

Dimension Theory

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PREFACE

Dimension theory developed from an initial effort to give a rigorous definition of dimension for the line and the plane. By 1941, as can be seen from Hurewicz and Wallman's "Dimension Theory," classical dimension theory for separable metric spaces had been largely completed. The highlights of this theory include Brouwer's invariance of domain theorem, Lebesgue's covering theorem, and the fundamental equalities connecting the three basic dimension functions: in our notation, $\text{ind} = \text{dim} = \text{Ind}$.

In spite of many efforts to extend dimension theory to nonseparable spaces there was almost no progress before Dieudonné and Stone established a foundation for the theory of open coverings. This theory is given in Chapter 1. In 1952–1954 Katětov and Morita used this theory to establish the equality $\text{dim} = \text{Ind}$ for general metric spaces. Since that time, many other interesting results have been discovered, some of which are new even for separable metric spaces.

It seems difficult at the present stage to go beyond metric spaces and still retain the equality $\text{dim} = \text{Ind}$. Then are we at a dead end? The answer is no. The gaps between the basic dimension functions, far from being obstructions, are very interesting in themselves. The author thinks of the present status of dimension theory as a symphony having basic rhythms, some of which are ind , dim , Ind , and metric-dependent dimension functions. Hence, in general, they play different parts. In this book the reader will look at some of the "music" that appears. And a study of the gaps has this advantage: If we have enough knowledge about the gaps, then by avoiding them carefully it may be easier to find a new class of spaces where a "harmonious" dimension theory could be developed.

The aim of this book is thus to study the theory not only for metric spaces but also for nonmetric spaces. So the present book has different features than Nagata's "Modern Dimension Theory," in which the main concern is with metric spaces. Another characteristic of the present book is the inclusion of many examples. Students who have only an elementary knowledge of general topology will be able to read the book (except the appendix) without difficulty. We expect the reader to be familiar with metric spaces, compact spaces, the Hilbert cube, and Tietze's extension theorem for normal spaces. Chapter 1 is designed to be used as a modern textbook for a second course in general topology. Except for Sections 5 and 7 this chapter may also be used as an introduction to algebraic topology. If the reader would like to have an outline of modern dimension theory rapidly, the author recommends Sections 1, 2, 3, and 6 in Chapter 1, Sections 8 and 9 in Chapter 2, and Section 12 in Chapter 3. The book is based on lectures given at Duke University, 1963–1965, and at the University of Pittsburgh, 1968–1969.

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Professor Y. Kodama has given us an exciting present: an appendix which will lead the reader rapidly to the present state of algebraic dimension theory. The so-called logarithmic law for dimension, in particular, requires algebraic treatment.

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Chapter 1

THEORY OF OPEN COVERINGS

Throughout this book *all spaces are T_1 -spaces*, i.e., spaces in which every point is a closed set. Our main concern is with normal spaces or spaces with stronger conditions than normality. It is also to be noted that coverings are not necessarily assumed to be open. Closed coverings often play an important role in this book. The index i always ranges countably, and usually it ranges over the positive integers.

This chapter is mainly concerned with the theory of open coverings and its basic applications. The normality of open coverings is a powerful tool and often reflects global properties of a space. Many important topological problems can be reduced to the question of the normality of certain types of open coverings. This technique is used throughout the book. Section 1 is a study of simplicial complexes having two kinds of topologies: the weak topology and the metric topology. In Section 2 the reader will see the problem of the normality of a given open covering of a space translated to the corresponding problem on a simplicial complex with the metric topology. Since it is easy and convenient to handle metric spaces, this technique changes difficult problems to easy ones. After establishing several criteria for the normality of open coverings in Section 2, we apply them in Sections 3, 4, and 5 to the modernized theory of extensors and absolute retracts and to metrization theorems. In Section 6 inverse limits and the bridge theorem are studied. The bridge theorem (Corollary 6-8) will play an important role in Section 28 in treating the dimension of the complement of a disjoint sum. The theory of inverse limits is also one of the leading ideas throughout the book. The advantage is to change a complicated problem to a simple one about each

coordinate space in the inverse limiting system. Section 7 is a study of totally normal spaces, a class of spaces well-designed for the large inductive dimension. Total normality is a generalization of the classical concept of perfect normality.

1. Complexes

1-1. DEFINITION. Let $K = \{s_\alpha\}$ be a set and $<$ be a relation defined for some pairs of elements of K such that:

- i. If $s_\beta < s_\alpha$ and $s_\gamma < s_\beta$, then $s_\gamma < s_\alpha$
- ii. ϕ (the empty set) is in K
- iii. For any $s_\alpha \in K$, $\phi < s_\alpha$

Then K with the relation $<$ is an *abstract complex*. If $s_\beta < s_\alpha$, then s_β is a *face* of s_α ; furthermore, if $s_\beta \neq s_\alpha$, $s_\beta \neq \phi$, then s_β is a *proper face* of s_α . Let L be a subset of K . If the statements $s_\alpha \in L$ and $s_\beta < s_\alpha$ imply that $s_\beta \in L$, then L is a *subcomplex* of K .

Let P be a set and $K(P)$ the set of all finite subsets of P . For any pair of elements of $K(P)$, say s_α and s_β , we define $s_\alpha < s_\beta$ if and only if s_α is a subset of s_β . The abstract complex $K(P)$ thus obtained is a *full complex* spanned by P , and P is the *vertex set* of $K(P)$. A point v of P is a *vertex* of $K(P)$. If $s_\alpha \in K(P)$ and $v \in s_\alpha$, then v is a vertex of s_α . If

$$s = \{v_0, \dots, v_m\}$$

then s is *spanned* by $v_0, \dots, v_m$ and the number m is the *combinatorial dimension* (or merely the dimension) of s , denoted by $\dim s$. If L is a subcomplex of $K(P)$, then L is a *simplicial complex*. An element of L is a *simplex* of L . An *n-simplex* is a simplex whose dimension is n . If we set

$$Q = \{v: v \in s, \quad s \in L\}$$

then Q is the *vertex set* of L and elements of Q are vertices of L . The *combinatorial dimension* of L , $\dim L$, is

$$\max\{n: \dim s = n, \quad s \in L\}$$

$\dim L$ may be infinite. Let $|Q|$ be the power of Q . When $|Q| < \infty$, $|Q| = \infty$, or $|Q| \leq \aleph_0$, L is, respectively, a *finite*, an *infinite*, or a *countable* simplicial complex. If each vertex of L is not a face of an infinite number of simplices of L , then L is *star-finite*. The *n-skeleton* of L is the subcomplex

$$\{s: s \in L, \quad \dim s \leq n\}$$

Let P be a set with $|P| > n$. Then the n -full complex spanned by P is the n -skeleton of the full complex spanned by P .

1-2. DEFINITION. Let L be a simplicial complex and Q the vertex set of L . Let $|L|$ be the aggregate of all forms

$$x = \sum_{v \in S} \lambda_v \cdot v, \quad s \in L$$

where the λ_v 's are nonnegative real numbers and $\sum_v \lambda_v = 1$. Let $|s|$ be the aggregate of all forms: $\sum_{v \in s} \lambda_v \cdot v$ in $|L|$, where $s \in L$. Let $p_0 = \langle 0, 0, \dots \rangle$, $p_1 = \langle 1, 0, \dots \rangle$, $p_2 = \langle 0, 1/2, 0, \dots \rangle$, etc., be points of the Hilbert cube I^ω . Then any finite points of them are in general position. Let e_n , a Euclidean n -simplex, be the closed convex hull spanned by $p_0, p_1, \dots, p_n$. Then every point p of e_n is expressed as $\sum_{i=0}^n \lambda_i \cdot p_i$ where the λ_i 's are the barycentric coordinates† with respect to $p_0, \dots, p_n$. For any simplex s spanned by $v_0, \dots, v_n$, define the transformation $\varphi_s: |s| \rightarrow e_n$ as follows:

$$\varphi_s\left(\sum_{i=0}^n \lambda_{v_i} \cdot v_i\right) = \sum_{i=0}^n \lambda_{v_i} \cdot p_i$$

Let us introduce a topology into $|L|$ in such a way that a subset G of $|L|$ is open if and only if, for each simplex $s \in L$, where $\dim s = n$, $\varphi_s(G \cap |s|)$ is open in e_n . Then $|L|$ is evidently a topological space, i.e., a T_1 -space. This topology is called the *weak topology* of $|L|$. $|L|$ with the weak topology is denoted by $|L|_w$ and is called a (simplicial) *polyhedron with the weak topology*. If v is a vertex of L , $1 \cdot v$ in $|L|$ is called a vertex of $|L|$ and is expressed simply as v .

Let us define another topology for $|L|$. Every point x of $|L|$ can be written as $\sum_{v \in Q} \lambda_v \cdot v$, where all but a finite number of λ_v 's are zero. Take two points $x = \sum \lambda_v \cdot v$ and $y = \sum \mu_v \cdot v$ expressed in this manner. Set

$$\rho(x, y) = [(\sum \lambda_v - \mu_v)^2]^{1/2}$$

Then ρ is a metric on $|L|$. The topology of $|L|$ defined by this metric is called the *metric topology* of $|L|$. $|L|$ with the metric topology is denoted by $|L|_{\text{met}}$ and is called a (simplicial) *polyhedron with the metric topology*. $|L|_w$ and $|L|_{\text{met}}$ are *geometric realizations* of L . In the sequel $|L|$ stands for $|L|_w$ or $|L|_{\text{met}}$. Sometimes a polyhedron $|L|$ implies a pair $(|L|, L)$. A *subpolyhedron* of $|L|$ is a subspace $|M|$ of $|L|$ corresponding to a subcomplex M of L . When L is full or n -full, $|L|$ is called

† Compare Hurewicz and Wallman [1, p. 70, footnote *].

full or n -full, respectively. For any n -simplex $s \in L$, $|s|$ with either the weak topology or the metric topology is homeomorphic to a Euclidean n -simplex e_n . $|s|$ with this unique topology is called an n -simplex again.

Let x be a point of $|L|$ with $x = \sum_{v \in Q} \lambda_v \cdot v$. Then the λ_v 's are *barycentric coordinates* of x and each λ_v is the *weight* on v of x . Let v be a vertex of $|L|$. Set the *star* of v :

$$\text{St}(v) = \{x: \text{the weight on } v \text{ of } x \text{ is positive}\}$$

1-3. REMARK. Let L be a simplicial complex, X a space, and f a transformation of $|L|_w$ to X . Then f is continuous if and only if for every simplex $s \in L$, $f|_{|s|}$, the restriction of f to $|s|$, is continuous. Therefore, the identity transformation $g: |L|_w \rightarrow |L|_{\text{met}}$ is continuous. In other words the weak topology is stronger (!) than the metric topology. If L is finite or star-finite, then g is a homeomorphism.

1-4. DEFINITION. Let X be a set and $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ a collection of subsets of X . Let $K(A)$ be the full complex spanned by A and let

$$N(\mathcal{G}) = \{s \in K(A): \cap \{G_\alpha: \alpha \in s\} \neq \emptyset\}$$

Then $N(\mathcal{G})$ is a subcomplex of $K(A)$ and is called the *nerve* of $\mathcal{G}$. A geometric realization of $N(\mathcal{G})$, $|N(\mathcal{G})|_w$ or $|N(\mathcal{G})|_{\text{met}}$, is a *geometric nerve*, or simply a *nerve*, of $\mathcal{G}$.

Let X be a topological space and $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ an open covering of X . $\mathcal{G}$ is *locally finite* if every point x in X has a neighborhood which meets at most a finite number of elements of $\mathcal{G}$. It is to be noted that a neighborhood is not necessarily open in this book.

1-5. THEOREM. Let X be a topological space and $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ a locally-finite open covering of X such that for every α there exists a continuous nonnegative function f_α defined on X with $G_\alpha = \{x: f_\alpha(x) > 0\}$,[†] then the transformation f defined by

$$f(x) = \sum_{\alpha \in A} [f_\alpha(x) / \sum_{\beta \in A} f_\beta(x)] \cdot \alpha, \quad x \in X$$

is a mapping[‡] of X to the nerve $|N(\mathcal{G})|_w$.

[†] This type of set is a *cozero set*. A set is a *zero set* if its complement is a cozero set.

[‡] A *mapping* is always a continuous transformation in this book.

PROOF. First f is clearly a transformation into $|N(\mathcal{G})|_w$. To prove the continuity of f , let x be an arbitrary point of X and U an open neighborhood of x such that

$$A' = \{\alpha: U \cap G_\alpha \neq \emptyset\}$$

is a finite set of indices. Set

$$L = \{s = \{\alpha_0, \dots, \alpha_n\} \in N(\mathcal{G}): U \cap (\bigcap_{i=0}^n G_{\alpha_i}) \neq \emptyset\}$$

Then L is a finite subcomplex of $N(\mathcal{G})$ whose vertex set is A' . $f|_U$ is a transformation of U to $|L|$. Since for each $y \in U$

$$f_\alpha(y) / \sum_{\beta \in A'} f_\beta(y) = f_\alpha(y) / \sum_{\beta \in A'} f_\beta(y)$$

and the right side is continuous on U , the weight of $f(y)$ on $\alpha \in A'$ is continuous. Thus $f|_U$ is continuous and hence f is continuous. $\square$

This mapping f is called the *canonical mapping* of $\mathcal{G}$ (with respect to the f_α 's). It is to be noted that, for every α , $f^{-1}(\text{St}(\alpha)) = G_\alpha$.

1-6. DEFINITION. A collection of sets $\mathcal{U}$ is *refined* by a collection of sets $\mathcal{V}$ if for any set $V \in \mathcal{V}$ there exists a set $U \in \mathcal{U}$ with $V \subset U$. $\mathcal{V} < \mathcal{U}$ is the abbreviation of this fact.

Let X be a space and $\mathcal{G}$ an open covering of X . If there exists a polyhedron $|L|$ and a mapping $f: X \rightarrow |L|$ such that $\mathcal{G}$ is refined by

$$\{f^{-1}(\text{St}(v)): v \in \text{the vertex set of } |L|\}$$

then $\mathcal{G}$ is said to be approximated by the polyhedron $|L|$.[†]

Let (X, ρ) be a metric space with a metric ρ . Let $S_\varepsilon(x) = \{y: \rho(x, y) < \varepsilon\}$. For a subset H of X , $S_\varepsilon(H) = \cup \{S_\varepsilon(x): x \in H\}$. If for every $\varepsilon > 0$, the ε -covering $\{S_\varepsilon(x): x \in X\}$ is approximated by a polyhedron, (X, ρ) is said to be *approximated by polyhedra*.

