remark 1.2.10 (Lemma 6.1.5) there exists a sequence $(U_1'', W_1''), (U_2'', W_2''), \ldots$ of pairs of disjoint open sets (open F_σ-sets) in X such that

$$A_i \subset U_i'', \quad B_i \subset W_i'', \quad M \cap U_i'' = U_i' \quad \text{and} \quad M \cap W_i'' = W_i'. \tag{1}$$

One easily sees that M is contained in the open set $\bigcup_{i=1}^{\infty}(U_i'' \cup W_i'')$; thus by Urysohn's lemma there exists a closed G_δ-set F such that

$$M \subset F \subset \bigcup_{i=1}^{\infty}(U_i'' \cup W_i''). \tag{2}$$

Let $X \setminus F = \bigcup_{j=1}^{\infty} F_j$, where F_j's are closed in X.

Represent the set N of all positive integers as the union $\bigcup_{j=1}^{\infty} N_j$ of infinite sets, where $N_j \cap N_k = \emptyset$ whenever $j \neq k$. Since for $j = 1, 2, \ldots$ the space $F_j \subset X \setminus M$ is weakly infinite-dimensional, there exists in F_j a family $\{K_i : i \in N_j\}$ of closed sets (closed G_δ-sets) such that K_i is a partition between $F_j \cap A_i$ and $F_j \cap B_i$, and $\bigcap\{K_i : i \in N_j\} = \emptyset$. The sets F and F_j being closed and disjoint, the union $(F \cap L_i'') \cup K_i$, where $L_i'' = X \setminus (U_i'' \cup W_i'')$, is a closed set (closed G_δ-set) in $F \cup F_j$ which is a partition in $F \cup F_j$ between $(F \cup F_j) \cap A_i$ and $(F \cup F_j) \cap B_i$ for each $i \in N_j$. By Lemma 1.2.9 and

Remark 1.2.10 (Lemma 6.1.5) there exists a sequence $(U_1, W_1), (U_2, W_2), \ldots$ of disjoint open sets (open F_σ-sets) in X such that $A_i \subset U_i$, $B_i \subset W_i$,

$$F_j \setminus K_i \subset U_i \cup W_i \quad \text{for each } i \in N_j \tag{3}$$

and

$$F \cap U_i = F \cap U_i'' \quad \text{and} \quad F \cap W_i = F \cap W_i'' \quad \text{for } i = 1, 2, \ldots \tag{4}$$

From the last two equalities and the two equalities in (1) it follows that $M \cap U_i = M \cap F \cap U_i = M \cap U_i'' = U_i'$ and $M \cap W_i = M \cap F \cap W_i = M \cap W_i'' = W_i'$. Since $\bigcap\{K_i : i \in N_j\} = \emptyset$, from (3) we obtain the inclusion $F_j \subset \bigcup\{U_i \cup W_i : i \in N_j\}$, whereas (2) and (4) yield the inclusion $F \subset \bigcup_{i=1}^{\infty}(U_i \cup W_i)$; thus $X = \bigcup_{i=1}^{\infty}(U_i \cup W_i)$. ∎

6.1.15. THE LOCALLY FINITE SUM THEOREM FOR WEAKLY INFINITE-DIMENSIONAL SPACES. *If a space X which is either hereditarily normal or normal and countably paracompact can be represented as the union of a locally finite family $\{F_s\}_{s \in S}$ of weakly infinite-dimensional closed subspaces, then X is a weakly infinite-dimensional space.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairs of disjoint closed subsets of a hereditarily normal (normal and countably paracompact) space X.

We shall consider the family $\mathcal{P}$ consisting of pairs $(T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty})$, where $T \subset S$ and $U_{T,1}, U_{T,2}, \ldots, W_{T,1}, W_{T,2}, \ldots$ are open sets (open F_σ-sets) in $X_T = \bigcup_{s \in T} F_s$ satisfying the following conditions:

$$X_T \cap A_i \subset U_{T,i}, \quad X_T \cap B_i \subset W_{T,i}. \tag{5}$$

$$U_{T,i} \cap W_{T,i} = \emptyset \quad \text{for } i = 1, 2, \ldots \tag{6}$$

and

$$\bigcup_{i=1}^{\infty}(U_{T,i} \cup W_{T,i}) = X_T. \tag{7}$$

By declaring that $(T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty}) \leq (T', \{(U_{T',i}, W_{T',i})\}_{i=1}^{\infty})$ if

$$T \subset T', \quad \text{and} \quad X_T \cap U_{T',i} = U_{T,i} \quad \text{and} \quad X_T \cap W_{T',i} = W_{T,i} \quad \text{for } i = 1, 2, \ldots,$$

we introduce an order in the family $\mathcal{P}$.

For each linearly ordered subfamily $\mathcal{P}_0$ of $\mathcal{P}$ let the pair $(T_0, \{(U_{T_0,i}, W_{T_0,i})\}_{i=1}^{\infty})$ be defined by setting

$$T_0 = \bigcup\{T : (T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty}) \in \mathcal{P}_0\}$$

and, for $i = 1, 2, \ldots$,

$$U_{T_0,i} = \bigcup\{U_{T,i} : (T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty}) \in \mathcal{P}_0\}$$

and

$$W_{T_0,i} = \bigcup\{W_{T,i} : (T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty}) \in \mathcal{P}_0\}.$$

It is not hard to check that $(T_0, \{(U_{T_0,i}, W_{T_0,i})\}_{i=1}^{\infty}) \in \mathcal{P}$ and that $(T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty}) \leq (T_0, \{(U_{T_0,i}, W_{T_0,i})\}_{i=1}^{\infty})$ for each $(T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty}) \in \mathcal{P}_0$. From the Kuratowski–Zorn lemma (see [GT], Section I.4) it follows that there exists a maximal element $(T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty})$ in $\mathcal{P}$; to complete the proof it suffices to show that $X_T = X$,

because conditions (5)–(7) imply that for $i = 1, 2, \ldots$ the set $L_i = X_T \setminus (U_{T,i} \cup W_{T,i})$ is then a partition in X between A_i and B_i, and that $\bigcap_{i=1}^{\infty} L_i = \emptyset$.

Assume that $X_T \neq X$. Consider a point $x \in X \setminus X_T$ and an $s \in S$ such that $x \in F_s$; clearly, $s \notin T$. Letting $T' = T \cup \{s\}$ and applying Lemma 6.1.14 to $M = X_T$ and $X = X_T \cup F_s$ we obtain a contradiction with the maximality of $(T, \{(U_{T,i}, W_{T,i})\}_{i=1}^{\infty})$. ■

It should be noted that if the space X in the last theorem is metrizable, the assumption that F_s's are closed can be dropped (see Problem 6.1.B(d)).

Let us state, for further use, a simple consequence of Lemma 6.1.14 (cf. Problem 6.1.C(a)).

6.1.16. PROPOSITION. *If a countably paracompact normal space X contains a closed weakly infinite-dimensional subspace M such that every closed subspace of X contained in $X \setminus M$ is weakly infinite-dimensional, then X is weakly infinite-dimensional.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairs of disjoint closed subsets of X. In the subspace M there exist partitions L_i' between $M \cap A_i$ and $M \cap B_i$, where $i = 1, 2, \ldots$, such that $\bigcap_{i=1}^{\infty} L_i' = \emptyset$; by Lemma 6.1.4 one can assume that L_i' is a closed G_δ-set in M. Consider disjoint open subsets V_i', W_i' of M such that $M \cap A_i \subset U_i'$, $M \cap B_i \subset W_i'$ and $M \setminus L_i' = U_i' \cup W_i'$. Since $U_i' = M \setminus (L_i' \cup W_i')$ and $W_i' = M \setminus (L_i' \cup U_i')$, they are F_σ-sets. To conclude the proof it suffices to apply Lemma 6.1.14. ■

In the counterpart of Proposition 6.1.16 for hereditarily normal spaces the assumption that M is closed can be dropped, i.e., the following proposition holds (cf. Problem 6.1.C(b)):

6.1.17. PROPOSITION. *If a hereditarily normal space X contains a weakly infinite-dimensional subspace M such that every closed subspace of X contained in $X \setminus M$ is weakly infinite-dimensional, then X is weakly infinite-dimensional.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairs of disjoint closed subsets of X. For $i = 0, 1, \ldots$ choose open sets $V_{i,1}, V_{i,2} \subset X$ such that $A_{2i+1} \subset V_{i,1}$, $B_{2i+1} \subset V_{i,2}$ and $\overline{V}_{i,1} \cap \overline{V}_{i,2} = \emptyset$. In the subspace M there exist partitions L_{2i+1}' between $M \cap \overline{V}_{i,1}$ and $M \cap \overline{V}_{i,2}$ such that $\bigcap_{i=0}^{\infty} L_{2i+1}' = \emptyset$. Applying Lemma 1.2.9 and Remark 1.2.10 we obtain closed subsets $L_1, L_3, \ldots$ of X such that L_{2i+1} is a partition between A_{2i+1} and B_{2i+1} and $M \cap L_{2i+1} \subset L_{2i+1}'$ for $i = 0, 1, \ldots$ The last relation implies that the intersection $L = \bigcap_{i=0}^{\infty} L_{2i+1}$ is contained in $X \setminus M$, and thus is weakly infinite-dimensional. In the subspace L there exist partitions L_{2i}' between $L \cap A_{2i}$ and $L \cap B_{2i}$ such that $\bigcap_{i=1}^{\infty} L_{2i}' = \emptyset$. Applying again Lemma 1.2.9 and Remark 1.2.10 we obtain closed subsets $L_2, L_4, \ldots$ of X such that L_{2i} is a partition between A_{2i} and B_{2i} and $L \cap L_{2i} \subset L_{2i}'$ for $i = 1, 2, \ldots$ Clearly, $L \cap \bigcap_{i=1}^{\infty} L_{2i} = \emptyset$, and thus $\bigcap_{i=1}^{\infty} L_i = \emptyset$; hence X is weakly infinite-dimensional. ■

Now we pass to the compactification theorem for weakly infinite-dimensional spaces. The theorem is a direct consequence of Lemma 5.3.1 and Theorem 6.1.6; we shall deduce it, however, from the following more general theorem that will also be used in Example 6.1.21.

6.1.18. THEOREM. *For every completely metrizable separable space X whose compact subspaces are all weakly infinite-dimensional there exist a weakly infinite-dimensional compact metrizable space Y and a homeomorphic embedding $f: X \to Y$.*

PROOF. It suffices to show that the space Y in Lemma 5.3.1 is weakly infinite-dimensional. This follows, however, from Proposition 6.1.17 applied to the space Y and its subspace $M = Y \setminus f(X)$. ■

6.1.19. THE COMPACTIFICATION THEOREM FOR WEAKLY INFINITE-DIMENSIONAL SPACES. *For every weakly infinite-dimensional completely metrizable separable space X there exists a weakly infinite-dimensional compactification, i.e., a weakly infinite-dimensional compact metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X.* ■

The assumption of complete metrizability is essential in the last theorem; indeed, it follows from Example 5.3.6 that each metrizable compactification of the weakly infinite-dimensional space K_ω (in fact, each completely metrizable space containing K_ω) contains a copy of the Hilbert cube, and thus it is strongly infinite-dimensional. It turns out that K_ω has no weakly infinite-dimensional compactification (see Problem 6.1.D(a)); in particular, βK_ω is not weakly infinite-dimensional.

As shown by the following example, even for a completely metrizable separable weakly infinite-dimensional space the Čech–Stone compactification is not necessarily weakly infinite-dimensional.

6.1.20. EXAMPLE. We shall show that the Čech–Stone compactification of the weakly infinite-dimensional space $S = \bigoplus_{n=1}^{\infty} I^n$ is strongly infinite-dimensional.

For $n = 1, 2, \ldots$ let $(A_{n,1}, B_{n,1}), (A_{n,2}, B_{n,2}), \ldots, (A_{n,n}, B_{n,n})$ be the sequence of pairs of opposite faces of I^n. The sets

$$A_i = \bigcup_{n=1}^{\infty} A_{n,i} \quad \text{and} \quad B_i = \bigcup_{n=1}^{\infty} B_{n,i},$$

where $i = 1, 2, \ldots$, are closed in S and disjoint, so that their closures $\overline{A}_i$ and $\overline{B}_i$ in βS are disjoint as well.

Assume that βS is weakly infinite-dimensional. Thus there exist closed sets L_1, $L_2, \ldots$ in βS such that L_i is a partition between $\overline{A}_i$ and $\overline{B}_i$ and $\bigcap_{i=1}^{\infty} L_i = \emptyset$. From the compactness of βS it follows that for some integer n we have $L_1 \cap L_2 \cap \ldots \cap L_n = \emptyset$. Since $A_{n,i} \subset A_i$ and $B_{n,i} \subset B_i$ if $i \leq n$, the intersection $I^n \cap L_i$ is a partition in I^n between $A_{n,i}$ and $B_{n,i}$ for $i = 1, 2, \ldots, n$; the intersection of these partitions being empty, we obtain a contradiction with Theorem 1.8.1. Thus βS is strongly infinite-dimensional. ■

To close our presentation of basic properties of weakly infinite-dimensional spaces let us note that there is no universal space for the class of all weakly infinite-dimensional compact metrizable space (see Problem 6.1.I(e)), and that under the assumption of the continuum hypothesis one can prove that there is no universal space for the class of all weakly infinite-dimensional separable metrizable spaces.

We conclude this section with an example of a compact metrizable weakly infinite-dimensional space Y which is not countable-dimensional. In this example we shall use

the fact that there exist separable completely metrizable spaces which are totally disconnected but not weakly infinite-dimensional. A construction of such a space, by no means easy, will be given in the next section.

6.1.21. POL'S EXAMPLE. Let X be a separable completely metrizable totally disconnected strongly infinite-dimensional space. Since total disconnectedness is a hereditary property, and the covering dimension of any compact totally disconnected space is equal to zero (see Theorem 1.4.5), each compact subspace of X is weakly infinite-dimensional. Thus by Theorem 6.1.18 there exists a weakly infinite-dimensional compact metric space Y that contains X. From Theorems 5.2.3 and 6.1.10 it follows that the space Y is not countable-dimensional.

Since the subspace X of Y is strongly infinite-dimensional, the same example shows that weak infinite dimensionality is not a hereditary property. ■

Historical and bibliographic notes

Weak infinite dimensionality was first considered by Alexandroff in [1948]. Proposition 6.1.3, Lemmas 6.1.4 and 6.1.5 (the latter implicitly) and Theorems 6.1.6, 6.1.7, 6.1.9 and 6.1.10 were proved by Levšenko in [1959]. Theorem 6.1.15 for countably paracompact normal spaces was established by Hadziivanov in [1971]; its version for hereditarily normal spaces was obtained by Polkowski in [1983] (under the weaker assumption that the family of closed sets is hereditarily closure preserving). Propositions 6.1.16 and 6.1.17 are due to Leĭbo [1971]. Theorem 6.1.18 is implicit in R. Pol's paper [1981a], Theorem 6.1.19 – in Lelek's paper [1965], and Example 6.1.20 – in Levšenko's paper [1959]. R. Pol proved in [1982a] that under the assumption of the continuum hypothesis there is no universal space for weakly infinite-dimensional separable metrizable spaces; his Example 6.1.21 was presented in [1981a].

Problems

6.1.A (Skljarenko [1959]). Prove that in each compact metrizable space X for every uncountable family $\mathcal{P}$ of pairs of disjoint closed sets there exist two pairs $(A_1, B_1), (A_2, B_2) \in \mathcal{P}$ and two disjoint closed sets $L_1, L_2 \subset X$ such that L_i is a partition between A_i and B_i for $i = 1, 2$.

Hint. Define a sequence $(U_1, W_1), (U_2, W_2), \ldots$ of pairs of open subsets of X such that $\overline{U}_i \cap \overline{W}_i = \emptyset$ for $i = 1, 2, \ldots$ and for each pair (A, B) of disjoint closed subsets of X the inclusions $A \subset U_i$ and $B \subset W_i$ hold for some integer i.

6.1.B. (a) Note that every open subset of a weakly infinite-dimensional perfectly normal space is weakly infinite-dimensional.

Remark. There exist weakly infinite-dimensional hereditarily normal spaces with open strongly infinite-dimensional subspaces (see Gruenhage and E. Pol [1983]).

(b) Note that if every subspace M of a normal space X which is a G_δ-set in X is weakly infinite-dimensional, then all subspaces of X are weakly infinite-dimensional.

(c) Note that in the realm of metrizable spaces Theorem 6.1.10 follows readily from Theorem 4.1.13.

(d) Prove that if a metrizable space X can be represented as the union of a locally finite family $\{A_s\}_{s\in S}$ of weakly infinite-dimensional subspaces, then X is a weakly infinite-dimensional space.

Hint. See the hint to Problem 5.2.E(a).

6.1.C. (a) (Čoban [1983]) Show that if a countably paracompact normal space X can be represented as the union of a sequence $K_1, K_2, \ldots$ of subspaces such that every closed subspace of X contained in some K_i is weakly infinite-dimensional and the union $\bigcup_{j\le i} K_j$ is closed for $i = 1, 2, \ldots$, then X is a weakly infinite-dimensional space.

(b) Show that if a hereditarily normal space X can be represented as the union $K_1, K_2, \ldots$ of subspaces such that every closed subspace of $\bigcup_{j\le i} K_j$ contained in K_i is weakly infinite-dimensional for $i = 1, 2, \ldots$, then X is weakly infinite-dimensional.

(c) Show that if a countably paracompact normal space X can be represented as the union of a family $\{F_s\}_{s\in S}$ of closed weakly infinite-dimensional subspaces, and if there exists a point-finite open cover $\{U_s\}_{s\in S}$ of the space X such that $F_s \subset U_s$ for $s \in S$, then X is a weakly infinite-dimensional space.

Hint. See the proof of Theorem 3.1.13.

(d) (Čoban [1983], Polkowski [1983a]) Show that if a weakly paracompact normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of open subspaces such that $\overline{U}_s$ is weakly infinite-dimensional for $s \in S$, then X is a weakly infinite-dimensional space.

(e) (Leĭbo [1971]) Show that if a weakly paracompact normal space X can be represented as the union of a locally countable family $\{F_s\}_{s\in S}$ of weakly infinite-dimensional closed subspaces, then X is a weakly infinite-dimensional space.

(f) (Yamada [1988]) Show that if a countably paracompact normal space X can be represented as the union of an increasing sequence of weakly infinite-dimensional subspaces $X_1 \subset X_2 \subset \ldots$, then X is a weakly infinite-dimensional space.

Hint. Apply Problem 2.2.E(b) to show that each normal space that can be represented as a countable union of dense weakly infinite-dimensional subspaces is weakly infinite-dimensional. Deduce that $\overline{X}_i$ is weakly infinite-dimensional for $i = 1, 2, \ldots$

6.1.D. (a) (Skljarenko [1959a]) Prove that the strongly countable-dimensional space $K_\omega \subset I^{\aleph_0}$ defined in Example 1.8.22(b) has no weakly infinite-dimensional compactification.

Hint. Assume that K_ω is a subspace of a weakly infinite-dimensional compact space X and define inductively an increasing sequence $n_1, n_2, \ldots$ of natural numbers and a decreasing sequence of real numbers in the interval $(0, 1]$ such that for $i = 1, 2, \ldots$ the sets $A_i = K_\omega \cap \{\{x_j\} \in I^{\aleph_0} : x_j \le a_j$ for $j = 1, 2, \ldots, i-1$, $x_i = 0$, and $x_j \le a_{i+1}$ for $j = i+1, i+2, \ldots, n_i\}$ and $B_i = K_\omega \cap \{\{x_j\} \in I^{\aleph_0} : x_j \le a_j$ for $j = 1, 2, \ldots, i-1$, $x_i = a_i$, and $x_j \le a_{i+1}$ for $j = i+1, i+2, \ldots, n_i\}$ have disjoint closures in X.

(b) Prove that for every weakly infinite-dimensional completely metrizable space X of weight $\le \mathbf{m} \ge \aleph_0$ there exists a weakly infinite-dimensional compactification of weight $\le \mathbf{m}$.

Hint. See Problem 5.3.B.

6.1.E. (a) (Levšenko [1959], Skljarenko, cited in Levšenko [1959]) Prove that the Čech–Stone compactification of a normal space X is weakly infinite-dimensional if and only if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and some finite collection of L_i's has an empty intersection.

Hint. First deduce from Lemma 3.3.8 that for any closed subsets $F_1, F_2, \ldots, F_m$ of a normal space X, in βX we have the equality $\overline{F_1 \cap F_2 \cap \ldots \cap F_m} = \overline{F}_1 \cap \overline{F}_2 \cap \ldots \cap \overline{F}_m$, and then apply Lemma 2.2.9.

Remark. A normal space X satisfying the partition condition in (a) is called *S-weakly infinite-dimensional* or *Smirnov weakly infinite-dimensional*; this notion, due to Ju. M. Smirnov, was first studied in Levšenko [1959] and Skljarenko [1959a]. Let us note here that, as proved in Skljarenko [1959a], a normal space X satisfies the inequality $\dim X \leq n$ if and only if for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $L_{i_1} \cap L_{i_2} \cap \ldots \cap L_{i_{n+1}} = \emptyset$ for some integers $i_1, i_2, \ldots, i_{n+1}$.

(b) (Skljarenko [1959a]) Show that if a normal space X contains a closed S-weakly infinite-dimensional subspace K such that each closed subspace $F \subset X$ disjoint from K is S-weakly infinite-dimensional, then X is weakly infinite-dimensional.

(c) (Skljarenko [1959a]) Show that a weakly paracompact normal space X is S-weakly infinite-dimensional if and only if X contains a weakly infinite-dimensional compact subspace K such that each closed subspace $F \subset X$ disjoint from K is finite-dimensional.

Hint. Apply Theorem 5.5.21.

(d) Show that a weakly paracompact normal space X is S-weakly infinite-dimensional if and only if X is weakly infinite-dimensional and contains no closed almost S-like subspace.

(e) (Skljarenko [1959a]) Show that if the Čech–Stone compactification βX of a metrizable space X is weakly infinite-dimensional, then X has a weakly infinite-dimensional compactification of weight $w(X)$.

Hint. Apply part (c) and Problem 3.4.A(c).

Remark. As proved in Pasynkov [1971], (e) holds for normal spaces.

6.1.F (Hadziivanov [1975]). Prove that if M is a weakly infinite-dimensional subspace of the Hilbert cube $I^{\aleph_0}$, then M does not cut $I^{\aleph_0}$, i.e., for every pair of points $x, y \in I^{\aleph_0} \setminus M$ there exists a continuum $C \subset I^{\aleph_0} \setminus M$ which contains x and y.

Hint. First note, modifying the proof of the universality of $I^{\aleph_0}$ ([GT], Theorem 2.3.23), that there exists a continuous mapping $f: I^{\aleph_0} \to I^{\aleph_0}$ which maps a pair (A, B) of opposite faces of $I^{\aleph_0}$ to x and y, respectively, and has the property that the restriction $f|[I^{\aleph_0} \setminus (A \cup B)]$ is a homeomorphic embedding. Then apply Remark 1.8.17 (cf. the proof of Theorem 1.8.18).

6.1.G. (a) (Rubin [1980]) Give an example of a strongly infinite-dimensional compact metric space X whose non-empty weakly infinite-dimensional subspaces are all zero-dimensional.

Hint (R. Pol [1986]). Observe that the construction in Example 5.2.23 yields such a space; check that each space $Y \subset \bigcap_{i\in N_0} L_i$ satisfying $K \subset p_{i_0}(Y)$ is strongly infinite-dimensional.

(b) (Rubin [1980a]) Prove that every strongly infinite-dimensional separable metrizable space Y contains a closed strongly infinite-dimensional subspace Z whose non-empty weakly infinite-dimensional subspaces are all zero-dimensional.

Hint. First, modify the construction of the space X in Example 5.2.23 by decomposing the set $\{3,5,7,\ldots\}$ into the sets $N_{j,k}$ and letting $X = \bigcap\{L_i : i = 3,5,7,\ldots\}$. Then define a homeomorphic embedding $h\colon Y \to I^{\aleph_0}$ such that every partition between A_1 and B_1 in $I^{\aleph_0}$ meets the set $X \cap h(Y)$, and finally let $Z = h^{-1}(X)$.

Remark. Henderson proved in [1967a] that every strongly infinite-dimensional compact metrizable space contains an infinite-dimensional closed subspace each non-empty finite-dimensional closed subspace of which is zero-dimensional. In [1974] Zarelua strengthened this result and proved that every strongly infinite-dimensional compact metrizable space contains a strongly infinite-dimensional closed subspace whose non-empty weakly infinite-dimensional closed subspaces are all zero-dimensional. Walsh in [1979] proved that every strongly infinite-dimensional compact metrizable space contains an infinite-dimensional closed subspace whose non-empty finite-dimensional subspaces are all zero-dimensional.

(c) (R. Pol [1986]) Check that every infinite-dimensional closed subspace Z of the uncountable-dimensional space Y defined in Example 6.1.21 contains a closed subspace M_n with $\dim M_n = n$ for $n = 1,2,\ldots$.

6.1.H. (a) (R. Pol [1982a] (announcement [1981a])) Use the space Y and its subspace X considered in Example 6.1.21 to define weakly infinite-dimensional separable metric spaces X_1 and X_2 such that the Cartesian product $X_1 \times X_2$ is strongly infinite-dimensional.

Hint. Apply Problem 1.5.I(a) to show that the subspace $Y \setminus X$ contains a Bernstein set Z_1 (i.e., a set Z_1 such that $|Z_1| = \mathbf{c}$ and every compact set contained either in Z_1 or in $Z_2 = (Y \setminus X) \setminus Z_1$ is countable), and let $X_i = X \cup Z_i$ for $i = 1,2$. In proving that X_i is weakly infinite-dimensional, make use of Theorem 1.2.14.

(b) (Lifanov [1968a]) Show that if X_i is a compact space with $\dim X_i > 0$ for $i = 1,2,\ldots$, then the Cartesian product $\prod_{i=1}^{\infty} X_i$ is strongly infinite-dimensional.

Hint. See Problem 1.8.K(a).

Remark. E. Pol defined in [1986] separable metric spaces X and Y such that X is weakly infinite-dimensional, $\dim Y = 0$, and the Cartesian product $X \times Y$ is strongly infinite-dimensional (under the assumption of the continuum hypothesis one can take for Y the space of irrational numbers).

6.1.I. (a) For every ordinal number α the αth *derived set* $A^{(\alpha)}$ of a subset A of a topological space X is defined inductively by the formulas

$$A^{(0)} = A, \quad A^{(\alpha+1)} = (A^{(\alpha)})^{\mathrm{d}} \quad \text{and} \quad A^{(\lambda)} = \bigcap_{\alpha<\lambda} A^{(\alpha)} \quad \text{if } \lambda \text{ is a limit number,}$$

where A^{d} is the set of all accumulation points of A.

Show that for every countable compact space X there exists a smallest countable ordinal number α such that the αth derived set $X^{(\alpha)}$ is finite; the ordinal number α is called the *Cantor–Bendixson index* of X.

Prove that for every countable ordinal number α there exists a countable compact subspace X of the space Q of rational numbers such that the Cantor–Bendixson index of X equals α.

(b) (implicitly, Hurewicz [1930]) Prove that for every continuous mapping $p\colon Z \to Q$ of a separable completely metrizable space Z to the space Q of rational numbers there exists a compact subspace Y of Q which is not the image of any compact subspace T of Z.

Hint. Apply the Baire category theorem to define a decreasing transfinite sequence $Z = F_0 \supset F_1 \supset \ldots \supset F_\alpha \supset \ldots$, $\alpha < \xi$, of closed subsets of Z such that $|p(F_\alpha \setminus F_{\alpha+1})| = 1$ for each $\alpha < \xi$, $F_\lambda = \bigcap_{\alpha<\lambda} F_\alpha$ if λ is a limit number, and $\bigcap_{\alpha<\xi} F_\alpha = \emptyset$. Observe that $\xi < \omega_1$, show that $(p(T))^{(\alpha)} \subset p(T \cap F_\alpha)$ for each compact subspace T of Z and each $\alpha < \xi$, and deduce that the Cantor–Bendixson indices of all images of compact subspaces of Z are less than ξ.

(c) (R. Pol [1983]) Define a continuous mapping $g\colon K \to C$ of a compact metrizable space K onto the Cantor set C such that $g^{-1}(t)$ is homeomorphic to a finite-dimensional cube if $t \in Q$, and to the Hilbert cube if $t \in C \setminus Q$, where Q is the set of the end-points of all intervals removed from I to obtain the Cantor set.

Hint. Map $K = (\{0\} \cup [1/2, 1])^{\aleph_0}$ onto $D^{\aleph_0}$ by letting $g(\{x_i\}) = \{t_i\}$, where $t_i = \min(2x_i, 1)$ for $i = 1, 2, \ldots$

(d) (R. Pol [1983]) Prove that every separable completely metrizable space X which topologically contains all compact metrizable spaces with countably many components each of which is homeomorphic to a finite-dimensional cube contains topologically the Hilbert cube $I^{\aleph_0}$.

Hint. Assume that $X \subset I^{\aleph_0}$, take the mapping $g\colon K \to C$ defined in part (c), consider the spaces $(I^{\aleph_0})^K$ and 2^C (see Sections 1.11 and 6.2), note that the Cartesian product $C \times 2^C \times (I^{\aleph_0})^K$ is completely metrizable and show that $Z = \{(t, Y, f) \in C \times 2^C \times (I^{\aleph_0})^K : t \in Y$ and $f|g^{-1}(Y)$ is a homeomorphic embedding of $g^{-1}(Y)$ in $X\}$ is a G_δ-set in it. Observe that for each compact subspace Y of the set $Q \subset C$ defined in (c) we have $Y = p(Y \times \{Y\} \times \{f\})$, where p is the projection of Z to C and $f\colon K \to I^{\aleph_0}$ embeds homeomorphically $g^{-1}(Y)$ in X; apply (b) and Problem 1.3.H(e) to deduce that $p(Z) \cap (C \setminus Q) \neq \emptyset$.

(e) (R. Pol [1983]) Deduce from (d) that there is no universal space for the class of all weakly infinite-dimensional compact metrizable spaces.

6.2. An example of a totally disconnected strongly infinite-dimensional space

This section is devoted to the construction of a separable completely metrizable totally disconnected space which is strongly infinite-dimensional. We start with recalling the definition of the Hausdorff metric ρ_H on the family of all bounded non-empty closed

subsets of a metric space (X, ρ); for two such sets $A, B \subset X$ we define

$$\rho_H(A, B) = \max\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\}.$$

Obviously, $\rho_H(A, B) = 0$ if and only if $A = B$, and $\rho_H(A, B) = \rho_H(B, A)$; hence to show that ρ_H is a metric it suffices to check that $\rho_H(A, C) \leq \rho_H(A, B) + \rho_H(B, C)$.