The study of metric spaces which can be approximated by polyhedra of dimension $\leq n$ has been one of the most serious problems in dimension theory. We shall discuss this problem in Chapter 7.

TERMINOLOGY. A collection $\mathcal{U}$ of subsets of a set X is *point-finite* if every point x of X is contained in at most a finite number of elements of $\mathcal{U}$.

[†] If $\mathcal{G}$ is approximated by a polyhedron with the weak topology, then $\mathcal{G}$ is approximated by the same polyhedron with the metric topology, because of Remark 1-3.

1-7. THEOREM (Lefschetz [1]). Every point-finite open covering $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ of a normal space X can be refined by a closed covering $\{F_\alpha: \alpha \in A\}$ in such a way that $F_\alpha \subset G_\alpha$ for each α . (This type of refinement is called a refinement in one-one correspondence.)

PROOF (by transfinite induction). Well-order the index set A as $A = \{0, 1, \dots\}$. By the normality of X there exists an open set H_0 such that

$$X - \bigcup_{0 < \alpha} G_\alpha \subset H_0 \subset \overline{H_0} \subset G_0$$

Then $\{H_0, G_1, G_2, \dots\}$ is an open covering of X . Suppose that, for α with $0 < \alpha \in A$, open sets $H_\beta, \beta < \alpha$, are defined in such a way that

$$\overline{H_\beta} \subset G_\beta, \quad \beta < \alpha$$

$$\{H_0, \dots, H_\beta, G_{\beta+1}, G_{\beta+2}, \dots\} \text{ covers } X, \quad \beta < \alpha$$

Then $\{H_\beta, G_\gamma: \beta < \alpha \leq \gamma\}$ covers X by the point-finiteness of $\mathcal{G}$. In the same way that H_0 was defined we can find an open set H_α such that $\overline{H_\alpha} \subset G_\alpha$ and $\{H_\beta, G_\gamma: \beta \leq \alpha < \gamma\}$ covers X . Again by the point-finiteness of $\mathcal{G}$, $\{H_\beta: \beta \in A\}$ covers X . Set

$$F_\alpha = \overline{H_\alpha}, \quad \alpha \in A$$

Then we get the desired closed covering $\{F_\alpha\}$. $\square$

An open covering of a space is called *shrinkable* if it can be refined by a closed covering in one-one correspondence, as $\mathcal{G}$ was in this theorem.

1-8. COROLLARY. Every locally finite open covering $\{G_\alpha\} = \mathcal{G}$ of a normal space X is approximated by a polyhedron with the weak topology.

PROOF. Since $\mathcal{G}$ is evidently point-finite, there exists by Theorem 1-7 a closed covering $\{F_\alpha\}$ such that $F_\alpha \subset G_\alpha$ for every α . Since X is normal there exists for every α a cozero set H_α with $F_\alpha \subset H_\alpha \subset G_\alpha$. Since $\{H_\alpha\}$ is locally finite, $\{H_\alpha\}$ is approximated by a polyhedron with the weak topology, as can easily be seen from the argument in the proof of Theorem 1-5. Hence, $\{G_\alpha\}$ is so approximated. $\square$

2. Normality of Open Coverings

2-1. DEFINITION. Let X be a topological space and $\mathcal{H} = \{H_\alpha\}$ be a collection of subsets of X . $\mathcal{H}$ is called *discrete* if $\overline{\mathcal{H}} = \{\overline{H_\alpha}\}$ is disjoint and

locally finite. $\mathcal{H}$ is called σ -discrete or σ -locally finite if there exists a sequence of subfamilies $\mathcal{H}_1, \mathcal{H}_2, \dots$, of $\mathcal{H}$ with $\mathcal{H} = \bigcup \mathcal{H}_i$ such that every $\mathcal{H}_i$ is respectively discrete or locally finite.

2-2. DEFINITION (Bing [1]). A topological space X is called *collectionwise normal* if, for any discrete collection $\{H_\alpha\}$ of subsets of X , there exists a disjoint open collection $\{G_\alpha\}$ such that $H_\alpha \subset G_\alpha$ for every α .

Clearly a collectionwise normal space is normal. It can easily be seen that a space X is collectionwise normal if and only if for any discrete collection $\{H_\alpha\}$ of subsets of X there exists a discrete collection $\{G_\alpha\}$ of open sets of X such that $H_\alpha \subset G_\alpha$ for every α .

2-3. EXAMPLE (Bing [1]). There exists a normal space which is not collectionwise normal. Let P be an uncountable set, Q the set of all subsets of P , and F the set of all functions f on Q having only 0 and 1 as values. To each point p of P associate the function f_p whose value $f_p(q) = 1$ if and only if $p \in q$. Let F_0 be the set of all such functions f_p . Let Q_0 be the set of all finite subsets of Q . The set F is topologized as follows: Any point f in $F - F_0$ is open. For each $f_p \in F_0$ and each $r \in Q_0$, set

$$(f_p: r) = \{f \in F: f(q) = f_p(q) \text{ whenever } q \in r\}$$

$\{(f_p: r): r \in Q_0\}$ is an open neighborhood base of f_p .

ASSERTION 1. F is normal.

PROOF. Obviously every point of F is closed. Let H_1 and H_2 be a disjoint pair of closed sets of F . Set for $i = 1, 2$,

$$A_i = H_i \cap F_0, \quad q_i = \{p: f_p \in A_i\}$$

If $A_1 = \phi$, then H_1 and $F - H_1$ are open sets containing H_1 and H_2 , respectively. We consider the case where neither A_1 nor A_2 is empty. Let

$$D_i = \{f \in F: f(q_i) = 1 \text{ and } f(q_j) = 0 \text{ } (j \neq i)\}$$

Then D_1 and D_2 are open sets with $D_i \supset A_i$, $i = 1, 2$, and $D_1 \cap D_2 = \phi$. Therefore, $H_1 \cup (D_1 - H_2)$ and $H_2 \cup (D_2 - H_1)$ are mutually exclusive open sets containing H_1 and H_2 , respectively. $\square$

ASSERTION 2. F is not collectionwise normal.

PROOF. Consider F_0 which is a discrete collection of points f_p . However, there does not exist a disjoint collection $\{(f_p: r_p): f_p \in F_0\}$. For suppose,

to the contrary, that there were such a collection. Since r_p is a finite set of Q , and P is uncountable, there exist an integer n and an uncountable subset P_0 of P such that r_p has exactly n elements for every p in P_0 . For any two elements a and b of P_0 , $r_a \cap r_b \neq \phi$, otherwise $(f_a: r_a) \cap (f_b: r_b) \neq \phi$. Hence there exist an element q_1 of Q and an uncountable subset P_1' of P_0 such that q_1 belongs to r_p for every p in P_1' . Moreover there exist a t_1 with value 0 or 1 and an uncountable subset P_1 of P_1' such that

$$f_p(q_1) = t_1 \quad \text{for every } p \in P_1$$

Similarly, there exist an element q_2 of Q with $q_2 \neq q_1$, a t_2 with value 0 or 1, and an uncountable subset P_2 of P_1 such that $q_2 \in r_p$ and $f_p(q_2) = t_2$ for every p in P_2 . Continuing recursively in this fashion we get

$$q_k, \quad t_k, \quad P_k \quad \text{for } k = 1, \dots, n$$

Let r be the set consisting of $q_1, \dots, q_n$ and D the set of all f with $f(q_k) = t_k$ for $k = 1, \dots, n$. Then

$$D = (f_p: r_p) = (f_p: r)$$

for all p in P_n , which provides a contradiction. $\square$

2-4. DEFINITION (These concepts come from Tukey [1]). Let X be a set and $\mathcal{H}$ a collection of subsets of X . If Y is a subset of X , set

$$S(Y, \mathcal{H}) = \cup \{H \in \mathcal{H} : Y \cap H \neq \phi\}$$

$$S^n(Y, \mathcal{H}) = S(S^{n-1}(Y, \mathcal{H}), \mathcal{H}), \quad n = 1, 2, \dots$$

$$\mathcal{H}^* = \{S(H, \mathcal{H}) : H \in \mathcal{H}\}, \quad \mathcal{H}^{**} = (\mathcal{H}^*)^*$$

$$\mathcal{H}^\Delta = \{S(x, \mathcal{H}) : x \in X\}, \quad \mathcal{H}^{\Delta\Delta} = (\mathcal{H}^\Delta)^\Delta$$

Then it is evident that

$$\mathcal{H}^\Delta < \mathcal{H}^* < \mathcal{H}^{\Delta\Delta}$$

Let $\mathcal{G}$ be another collection of subsets of X . When $\mathcal{G}^\Delta < \mathcal{H}$, $\mathcal{G}$ is a Δ -refinement of $\mathcal{H}$ or $\mathcal{G}$ Δ -refines $\mathcal{H}$. In such a case we write

$$\mathcal{G} \triangleleft \mathcal{H} \quad \text{or} \quad \mathcal{H} \triangleright \mathcal{G}$$

When $\mathcal{G}^* < \mathcal{H}$, $\mathcal{G}$ is a $*$ -refinement of $\mathcal{H}$ or $\mathcal{G}$ $*$ -refines $\mathcal{H}$. In such a case we write

$$\mathcal{G}^* \triangleleft \mathcal{H} \quad \text{or} \quad \mathcal{H}^* \triangleright \mathcal{G}$$

Let X be a space and $\mathcal{G}$ an open covering of X . If there exists a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open coverings of X with

$$\mathcal{G} \supset \mathcal{G}_1 \supset \mathcal{G}_2 \supset \dots$$

$\mathcal{G}$ is called *normal*. If every open covering of a space X is Δ -refined by an open covering, then X is called a *fully normal* space. It is evident that a space X is fully normal if and only if every open covering of X is normal.

2-5. THEOREM. A fully normal space X is collectionwise normal.

PROOF. Let $\{H_\alpha\}$ be a discrete collection of subsets of X . Set

$$U_\alpha = X - \bigcup_{\beta \neq \alpha} \bar{H}_\beta$$

By the local finiteness of $\{\bar{H}_\alpha\}$, U_α is open. Consider the nontrivial case when $\{H_\alpha\}$ contains at least two elements. Then $\mathcal{U} = \{U_\alpha\}$ is an open covering of X . Let $\mathcal{V}$ be an open covering of X which Δ -refines $\mathcal{U}$. Set

$$G_\alpha = S(\bar{H}_\alpha, \mathcal{V})$$

Then G_α is an open set with $H_\alpha \subset G_\alpha$. Assume that there exist two indices α and β with $\alpha \neq \beta$ and with $G_\alpha \cap G_\beta \neq \emptyset$; then take a point x from $G_\alpha \cap G_\beta$. There exist elements V_1 and V_2 of $\mathcal{V}$ with

$$\begin{aligned} x &\in V_1, & x &\in V_2 \\ V_1 \cap \bar{H}_\alpha &\neq \emptyset, & V_2 \cap \bar{H}_\beta &\neq \emptyset \end{aligned}$$

Hence $S(x, \mathcal{V})$ meets $\bar{H}_\alpha$ and $\bar{H}_\beta$. Since $\mathcal{V} \triangleleft \mathcal{U}$, there exists a U_γ with $S(x, \mathcal{V}) \subset U_\gamma$. But U_γ does not meet any $\bar{H}_\delta$ but $\bar{H}_\gamma$, which is a contradiction. Hence $\{G_\alpha\}$ is disjoint. $\square$

2-6. THEOREM (Tukey [1]). A metric space (X, ρ) is fully normal.

PROOF. Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be an open covering of X . For every point x of X , there exists an index $\alpha(x)$ such that $x \in G_{\alpha(x)}$. Set

$$\rho(x) = \min\{1, \rho(x, X - G_{\alpha(x)})/4\}$$

Then it can easily be seen that $\{S_{\rho(x)}(x) : x \in X\}$ is an open covering of X which Δ -refines $\mathcal{G}$. $\square$

TERMINOLOGY. A collection $\mathcal{U}$ of subsets of a set X is called *star-finite* if every element of $\mathcal{U}$ meets at most a finite number of other elements of

$\mathcal{U}$. Of course a star-finite open covering is locally finite. A space is said to have the *star-finite property* if every open covering is refined by a star-finite open covering.

2-7. THEOREM. (This is essentially proved by Morita [1]; also compare Dowker [2].) Let $\mathcal{G} = \{G_1, G_2, \dots\}$ be a countable covering of a space X such that every G_i is a cozero set. Then there exists a countable star-finite covering $\mathcal{H}$ of X such that (i) $\mathcal{H} < \mathcal{G}$, (ii) every element of $\mathcal{H}$ is a cozero set.

PROOF. Let

$$G_i = \bigcup_{j=1}^{\infty} F_{ij} = \bigcup_{j=1}^{\infty} V_{ij}, \quad i = 1, 2, \dots$$

where F_{ij} are zero sets and V_{ij} are cozero sets such that

$$F_{ij} \subset V_{ij} \subset F_{i,j+1}$$

for each i and j . Set

$$F_i = \bigcup_{j \leq i} F_{ji}, \quad V_i = \bigcup_{j \leq i} V_{ji}$$

Then, as can easily be seen,

$$\mathcal{H} = \{(V_i - F_{i-1}) \cap V_{ji} : j \leq i, \quad i = 1, 2, \dots\}$$

where $F_{-1} = \phi$, has the desired property. $\square$

2-8. COROLLARY. A countable open covering of a perfectly normal[†] space X can be refined by a star-finite open covering.

This is evident since every open set of X is a cozero set.

2-9. COROLLARY. A countable covering $\{G_i\}$ of a space X such that every G_i is a cozero set is refined by a countable locally-finite covering $\{U_i\}$ such that every U_i is a cozero set and such that $U_i \subset G_i$ for every i .

PROOF. Let $\{H_i\}$ be a countable star-finite covering with $\{H_i\} < \{G_i\}$ such that every H_i is a cozero set. Set

$$U_i = \bigcup \{H_j : H_j \subset G_i, \quad H_j \not\subset G_j \text{ for } j < i\}, \quad i = 1, 2, \dots$$

[†] A space is *perfectly normal* if it is normal and every open set is F_σ .

Then $\{U_i\}$ is a locally finite covering such that $U_i \subset G_i$ for every i . Since a countable sum of cozero sets is also a cozero set, every U_i is a cozero set. $\square$

2-10. COROLLARY (Morita [1]). A regular Lindelöf[†] space X has the star-finite property.

PROOF. To prove the normality of X , let F and H be a disjoint pair of closed sets. For every point x in F , let $U(x)$ be an open neighborhood of x with $\overline{U(x)} \cap H = \phi$. Since a closed subset of X is also a Lindelöf space, there exists a sequence

$$U_1 = U(x_1), \quad U_2 = U(x_2), \dots \quad \text{with} \quad \bigcup U_i \supset F$$

Similarly, there exists a sequence $V_1, V_2, \dots$ of open sets of X such that $\bigcup V_i \supset H$ and $\overline{V_i} \cap F = \phi$ for every i . Set

$$W = \bigcup_i (U_i - \bigcup_{j < i} \overline{V_j})$$

Then W is an open set with $F \subset W$ and $\overline{W} \cap H = \phi$.

Let $\mathcal{G}$ be an arbitrary open covering of X . For every $x \in X$, let U_x be an open neighborhood such that $\overline{U_x}$ refines $\mathcal{G}$. Let $\{U_1, U_2, \dots\}$ be a countable subcovering of $\{U_x: x \in X\}$. Let G_i be an element of $\mathcal{G}$ with $\overline{U_i} \subset G_i$. Since X is already normal, there exists a cozero set H_i with $\overline{U_i} \subset H_i \subset G_i$ for every i . By Theorem 2-7 there exists a star-finite open covering $\mathcal{H}$ with $\mathcal{H} < \{H_i\}$ and hence with $\mathcal{H} < \mathcal{G}$. $\square$

2-11. PROBLEM. Can an arbitrary countable open covering of a normal space be refined by a locally finite open covering? This problem, which is due to Dowker [2], is one of the most difficult problems in general topology. If a countable open covering $\mathcal{G}$ of a normal space is shrinkable, then $\mathcal{G}$ can be refined by a locally finite open covering by the method in the proof of Corollary 1-8 and by Theorem 2-7. Rudin [1] proved that if Souslin's problem is solved in the negative, then Dowker's problem is solved in the negative. A space where every countable open covering is refined by a locally finite open covering is called *countably paracompact*.

2-12. DEFINITION. A space X is *paracompact* if every open covering of the space is refined by a locally finite open covering.

[†] A space is called a *Lindelöf* space if every open covering of the space admits a countable subcovering.

2-13. THEOREM (Stone [1]). A metric space X is paracompact.

PROOF. Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$ be an arbitrary open covering indexed by a well-ordered set A . For every α and every positive integer i , set

$$F_{i\alpha} = (G_\alpha - S_{1/i}(X - G_\alpha)) - \bigcup_{\beta < \alpha} G_\beta$$

Then $\mathcal{F}_i = \{F_{i\alpha} : \alpha \in A\}$ is a discrete collection of closed sets. Moreover, $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ is a covering of X which refines $\mathcal{G}$. Set

$$\mathcal{G}_i = \{S_{1/(3i)}(F_{i\alpha}) : \alpha \in A\}$$

Then $\mathcal{G}_i$ is a discrete collection of open sets which refines $\mathcal{G}$. Set

$$U_i = \bigcup \{G : G \in \mathcal{G}_i\}, \quad i = 1, 2, \dots$$

Then by Corollary 2-9 there exists a locally finite open covering $\{H_i\}$ such that $H_i \subset U_i$ for every i

$$\{H_i \cap G : G \in \mathcal{G}_i, \quad i = 1, 2, \dots\}$$

is a locally finite open covering of X which refines $\mathcal{G}$. $\square$

2-14. THEOREM. A space X is normal if and only if every (locally) finite open covering $\mathcal{G}$ of X is normal.

PROOF. Sufficiency is trivial. Necessity: By Corollary 1-8 and by Remark 1-3, there exists a polyhedron $|L|_{\text{met}}$ and a mapping $f: X \rightarrow |L|_{\text{met}}$ such that

$$\mathcal{G}_0 = \{f^{-1}(\text{St}(v)) : v \in \text{the vertex set of } L\}$$

refines $\mathcal{G}$. Since $\{\text{St}(v)\}$ is an open covering of $|L|_{\text{met}}$, $\mathcal{G}_0$ is an open covering of X . Since $|L|_{\text{met}}$ is fully normal by Theorem 2-6, there exists a sequence of open coverings $\mathcal{S}_1, \mathcal{S}_2, \dots$ of $|L|_{\text{met}}$ such that

$$\{\text{St}(v)\} \supset \mathcal{S}_1 \supset \mathcal{S}_2 \supset \dots$$

Let $f^{-1}(\mathcal{S}_i) = \{f^{-1}(s) : s \in \mathcal{S}_i\}$, then since

$$\mathcal{G} \supset \mathcal{G}_0 \supset f^{-1}(\mathcal{S}_1) \supset f^{-1}(\mathcal{S}_2) \supset \dots$$

$\mathcal{G}$ is normal. $\square$

2-15. COROLLARY (Dieudonne [1]). A paracompact Hausdorff space X is fully normal.

PROOF. By virtue of Theorem 2-14 it suffices to prove that X is a normal space. First, to prove the regularity of X , let x be a point of X and F a closed set of X with $x \notin F$. For every point y of F , there is an open neighborhood $U(y)$ of y such that $x \notin \overline{U(y)}$. An open covering $\{X - F, U(y): y \in F\}$ of X can be refined by a locally finite open covering $\mathcal{V}$ of X . Set

$$\mathcal{V}_0 = \{V \in \mathcal{V}: V \cap F \neq \emptyset\}, \quad W = \bigcup \{V: V \in \mathcal{V}_0\}$$

Then $\overline{W} = \bigcup \{\overline{V}: V \in \mathcal{V}_0\}$ by the local finiteness of $\mathcal{V}$. Since every V of $\mathcal{V}_0$ is contained in some $U(y)$, $x \notin \overline{V}$. Hence $x \notin \overline{W}$, which proves the regularity of X .

Take two disjoint closed sets H and K of X . For every x of K , there exists an open neighborhood $U(x)$ with $\overline{U(x)} \cap H = \emptyset$. Starting with an open covering

$$\{X - K, U(x): x \in K\}$$

of X , we can find an open set D with $\overline{D} \cap H = \emptyset$ and $K \subset D$ by the same technique as in the above. $\square$

2-16. THEOREM. Let X be a space and $\mathcal{U}_1, \mathcal{U}_2, \dots$, a sequence of open coverings of X which satisfies condition (a):

For every i , if U_1 and U_2 are elements of $\mathcal{U}_{i+1}$ with $U_1 \cap U_2 \neq \emptyset$, then $U_1 \cup U_2$ refines $\mathcal{U}_i$.[†]

Then there exists a pseudometric d on X such that

$$\mathcal{U}_{i+3}^\Delta \subset \{S_{2^{-i}}(x) = \{y: d(x, y) < 2^{-i}\}: x \in X\} \subset \mathcal{U}_i^\Delta$$

for every i .

PROOF. For any pair of points x, y of X , define the function D as follows:

$$\begin{aligned} D(x, y) &= 4 && \text{if } \{x, y\} \text{ is not contained in any element of } \mathcal{U}_1 \\ D(x, y) &= 1/2^{i-2} && \text{if } \{x, y\} \text{ is contained in an element of } \mathcal{U}_i \text{ and is not} \\ &&& \text{contained in any element of } \mathcal{U}_j, j > i \\ D(x, y) &= 0 && \text{if } \{x, y\} \text{ is contained in an element of } \mathcal{U}_i \text{ for every } i \end{aligned}$$

Then D satisfies the following four conditions for any x, y, z in X and any $\varepsilon > 0$:

$$D(x, x) = 0 \tag{1}$$

$$D(x, y) = D(y, x) \tag{2}$$

$$D(x, y) \leq 2 \max\{D(x, z), D(z, y)\} \tag{3}$$

$$\{x': D(x, x') < \varepsilon\} \text{ is open} \tag{4}$$

[†] A $\mathcal{U}_{i+1}$ satisfying condition (a) is said to be a 2-refinement of $\mathcal{U}_i$.

Set

$$d(x, y) = \inf\{D(x, x_1) + D(x_1, x_2) + \cdots + D(x_n, y)\}$$

where the infimum is taken over all possible choices of finite points $x_1, \dots, x_n \in X$. Then d satisfies the following four conditions for any x, y, z in X and any $\varepsilon > 0$:

$$d(x, x) = 0 \quad (5)$$

$$d(x, y) = d(y, x) \quad (6)$$

$$d(x, y) \leq d(x, z) + d(z, y) \quad (7)$$

$$\{x' : d(x, x') < \varepsilon\} \text{ is open} \quad (8)$$

Thus d is a pseudometric on X satisfying

$$d(x, y) \leq D(x, y) \quad (9)$$

To prove the inequality

$$\begin{aligned} D(x, y) &\leq 2D(x, x_1) + 4D(x_1, x_2) + 4D(x_2, x_3) + \cdots \\ &\quad + 4D(x_{n-1}, x_n) + 2D(x_n, y) \end{aligned} \quad (10)$$

assume the contrary. Then there exists a minimum number N for which

$$D(x, y) > 2D(x, x_1) + 4D(x_1, x_2) + \cdots + 4D(x_{N-1}, x_N) + 2D(x_N, y) \quad (11)$$

Then $N > 1$ and for any x_r with $1 \leq r \leq N$, either

$$D(x, y) \leq 2D(x, x_r) \quad \text{or} \quad D(x, y) \leq 2D(x_r, y)$$

by virtue of inequality (3). Set

$$k = \max\{r : D(x, y) \leq 2D(x_r, y)\}$$

Then $1 \leq k < N$, and we have

$$D(x, y) \leq 2D(x_k, y) \quad (12)$$

$$D(x, y) \leq 2D(x, x_{k+1}) \quad (13)$$

By inequalities (12) and (13) we have

$$\begin{aligned} D(x, y) &\leq D(x, x_{k+1}) + D(x_k, y) \\ &\leq (2D(x, x_1) + 4D(x_1, x_2) + \cdots \\ &\quad + 4D(x_{k-1}, x_k) + 2D(x_k, x_{k+1})) \\ &\quad + (2D(x_k, x_{k+1}) + 4D(x_{k+1}, x_{k+2}) + \cdots \\ &\quad + 4D(x_{N-1}, x_N) + 2D(x_N, y)) \end{aligned}$$

which contradicts inequality (11). By inequality (10) we obtain

$$D(x, y)/4 \leq d(x, y) \quad (14)$$

By inequalities (9) and (14) we obtain at once the inequality

$$S(x, \mathcal{U}_{i+3}) \subset S_{2^{-i}}(x) \subset S(x, \mathcal{U}_i)$$

As a consequence we obtain

$$\mathcal{U}_{i+3}^\Delta < \{S_{2^{-i}}(x): x \in X\} < \mathcal{U}_i^\Delta \quad (15)$$

2-17. COROLLARY. Assume the same hypotheses as in Theorem 2-16, then there exist a metric space (X^*, d^*) and a mapping f of X onto X^* such that

$$f^{-1}\{S_{1/2}(x^*): x^* \in X^*\} < \mathcal{U}_1^\Delta$$

PROOF. For every $x \in X$ let x^* be the set $\{x': d(x, x') = 0\}$ where d is a pseudometric on X constructed as in the preceding proof. Set

$$X^* = \{x^*: x \in X\}, \quad f(x) = x^*$$

$$d^*(x^*, y^*) = d(x, y), \quad x \in x^*, \quad y \in y^*$$

Then (X^*, d^*) and f are the desired space and mapping. $\square$

Since X^* is fully normal by Theorem 2-6, $\{S_{1/2}(x^*)\}$ is normal and hence $f^{-1}(\{S_{1/2}(x^*)\})$ is. Therefore we can conclude that $\mathcal{U}_1^\Delta$ is normal. This fact is of some interest.

TERMINOLOGY. A *neighborhood base* of x is a system of neighborhoods $\{U(x)\}$ of x such that for any neighborhood G of x there exists a $U(x)$ with $U(x) \subset G$.

A sequence of open coverings satisfying condition (b) is called a *development* of X . A space is called *developable* if it has a development. This notion will be used in Section 5.

2-18. COROLLARY (metrization theorem of Alexandroff and Urysohn [1]). In order that a space X be metrizable it is necessary and sufficient that there exists a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open coverings of X which satisfies condition (a) in Theorem 2-16 and the following condition (b):

$$\{S(x, \mathcal{U}_i): i = 1, 2, \dots\} \text{ is a neighborhood base of } x \in X$$

PROOF. The necessity is almost evident. If conditions (a) and (b) are satisfied, then f in the proof of Corollary 2-17 is actually a homeomorphism by inequality (15) and condition (b). $\square$

2-19. COROLLARY (Stone [1]). A fully normal space X is paracompact.

PROOF. Let $\mathcal{G}$ be an arbitrary open covering of X and $\mathcal{U}_1, \mathcal{U}_2, \dots$ a sequence of open coverings with

$$\mathcal{G} \supset \mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots$$

Take the same (X^*, d^*) and $f: X \rightarrow X^*$ as in Corollary 2-17. Then $f^{-1}(\{S_{1/2}(x^*)\})$ refines $\mathcal{G}$. Since a metric space is already paracompact by Theorem 2-13, there exists a locally finite open covering $\mathcal{H}$ of X^* which refines $\{S_{1/2}(x^*)\}$. Then $f^{-1}(\mathcal{H})$ is a locally finite open covering of X which refines $\mathcal{G}$. $\square$

This argument contains the assertion: A normal open covering of an arbitrary space can be refined by a locally finite open covering. Moreover, in consideration of Theorem 1-5 we can say that an open covering of an arbitrary space X is normal if and only if it can be refined by a locally finite covering each element of which is a cozero set of X .

2-20. EXAMPLE. There exists a collectionwise normal space which is not paracompact. Let ω_1 be the first uncountable ordinal and J the set of all countable ordinals less than ω_1 . Introduce the interval topology to J in such a way that the collection

$$\{0, (\alpha, \beta] = \{x: \alpha < x \leq \beta\} : \alpha < \beta < \omega_1\}$$

is a base. Then J is a topological space.