For all $a \in A$ and $b \in B$ we have

$$\rho(a, C) \leq \rho(a, b) + \rho(b, C) \leq \rho(a, b) + \rho_H(B, C),$$

so that for each $a \in A$,

$$\rho(a, C) \leq \rho(a, B) + \rho_H(B, C) \leq \rho_H(A, B) + \rho_H(B, C).$$

Similarly, for each $c \in C$,

$$\rho(c, A) \leq \rho_H(A, B) + \rho_H(B, C)$$

and thus $\rho_H(A, C) \leq \rho_H(A, B) + \rho_H(B, C)$.

If (X, ρ) is a compact metric space the Hausdorff metric ρ_H is defined on the family 2^X of all non-empty closed subsets of X. One easily checks that in this case for all $A, A' \in 2^X$ and any $\epsilon > 0$ we have $\rho_H(A, A') < \epsilon$ if and only if $A \subset B(A', \epsilon)$ and $A' \subset B(A, \epsilon)$, where $B(A', \epsilon)$ and $B(A, \epsilon)$ are open ϵ-balls about A' and A in the space (X, ρ).

In our construction we shall use a corollary to the following theorem.

6.2.1. THEOREM. *For every compact metric space (X, ρ) the space $(2^X, \rho_H)$ is compact.*

PROOF. It suffices to prove that the space $(2^X, \rho_H)$ is totally bounded and complete (see [GT], Theorem 4.3.29).

We shall show first that $(2^X, \rho_H)$ is totally bounded. To that end, consider an $\epsilon > 0$, take a finite cover $\{X_i\}_{i=1}^k$ of the space X by non-empty sets of diameter less than ϵ, and for each $i \leq k$ pick a point $x_i \in X_i$. One easily checks that for every set $A \in 2^X$ and the set $X_A = \{x_i : A \cap X_i \neq \emptyset\}$ we have $\rho_H(A, X_A) < \epsilon$. Since there are only finitely many distinct X_A's, the space 2^X can be covered by finitely many ϵ-balls, and thus $(2^X, \rho_H)$ is totally bounded.

To prove that $(2^X, \rho_H)$ is complete, we shall show that every Cauchy sequence $A_1, A_2, \ldots$ in 2^X converges to $A = \bigcap_{m=1}^{\infty} \overline{\bigcup_{n \geq m} A_n} \in 2^X$. Consider an $\epsilon > 0$. By the compactness of X, there exists an n_1 such that $\overline{\bigcup_{n \geq n_1} A_n} \subset B(A, \epsilon)$, and thus

$$A_n \subset B(A, \epsilon) \quad \text{if } n \geq n_1. \tag{1}$$

On the other hand, since $A_1, A_2, \ldots$ is a Cauchy sequence, there exists an n_2 such that $\rho_H(A_n, A_m) < \epsilon/2$ if $n, m \geq n_2$; thus we have $\bigcup_{m \geq n_2} A_m \subset B(A_n, \epsilon/2)$ if $n \geq n_2$. The last inclusion implies that

$$A \subset \overline{\bigcup_{m \geq n_2} A_m} \subset \overline{B(A_n, \epsilon/2)} \subset B(A_n, \epsilon) \quad \text{if } n \geq n_2, \tag{2}$$

and from (1) and (2) it follows that $\rho_H(A, A_n) < \epsilon$ if $n \geq \max(n_1, n_2)$; thus $A = \dim A_n$. ■

6.2.2. COROLLARY. *For every compact metric space X and any closed sets $A, B \subset X$ the family of all continua $C \in 2^X$ such that $A \cap C \neq \emptyset \neq B \cap C$ is a compact subspace of 2^X.*

PROOF. One easily checks that the family of all continua contained in X is closed in 2^X. Similarly, for every closed subset of X the family of all elements of 2^X that meet this subset is closed in 2^X. ■

The following selection theorem will also be used in our construction.

6.2.3. THEOREM. *For every continuous mapping $p: Y \to Z$ of a compact subspace Y of the Hilbert cube $I^{\aleph_0}$ onto a metrizable space Z there exists a G_δ-set $X \subset Y$ which intersects each fibre $p^{-1}(z)$ of p in exactly one point.*

PROOF. Let us recall the definition of the lexicographic order $\prec$ on $I^{\aleph_0}$: for two distinct points $x, y \in I^{\aleph_0}$ we let $x \prec y$ if the first non-zero term of the sequence $p_1(y) - p_1(x), p_2(y) - p_2(x), \ldots$, where p_i denotes the projection of $I^{\aleph_0} = \prod_{i=1}^{\infty} I_i$ onto $I_i = I$, is positive. One easily checks that each compact subspace A of $I^{\aleph_0}$ has a smallest element with respect to $\prec$; we shall denote it by $s(A)$.

For each $x \in Y$ the fibre $p^{-1}p(x)$ is compact, and thus the point $s(x) = s(p^{-1}p(x)) \in p^{-1}p(x)$ is defined. Obviously, the set $X = \{s(x) : x \in Y\}$ intersects each fibre of p in exactly one point; we shall prove that X is a G_δ-set in Y. For $j, k = 1, 2, \ldots$ let

$$A_{j,k} = \{x \in Y : \text{there exists an } x' \in p^{-1}p(x) \text{ such that } p_i(x) = p_i(x') \text{ for } i < j \text{ and } p_j(x) - p_j(x') \geq 1/k\}.$$

Clearly $\bar{Y} \setminus X \subset \bigcup_{j,k=1}^{\infty} A_{j,k}$, and to conclude the proof it suffices to show that $\bar{A}_{j,k} \cap X = \emptyset$ for $j, k = 1, 2, \ldots$

Consider a point $x \in \bar{A}_{j,k}$ and a sequence $x_1, x_2, \ldots$ of elements of $A_{j,k}$ with $\lim x_n = x$. For $n = 1, 2, \ldots$ choose a point $x'_n \in p^{-1}p(x_n)$ such that

$$p_i(x_n) = p_i(x'_n) \quad \text{for } i < j \quad \text{and} \quad p_j(x_n) - p_j(x'_n) \geq 1/k. \tag{3}$$

By the compactness of Y the sequence $x'_1, x'_2, \ldots$ contains a subsequence converging to a point $x' \in Y$; without loss of generality one can assume that $x' = \lim x'_n$. Since p is continuous and $p(x'_n) = p(x_n)$, we have $p(x') = p(x)$; thus $x' \in p^{-1}p(x)$. The projections being continuous, (3) implies that

$$p_i(x) = p_i(x') \quad \text{for } i < j \quad \text{and} \quad p_j(x) - p_j(x') \geq 1/k,$$

so that $x' \prec x$. Hence $s(x) \prec x$, and $x \notin X$. ■

Let us note that the last theorem holds for mappings defined on any compact metrizable space Y, because each such space can be embedded in $I^{\aleph_0}$.

6.2.4. EXAMPLE. We shall now describe a completely metrizable, totally disconnected, and yet strongly infinite-dimensional subspace X of the Hilbert cube $I^{\aleph_0} = \prod_{i=1}^{\infty} I_i$.

Let $A = \{x \in I^{\aleph_0} : p_1(x) = 0\}$ and $B = \{x \in I^{\aleph_0} : p_1(x) = 1\}$, where $p_1: I^{\aleph_0} \to I_1 = I$ denotes the projection. By Corollary 6.2.2 the subspace S of $2^{I^{\aleph_0}}$ consisting of all continua intersecting A and B is compact, so that there exists a continuous mapping f of the Cantor set $C \subset I_1$ onto S (see Problem 1.3.D(b)). We shall show that the set

$$Y = \bigcup \{p_1^{-1}(z) \cap f(z) : z \in C\}$$

is closed in $I^{\aleph_0}$. Consider a sequence $y_1, y_2, \ldots$ of points in Y converging to $y \in I^{\aleph_0}$ and let $y_i \in p_1^{-1}(z_i) \cap f(z_i)$ with $z_i \in C$. Obviously, there exists a subsequence $y_{n_1}, y_{n_2}, \ldots$ such that $\lim z_{n_i} = z \in C$. We have $p_1(y) = \lim p_1(y_{n_i}) = \lim z_{n_i} = z$, i.e., $y \in p_1^{-1}(z)$. We also have $y \in f(z)$, since otherwise we could find an $\epsilon > 0$ such that $y \notin B(f(z), \epsilon)$, which is impossible because $f(z) = \lim f(z_{n_i})$ and $y_{n_i} \in f(z_{n_i})$. Thus $y \in Y$, which shows that Y is closed in $I^{\aleph_0}$.

By Theorem 6.2.3 applied to $p = p_1|Y: Y \to C$ there exists a G_δ-set $X \subset Y$ such that $|X \cap p^{-1}(z)| = 1$ for each $z \in C$. The subspace X of $I^{\aleph_0}$ has all the required properties. Indeed, X is completely metrizable by Lemma 1.3.12. Then, since $p|X: X \to C$ is a one-to-one continuous mapping, X is totally disconnected. Finally, X meets every continuum in $I^{\aleph_0}$ which intersects A and B, because every such continuum is of the form $f(z)$ with $z \in C$, and $\emptyset \neq X \cap p^{-1}(z) \subset f(z)$; hence X is strongly infinite-dimensional by Remark 1.8.17. ■

Historical and bibliographic notes

Theorem 6.2.1 is due to Ważewski [1923]. Theorem 6.2.3 was proposed as an exercise by Bourbaki in [1958], p. 144; its proofs based on the use of the lexicographic order were given independently by van Mill in [1989] and Kulesza in [1990]. Example 6.2.4 is implicit in Rubin, Schori and Walsh's paper [1979]; it was described explicitly by R. Pol in [1981a], and applied to the construction of a weakly infinite-dimensional compact metrizable space which is not countable-dimensional (see Example 6.1.21).

Problems

6.2.A (Mazurkiewicz [1927]). For $n = 1, 2, \ldots$ give an example of a separable completely metrizable totally disconnected space X such that $\dim X = n$.

Hint (Rubin, Schori and Walsh [1979], R. Pol [1981]). See Example 6.2.4.

Remark. A related construction of such spaces can be found in Lelek [1971].

6.2.B (van Mill and R. Pol [1991]). Apply the construction in Example 6.2.4 to obtain, for $n = 1, 2, \ldots$, a separable completely metrizable weakly n-dimensional space (see Problem 1.5.H(c)).

Hint. Let $A = \{x \in I^{n+1} : p(x) = 0\}$ and $B = \{x \in I^{n+1} : p(x) = 1\}$, where $p: I^{n+1} = I \times I^n \to I$ denotes the projection. Consider a continuous mapping f of the Cantor set $C \subset I$ onto the subspace of $2^{I^{n+1}}$ consisting of all continua intersecting A and B and let $Y = \bigcup\{p^{-1}(z) \cap f(z) : z \in C\}$. Show that if $p(T) = C$ for a $T \subset Y$, then $\dim T = n$, check that the dimensional kernel K of Y is disjoint from each zero-dimensional set of the form $p^{-1}(z) \cap f(z)$, find an F_σ-set $L \subset Y$ with $\dim L = 0$ and $\dim(K \setminus L) \leq n - 1$, and conclude that $X = Y \setminus L$ is weakly n-dimensional.

6.2.C (Kulesza [1990]). Apply the construction in Example 6.2.4 to obtain, for $n = 1, 2, \ldots$, a separable completely metrizable space $K(n) \subset I^{n+1}$ such that $\dim[K(n)]^m = \dim[K(n)]^{\aleph_0} = n$ for $m = 1, 2, \ldots$

Hint. Consider the closed subspace Y of $C \times I^n$ defined in the hint to Problem 6.2.B, to each $z \in C$ assign the point $s(z) \in I^n$ such that $(z, s(z))$ is the smallest

element of $p^{-1}(z) \cap f(z)$ with respect to the lexicographic order $\prec$ on R^{n+1} and show that $K(n) = \{(z, s(z)) : z \in C\} \subset C \times I^n$ is a G_δ-set in $C \times I^n$. Check that $\dim K(n) = n$ and prove that for $m = 2, 3, \ldots$ the mapping $h\colon [K(n)]^m \to C^m \times R^n$ defined by letting $h((z_1, s(z_1)), (z_2, s(z_2)), \ldots, (z_m, s(z_m))) = ((z_1, z_2, \ldots, z_m), s(z_1) + s(z_2) + \ldots + s(z_m))$, the addition being coordinatewise, is a homeomorphic embedding. To that end, observe that if $x_1, x_2, \ldots, x_m$, $y_1, y_2, \ldots, y_m \in R^n$ such that $x_i \preceq y_i$ for $i = 1, 2, \ldots, m$ satisfy the equality $x_1 + x_2 + \ldots + x_m = y_1 + y_2 + \ldots + y_m$, then $x_i = y_i$ for $i = 1, 2, \ldots, m$. To evaluate the dimension of $[K(n)]^{\aleph_0}$ apply Theorem 4.1.22.

Remark. Separable metrizable spaces with similar properties were described by Anderson and Keisler in [1967].

6.2.D. (a) Give an example of an open mapping with countable fibres which maps a strongly infinite-dimensional completely metrizable space onto the Cantor set.

Hint. Consider the projection of the space $X \subset C \times I^{\aleph_0}$ defined in Example 6.2.4 onto C and apply Problems 1.12.G(b) and 1.12.F.

(b) (R. Pol [1983]) Give an example of an open mapping with strongly countable-dimensional fibres which maps an uncountable-dimensional compact metric space onto the Cantor set.

Hint. Consider the space $X \subset C \times I^{\aleph_0}$ defined in Example 6.2.4 and, applying the argument in the proof of Lemma 5.3.1, find a continuous mapping $F\colon C \times I^{\aleph_0} \to I^{\aleph_0}$ such that $F^{-1}(I^{\aleph_0} \setminus K_\omega) \subset X$ and $F|X$ is a homeomorphic embedding. Show that the restriction of the projection $p\colon C \times I^{\aleph_0} \to C$ to the compact uncountable-dimensional space $\{(z, F(z, x)) : (z, x) \in C \times I^{\aleph_0}\}$ of $C \times I^{\aleph_0}$ has strongly countable-dimensional fibres. To obtain an open mapping with similar properties apply Problem 1.12.G(b).

6.3. Weak infinite dimensionality and mappings

In this section we shall study the behaviour of weakly infinite-dimensional spaces under mappings. First we shall prove two invariance theorems, counterparts of the theorem on dimension-raising mappings. We start with a lemma that will make up for the lack of heredity of the weak infinite dimensionality.

6.3.1. LEMMA. *If $f\colon X \to Y$ is a continuous mapping of a weakly infinite-dimensional space X onto a hereditarily normal space Y, then for each subspace A of X such that $f|A\colon A \to f(A)$ is a homeomorphism the space $C = f(A)$ has the following property:*

(δ) *for every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of Y there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and $C \cap \bigcap_{i=1}^{\infty} L_i = \emptyset$.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairs of disjoint closed subsets of Y. For $i = 1, 2, \ldots$ choose open sets $V_{i,1}, V_{i,2} \subset Y$ such that $A_i \subset V_{i,1}$, $B_i \subset V_{i,2}$ and $\overline{V}_{i,1} \cap \overline{V}_{i,2} = \emptyset$. Since X is weakly infinite-dimensional there exist closed sets $K_1, K_2, \ldots$ such that K_i is a partition in X between $f^{-1}(\overline{V}_{i,1})$ and $f^{-1}(\overline{V}_{i,2})$ and $\bigcap_{i=1}^{\infty} K_i = \emptyset$. As f maps homeomorphically A onto C, for each $i \in N$ the set $L'_i = f(A \cap K_i)$ is a partition in C between $C \cap \overline{V}_{i,1}$, and $C \cap \overline{V}_{i,2}$ and $\bigcap_{i=1}^{\infty} L'_i = \emptyset$. By

virtue of Lemma 1.2.9 and Remark 1.2.10 for each $i \in N$ there exists a partition L_i in the space Y between A_i and B_i such that $C \cap L_i \subset L_i'$. As clearly $C \cap \bigcap_{i=1}^{\infty} L_i = \emptyset$, the space C has property (δ). ■

6.3.2. THEOREM. *If $f: X \to Y$ is a closed mapping of a weakly infinite-dimensional metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point or is of cardinality less than* $\mathbf{c}$, *then Y is weakly infinite-dimensional.*

PROOF. As shown in the first part of the proof of Theorem 5.4.3, it suffices to prove our theorem under the additional assumption that f is a perfect mapping. Under this additional assumption for every $y \in Y$ the fibre $f^{-1}(y)$ has an isolated point.

Let $\mathcal{B} = \{U_s\}_{s \in S}$ be a base for the space X such that $S = \bigcup_{i=1}^{\infty} S_i$, $S_i \cap S_j = \emptyset$ whenever $i \neq j$, and the family $\{U_s\}_{s \in S_i}$ is discrete for $i = 1, 2, \ldots$ And let $\{A_s\}_{s \in S}$ be a family of subsets of X which satisfy (i) and (ii) in Lemma 5.4.2; since each fibre has an isolated point, we have $Y = \bigcup_{s \in S} f(A_s)$.

Since by Lemma 5.4.1 the family $\{f(A_s)\}_{s \in S}$ is σ-locally finite, it suffices to prove that the family $\mathcal{C}$ of subspaces $C \subset Y$ which have property (δ) in Lemma 6.3.1 is closed with respect to the unions of locally finite families as well as the unions of countable families.

The latter can be easily proved by the standard device of dividing the set of positive intgers into countably many pairwise disjoint infinite sets (cf. the proof of Theorem 6.1.6). Thus, it remains to show that if the family $\{C_s\}_{s \in S}$ is locally finite and $C_s \in \mathcal{C}$ for every $s \in S$, then $C = \bigcup_{s \in S} C_s \in \mathcal{C}$. To that end, for every point $y \in Y$ take a neighbourhood W_y of y that meets only finitely many C_s's and consider a σ-discrete open refinement $\mathcal{V}$ of the cover $\{W_y\}_{y \in Y}$ (see [GT], Theorem 4.4.1). Since $\mathcal{C}$ is closed under finite unions, $V \cap C \in \mathcal{C}$ for every $V \in \mathcal{V}$; the family $\{V \cap C : V \in \mathcal{V}\}$ being σ-discrete, it suffices to show that $\mathcal{C}$ is closed with respect to the unions of discrete families.

Consider a discrete family $\{C_s\}_{s \in S}$ of members of $\mathcal{C}$, its union $C = \bigcup_{s \in S} C_s$, and a sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of Y. For every $s \in S$ take closed sets $L_{s,1}, L_{s,2}, \ldots$ such that $L_{s,i}$ is a partition between A_i and B_i and $C_s \cap \bigcap_{i=1}^{\infty} L_{s,i} = \emptyset$. Clearly, for $i = 1, 2, \ldots$ the set $L_i' = \bigcup_{s \in S} \overline{C}_s \cap L_{s,i}$ is a partition in $\overline{C} = \bigcup_{s \in S} \overline{C}_s$ between $\overline{C} \cap A_i$ and $\overline{C} \cap B_i$, so that by Lemma 1.2.9 there exists a partition L_i between A_i and B_i which satisfies the inclusion $\overline{C} \cap L_i \subset L_i'$. From the obvious equality $C \cap \bigcap_{i=1}^{\infty} L_i = \emptyset$ it follows that $C \in \mathcal{C}$. ■

We now pass to the second invariance theorem.

6.3.3. LEMMA. *If $f: X \to Y$ is a closed mapping of a weakly infinite-dimensional countably paracompact normal space X onto a normal strongly infinite-dimensional space Y, then there exist a strongly infinite-dimensional subspace Y_0 of Y and disjoint closed subsets X_0, X_1 of X such that $f(X_0) = f(X_1) = Y_0$.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairs of disjoint closed subsets of Y with the property that for any closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i we have $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. Since X is weakly infinite-

dimensional, by Lemma 6.1.4 there exist in X closed G_δ-sets $K_1, K_2, \ldots$ such that K_i is a partition between $f^{-1}(A_{2i})$ and $f^{-1}(B_{2i})$ and $\bigcap_{i=1}^{\infty} K_i = \emptyset$. Clearly, the disjoint open sets $U_i, W_i \subset X$ satisfying

$$f^{-1}(A_{2i}) \subset U_i, \quad f^{-1}(B_{2i}) \subset W_i \quad \text{and} \quad X \setminus K_i = U_i \cup W_i$$

are F_σ-sets in X for $i = 1, 2, \ldots$ One easily sees that $L_{2i} = f(X \setminus U_i) \cap f(X \setminus W_i)$ is a partition in Y between A_{2i} and B_{2i} for $i = 1, 2, \ldots$ The space Y, being a closed image of a countably paracompact normal space, is countably paracompact (see [GT], Exercise 5.2.G(e)), and so is its closed subspace $L = \bigcap_{i=1}^{\infty} L_{2i}$. If the space L were weakly infinite-dimensional, then – by virtue of Lemmas 6.1.4 and 6.1.5 – we could find closed sets $L_1, L_3, \ldots$ such that L_{2i+1} is a partition in Y between A_{2i+1} and B_{2i+1} and $L \cap \bigcap_{i=1}^{\infty} L_{2i+1} = \emptyset$, and this contradicts the choice of the sequence $(A_1, B_1), (A_2, B_2), \ldots$ Thus the space L is strongly infinite-dimensional.

Now, since $X = \bigcup_{i=1}^{\infty}(U_i \cup W_i)$, we have

$$L = L \cap \bigcup_{i=1}^{\infty}(f(U_i) \cup f(W_i)) = \bigcup_{i=1}^{\infty}(L \cap f(U_i)) \cup \bigcup_{i=1}^{\infty}(L \cap f(W_i)). \tag{1}$$

As the mapping f is closed and both U_i and W_i are F_σ-sets, (1) and Theorem 6.1.7 yield the existence of a closed set $F \subset X$, contained in a U_i or in a W_i, such that the subspace $Y_0 = L \cap f(F)$ of Y is strongly infinite-dimensional. If $F \subset U_i$, we let

$$X_0 = f^{-1}(L) \cap F \quad \text{and} \quad X_1 = (X \setminus U_i) \cap f^{-1}(L \cap f(F)),$$

and if $F \subset W_i$, we let

$$X_0 = f^{-1}(L) \cap F \quad \text{and} \quad X_1 = (X \setminus W_i) \cap f^{-1}(L \cap f(F)). \blacksquare$$

6.3.4. THEOREM. *If $f\colon X \to Y$ is a continuous mapping of a weakly infinite-dimensional compact space X onto a normal space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$, then Y is weakly infinite-dimensional.*

PROOF. Suppose that Y is strongly infinite-dimensional; we shall obtain a contradiction by showing that $|f^{-1}(y)| \geq \mathbf{c}$ for some point $y \in Y$.

To that end, for every positive integer i and every sequence $k_1, k_2, \ldots, k_i$ of zeros and ones we shall define inductively a strongly infinite-dimensional subspace T_i of Y and closed subsets $S_{k_1 k_2 \ldots k_i}$ of X such that

(2) $S_{k_1 k_2 \ldots k_i} \cap S_{m_1 m_2 \ldots m_i} = \emptyset$ whenever $(k_1, k_2, \ldots, k_i) \neq (m_1, m_2, \ldots, m_i)$,

(3) $S_{k_1 k_2 \ldots k_{i-1} 0} \cup S_{k_1 k_2 \ldots k_{i-1} 1} \subset S_{k_1 k_2 \ldots k_{i-1}}$ if $i > 1$,

(4) $f(S_{k_1 k_2 \ldots k_i}) = T_i$.

The subspace T_1 and the sets S_0 and S_1 are defined by an application of Lemma 6.3.3; thus the induction is started.

Assume now that $i > 1$ and that a strongly infinite-dimensional subspace T_{i-1} of Y and closed subsets $S_{k_1 k_2 \ldots k_{i-1}}$ of X with the required properties are already defined. Arrange all sequences $k_1, k_2, \ldots, k_{i-1}$ of zeros and ones into a sequence $\sigma_1, \sigma_2, \ldots, \sigma_{2^{i-1}}$. By virtue of Lemma 6.3.3 there exist a strongly infinite-dimensional subspace Y_1 of T_{i-1} and disjoint closed subsets $X_{\sigma_1 0}, X_{\sigma_1 1}$ of S_{σ_1} such that $f(X_{\sigma_1 0}) = f(X_{\sigma_1 1}) = Y_1$. The same lemma yields a strongly infinite-dimensional subspace Y_2 of Y_1 and disjoint closed

subsets $X_{\sigma 20}, X_{\sigma 21}$ of $S_{\sigma_2} \cap f^{-1}(Y_1)$ such that $f(X_{\sigma 20}) = f(X_{\sigma 21}) = Y_2$. Repeating this procedure we obtain strongly infinite-dimensional subspaces $Y_1 \supset Y_2 \supset \ldots \supset Y_{2^{i-1}}$ of T_{i-1} and disjoint closed subsets $X_{\sigma_j 0}, X_{\sigma_j 1}$ of $S_{\sigma_j} \cap f^{-1}(Y_{j-1})$, where $j = 1, 2, \ldots, 2^{i-1}$ and $Y_0 = Y$, such that $f(X_{\sigma_j 0}) = f(X_{\sigma_j 1}) = Y_j$. The subspace $T_i = Y_{2^{i-1}}$ of Y and the sets $S_{\sigma_j k} = X_{\sigma_j k} \cap f^{-1}(T_i)$ for $j = 1, 2, \ldots, 2^{i-1}$ and $k \in \{0, 1\}$ have all the required properties.

As the space X is compact, there exists a point $x \in S_0 \cap S_{00} \cap \ldots$; let $f(x) = y$. From (3) and (4) it readily follows that for each infinite sequence $k_1, k_2, \ldots$ of zeros and ones the intersection $f^{-1}(y) \cap S_{k_1} \cap S_{k_1 k_2} \cap \ldots$ is non-empty, and from (1) it follows that such intersections corresponding to different sequences are disjoint. Thus $|f^{-1}(y)| \geq \mathbf{c}$. ■

Let us state one more consequence of Lemma 6.3.3.

6.3.5. THEOREM. *If $f: X \to Y$ is a closed mapping of a weakly infinite-dimensional countably paracompact normal space X onto a normal space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then Y is weakly infinite-dimensional.*

PROOF. If Y were strongly infinite-dimensional, we could, by applying k times Lemma 6.3.3, find a point $y \in Y$ with $|f^{-1}(y)| \geq k + 1$. ■

It is not known if the last theorem holds for closed mappings with finite fibres.

Now we pass to a counterpart of the theorem on dimension-lowering mappings for weakly infinite-dimensional spaces. In the counterpart a new type of infinite-dimensional spaces occur, the so-called C-spaces; they form a class which contains all countable-dimensional hereditarily paracompact spaces and is contained in the class of weakly infinite-dimensional spaces. However, while there exist uncountable-dimensional compact metrizable C-spaces (see Problem 6.3.D(b)), no example of a metrizable or compact weakly infinite-dimensional space that is not a C-space is known (cf. Problem 6.3.D(c)).

6.3.6. DEFINITION. A normal space X is a *C-space* if for every sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of X there exists a sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots$ of families of pairwise disjoint open subsets of X such that for $i = 1, 2, \ldots$ each member of $\mathcal{H}_i$ is contained in a member of $\mathcal{G}_i$ and the union $\bigcup_{i=1}^{\infty} \mathcal{H}_i$ is a cover of X.

We shall now prove that hereditarily paracompact countable-dimensional spaces are C-spaces. In the proof Ostrand's characterization of the covering dimension (see Theorem 3.2.4) will be used; for metrizable spaces the proof is much simpler (cf. Problem 6.3.D(a)).

6.3.7. LEMMA. *Every paracompact finite-dimensional space is a C-space.*

PROOF. Let X be a paracompact space with $\dim X = n$. Consider a sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of open covers of X and for $i = 1, 2, \ldots$ take a locally finite open refinement $\mathcal{U}_i$ of the cover $\mathcal{G}_i$. Applying Theorem 3.2.4 to the locally finite open cover $\mathcal{U} = \mathcal{U}_1 \wedge \mathcal{U}_2 \wedge \ldots \wedge \mathcal{U}_{n+1}$ of X we obtain a refinement $\mathcal{V}$ of the cover $\mathcal{U}$ which can be represented as the union of $n + 1$ families $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{n+1}$ of pairwise disjoint open sets. Letting

$$\mathcal{H}_i = \mathcal{V}_i \quad \text{for } i \leq n + 1 \quad \text{and} \quad \mathcal{H}_i = \{\emptyset\} \quad \text{for } i > n + 1$$

we get the required sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots$ ■

6.3.8. THEOREM. *Every hereditarily paracompact countable-dimensional space is a C-space.*

PROOF. By virtue of the lemma it suffices to prove that if a hereditarily paracompact space X can be represented as the union of a sequence of subspaces $X_1, X_2, \ldots$, where each X_j is a C-space, then X is a C-space.

Let $\mathcal{G}_1, \mathcal{G}_2, \ldots$ be a sequence of open covers of X. Represent the set N of all positive integers as the union $\bigcup_{j=1}^{\infty} N_j$ of infinite sets, where $N_j \cap N_k = \emptyset$ whenever $j \neq k$. As X_j is a C-space, there exists a class $\{\mathcal{H}'_i : i \in N_j\}$ of families of pairwise disjoint open subsets of X_j such that for $i \in N_j$ each member of $\mathcal{H}'_i$ is contained in a member of $\mathcal{G}_i$ and the union $\bigcup\{\mathcal{H}'_i : i \in N_j\}$ is a cover of X_j. Since each of the families $\mathcal{H}'_i$ with $i \in N_j$ is discrete in the union $\bigcup \mathcal{H}'_i$, and since X is hereditarily collectionwise normal (see [GT], Theorem 5.1.18), the members of $\mathcal{H}'_i$ can be enlarged to pairwise disjoint open subsets of X (see [GT], Problem 5.5.1(a)). Without loss of generality one can assume that each member of the family $\mathcal{H}_i$ obtained in this way is contained in a member of $\mathcal{G}_i$, and thus the sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots$ has all the required properties. ■

The fact that every C-space is weakly infinite-dimensional will follow from the counterpart of the theorem on dimension-lowering mappings that we are now going to prove.