ASSERTION 1. J is collectionwise normal.

TERMINOLOGY. A set A is a *directed set* if for some pairs of points of A the relation $<$ is defined satisfying the following conditions:

- i. The statements $\alpha < \beta$ and $\beta < \gamma$ imply $\alpha < \gamma$
- ii. For any α, β in A there exists a $\gamma \in A$ with $\alpha < \gamma$ and $\beta < \gamma$

A subset B of a directed set A is *cofinal* if for any $\alpha \in A$, there exists a $\beta \in B$ with $\alpha < \beta$.

A subset C of a directed set A is *equifinal* if there exists an element $\alpha \in A$ such that C contains any β with $\alpha < \beta$.

PROOF. Let us first prove that J is normal. Let F and H be two disjoint closed sets of J . Suppose that both F and H are cofinal in J . Then there exists a sequence $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots$ of points of J such that every α_i is in F and β_i is in H . Let α be the limit point of the sequence. Then $\alpha \in F \cap H$, which is a contradiction. Hence we can assume without loss of generality that $J - F$ is equifinal. Take an isolated ordinal β with $F \cap [\beta, \omega_1) = \emptyset$. Since $[0, \beta)$ is a perfectly separable regular space, it is metrizable. Hence there exist open sets U and V of $[0, \beta)$ such that $F \subset U$, $H \cap [0, \beta) \subset V$ and $U \cap V = \emptyset$. Setting $W = V \cup [\beta, \omega_1)$, W is an open set with $H \subset W$ and $U \cap W = \emptyset$. Thus the normality of J is established.

Let $\{H_\alpha\}$ be a discrete collection of J . Then the collection contains at most a finite number of elements because of the local finiteness of $\{\bar{H}_\alpha\}$. By the normality of J we can find disjoint open sets G_α with $\bar{H}_\alpha \subset G_\alpha$ for every α . Thus the collectionwise normality is proved. $\square$

ASSERTION 2. J is not paracompact.

PROOF. Set

$$\mathcal{G} = \{[0, \alpha] : \alpha \in J\}$$

Then $\mathcal{G}$ is an open covering of J . If $\mathcal{G}$ is refined by a locally finite open covering $\mathcal{G}_0$, $\mathcal{G}_0$ is finite because of the local finiteness. Then some element of $\mathcal{G}_0$ must be cofinal, which gives a contradiction. Hence J is not paracompact. $\square$

2-21. LEMMA. If every open covering of a regular space X is refined by a σ -locally finite open covering, then X is collectionwise normal.

PROOF. Let $\mathcal{F} = \{F_\alpha\}$ be a discrete collection of closed sets of X . Let $\mathcal{G}$ be an open covering of X such that the closure of each element of $\mathcal{G}$ does not meet two elements of $\mathcal{F}$. Then $\mathcal{G}$ can be refined by an open covering $\cup \mathcal{G}_i$ such that $\mathcal{G}_i$ is locally finite for $i = 1, 2, \dots$. Set

$$G_{i\alpha} = S(F_\alpha, \mathcal{G}_i)$$

$$U_\alpha = \bigcup_i (G_{i\alpha} - \bigcup \{\bar{G}_{j\beta} : j \leq i, \beta \neq \alpha\})$$

Then U_α is an open set such that (i) $F_\alpha \subset U_\alpha$ for every α and (ii) $\{U_\alpha\}$ is disjoint. $\square$

2-22. THEOREM. In a regular space X the following conditions are equivalent:

1. X is paracompact
2. Every open covering of X is refined by a σ -discrete open covering
3. Every open covering of X is refined by a σ -locally finite open covering

PROOF. That (1) implies (2) (Stone [1]): Let $\mathcal{G}$ be an arbitrary open covering of X . Since X is fully normal by Corollary 2-15, $\mathcal{G}$ is normal. By Corollary 2-17 there exist a metric space X^* , an open covering $\mathcal{U}$ of X^* and a mapping $f: X \rightarrow X^*$ such that $f^{-1}(\mathcal{U})$ refines $\mathcal{G}$. There exists a σ -discrete open covering $\mathcal{V}$ of X^* with $\mathcal{V} < \mathcal{U}$. The existence of such a $\mathcal{V}$ was essentially proved in the proof of Theorem 2-13. $f^{-1}(\mathcal{V})$ is a σ -discrete open covering of X refining $\mathcal{G}$.

That (2) implies (3) is evident.

That (3) implies (1) (Michael [1]): Let $\mathcal{G}$ be an arbitrary open covering of X . Let $\cup \mathcal{G}_i$ be an open covering with $\cup \mathcal{G}_i < \mathcal{G}$ such that $\mathcal{G}_i$ is locally finite for $i = 1, 2, \dots$. Let $\mathcal{H} = \cup \mathcal{H}_i$ be an open covering with $\overline{\mathcal{H}} < \cup \mathcal{G}_i$ such that $\mathcal{H}_i$ is locally finite for $i = 1, 2, \dots$. Let

$$G_i = \cup \{G : G \in \mathcal{G}_i\}, \quad i = 1, 2, \dots$$

For $i = 1, 2, \dots$ and $j = 1, 2, \dots$, let

$$F_{ij} = \cup \{\overline{H} : H \in \mathcal{H}_j, \overline{H} < G_i\}$$

By Lemma 2-21 there exists a cozero set V_{ij} with

$$F_{ij} \subset V_{ij} \subset G_i$$

Set $V_i = \cup_j V_{ij}$. Then V_i is a cozero set with $V_i \subset G_i$. Since $\{F_{ij} : i, j = 1, 2, \dots\}$ covers X , $\{V_i\}$ covers X . By Corollary 2-9 there exists a locally finite open covering $\{W_i\}$ such that $W_i \subset V_i$ for every i . Then

$$\{W_i \cap G : G \in \mathcal{G}_i, \quad i = 1, 2, \dots\}$$

is a locally finite open covering which refines $\mathcal{G}$. $\square$

2-23. THEOREM (proved by Nagami [1] and by Michael [2]). Every point-finite open covering of a collectionwise normal space X is normal.

TERMINOLOGY. Let $\mathcal{H}$ be a collection of subsets of a set X . For a point x in X the *order of $\mathcal{H}$ at x* , say $\text{ord}_x \mathcal{H}$, is the number of elements of $\mathcal{H}$ which contain x . The *order of $\mathcal{H}$* , say $\text{ord } \mathcal{H}$, is the supremum of $\text{ord}_x \mathcal{H}$, x running through X .

PROOF. Let $\mathcal{G}_1 = \{G_{\alpha_1} : \alpha_1 \in A_1\}$ be an arbitrary point-finite open covering of X . For $i = 2, 3, \dots$ let A_i be the set of all subsets of A_1 the number of whose elements is exactly i . Let

$$\mathcal{G}_i = \{G_{\alpha_i} = \cap \{G_{\alpha_1} : \alpha_1 \in \alpha_i\} : \alpha_i \in A_i\}$$

$$H_i = \{x : \text{ord}_x \mathcal{G}_i = 1\}$$

Then $\{H_1 \cap G_{\alpha_1} : \alpha_1 \in A_1\}$ is a discrete collection of closed sets. Take an open set U_{α_1} for each $\alpha_1 \in A_1$ such that:

- i. $H_1 \cap G_{\alpha_1} \subset U_{\alpha_1} \subset G_{\alpha_1}$
- ii. $\{U_{\alpha_1} : \alpha_1 \in A_1\}$ is discrete
- iii. $U_1 = \cup \{U_{\alpha_1} : \alpha_1 \in A_1\}$ is a cozero set

$\{(H_2 - U_1) \cap G_{\alpha_2} : \alpha_2 \in A_2\}$ is a discrete collection. Take an open set U_{α_2} for each $\alpha_2 \in A_2$ such that:

- i. $(H_2 - U_1) \cap G_{\alpha_2} \subset U_{\alpha_2} \subset G_{\alpha_2}$
- ii. $\{U_{\alpha_2} : \alpha_2 \in A_2\}$ is discrete
- iii. $U_2 = \cup \{U_{\alpha_2} : \alpha_2 \in A_2\}$ is a cozero set

Continuing recursively in this fashion we get a sequence

$$\{U_{\alpha_i} : \alpha_i \in A_i\}, \quad i = 1, 2, \dots$$

of discrete collections of open sets such that for each i

$$U_i = \cup \{U_{\alpha_i} : \alpha_i \in A_i\}$$

is a cozero set, and

$$(H_i - \cup_{j < i} U_j) \cap G_{\alpha_i} \subset U_{\alpha_i} \subset G_{\alpha_i}, \quad \alpha_i \in A_i$$

Since $\text{ord}_x \mathcal{G}_i = 1$ if and only if $\text{ord}_x \mathcal{G}_1 = i$, $\{H_i\}$ covers X . Hence $\{U_i\}$ covers X . By Corollary 2-9 there is a locally finite open covering $\{V_i\}$ of X such that $V_i \subset U_i$ for each i . Then

$$\mathcal{H} = \{U_{\alpha_i} \cap V_i : \alpha_i \in A_i, \quad i = 1, 2, \dots\}$$

is a locally finite open covering which refines $\mathcal{G}_1$. Since $\mathcal{H}$ is normal by Theorem 2-14, $\mathcal{G}_1$ is normal. $\square$

2-24. PROBLEM. Is every point-finite open covering of a normal space normal? By Theorem 2-14 we already know that a space is normal if and only if every locally finite open covering is normal. By the definition itself a space is fully normal if and only if every open covering is normal.

In consideration of these facts and the present theorem it is natural to ask: If every point-finite open covering of a space X is normal, is X collectionwise normal? But the answer is no. In fact Bing's space in Example 2-3 gives a counterexample.

2-25. LEMMA. Let X be a space, F a closed set of X and G a cozero set of X with $G \subset F$. If H is a cozero set of F , then $G \cap H$ is a cozero set of X .

PROOF. Set

$$G = \{x: g(x) > 0\}, \quad H = \{x: h(x) > 0\}$$

where g and h are nonnegative continuous functions defined on X and F respectively. Set

$$\begin{aligned} f(x) &= g(x)h(x), & x \in F \\ f(x) &= 0, & x \in X - F \end{aligned}$$

Then f is continuous on X and $G \cap H = \{x: f(x) > 0\}$. $\square$

2-26. THEOREM (Morita [9]). Let X be a space, $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ a normal covering of X and $\mathcal{G}_\alpha$ a normal open covering of G_α for each α . Then $\cup \mathcal{G}_\alpha$ is a normal open covering of X .

PROOF. By Theorem 2-16 there exist locally finite coverings $\{F_\alpha: \alpha \in A\}$ and $\{H_\alpha: \alpha \in A\}$ such that:

- i. $H_\alpha \subset F_\alpha \subset G_\alpha$ for each α
- ii. F_α is closed for each α
- iii. H_α is a cozero set of X for each α

Let, for each α , $\mathcal{H}_\alpha$ be a locally finite (in G_α) covering of G_α with $\mathcal{H}_\alpha < \mathcal{G}_\alpha$ such that each element of $\mathcal{H}_\alpha$ is a cozero set of G_α . Then

$$\mathcal{H} = \{H \cap H_\alpha: H \in \mathcal{H}_\alpha, \alpha \in A\}$$

is a locally finite covering of X which refines $\cup \mathcal{G}_\alpha$. By the lemma each element $H \cap H_\alpha$ of $\mathcal{H}$ is a cozero set of X . By the remark at the end of Section 2-19, $\mathcal{H}$ is normal and hence $\mathcal{G}$ is normal. $\square$

2-27. THEOREM (Morita [10]). Let $\mathcal{G}$ be an open covering of a space X . Then the following conditions are equivalent:

- 1. $\mathcal{G}$ is normal

2. $\mathcal{G}$ is refined by a covering $\cup \mathcal{G}_i$ such that (i) every element of $\cup \mathcal{G}_i$ is a cozero set of X and (ii) $\mathcal{G}_i$ is discrete for $i = 1, 2, \dots$

3. $\mathcal{G}$ is refined by a covering $\cup \mathcal{G}_i$ such that (i) every element of $\cup \mathcal{G}_i$ is a cozero set of X and (ii) $\mathcal{G}_i$ is locally finite for $i = 1, 2, \dots$

PROOF. That (1) implies (2): If $\mathcal{G}$ is normal, there exist a metric space X^* and a mapping $f: X \rightarrow X^*$ such that

$$f^{-1}(\{S_{1/2}(x^*): x^* \in X^*\}) < \mathcal{G}$$

by Corollary 2-17. Since X^* is paracompact by Theorem 2-13, $\{S_{1/2}(x^*)\}$ is refined by an open covering $\cup \mathcal{H}_i$ such that $\mathcal{H}_i$ is discrete for $i = 1, 2, \dots$, by Theorem 2-22. Setting $\mathcal{G}_i = f^{-1}(\mathcal{H}_i)$, the $\mathcal{G}_i$'s satisfy condition (2).

The implication of (3) by (2) is evident.

That (3) implies (1): Let the $\mathcal{G}_i$ be collections of sets satisfying condition (3). Set

$$G_i = \cup \{G: G \in \mathcal{G}_i\}$$

Since as can easily be seen G_i is a cozero set of X , $\{G_i\}$ is refined by a star-finite covering $\mathcal{H}$ each element of which is a cozero set of X by Theorem 2-7. By Theorem 1-5, $\mathcal{H}$ is approximated by a star-finite polyhedron $|L|_w = |L|_{\text{met}}$. Let $\{v\}$ be its vertex set. Since $|L|_{\text{met}}$ is fully normal by Theorem 2-6, $\{\text{St}(v)\}$ is normal. Hence $\mathcal{H}$ is normal. Therefore $\{G_i\}$ is normal. In this fashion we know that every $\mathcal{G}_i$ is normal. Hence by Theorem 2-26, $\mathcal{G}$ itself is normal. $\square$

3. Extension of Coverings and Extensors

3-1. DEFINITION. Let X be a space, F a closed set of X and $\mathcal{H}$ a relatively open covering of F . $\mathcal{H}$ is called *neighborhood-extendable* if there exist an open set U with $U \supset F$ and an open covering $\mathcal{G}$ of U such that $\mathcal{G} \upharpoonright F$ ($\mathcal{G} \upharpoonright F$ is the restriction of $\mathcal{G}$ to F ; $\mathcal{G} \upharpoonright F = \{G \cap F: G \in \mathcal{G}\}$) refines $\mathcal{H}$ and $\mathcal{G}$ is normal. Such a $\mathcal{G}$ is called a *neighborhood-extension* of $\mathcal{H}$ (to U). If there exists an open covering $\mathcal{V}$ of X with $\mathcal{V} \upharpoonright F < \mathcal{H}$ such that $\mathcal{V}$ is normal, then $\mathcal{H}$ is *extendable*. Such a $\mathcal{V}$ is an *extension* of $\mathcal{H}$.

Let $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ be a locally finite relatively open covering of F . If there exist an open set U with $F \subset U$ and a locally finite (in U) open covering $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ of U such that $G_\alpha \cap F = H_\alpha$ for each $\alpha \in A$, then $\mathcal{H}$ is *strictly neighborhood-extendable* and $\mathcal{G}$ is a *strict neighborhood-extension* of $\mathcal{H}$ (to U). If there exists a locally finite open covering $\mathcal{V} = \{V_\alpha: \alpha \in A\}$ of X such that $V_\alpha \cap F = H_\alpha$ for each $\alpha \in A$, then $\mathcal{H}$ is *strictly extendable* and $\mathcal{V}$ is a *strict extension* of $\mathcal{H}$.

3-2. THEOREM. The following conditions of a space X are equivalent:

1. X is normal
2. Any countable, normal, relatively open covering of any closed set is extendable
3. Any binary, normal, relatively open covering of any closed set is neighborhood-extendable

PROOF. That (1) implies (2): Let F be an arbitrary closed set of X and $\mathcal{G}$ be an arbitrary, countable, normal, relatively open covering of F . There exists a countable covering $\{H_i\}$ of F with $\{H_i\} < \mathcal{G}$ such that H_i is a cozero set of F for each i . Set

$$H_i = \{x: f_i(x) > 0\}$$

where f_i is a continuous function on F . By Tietze's extension theorem f_i has an extension g_i over X . Set

$$U_i = \{x: g_i(x) > 0\}$$

Then U_i is a cozero set of X with $U_i \cap F = H_i$. By the normality of X there exists a cozero set U of X with

$$F \subset X - U \subset \cup U_i$$

Set $V_i = U_i \cup U$. Then $\{V_i\}$ is normal by Theorem 2-27.

To show that (2) implies (3) is trivial.

That (3) implies (1): Let F and H be a disjoint pair of closed sets of X . $\{F, H\}$ is a normal binary open covering of $F \cup H$. Let $\mathcal{G}$ be a neighborhood-extension of $\{F, H\}$ to an open neighborhood U of $F \cup H$. Let $\mathcal{H}$ be an open covering of U which Δ -refines $\mathcal{G}$. Then

$$S(F, \mathcal{H}) \cap S(H, \mathcal{H}) = \phi \quad \square$$

3-3. THEOREM. The following conditions on a space X are equivalent:

1. X is collectionwise normal
2. Any normal open covering of any closed subset is neighborhood-extendable
3. Any normal open covering of any closed subset is extendable

PROOF. That (1) implies (2): Let F be an arbitrary closed set of X and $\mathcal{G}$ an arbitrary normal (relatively) open covering of F . By Theorem 2-27 there exists a closed covering $\cup \mathcal{F}_i$ of F with $\cup \mathcal{F}_i < \mathcal{G}$ such that $\mathcal{F}_i = \{F_\alpha: \alpha \in A_i\}$ is discrete in F and hence in X . Let $\mathcal{G}_i = \{G_\alpha: \alpha \in A_i\}$

be a disjoint collection of open sets of X such that (i) $F_\alpha \subset G_\alpha$ for each $\alpha \in A_i$, (ii) $\mathcal{G}_i \mid F$ refines $\mathcal{G}$, and (iii) $G_i = \cup \{G_\alpha : \alpha \in A_i\}$ is a cozero set of X . Since $\mathcal{G}_i$ is normal in G_i and $\{G_i\}$ is normal, $\cup \mathcal{G}_i$ is normal by Theorem 2-26.

That (2) implies (3): Let F and $\mathcal{G}$ be the same as in the above. Let $\mathcal{H}$ be a neighborhood-extension of $\mathcal{G}$. Since $\mathcal{H}$ is normal, we suppose that $\mathcal{H}$ is point-finite. Then

$$\mathcal{H}_1 = \mathcal{H} \cup \{X - F\}$$

is a point-finite open covering of X . Since $\mathcal{H}_1$ is normal by Theorem 2-23, $\mathcal{H}_1$ is an extension of $\mathcal{G}$.

That (3) implies (1): Let $\{H_\alpha\}$ be a discrete collection of closed sets of X . Then $\{H_\alpha\}$ itself is a normal open covering of the union $\cup H_\alpha$, which is closed. Let $\mathcal{H}$ be an extension of $\{H_\alpha\}$ and $\mathcal{H}_1$ an open covering of X which Δ -refines $\mathcal{H}$. Then $\{S(H_\alpha, \mathcal{H}_1)\}$ is a mutually exclusive collection of open sets such that $H_\alpha \subset S(H_\alpha, \mathcal{H}_1)$ for each α . $\square$

3-4. THEOREM (Hanner [1]). Let X be a fully normal space, F a closed set of X and $\mathcal{H} = \{H_\alpha\}$ a locally finite (relatively) open covering of F . Then $\mathcal{H}$ is strictly extendable.

PROOF. For each point x in X there is an open neighborhood $U(x)$ of x which meets at most a finite number of elements of $\mathcal{H}$. Let $\mathcal{G}$ be an open covering of X which $*$ -refines $\{U(x) : x \in X\}$. Fix some index α_0 . Set

$$\begin{aligned} G_{\alpha_0} &= (X - F) \cup H_{\alpha_0} \\ G_\alpha &= (S(H_\alpha, \mathcal{G}) - F) \cup H_\alpha, \quad \alpha \neq \alpha_0 \end{aligned}$$

It is easy to see that G_α is an open set with $G_\alpha \cap F = H_\alpha$. To show the local finiteness of $\{G_\alpha : \alpha \neq \alpha_0\}$ let x be an arbitrary point of X . Take an element G of $\mathcal{G}$ with $x \in G$. This G meets at most a finite number of G_α 's. Let y be a point of X with $S(G, \mathcal{G}) \subset U(y)$. Then if $G \cap G_\alpha \neq \phi$, then $U(y) \cap H_\alpha \neq \phi$. Thus $\{G_{\alpha_0}, G_\alpha\}$ is a strict extension of $\mathcal{H}$. $\square$

3-5. DEFINITION. Let Ω be a class of spaces. A space X is a *neighborhood-extensor* for Ω -spaces, an NES (Ω), if for any space $Y \in \Omega$, for any closed set F of Y and for any mapping $f: F \rightarrow X$, there exist an open set U with $U \supset F$ and a mapping $g: U \rightarrow X$ with $g \mid F = f$.

A space X is an *extensor* for Ω -spaces, an ES (Ω), if U is replaced with the whole space X in the above definition.

3-6. THEOREM (Hanner [1]). A full polyhedron with the weak topology is an extensor for metric spaces, i.e. an ES (metric).

PROOF. Let $Q = \{v\}$ be the vertex set of a full polyhedron $|L|_w$. Let X be a metric space, F a closed set of X and $f: F \rightarrow |L|_w$ a mapping. To prove that $\{f^{-1}(\text{St}(v))\}$ is locally finite, assume the contrary. Then there would exist a point ξ of F such that any neighborhood of ξ meets infinitely many $f^{-1}(\text{St}(v))$. Since $\{f^{-1}(\text{St}(v))\}$ is point-finite, the set of vertices

$$Q_1 = \{v: \xi \in f^{-1}(\text{St}(v))\}$$

is finite.

There is a sequence $x_1, x_2, \dots$ of points of F such that (i) $\lim x_i = \xi$, (ii) $x_i \in f^{-1}(\text{St}(v_i))$, (iii) $v_i \neq v_j$ whenever $i \neq j$, and (iv) $\{v_1, v_2, \dots\} \cap Q_1 = \emptyset$.

Set

$$U = |L|_w - \{f(x_i): i = 1, 2, \dots\}$$

$f(x_i)$ has positive weight on v_i , while $f(\xi)$ has zero weight on v_i , since $v_i \notin Q_1$. Hence $f(x_i) \neq f(\xi)$ for any i . Let $|s|$ be an arbitrary simplex of $|L|_w$ and Q_2 its vertex set. Every point in $|s|$ has positive weight on v only if $v \in Q_2$. Hence $|s|$ contains only a finite number of $f(x_i)$'s. Therefore U is an open neighborhood of $f(\xi)$. By the continuity of f , $\{f(x_i)\}$ should converge to $f(\xi)$, which is a contradiction.

By Theorem 3-4 there is a strict extension $\{G_v\}$ of $\{f^{-1}(\text{St}(v))\}$ with $G_v \cap F = f^{-1}(\text{St}(v))$ for each v . Let $f_v: F \rightarrow [0, 1]$ be a function defined by

$$f_v(x) = \text{the weight of } f(x) \text{ on } v$$

Then f_v is continuous. Define a function $g_v: F \cup (X - G_v) \rightarrow [0, 1]$ by

$$\begin{aligned} g_v|_F &= f_v \\ g_v(x) &= 0 \quad \text{if } x \in X - (G_v \cup F) \end{aligned}$$

Then g_v is continuous. By Tietze's extension theorem there is a mapping $h_v: X \rightarrow [0, 1]$ with

$$h_v|_{F \cup (X - G_v)} = g_v$$

Set

$$H_v = \{x: h_v(x) > 0\}$$

Then the collection $\{H_v\}$ is a strict neighborhood-extension of

$\{f^{-1}(\text{St}(v))\}$. Fix some vertex v_0 . Let $h'_{v_0}: X \rightarrow [0, 1]$ be a mapping such that

$$\begin{aligned} h'_{v_0}(x) &= 0 & \text{if } x \in F \\ h'_{v_0}(x) &= 1 & \text{if } x \in X - \bigcup H_v \end{aligned}$$

Let

$$k_{v_0} = \max\{h_{v_0}, h'_{v_0}\}, \quad k_v = h_v \quad \text{if } v \neq v_0$$

Setting $K_v = \{x: k_v(x) > 0\}$, the collection $\{K_v\}$ is a strict extension of $\{f^{-1}(\text{St}(v))\}$. Moreover, $k_v|_F = f_v$ for each v . Let k be a canonical mapping of $\{K_v\}$ with respect to the k_v 's. It is evident that $k|_F = f$. $\square$

3-7. THEOREM. A finite full polyhedron is an extensor for normal spaces, i.e., an ES (normal).

This is proved in the same fashion as that in the above proof. But if we use Tietze's extension theorem, the proof is quite simple.

TERMINOLOGY. Let X be a space and Y a subset. Y is a *retract* of X if there exists a mapping $f: X \rightarrow Y$ such that $f|_Y$ is the identity mapping. Y is a *neighborhood retract* of X if there exists an open neighborhood U of Y such that Y is a retract of U . Such an f is a *retraction*.

3-8. LEMMA (Hanner [1]). Let L be a simplicial complex and P its vertex set, then $|L|_w$ is a neighborhood-retract of the full polyhedron $|K(P)|_w$ spanned by P .

PROOF. Let $\{|s_\alpha|: \alpha \in A\}$ be the collection of all simplices of $|K(P)|$ which are not contained in $|L|$. Let S_α be the surface of $|s_\alpha|$, i.e.,

$$S_\alpha = \bigcup \{|s_\beta|: s_\beta \text{ is a proper face of } s_\alpha\}$$

Let p_α be the barycenter[†] of $|s_\alpha|$ and $f_\alpha: |s_\alpha| - \{p_\alpha\} \rightarrow s_\alpha$ the mapping defined by

$$f_\alpha(x) = \text{the intersection of } S_\alpha \text{ and the directed line from } p_\alpha \text{ to } x$$

Let F be the sum of all possible closed convex hulls[†] spanned by $p_{\alpha_0}, \dots, p_{\alpha_n}$ such that:

- i. $\dim s_{\alpha_i} = \dim s_{\alpha_{i+1}} + 1$ for $i = 0, \dots, n-1$
- ii. $s_{\alpha_i} > s_{\alpha_{i+1}}$ for $i = 0, \dots, n-1$

[†] These are understood respectively to be the natural images of a barycenter and a closed convex hull of corresponding Euclidean simplices.

It is evident that every p_α is on the outside of $|L|$. Hence $F \cap |L| = \phi$. Set

$$U = |K(P)|_w - F$$

Then U is an open neighborhood of $|L|_w$. Let us define a transformation $f: U \rightarrow |L|_w$. Let x be a point of $U - |L|_w$. Then there exists a sequence of simplices $s_{\alpha_1}, s_{\alpha_2}, \dots, s_{\alpha_m}$ such that

$$\begin{aligned} s_{\alpha_1} &> s_{\alpha_2} > \dots > s_{\alpha_m} \\ \dim s_{\alpha_i} - \dim s_{\alpha_{i+1}} &= 1 \quad \text{for } i = 1, \dots, m-1 \\ x &\in |s_{\alpha_1}| - (S_{\alpha_1} \cup p_{\alpha_1}) \\ f_{\alpha_1}(x) &\in |s_{\alpha_2}| - (S_{\alpha_2} \cup p_{\alpha_2}) \\ &\dots \\ f_{\alpha_{m-1}} \dots f_{\alpha_1}(x) &\in |s_{\alpha_m}| - (S_{\alpha_m} \cup p_{\alpha_m}) \\ f_{\alpha_m} f_{\alpha_{m-1}} \dots f_{\alpha_1}(x) &\in |L|_w \end{aligned}$$

Such a sequence $\alpha_1, \dots, \alpha_m$ is unique. Let

$$f(x) = f_{\alpha_m} f_{\alpha_{m-1}} \dots f_{\alpha_1}(x)$$

When $x \in |L|_w$, then we let $f(x) = x$. Then f is a retraction of U to $|L|_w$. $\square$

Lemma 3-8 can be generalized as follows: If L is a subcomplex of a simplicial complex K , then $|L|_w$ is a neighborhood retract of $|K|_w$. A proof can be carried out in an analogous way if we adopt collapsing transformations about vertices of K which are not those of L , step by step.