6.3.9. THEOREM. *If $f: X \to Y$ is a closed mapping of a hereditarily normal or normal and countably paracompact space X to a C-space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is weakly infinite-dimensional, then X is weakly infinite-dimensional.*

PROOF. Let $(A_1, B_1), (A_2, B_2), \ldots$ be an arbitrary sequence of pairwise disjoint closed subsets of X. For $i = 1, 2, \ldots$ choose open sets $U'_i, W'_i \subset X$ such that $A_i \subset U'_i$, $B_i \subset W'_i$ and $\overline{U'_i} \cap \overline{W'_i} = \emptyset$. Represent the set N of all positive integers as the union $\bigcup_{j=1}^{\infty} N_j$ of infinite sets, where $N_j \cap N_k = \emptyset$ whenever $j \neq k$. Since for every $y \in Y$ the fibre $f^{-1}(y)$ is weakly infinite-dimensional, by virtue of Lemma 1.2.9 and Remark 1.2.10 or by Lemmas 6.1.4 and 6.1.5 for each j there exist open sets $U_{i,y}, W_{i,y}$, where $i \in N_j$, such that

$$U_{i,y} \cap W_{i,y} = \emptyset, \tag{6}$$

$$\overline{U'_i} \subset U_{i,y}, \quad \overline{W'_i} \subset W_{i,y}, \tag{7}$$

and

$$f^{-1}(y) \subset V_{j,y} = \bigcup\{U_{i,y} \cup W_{i,y} : i \in N_j\}. \tag{8}$$

The mapping f being closed, for every $y \in Y$ and each j there exists a neighbourhood $G_{j,y}$ of y such that

$$f^{-1}(G_{j,y}) \subset V_{j,y}; \tag{9}$$

the family $\mathcal{G}_j = \{G_{j,y}\}_{y \in Y}$ is an open cover of Y. As Y is a C-space, for the sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ there exists a sequence $\mathcal{H}_1, \mathcal{H}_2, \ldots$ of families of pairwise disjoint open subsets of Y such that the union $\bigcup_{j=1}^{\infty} \mathcal{H}_j$ is a cover of Y and each set $H \in \mathcal{H}_j$ is contained in a set $G_{j,y(H)} \in \mathcal{G}_j$. For each $i \in N_j$ consider the open sets

$$U_i = U'_i \cup \bigcup\{f^{-1}(H) \cap U_{i,y(H)} : H \in \mathcal{H}_j\}$$

and

$$W_i = W_i' \cup \bigcup\{f^{-1}(H) \cap W_{i,y(H)} : H \in \mathcal{H}_j\};$$

clearly, $A_i \subset U_i$ and $B_i \subset W_i$. The inverse images $f^{-1}(H)$ with $H \in \mathcal{H}_j$ being pairwise disjoint, it follows from (6) and (7) that $U_i \cap W_i = \emptyset$. Hence $L_i = X \setminus (U_i \cup W_i)$ is a partition in X between A_i and B_i.

To conclude the proof it suffices to show that $\bigcap_{i=1}^{\infty} L_i = \emptyset$, or – equivalently – that $\bigcup_{i=1}^{\infty}(U_i \cup W_i) = X$. Consider an arbitrary point $x \in X$. There exist an integer j and a set $H \in \mathcal{H}_j$ such that $f(x) = y \in H$; by (9) and (8) we have

$$x \in f^{-1}(y) \subset f^{-1}(H) \subset f^{-1}(G_{j,y(H)}) \subset V_{j,y(H)} = \bigcup\{U_{i,y(H)} \cup W_{i,y(H)} : i \in N_j\},$$

so that $x \in \bigcup\{U_i \cup W_i : i \in N_j\} \subset \bigcup_{i=1}^{\infty}(U_i \cup W_i)$. ■

It is readily seen that if for some $y \in Y$ the fibre $f^{-1}(y)$ is a one-point set, then one can define the open sets $U_{i,y}$ and $W_{i,y}$ satisfying (6)–(8) using only normality of the space X (cf. Problem 6.3.E). Hence applying Theorem 6.3.9 to the identity mapping id_X, we obtain

6.3.10. THEOREM. *Every C-space is weakly infinite-dimensional.* ■

Since the Cartesian product $X \times Y$ of a compact space X and a countably paracompact space Y is countably paracompact (see [GT], Exercise 5.2.G(b)), the following result is a direct consequence of Theorem 6.3.9 and the fact that the projection parallel to a compact factor is a closed mapping (see [GT], Theorem 3.1.16).

6.3.11. THEOREM. *If the Cartesian product $X \times Y$ of a compact weakly infinite-dimensional space X and a countably paracompact C-space Y is normal, then it is weakly infinite-dimensional.* ■

Let us note that under an additional set-theoretic assumption one can define an open-and-closed mapping $f: X \to Y$ of a strongly infinite-dimensional countably paracompact normal space X onto a weakly infinite-dimensional space Y such that $|f^{-1}(y)| < \mathfrak{c}$ for every $y \in Y$. It is an open problem, though, if Theorem 6.3.9 holds in the realm of metrizable spaces under the weaker assumption that Y is a weakly infinite-dimensional space.

We conclude this section with a theorem on open mappings.

6.3.12. LEMMA. *If all fibres of an open mapping $f: X \to Y$ defined on a Hausdorff space X are finite and have the same cardinality, then f is closed.*

PROOF. Consider a closed set $A \subset X$ and take a point $y \in Y \setminus f(A)$; let $f^{-1}(y) = \{x_1, x_2, \ldots, x_j\}$. Choose pairwise disjoint open sets $U_1, U_2, \ldots, U_j \subset X$ disjoint from A and such that $x_i \in U_i$ for $i = 1, 2, \ldots, j$. One easily checks that the intersection $\bigcap_{i=1}^{j} f(U_i)$ is a neighbourhood of y and is disjoint from $f(A)$. Hence the set $f(A)$ is closed. ■

6.3.13. THEOREM. *If $f: X \to Y$ is an open mapping of a weakly infinite-dimensional normal space X onto a countably paracompact normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then Y is weakly infinite-dimensional.*

PROOF. Let $K_j = \{y \in Y : |f^{-1}(y)| = j\}$ for $j = 1, 2, \ldots$; one readily sees that the union $\bigcup_{j \leq i} K_j$ is closed in Y for $i = 1, 2, \ldots$ Note that every closed subspace Z of Y contained in a set K_j is weakly infinite-dimensional. Indeed, by the lemma the restriction $f_Z: f^{-1}(Z) \to Z$ is closed, and since the inverse image of a countably paracompact space under a closed mapping with finite fibres is countably paracompact (see [GT], Exercise 5.2.G(a)), the weak infinite dimensionality of Z follows from Theorem 6.3.5. Applying Proposition 6.1.16 one can show by an easy induction that the union $\bigcup_{j \leq i} K_j$ is weakly infinite-dimensional for $i = 1, 2, \ldots$ Thus Y is weakly infinite-dimensional by virtue of Theorem 6.1.7 (cf. Problem 6.1.C(a)). ■

From the comments following Theorem 5.4.11 we gather that in the last theorem the assumption that the fibers of f are finite cannot be relaxed to their being countable.

Let us add that if a countably paracompact normal space X can be mapped by an open mapping with finite fibers onto a weakly infinite-dimensional normal space Y, then X is weakly infinite-dimensional (see Problem 6.3.G).

Historical and bibliographic notes

Lemma 6.3.1 and Theorem 6.3.2 were established by Olszewski in [1993] (in the case of a closed mapping with fibres of cardinality less than $\mathbf{c}$ defined on a completely metrizable space X Theorem 6.3.2 was obtained by Skljarenko in [1962a]). Lemma 6.3.3 was proved by Polkowski in [1983a]; it is a variant of a lemma from Skljarenko [1962a], where Theorem 6.3.4 was established. Theorem 6.3.5 is implicit in Polkowski [1983a]. The class of C-spaces was introduced by Addis and Gresham in [1978]; in [1973] Haver studied a related class of metrizable spaces. Lemma 6.3.7, Theorem 6.3.8 (implicitly), and Theorem 6.3.10 were given by Addis and Gresham in [1978]. Theorems 6.3.9 and 6.3.11 were proved by Hattori and Yamada in [1989] (the latter was obtained independently by Rohm, cited in Hattori and Yamada [1989]; cf. Rohm [1991]); Levšenko proved in [1959] that the Cartesian product $X \times Y$ of a metrizable compact weakly infinite-dimensional space X and a metrizable countable-dimensional space Y is weakly infinite-dimensional. The example of an open-and-closed mapping $f: X \to Y$ of a strongly infinite-dimensional countably paracompact normal space X onto a weakly infinite-dimensional space Y such that $|f^{-1}(y)| < \mathbf{c}$ for every $y \in Y$ (defined under an additional set-theoretic assumption) is due to H. Ohta (cited in Hattori and Yamada [1989]). Theorem 6.3.13 was proved by Polkowski in [1983a] under the slightly stronger assumption that Y is a weakly paracompact normal space.

Problems

6.3.A. (a) Prove that if $f: X \to Y$ is a closed mapping of a weakly infinite-dimensional metrizable space X onto a strongly infinite-dimensional metrizable space Y, then the subspace $\{y \in Y: |f^{-1}(y)| \geq \mathbf{c}\}$ of Y is strongly infinite-dimensional.

Hint. Apply Proposition 6.1.17.

(b) Observe that in Theorem 6.3.2 the assumption on the fibres can be replaced by the assumption that for every $y \in Y$ the set $\operatorname{Fr} f^{-1}(y)$ has an isolated point or is of cardinality less than $\mathbf{c}$.

(c) Prove that Theorem 6.3.2 holds under the assumption that Y is a normal space.

Hint. See the hint to Problem 5.4.A(c).

6.3.B. Check that Lemma 6.3.3 holds under the assumption that X is a hereditarily normal space.

6.3.C. (a) (Polkowski [1983a]) Prove that if $f: X \to Y$ is a closed mapping of a metrizable space X to a weakly infinite-dimensional paracompact space Y and there exists an integer $k \geq 1$ such that $|f^{-1}(y)| \leq k$ for every $y \in Y$, then X is weakly infinite-dimensional.

It is an open problem if the assumption on the fibres can be relaxed to their being finite (cf. part (c)).

Hint. Show first that if all fibres of f have the same cardinality j, then there exists a σ-locally finite closed cover $\mathcal{F}$ of Y such that for each $F \in \mathcal{F}$ the inverse image $f^{-1}(F)$ can be represented as the union of j disjoint closed subsets with the property that the restriction of f to each of them is a homeomorphism, so that X is weakly infinite-dimensional. To that end, find an appropriate locally finite closed cover of the set $\{y \in Y : \rho(x, x') \geq 1/i$ whenever $x, x' \in f^{-1}(y)$ and $x \neq x'\}$ for $i = 1, 2, \ldots$ Then note that $\{y \in Y : |f^{-1}(y)| = k\}$ is an F_σ-set in Y and apply induction, using Problem 6.1.C(b)).

(b) (R. Pol [1995]) Prove that if $f: X \to Y$ is a continuous mapping of a strongly infinite-dimensional compact metrizable space X to a weakly infinite-dimensional metrizable space Y, then there exist disjoint closed subsets X_0, X_1 of X such that X_0 is strongly infinite-dimensional and $f(X_0) = f(X_1)$. Observe that if in addition $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$ and each non-empty weakly infinite-dimensional closed subspace of X is zero-dimensional, then X_1 is strongly infinite-dimensional as well.

Hint. Fix a metric on the space X and deduce from Proposition 6.1.17 that the set $\{x \in X : \delta(f^{-1}f(x)) > 0\}$ is strongly infinite-dimensional. Then note that for some integer i the closed set $F_i = \{x \in X : \delta(f^{-1}f(x)) \geq 1/i\}$ is strongly infinite-dimensional, find a closed strongly infinite-dimensional subset X_0 of F_i such that $\delta(X_0) \leq 1/(4i)$, and let $X_1 = (X \setminus B(X_0, 1/(4i))) \cap f^{-1}f(X_0)$.

(c) (R. Pol [1995]) Prove that if $f: X \to Y$ is a continuous mapping of a compact metrizable space X to a weakly infinite-dimensional metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then X is weakly infinite-dimensional.

Deduce that if $f: X \to Y$ is a continuous mapping of a compact metrizable space X onto a metrizable space Y such that $|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then X is weakly infinite-dimensional if and only if Y is weakly infinite-dimensional.

Hint. Assume that X is strongly infinite-dimensional and obtain a contradiction by finding a point $y \in Y$ such that $|f^{-1}(y)| = \mathbf{c}$. First observe that without loss of generality one can assume that each non-empty weakly infinite-dimensional closed subspace of X is zero-dimensional (see Problem 6.1.G(b)), and then modify the proof of Theorem 6.3.4 applying part (b).

(d) (R. Pol [1995]) Prove that if $f: X \to Y$ is a continuous mapping of a strongly infinite-dimensional compact metrizable space X to a metrizable space Y such that

$|f^{-1}(y)| \leq \aleph_0$ for every $y \in Y$, then there exists a strongly infinite-dimensional closed subspace F of X on which f is a homeomorphic embedding.

Hint. Analyze the arguments in the hints to parts (b) and (c).

6.3.D. (a) (Haver [1973]) Show that every metrizable countable-dimensional space is a C-space.

Hint. Apply Propositions 5.1.3 and 3.2.2 and note that Lemma 1.7.4 holds as well for infinite families of sets.

(b) Check that the compact metrizable uncountable-dimensional space Y defined in Example 6.1.21 is a C-space.

(c) (Addis and Gresham [1978]) Show that the space W_0 defined in Example 2.1.6 is weakly infinite-dimensional but is not a C-space.

6.3.E. (a) (Hattori and Yamada [1989]) Show that if f is a perfect mapping, then the assertion of Theorem 6.3.9 holds under the assumption that X is a normal space.

(b) Show that if $\dim f^{-1}(y) < \infty$ for every $y \in Y$, then the assertion of Theorem 6.3.9 holds under the assumption that X is a normal space.

6.3.F (Polkowski [1983a]). (a) Show that if $f: X \to Y$ is an open mapping of a metrizable space X to a weakly infinite-dimensional metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a discrete subspace of X, then X is weakly infinite-dimensional (cf. Problem 4.3.C(b)).

Hint. Apply Lemma 1.12.5.

(b) Show that if $f: X \to Y$ is an open mapping of a weakly infinite-dimensional metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is separable and has an isolated point, then Y is weakly infinite-dimensional.

Deduce that if $f: X \to Y$ is an open mapping of a metrizable space X onto a metrizable space Y such that for every $y \in Y$ the fibre $f^{-1}(y)$ is a countable discrete space, then X is weakly infinite-dimensional if and only if Y is weakly infinite-dimensional.

Hint. See the hint to Problem 4.3.C(c).

6.3.G (Polkowski [1983a] for weakly paracompact normal X). Prove that if $f: X \to Y$ is an open mapping of a countably paracompact normal space X onto a weakly infinite-dimensional normal space Y such that $|f^{-1}(y)| < \aleph_0$ for every $y \in Y$, then X is weakly infinite-dimensional (cf. Problem 6.2.D(a)).

Chapter 7

Transfinite dimensions

In the present chapter we show how the "degree of infinite dimensionality" of spaces can be measured by various dimension functions taking their values in the set of ordinal numbers, finite and infinite. Such functions introduce further stratification in the classes of strongly countable-dimensional, countable-dimensional and weakly infinite-dimensional spaces and permit to classify infinite-dimensional spaces more precisely.

We shall assume that the reader has a basic knowledge of ordinal numbers. In particular, the operations of addition and multiplication of ordinal numbers will be used, as well as their elementary properties, e.g., the property that if $\alpha < \beta$, then $\gamma + \alpha < \gamma + \beta$ and $\omega_0 \cdot \alpha < \omega_0 \cdot \beta$. We shall also use the fact that each ordinal number α has a unique representation in the form $\alpha = \lambda + n$, where λ is a limit number and n a non-negative integer (we consider 0 to be a limit number).

The material of Chapters 2 and 3 – with the exception of Sections 2.1 and 3.1 and Theorems 2.2.5, 2.3.6 and 2.3.14 – is not used here. The reader interested only in metrizable spaces should skip Lemmas 7.2.4 and 7.2.5, Theorems 7.2.6 and 7.2.7 as well as Corollaries 7.2.8 and 7.2.9, and in the proof of Theorem 7.2.10 use Theorem 7.2.14 rather than Corollary 7.2.9.

In Section 7.1 we consider transfinite extensions of the small and large inductive dimensions; they are respectively denoted by trind and trInd and are called small and large transfinite dimension. Several basic properties of those dimensions are established and their relations with countable dimensionality are studied. We conclude with a description of Smirnov's spaces S_α, a kind of "unit α-cubes", and show that $\operatorname{trInd} S_\alpha = \alpha$ for every $\alpha < \omega_1$, while it may happen that $\operatorname{trind} S_\alpha < \alpha$.

Section 7.2 is devoted to a further study of the transfinite dimensions trind and trInd. We prove there the separation, sum, enlargement and completion theorems for these dimensions, as well as the universal space theorem for trind.

In Section 7.3 we study the transfinite kernel dimension trker, and discuss its relations with strong countable dimensionality as well as with trind and trInd. We also show that trker coincides with the D-dimension D. We close the section with some information on the transfinite covering dimension trdim which provides a classification of compact weakly infinite-dimensional spaces.

7.1. Definitions and basic properties of the transfinite dimensions trind and trInd

The transfinite dimension trind that we are now going to define introduces a stratification in the class of all spaces with ind equal to ∞. To some of these spaces an

ordinal number will be assigned which gauges their "degree of infinite dimensionality". The transfinite dimension trInd introduces a similar stratification in the class of all spaces with Ind equal to ∞. It is convenient to consider also transfinite dimensions trind and trInd of spaces with finite ind or Ind; their values are finite ordinal numbers and -1 (i.e., integers larger than or equal to -1), and they are equal to ind and Ind respectively.

7.1.1. DEFINITION. To every regular space X one assigns the *small transfinite dimension* of X, denoted by $\operatorname{trind} X$, which is the integer -1, an ordinal number, or the "infinite number" ∞; the definition of the dimension function trind consists in the following conditions:

(ST1) $\operatorname{trind} X = -1$ if and only if $X = \emptyset$;

(ST2) $\operatorname{trind} X \leq \alpha$, where α is an ordinal number, if for every point $x \in X$ and each neighbourhood V of the point x there exists an open set $U \subset X$ such that

$$x \in U \subset V \quad \text{and} \quad \operatorname{trind} \operatorname{Fr} U < \alpha;$$

(ST3) $\operatorname{trind} X = \alpha$ if $\operatorname{trind} X \leq \alpha$ and the inequality $\operatorname{trind} X \leq \beta$ holds for no $\beta < \alpha$;

(ST4) $\operatorname{trind} X = \infty$ if $\operatorname{trind} X \leq \alpha$ holds for no ordinal number α.

If $\operatorname{trind} X$ is not equal to ∞, we say that X *has small transfinite dimension*; otherwise, we say that X *does not have small transfinite dimension*.

Applying transfinite induction with respect to $\operatorname{trind} X$, one easily verifies that whenever regular spaces X and Y are homeomorphic, then $\operatorname{trind} X = \operatorname{trind} Y$, i.e., the small transfinite dimension is a topological invariant.

In order to simplify the statements of certain results proved in this chapter, we shall assume that for each ordinal number α we have $-1 < \alpha < \infty$; for ordinal numbers we shall consider the usual linear order $<$. By ω_0 and ω_1 we denote respectively the first infinite and the first uncountable ordinal number.

7.1.2. PROPOSITION. *For every regular space* X, *if* $\operatorname{ind} X < \infty$, *then* $\operatorname{trind} X = \operatorname{ind} X$, *and if* $\operatorname{trind} X < \omega_0$, *then* $\operatorname{ind} X = \operatorname{trind} X$. ■

Since every subspace M of a regular space X is itself regular, it follows that if the transfinite dimension trind is defined for a space X, it is also defined for every subspace M of X. Applying transfinite induction we easily obtain (cf. Theorem 1.1.2)

7.1.3. THE SUBSPACE THEOREM FOR trind. *For every subspace* M *of a regular space* X *we have* $\operatorname{trind} M \leq \operatorname{trind} X$. ■

7.1.4. THEOREM. *If* X *is a regular space and* $\operatorname{trind} X = \alpha$, *then for each ordinal number* $\beta < \alpha$ *the space* X *contains a closed subspace* M_β *such that* $\operatorname{trind} M_\beta = \beta$.

PROOF. We shall apply transfinite induction with respect to α. For $\alpha = 0$ the theorem is obvious. Assume that the theorem holds for all $\alpha < \alpha_0$ and consider a regular space X with $\operatorname{trind} X = \alpha_0$ as well as an ordinal number $\beta < \alpha_0$. Suppose that X contains no closed subspace M with $\operatorname{trind} M = \beta$. By the inductive assumption X contains no closed subspace M' which satisfies $\beta \leq \operatorname{trind} M' < \alpha_0$. Thus for every point

$x \in X$ and each neighbourhood V of the point x there exists an open set U such that $x \in U \subset V$ and $\operatorname{trind} \operatorname{Fr} U < \beta$. This contradicts, however, the equality $\operatorname{trind} X = \alpha_0$, so that X contains a closed subspace M_β with $\operatorname{trind} M_\beta = \beta$. ■

From Proposition 7.1.2 and the definition of trind we obtain

7.1.5. PROPOSITION. *A regular space X satisfies the equality* $\operatorname{trind} X = \omega_0$ *if and only if* $\operatorname{ind} X = \infty$ *and for every point $x \in X$ and each neighbourhood V of the point x there exists an open set $U \subset X$ such that $x \in U \subset V$ and* $\operatorname{ind} \operatorname{Fr} U < \infty$. ■

The last proposition implies that for the space $S = \bigoplus_{n=1}^{\infty} I^n$ we have $\operatorname{trind} S = \omega_0$. Similarly, for the one-point compactification ωS of S we have $\operatorname{trind} \omega S = \omega_0$.

7.1.6. THEOREM. *If a regular space X with weight $w(X) \leq \aleph_\alpha$ has small transfinite dimension, then it satisfies the inequality* $\operatorname{trind} X < \omega_{\alpha+1}$.

PROOF. Suppose that there exists a regular space X such that $w(X) \leq \aleph_\alpha$ and $\operatorname{trind} X \geq \omega_{\alpha+1}$. By virtue of Theorem 7.1.4 we can assume that $\operatorname{trind} X = \omega_{\alpha+1}$. Thus X has a base $\mathcal{B}$ such that $\operatorname{trind} \operatorname{Fr} U < \omega_{\alpha+1}$ for every $U \in \mathcal{B}$. From the inequality $w(X) \leq \aleph_\alpha$ it follows (cf. Lemma 1.1.5) that X has a base $\mathcal{B}_0$ such that $|\mathcal{B}_0| \leq \aleph_\alpha$ and $\mathcal{B}_0 \subset \mathcal{B}$. Since the set of all ordinal numbers less than $\omega_{\alpha+1}$ contains no cofinal subset of cardinality $\aleph_\alpha$, there exists an ordinal number $\beta < \omega_{\alpha+1}$ such that $\operatorname{trind} \operatorname{Fr} U < \beta$ for every $U \in \mathcal{B}_0$; thus $\operatorname{trind} X \leq \beta < \omega_{\alpha+1}$, a contradiction. ■

7.1.7. COROLLARY. *If a separable metrizable space X has small transfinite dimension, then it satisfies the inequality* $\operatorname{trind} X < \omega_1$. ■

It turns out, however, that metrizable spaces can have arbitrarily large small transfinite dimension (see Problem 7.2.C).

We shall now show that in the realm of metrizable spaces there exists a close relationship between the existence of small transfinite dimension and countable dimensionality.

7.1.8. THEOREM. *Every separable metrizable space which has small transfinite dimension is countable-dimensional.*

PROOF. We shall apply transfinite induction with respect to the small transfinite dimension of the space X under consideration. If $\operatorname{trind} X = -1$ the theorem holds. Assume that the theorem is established for all separable metrizable spaces with trind less than $\alpha \geq 0$ and consider a space X with $\operatorname{trind} X = \alpha$. Thus X has a base $\mathcal{B}$ such that $\operatorname{trind} \operatorname{Fr} U < \alpha$ for every $U \in \mathcal{B}$. Without loss of generality one can assume that $|\mathcal{B}| \leq \aleph_0$ (see Lemma 1.1.5). The space $Y = \bigcup\{\operatorname{Fr} U : U \in \mathcal{B}\}$ is countable-dimensional by the inductive assumption, and the space $Z = X \setminus Y$ satisfies the equality $\dim Z = \operatorname{ind} Z = 0$ (see Proposition 1.2.4 and Lemma 1.7.4), so that the space X is countable-dimensional. ■

The last theorem implies that the Hilbert cube $I^{\aleph_0}$ does not have small transfinite dimension (cf. Theorem 1.8.21). Since for the countable-dimensional compact space Y which is not strongly countable-dimensional, defined in Example 5.1.7, we have $\operatorname{trind} Y = \omega_0$ (see Problem 5.1.D), there exist compact metrizable spaces which have small transfinite dimension and are not strongly countable-dimensional.

Let us note that we know no example of a metrizable space which has small transfinite dimension and yet is not countable-dimensional; there exist, though, such spaces which are perfectly normal.

7.1.9. THEOREM. *Every completely metrizable countable-dimensional space has small transfinite dimension.*

PROOF. Assume that a completely metrizable countable-dimensional space X does not have small transfinite dimension and consider a complete metric ρ on the space X. Let $X = \bigcup_{i=1}^{\infty} X_i$, where $\operatorname{Ind} X_i \leq 0$ for $i = 1, 2, \ldots$ (see Proposition 5.1.3 and Theorem 4.1.3). By our assumption, there exist a point $x \in X$ and a neighbourhood $V \subset X$ of the point x such that for every open set U satisfying $x \in U \subset V$ we have $\operatorname{trind} \operatorname{Fr} U = \infty$. By Theorem 4.1.13 there exists an open set U_1 such that $x \in U_1 \subset V \cap B(x, 1/2)$ and $(\operatorname{Fr} U_1) \cap X_1 = \emptyset$. The closed set $F_1 = \operatorname{Fr} U_1$ satisfies

$$F_1 \cap X_1 = \emptyset, \quad \delta(F_1) \leq 1, \quad \text{and} \quad \operatorname{trind} F_1 = \infty.$$

Since the subspace F_1 of X does not have small transfinite dimension, we can repeat the above argument and, by induction, obtain a decreasing sequence $F_1 \supset F_2 \supset \ldots$ of closed subsets of X such that

$$F_i \cap X_i = \emptyset, \quad \delta(F_i) \leq 1/i, \quad \text{and} \quad \operatorname{trind} F_i = \infty \quad \text{for } i = 1, 2, \ldots$$

As the sets F_i are non-empty, by the completeness of X we have $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$, but

$$\bigcap_{i=1}^{\infty} F_i \subset \bigcap_{i=1}^{\infty} (X \setminus X_i) = X \setminus \bigcup_{i=1}^{\infty} X_i = \emptyset;$$

the contradiction shows that X has small transfinite dimension. ■

7.1.10. COROLLARY. *A separable completely metrizable space has small transfinite dimension if and only if it is countable-dimensional.* ■

The assumption of complete metrizability in Theorem 7.1.9 and Corollary 7.1.10 is essential. Indeed, the strongly countable-dimensional space K_ω defined in Example 1.8.22(b) does not have small transfinite dimension. This follows from the fact that for every subspace M of the Hilbert cube $I^{\aleph_0}$ satisfying the inequality $\operatorname{trind} M \leq \alpha$ there exists a G_δ-set M^* in $I^{\aleph_0}$ such that $M \subset M^*$ and $\operatorname{trind} M^* \leq \alpha$ (see Theorem 7.2.17), and from Example 5.3.6, because no space containing a subspace homeomorphic to $I^{\aleph_0}$ has small transfinite dimension. The space N_ω defined in Example 1.8.22(c) does not have small transfinite dimension either; indeed, as one easily sees, N_ω contains a subspace homeomorphic to K_ω (cf. Theorem 5.3.7).

We now pass to the large transfinite dimension.

7.1.11. DEFINITION. To every normal space X one assigns the *large transfinite dimension* of X, denoted by $\operatorname{trInd} X$, which is the integer -1, an ordinal number, or the "infinite number" ∞; the definition of the dimension function trInd consists in the following conditions:

(LT1) $\operatorname{trInd} X = -1$ if and only if $X = \emptyset$;

(LT2) $\operatorname{trInd} X \leq \alpha$, where α is an ordinal number, if for every closed set $A \subset X$ and each open set $V \subset X$ which contains the set A there exists an open set $U \subset X$ such that

$$A \subset U \subset V \quad \text{and} \quad \operatorname{trInd} \operatorname{Fr} U < \alpha;$$

(LT3) $\operatorname{trInd} X = \alpha$ if $\operatorname{trInd} X \leq \alpha$ and the inequality $\operatorname{trInd} X \leq \beta$ holds for no $\beta < \alpha$;

(LT4) $\operatorname{trInd} X = \infty$ if $\operatorname{trInd} X \leq \alpha$ holds for no ordinal number α.

If $\operatorname{trInd} X$ is not equal to ∞, we say that X *has large transfinite dimension*; otherwise, we say that X *does not have large transfinite dimension.*

Applying transfinite induction, one easily verifies that the large transfinite dimension is a topological invariant.

7.1.12. PROPOSITION. *A normal space X satisfies the inequality* $\operatorname{trInd} X \leq \alpha$ *if and only if for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that* $\operatorname{trInd} L < \alpha$. ■

Exactly as the small transfinite dimension, trInd has the following four properties.

7.1.13. PROPOSITION. *For every normal space X, if* $\operatorname{Ind} X < \infty$, *then* $\operatorname{trInd} X = \operatorname{Ind} X$, *and if* $\operatorname{trInd} X < \omega_0$, *then* $\operatorname{Ind} X = \operatorname{trInd} X$. ■

7.1.14. THE SUBSPACE THEOREM FOR trInd. *For every closed subspace M of a normal space X we have* $\operatorname{trInd} M \leq \operatorname{trInd} X$. ■

7.1.15. THEOREM. *If X is a normal space and* $\operatorname{trInd} X = \alpha$, *then for each ordinal number $\beta < \alpha$ the space X contains a closed subspace M_β such that* $\operatorname{trInd} M_\beta = \beta$. ■

7.1.16. PROPOSITION. *A normal space X satisfies the equality* $\operatorname{trInd} X = \omega_0$ *if and only if* $\operatorname{Ind} X = \infty$ *and for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that* $\operatorname{Ind} L < \infty$. ■

7.1.17. THEOREM. *If a compact space X with weight $w(X) \leq \aleph_\alpha$ has large transfinite dimension, then it satisfies the inequality* $\operatorname{trInd} X < \omega_{\alpha+1}$.