3-9. THEOREM (Hanner [1]). A simplicial polyhedron with the weak topology is a neighborhood-extensor for metric spaces, i.e., an NES (metric).

PROOF. Let X be a metric space, F a closed set of X and f a mapping of F to $|L|_w$. Let K be the full complex spanned by the vertex set of L . By Theorem 3-6 there is a mapping $g: X \rightarrow |K|_w$ with $g|_F = f$. Since $|L|_w$ is a neighborhood retract of $|K|_w$ by Lemma 3-8, there exist an open neighborhood U of $|L|_w$ and a retraction $r: U \rightarrow |L|_w$. Set $V = g^{-1}(U)$. Then $r \cdot (g|_V)$ maps V to $|L|_w$. $\square$

3-10. THEOREM. A finite simplicial polyhedron is a neighborhood-extensor for normal spaces i.e. an NES (normal).

The proof is quite analogous to that above.

3-11. PROBLEM. Is a simplicial polyhedron with the weak topology $|L|_w$ an NES (paracompact Hausdorff)? It is to be noted that $|L|_w$ is a paracompact Hausdorff space. The normality of $|L|_w$ is proved in a manner analogous to that in the proof of Lemma 5-5, considering the closed covering $\{|s|\}$. The paracompactness is also proved in a way analogous to the proof of Lemma 5-5 again by induction on n -skeletons L_n , $n = 0, 1, \dots$.

3-12. THEOREM. Let X be a normal space or a collectionwise normal space, then an F_σ set H of X is normal or collectionwise normal, respectively.

PROOF. Let X be a collectionwise normal space. Set $H = \cup H_i$, where H_i 's are closed sets of X . Let F be a relatively closed set of H . Then $F \cap H_i$ is closed in X for $i = 1, 2, \dots$. Let $\mathcal{G}$ be an arbitrary, normal, relatively open covering of F . Then $\mathcal{G}|F \cap H_i$ has a neighborhood-extension $\mathcal{G}_i$ in X . We can assume that (i) $\mathcal{G}_i|F$ refines $\mathcal{G}$ and (ii) $G_i = \cup \{G: G \in \mathcal{G}_i\}$ is a cozero set of X . Since $\{G_i \cap H\}$ is normal by Theorem 2-27 and since every $\mathcal{G}_i|H \cap G_i$ is normal, $\cup \mathcal{G}_i|H$ is normal by Theorem 2-26. $\cup \mathcal{G}_i|H$ is a neighborhood-extension of $\mathcal{G}$. Hence by Theorem 3-3, H is collectionwise normal.

When X is a normal space, we can verify the assertion in an analogous way, starting from a finite open covering of F . $\square$

3-13. THEOREM (Michael [1]). An F_σ set H of a fully normal space X is fully normal.

PROOF. Set $H = \cup H_i$ where H_i is closed for $i = 1, 2, \dots$. Let $\mathcal{G}$ be an arbitrary open covering of H . Since H_i is fully normal, $\mathcal{G}|H_i$ is normal. Since X is collectionwise normal, $\mathcal{G}|H_i$ has a neighborhood-extension $\mathcal{G}_i$ such that:

- i. $G_i = \cup \{G: G \in \mathcal{G}_i\}$ is a cozero set of X
- ii. $\mathcal{G}_i|H$ refines $\mathcal{G}$

Then by Theorem 2-26, $\cup \mathcal{G}_i|H$ is a normal open covering of H which refines $\mathcal{G}$. $\square$

4. Identification Spaces and Absolute Retracts†

4-1. DEFINITION. Let X, Y be spaces, F a closed set of X and f a mapping of F to Y . Let the set Z be the disjoint sum of $X - F$ and Y . Let $k: X \rightarrow Z$ be a transformation defined by

$$\begin{aligned} k(x) &= x, & x \in X - F \\ k(x) &= f(x), & x \in F \end{aligned}$$

We give Z a topology as follows: A subset G of Z is open if and only if $k^{-1}(G)$ is open in X and $G \cap Y$ is open in Y . Then

$$Z = Z(X, F, f, Y)$$

is a topological space and is called the *identification space*. It is evident that Y is a closed set of Z .

Let us consider the following six classes of spaces:

[nor] is the class of all normal spaces

[col] is the class of all collectionwise normal spaces

[par] is the class of all paracompact Hausdorff spaces

[Lin] is the class of all regular Lindelöf spaces

[com] is the class of all compact Hausdorff spaces

[cme] is the class of all compact metric spaces

4-2. THEOREM. Let Ω be one of the six classes, then if X and Y are in Ω , Z is in Ω .

PROOF. Let X and Y be in [col]. Let $\{F_\alpha\}$ be a discrete collection of closed sets of Z . There exists an open collection $\{U_\alpha\}$ of Y such that

$$U_\alpha \supset F_\alpha \cap Y, \quad \{\bar{U}_\alpha\} \text{ is discrete}$$

Since $\{k^{-1}(F_\alpha \cup \bar{U}_\alpha)\}$ is discrete, there exist open sets V_α of X such that

$$V_\alpha \supset k^{-1}(F_\alpha \cup \bar{U}_\alpha), \quad \{V_\alpha\} \text{ is disjoint}$$

Set $W_\alpha = k(V_\alpha - F) \cup U_\alpha$. Then $\{W_\alpha\}$ is a disjoint open collection of Z such that $W_\alpha \supset F_\alpha$ for each α . Hence Z is in [col].

When X and Y are in [nor], Z is in [nor] by an analogous argument. Since the six classes are contained in [nor], Z is normal whichever class Ω is. The proofs for the classes [par], [Lin], and [com] are easy if we use Theorem 3-4, and are omitted.

† This section is due to Hanner [1].

Let us go to the last class, [cme]. Let X and Y be compact metric spaces. Then we already know that Z is a compact Hausdorff space. It suffices to prove that Z has a countable base. Let $\mathcal{U}$ be a countable base of $X - F$. Let $\mathcal{V}$ be a countable base of Y . For $i = 1, 2, \dots$ and V in $\mathcal{V}$ let

$$V_i = k(S_{1/i}(f^{-1}(V)) - F) \cup V$$

Then V_i is open in Z . To show that

$$\mathcal{U} \cup \{V_i : V \in \mathcal{V}, i = 1, 2, \dots\}$$

is a base of Z , let x be a point of Z and G an open neighborhood of x in Z . When $x \in X - F$, we can find a U from $\mathcal{U}$ with $x \in U \subset G - F$. U is of course open in Z . When $x \in Y - f(F)$, we can find a V from $\mathcal{V}$ with $x \in V \subset G - f(F)$ because of the compactness of $f(F)$. For this V , $V = V_i$ for each i .

When $x \in f(F)$, we can find a V from $\mathcal{V}$ such that

$$x \in V \subset \bar{V} \subset G$$

Since $X - k^{-1}(G)$ and $f^{-1}(\bar{V})$ have a positive distance, there exists a positive integer i with

$$S_{1/i}(f^{-1}(\bar{V})) \subset k^{-1}(G)$$

For this i we have $x \in V_i \subset G$. $\square$

4.3. DEFINITION. Let Ω be a class of spaces. A space X is an *absolute retract* for the class Ω , an **AR** (Ω), if the following conditions are satisfied:

- i. $X \in \Omega$
- ii. If $Y \in \Omega$ and Y contains X as a closed subset, then X is a retract of Y

A space X_1 is an *absolute neighborhood retract* for the class Ω , an **ANR** (Ω), if the following conditions are satisfied:

- i. $X_1 \in \Omega$
- ii. If $Y \in \Omega$ and Y contains X_1 as a closed subset, then X_1 is a neighborhood retract of Y

4.4. THEOREM. Let Ω be one of the six classes, then a space Y is an **AR** (Ω) if and only if $Y \in \Omega$ and Y is an **ES** (Ω). A space Y_1 is an **ANR** (Ω) if and only if $Y_1 \in \Omega$ and Y_1 is an **NES** (Ω).

PROOF. Sufficiency in both parts is evident. Suppose that Y is an **AR** (Ω). Let X be an arbitrary space in Ω , F an arbitrary closed set of X and f

an arbitrary mapping of F to Y . Let $Z = Z(X, F, f, Y)$ be the identification space. Then by Theorem 4-2, Z is in Ω and contains Y as a closed subset. So there is a retraction $r: Z \rightarrow Y$. Define $g: X \rightarrow Y$ by

$$\begin{aligned} g(x) &= r(x), & x \in X - F \\ g(x) &= f(x), & x \in F \end{aligned}$$

Then g is continuous. Since $g|_F = f$, Y is an ES (Ω). The remaining case is proved similarly. $\square$

5. Metrization Theorems

5-1. THEOREM. A space X is metrizable if and only if X is a fully normal, developable space.

PROOF. The necessity is evident. To prove the sufficiency let $\mathcal{U}_i$, $i = 1, 2, \dots$, be a development of X . Let $\mathcal{U}_1'$ be an open covering which Δ -refines $\mathcal{U}_1$. Let

$$\mathcal{V}_2 = \mathcal{U}_1' \wedge \mathcal{U}_2^\dagger$$

Then $\mathcal{V}_2$ is an open covering with $\mathcal{U}_1 \supset \mathcal{V}_2$ and $\mathcal{U}_2 \supset \mathcal{V}_2$. In this fashion we get open coverings $\mathcal{V}_3, \mathcal{V}_4, \dots$ successively such that

$$\begin{aligned} \mathcal{V}_i \text{ refines } \mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_i \\ \mathcal{V}_2 \supset \mathcal{V}_3 \supset \dots \end{aligned}$$

Then the sequence $\mathcal{V}_2, \mathcal{V}_3, \dots$ satisfies the condition of the Alexandroff-Urysohn metrization theorem, Section 2-18. $\square$

5-2. COROLLARY (Bing [1]). Let X be a collectionwise normal, developable space. Then X is metrizable.

PROOF. By the above theorem it suffices to prove the full normality, or equivalently the paracompactness, of X . Let $\mathcal{G} = \{G_\alpha\}$ be an arbitrary open covering of X indexed by a well-ordered set. Let $\mathcal{U}_i$, $i = 1, 2, \dots$, be a development of X . Set

$$H_{\alpha i} = \{x: x \notin \bigcup_{\beta < \alpha} G_\beta, \quad S(x, \mathcal{U}_i) \subset G_\alpha\}$$

$\dagger$ Let $\mathcal{U}_\alpha$, $\alpha \in A$, be collections of subsets of a set Y . Then $\wedge \mathcal{U}_\alpha$ denotes the collection of all sets of type $\cap \{U_\alpha: \alpha \in A\}$ where $U_\alpha \in \mathcal{U}_\alpha$.

Then $\mathcal{H}_i = \{H_{\alpha i}\}$ is a discrete collection of closed sets of X such that $\cup \mathcal{H}_i$ covers X . Let $V_{\alpha i}$ be open sets of X such that:

- i. $H_{\alpha i} \subset V_{\alpha i} \subset G_\alpha$ for each α and i
- ii. $\{V_{\alpha i}\}$ is discrete for each i

Then the collection of all $V_{\alpha i}$'s is a σ -discrete open covering which refines $\mathcal{G}$. By Theorem 2-22, X is paracompact. $\square$

5-3. THEOREM (Nagata-Smirnov metrization theorem).[†] A regular space X is metrizable if and only if X has a σ -locally finite base.

PROOF. Let X be a metric space. For any $i, i = 1, 2, \dots$, there exists a σ -discrete open covering $\mathcal{U}_i$ such that $\mathcal{U}_i$ refines $\{S_{1/i}(x): x \in X\}$ by Theorem 2-22. Then $\cup \mathcal{U}_i$ is a σ -discrete, and hence a σ -locally finite, base of X .

Suppose that a regular space X has a σ -locally finite base $\cup \mathcal{U}_i$, where each $\mathcal{U}_i = \{U_\alpha: \alpha \in A_i\}$ is a locally finite collection. Let $\mathcal{G}$ be an arbitrary open covering of X . Let $\mathcal{V}_i$ be the subcollection of $\mathcal{U}_i$ consisting of all elements of $\mathcal{U}_i$ which refine $\mathcal{G}$. Then $\cup \mathcal{V}_i$ refines $\mathcal{G}$ and covers X . By Lemma 2-21, X is normal. Take U_α from $\mathcal{U}_i$. Let

$$F_{\alpha j} = \cup \{\bar{U}: U \in \mathcal{U}_j, \bar{U} \subset U_\alpha\}$$

Let $f_{\alpha j}$ be a continuous function defined on X such that

$$\begin{aligned} f_{\alpha j}(x) &= 1, & x \in F_{\alpha j} \\ f_{\alpha j}(x) &= 0, & x \in X - U_\alpha \\ 0 \leq f_{\alpha j}(x) &\leq 1 & \text{everywhere} \end{aligned}$$

Set

$$\begin{aligned} \rho_{ij}(x, y) &= \sum_{\alpha \in A_i} |f_{\alpha j}(x) - f_{\alpha j}(y)| \\ \rho(x, y) &= \sum \sum [\rho_{ij}(x, y) / (2^{i+j}(1 + \rho_{ij}(x, y)))] \end{aligned}$$

Then ρ is a metric on X agreeing with the topology of X . $\square$

5-4. LEMMA. Let X be a space which is the sum of two closed metrizable sets F_1 and F_2 . Then X is metrizable.

PROOF. X is nothing but an identification space

$$Z(F_1, F_1 \cap F_2, f, F_2)$$

[†] Nagata [1] and Smirnov [1] proved this theorem independently.

as described in Definition 4-1, where $f: F_1 \cap F_2 \rightarrow F_2$ is the identity mapping. Hence X is a paracompact Hausdorff space by Theorem 4-2. By Theorem 5-1, there exist sequences $\mathcal{U}_1, \mathcal{U}_2, \dots$ and $\mathcal{V}_1, \mathcal{V}_2, \dots$ such that the former is a development of F_1 and the latter is a development of F_2 . Since each member of both sequences is extendable by Theorem 3-3, there exist extensions $\mathcal{U}_i'$ of $\mathcal{U}_i$ and $\mathcal{V}_i'$ of $\mathcal{V}_i$. It is easy to see that

$$\mathcal{U}_i' \wedge \mathcal{V}_j', \quad i, j = 1, 2, \dots$$

is a development of X . Again by Theorem 5-1, X is metrizable. $\square$

5-5. LEMMA. If a space X has a locally finite closed covering $\{F_\alpha\}$ each element of which is normal or paracompact Hausdorff, then X is normal or paracompact Hausdorff, respectively.