PROOF. Suppose that there exists a compact space X such that $w(X) \leq \aleph_\alpha$ and $\operatorname{trInd} X \geq \omega_{\alpha+1}$. We can assume that $\operatorname{trInd} X = \omega_{\alpha+1}$. Take a base $\mathcal{B}$ for the space X such that $|\mathcal{B}| \leq \aleph_\alpha$ and consider the family $\mathcal{W}$ consisting of all finite unions of sets in $\mathcal{B}$; clearly, $|\mathcal{W}| \leq \aleph_\alpha$. Now, for all $W, W' \in \mathcal{W}$ satisfying the inclusion $\overline{W} \subset W'$ choose an open set $U \subset X$ such that $\overline{W} \subset U \subset W'$ and $\operatorname{trInd} \operatorname{Fr} U < \omega_{\alpha+1}$, and denote by $\mathcal{U}$ the family of all U's obtained in this way. Since $|\mathcal{U}| \leq \aleph_\alpha$, there exists an ordinal number $\beta < \omega_{\alpha+1}$ such that $\operatorname{trInd} \operatorname{Fr} U < \beta$ for every $U \in \mathcal{U}$. One easily checks, applying the compactness of X, that for every closed set $A \subset X$ and each open set $V \subset X$ that contains A there exist $W, W' \in \mathcal{W}$ with $A \subset \overline{W} \subset W' \subset V$. Thus we have $\operatorname{trInd} X \leq \beta < \omega_{\alpha+1}$, a contradiction. ■

7.1.18. COROLLARY. *If a compact metrizable space X has large transfinite dimension, then it satisfies the inequality* $\operatorname{trInd} X < \omega_1$. ■

We shall show later (see Corollary 7.1.27) that in the last corollary the assumption of compactness can be dropped.

Our next theorem can be easily proved by transfinite induction.

7.1.19. THEOREM. *For every normal space X we have* $\operatorname{trind} X \leq \operatorname{trInd} X$. ■

In particular, if a normal space X has large transfinite dimension, then X also has small transfinite dimension. The reverse implication does not hold. Indeed, for the space $S = \bigoplus_{n=1}^{\infty} I^n$ we have $\operatorname{trind} S = \omega_0$, while – as we are going to show – S does not have large transfinite dimension.

We shall prove a little more, viz., that no space which is in some way similar to S has large transfinite dimension.

7.1.20. DEFINITION. A normal space X is *nearly S-like* if $X = \bigoplus_{n=1}^{\infty} X_n$, where $\operatorname{Ind} X_n \geq n$ for $n = 1, 2, \ldots$

From Theorem 3.1.28 it follows that every almost S-like space is nearly S-like (cf. Definition 5.5.18), and Theorem 4.1.3 implies that in the realm of metrizable spaces nearly S-like and almost S-like spaces coincide.

7.1.21. PROPOSITION. *No normal nearly S-like space has large transfinite dimension.*

PROOF. Consider a nearly S-like normal space $X = \bigoplus_{n=1}^{\infty} X_n$, where $\operatorname{Ind} X_n \geq n$. Observe first that the inequality $\operatorname{trInd} X \leq \omega_0$ cannot hold. Indeed, for $n = 1, 2, \ldots$ in the space X_n there exists a pair A_n, B_n of disjoint closed subsets with the property that no partition in X_n between these sets has Ind smaller than $n - 1$. This implies that in X no partition L between the closed sets $A = \bigcup_{n=1}^{\infty} A_n$ and $B = \bigcup_{n=1}^{\infty} B_n$ satisfies $\operatorname{Ind} L < \infty$, so that $\operatorname{trInd} X > \omega_0$ by Proposition 7.1.16.

Suppose now that $\operatorname{trInd} X$ is equal to $\alpha > \omega_0$. From Theorem 7.1.15 it follows that X contains a subspace M with $\operatorname{trInd} M = \omega_0$. Since the sequence $\operatorname{Ind}(M \cap X_1)$, $\operatorname{Ind}(M \cap X_2), \ldots$ cannot be bounded, M is a nearly S-like space, a contradiction. ■

From Theorem 7.1.19 it follows that the spaces K_ω and N_ω do not have large transfinite dimension. This can also be deduced from the fact that both K_ω and N_ω contain a closed subspace homeomorphic to S. On the other hand, from Proposition 7.1.22 below it follows that the one-point compactification ωS of the space S and our countable-dimensional compact space Y which is not strongly countable-dimensional (see Example 5.1.7) satisfy the equality $\operatorname{trInd} \omega S = \operatorname{trInd} Y = \omega_0$.

7.1.22. PROPOSITION. *For every compact metrizable space X with* $\operatorname{trind} X = \omega_0$ *we have* $\operatorname{trInd} X = \omega_0$.

PROOF. Consider a pair A, B of disjoint closed subsets of X. Each point $x \in A$ has a neighbourhood $U_x \subset X \setminus B$ with $\operatorname{ind} \operatorname{Fr} U_x < \infty$; since A is compact, there exist in A finitely many points $x_1, x_2, \ldots, x_m$ such that $A \subset U = U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_m}$, and since $\operatorname{Fr} U \subset \operatorname{Fr} U_{x_1} \cup \operatorname{Fr} U_{x_2} \cup \ldots \cup \operatorname{Fr} U_{x_m}$, by Theorems 1.5.3 and 1.1.2 we have $\operatorname{ind} \operatorname{Fr} U < \infty$. Thus $\operatorname{Ind} \operatorname{Fr} U < \infty$ by Theorem 1.6.4, and by Proposition 7.1.16 we have $\operatorname{trInd} X = \omega_0$. ■

Let us note that in the last proposition the assumption of compactness of X can be weakened to that of separability, if we additionally assume that X has large transfinite dimension (see Problem 7.1.C(a)). Let us also note that there are compact metrizable spaces X with $\operatorname{trind} X = \omega_0 + 1$ and $\operatorname{trInd} X \geq \omega_0 + 2$ (see Problem 7.1.G). Finally, let us observe that we have no example to show that the assumption of metrizability in Proposition 7.1.22 cannot be dropped (cf. Problem 7.1.C(b)).

For every normal space X we shall now define a subspace $S(X)$ of X which plays an important role in the study of large transfinite dimension.

For each normal space X and $n = 0, 1, \ldots$ we let

$$I_n(X) = \bigcup\{U \subset X : U \text{ is open and } \operatorname{Ind} \overline{U} \leq n\}$$

and

$$S(X) = X \setminus \bigcup_{n=0}^{\infty} I_n(X);$$

clearly, $I_0(X) \subset I_1(X) \subset \ldots$, the sets $I_n(X)$ are open, and $S(X)$ is closed.

7.1.23. THEOREM. *If a weakly paracompact strongly hereditarily normal space X has large transfinite dimension, then the subspace $S(X)$ of X is compact and* $\operatorname{Ind} F < \infty$ *for each closed subspace $F \subset X$ disjoint from $S(X)$.*

PROOF. Since the subspace $S(X)$ is weakly paracompact, to prove that it is compact it suffices to prove that it is countably compact (see [GT], Theorem 5.3.2), i.e., that every countably infinite set $\{x_1, x_2, \ldots\} \subset S(X)$ has an accumulation point in $S(X)$ (see [GT], Theorem 3.10.3); we can obviously assume that all points x_n are distinct. Suppose that the set $\{x_1, x_2, \ldots\}$ has no accumulation point. As $S(X)$ is closed in X, this means that the family $\{\{x_n\}\}_{n=1}^{\infty}$ is discrete, and by Lemma 5.5.20 there exists a family $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that $x_n \in U_n$ for $n = 1, 2, \ldots$ and the family $\{\overline{U}_n\}_{n=1}^{\infty}$ is discrete. As $x_n \notin I_n(X)$, we have $\operatorname{Ind} \overline{U}_n > n$, so that $\bigcup_{n=1}^{\infty} \overline{U}_n$ is a closed nearly S-like subspace of X. The contradiction with Theorem 7.1.14 and Proposition 7.1.21 shows that $S(X)$ is compact.

Consider now a closed subspace $F \subset X$ disjoint from $S(X)$ and suppose that $\operatorname{Ind} F = \infty$. From Theorems 2.3.6 and 2.3.14 it follows that $F \setminus I_n(X) \neq \emptyset$ for $n = 1, 2, \ldots$; for each n choose a point $x_n \in F \setminus I_n(X)$. One easily defines an increasing sequence $n_1 < n_2 < \ldots$ such that all points x_{n_k} are distinct, and one easily checks that the set $\{x_{n_1}, x_{n_2}, \ldots\}$ has no accumulation point in X. Arguing as in the first part of the proof one obtains a nearly S-like closed subspace of X, and a contradiction. Thus $\operatorname{Ind} F < \infty$. ∎

7.1.24. LEMMA. *If a hereditarily normal space X contains a subspace K such that* $\operatorname{trInd} K \leq \alpha \geq 0$ *and* $\operatorname{Ind} F < \infty$ *for each closed subspace $F \subset X$ disjoint from K, then* $\operatorname{trInd} X \leq \omega_0 + \alpha$.

If, moreover, $\operatorname{Ind}(X \setminus K) < \infty$, *then* $\operatorname{trInd} X < \omega_0$ *if* $\alpha < \omega_0$, *and* $\operatorname{trInd} X \leq \alpha$ *if* $\alpha \geq \omega_0$.

PROOF. We apply transfinite induction with respect to α. Assume that $\alpha_0 = 0$ or $\alpha_0 > 0$ and the first part of the lemma holds for all $\alpha < \alpha_0$; we shall show that it also holds for $\alpha = \alpha_0$. Consider a pair A, B of disjoint closed subsets of X, and enlarge

them to open sets $V_1, V_2 \subset X$ such that $\overline{V}_1 \cap \overline{V}_2 = \emptyset$. Take a partition L' in K between $K \cap \overline{V}_1$ and $K \cap \overline{V}_2$ with $\operatorname{trInd} L' = \alpha < \alpha_0$, and apply Lemma 1.2.9 and Remark 1.2.10 to obtain a partition L in the space X between A and B such that $K \cap L \subset L'$. If $\alpha_0 = 0$ we have $L' = \emptyset$, so that L is disjoint from K and thus $\operatorname{Ind} L < \infty$; hence $\operatorname{trInd} X \leq \omega_0 = \omega_0 + \alpha_0$. Assume now that $\alpha_0 > 0$. Since $K \cap L$ is a closed subspace of K, and thus of L', we have $\operatorname{trInd}(K \cap L) \leq a < \alpha_0$. By the inductive assumption applied to the space L and subspace $K \cap L$ we have $\operatorname{trInd} L \leq \omega_0 + \alpha < \omega_0 + \alpha_0$, so that $\operatorname{trInd} X \leq \omega_0 + \alpha_0$. Thus the first part of the lemma is proved.

Assume now that $\operatorname{Ind}(X \setminus K) < \infty$. If $\alpha < \omega_0$, we have $\operatorname{trInd} X < \omega_0$ by virtue of Theorem 2.2.5. If $\alpha \geq \omega_0$, the inequality $\operatorname{trInd} X \leq \alpha$ is obtained by transfinite induction starting with $\alpha = \omega_0$ and following the pattern of the first part of our proof. ■

Theorem 7.1.23 and Lemma 7.1.24 yield the following two theorems.

7.1.25. THEOREM. *A weakly paracompact strongly hereditarily normal space X has large transfinite dimension if and only if X contains a compact subspace K such that* $\operatorname{trInd} K < \infty$ *and* $\operatorname{Ind} F < \infty$ *for each closed subspace $F \subset X$ disjoint from K.* ■

7.1.26. THEOREM. *If a weakly paracompact strongly hereditarily normal space X has large transfinite dimension, then* $\operatorname{trInd} X < \omega_0 + \operatorname{trInd} S(X)$ *if* $S(X) \neq \emptyset$, *and* $\operatorname{trInd} X < \omega_0$ *if* $S(X) = \emptyset$. ■

From Theorems 7.1.23 and 7.1.26 and Corollary 7.1.18 we obtain

7.1.27. COROLLARY. *If a metrizable space X has large transfinite dimension, then it satisfies the inequality* $\operatorname{trInd} X < \omega_1$. ■

We shall now show that to establish the inequality $\operatorname{trInd} X \leq \alpha$ it often suffices to consider pairs of disjoint closed subsets of $S(X)$ rather than of X itself.

7.1.28. PROPOSITION. *If a weakly paracompact strongly hereditarily normal space X has large transfinite dimension, and for every pair A, B of disjoint closed subsets of $S(X)$ there exists in X a partition L between A and B such that* $\operatorname{trInd} L < \alpha \geq \omega_0$, *then* $\operatorname{trInd} X \leq \alpha$.

PROOF. Consider an arbitrary pair A_1, B_1 of disjoint closed subsets of X and take a partition L between $A = A_1 \cap S(X)$ and $B = B_1 \cap S(X)$ satisfying $\operatorname{trInd} L = \beta < \alpha$. Let U, W be open subsets of X such that

$$A \subset U, \quad B \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad X \setminus L = U \cup W.$$

Since $S(X) \cap (A_1 \setminus U) = \emptyset = S(X) \cap (B_1 \setminus W)$, there exists an open set $V \subset X$ satisfying

$$S(X) \subset V \subset \overline{V} \subset X \setminus [(A_1 \setminus U) \cup (B_1 \setminus W)].$$

Clearly, $L \cap \overline{V}$ is a partition in $\overline{V}$ between $A_1 \cap \overline{V}$ and $B_1 \cap \overline{V}$, and by virtue of Lemma 1.2.9 and Remark 1.2.10 there exists in X a partition L_1 between A_1 and B_1 such that $L_1 \cap \overline{V} \subset L \cap \overline{V} \subset L$. From the last inclusion it follows that $\operatorname{trInd}(L_1 \cap \overline{V}) \leq \beta$, and since $L_1 \setminus V$ is a closed subset of X disjoint from $S(X)$, we have $\operatorname{Ind}(L_1 \setminus V) < \infty$ by virtue of Theorem 7.1.23. As $L_1 \setminus (L_1 \cap \overline{V}) = L_1 \setminus \overline{V} \subset L_1 \setminus V$, it follows from the second

part of Lemma 7.1.24 that $\operatorname{trInd} L_1 < \omega_0$ if $\beta < \omega_0$, and $\operatorname{trInd} L_1 \leq \beta$ if $\beta \geq \omega_0$. In both cases $\operatorname{trInd} L_1 < \alpha$, so that $\operatorname{trInd} X \leq \alpha$. ■

From the last proposition it follows that in weakly paracompact perfectly normal spaces the subspace theorem for trInd can be strengthened.

7.1.29. THEOREM. *If a weakly paracompact perfectly normal space X and a subspace M of X both have large transfinite dimension, then* $\operatorname{trInd} M \leq \operatorname{trInd} X$.

PROOF. We can assume that $\alpha = \operatorname{trInd} X \geq \omega_0$ (see Theorem 2.3.6). Since every weakly paracompact perfectly normal space is hereditarily weakly paracompact (see Problem 2.1.D(b)), by virtue of Proposition 7.1.28 it suffices to show that for every pair A, B of disjoint closed subsets of $S(M)$ there exists in M a partition L between A and B such that $\operatorname{trInd} L < \alpha$. This follows, however, from the equality $\operatorname{trInd} X = \alpha$, because A and B, being closed in the compact space $S(M)$, are closed in X (see Theorem 7.1.23). ■

It is not known if in the last theorem the assumption of weak paracompactness can be dropped.

Now we shall discuss the relationship between the existence of large transfinite dimension and countable dimensionality.

Theorems 7.1.23, 5.5.10 and 7.1.8 yield

7.1.30. THEOREM. *Every metrizable space which has large transfinite dimension is countable-dimensional.* ■

By introducing slight modifications in the proof of Theorem 7.1.9 we obtain (cf. Problem 7.2.E(b))

7.1.31. THEOREM. *Every countable-dimensional compact metrizable space has large transfinite dimension.* ■

The last two theorems and Corollary 7.1.10 yield

7.1.32. COROLLARY. *For every compact metrizable space X the following conditions are equivalent:*

(i) *The space X has small transfinite dimension.*

(ii) *The space X has large transfinite dimension.*

(iii) *The space X is countable-dimensional.* ■

We conclude this section with a description of Smirnov's spaces S_α, strongly countable-dimensional compact metrizable spaces with $\operatorname{trInd} S_\alpha = \alpha$ for $\alpha < \omega_1$.

7.1.33. EXAMPLE. *Smirnov's spaces* $S_0, S_1, \ldots, S_\alpha, \ldots$, $\alpha < \omega_1$, are defined by transfinite induction: $S_0 = \{0\}$ is a one-point space, $S_\alpha = S_\beta \times I$ for $\alpha = \beta + 1$, and if α is a limit ordinal number, then $S_\alpha = \omega \bigoplus_{\beta<\alpha} S_\beta$ is the one-point compactification of the sum of all previously defined S_β's. Obviously, Smirnov's spaces are compact, metrizable and strongly countable-dimensional. We shall now prove that for every $\alpha < \omega_1$ the equality $\operatorname{trInd} S_\alpha = \alpha$ holds.

Recall that for each ordinal number α there exist a uniquely determined limit number $\lambda(\alpha) \geq 0$ and an integer $n(\alpha) \geq 0$ such that $\alpha = \lambda(\alpha) + n(\alpha)$. We shall show

first that for each ordinal number α and for each compact metrizable space Y with $\operatorname{Ind} Y \leq n(\alpha)$ we have

$$\operatorname{trInd}(S_{\lambda(\alpha)} \times Y) \leq \alpha. \tag{1}$$

Since S_α is homeomorphic to $S_{\lambda(\alpha)} \times I^{n(\alpha)}$, where $I^0 = \{0\}$, from (1) it follows that $\operatorname{trInd} S_\alpha \leq \alpha$ for $\alpha < \omega_1$.

For $\alpha < \omega_0$ we have $\lambda(\alpha) = 0$ and $S_{\lambda(\alpha)} = \{0\}$, so that in this case for each compact metrizable space Y with $\operatorname{Ind} Y \leq n(\alpha)$ we have $\operatorname{Ind}(S_{\lambda(\alpha)} \times Y) \leq \alpha$, i.e., (1) holds. Assume now that (1) holds for all $\alpha < \alpha_0 \geq \omega_0$; let $\lambda(\alpha_0) = \lambda_0 \geq \omega_0$ and $n(\alpha_0) = n_0$. Consider a compact metrizable space Y with $\operatorname{Ind} Y \leq n_0$, a closed set $A \subset S_{\lambda_0} \times Y$, and an open set $V \subset S_{\lambda_0} \times Y$ which contains the set A. The space S_{λ_0} is the one-point compactification of the sum $\bigoplus_{\beta<\lambda_0} S_\beta$; let x_0 denote the compactifying point of S_{λ_0}. Since the subspace $A \cap (\{x_0\} \times Y)$ of $S_{\lambda_0} \times Y$ is compact, there exist open sets $W_0 \subset S_{\lambda_0}$ and $W \subset Y$ such that

$$A \cap (\{x_0\} \times Y) \subset W_0 \times W \subset V;$$

without loss of generality we can assume that $W_0 = S_{\lambda_0} \setminus (S_{\beta_1} \cup S_{\beta_2} \cup \ldots \cup S_{\beta_k})$, where $\beta_i < \lambda_0$ for $i \leq k$, and that $\operatorname{Ind} \operatorname{Fr} W \leq n_0 - 1$. Removing from W_0 further S_β's if necessary, we can secure that $A \cap (W_0 \times Y) \subset W_0 \times W$. The subspace $Z = (S_{\lambda_0} \setminus W_0) \times Y = (S_{\beta_1} \cup S_{\beta_2} \cup \ldots \cup S_{\beta_k}) \times Y$ is open-and-closed in $S_{\lambda_0} \times Y$, and since $\operatorname{trInd} Z < \lambda_0$ by virtue of the inductive assumption, there exists an open set $U_1 \subset S_{\lambda_0} \times Y$ such that $A \cap Z \subset U_1 \subset V \cap Z$ and $\operatorname{trInd} \operatorname{Fr} U_1 < \lambda_0$. For $U_2 = W_0 \times W$ we have $\operatorname{Fr} U_2 = W_0 \times \operatorname{Fr} W \subset S_{\lambda_0} \times \operatorname{Fr} W$, so that $\operatorname{trInd} \operatorname{Fr} U_2 \leq \operatorname{trInd}(S_{\lambda_0} \times \operatorname{Fr} W) \leq \lambda_0 + (n_0 - 1) < \alpha_0$ if $n_0 \geq 1$, and $\operatorname{Fr} U_2 = \emptyset$ if $n_0 = 0$. The open set $U = U_1 \cup U_2 = U_1 \oplus U_2 \subset S_{\lambda_0} \times Y$ satisfies

$$A \subset U \subset V \quad \text{and} \quad \operatorname{trInd} \operatorname{Fr} U < \alpha_0,$$

so that $\operatorname{trInd} S_{\lambda_0} \times Y \leq \alpha_0$, and (1) is established.

It remains to show that $\operatorname{trInd} S_\alpha \geq \alpha$ for every $\alpha < \omega_1$. To that end, we shall establish by transfinite induction with respect to α the following property of Smirnov's spaces, where $(A_1^n, B_1^n), (A_2^n, B_2^n), \ldots, (A_n^n, B_n^n)$ denote the pairs of opposite faces of I^n:

(2) For each $n \geq 1$ and any closed sets $L_1, L_2, \ldots, L_n$, where L_i is a partition in $S_\alpha \times I^n$ between $S_\alpha \times A_i^n$ and $S_\alpha \times B_i^n$, we have $\operatorname{trInd}(\bigcap_{i=1}^n L_i) \geq \alpha$.

From (2) it follows that $\operatorname{trInd} S_\alpha \geq \alpha$ for $\alpha < \omega_1$. Indeed, (2) implies that for every β and any partition L in $S_\beta \times I$ between the disjoint closed subsets $S_\beta \times \{0\}$ and $S_\beta \times \{1\}$ we have $\operatorname{trInd} L \geq \beta$, so that $\operatorname{trInd} S_{\beta+1} > \beta$. Thus if $\alpha = \beta + 1$ we have $\operatorname{trInd} S_\alpha \geq \alpha$; if α is a limit number, then for every $\beta < \alpha$ we have $\operatorname{trInd} S_\alpha \geq \operatorname{trInd} S_{\beta+1} > \beta$, because the space $S_{\beta+1}$ is a closed subspace of S_α, so that also in this case $\operatorname{trInd} S_\alpha \geq \alpha$.

The validity of (2) for $\alpha = 0$ follows from Theorem 1.8.1. Assume now that (2) holds for all $\alpha < \alpha_0 \geq 1$. Consider an $n \geq 1$ and suppose that for $i = 1, 2, \ldots, n$ there exists a partition L_i in $S_{\alpha_0} \times I^n$ between $S_{\alpha_0} \times A_i^n$ and $S_{\alpha_0} \times B_i^n$ such that $\operatorname{trInd}(\bigcap_{i=1}^n L_i) = \alpha < \alpha_0$. If α_0 is a limit number, then $\alpha + 1 < \alpha_0$ and in $S_{\alpha+1} \times I^n \subset S_{\alpha_0} \times I^n$ there are partitions $L_i' = (S_{\alpha+1} \times I^n) \cap L_i$ between $S_{\alpha+1} \times A_i^n$ and $S_{\alpha+1} \times B_i^n$ such that $\operatorname{trInd}(\bigcap_{i=1}^n L_i') \leq \alpha < \alpha + 1$, a contradiction with the inductive assumption. If $\alpha_0 = \beta + 1$, we have $S_{\alpha_0} \times I^n = S_\beta \times (I \times I^n) = S_\beta \times I^{n+1}$, and in $M = \bigcap_{i=1}^n L_i$ there exists

a partition L_0' between $M \cap A$ and $M \cap B$, where $A = S_\beta \times \{0\} \times I^n$ and $B = S_\beta \times \{1\} \times I^n$, such that $\operatorname{trInd} L_0' < \alpha$. By Lemma 1.2.9 there exists in $S_{\alpha_0} \times I^n = S_\beta \times I^{n+1}$ a partition L_0 between A and B such that $M \cap L_0 \subset L_0'$, and we have $\operatorname{trInd}(\bigcap_{i=0}^n L_i) < \alpha \leq \beta$, which also contradicts the inductive assumption. Thus (2) holds for $\alpha = \alpha_0$, i.e., the proof of (2) is complete.

Let us note that by Theorem 7.1.19 we have $\operatorname{trind} S_\alpha \leq \alpha$ for every $\alpha < \omega_1$. There exist, however, α's such that $\operatorname{trind} S_\alpha < \alpha$ (see the remark to Problem 7.1.G(e)). In Example 7.2.12 we shall show that for every ordinal number $\alpha_0 < \omega_1$ there exist α's such that $\operatorname{trind} S_\alpha > \alpha_0$. ■

Historical and bibliographic notes

In [1925] Urysohn hinted at the possibility of extending the small transfinite dimension ind by letting it assume ordinal values; a formal definition appeared in Hurewicz [1928], where trind was first studied. Theorems 7.1.3, 7.1.4 and 7.1.6 can be found in Toulmin [1954], Corollary 7.1.7 – in Hurewicz [1928]. Hurewicz's paper also contains Theorems 7.1.8 and 7.1.9 (the latter is proved under the assumption of compactness but the proof works for completely metrizable spaces). An example of a perfectly normal space which has small transfinite dimension and yet is not countable-dimensional was given by E. Pol in [1991].

The large transfinite dimension was introduced by Smirnov in [1959]; in that paper Theorem 7.1.17, Theorem 7.1.19 and Proposition 7.1.21 were proved. Theorems 7.1.14 and 7.1.15 can be found in Landau [1969]. Proposition 7.1.22 was stated in Wenner [1972], it is implicit in Levšenko's paper [1965]. Theorem 7.1.23 and Lemma 7.1.24 (for metrizable spaces), as well as Corollary 7.1.27, were established by Smirnov in [1962] (announcement in [1961]); Lemma 7.1.24 is formulated there in a slightly weaker form, the present formulation is due to Engelking [1980] and Fedorčuk [1987]. Proposition 7.1.28 and Theorem 7.1.29 were announced by Luxemburg in [1973] (for metrizable spaces). Theorems 7.1.30 and 7.1.31 are due to Smirnov, the former was proved in [1962] (announcement in [1961]) and the latter in [1959]. Example 7.1.33 is taken from Smirnov's paper [1959].

Problems

7.1.A. (a) Show that if a weakly paracompact strongly hereditarily normal space X contains no closed nearly S-like subspace, then $S(X)$ is compact and $\operatorname{Ind} F < \infty$ for each closed subspace $F \subset X$ disjoint from $S(X)$.

(b) Show that if a separable metrizable space X which has small transfinite dimension contains a closed nearly S-like subspace, then X also contains a closed S-like subspace.

7.1.B. (a) Show that a metrizable space X has large transfinite dimension if and only if X is countable-dimensional and contains no closed nearly S-like subspace.

Deduce that a metrizable space X has large transfinite dimension if and only if X contains a countable-dimensional compact subspace K such that each closed subspace $F \subset K$ disjoint from K is finite-dimensional.

Hint. Apply Theorem 5.5.21.

(b) Note that if M is a subspace of a metrizable space X which has large transfinite dimension, then M has large transfinite dimension if and only if it contains no closed nearly S-like subspace.

(c) Show that a separable metrizable space X has large transfinite dimension if and only if X has small transfinite dimension and contains no closed S-like subspace.

7.1.C. (a) Prove that if a separable metrizable space X has large transfinite dimension and $\operatorname{trind} X = \omega_0$, then $\operatorname{trInd} X = \omega_0$.

Hint. Apply Proposition 7.1.28.

(b) Show that for every compact strongly hereditarily normal space X with $\operatorname{trind} X = \omega_0$ we have $\operatorname{trInd} X = \omega_0$.

Hint. Apply Theorem 2.4.4.

7.1.D (Luxemburg [1973]). Show that if a weakly paracompact strongly hereditarily normal space X has large transfinite dimension, and for every point $x \in S(X)$ and each closed set $B \subset S(X)$ such that $x \notin B$ there exists in X a partition L between x and B such that $\operatorname{trind} L < \alpha \geq \omega_0$, then $\operatorname{trind} X \leq \alpha$.

7.1.E. (a) (Zarelua [1963] (announcement [1961])) Prove that every open cover $\mathcal{U}$ of a strongly metrizable space X has an open σ-discrete refinement $\mathcal{V}$ such that for each $V \in \mathcal{V}$ one can find a $U \in \mathcal{U}$ with $\operatorname{Fr} V \subset \operatorname{Fr} U$.

(b) (Smirnov [1962] (announcement [1961])) Show that every strongly metrizable space which has small transfinite dimension is countable-dimensional.

7.1.F. (a) (Levšenko [1961]) Prove that every strongly paracompact space X which has small transfinite dimension is weakly infinite-dimensional.

Hint. Apply transfinite induction. When $\operatorname{trind} X = 0$ use Theorem 2.4.2. Then take an X with $\operatorname{trind} X = \alpha > 0$ and note that by virtue of Lemmas 6.1.4 and 6.1.5 it suffices to show that for every pair (A_1, B_1) of disjoint closed subsets of X there exists a partition L_1 between A_1 and B_1 with $\operatorname{trind} L_1 < \alpha$. For each $x \in X$ choose open sets $U_x, V_x \subset X$ with $\operatorname{trind} \operatorname{Fr} U_x < \alpha$ and $\operatorname{trind} \operatorname{Fr} V_x < \alpha$ such that $x \in U_x \subset \overline{U}_x \subset V_x$ and $\overline{U}_x \cap A_1 = \emptyset$ or $\overline{U}_x \cap B_1 = \emptyset$. Observe that X can be represented as the union of open-and-closed subspaces X each of which is contained in the union of countably many V_x's; it suffices to prove that in each X_s there exists a partition L_s between $A_1 \cap X_s$ and $B_1 \cap X_s$ with $\operatorname{trind} L_s < \alpha$, so that one can assume that for a sequence $x_1, x_2, \ldots$ of points of X we have $X = \bigcup_{i=1}^{\infty} V_{x_i}$. Check that the open sets $W_i = V_{x_i} \setminus \bigcup_{j<i} \overline{U}_{x_j}$), where $i = 1, 2, \ldots$, constitute a locally finite cover of the space X and that the boundary L_1 of $\bigcup\{W_i : A_1 \cap W_i \neq \emptyset\}$ is a partition between A_1 and B_1 which is contained in the union $\bigcup_{i=1}^{\infty}(\operatorname{Fr} U_{x_i} \cup \operatorname{Fr} V_{x_i})$.

(b) (Čigogidze [1982]; for hereditarily normal spaces, Smirnov [1962] (announcement [1961])) Prove that every normal space X which has large transfinite dimension is weakly infinite-dimensional.

Hint. Show inductively that X is S-weakly infinite-dimensional (see Problem 6.1.E); apply Lemmas 6.1.4 and 6.1.5.

7.1.G (Luxemburg [1973a]). Note that there exists a base $\mathcal{B} = \{U_i\}_{i=1}^{\infty}$ for the cube I^3, where U_i's are open cubes with $\delta(U_i) < 1$ and faces paralled to the faces of I^3, such that $\operatorname{ord}(\{\operatorname{Fr} U_i\}_{i=1}^{\infty}) \leq 2$ and $\lim \delta(U_i) = 0$.

(b) Let (A, B) be a pair of opposite faces of I^3. For $n = 1, 2, \ldots$ define a partition L_n between A and B such that for each $i \leq n$ the intersection $L_n \cap \operatorname{Fr} U_i$, where $\mathcal{B} = \{U_i\}_{i=1}^{\infty}$ is the base of part (a), can be represented as the union of finitely many pairwise disjoint closed sets of diameter less than $1/n$.

Hint (R. Pol [1982]). For each $i \leq n$ take a finite cover $\mathcal{B}_i \subset \mathcal{B} \setminus \{U_1, U_2, \ldots, U_n\}$ of I^3 with $\operatorname{mesh} \mathcal{B}_i < 1/n$ such that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ whenever $i \neq j$ and note that the sets $C_i = \operatorname{Fr} U_i \cap \bigcup\{\operatorname{Fr} U_k : U_k \in \mathcal{B}_i\}$, where $i \leq n$, are pairwise disjoint and none of them intersects both A and B. Take for L_n an arbitrary partition between the disjoint closed sets

$$A \cup \bigcup\{C_i : C_i \cap A \neq \emptyset\} \quad \text{and} \quad B \cup \{C_i : C_i \cap A = \emptyset\}.$$

(c) Show that if a metric space X can be represented as the union $F \cup \bigcup_{n=1}^{\infty} E_n$, where F is closed with $\operatorname{ind} F < \infty$, and E_n's are open-and-closed with $\operatorname{ind} E_n < \infty$ and $\lim \delta(E_n) = 0$, then $\operatorname{trind} X \leq \omega_0$.

(d) Prove that the subspace $L = (\{x_0\} \times I^3) \cup \bigcup_{n=1}^{\infty}(I^n \times L_n)$ of S_{ω_0+3}, where L_n's are the partitions of part (b), satisfies the equality $\operatorname{trind} L = \omega_0 + 1$.

Hint. In the proof that $\operatorname{trind} L \leq \omega_0 + 1$ apply part (c).

(e) Prove that the space L defined in part (d) satisfies the inequality $\operatorname{trInd} L \geq \omega_0 + 2$.

Hint. Show that if K_i is a partition in L between $L \cap (S_{\omega_0} \times A_i)$ and $L \cap (S_{\omega_0} \times B_i)$ for $i = 1, 2$, where (A_1, B_1) and (A_2, B_2) are the two pairs of opposite faces of I^3 distinct from (A, B), then $\operatorname{trInd}(K_1 \cap K_2) \geq \omega_0$.

Remark. Luxemburg proved in [1973a] that $\operatorname{trind} S_{\omega_0+3} = \omega_0 + 2$.

7.1.H (Hattori [1987]). We shall say that a family $\mathcal{A}$ of subsets of a topological space X is *d-point-finite* if $\operatorname{ord} \mathcal{A}|F$ is finite for each discrete closed subspace F of X. A mapping $f: X \to Y$ is of *d-finite order* if for each discrete closed subspace F of Y there exists an integer $n_F \geq 1$ such that $|f^{-1}(y)| \leq n_F$ for every $y \in F$.

Prove that for every metrizable space X the following conditions are equivalent:

(1) The space X has large transfinite dimension.

(2) For every sequence $(A_1, B_1), (A_2, B_2), \ldots$ of pairs of disjoint closed subsets of X there exist closed sets $L_1, L_2, \ldots$ such that L_i is a partition between A_i and B_i and the family $\{L_i\}_{i=1}^{\infty}$ is d-point-finite.

(3) The space X has a σ-discrete base $\mathcal{B}$ such that the family $\{\operatorname{Fr} U : U \in \mathcal{B}\}$ is d-point-finite.

(4) The space X is the image of a metrizable space Z with $\operatorname{Ind} Z \leq 0$ and $w(Z) \leq w(X)$ under a closed mapping of d-finite order.

Hint. To prove the implication (1)⇒(2) use Theorem 7.1.23 and Lemma 4.2.9. The implication (2)⇒(3) follows from Lemma 5.1.12. In the proof of the implication

(3)⇒(4) apply the construction from the proof of Lemma 5.4.13, and in the proof of the implication (4)⇒(1) apply Problem 7.1.B(a).

Remark. A characterization of metrizable spaces which have large transfinite dimension similar to condition (2) was given by Engelking and R. Pol in [1983]; one can also find there a characterization of separable metrizable spaces which have small transfinite dimension similar to condition (3).

7.1.I (Smirnov [1959]). Apply Smirnov's spaces to prove that there is no universal space either for the class of all countable-dimensional compact metrizable spaces or for the class of all strongly countable-dimensional compact metrizable spaces (cf. Problem 6.1.I(d)).

7.1.J (R. Pol [1983]). Prove that every separable completely metrizable space which topologically contains all Smirnov's spaces S_α contains topologically the Hilbert cube $I^{\aleph_0}$.

Hint. By Problem 6.1.I(d) it suffices to prove that every compact metrizable space X with countably many components each of which is homeomorphic to a finite-dimensional cube is embeddable in a Smirnov's space S_α. To that end, first note that by identifying each component of X to a point one obtains a continuous mapping with finite-dimensional fibres which maps X onto a countable compact space X' (cf. [GT], Exercise 6.2.F). Then apply transfinite induction with respect to the Cantor–Bendixson index of X'.

7.2. The separation, sum, enlargement, completion and universal space theorems for the transfinite dimensions trind and trInd

In this section we continue our study of transfinite dimensions and prove the theorems stated in the title. They hold both for trind and trInd, with the exception of the universal space theorem which holds only for the small transfinite dimension.

We start with the separation theorems.

7.2.1. THE SEPARATION THEOREM FOR trind. *If X is a hereditarily normal space and M is a subspace of X with* $\operatorname{trind} M \leq \alpha \geq 0$, *then for every point $x \in M$ and each closed set $B \subset X$ with $x \notin B$ there exists a partition L between x and B such that* $\operatorname{trind}(M \cap L) < \alpha$.

PROOF. Take an open set $V_2 \subset X$ such that $x \notin \overline{V}_2$ and $B \subset V_2$ and consider a partition L' in M between x and $M \cap \overline{V}_2$ with $\operatorname{trind} L' < \operatorname{trind} M \leq \alpha$. Let U', W' be open subsets of M satisfying the conditions

$$x \in U', \quad M \cap \overline{V}_2 \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad M \setminus L' = U' \cup W'.$$

Since $x \notin \overline{M \setminus U'}$, there exists an open set $V_1 \subset X$ such that $x \in V_1 \subset \overline{V}_1 \subset X \setminus \overline{V}_2$ and $M \cap \overline{V}_1 \subset U'$. Clearly L' is a partition in M between $M \cap \overline{V}_1$ and $M \cap \overline{V}_2$, so that by Lemma 1.2.9 and Remark 1.2.10 there exists a partition L in the space X between x and B which satisfies the inclusion $M \cap L \subset L'$. From Theorem 7.1.3 it follows that $\operatorname{trind}(M \cap L) \leq \operatorname{trind} L' < \alpha$. ■

7.2.2. REMARK. One easily sees that if the subspace M in Theorem 7.2.1 is closed, then the assumption that $x \in M$ can be dropped (see Problem 7.2.A). ■

The following theorem is a consequence of Lemma 1.2.9, Remark 1.2.10 and Theorem 7.1.14.

7.2.3. THE SEPARATION THEOREM FOR trInd. *If X is a hereditarily normal space and M is a subspace of X with* $\operatorname{trInd} M \leq \alpha \geq 0$, *then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that* $\operatorname{trInd}(M \cap L) < \alpha$. ■

We now pass to sum theorems. Observe first that since Smirnov's spaces S_α defined in Example 7.1.33 (see also Example 7.2.12 below) are strongly countable-dimensional, there are no countable sum theorems for transfinite dimensions. There are, however, sum theorems for finite unions, and we are now going to prove them. To make the statements simpler, we shall consider only the case of two subspaces.

In the proofs the following lemmas will be used.

7.2.4. LEMMA. *If a regular space X is the union of two closed subspaces F_1 and F_2, where* $\operatorname{trind} F_2 \leq \alpha \geq 0$, *then for every point $x \in X$ and each closed set $B \subset X$ with $x \notin B$ there exists a partition L between x and B which is the union of two subspaces F_1' and F_2' such that F_i' is a closed subspace of F_i for $i = 1, 2$, and* $\operatorname{trind} F_2' < \alpha$.

PROOF. The lemma is obvious if $x \notin F_2$. Assume that $x \in F_2$, take an open set $V_2 \subset X$ such that $x \notin \overline{V}_2$ and $B \subset V_2$, and consider a partition F_2' in F_2 between x and $F_2 \cap \overline{V}_2$ with $\operatorname{trind} F_2' < \operatorname{trind} F_2 \leq \alpha$. Let U', W' be open subsets of F_2 satisfying the conditions

$$x \in U', \quad F_2 \cap \overline{V}_2 \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad F_2 \setminus F_2' = U' \cup W'.$$

Since $x \notin F_2 \setminus U'$, there exists an open set $V_1 \subset X$ such that $x \in V_1 \subset X \setminus \overline{V}_2$ and $F_2 \cap V_1 \subset U'$. One easily checks that the sets

$$U = V_1 \cup [(X \setminus F_1) \cap U'] \quad \text{and} \quad W = V_2 \cup [(X \setminus F_1) \cap W']$$

are open in X and disjoint. Since $U \cup W = V_1 \cup V_2 \cup [(X \setminus F_1) \setminus F_2']$, the set $L = F_1' \cup F_2'$, where $F_1' = F_1 \setminus (V_1 \cup V_2)$, satisfies $(U \cup W) \cap L = \emptyset$ and $U \cup W \cup L = X$. Thus L is the required partition. ■

7.2.5. LEMMA. *If a normal space X is the union of two subspaces F_1 and F_2, where F_1 is closed, F_2 is normal and* $\operatorname{trInd} F_2 \leq \alpha \geq 0$, *then for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B which is the union of two subspaces F_1' and F_2' such that F_i' is a closed subspace of F_i for $i = 1, 2$ and* $\operatorname{trInd} F_2' < \alpha$.

PROOF. Take open sets $V_1, V_2 \subset X$ such that $A \subset V_1, B \subset V_2$ and $\overline{V}_1 \cap \overline{V}_2 = \emptyset$ and consider a partition F_2' in F_2 between $F_2 \cap \overline{V}_1$ and $F_2 \cap \overline{V}_2$ with $\operatorname{trInd} F_2' < \operatorname{trInd} F_2 \leq \alpha$. Let U', W' be open subsets of F_2 satisfying the conditions

$$F_2 \cap \overline{V}_1 \subset U', \quad F_2 \cap \overline{V}_2 \subset W', \quad U' \cap W' = \emptyset \quad \text{and} \quad F_2 \setminus F_2' = U' \cup W'.$$

As in the proof of Lemma 7.2.4, one easily checks that the sets

$$U = V_1 \cup [(X \setminus F_1) \cap U'] \quad \text{and} \quad W = V_2 \cup [(X \setminus F_1) \cap W']$$

are open and disjoint, and that $L = X \setminus (U \cup W) = F_1' \cup F_2'$, where $F_1' = F_1 \setminus (V_1 \cup V_2)$, is the required partition. ■

The first sum theorem for trind and the first sum theorem for trInd can be proved in a strictly parallel way, with an application of Lemma 7.2.4 and Lemma 7.2.5 respectively. We shall only give a proof for the case of the small transfinite dimension; the reader will easily find out the modifications necessary to obtain a proof for the case of the large transfinite dimension.

7.2.6. THE FIRST SUM THEOREM FOR trind. *If a regular space X is the union of two closed subspaces F_1 and F_2 such that* $\operatorname{trind} F_1 \leq \alpha_1 = \lambda_1 + n_1$ *and* $\operatorname{trind} F_2 \leq \alpha_2 = \lambda_2 + n_2 \geq \alpha_1$, *where λ_i is a limit number and n_i is a non-negative integer for $i = 1, 2$, then*

$$\operatorname{trind} X \leq \begin{cases} \lambda_2 + n_2 & \text{if } \lambda_1 < \lambda_2, \\ \lambda_2 + n_2 + n_1 + 1 & \text{if } \lambda_1 = \lambda_2. \end{cases} \tag{1}$$

PROOF. Suppose that the theorem does not hold and consider the smallest α_2 such that it is not valid for some $\alpha_1 \leq \alpha_2$ and a regular space X. Taking next the smallest possible α_1 we obtain ordinal numbers β_1, β_2 such that the theorem holds whenever $\alpha_2 < \beta_2$ or $\alpha_2 = \beta_2$ and $\alpha_1 < \beta_1$, but there exists a regular space $X = F_1 \cup F_2$ with F_1, F_2 closed, $\operatorname{trind} F_1 \leq \beta_1 = \lambda_1 + n_1$ and $\operatorname{trind} F_2 \leq \beta_2 = \lambda_2 + n_2 \geq \beta_1$ such that (1) does not hold.

Note first that from Lemma 7.2.4 it follows that $\beta_2 \geq 1$. Now, consider a point $x \in X$ and a closed set $B \subset X$ with $x \notin B$. By Lemma 7.2.4 there exists a partition $L = F_1' \cup F_2'$ between x and B such that F_i' is a closed subspace of F_i for $i = 1, 2$ and $\operatorname{trind} F_2' \leq \alpha' = \lambda' + n' < \beta_2$, where λ' is a limit number and n' a non-negative integer. Clearly $\operatorname{trind} F_1' \leq \beta_1$.

Assume that $\lambda_1 < \lambda_2$. If $n_2 = 0$ then $\lambda' < \lambda_2$, so that $\operatorname{trind} L < \max(\lambda_1, \lambda') + \omega_0 \leq \lambda_2 = \lambda_2 + n_2$. If $n_2 \geq 1$ we have $\alpha' \leq \lambda_2 + (n_2 - 1)$, so that $\operatorname{trind} L \leq \lambda_2 + (n_2 - 1) < \lambda_2 + n_2$. Thus, if $\lambda_1 < \lambda_2$ we have $\operatorname{trind} X \leq \lambda_2 + n_2$.

Assume now that $\lambda_1 = \lambda_2$. If $n_2 = 0$ we have $n_1 = 0$ and $\lambda' < \lambda_2 = \lambda_1$, so that $\operatorname{trind} L \leq \lambda_1 + n_1 = \lambda_2 + n_2 + n_1$. If $n_2 \geq 1$ we have $\alpha' \leq \lambda_2 + (n_2 - 1)$, so that $\operatorname{trind} L \leq \lambda_2 + (n_2 - 1) + n_1 + 1 = \lambda_2 + n_2 + n_1$. Thus, if $\lambda_1 = \lambda_2$ we have $\operatorname{trind} X \leq \lambda_2 + n_2 + n_1 + 1$.

It follows that (1) holds, i.e., our supposition leads to a contradiction. ■

7.2.7. THE FIRST SUM THEOREM FOR trInd. *If a normal space X is the union of two closed subspaces F_1 and F_2 such that* $\operatorname{trInd} F_1 \leq \alpha_1 = \lambda_1 + n_1$ *and* $\operatorname{trInd} F_2 \leq \alpha_2 = \lambda_2 + n_2 \geq \alpha_1$, *where λ_i is a limit number and n_i is a non-negative integer for $i = 1, 2$, then*

$$\operatorname{trInd} X \leq \begin{cases} \lambda_2 + n_2 & \text{if } \lambda_1 < \lambda_2, \\ \lambda_2 + n_2 + n_1 + 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$ ■

The evaluations of $\operatorname{trind} X$ and $\operatorname{trInd} X$ in Theorems 7.2.6 and 7.2.7 cannot be improved; indeed, for Smirnov's space S_{ω_0+1} we have $\operatorname{trind} S_{\omega_0+1} = \operatorname{trInd} S_{\omega_0+1} = \omega_0 + 1$ (see Example 7.1.33 and Proposition 7.1.22), and yet S_{ω_0+1} can be represented as the union of two closed subspaces F_1 and F_2 such that $\operatorname{trind} F_i = \operatorname{trInd} F_i = \omega_0$ for $i = 1, 2$ (see Problem 7.2.B; cf. Problems 4.1.B and 2.2.C(c)).

Theorems 7.2.6 and 7.2.7 yield the following two corollaries.

7.2.8. COROLLARY. *If a regular space X can be represented as the union $F_1 \cup F_2 \cup \ldots \cup F_m$ of closed subspaces such that* $\operatorname{trind} F_i < \lambda$ *for* $i = 1, 2, \ldots, m$, *where λ is a limit number, then* $\operatorname{trind} X < \lambda$. ■

7.2.9. COROLLARY. *If a normal space X can be represented as the union $F_1 \cup F_2 \cup \ldots \cup F_m$ of closed subspaces such that* $\operatorname{trInd} F_i < \lambda$ *for* $i = 1, 2, \ldots, m$, *where λ is a limit number, then* $\operatorname{trInd} X < \lambda$. ■

In the next theorem we give a simple evaluation of $\operatorname{trInd} X$ in terms of $\operatorname{trind} X$ for a compact space X. Let us recall that for an ordinal number α the product $\omega_0 \cdot \alpha$ is defined as the order type of the set $N \times A$, where N is the set of positive integers and A is the set of all ordinal numbers less than α (both with their natural order), ordered anti-lexicographically; one easily sees that $\omega_0 \cdot \alpha$ is a limit number; in addition, we define $\omega_0 \cdot \infty = \infty$.

7.2.10. THEOREM. *For every compact space X we have* $\operatorname{trInd} X \leq \omega_0 \cdot \operatorname{trind} X$.

PROOF. The theorem is obvious if $\operatorname{trind} X = \infty$. We shall show by transfinite induction that if $\operatorname{trind} X \leq \alpha$, then $\operatorname{trInd} X \leq \omega_0 \cdot \alpha$.

For $\alpha = 0$ the statement follows from Theorem 1.6.5. Assume that for every compact space F with $\operatorname{trind} F < \alpha \geq 1$ we have $\operatorname{trInd} F \leq \omega_0 \cdot \operatorname{trind} F$ and consider a compact space X such that $\operatorname{trind} X = \alpha$. Let A, B be a pair of disjoint closed subsets of X. Each $x \in A$ has a neighbourhood $U_x \subset X \setminus B$ with $\operatorname{trind} \operatorname{Fr} U_x < \alpha$; by the inductive assumption we have $\operatorname{trInd} \operatorname{Fr} U_x \leq \omega_0 \cdot \operatorname{trind} \operatorname{Fr} U_x < \omega_0 \cdot \alpha$. Since A is compact, there exist in A finitely many points $x_1, x_2, \ldots, x_m$ such that $A \subset U = U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_m}$, and since $\operatorname{Fr} U \subset \operatorname{Fr} U_{x_1} \cup \operatorname{Fr} U_{x_2} \cup \ldots \cup \operatorname{Fr} U_{x_m}$, by Corollary 7.2.9 and Theorem 7.1.14 we have $\operatorname{trInd} \operatorname{Fr} U < \omega_0 \cdot \alpha$, so that $\operatorname{trInd} X \leq \omega_0 \cdot \alpha$. ■

Theorems 7.1.19 and 7.2.10 yield

7.2.11. THEOREM. *A compact space X has large transfinite dimension if and only if X has small transfinite dimension.* ■

Now we return to Smirnov's spaces S_α.

7.2.12. EXAMPLE. We shall show that for every ordinal number $\alpha_0 < \omega_1$ there exist α's such that $\operatorname{trind} S_\alpha > \alpha_0$.

Indeed, by Theorem 7.2.10 the assumption that $\operatorname{trind} S_\alpha \leq \alpha_0$ for each $\alpha < \omega_1$ implies that $\operatorname{trInd} S_\alpha \leq \omega_0 \cdot \alpha_0 < \omega_1$, and this contradicts the equality $\operatorname{trInd} S_\alpha = \alpha$ established in Example 7.1.33. ■

In the class of hereditarily normal spaces Theorems 7.2.6 and 7.2.7 can be strengthened to sum theorems which take into account the dimension of the intersection $F_1 \cap F_2$. These stronger theorems can be proved in a parallel way, with an application of Theorem 7.2.1 and Theorem 7.2.3 respectively. We shall only give a proof for the case of the small transfinite dimension.

7.2.13. THE SECOND SUM THEOREM FOR trind. *If a hereditarily normal space X is the union of two closed subspaces F_1 and F_2 such that* $\operatorname{trind} F_1 \le \operatorname{trind} F_2 \le \alpha_2 = \lambda_2 + n_2$ *and* $\operatorname{trind}(F_1 \cap F_2) \le \alpha_1 = \lambda_1 + n_1 \le \alpha_2$, *where λ_i is a limit number and n_i is a non-negative integer for $i = 1, 2$, then*

$$\operatorname{trind} X \le \begin{cases} \lambda_2 + n_2 & \text{if } \lambda_1 < \lambda_2, \\ \lambda_2 + n_2 + n_1 + 1 & \text{if } \lambda_1 = \lambda_2. \end{cases} \tag{2}$$

PROOF. Suppose that the theorem does not hold. Arguing as in the proof of Theorem 7.2.6 we find ordinal numbers β_1, β_2 such that the theorem holds whenever $\alpha_2 < \beta_2$ or $\alpha_2 = \beta_2$ and $\alpha_1 < \beta_1$, but there exists a hereditarily normal space $X = F_1 \cup F_2$ with F_1, F_2 closed, $\operatorname{trind} F_1 \le \operatorname{trind} F_2 \le \beta_2 = \lambda_2 + n_2$ and $\operatorname{trind}(F_1 \cap F_2) \le \beta_1 = \lambda_1 + n_1 \le \beta_2$ such that (2) does not hold.

Note first that from Theorem 7.2.1 and Remark 7.2.2 it follows that $\beta_2 \ge 1$. Now, consider a point $x \in X$ and a closed set $B \subset X$ with $x \notin B$.

Assume that $\lambda_1 < \lambda_2$. By Theorem 7.2.1 and Remark 7.2.2 there exists a partition L_1 between x and B such that $\operatorname{trind}(F_1 \cap L_1) < \beta_2$. Let U_1, W_1 be open subsets of X satisfying the conditions

$$x \in U_1, \quad B \subset W_1, \quad U_1 \cap W_1 = \emptyset \quad \text{and} \quad X \setminus L_1 = U_1 \cup W_1.$$

Since $x \notin L_1 \cup W_1$, there also exists a partition L_2 between x and $L_1 \cup W_1$ such that $\operatorname{trind}(F_2 \cap L_2) < \beta_2$. Let U_2, W_2 be open subsets of X satisfying the conditions

$$x \in U_2, \quad L_1 \cup W_1 \subset W_2, \quad U_2 \cap W_2 = \emptyset \quad \text{and} \quad X \setminus L_2 = U_2 \cup W_2;$$

without loss of generality we can assume that, moreover,

$$L_1 \subset \overline{U}_1 \quad \text{and} \quad L_2 \subset \overline{W}_2. \tag{3}$$

The intersection $F_1' = F_1 \cap F_2 \cap \overline{U}_1 \cap \overline{W}_2$ clearly satisfies the inequality $\operatorname{trind} F_1' \le \beta_1 = \lambda_1 + n_1 < \lambda_2 + n_2 = \beta_2$. And since $L_1 \cap L_2 = \emptyset$, the union $F_2' = (F_1 \cap L_1) \cup (F_2 \cap L_2)$ satisfies the inequality $\operatorname{trind} F_2' < \beta_2$. Let $\max(\operatorname{trind} F_1', \operatorname{trind} F_2', \beta_1) = \alpha' = \lambda' + n' < \beta_2$, where λ' is a limit number and n' is a non-negative integer; note that $\operatorname{trind}(F_1' \cap F_2') \le \beta_1 = \lambda_1 + n_1 \le \alpha'$. We shall consider the set $L = F_1' \cup F_2'$. From the definition of β_2 and β_1 it follows that if $\lambda_1 < \lambda'$ we have $\operatorname{trind} L \le \lambda' + n' < \beta_2$, and if $\lambda_1 = \lambda'$ we have $\operatorname{trind} L \le \lambda' + n' + n_1 + 1 = \lambda_1 + n_1 + n' + 1 < \lambda_2 \le \beta_2$. Thus to show that if $\lambda_1 < \lambda_2$ we have $\operatorname{trind} X \le \beta_2 = \lambda_2 + n_2$, it suffices to verify that L is a partition between x and B.

One easily checks that the sets

$$U = U_1 \setminus (F_2 \cap \overline{W}_2) \quad \text{and} \quad W = W_2 \setminus (F_1 \cap \overline{U}_1)$$

are open, disjoint, and contain x and B respectively. Moreover, by virtue of (3) we have

$$\begin{aligned} X \setminus (U \cup W) &= (X \setminus U) \cap (X \setminus W) \\ &= [X \setminus (U_1 \setminus (F_2 \cap \overline{W}_2))] \cap [X \setminus (W_2 \setminus (F_1 \cap \overline{U}_1))] \\ &= [(X \setminus U_1) \cup (F_2 \cap \overline{W}_2)] \cap [(X \setminus W_2) \cup (F_1 \cap \overline{U}_1)] \\ &= [(L_1 \cup W_1) \cup (F_2 \cap \overline{W}_2)] \cap [(L_2 \cup U_2) \cup (F_1 \cap \overline{U}_1)] \\ &= (F_1 \cap L_1) \cup (F_2 \cap L_2) \cup (F_1 \cap F_2 \cap \overline{U}_1 \cap \overline{W}_2) \\ &= F_1' \cup F_2' = L, \end{aligned}$$

so that L is a partition between x and B.

Assume now that $\lambda_1 = \lambda_2$. By Theorem 7.2.1 and Remark 7.2.2 there exists a partition L between x and B such that $\operatorname{trind}(F_1 \cap F_2 \cap L) < \beta_1$. Let $F_1' = F_1 \cap L$ and $F_2' = F_2 \cap L$; we shall consider the set $L = F_1' \cup F_2'$. Clearly, $\operatorname{trind} F_i' \leq \beta_2 = \lambda_2 + n_2$ for $i = 1, 2$, and $\operatorname{trind}(F_1' \cap F_2') \leq \alpha' = \lambda' + n' < \beta_1 \leq \beta_2$, where λ' is a limit number and n' is a non-negative integer. Now, if $\lambda' < \lambda_2$ we have $\operatorname{trind} L \leq \lambda_2 + n_2$, and if $\lambda' = \lambda_2 = \lambda_1$ we have $n' \leq n_1 - 1$ and $\operatorname{trind} L \leq \lambda_2 + n_2 + n' + 1 \leq \lambda_2 + n_2 + n_1$. Thus if $\lambda_1 = \lambda_2$ we have $\operatorname{trind} X \leq \lambda_2 + n_2 + n_1 + 1$.

It follows that (2) holds, i.e., our supposition leads to a contradiction. ■

7.2.14. THE SECOND SUM THEOREM FOR trInd. *If a hereditarily normal space X is the union of two closed subspaces F_1 and F_2 such that* $\operatorname{trInd} F_1 \leq \operatorname{trInd} F_2 \leq \alpha_2 = \lambda_2 + n_2$ *and* $\operatorname{trInd}(F_1 \cap F_2) \leq \alpha_1 = \lambda_1 + n_1 \leq \alpha_2$, *where λ_i is a limit number and n_i is a non-negative integer for $i = 1, 2$, then*

$$\operatorname{trInd} X \leq \begin{cases} \lambda_2 + n_2 & \text{if } \lambda_1 < \lambda_2, \\ \lambda_2 + n_2 + n_1 + 1 & \text{if } \lambda_2 = \lambda_2. \end{cases} \blacksquare$$

Theorems 7.2.13 and 7.2.14 yield the following two corollaries.

7.2.15. COROLLARY. *If a hereditarily normal space X can be represented as the union of two closed subspaces F_1 and F_2 such that* $\operatorname{trind} F_i \leq \alpha \geq \omega_0$ *for $i = 1, 2$ and* $\operatorname{ind}(F_1 \cap F_2)$ *is finite, then* $\operatorname{trind} X \leq \alpha$. ■

7.2.16. COROLLARY. *If a hereditarily normal space X can be represented as the union of two closed subspaces F_1 and F_2 such that* $\operatorname{trInd} F_i \leq \alpha \geq \omega_0$ *for $i = 1, 2$ and* $\operatorname{Ind}(F_1 \cap F_2)$ *is finite, then* $\operatorname{trInd} X \leq \alpha$. ■

We now pass to the enlargement theorems for transfinite dimensions.

7.2.17. THE ENLARGEMENT THEOREM FOR trind. *For every subspace M of a separable metrizable space X satisfying the inequality* $\operatorname{trind} M \leq \alpha$ *there exists a G_δ-set M^* in X such that $M \subset M^*$ and* $\operatorname{trind} M^* \leq \alpha$.

PROOF. We shall apply transfinite induction with respect to α. If $\alpha = 0$ our theorem follows from Theorem 1.2.14. Assume that the theorem is established for subspaces with small transfinite dimension less than $\alpha \geq 0$ and consider a separable metrizable space X and a subspace M of X such that $\operatorname{trind} M \leq \alpha$.

For $i = 1, 2, \ldots$ by virtue of Theorem 7.2.1 there exists a family $\mathcal{U}_i$ of open subsets of X such that $M \subset M_i = \bigcup \mathcal{U}_i$, $\operatorname{mesh} \mathcal{U}_i < 1/i$ and $\operatorname{trind}(M \cap \operatorname{Fr} U) < \alpha$ for every $U \in \mathcal{U}_i$. Since X has a countable base, one can assume that $\mathcal{U}_i$ is countable; let $\mathcal{U}_i = \{U_{i,j}\}_{j=1}^{\infty}$. By the inductive assumption for $i, j = 1, 2, \ldots$ there exists a G_δ-set $F_{i,j}^* \subset X$ such that $M \cap \operatorname{Fr} U_{i,j} \subset F_{i,j}^*$ and $\operatorname{trind} F_{i,j}^* < \alpha$. The intersection $M^* = (\bigcap_{i=1}^{\infty} M_i) \cap \bigcap_{i,j=1}^{\infty} [F_{i,j}^* \cup (X \setminus \operatorname{Fr} U_{i,j})]$ is a G_δ-set which contains M and, as one easily sees, satisfies the inequality $\operatorname{trind} M^* \leq \alpha$. ■

7.2.18. THE ENLARGEMENT THEOREM FOR trInd. *For every subspace M of a metrizable space X satisfying the inequality* $\operatorname{trInd} M \leq \alpha$ *there exists a G_δ-set M^* in X such that $M \subset M^*$ and* $\operatorname{trInd} M^* \leq \alpha$.