PROOF. Suppose that each F_α is normal. Let F be an arbitrary closed set of X and $f: F \rightarrow [0, 1]$ a mapping. Well-order the indices α . $f|_{F_0 \cap F}$ has an extension f_0 defined on F_0 whose values are in $[0, 1]$. In this fashion we get a transfinite sequence $f_0, f_1, \dots, f_\alpha, \dots$ of functions such that:

- i. f_α is defined on $\cup_{\beta \leq \alpha} F_\beta$
- ii. The values of f_α are in $[0, 1]$
- iii. $f_\alpha|_{F_\beta} = f_\beta|_{F_\beta}$ if $\beta < \alpha$
- iv. f_α is an extension of $f|_{(\cup_{\beta < \alpha} F_\beta) \cap F}$

Let g be a function with

$$g|_{F_\alpha} = f_\alpha|_{F_\alpha} \quad \text{for each } \alpha$$

Then g is an extension of f . Thus the normality of X is proved.

Suppose that each F_α is paracompact Hausdorff. Let $\mathcal{G}$ be an arbitrary open covering of X . Well-order the indices again. Let $\mathcal{G}_0$ be a locally finite open covering of F_0 with $\mathcal{G}_0 < \mathcal{G}$. By Theorem 3-4, $\mathcal{G}_0|_{F_0 \cap F_1}$ has a strict neighborhood-extension in F_1 which refines $\mathcal{G}$. Hence $\mathcal{G}_0|_{F_0}$ has a strict neighborhood-extension $\mathcal{G}_1'$ to a set U , open in $F_0 \cup F_1$, with $F_0 \subset U$ such that $\mathcal{G}_1'$ refines $\mathcal{G}$. Choose sets U_1 and U_2 such that

$$F_0 \subset U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U$$

where U_1 and U_2 are open in $F_0 \cup F_1$. Let $\mathcal{G}_1''$ be a locally finite open covering of $(F_0 \cup F_1) - U_1$ which refines $\mathcal{G}$.

If we set

$$\mathcal{G}_1 = (\mathcal{G}_1' | U_2) \cup (\mathcal{G}_1'' | (F_0 \cup F_1) - \bar{U}_1)$$

then $\mathcal{G}_1$ is a strict extension of $\mathcal{G}_0$ to $F_0 \cup F_1$ which refines $\mathcal{G}$. Continuing this argument transfinitely we get a locally finite open covering of X which refines $\mathcal{G}$. Thus X is paracompact. Since X is already normal, X is a Hausdorff space. $\square$

5-6. THEOREM.[†] If a space X has a locally finite closed covering each element of which is metrizable, then X is metrizable.

PROOF. By Lemma 5-4, X is locally metrizable. By Lemma 5-5, X is a paracompact Hausdorff space. Hence there exists a locally finite open covering $\{G_\alpha\}$ of X such that each G_α is metrizable. Choose a development $\mathcal{U}_{\alpha 1}, \mathcal{U}_{\alpha 2}, \dots$ of G_α such that $\mathcal{U}_{\alpha i}$ is refined by $\mathcal{U}_{\alpha i+1}$ for each i . Since

$$\bigcup_{\alpha} \mathcal{U}_{\alpha 1}, \quad \bigcup_{\alpha} \mathcal{U}_{\alpha 2}, \quad \dots$$

is a development of X , X is metrizable by Theorem 5-1. $\square$

5-7. THEOREM (Frink [1]). A space X is metrizable if and only if there exists a countable open neighborhood base $\{V_i(x) : i = 1, 2, \dots\}$ for each point x in X which satisfies the following condition:

For each point x in X and each number i there exists a number $j = j(x, i)$ such that $V_j(x) \cap V_j(y) \neq \emptyset$ implies $V_j(y) \subset V_i(x)$.

PROOF. It suffices to prove the sufficiency. Set

$$W_i(x) = \bigcap_{j \leq i} V_j(x)$$

Take an arbitrary point x in X and an arbitrary number i . Set

$$j_1 = j(x, 1), \dots, j_i = j(x, i)$$

If $j_0 = \max\{j_1, \dots, j_i\}$, then as can easily be seen

$$W_{j_0}(x) \cap W_{j_0}(y) \neq \emptyset \quad \text{implies} \quad W_{j_0}(y) \subset W_i(x)$$

Therefore we can assume without loss of generality that the original $\{V_i(x)\}$ is monotone: $V_1(x) \supset V_2(x) \supset \dots$.

For any point x let

$$1(x) = 1 < 2(x) = j(x, 1(x)) < 3(x) = j(x, 2(x)) < \dots$$

[†] Nagata [1] and Smirnov [1] proved this theorem independently.

Set

$$U_i(x) = V_{i(x)}(x),$$

$$\mathcal{U}_i = \{U_i(x) : x \in X\}, \quad i = 1, 2, \dots$$

If $j = (i + 1)(x)$, $V_j(x) \cap V_j(y) \neq \phi$ implies $V_j(y) \subset V_{i(x)}(x) = U_i(x)$. Since $U_j(y) \subset V_j(y)$, $S(x, \mathcal{U}_j) \subset U_i(x)$. Thus the sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ is a development of X .

To prove that $\mathcal{U}_{i+1}$ is a 2-refinement of $\mathcal{U}_i$ assume that

$$U_{i+1}(x) \cap U_{i+1}(y) \neq \phi$$

Set

$$s = (i + 1)(x), \quad t = (i + 1)(y)$$

When $s \leq t$,

$$U_{i+1}(y) = V_t(y) \subset V_s(y) = V_{(i+1)(x)}(y) \subset V_{i(x)}(x) = U_i(x)$$

Hence $U_{i+1}(x) \cup U_{i+1}(y) \subset U_i(x)$. When $s \geq t$, we have $U_{i+1}(x) \cup U_{i+1}(y) \subset U_i(y)$ analogously. Now by the Alexandroff-Urysohn metrization theorem, Section 2-18, X is metrizable. $\square$

5-8. THEOREM (Morita [4]). A space X is metrizable if and only if X has a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open coverings of X which satisfies the condition:

For each point x in X , $\{S(S(x, \mathcal{U}_i), \mathcal{U}_j) : i, j = 1, 2, \dots\}$ is a neighborhood base of x .

PROOF. It suffices to prove the sufficiency. Set

$$\mathcal{V}_i = \bigwedge_{j \leq i} \mathcal{U}_j$$

Then $\mathcal{V}_1 > \mathcal{V}_2 > \dots$ and $\{S^2(x, \mathcal{V}_i)\}$ forms a neighborhood base of x . There exist a sequence $1 = 1(x) < 2(x) < \dots$ and a sequence $V_1(x), V_2(x), \dots$ such that:

- i. Each $V_i(x)$ is an element of $\mathcal{V}_{i(x)}$
- ii. $x \in V_{i+1}(x) \subset V_i(x)$ for each i

Let us show the collection $\{V_i(x)\}$ satisfies the condition of Frink's metrization theorem, Theorem 5-7. Let x be an arbitrary point of X and i an arbitrary number, then there exists a number $j = j(x, i)$ such that $S^2(x, \mathcal{V}_j) \subset V_i(x)$. Suppose that $V_j(x) \cap V_j(y) \neq \phi$. Since

$$V_j(x) \in \mathcal{V}_{j(x)} < \mathcal{V}_j \quad \text{and} \quad V_j(y) \in \mathcal{V}_{j(y)} < \mathcal{V}_j$$

$$V_j(y) \subset S^2(x, \mathcal{V}_j) \subset V_i(x) \square$$

5-9. THEOREM (Nagata [2]). A space X is metrizable if and only if for each point x in X there exist three systems of neighborhoods of x , $\{U_i(x): i = 1, 2, \dots\}$, $\{U_i^1(x): i = 1, 2, \dots\}$, $\{U_i^2(x): i = 1, 2, \dots\}$ such that:

- i. $\{U_i(x)\}$ is a neighborhood base of x
- ii. $y \notin U_i(x)$ implies $U_i^1(x) \cap U_i^2(y) = \emptyset$
- iii. $y \in U_i^1(x)$ implies $U_i^2(y) \subset U_i(x)$

PROOF.† Taking finite intersections if necessary, we can assume without loss of generality that $U_i^2(x) \supset U_{i+1}^2(x)$ for each x and each i . Set

$$W_i(x) = \text{the interior of } U_i(x) \cap U_i^1(x) \cap U_i^2(x)$$

$$\mathcal{W}_i = \{W_i(x): x \in X\}$$

Let us show that the sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ satisfies the condition of Theorem 5-8. Let a point x and a number i be given. Take $j = j(x, i)$ such that $j > i$ and $U_j(x) \subset U_i^1(x)$. Then

$$U_j^1(x) \cap U_j^2(y) \neq \emptyset \quad \text{implies} \quad y \in U_j(x)$$

Therefore $y \in U_i^1(x)$ and hence $U_i^2(y) \subset U_i(x)$. Since $U_j^2(y) \subset U_i^2(y)$, $U_j^2(y) \subset U_i(x)$. Thus we have

$$S(W_j(x), \mathcal{W}_j) \subset S(U_j^1(x), \mathcal{W}_j) \subset U_i(x)$$

which implies that

$$\{S(S(x, \mathcal{W}_s), \mathcal{W}_t): s, t = 1, 2, \dots\}$$

forms a neighborhood base of x . $\square$

5-10. COROLLARY.‡ A space X is metrizable if and only if there exists a sequence $\mathcal{H}_1, \mathcal{H}_2, \dots$ of coverings of X such that for each point x in X

$$\{S(S(x, \mathcal{H}_i), \mathcal{H}_j): i, j = 1, 2, \dots\}$$

forms a neighborhood base of x .

† The present proof is due to Morita.

‡ This is due to Nagami [4]. It is a generalization of Morita's metrization theorem, Section 5-8.

PROOF. It suffices to prove the sufficiency. When x is in the outside of the closure of $S(X - S(x, \mathcal{H}_i), \mathcal{H}_j)$, let

$$\begin{aligned} U_{ij}(x) &= S(x, \mathcal{H}_i) \\ U_{ij}^1(x) &= X - S(X - S(x, \mathcal{H}_i), \mathcal{H}_j) \\ U_{ij}^2(x) &= S(x, \mathcal{H}_j) \end{aligned}$$

In the other case, let

$$\begin{aligned} U_{ij}(x) &= U_{ij}^1(x) = X \\ U_{ij}^2(x) &= S(x, \mathcal{H}_i) \end{aligned}$$

Then the systems $\{U_{ij}(x)\}, \{U_{ij}^k(x)\}, k = 1, 2$, satisfy the three conditions of Nagata's general metrization theorem, Theorem 5-9. $\square$

5-11. COROLLARY (Morita and Hanai [1]; Stone [2]). If a space X is the image of a metric space Y under a closed mapping[†] f such that $f^{-1}(x)$ is compact for each point x in X , then X is metrizable.

PROOF. Set

$$\begin{aligned} U_n(x) &= f(S_{1/n}(f^{-1}(x))) \\ U_n^1(x) &= U_n^2(x) = X - f(Y - S_{1/(2n)}(f^{-1}(x))) \end{aligned}$$

Then $\{U_n(x): n = 1, 2, \dots\}$ is a neighborhood base of x . If $y \in U_n^1(x)$, $f^{-1}(y) \subset S_{1/(2n)}(f^{-1}(x))$. Hence

$$S_{1/(2n)}(f^{-1}(y)) \subset S_{1/n}(f^{-1}(x))$$

which implies $U_n^2(y) \subset U_n(x)$.

If $y \notin U_n(x)$, $f^{-1}(y) \cap S_{1/n}(f^{-1}(x)) = \phi$. Hence

$$S_{1/(2n)}(f^{-1}(y)) \cap S_{1/(2n)}(f^{-1}(x)) = \phi$$

which implies $U_n^2(y) \cap U_n^1(x) = \phi$. Therefore all conditions of Nagata's general metrization theorem are satisfied. $\square$

When A is a subset of a space X , let $B(A)$ denote the boundary of A . Then this corollary is generalized as follows: If a space X is the image of a metric space Y under a closed mapping f such that $B(f^{-1}(x))$ is

[†] A transformation is called *closed* (or *open*) if every closed set (or open set) has a closed (or open, respectively) image.

compact for each $x \in X$, then X is metrizable. The reason is: Since $\cup B(f^{-1}(x))$ is closed in Y , the restriction of f to this set is also closed.

If X and Y are metric and $f: Y \rightarrow f(Y) = X$ is closed, $B(f^{-1}(x))$ has to be compact for each $x \in X$. This is proved in Section 13-10.

6. Inverse Limits and Compact Spaces

6-1. DEFINITION. $\{X_\alpha, f_\alpha^\beta: \alpha \in A\}$ is called an *inverse limiting system* if each X_α is a space indexed by a directed set A and for each ordered pair $\alpha < \beta$ in A there exists a mapping $f_\alpha^\beta: X_\beta \rightarrow X_\alpha$ satisfying the following condition:

$$f_\alpha^\gamma = f_\alpha^\beta f_\beta^\gamma, \quad \text{whenever } \alpha < \beta < \gamma$$

Let $\pi_\alpha': \Pi X_\alpha \rightarrow X_\alpha$ be the projection of the product of all X_α 's to X_α . Let

$$\text{inv lim } X_\alpha = \text{inv lim } \{X_\alpha, f_\alpha^\beta: \alpha \in A\}$$

be the aggregate of all points x in ΠX_α such that

$$\pi_\alpha'(x) = f_\alpha^\beta \pi_\beta'(x), \quad \text{whenever } \alpha < \beta$$

The set $\text{inv lim } X_\alpha$ is topologized in such a way that

$$\{\pi_\alpha^{-1}(G): G \text{ is open in } X_\alpha, \alpha \in A\}$$

is its base, where $\pi_\alpha = \pi_\alpha' | \text{inv lim } X_\alpha$. The space $\text{inv lim } X_\alpha$ thus topologized is the *inverse limit* of the system $\{X_\alpha, f_\alpha^\beta\}$.