PROOF. We shall first prove that there exists a G_δ-set M' in X such that $M \subset M'$ and $S(M') = S(M)$.

Since by Theorem 7.1.23 the subspace $S(M)$ of X is compact, there exists a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of X which all contain $S(M)$ such that for every open set U which contains $S(M)$ one can find a U_i with $U_i \subset U$; clearly $\bigcap_{i=1}^{\infty} U_i = S(M)$. Again by Theorem 7.1.23 for $i = 1, 2, \ldots$ we have $\operatorname{Ind}(M \setminus U_i) < \infty$, so that by virtue of Theorem 4.1.19 there exists a G_δ-set G_i in X such that $M \setminus U_i \subset G_i$ and $\operatorname{Ind} G_i < \infty$. The intersection $M' = \bigcap_{i=1}^{\infty}(U_i \cup G_i)$ is a G_δ-set which contains M. One easily checks that $M' = S(M) \cup \bigcup_{i=1}^{\infty}(M' \setminus U_i)$, and since $\operatorname{Ind}(M' \setminus U_i) < \infty$ for $i = 1, 2, \ldots$, we have the inclusion $S(M') \subset S(M)$. The reverse inclusion is obvious, so that $S(M') = S(M)$.

By the first part of the proof we can assume without loss of generality that the subspace M of X satisfies the equality $S(M) = S(X)$.

Now we shall apply transfinite induction with respect to α. If $\alpha < \omega_0$ our theorem follows from Theorem 4.1.19. Assume that the theorem is established for subspaces with large transfinite dimension less than $\alpha \geq \omega_0$ and consider a metrizable space X and a subspace M of X such that $\operatorname{trInd} M \leq \alpha$.

From the compactness of $S(M)$ it follows that the countable family $\mathcal{W}$ of all finite unions of sets in a countable base for $S(M)$ has the property that for every pair A, B of disjoint closed subsets of $S(M)$ there exist $W, W' \in \mathcal{W}$ with $A \subset \overline{W} \subset W' \subset X \setminus B$. By virtue of Theorem 7.2.3 for each pair $W, W' \in \mathcal{W}$ with $\overline{W} \subset W'$ we can choose an open set $U \subset X$ such that $\overline{W} \subset U \subset \overline{U} \subset W'$ and $\operatorname{trInd}(M \cap \operatorname{Fr} U) < \alpha$; let $\{U_i\}_{i=1}^{\infty}$ be the family of all U's obtained in this way. By the inductive assumption for $i = 1, 2, \ldots$ there exists a G_δ-set $F_i^* \subset X$ such that $M \cap \operatorname{Fr} U_i \subset F_i^*$ and $\operatorname{trInd} F_i^* < \alpha$. The intersection $M^* = \bigcap_{i=1}^{\infty}[F_i^* \cup (X \setminus \operatorname{Fr} U_i)]$ is a G_δ-set which contains M. As $S(M) \subset S(M^*) \subset S(X) = S(M)$, we have $S(M) = S(M^*) = S(X)$. From the equality $S(M^*) = S(M)$ it follows that for every pair A, B of disjoint closed subsets of $S(M^*)$ there exists in M^* a partition L of the form $M^* \cap \operatorname{Fr} U_i$ between A and B; since $M^* \cap \operatorname{Fr} U_i \subset F_i^*$, we have $\operatorname{trInd} L < \alpha$. And from the equality $S(M^*) = S(X)$ it follows that for each closed subspace F of M^* disjoint from $S(M^*)$ we have $\overline{F} \cap S(X) = \emptyset$, so that $\operatorname{Ind} F < \infty$; thus by Theorem 7.1.25 the space M^* has large transfinite dimension. Proposition 7.1.28 implies that $\operatorname{trInd} M^* \leq \alpha$. ■

It is an open problem if the assumption of separability can be omitted in Theorem 7.2.17 (see Problem 7.2.G, cf. the remark to Problem 4.1.B).

From Theorems 7.2.17 and 7.2.18, applying Lemma 1.3.12, Theorems 7.1.3 and 7.1.29, and the fact that each metrizable space is homeomorphic to a subspace of a completely metrizable space (see [GT], Corollary 4.3.15), we obtain the following two completion theorems.

7.2.19. THE COMPLETION THEOREM FOR trind. *For every separable metrizable space X there exists a separable completely metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the equality* $\operatorname{trind} \widetilde{X} = \operatorname{trind} X$. ■

7.2.20. THE COMPLETION THEOREM FOR trInd. *For every metrizable space X there exists a completely metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the equalities* $\operatorname{trInd} \widetilde{X} = \operatorname{trInd} X$ *and* $w(\widetilde{X}) = w(X)$. ■

We conclude this section with a discussion of compactification and universal space theorems for transfinite dimensions.

It turns out that there is no compactification theorem for the small transfinite dimension (see Problem 7.2.H); in the case of the large transfinite dimension the situation is better. For every separable metrizable space X with $\operatorname{trInd} X < \infty$ there exists a metrizable compactification $\widetilde{X}$ such that $\operatorname{trInd} \widetilde{X} = \operatorname{trInd} X$ (one can moreover get $\operatorname{trind} \widetilde{X} = \operatorname{trind} X$). More generally, for every normal space X with $\operatorname{trInd} X < \infty$ there exists a compactification $\widetilde{X}$ such that $\operatorname{trInd} \widetilde{X} \leq \operatorname{trInd} X$ and $w(\widetilde{X}) \leq w(X)$ (if X is metrizable one can moreover get $\operatorname{trind} \widetilde{X} \leq \omega_0 + \operatorname{trind} X$). The proofs of these results are too difficult to be presented here.

We shall now prove two theorems that characterize the separable metrizable spaces which have transfinite dimension in terms of their compactifications.

7.2.21. THEOREM. *A separable metrizable space X has small transfinite dimension if and only if X has a countable-dimensional metrizable compactification, i.e., there exists a countable-dimensional compact metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X.*

PROOF. By Theorems 7.1.9 and 7.1.3 it suffices to show that every separable metrizable space X with $\operatorname{trind} X < \infty$ has a countable-dimensional compactification, and by Theorem 7.2.19 one can moreover assume that X is completely metrizable. The existence of the required compactification now follows from Theorems 7.1.8 and 5.3.2. ■

7.2.22. THEOREM. *A separable metrizable space X has large transfinite dimension if and only if there exists a countable-dimensional compact metrizable space $\widetilde{X}$ which contains X as a dense subspace and has the property that $S(\widetilde{X}) = S(X)$.*

PROOF. Theorems 7.1.31 and 7.1.23 and Lemma 7.1.24 imply that if for some X there exists such a space $\widetilde{X}$, then X has large transfinite dimension.

Consider now a separable metrizable space X which has large transfinite dimension. Since by Theorem 7.1.23 the subspace $S(X)$ is compact, there exists a family $\{U_i\}_{i=1}^{\infty}$ of open subsets of X which all contain $S(X)$ such that for every closed set $F \subset X$ disjoint from $S(X)$ one can find a U_i with $F \cap U_i = \emptyset$. Let $F_i = X \setminus U_i$ for $i = 1, 2, \ldots$; clearly $\operatorname{Ind} F_i < \infty$. By virtue of Lemma 5.3.3 there exists a compact metrizable space $\widetilde{X}$ which contains X as a dense subspace such that for $i = 1, 2, \ldots$ the closure $\overline{F}_i$ of F_i in $\widetilde{X}$ satisfies the inequality $\operatorname{Ind} \overline{F}_i < \infty$. Now it suffices to check that every point $x \in \widetilde{X} \setminus X$ has a neighbourhood U contained in one of $\overline{F}_i$'s. Suppose that this is not true. Then we can easily define a sequence $x_1, x_2, \ldots$ converging to x such that $x_i \in U_i \setminus S(X)$ for $i = 1, 2, \ldots$ The set $F = \{x_1, x_2, \ldots\} \subset X$ is closed in X and disjoint from $S(X)$, so that one can find a U_i such that $F \cap U_i = \emptyset$, a contradiction. ■

Now we pass to the problem of the existence of universal spaces. We shall establish

7.2.23. THE UNIVERSAL SPACE THEOREM FOR trind. *For every countable ordinal number α there exists a universal space P_α for the class of all separable metrizable spaces whose small transfinite dimension is not larger than α.*

PROOF. For every subset P of the Cartesian product $C \times I^{\aleph_0}$ of the Cantor set C and the Hilbert cube $I^{\aleph_0}$ and for every $t \in C$ we let

$$P(t) = \{x \in I^{\aleph_0} : (t, x) \in P\};$$

clearly, the space $P(t)$ is homeomorphic to the subspace $\{t\} \times P(t)$ of P. Thus, by virtue of Theorem 7.2.17, to establish our theorem it suffices to define, for each $\alpha < \omega_1$, a subspace P_α of $C \times I^{\aleph_0}$ such that

$$\operatorname{trind} P_\alpha \leq \alpha \tag{4}$$

and

(5) if X is a G_δ-set in $I^{\aleph_0}$ with $\operatorname{trind} X \leq \alpha$, then $X = P_\alpha(t)$ for a $t \in C$.

The spaces P_α will be defined by transfinite induction. Let $P_{-1} = \emptyset$ and assume that the spaces P_α with the required properties are defined for all $\alpha < \alpha_0 \geq 0$. Now let

$$A_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 0\}, \quad B_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 1\},$$
$$L_i = \{\{x_j\} \in I^{\aleph_0} : x_i = 1/2\},$$

for $i = 1, 2, \ldots$, and $Y = I^{\aleph_0} \setminus \bigcup_{i=0}^{\infty}(A_{2i+1} \cup B_{2i+1})$. Represent the set E of all even positive integers as the union $E_{-1} \cup \bigcup_{\alpha<\alpha_0} E_\alpha$ of pairwise disjoint infinite subsets, and in the function space of all continuous mappings of $I^{\aleph_0}$ into itself (see Section 1.11) consider the space Z consisting of all homeomorphic embeddings $h\colon I^{\aleph_0} \to I^{\aleph_0}$ satisfying the following two conditions:

(6) For every point $x \in h^{-1}(Y)$ and each closed set $F \subset I^{\aleph_0}$ with $x \notin F$ there exists a $j \in E$ such that $h(x) \in A_j$ and $h(F) \subset B_j$.

(7) $\operatorname{trind} h^{-1}(Y \cap L_j) \leq \alpha$ for every $j \in E_\alpha$.

First we shall prove that

(8) if X is a G_δ-set in $I^{\aleph_0}$ with $\operatorname{trind} X \leq \alpha_0$ then there exists an $h \in Z$ such that $X = h^{-1}(Y)$.

Consider a countable base $\mathcal{B}$ for $I^{\aleph_0}$ and arrange into a sequence (U_1, V_1), $(U_2, V_2), \ldots$ the family $\mathcal{P}$ of all pairs (U, V) of members of $\mathcal{B}$ such that $\overline{U} \subset V$ and there exists a partition L between $\overline{U}$ and $I^{\aleph_0} \setminus V$ with $\operatorname{trind}(X \cap L) = \alpha < \alpha_0$. For $i = 1, 2, \ldots$ denote by α_i the smallest possible α in the last equality, where L is a partition between $\overline{U}_i$ and $I^{\aleph_0} \setminus V_i$, and take a continuous function $f_i\colon I^{\aleph_0} \to I$ satisfying

$$f_i(U_i) \subset \{0\}, \quad f_i(I^{\aleph_0} \setminus V_i) \subset \{1\} \quad \text{and} \quad \operatorname{trind}(X \cap f_i^{-1}(1/2)) = \alpha_i. \tag{9}$$

Now consider closed subspaces $X_1, X_2, \ldots$ of $I^{\aleph_0}$ such that $X = I^{\aleph_0} \setminus \bigcup_{i=1}^{\infty} X_i$ and for $i = 1, 2, \ldots$ take a continuous function $g_i\colon I^{\aleph_0} \to (0, 1]$ with $g_i^{-1}(1) = X_i$. Finally, take one-to-one mappings $\varphi_1\colon O_1 \to N$ and $\varphi_2\colon O_2 \to N$ of $O_1 = \{1, 5, 9, \ldots\}$ and $O_2 = \{3, 7, 11, \ldots\}$ onto the set N of all positive integers, and a one-to-one mapping $\psi\colon N \to E$ such that $\psi(i) \in E_{\alpha_i}$. We shall show that the mapping $h\colon I^{\aleph_0} \to I^{\aleph_0}$ defined by letting $h(\{x_j\}) = \{y_j\}$, where

$$y_j = \begin{cases} g_{\varphi_1(j)}(\{x_j\}) & \text{if } j \in O_1, \\ \frac{1}{2}x_{\varphi_2(j)} + \frac{1}{4} & \text{if } j \in O_2, \\ f_i(\{x_j\}) & \text{if } j = \psi(i), \\ 0 & \text{if } j \in E \setminus \psi(N), \end{cases}$$

belongs to Z and satisfies (8). Clearly, h is continuous, and from the definition of y_j for $j \in O_1 \cup O_2$ it follows that $X = h^{-1}(Y)$ and that h is a homeomorphic embedding. It remains to check that h satisfies (6) and (7). Consider a point $x \in X = h^{-1}(Y)$ and a closed set $F \subset I^{\aleph_0}$ with $x \notin F$; let $V \in \mathcal{B}$ satisfy $x \in V \subset I^{\aleph_0} \setminus F$. Since $\operatorname{trind} X \leq \alpha_0$, by virtue of Theorem 7.2.1 there exists a partition L between x and $I^{\aleph_0} \setminus V$ with $\operatorname{trind}(X \cap L) < \alpha_0$; there also exists a $U \in \mathcal{B}$ such that $(U, V) = (U_i, V_i) \in \mathcal{P}$. From the definition of y_j for $j = \psi(i)$ and from the inclusions in (9) it follows that $h(x) \in A_j$ and $h(F) \subset h(I^{\aleph_0} \setminus V_i) \subset B_j$. Thus h satisfies (6). Now, for $j = \psi(i) \in E_\alpha \cap \psi(N)$ we have $\alpha = \alpha_i$ and $h^{-1}(Y \cap L_j) = X \cap f_i^{-1}(1/2)$, so that $\operatorname{trind} h^{-1}(Y \cap L_j) = \alpha_i = \alpha$ by the last part of (9). And for $j \in E_\alpha \setminus \psi(N)$ we have $y_j = 0$, so that $h^{-1}(Y \cap L_j) = \emptyset$ and $\operatorname{trind} h^{-1}(Y \cap L_j) = -1 < \alpha$. Thus h satisfies (7) as well.

Now we proceed to the definition of P_{α_0}. For every $j = 2i \in E$ and any $h \in Z$ we have $\operatorname{trind} h^{-1}(Y \cap L_{2i}) \leq \alpha(i) < \alpha_0$, where $2i \in E_{\alpha(i)}$; thus by the inductive assumption there exists a $t_i \in C$ such that

$$h^{-1}(Y \cap L_{2i}) = P_{\alpha(i)}(t_i). \tag{10}$$

Let $S \subset Z \times C \times C \times \ldots$ consist of all sequences $(h, t_1, t_2, \ldots)$ such that (10) is satisfied for $i = 1, 2, \ldots$ Since Z is metrizable and separable (see [GT], Corollary 4.2.18), so is S, and thus there exists a continuous mapping u of a subspace A of C onto the space S (see Problem 1.3.D(b)). Clearly $u(t) = (u_0(t), u_1(t), u_2(t), \ldots)$, where the mappings $u_0 \colon A \to Z$ and $u_1 \colon A \to C, u_2 \colon A \to C, \ldots$ are continuous. One easily checks that by letting

$$k(t, x) = [u_0(t)](x) \quad \text{for } t \in A \text{ and } x \in I^{\aleph_0}$$

we define a continuous mapping $k \colon A \times I^{\aleph_0} \to I^{\aleph_0}$. We shall show that the set

$$P_{\alpha_0} = k^{-1}(Y) = \{(t, x) \in A \times I^{\aleph_0} : [u_0(t)](x) \in Y\} \subset C \times I^{\aleph_0}$$

satisfies (4) and (5) for $\alpha = \alpha_0$.

To begin, we shall show that the set

$$K_j = k^{-1}(L_j) = \{(t, x) \in A \times I^{\aleph_0} : [u_0(t)](x) \in L_j\}$$

satisfies the inequality

$$\operatorname{trind}(P_{\alpha_0} \cap K_j) < \alpha_0 \quad \text{for every } j \in E. \tag{11}$$

Let $j = 2i$; by the definition of S for every $t \in A$ we have $[u_0(t)]^{-1}(Y \cap L_j) = P_{\alpha(i)}(u_i(t))$, so that

$$(P_{\alpha_0} \cap K_j)(t) = [u_0(t)]^{-1}(Y \cap L_j) = P_{\alpha(i)}(u_i(t)).$$

From the last equality it follows that the mapping $h_i \colon A \times I^{\aleph_0} \to A \times C \times I^{\aleph_0}$ defined by letting $h_i(t, x) = (t, u_i(t), x)$, which is a homeomorphic embedding, maps $P_{\alpha_0} \cap K_j$ into $A \times P_{\alpha(i)}$, so that $\operatorname{trind}(P_{\alpha_0} \cap K_j) \leq \operatorname{trind}(A \times P_{\alpha(i)}) = \operatorname{trind} P_{\alpha(i)} \leq \alpha(i) < \alpha_0$, and (11) holds (cf. Problem 7.2.F(a)).

Now we shall show that $\operatorname{trind} P_{\alpha_0} \leq \alpha_0$. Consider a point $(t, x) \in P_{\alpha_0}$ and a closed set $F \subset A \times I^{\aleph_0}$ with $(t, x) \notin F$. By the definition of P_{α_0} we have $x \in [u_0(t)]^{-1}(Y)$; since $x \notin F(t)$, by (6) there exists a $j \in E$ such that $k(t, x) = [u_0(t)](x) \in A_j$ and

$k(\{t\} \times F(t)) = [u_0(t)](F(t)) \subset B_j$. Thus K_j is a partition in $A \times I^{\aleph_0}$ between (t,x) and $F' = \{t\} \times F(t) = (\{t\} \times I^{\aleph_0}) \cap F$; let the open sets $U, W \subset A \times I^{\aleph_0}$ satisfy the conditions

$$(t,x) \in U, \quad F' \subset W, \quad U \cap W = \emptyset \quad \text{and} \quad (A \times I^{\aleph_0}) \setminus K_j = U \cup W.$$

Since the projection $p\colon A \times I^{\aleph_0} \to A$ parallel to the compact space $I^{\aleph_0}$ is closed (see [GT], Theorem 3.1.16), the set $p((K_j \cup U) \cap F)$ is closed, and thus there exists an open-and-closed set $V \subset A$ such that $t \in V$ and $V \cap p((K_j \cup U) \cap F) = \emptyset$. As $\operatorname{Fr}(U \cap (V \times I^{\aleph_0})) \subset K_j \cap (V \times I^{\aleph_0})$, the last intersection is a partition in $A \times I^{\aleph_0}$ between the point (t,x) and the set F, so that by (11) we have $\operatorname{trind} P_{\alpha_0} \leq \alpha_0$.

It remains to show that P_{α_0} satisfies (5) for $\alpha = \alpha_0$. Consider a G_δ-set $X \subset I^{\aleph_0}$ with $\operatorname{trind} X \leq \alpha_0$. By (8) there exists an $h \in Z$ such that $X = h^{-1}(Y)$, and since $u_0(A) = Z$, we have $h = u_0(t)$ for some $t \in A$. Thus by the definition of P_{α_0},

$$P_{\alpha_0}(t) = \{x \in I^{\aleph_0} : [u_0(t)](x) \in Y\} = h^{-1}(Y) = X. \blacksquare$$

It was recently proved that there is no universal space for the class of all separable metrizable spaces whose large transfinite dimension is not larger than ω_0 (see Problem 7.2.J). This result together with the compactification theorem for trInd implies that there is no universal space for the class of all compact metrizable spaces whose large transfinite dimension is not larger than ω_0. And thus by Proposition 7.1.22 there is no universal space for the class of all compact metrizable spaces whose small transfinite dimension is not larger than ω_0.

Historical and bibliographic notes

Theorem 7.2.1 was proved by Toulmin in [1954] (cf. Problem 2.2.A(a)), and Theorem 7.2.3 is implicit in Smirnov's paper [1962]. Theorem 7.2.6 is a counterpart of Theorem 7.2.7 obtained by Hattori in [1983a]. Corollary 7.2.8 was first stated in Toulmin [1954] for hereditarily normal spaces; Corollary 7.2.9 and Theorem 7.2.10 – obtained by Hattori in [1983a] – were first stated in Levšenko [1965] for hereditarily normal spaces. Theorem 7.2.11 appeared in Fedorčuk's paper [1978]; for hereditarily normal spaces it was obtained by Levšenko in [1965], this last paper contains also Example 7.2.12. Theorem 7.2.13 was proved by Toulmin in [1954], and Theorem 7.2.14 independently by Landau in [1969] and Pears in [1971] (Corollary 7.2.16 for hereditarily paracompact spaces was obtained by Henderson in [1968]). Theorems 7.2.17–7.2.20 were proved by Luxemburg in [1982] (announcement in [1973], a proof of 7.2.17 in Engelking [1980]). The results on compactifications preserving the large inductive dimension of metrizable spaces were also proved by Luxemburg in [1982] (announcement in [1973]); the corresponding result for normal spaces was announced by Pasynkov in [1971] (where it is explained that Luxemburg obtained his theorems independently). Theorem 7.2.21 was announced by Hurewicz in [1928], Theorem 7.2.22 is due to Przymusiński (cited in Engelking [1980]) and Luxemburg [1982]; a characterization of separable metrizable spaces which have a weakly infinite-dimensional metrizable compactification was given by Borst in [1985]. Theorem 7.2.23 was established by R. Pol in [1986].

Problems

7.2.A. Give an example of a compact metrizable space X and a subspace M of X such that for some point $x \in X \setminus M$ and a closed set $B \subset X$ with $x \notin B$ there exists no partition L between x and B such that $\operatorname{trind}(M \cap L) < \operatorname{trind} M$ (see Proposition 1.5.14).

7.2.B (Levšenko [1965]). Show that Smirnov's space S_{ω_0+1} can be represented as the union of two closed subspaces F_1 and F_2 such that $\operatorname{trind} F_i = \operatorname{trInd} F_i = \omega_0$ for $i = 1, 2$.

Hint. For $n = 1, 2, \ldots$ consider the subsets

$$C_n = \bigcup_{i=0}^{2^n-1} [2i/2^{n+1}, (2i+1)/2^{n+1}] \quad \text{and} \quad D_n = \bigcup_{i=0}^{2^n-1} [(2i+1)/2^{n+1}, (2i+2)/2^{n+1}]$$

of the unit interval I and take the closures F_1 and F_2 of the sets $\bigcup_{n=1}^{\infty}(I^n \times C_n)$ and $\bigcup_{n=1}^{\infty}(I^n \times D_n)$ in $S_{\omega_0+1} = \omega(\bigoplus_{n=1}^{\infty} I^n) \times I = (\{x_0\} \cup \bigoplus_{n=1}^{\infty} I^n) \times I$, where x_0 is the compactifying point.

Remark. The first example of a separable metrizable space X with $\operatorname{trind} X = \operatorname{trInd} X = \omega_0 + 1$ which can be represented as the union of two closed subspaces F_1 and F_2 such that $\operatorname{trind} F_i = \operatorname{trInd} F_i = \omega_0$ for $i = 1, 2$ was given by Toulmin in [1954].

7.2.C (Pasynkov [1982], Hattori [1983a]). Prove that for every ordinal number α there exists a metrizable space X such that $\operatorname{trind} X = \alpha$.

Hint. Consider a function θ defined in the same way as trInd with the only difference that in the second condition one considers only compact sets A and those neighbourhoods V for which $X \setminus V$ is compact. Check that the counterpart of Theorem 7.2.7 for θ is valid and that for every normal space X we have $\theta(X) \leq \omega_0 \cdot \operatorname{trind} X$. Then for each ordinal number α define a metrizable space M_α such that $\theta(M_\alpha) \geq \alpha$. To that end take $M_0 = \{0\}$, let $M_\alpha = M_\beta \times I$ if $\alpha = \beta + 1$, and for a limit number α let $M_\alpha = (\bigoplus_{\beta<\alpha} M_\beta) \cup \{x_\alpha\}$, where the subspace $\bigoplus_{\beta<\alpha} M_\beta$ has the sum topology and the sets $U_i(x_\alpha) = \{x_\alpha\} \cup \bigoplus\{M_{\lambda+n} : \lambda + n < \alpha, \lambda \text{ is a limit number and } n \geq i\}$, with $i = 1, 2, \ldots$, form a base at $\{x_\alpha\}$.

7.2.D. (a) (Levšenko [1965]) Show that if a hereditarily normal space X is the union of two subspaces X_1 and X_2 such that $\operatorname{trInd} X_1 \leq \alpha_1 = \lambda_1 + n_1$ and $\operatorname{trInd} X_2 \leq \alpha_2 = \lambda_2 + n_2 \geq \alpha_1$, where λ_i is a limit number and n_i is a non-negative integer for $i = 1, 2$, then

$$\operatorname{trInd} X \leq \begin{cases} \lambda_2 + n_2 & \text{if } \lambda_1 < \lambda_2, \\ \lambda_2 + n_2 + n_1 + 1 & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Hint. Follow the proof of Theorem 7.2.7.

(b) Give an example of a compact metrizable space X such that $\operatorname{trind} X = \omega_0 + 1$, and yet X is the union of two subspaces X_1 and X_2, where X_1 is closed, $\operatorname{ind} X_1 = 1$ and $\operatorname{trind} X_2 = \omega_0$.

Remark. An evaluation of the small transfinite dimension of a hereditarily normal space $X = X_1 \cup X_2$ in terms of $\operatorname{trind} X_1$ and $\operatorname{trind} X_2$ was given by Toulmin in [1954].

7.2.E. (a) (Toulmin [1954]) Prove that if a regular space X is the union of two closed subspaces F_1 and F_2 such that $\operatorname{trind} F_1 \leq \alpha$ and there exists a homeomorphism h of F_1 onto F_2 satisfying $h(x) = x$ for $x \in F_1 \cap F_2$, then $\operatorname{trind} X \leq \alpha$.

(b) (Fedorčuk [1978]) Show that every hereditarily normal compact space which can be represented as the union of countably many subspaces which all have large transfinite dimension has itself large transfinite dimension.

(c) (Fedorčuk [1978]) Deduce from (b) that every hereditarily normal compact space which can be represented as the union of countably many closed subspaces which all have small transfinite dimension has itself small transfinite dimension.

Remark. It is not known if in (b) and (c) the assumption of hereditary normality, and in (c) the assumption that the subspaces are closed, can be dropped.

7.2.F. (a) Note that for every regular space X and every regular space Y with $\operatorname{ind} Y = 0$ we have $\operatorname{trind}(X \times Y) = \operatorname{trind} X$.

(b) (announcement Luxemburg [1973a]; for metrizable spaces, Toulmin [1954]) Show that for every completely regular space X we have $\operatorname{trind}(X \times I) \leq \operatorname{trind} X + 1$.

Hint. Apply Problem 7.2.E(a).

Remark. It may happen that $\operatorname{trind}(X \times I) < \operatorname{trind} X + 1$ (see the remark to Problem 7.1.G(e)). It is not known if for a metrizable space X we always have $\operatorname{trInd}(X \times I) \leq \operatorname{trInd} X + 1$.

(c) (Fedorčuk [1978] for normal spaces; for hereditarily normal spaces, Toulmin [1954]) Show that if regular spaces X and Y have small transfinite dimension, then the Cartesian product $X \times Y$ also has small transfinite dimension.

Remark. Toulmin in [1954] gives an evaluation for $\operatorname{trind}(X \times Y)$.

(d) (Fedorčuk [1978]) Show that if compact spaces X and Y have large transfinite dimension, then the Cartesian product $X \times Y$ also has large transfinite dimension.

7.2.G (E. Pol, announcement in Engelking [1980]). Show that Theorem 7.2.17 is valid for strongly metrizable spaces.

7.2.H (Luxemburg [1973]). Give an example of a separable completely metrizable space X such that $\operatorname{trind} X = \omega_0$, and yet for every metrizable compactification $\widetilde{X}$ of the space X we have $\operatorname{trind} \widetilde{X} > \omega_0$.

Hint. In the space $S = \bigoplus_{n=1}^{\infty} I^n$ pick a pair of points a_n, b_n from a pair of opposite faces of I^n for $n = 1, 2, \ldots$ For $m = 1, 2, \ldots$ consider a copy S_m of the space S and denote by $a_{n,m}$ and $b_{n,m}$ the points of S_m corresponding to a_n and b_n. In the sum $Z = \bigoplus_{m=1}^{\infty} S_m$ identify points $b_{n,m}$ and $a_{n,m+1}$ for $n, m = 1, 2, \ldots$; let Y be the space obtained by these identifications and let $q\colon Z \to Y$ be the corresponding quotient mapping. Take a point $x_0 \notin Y$ and consider the space $X = Y \cup \{x_0\}$, where the subspace Y has its original topology and the sets $U_i(x_0) = Y \setminus q(\bigoplus_{m=1}^{i} S_m)$, with $i = 1, 2, \ldots$, form a base at x_0.

7.2.I. (a) (Pasynkov [1971]) Show that for every normal space X we have $\operatorname{trInd} \beta X = \operatorname{trInd} X$.

(b) Give an example of a separable metrizable space X with $\operatorname{trind} X = \omega_0$ such that $\operatorname{trind} \beta X = \infty$.

(c) Show that every separable metrizable space X with $\operatorname{trind} X < \infty$ has a metrizable compactification $\widetilde{X}$ with $\operatorname{trind} \widetilde{X} < \infty$.

Remark. It is an open problem to evaluate the increase of small transfinite dimension in the process of compactifying a separable metrizable space.

7.2.J (Olszewski [1994]). (a) Show that for every totally bounded metric space (X,ρ) with $\operatorname{trInd} X \leq \omega_0$ and every $\epsilon > 0$ there exists a positive integer m such that for each $A \subset X$ with $\delta(A) > \epsilon$ a set $L \subset X$ satisfying $\operatorname{Ind} L \leq m$ separates X between a pair of points $x, y \in A$.

(b) Let T_k, for $k = 1, 2, \ldots$, be the set of all pairs (i,k), where $i = 1, 2, \ldots, 2^k$, and let $T = \{0\} \cup \bigcup_{k=1}^{\infty} T_k$. For each sequence $s = (s(1), s(2), \ldots)$ of integers ≥ 2 define a compact metrizable space $C(s)$ in the following way: consider the interval $C_0 = [0,1]$ and for each pair $t = (i,k) \in T$ take an $s(k)$-dimensional cube C_t whose edges have length $1/2^k$ and glue it along one of its edges to the interval $L_t = [(i-1)/2^k, i/2^k] \subset C_0$ in such a way that $(C_t \setminus L_t) \cap (C_{t'} \setminus L_{t'}) = \emptyset$ whenever $t \neq t'$. Show that in $C(s)$ between any two points there exists a finite-dimensional partition and deduce that $\operatorname{trInd} C(s) \leq \omega_0$. Then check that for every metric space X and any continuous mapping $h\colon C(s) \to X$ we have $\delta(h(C_0)) \leq 2^k \max\{\delta(h(C_t)) : t \in T_k\}$ for $k = 1, 2, \ldots$

(c) Prove that there is no universal space for the class of all separable metrizable spaces whose large transfinite dimension is not larger than ω_0.

Hint. Using part (a), for every totally bounded metric space (X,ρ) with $\operatorname{trInd} X \leq \omega_0$ define a sequence s of integers ≥ 2 such that the space $C(s)$ is not embeddable in X.

Remark. Olszewski proved in [1994] that for every limit ordinal number $\lambda < \omega_1$ there is no universal space either for the class of all compact metrizable spaces whose small transfinite dimension is not larger than λ or for the class of all separable metrizable spaces whose large transfinite dimension is not larger than λ.

7.3. Transfinite dimensions trker and trdim

In Section 5.5 for each normal space X we considered the subset $K(X) = X \setminus \bigcup_{n=1}^{\infty} D_n(X)$ of the space X, where

$$D_n(X) = \bigcup\{U \subset X : U \text{ is open and } \dim \overline{U} \leq n\}, \tag{1}$$

and observed that X is locally finite-dimensional if and only if $K(X) = \emptyset$, i.e., if X can be emptied by scooping out finite-dimensional sets. In this section we shall extend the process of scooping out finite-dimensional subspaces of a space X by applying it next to $K(X)$, then to $K(K(X))$ etc. Supposing that an infinite-dimensional space can be emptied in this way, the number of steps necessary to empty it can be considered as a measure of its "degree of infinite dimensionality". We shall make these considerations precise in the following definitions where – as in the whole of this section – for an ordinal number α we denote respectively by $\lambda(\alpha)$ and $n(\alpha)$ the limit number and the non-negative integer satisfying the equality $\alpha = \lambda(\alpha) + n(\alpha)$.

7.3.1. DEFINITION. To every normal space X one assigns the *transfinite kernel dimension* of X, denoted by $\operatorname{trker} X$, which is the integer -1, an ordinal number, or the "infinite number" ∞; to this end for the smallest ordinal number $\xi(X)$ whose cardinality is larger than the cardinality of the family of all subsets of X one defines inductively a decreasing transfinite sequence $X_0 \supset X_1 \supset \ldots \supset X_\alpha \supset \ldots$, $\alpha < \xi(X)$, of subsets of X and a transfinite sequence $D_0(X), D_1(X), \ldots, D_\alpha(X), \ldots$, $\alpha < \xi(X)$, of subsets of X

by letting

$$X_\alpha = X \setminus \bigcup_{\beta<\alpha} D_\beta(X) \quad \text{and} \quad D_\alpha(X) = D_{n(\alpha)}(X_{\lambda(\alpha)}) \quad \text{for } \alpha < \xi(X),$$

where $D_{n(\alpha)}(X_{\lambda(\alpha)})$ is defined in (1). From the definition of $\xi(X)$ it follows that there exists a $\beta_0 < \xi(X)$ such that $X_\beta = X_{\beta_0}$ for every $\beta \geq \beta_0$; this common value of X_β's with large indices is denoted by $k(X)$ and called the *transfinite dimensional kernel* of the space X. Now, if $k(X) = \emptyset$ we denote by $\operatorname{trker} X$ the smallest $\alpha < \xi(X)$ such that $X_\beta = \emptyset$ for $\beta > \alpha$, and if $k(X) \neq \emptyset$ we let $\operatorname{trker} X = \infty$.

If $\operatorname{trker} X \neq \infty$ we say that the space X is *shallow*; if $\operatorname{trker} X = \infty$, we say that X is *deep*.

Note that since $X_0 = X$, for $\alpha = n = 0, 1, \ldots$ we have $D_\alpha(X) = D_{n(\alpha)}(X_{\lambda(\alpha)}) = D_n(X)$, so that the notation in the definition is consistent with (1).

One easily sees that whenever normal spaces X and Y are homeomorphic, then $\operatorname{trker} X = \operatorname{trker} Y$, i.e., the transfinite kernel dimension is a topological invariant.

From Theorem 3.1.14 we obtain

7.3.2. PROPOSITION. *For every weakly paracompact normal space X, if* $\dim X < \infty$, *then* $\operatorname{trker} X = \dim X$, *and if* $\operatorname{trker} X < \omega_0$, *then* $\dim X = \operatorname{trker} X$. ■

Let us note that if λ is a limit number, then

$$(2) \qquad X_\lambda = \bigcap_{\beta<\lambda} X_\beta, \quad \text{and} \quad X_{\alpha+1} = X_\alpha \setminus D_\alpha(X) = X_\alpha \cap (X_{\lambda(\alpha)} \setminus D_\alpha(X));$$

since $D_\alpha(X)$ is an open subset of $X_{\lambda(\alpha)}$, from (2) by an easy transfinite induction it follows that the sets X_α are closed in X.

Let us also note that $X_{\omega_0} = K(X)$, so that we have the following

7.3.3. THEOREM. *Every locally finite-dimensional normal space X is shallow and satisfies the inequality* $\operatorname{trker} X \leq \omega_0$. ■

Finally, let us note that if $\operatorname{trker} X = \alpha = \beta + 1$, then clearly $X_\alpha \neq \emptyset$, but if $\operatorname{trker} X = \alpha$ is a limit number, then either $X_\alpha \neq \emptyset$ or $X_\alpha = \emptyset$. Indeed, for the space $X = S = \bigoplus_{n=1}^\infty I^n$ we have $\operatorname{trker} S = \omega_0$ and $X_{\omega_0} = \emptyset$, while for its one-point compactification $X = \omega S$ we have $\operatorname{trker} X = \omega_0$ and $X_{\omega_0} \neq \emptyset$. One proves that for Smirnov's spaces S_α defined in Example 7.1.33 we have $\operatorname{trker} S_\alpha = \alpha$ for $\alpha < \omega_1$ (see Problem 7.3.D(a)).

To establish an evaluation for $\operatorname{trker} X$ in terms of the weight $w(X)$ of the space X we need the following

7.3.4. LEMMA. *If a topological space X satisfies the inequality $w(X) \leq \aleph_\alpha$, then for every decreasing transfinite sequence $F_0 \supset F_1 \supset \ldots \supset F_\beta \supset \ldots$, $\beta < \omega_{\alpha+1}$, of closed subsets of X there exists a $\beta_0 < \omega_{\alpha+1}$ such that $X_\beta = X_{\beta_0}$ for every $\beta \geq \beta_0$.*

PROOF. If no such β_0 exists, then one can define a set B cofinal in the set of all ordinal numbers less than $\omega_{\alpha+1}$ such that $F_\beta \setminus F_{\beta'} \neq \emptyset$ whenever $\beta, \beta' \in B$ and $\beta < \beta'$. From the cofinality it follows that $|B| = \aleph_{\alpha+1}$. Consider an arbitrary base $\mathcal{B}$ for the space X and for each $\beta \in B$ take a member U_β of $\mathcal{B}$ such that $U_\beta \cap F_\beta \neq \emptyset$ and

$U_\beta \cap F_{\beta+} = \emptyset$, where β^+ is the smallest element of the set $\{\beta' \in B : \beta < \beta'\}$. Clearly $U_\beta \neq U_{\beta'}$ whenever $\beta \neq \beta'$, so that $\aleph_{\alpha+1} = |B| \leq |\mathcal{B}|$, and $w(X) > \aleph_\alpha$. ■

The lemma immediately yields

7.3.5. THEOREM. *Every shallow space X with weight $w(X) = \aleph_\alpha$ satisfies the inequality* $\operatorname{trker} X < \omega_{\alpha+1}$. ■

7.3.6. COROLLARY. *Every separable metrizable shallow space X satisfies the inequality* $\operatorname{trker} X < \omega_1$. ■

We shall now prove a useful lemma on relations between the sets X_α and the sets M_α defined for a subspace M of the space X. It will be convenient to assume in the sequel that for every normal space X the sets X_α are also defined for $\alpha \geq \xi(X)$ and are equal to $k(X)$.

7.3.7. LEMMA. *If M is a subspace of a normal space X, then for every α we have*

(i) $M_\alpha \subset X_\alpha$ *if M is closed in X,*
(ii) $\bar{M}_\alpha = \bar{M} \cap X_\alpha$ *if $\bar{M}$ is open in X and normal,*
(iii) $M_\alpha \subset X_\alpha$ *if X is strongly hereditarily normal.*

PROOF. We shall first prove by transfinite induction that the inclusion $M_\alpha \subset X_\alpha$ holds under the assumptions in (i), (ii) or (iii). Clearly $M_0 \subset X_0$; assume that $M_\beta \subset X_\beta$ for $\beta < \alpha$ and that there exists a point $x \in M_\alpha \setminus X_\alpha$. Since $x \notin X_\alpha$ there exists a $\beta < \alpha$ such that $x \in D_\beta(X) = D_{n(\beta)}(X_{\lambda(\beta)})$, and thus there exists a neighbourhood U of the point x in $X_{\lambda(\beta)}$ with $\dim(\overline{U} \cap X_{\lambda(\beta)}) \leq n(\beta)$. Since by the inductive assumption $M_{\lambda(\beta)} \subset X_{\lambda(\beta)}$, the intersection $U \cap M_{\lambda(\beta)}$ is a neighbourhood of x in $M_{\lambda(\beta)} \supset M_\alpha$, and its closure $C = \overline{U \cap M_{\lambda(\beta)}} \cap M_{\lambda(\beta)}$ in $M_{\lambda(\beta)}$ is contained in $\overline{U} \cap X_{\lambda(\beta)}$. From Theorems 3.1.3 and 3.1.19 it follows that under the assumptions in (i) or (iii) we have $\dim C \leq n(\beta)$, so that $x \in D_{n(\beta)}(M_{\lambda(\beta)}) = D_\beta(M) \subset X \setminus M_\alpha$, a contradiction. Under the assumption in (ii) we can assume that moreover $\overline{U} \subset M$, so that $\overline{U} \cap X_{\lambda(\beta)} \subset M$; since C is a closed subset of M, it is a closed subset of $\overline{U} \cap X_{\lambda(\beta)}$ as well, and thus by Theorem 3.1.3 we also have the inequality $\dim C \leq n(\beta)$, which yields a contradiction. Thus $M_\alpha \subset X_\alpha$.

To conclude the proof we have to show that if M is open in X and normal, then $M \cap X_\alpha \subset M_\alpha$. Again we shall apply transfinite induction. Clearly $M \cap X_0 \subset M_0$; assume that $M \cap X_\beta \subset M_\beta$ for $\beta < \alpha$. If α is a limit number the inclusion $M \cap X_\alpha \subset M_\alpha$ follows from the inductive assumption and the first part of (2). Consider the case when $\alpha = \beta + 1$. Since $M_\alpha = M_\beta \setminus D_{n(\beta)}(M_{\lambda(\beta)})$, to establish the inclusion $M \cap X_\alpha \subset M_\alpha$ it suffices to prove that $X_\alpha \cap D_{n(\beta)}(M_{\lambda(\beta)}) = \emptyset$. Assume that there exists a point x in the last intersection, consider a neighbourhood U of the point x in $M_{\lambda(\beta)}$ such that $\dim(\overline{U} \cap M_{\lambda(\beta)}) \leq n(\beta)$ and take a neighbourhood V of x in the space X with $\overline{V} \cap M_{\lambda(\beta)} \subset U$; since M is open we can assume that $\overline{V} \subset M$. The intersection $V \cap X_{\lambda(\beta)}$ is a neighbourhood of x in $X_{\lambda(\beta)}$ and its closure $C = \overline{V \cap X_{\lambda(\beta)}} \subset \overline{V} \cap X_{\lambda(\beta)} \subset \overline{V} \cap M \cap X_{\lambda(\beta)} \subset \overline{V} \cap M_{\lambda(\beta)}$ is contained in $\overline{U} \cap M_{\lambda(\beta)}$, so that $x \in D_{n(\beta)}(X_{\lambda(\beta)}) \subset X \setminus X_\alpha$, and we have a contradiction. Thus $M \cap X_\alpha \subset M_\alpha$. ■

The last lemma yields the following two theorems.

7.3.8. THEOREM. *If a subspace M of a normal space X is closed, or open and normal, then* $\operatorname{trker} M \leq \operatorname{trker} X$. ■

7.3.9. THE SUBSPACE THEOREM FOR trker. *For every subspace M of a strongly hereditarily normal space X we have* $\operatorname{trker} M \leq \operatorname{trker} X$. ■

We are now going to prove the sum and enlargement theorems for trker. Let us start with the observation that since Smirnov's spaces S_α are strongly countable-dimensional and $\operatorname{trker} S_\alpha = \alpha$ for $\alpha < \omega_1$, there is no countable sum theorem for the transfinite kernel dimension. We have, however,

7.3.10. THE LOCALLY FINITE SUM THEOREM FOR trker. *If a normal space X can be represented as the union of a locally finite family $\{F_s\}_{s\in S}$ of closed subspaces such that* $\operatorname{trker} F_s \leq \alpha$ *for $s \in S$, then* $\operatorname{trker} X \leq \alpha$.

PROOF. For each $s \in S$ and each ordinal number β we shall denote by $F_{s,\beta}$ the set $(F_s)_\beta \subset X_\beta$. It suffices to show that for every β,

$$X_\beta \subset \bigcup_{s\in S} F_{s,\beta}. \tag{3}$$

We shall apply transfinite induction. Clearly (3) holds for $\beta = 0$. Assume that it holds for all $\beta < \gamma$.

If γ is a limit number then by the inductive assumption we have $X_\gamma = \bigcap_{\beta<\gamma} X_\beta \subset \bigcap_{\beta<\gamma} \bigcup_{s\in S} F_{s,\beta}$, and the last intersection is contained in $\bigcup_{s\in S} F_{s,\gamma}$. Indeed, if $x \notin \bigcup_{s\in S} F_{s,\gamma}$, then for each s in the finite set $S_0 = \{s \in S : x \in F_s\}$ there exists a $\beta(s) < \gamma$ such that $x \in D_{\beta(s)}(F_s)$, and for $\beta_0 = \max\{\beta(s) : s \in S_0\} + 1 < \gamma$ we have $x \notin \bigcup_{s\in S} F_{s,\beta_0}$. Thus (3) holds for $\beta = \gamma$.

Suppose now that $\gamma = \delta + 1$ and consider a point $x \notin \bigcup_{s\in S} F_{s,\gamma}$. For $\lambda = \lambda(\gamma) = \lambda(\delta)$ the set $S_1 = \{s \in S : x \in F_{s,\lambda}\}$ is finite; let $S_1 = \{s_1, s_2, \ldots, s_k\}$, and for each $i \leq k$ take a neighbourhood U_i of x in $F_{s_i,\lambda}$ such that $\dim(\overline{U}_i \cap F_{s_i,\lambda}) \leq n(\delta)$. Since the family $\{F_{s,\lambda}\}_{s\in S}$ is locally finite and consists of closed sets, there exists a neighbourhood U of x in X such that $\overline{U} \cap F_{s,\lambda} = \emptyset$ whenever $s \notin S_1$ and $\overline{U} \cap F_{s_i,\lambda} \subset U_i$ for $i = 1, 2, \ldots, k$. By the inductive assumption $X_\lambda \subset \bigcup_{s\in S} F_{s,\lambda}$, so that

$$\overline{U} \cap X_\lambda \subset \bigcup_{s\in S} \overline{U} \cap F_{s,\lambda} = \bigcup_{i=1}^{k} \overline{U} \cap F_{s_i,\lambda} \subset \bigcup_{i=1}^{k} \overline{U}_i \cap F_{s_i,\lambda}.$$

Since $\dim(\bigcup_{i=1}^k \overline{U}_i \cap F_{s_i,\lambda}) \leq n(\delta)$ by Theorem 3.1.8, it follows from Theorem 3.1.3 that $\dim(\overline{U} \cap X_\lambda) \leq n(\delta)$; thus $x \in D_{n(\delta)}(X_\lambda)$ and $x \notin X_\gamma$. Therefore (3) holds for $\beta = \gamma$ also in the case when $\gamma = \delta + 1$. Thus the proof is complete. ■

7.3.11. THEOREM. *If a normal space X can be represented as the union of a family $\{U_s\}_{s\in S}$ of open normal subspaces such that* $\operatorname{trker} U_s \leq \alpha$ *for $s \in S$, then* $\operatorname{trker} X \leq \alpha$.

PROOF. By virtue of Lemma 7.3.7 for every $\gamma > \alpha$ we have

$$X_\gamma = \bigcup_{s\in S} U_s \cap X_\gamma = \bigcup_{s\in S} (U_s)_\gamma = \emptyset,$$

so that $\operatorname{trker} X \leq \alpha$. ■

7.3.12. THE ENLARGEMENT THEOREM FOR trker. *For every subspace M of a metrizable space X satisfying the inequality* $\operatorname{trker} M \leq \alpha$ *there exists a G_δ-set M^* in X such that $M \subset M^*$ and* $\operatorname{trker} M^* \leq \alpha$.

PROOF. We shall apply transfinite induction with respect to α. If $\alpha = 0$ our theorem follows from Proposition 7.3.2 and Theorems 1.6.10 and 4.1.19. Assume that the theorem is established for subspaces with transfinite kernel dimension less than $\alpha > 0$ and consider a metrizable space X and a subspace M of X such that $\operatorname{trker} M = \alpha$.

First we shall prove that in the case when $M_\alpha = \emptyset$ there exists a G_δ-set M^* in X such that $M \subset M^*$ and $M^*_\alpha = \emptyset$. In this case α is a limit number, so that $\bigcap_{\beta<\alpha} M_\beta = M_\alpha = \emptyset$, and for every $x \in M$ there exists an ordinal number $\beta(x) < \alpha$ and a neighbourhood $U(x) \subset M$ such that $U(x) \cap M_{\beta(x)} = \emptyset$. By Lemma 7.3.7 we have $\operatorname{trker} U(x) \leq \beta(x) < \alpha$. Now for every $x \in M$ choose a neighbourhood $V(x)$ of x in X such that $U(x) = M \cap V(x)$ and consider an open locally finite refinement $\{W_s\}_{s\in S}$ of the open cover $\{V(x)\}_{x\in M}$ of $V = \bigcup_{x\in M} V(x)$. For every $s \in S$ the intersection $M \cap W_s$ is contained in a $U(x)$, so that $\operatorname{trker}(M \cap W_s) < \alpha$ by Theorem 7.3.9, and by the inductive assumption in V there exists a G_δ-set W^*_s such that

$$M \cap \bar{W}_s \subset \bar{W}^*_s \subset \bar{W}_s \quad \text{and} \quad \operatorname{trker} \bar{W}^*_s < \alpha. \tag{4}$$

The family $\{W_s \setminus W^*_s\}_{s\in S}$ is locally finite in V and consists of F_σ-sets, so that its union is an F_σ-set in V, and $M^* = V \setminus \bigcup_{s\in S}(W_s \setminus W^*_s)$ is a G_δ-set in V, and also in X. From the first part of (4) it follows that $M \subset M^*$, and since $M^* \cap W^*_s = M^* \cap W_s$ for each $s \in S$, the family $\{M^* \cap W^*_s\}_{s\in S}$ is an open cover of M^*. By Lemma 7.3.7 and the second part of (4) we have

$$M^*_\alpha = \bigcup_{s\in S}(M^* \cap W^*_s) \cap M^*_\alpha = \bigcup_{s\in S}(M^* \cap W^*_s)_\alpha \subset \bigcup_{s\in S}(W^*_s)_\alpha = \emptyset.$$

Since the equality $M^*_\alpha = \emptyset$ implies that $\operatorname{trker} M^* \leq \alpha$, our theorem holds if $M_\alpha = \emptyset$.

Now consider the case when $M_\alpha \neq \emptyset$. Since $M_{\alpha+1} = \emptyset$, by (2) we have $M_\lambda = D_n(M_\lambda)$, where $\lambda = \lambda(\alpha)$ and $n = n(\alpha)$. Take an open set $U \subset X$ such that $M \cap U = M \setminus M_\lambda$. By Lemma 7.3.7 we have $(M \cap U)_\lambda = \emptyset$, so that by the first part of the proof there exists in U a G_δ-set K such that

$$M \cap U \subset K \subset U \quad \text{and} \quad K_\lambda = \emptyset. \tag{5}$$

Since $\dim M_\lambda \leq n$ by virtue of Theorem 3.1.14, and since $M_\lambda \subset F = X \setminus U$, by Theorems 4.1.3 and 4.1.19 there exists in F a G_δ-set L such that

$$M \setminus U = M_\lambda \subset L \subset F \quad \text{and} \quad \dim L \leq n. \tag{6}$$

As K and L are G_δ-sets in X, the differences $A = U \setminus K$ and $B = F \setminus L$ are F_σ-sets in X, so that $M^* = (K \cup L) \setminus (A \cup B)$ is a G_δ-set in X. One easily checks that $M \subset M^*$ and $U \cap M^* \subset K$, so that $U \cap M^*_\lambda = (U \cap M^*)_\lambda \subset K_\lambda = \emptyset$ by the second part of (5). This implies that $M^*_\lambda \subset F \setminus B = L$, and from the second part of (6) it follows that $\dim M^*_\lambda \leq n$. Hence $M^*_\alpha \subset M^*_\lambda = D_n(M^*_\lambda) = D_\alpha(M^*)$, and $M^*_{\alpha+1} = M^*_\alpha \setminus D_\alpha(M^*) = \emptyset$, i.e., $\operatorname{trker} M^* \leq \alpha$; thus our theorem holds also if $M_\alpha \neq \emptyset$. ■

It turns out that there is no compactification theorem for the transfinite kernel dimension (see Problem 7.3.C). One can prove, however, that for every metrizable space

X there exists a compact space $\widetilde{X}$ which contains a dense subspace homeomorphic to X and satisfies the inequalities $\operatorname{trker}\widetilde{X} \leq \operatorname{trker} X + 1$ and $w(\widetilde{X}) \leq w(X)$. One also proves that for every countable ordinal number α there exists a universal space for the class of all compact metrizable spaces whose transfinite kernel dimension is not larger than α. Similarly, for every ordinal number α and every cardinal number $\mathbf{m} \geq \aleph_0$ there exists a universal space for the class of metrizable spaces whose transfinite kernel dimension is not larger than α and whose weight is not larger than $\mathbf{m}$ (cf. Problem 7.3.D(b)). Finally, one proves that if X and Y are metrizable spaces with $\operatorname{trker} X = \alpha$ and $\operatorname{trker} Y = \beta$, then $\operatorname{trker}(X \times Y) \leq \alpha \oplus \beta$, where $\oplus$ denotes the so-called natural sum of ordinal numbers. Let us note that the above results on compactifications, Cartesian products and universal spaces were proved not for trker but for a transfinite dimension D, defined in a different way, which coincides with trker in the realm of metrizable spaces (see Theorem 7.3.18).

In the next four theorems we shall discuss the relations between shallow spaces and strongly countable-dimensional spaces as well as the relations between the transfinite dimensions trker, trind and trInd.

7.3.13. THEOREM. *Every weakly paracompact perfectly normal shallow space is strongly countable-dimensional.*

PROOF. Consider a weakly paracompact perfectly normal shallow space X and let $\operatorname{trker} X = \alpha$. Since $X_{\alpha+1} = \emptyset$, we have $X = \bigcup_{\beta<\alpha+1} D_\beta(X)$. We shall prove by transfinite induction that $\bigcup_{\beta<\gamma} D_\beta(X)$ is strongly countable-dimensional for each $\gamma \leq \alpha + 1$.

Since X is hereditarily weakly paracompact (see Problem 2.1.D(b)), by virtue of Theorem 3.1.14 we have $\dim D_0(X) \leq 0$, so that $D_0(X)$ is strongly countable-dimensional. Assume now that $\bigcup_{\beta<\gamma} D_\beta(X)$ is strongly countable-dimensional for every $\gamma < \delta \leq \alpha$.

If δ is a limit number we have $\bigcup_{\beta<\delta} D_\beta(X) = \bigcup_{\gamma<\delta} U_\gamma$, where $U_\gamma = \bigcup_{\beta<\gamma} D_\beta(X)$, and by the inductive assumption U_γ is strongly countable-dimensional for every $\gamma < \delta$. Since $U_\gamma = X \setminus X_\gamma$ is an open subset of X, it follows from Theorem 5.2.17 that the union $\bigcup_{\beta<\delta} D_\beta(X)$ is strongly countable-dimensional.

If $\delta = \gamma + 1$, we have $\bigcup_{\beta<\delta} D_\beta(X) = \bigcup_{\beta<\gamma} D_\beta(X) \cup D_\gamma(X)$, and since both terms of the last union are F_σ-sets in X and are strongly countable-dimensional, the union itself is strongly countable-dimensional. ■

7.3.14. COROLLARY. *Every metrizable shallow space is strongly countable-dimensional.* ■

7.3.15. THEOREM. *Every completely metrizable strongly countable-dimensional space is shallow.*

PROOF. Consider a completely metrizable strongly countable-dimensional space X. Assume that the closed subspace $k(X)$ is non-empty. Since $k(X)$ is also completely metrizable and strongly countable-dimensional, by the Baire category theorem $k(X)$ contains a non-empty finitely-dimensional open subspace. This contradiction with the definition of $k(X)$ shows that $k(X) = \emptyset$, i.e., that X is a shallow space. ■

Let us note that in the last theorem the assumption that X is completely metrizable can be weakened to the assumption that the Baire category theorem holds in X; thus the theorem remains valid for normal Čech complete spaces (see [GT], Section 3.9), and in particular for compact spaces.

7.3.16. THEOREM. *For every metrizable space X we have* $\operatorname{trind} X \leq \operatorname{trker} X + 1$.

PROOF. The inequality holds if $\operatorname{trker} X = \infty$, so that we can suppose that $\operatorname{trker} X < \infty$ and apply transfinite induction with respect to $\operatorname{trker} X$. If $\operatorname{trker} X = -1$ the space X is empty and $\operatorname{trind} X \leq \operatorname{trker} X + 1$. Assume that the inequality under consideration holds for all metrizable spaces with trasnfinite kernel dimension less than α and consider a space X with $\operatorname{trker} X = \alpha$.

First we shall prove that in the case when $X_\alpha = \emptyset$ we have $\operatorname{trind} X \leq \alpha$. In this case α is a limit number, so that $\bigcap_{\beta<\alpha} X_\beta = X_\alpha = \emptyset$. Take a point $x \in X$ and a neighbourhood V of x. By the last equality there exists an ordinal number $\beta < \alpha$ and neighbourhood U of the point x such that $U \subset V$ and $\overline{U} \cap X_\beta = \emptyset$. From Lemma 7.3.7 it follows that $(\operatorname{Fr} U)_\beta = \emptyset$, so that $\operatorname{trker} \operatorname{Fr} U \leq \beta < \alpha$. By the inductive assumption we have $\operatorname{trind} \operatorname{Fr} U \leq \beta + 1 < \alpha$, so that indeed $\operatorname{trind} X \leq \alpha$, and our theorem holds if $X_\alpha = \emptyset$.

Now consider the case when $X_\alpha \neq \emptyset$. Since $X_{\alpha+1} = \emptyset$, we have $X_\lambda = D_n(X_\lambda)$, where $\lambda = \lambda(\alpha)$ and $n = n(\alpha)$, so that $\operatorname{Ind} X_\lambda = \dim X_\lambda \leq n$ by Theorems 4.1.3 and 3.1.14. Take a point x and a neighbourhood V of x. Applying Theorem 4.1.13 we obtain a neighbourhood U of the point x such that $U \subset V$ and $\operatorname{Ind}(X_\lambda \cap \operatorname{Fr} U) \leq n - 1$; the inclusion $(\operatorname{Fr} U)_\lambda \subset X_\lambda$ implies that $\operatorname{Ind}(\operatorname{Fr} U)_\lambda \leq n - 1$. If $n = 0$ we have $(\operatorname{Fr} U)_\lambda = \emptyset$, so that by the first part of the proof $\operatorname{trind} \operatorname{Fr} U \leq \lambda \leq \alpha$. If $n > 0$ we have $\operatorname{trker} \operatorname{Fr} U \leq \lambda + (n-1) < \alpha$, so that by the inductive assumption $\operatorname{trind} \operatorname{Fr} U \leq \lambda + (n-1) + 1 \leq \alpha$. Thus $\operatorname{trind} X \leq \alpha + 1 = \operatorname{trker} X + 1$. ■

Let us note that the evaluation in the last theorem cannot be improved (see Problem 7.3.E).

7.3.17. THEOREM. *If a metrizable space X has large transfinite dimension, then* $\operatorname{trInd} X \leq \operatorname{trker} X$.

PROOF. The inequality holds if $\operatorname{trker} X = \infty$, so that we can suppose that $\operatorname{trker} X < \infty$ and apply transfinite induction with respect to $\operatorname{trker} X$. If $\operatorname{trker} X < \omega_0$, the inequality $\operatorname{trInd} X \leq \operatorname{trker} X$ holds by virtue of Theorem 4.1.3 and Propositions 7.1.13 and 7.3.2. Assume that it holds for all metrizable spaces which have trInd and whose transfinite kernel dimension is less than $\alpha \geq \omega_0$. Consider now a metrizable space X which has trInd and satisfies $\operatorname{trker} X = \alpha = \lambda + n$, where $\lambda = \lambda(\alpha)$ and $n = n(\alpha)$, and note that by Theorem 7.1.23 the subspace $X_{\omega_0} = K(X) = S(X)$ of X is compact.

First consider the case when $n = 0$, i.e., $\alpha = \lambda$. Since $X_{\lambda+1} = \emptyset$, we have $\operatorname{Ind} X_\lambda = \dim X_\lambda \leq 0$. By Theorem 4.1.13 for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $L \cap X_\lambda = \emptyset$. If $\lambda = \omega_0$ we have $\operatorname{Ind} L = \dim L < \infty$ by virtue of Theorem 7.1.23. If $\lambda > \omega_0$ we have $\emptyset = L \cap X_\lambda = \bigcap_{\omega_0 \leq \beta < \lambda} L \cap X_\beta$, so that by the compactness of X_{ω_0} there exists an ordinal number $\beta < \lambda$

such that $L \cap X_\beta = \emptyset$. By Lemma 7.3.7 we have $L_\beta = \emptyset$, and this implies that $\operatorname{trker} L \leq \beta$. By the inductive assumption $\operatorname{trInd} L \leq \beta < \alpha$. Thus $\operatorname{trInd} X \leq \alpha = \operatorname{trker} X$.

Now consider the case when $n > 0$. We have $X_{\lambda+n+1} = \emptyset$, so that $\operatorname{Ind} X_\lambda = \dim X_\lambda \leq n$. By Theorem 4.1.13 for every pair A, B of disjoint closed subsets of X there exists a partition L between A and B such that $\operatorname{Ind}(L \cap X_\lambda) \leq n-1$. By Lemma 7.3.7 this implies that $\operatorname{Ind} L_\lambda \leq n-1$, so that $\operatorname{trker} L \leq \lambda + n - 1 < \alpha$, and by the inductive assumption $\operatorname{trInd} L < \alpha$. Thus $\operatorname{trInd} X \leq \alpha = \operatorname{trker} X$ also in the case when $n > 0$. ■

Corollary 7.3.14 and Theorem 7.1.31 imply that if the space X in the last theorem is compact, then the assumption that it has large transfinite dimension can be dropped.

From Corollary 7.3.14 it follows that the countable-dimensional space Y which is not strongly countable-dimensional defined in Example 5.1.7 is a deep space with $\operatorname{trind} Y = \operatorname{trInd} Y = \omega_0$ (see Problem 5.1.D and Proposition 7.1.22). One easily checks that the subspace of Y consisting of all the topological cubes contained in Y is a strongly countable-dimensional deep space which has small transfinite dimension. On the other hand, every strongly countable-dimensional metrizable space which has large transfinite dimension is shallow (see Problem 7.3.F).

We shall now define the D-dimension D and prove that it coincides with the transfinite kernel dimension in the realm of metrizable spaces.

We let $D(\emptyset) = -1$, and for every non-empty normal space X we define $D(X)$ as the smallest ordinal number α such that there exists a closed cover $\{A_\beta\}_{\beta \leq \lambda(\alpha)}$ of the space X satisfying the following conditions:

(D1) The union $\bigcup\{A_\beta : \delta \leq \beta \leq \lambda(\alpha)\}$ is closed for every $\delta \leq \lambda(\alpha)$;
(D2) For every $x \in X$ the set $\{\beta \leq \lambda(\alpha) : x \in A_\beta\}$ has a largest element;
(D3) $\dim A_\beta < \infty$ for every $\beta < \lambda(\alpha)$, and $\dim A_{\lambda(\alpha)} \leq n(\alpha)$;

if no such ordinal number exists, we let $D(X) = \infty$. We call $D(X)$ the *D-dimension* of the space X; one easily sees that this is a topological invariant.

Let us note that we can assume that, rather than (D3), the cover $\{A_\beta\}_{\beta \leq \lambda(\alpha)}$ satisfies the following stronger condition:

(D3′) $\dim A_\beta \leq n(\beta)$ for every $\beta < \lambda(\alpha)$, and $\dim A_{\lambda(\alpha)} \leq n(\alpha)$.

Indeed, it suffices to replace the sets A_β by the sets A'_β, where for every limit number $\lambda < \lambda(\alpha)$ we define $A'_{\lambda+k+n_0+n_1+\ldots+n_k} = A_{\lambda+k}$ for $k = 0, 1, \ldots$, where $n_i = \max(0, \dim A_{\lambda+i})$, and we let the remaining sets $A'_{\lambda+n}$ be empty.

7.3.18. THEOREM. *For every weakly paracompact perfectly normal space X we have $D(X) = \operatorname{trker} X$.*

PROOF. First we shall show that $D(X) \leq \operatorname{trker} X$. Let $\operatorname{trker} X = \alpha$ and let Λ be the set of all limit numbers less than $\lambda(\alpha)$. We clearly have

$$X = \bigcup\{X_\lambda \setminus X_{\lambda+\omega_0} : \lambda \in \Lambda\} \cup X_{\lambda(\alpha)}, \tag{7}$$

and $\dim X_{\lambda(\alpha)} \leq n(\alpha)$ by Problem 2.1.D(b) and Theorem 3.1.14. For every $\lambda \in \Lambda$ the sets $D_\lambda(X) \subset D_{\lambda+1}(X) \subset \ldots$ form an open cover of the subspace $D_\lambda = X_\lambda \setminus X_{\lambda+\omega_0}$; let $W_{\lambda,n}$ be open subsets of X_λ such that $W_{\lambda,0} \supset W_{\lambda,1} \supset \ldots$ and $X_{\lambda+\omega_0} = \bigcap_{n=0}^\infty W_{\lambda,n} = \bigcap_{n=0}^\infty \overline{W}_{\lambda,n}$. One easily checks that the sets $U_{\lambda,n} = D_{\lambda+n}(X) \setminus \overline{W}_{\lambda,n} \subset X_\lambda \setminus \overline{W}_{\lambda,n}$,

where $n = 0, 1, \ldots$, form an open cover of D_λ and that the closure $\overline{U}_{\lambda,n}$ of $U_{\lambda,n}$ in X coincides with its closure in D_λ; obviously each subset of $U_{\lambda,n}$ also has the latter property. Since perfect normality is a hereditary property and each perfectly normal space is countably paracompact, there exists a locally finite open cover $\{V_{\lambda,n}\}_{n=0}^{\infty}$ of D_λ such that $\overline{V}_{\lambda,n} \subset U_{\lambda,n}$ for $n = 0, 1, \ldots$ (see [GT], Theorem 5.2.3). Now, for every $\beta < \lambda(\alpha)$ define

$$A_\beta = \overline{V}_{\lambda(\beta),n(\beta)} \quad \text{and let} \quad A_{\lambda(\alpha)} = X_{\lambda(\alpha)};$$

by (7) the family $\{A_\beta\}_{\beta \le \lambda(\alpha)}$ is a closed cover of the space X. From the local finiteness of $\{\overline{V}_{\lambda,n}\}_{n=0}^{\infty}$ in D_λ it follows that for every n the union $X_{\lambda+\omega_0} \cup \bigcup_{m=n}^{\infty} \overline{V}_{\lambda,m}$ is closed in X, so that condition (D1) is satisfied. By (7) and the local finiteness of $\{\overline{V}_{\lambda,n}\}_{n=1}^{\infty}$ in D_λ condition (D2) is also satisfied. Finally, for every $\beta < \lambda(\alpha)$ we have $A_\beta = \overline{V}_{\lambda(\beta),n(\beta)} \subset U_{\lambda(\beta),n(\beta)} \subset D_\beta(X)$, so that $\dim A_\beta \le n(\beta) < \infty$ by Theorem 3.1.14, and thus condition (D3) is satisfied as well. Hence $D(X) \le \alpha = \operatorname{trker} X$.

Now we shall show that $\operatorname{trker} X \le D(X)$. Let $D(X) = \alpha$ and $\{A_\beta\}_{\beta \le \lambda(\alpha)}$ be a closed cover of the space X satisfying conditions (D1), (D2) and (D3′). To prove that $\operatorname{trker} X \le \alpha = D(X)$ it suffices to show by transfinite induction that for every $\delta \le \lambda(\alpha)$,

$$X_\delta \subset \bigcup_{\delta \le \beta \le \lambda(\alpha)} A_\beta, \tag{8}$$

because the inclusion $X_{\lambda(\alpha)} \subset A_{\lambda(\alpha)}$ and the second part of (D3′) imply that $X_{\alpha+1} = X_{\lambda(\alpha)+n(\alpha)+1} = \emptyset$. Clearly (8) holds for $\delta = 0$. Assume that it holds for all $\delta < \gamma$.

Consider first the case when $\gamma = \delta + 1$. By the inductive assumption we have $X_\delta \subset \bigcup_{\delta \le \beta \le \lambda(\alpha)} A_\beta$. The set $U = X \setminus \bigcup_{\gamma \le \beta \le \lambda(\alpha)} A_\beta$ is open by (D1), and by the last inclusion $U \cap X_\delta \subset A_\delta$. We have

$$U \cap X_{\lambda(\delta)} = \bigcup_{n=0}^{n(\delta)-1} (U \cap D_{\lambda(\delta)+n}(X)) \cup (U \cap X_\delta),$$

and by Theorem 3.1.14, condition (D3′) and Theorem 3.1.19 all terms on the right-hand side of the last equality have covering dimension $\le n(\delta)$. Since all these terms are F_σ-sets in X, we have $\dim(U \cap X_{\lambda(\delta)}) \le n(\delta)$. Thus $U \cap X_\gamma = \emptyset$, which implies that $X_\gamma \subset X \setminus U = \bigcup_{\gamma \le \beta \le \lambda(\alpha)} A_\beta$, i.e., (8) holds for γ if γ has a predecessor.

In the case when γ is a limit number by the inductive assumption we have

$$X_\gamma = \bigcap_{\delta < \gamma} X_\delta \subset \bigcap_{\delta < \gamma} \bigcup_{\delta \le \beta \le \lambda(\alpha)} A_\beta = \bigcup_{\gamma \le \beta \le \lambda(\alpha)} A_\beta,$$

where the last equality is a consequence of condition (D2). Thus (8) holds for γ also in the case when γ is a limit number. This completes the proof. ■

We conclude this section with a brief presentation of the recently introduced transfinite covering dimension. An auxiliary notion of large order will first be defined. For every set S denote by $\mathcal{F}(S)$ the family of all non-empty finite subsets of S and for each $s \in S$ and $\mathcal{F} \subset \mathcal{F}(S)$ let

$$\mathcal{F}^s = \{F \in \mathcal{F}(S) : s \notin F \text{ and } F \cup \{s\} \in \mathcal{F}\}.$$

For every subfamily $\mathcal{F}$ of $\mathcal{F}(S)$ the *large order* $\operatorname{Ord} \mathcal{F}$ of $\mathcal{F}$, which is an ordinal number or the "infinite number" ∞, is defined by the following conditions:

(O1) $\operatorname{Ord}\mathcal{F} = 0$ if and only if $\mathcal{F} = \emptyset$;
(O2) $\operatorname{Ord}\mathcal{F} \leq \alpha$, where α is an ordinal number > 0, if $\operatorname{Ord}\mathcal{F}^s < \alpha$ for every $s \in S$;
(O3) $\operatorname{Ord}\mathcal{F} = \alpha$ if $\operatorname{Ord}\mathcal{F} \leq \alpha$ and the inequality $\operatorname{Ord}\mathcal{F} \leq \beta$ holds for no $\beta < \alpha$;
(O4) $\operatorname{Ord}\mathcal{F} = \infty$ if $\operatorname{Ord}\mathcal{F} \leq \alpha$ holds for no ordinal number α.

Now we are ready to define the transfinite covering dimension.

7.3.19. DEFINITION. For every normal space X we denote by $S(X)$ the set of all pairs (A, B) of disjoint closed subsets of X and by $\mathcal{F}(X)$ the class of all finite subfamilies $\{(A_i, B_i)\}_{i=1}^n$ of $S(X)$ such that if L_i is a partition between A_i and B_i for $i = 1, 2, \ldots, n$, then $\bigcap_{i=1}^n L_i \neq \emptyset$. The number $\operatorname{Ord}\mathcal{F}(X)$ is denoted by $\operatorname{trdim} X$ and called the *transfinite covering dimension* of the space X.

If $\operatorname{trdim} X$ is not equal to ∞, we say that X *has transfinite covering dimension*; otherwise we say that X *does not have transfinite covering dimension*.

One easily sees that whenever normal spaces X and Y are homeomorphic, then $\operatorname{trdim} X = \operatorname{trdim} Y$, i.e., the transfinite covering dimension is a topological invariant.

It turns out that for every normal space X, if $\dim X < \infty$, then $\operatorname{trdim} X = \dim X$, and if $\operatorname{trdim} X < \omega_0$, then $\dim X = \operatorname{trdim} X$ (see Problem 7.3.J(a)). From the characterization of normal spaces which have transfinite covering dimension stated in Problem 7.3.J(b) it follows that for every compact weakly infinite-dimensional space X we have $\operatorname{trdim} X < \infty$. One also shows that for every normal space X we have the inequality $\operatorname{trdim} X \leq \operatorname{trInd} X$ (see Problem 7.3.L), and that if a metrizable space X has transfinite covering dimension, then $\operatorname{trdim} X < \omega_1$ (see Problem 7.3.M(b)).

It has been proved that the transfinite covering dimension is monotone with respect to closed subsets (see Problem 7.3.K), satisfies a sum theorem analogous to Theorem 7.2.14, that for every normal space X we have the equality $\operatorname{trdim} \beta X = \operatorname{trdim} X$, and that every normal space X has a compactification which preserves the transfinite covering dimension and the weight of X.

Historical and bibliographic notes

The definition of the transfinite kernel dimension is related to the "kernel constructions" common in general topology (see Stone [1963]). In the context of infinite-dimensional spaces this construction appeared first in Shmuely [1971], and then in Kozlovskiĭ [1972], where Shmuely's ideas were generalized; further study was done by Polkowski in [1981]. In [1981] Borst (who cites Polkowski's preprint) introduced a transfinite dimension sind, which he called the *strong small transfinite dimension*; our trker, which appears implicitly in Kozlovskiĭ [1972], is a slight modification of sind (if $\operatorname{sind} X$ is a limit ordinal, then $\operatorname{sind} X = \operatorname{trker} X$; otherwise $\operatorname{sind} X = \operatorname{trker} X + 1$).

Theorem 7.3.5 was proved by Henderson in [1968a] for metrizable spaces and the D-dimension. Lemma 7.3.7 in the case when α is a limit number was announced by Kozlovskiĭ in [1972] and proved by Polkowski in [1981] (they considered only sets X_α with limit indices), where some variants of Theorems 7.3.8, 7.3.9 and 7.3.10 are given. The subspace theorem and the locally finite sum theorem for the D-dimension in

metrizable spaces were proved by Henderson in [1968a]. Theorem 7.3.12 (in a slightly different form) was announced by Kozlovskiĭ in [1972] and proved by Polkowski in [1981]; for the D-dimension it was proved by Luxemburg in [1982].

The fact that every metrizable space X has a compactification $\widetilde{X}$ such that $\operatorname{trker}\widetilde{X} \leq \operatorname{trker} X + 1$ and $w(\widetilde{X}) \leq w(X)$ was established by Olszewski in [1991] (in the case of a separable metrizable space by Luxemburg in [1982] (announcement in Kozlovskiĭ [1972])). The existence of a universal space for the class of all compact metrizable spaces whose transfinite kernel dimension is not larger than α was established by Luxemburg in [1984] (announcement in [1972]), and the existence of a universal space for the class of of all metrizable spaces whose transfinite kernel dimension is not larger than α and whose weight is not larger than $\mathbf{m} \geq \aleph_0$ – by Olszewski in [1991] (for $\mathbf{m} = \aleph_0$ by Luxemburg in [1984]). The inequality $\operatorname{trker}(X \times Y) \leq \alpha \oplus \beta$, where X and Y are metrizable spaces such that $\operatorname{trker} X = \alpha$ and $\operatorname{trker} Y = \beta$, and $\oplus$ denotes the natural sum of ordinal numbers, was established by Henderson in [1968a]. As we already stated, these results were formulated and proved not for trker but for the D-dimension.

Theorem 7.3.13 was proved by Polkowski in [1981]; when X is a perfectly normal paracompact space, it follows from a general result on kernel constructions given by Stone in [1963]. Corollary 7.3.14 was announced by Kozlovskiĭ in [1972]); for separable metrizable spaces it can be found in Henderson [1968a]. Theorem 7.3.15 was proved by Kozlovskiĭ in [1972]. Theorems 7.3.16 and 7.3.17 with $\operatorname{trker} X$ replaced by $D(X)$ were obtained respectively by Luxemburg in [1981] (announcement in [1972]) and by Henderson in [1968b].

The D-dimension was introduced and studied by Henderson in [1968a] and [1968b]. The equality $D(X) = \operatorname{trker} X$ was announced for metrizable X by Kozlovskiĭ in [1972], a proof of a slightly more general Theorem 7.3.18 (stated in terms of sind) was given by Cuchillo-Ibáñez in [1991] (the inequality $D(X) \leq \operatorname{sind} X$ was proved by Criado and Tarrés in [1989]).

The transfinite covering dimension was introduced and studied by Borst in [1988] and [1988a]. The sum theorem for trdim mentioned at the end of this section was obtained by Borst in [1988], and the results on compactifications were established independently by Chatyrko in [1991] and Yokoi in [1991]. R. Pol introduced in [1983] a classification of compact metrizable weakly infinite-dimensional spaces by means of the Lusin–Sierpiński index which in principle coincides with the classification by means of the transfinite covering dimension.

Problems

7.3.A (Stone [1963], Henderson [1968b], Shmuely [1971]). Show that a normal space X is shallow if and only if every non-empty closed subspace F of X contains a non-empty normal subspace U open in F with $\dim U < \infty$, or – equivalently – every non-empty closed subspace F of X contains a dense normal locally finite-dimensional subspace open in F.

Note that if the space X is strongly hereditarily normal, then the assumption that F is closed can be dropped.

7.3.B (Shmuely [1971]). Prove that a separable metrizable space X is shallow if and only if X has a strongly countable-dimensional metrizable compactification, i.e., there exists a strongly countable-dimensional compact metrizable space $\widetilde{X}$ which contains a dense subspace homeomorphic to X.

Hint. Apply Theorem 5.3.5.

7.3.C (Henderson [1968b]). Give an example of a separable metrizable space X such that $\operatorname{trker} X = \omega_0$, and yet X has no compactification whose transfinite kernel dimension is equal to ω_0.

7.3.D (Henderson [1968b]). (a) Show that Smirnov's spaces S_α satisfy the equality $\operatorname{trker} S_\alpha = \alpha$ for every $\alpha < \omega_1$.

Hint. Apply Theorem 7.3.17 and Example 7.1.33.

(b) Prove that for every ordinal number α there exists a metrizable space X such that $\operatorname{trker} X = \alpha$.

Hint. See the hint to Problem 7.2.C.

7.3.E. (a) Give an example of a separable metrizable space X such that $\operatorname{trker} X = \omega_0$ and $\operatorname{trind} X_0 = \omega_0 + 1$. Note that necessarily $\operatorname{trInd} X = \infty$.

(b) Show that if a compact metrizable space X satisfies the equality $\operatorname{trker} X = \omega_0$, then $\operatorname{trind} X = \operatorname{trInd} X = \omega_0$.

(c) Give an example of a compact metrizable space X such that $\operatorname{trker} X = \omega_0 + 1$ and $\operatorname{trind} X = \operatorname{trInd} X = \omega_0$.

Hint. See the hint to Problem 7.2.B.

7.3.F (Polkowski [1981]). Show that every strongly countable-dimensional metrizable space which has large transfinite dimension is shallow.

7.3.G (Hattori [1985]). Show that a metrizable shallow space X has large transfinite dimension if and only if X is strongly countable-dimensional and contains no closed nearly S-like subspace.

7.3.H. (a) (Hattori [1983]) Show that if a normal space X contains a closed shallow space F such that every closed subspace of X disjoint from F is shallow, then X is a shallow space.

Hint. Apply Problem 7.3.A.

Remark. As shown by Henderson in [1968a], for every closed subspace F of a metrizable space X we have the inequality $D(X) \leq D(X \setminus F) + D(F)$, i.e., $\operatorname{trker} X \leq \operatorname{trker}(X \setminus F) + \operatorname{trker} F$ (where we assume that $\infty + \alpha = \alpha + \infty = \infty$ for every ordinal number α).

(b) (Shmuely [1971]) Show that if A is a normal subspace of a space X and $\operatorname{trker} A = \alpha$, then there exists a transfinite sequence $D_0^A, D_1^A, \ldots, D_\gamma^A, \ldots$, $\gamma \leq \alpha$, of subsets of X such that for every $\gamma \leq \alpha$ we have $D_\gamma^A \cap A = D_\gamma(A)$, where $D_\gamma(A)$ are the sets from the definition of trker, and D_γ^A is an open subset of $X \setminus \bigcup_{\mu<\lambda(\gamma)} D_\mu^A$. Note that the union $\bigcup_{\gamma\leq\alpha} D_\gamma^A$ is open in X.

(c) (Hattori [1983] for metrizable spaces (announcement Kozlovskiĭ [1972]); for separable metrizable spaces, Shmuely [1971]) Prove that if a strongly hereditarily normal space X can be represented as a finite union of shallow subspaces, then X is a shallow space.

Hint. It suffices to consider the case when X is the union of two shallow subspaces A and B. Let $\operatorname{trker} A = \alpha$ and $\operatorname{trker} B = \beta$, and consider the transfinite sequences $D_0^A, D_1^A, \ldots, D_\gamma^A, \ldots$, $\gamma \leq \alpha$, and $D_0^B, D_1^B, \ldots, D_\delta^B, \ldots$, $\delta \leq \beta$, from part (b). Let $U = \bigcup_{\gamma\leq\alpha} D_\gamma^A$ and $V = \bigcup_{\delta\leq\beta} D_\delta^B$. First show by transfinite induction with respect to β that for every $\gamma \leq \alpha$ the intersection $D_\gamma^A \cap V$ is shallow. Then show by transfinite induction that for every $\kappa \leq \alpha$ the union $\bigcup_{\gamma\leq\kappa} D_\gamma^A$ is shallow and deduce that U is a shallow space.

(d) Deduce from (c) that if a strongly hereditarily normal space X can be represented as the union of a locally finite family of shallow subspaces, then X is a shallow space.

7.3.I. Show that there is no universal space either for the class of all compact metrizable shallow spaces or for the class of all separable metrizable shallow spaces.

7.3.J (Borst [1988]). (a) Show that for every normal space X, if $\dim X < \infty$, then $\operatorname{trdim} X = \dim X$, and if $\operatorname{trdim} X < \omega_0$, then $\dim X = \operatorname{trdim} X$.

Hint. Check that $\operatorname{Ord} \mathcal{F} \leq n$ if and only if for every $F \in \mathcal{F}$ we have $|F| \leq n$.

(b) Prove that a normal space X has transfinite covering dimension if and only if X is an S-weakly infinite-dimensional space (see Problem 6.1.E).

Hint. Let $\mathcal{F} \subset \mathcal{F}(S)$ have the property that if $F \in \mathcal{F}$ and $F' \subset F$, then $F' \in \mathcal{F}$; show that $\operatorname{Ord} \mathcal{F} = \infty$ if and only if there exists a sequence $s_1, s_2, \ldots$ of distinct elements of S such that $\{s_1, s_2, \ldots, s_n\} \in \mathcal{F}$ for $n = 1, 2, \ldots$

7.3.K (Borst [1988]). Show that for every closed subspace M of a normal space X we have $\operatorname{trdim} M \leq \operatorname{trdim} X$.

Remark. In [1992] Borst defined a compact metrizable space X and a subspace M of X such that $\operatorname{trdim} X = \omega_0$, and yet $\operatorname{trdim} M = \omega_0 + 1$.

7.3.L (Borst [1988]). Prove that for every normal space X we have $\operatorname{trdim} X \leq \operatorname{trInd} X$ (cf. Problem 7.3.N(b)).

Hint. Prove that if for every pair A, B of disjoint closed subsets of a normal space X there exists a partition L between A and B such that $\operatorname{trdim} L < \alpha$, then $\operatorname{trdim} X \leq \alpha$.

Remark. Borst gave in [1988a] an example of a compact metrizable space X such that $\operatorname{trdim} X = \omega_0$ and $\operatorname{trInd} X = \omega_0 + 1$.

7.3.M (Hattori [1986]). (a) Prove that if a normal space X contains a closed subspace K such that $\operatorname{trdim} F \leq \alpha$ for every closed subspace $F \subset X$ disjoint from K, then $\operatorname{trdim} X \leq \alpha + \operatorname{trdim} K$.

(b) Prove that if a metrizable space X has transfinite covering dimension, then it satisfies the inequality $\operatorname{trdim} X < \omega_1$.

Hint. First prove the inequality under consideration for compact metrizable spaces, then apply Problems 7.3.J(b), 6.1.E(c) and part (a).

7.3.N (Hattori [1986]). Prove that Smirnov's spaces S_α satisfy the equality $\operatorname{trdim} S_\alpha = \alpha$ for every $\alpha < \omega_1$.

Hint (Borst [1988a]). For every $\alpha < \omega_1$ define inductively the family $\mathcal{P}_\alpha$ of "opposite faces" of S_α, let $\mathcal{F}_\alpha$ consist of those families in $\mathcal{F}(S_\alpha)$ whose members belong to $\mathcal{P}_\alpha$, and prove that if $\{(A_i, B_i)\}_{i=1}^n \in \mathcal{F}_\alpha$, then there exists a (topological) cube $I^n \subset S_\alpha$ such that $\{(I^n \cap A_i, I^n \cap B_i)\}_{i=1}^n$ is the family of all pairs of opposite faces of I^n. Then prove by transfinite induction that $\operatorname{Ord} \mathcal{F}_\alpha = \alpha$.

Remark. From the remark to Problem 7.1.G(e) it follows that $\operatorname{trind} S_{\omega_0+3} < \operatorname{trdim} S_{\omega_0+3}$.

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List of special symbols

Symbols defined in the text

$\operatorname{ind} X$	2	$\lvert\mathcal{K}\rvert$	81
$\operatorname{ind}_x X$	7	$\operatorname{St}_{\mathcal{K}}(p_i)$	81
$\lVert x\rVert$	12, 80	$\mathcal{N}(\mathcal{U})$	82
Q_k^n	17	$N(\mathcal{U})$	82
Q_k	18	$\widehat{\rho}(f,g)$	94
N_k^n	34	Y^X	94
L_k^n	34	M_n^m	96
N_n	34	$\operatorname{St}(x,\mathcal{U})$	131
$\operatorname{Ind} X$	40	$\operatorname{rd}_X A$	170
$\operatorname{ord}\mathcal{A}$	41	$\operatorname{St}(B,\mathcal{B})$	193
$\dim X$	42	$B(\mathbf{m})$	225
$\operatorname{mesh}\mathcal{A}$	43	$J(\mathbf{m})$	233
$\mu\dim(X,\rho)$	44	$N_n(\mathbf{m})$	234
$\mathcal{A}_1\wedge\mathcal{A}_2\wedge\ldots\wedge\mathcal{A}_k$	47	$K(X)$	290
$f^{-1}(\mathcal{A})$	47	$\rho_H(A,B)$	313
$\mathcal{A}\lvert M$	47	$\operatorname{trind} X$	326
S	64	$\operatorname{trInd} X$	328
K_ω	64	$S(X)$	331
N_ω	64	S_α	333
$\dim_G X$	75	$\operatorname{trker} X$	351
$p+q$	80	$k(X)$	352
λp	80	$D(X)$	358
$p_0p_1\ldots p_n$	80	$\operatorname{trdim} X$	360

Symbols from set theory and general topology

$x\in A$, $x\notin A$ – x belongs to A, x does not belong to A
$\emptyset$ – empty set
$A\subset B$, $B\supset A$ – A is contained in B
$\{x\in X:\varphi(x)\}$, $\{x:\varphi(x)\}$ – the set of x's which satisfy φ
$A\cup B$, $\bigcup_{s\in S}A_s$, $\bigcup_{i=1}^{\infty}A_i$, $\bigcup\mathcal{A}$ – union of sets
$A\cap B$, $\bigcap_{s\in S}A_s$, $\bigcap_{i=1}^{\infty}A_i$, $\bigcap\mathcal{A}$ – intersection of sets
$A\setminus B$ – difference of sets
$X\times Y$, $\prod_{s\in S}X_s$, $\prod_{i=1}^{\infty}X_i$ – Cartesian product of sets and spaces
$\{x_1,x_2,\ldots,x_k\}$ – set consisting of points $x_1,x_2,\ldots,x_k$
$x_1,x_2,\ldots$, $\{x_i\}$ – infinite sequence
$\{x_1,x_2,\ldots\}$ – set consisting of points $x_1,x_2,\ldots$
id_X – the identity mapping of X onto itself
$f(A)$, $f^{-1}(B)$ – image and inverse image of a set under a mapping
gf – composition of mappings
$\lvert A\rvert$ – cardinality of a set
$\aleph_0$ – cardinality of the set of natural numbers (aleph zero)
$\mathfrak{c}$ – cardinality of the set of real numbers (continuum)

$A_1, A_2, \ldots, A_\alpha, \ldots, \alpha < \xi$ – transfinite sequence of sets
R, I – the real line and the closed unit interval
R^n – Euclidean n-space
I^n – unit n-cube in R^n
S^{n-1}, B^n – unit $(n-1)$-sphere and unit n-ball in R^n
$I^{\aleph_0}$ – Hilbert cube
$\rho(x, y)$ – distance between points
$\rho(x, A)$ – distance from a point to a set
$\rho(A, B)$ – distance between sets
$B(x, r)$, $B(A, r)$ – balls in a metric space
$\delta(A)$ – diameter of a set
$f: X \to Y$ – continuous mapping
$f|A$, f_B – restrictions of a mapping
$\bar{A}$, $\operatorname{Int} A$, $\operatorname{Fr} A$ – closure, interior and boundary of a set
A^{d} – set of accumulation points of a set
$X^{\mathbf{m}}$ – Cartesian power of a space
X/E – quaotient space
$\{X_i, \pi_j^i\}$, $\{X_\sigma, \pi_\rho^\sigma, \Sigma\}$ – inverse sequence and inverse system of spaces
$\varprojlim \mathbf{S}$ – limit of an inverse sequence or an inverse system
$X \oplus Y$, $\bigoplus_{s \in S} X_s$ – sum of spaces
βX – Čech–Stone compactification
$w(X)$ – weight of a space

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