ASSERTION 1. This topology, τ_1 , of $\text{inv lim } X_\alpha$ coincides with its relative topology, τ_2 , of the product topology of ΠX_α .

PROOF. τ_2 is evidently stronger than τ_1 . Take a point x from $\text{inv lim } X_\alpha$. A basic neighborhood V of x in τ_2 has the form

$$V = \bigcap_{\alpha \in B} \pi_\alpha^{-1}(G_\alpha)$$

where B is a finite subset of A and G_α is an open neighborhood of $\pi_\alpha(x)$. Take β such that $\alpha < \beta$ for any $\alpha \in B$. Take an open neighborhood G_β of $\pi_\beta(x)$ such that $f_\alpha^\beta(G_\beta) \subset G_\alpha$ for any $\alpha \in B$. Then $\pi_\beta^{-1}(G_\beta) \subset V$. $\square$

ASSERTION 2. When each X_α is a Hausdorff space, then $\text{inv lim } X_\alpha$ is closed in ΠX_α .

PROOF. Let $x \in \Pi X_\alpha - \text{inv lim } X_\alpha$. Then there exists an ordered pair $\alpha < \beta$ such that $\pi_\alpha'(x) \neq f_\alpha^\beta \pi_\beta'(x)$. Take open sets G_1 and G_2 in X_α and

an open set G_3 in X_β such that:

- (i) $\pi'_\alpha(x) \in G_1$, (ii) $f_\alpha^\beta \pi'_\beta(x) \in G_2$, (iii) $\pi'_\beta(x) \in G_3$
 (iv) $G_1 \cap G_2 = \phi$, (v) $f_\alpha^\beta(G_3) \subset G_2$

Then $\pi'^{-1}_\alpha(G_1) \cap \pi'^{-1}_\beta(G_3)$ is an open neighborhood of x which does not meet $\text{inv lim } X_\alpha$. $\square$

ASSERTION 3. When each X_α is a nonempty compact Hausdorff space, $\text{inv lim } X_\alpha$ is compact and nonempty.

PROOF. Since $\text{inv lim } X_\alpha$ is closed, it is compact by the compactness of ΠX_α . Let $\mathcal{B} = \{B\}$ be the collection of all finite subsets of A . For each B in $\mathcal{B}$ let Y_B be the set of all points x in ΠX_α such that if $\alpha < \beta$, $\alpha \in B$, $\beta \in B$, then $\pi'_\alpha(x) = f_\alpha^\beta \pi'_\beta(x)$. Then Y_B is closed by the continuity of f_α^β . To see the finite intersection property of the collection $\{Y_B: B \in \mathcal{B}\}$ let $B_1, \dots, B_n$ be a finite number of elements of $\mathcal{B}$. Take γ such that $\alpha < \gamma$ for any α in $\cup B_i$. Take a point x_γ from X_γ . Then

$$\phi \neq \langle f_\alpha^\gamma(x_\gamma): \alpha \in \cup B_i \rangle \times \Pi \{X_\beta: \beta \in A - \cup B_i\} \subset Y_{B_j}$$

for $j = 1, 2, \dots, n$

Therefore $\cap \{Y_B: B \in \mathcal{B}\}$, which is nothing but $\text{inv lim } X_\alpha$, is not empty. $\square$

6-2. DEFINITION. Let X be a completely regular space. The *Stone-Čech compactification* βX of X is a compact Hausdorff space containing X as a dense subset such that each real valued, bounded, continuous function defined on X has an extension over βX .

A construction of βX is as follows: Let $\{f_\alpha\}$ be the set of all real valued, bounded mappings of X . Let I_α be the closed interval whose end points are $\inf f_\alpha(x)$ and $\sup f_\alpha(x)$. Define $\varphi: X \rightarrow \Pi I_\alpha$ by

$$\varphi(x) = \langle f_\alpha(x) \rangle$$

Then φ is an embedding. Identify X with $\varphi(X)$. Then βX is the closure of X in ΠI_α . An extension f_α^* of f_α is defined by

$$f_\alpha^*(x) = \pi_\alpha(x)$$

where $\pi_\alpha: \Pi I_\alpha \rightarrow I_\alpha$ is the projection.

6-3. DEFINITION. A space X is *connected* if there is no open-closed,[†] proper, nonempty set in X . A *compactum* is a connected compact Hausdorff space. If Y is a subset of X such that $X - Y$ is not connected, then Y *separates* X . Let H and K be sets of X with $H \cap K = \phi$. If Y is a separating set of X such that there exists a relatively open-closed set G of $X - Y$ with $H \subset G$ and $G \cap K = \phi$, then Y *separates* H and K and H is *separated by* Y *from* K . If D and E are open-closed sets of a space X such that $D \neq \phi$, $E \neq \phi$, $D \cap E = \phi$, $D \cup E = X$, then the pair D, E is a *separation* of X .

6-4. DEFINITION. Let $\{X_\alpha: \alpha \in A\}$ be a family of sets of a space X indexed by a directed set A . $\limsup X_\alpha$ is the set of points x such that for any neighborhood U of x , $\{\alpha: U \cap X_\alpha \neq \phi\}$ is cofinal. $\liminf X_\alpha$ is the set of points x such that for any neighborhood U of x , $\{\alpha: U \cap X_\alpha \neq \phi\}$ is equifinal.

It can easily be seen that:

- i. Both $\liminf X_\alpha$ and $\limsup X_\alpha$ are closed in X and

$$\liminf X_\alpha \subset \limsup X_\alpha$$

- ii. If B is equifinal in A , then

$$\liminf\{X_\alpha: \alpha \in A\} = \liminf\{X_\alpha: \alpha \in B\}$$

- iii. If C is cofinal in A , then

$$\limsup\{X_\alpha: \alpha \in A\} \supset \limsup\{X_\alpha: \alpha \in C\}$$

6-5. DEFINITION. Let $\mathcal{G}$ be an open covering of a space X . A finite sequence of points $x_1, \dots, x_n$ in X is a $\mathcal{G}$ -*chain* between x_1 and x_n if for each i there is an element G of $\mathcal{G}$ with $\{x_i, x_{i+1}\} \subset G$. A subset Y of X is $\mathcal{G}$ -*enchained* if for each pair of points x, y in Y there is a $\mathcal{G} \upharpoonright Y$ -chain between x and y .

6-6. THEOREM. Let X be a compact Hausdorff space and $\{\mathcal{G}_\alpha: \alpha \in A\}$ be the family of all open coverings of X , where A is directed in such a way that $\alpha < \beta$ if and only if $\mathcal{G}_\alpha > \mathcal{G}_\beta$. Let $\{M_\alpha: \alpha \in A\}$ be a family of sets of X such that each M_α is $\mathcal{G}_\alpha$ -enchained. If $\liminf M_\alpha \neq \phi$, then $\limsup M_\alpha$ is a compactum.

[†] An *open-closed* set is an open set which is closed at the same time.

PROOF. Set

$$\liminf M_\alpha = P \quad \text{and} \quad \limsup M_\alpha = Q$$

Assume that Q were not connected. Then there would exist two nonempty compact sets Q_1 and Q_2 such that $Q_1 \cap Q_2 = \emptyset$ and $Q_1 \cup Q_2 = Q$. Choose open sets G_1, G_2, H_1, H_2 such that:

- i. $Q_i \subset G_i \subset \bar{G}_i \subset H_i, \quad i = 1, 2$
- ii. $H_1 \cap H_2 = \emptyset$

Set

$$B = \{\alpha: M_\alpha \subset G_1 \cup G_2\}$$

Then B is equifinal. Let

$$\mathcal{G}_{\alpha_0} = \{H_1, H_2, X - \bar{G}_1 \cup \bar{G}_2\}$$

If $x \in G_1$ and $y \in G_2$, then there exists no $\mathcal{G}_{\alpha_0} \mid M_\alpha$ -chain between x and y for any α in B . Therefore if we set

$$B_i = \{\alpha: \alpha \in B, \quad M_\alpha \subset G_i\}, \quad i = 1, 2$$

then both B_1 and B_2 are cofinal and $B_1 \cup B_2$ is equifinal. Hence we have $\liminf\{M_\alpha: \alpha \in B_1 \cup B_2\} = \emptyset$, which contradicts

$$\liminf\{M_\alpha: \alpha \in B_1 \cup B_2\} = \liminf\{M_\alpha: \alpha \in A\} \neq \emptyset \quad \square$$

6-7. THEOREM. Let X be a compact Hausdorff space, H a closed set of X and p a point with $p \in X - H$. If H cannot be separated by the empty set from p , then there exists a compactum K such that $p \in K$ and $K \cap H \neq \emptyset$.

PROOF. Let $\{\mathcal{G}_\alpha: \alpha \in A\}$ be the family of all open coverings of X where A is directed in the usual manner. By the condition we can find a $\mathcal{G}_\alpha$ -chain M_α for each $\mathcal{G}_\alpha$ such that $p \in M_\alpha$ and $M_\alpha \cap H \neq \emptyset$. Let $K = \limsup M_\alpha$. Since p is in $\liminf M_\alpha$, K is a compactum by Theorem 6-6. To prove $K \cap H \neq \emptyset$, assume the contrary. Then for any point x in H we could find an open neighborhood $U(x)$ such that

$$\{\alpha: U(x) \cap M_\alpha \neq \emptyset\}$$

is not cofinal. Find a finite number of such $U(x)$'s, say $U(x_1), U(x_2), \dots, U(x_n)$, with $\cup U(x_i) \supset H$. Then $\cup U(x_i)$ could not meet $\{M_\alpha\}$ cofinally, which is a contradiction. $\square$

6-8. COROLLARY. Let X be a compact Hausdorff space, F and H closed sets of X with $F \cap H = \phi$. If there is no connected set meeting F and H , then the empty set separates F and H .

PROOF. Let x be an arbitrary point of F . By Theorem 6-7, there is an open-closed neighborhood $U(x)$ of x which does not meet H . Let $U(x_1), \dots, U(x_n)$ be such neighborhoods whose sum G contains F . Of course, $G \cap H = \phi$. $\square$

6-9. LEMMA. Let K be a compactum in a Hausdorff space X . Let U be an open set with $U \cap K \neq \phi$ and $(X - U) \cap K \neq \phi$. Then for any point p in $U \cap K$ there exists a compactum H such that: (i) $p \in H$, (ii) $H \cap B(U) \neq \phi$, and (iii) $H \subset \bar{U} \cap K$.

PROOF. If there were not such a compactum, then $\bar{U} \cap K$ would be decomposed, applying Theorem 6-7, into two closed sets K_1 and K_2 such that

$$K_1 \cap K_2 = \phi, \quad p \in K_1, \quad B(U) \cap K \subset K_2$$

Then the pair K_1 and $K - K_1$ is a separation of K , which is a contradiction. $\square$

6-10. THEOREM (Sierpinski's theorem). Let X be a compact Hausdorff space. If X admits a countable, disjoint, closed covering $\{F_i\}$ such that $F_i \neq \phi$ for any i , then X is not connected.

PROOF. Assume that X were connected. Take an integer $i_1 > 1$ and an open set U_1 such that (i) $\bar{U}_1 \cap F_i = \phi$ for $i < i_1$ and (ii) $F_{i_1} \subset U_1$. Then by the lemma there exists a compactum X_1 such that

$$X_1 \cap F_{i_1} \neq \phi, \quad X_1 \subset \bar{U}_1, \quad B(U_1) \cap X_1 \neq \phi$$

Take an integer i_2 such that $B(U_1) \cap X_1 \cap F_{i_2} \neq \phi$. Then $i_2 > i_1$. Let U_2 be an open set such that

$$\bar{U}_2 \cap F_i = \phi \quad \text{for } i < i_2 \quad \text{and} \quad F_{i_2} \subset U_2$$

Then there exists a compactum $X_2 \subset X_1$ such that

$$X_2 \cap F_{i_2} \neq \phi, \quad X_2 \subset \bar{U}_2, \quad B(U_2) \cap X_2 \neq \phi$$

Continuing recursively in this fashion we have a sequence of nonempty compactums $X_1, X_2, \dots$ such that $X_i \cap F_j \neq \phi$ for any i and any $j \leq i$, which contradicts the inequality $\cap X_i \neq \phi$. $\square$

7. Totally Normal Spaces

7-1. DEFINITION. A space X is a *totally normal space*[†] if X is normal and every open set G in X has an open covering $\mathcal{G}$ such that:

- i. Each element of $\mathcal{G}$ is a cozero set of X
- ii. $\mathcal{G}$ is locally finite in G

7-2. PROPOSITION. A perfectly normal space is totally normal.

PROOF. Every open set of a perfectly normal space is a cozero set of X . $\square$

TERMINOLOGY. A space is *hereditarily paracompact* if every subset of the space is paracompact. Let $\mathcal{Q}$ be a class of spaces. A space X is a hereditarily $\mathcal{Q}$ -space if $Y \subset X$ implies $Y \in \mathcal{Q}$.

7-3. PROPOSITION. A hereditarily paracompact Hausdorff space X is totally normal.

PROOF. Let G be an open set of X . For every point x in G let $U(x)$ be neighborhood of x such that $U(x) \subset G$ and $U(x)$ is cozero in X . Let $\mathcal{G}$ be an open covering of G such that:

- i. $\mathcal{G} < \{U(x) : x \in G\}$
- ii. $\mathcal{G}$ is locally finite in G
- iii. Every element of $\mathcal{G}$ is cozero in G

Such a $\mathcal{G}$ does exist by the remark at the end of Section 2-19. Since every element G_α of $\mathcal{G}$ is contained in some $U(x)$ which is cozero in X , then G_α is cozero in X . $\square$

7-4. PROPOSITION. A totally normal space X is hereditarily normal.

PROOF. Let Y be an arbitrary subset of X and $\mathcal{G} = \{G_i\}$ an arbitrary finite (relatively) open covering of Y . Let H_i be an open set of X with $H_i \cap Y = G_i$. Set $H = \cup H_i$. Let $\{D_\alpha\}$ be an open covering of H which is locally finite in H such that every D_α is cozero in X . Then by Theorem 3-12 every D_α is normal. Let $\mathcal{H}_\alpha$ be the restriction of $\{H_i\}$ to D_α . Then by Theorem 2-14 $\mathcal{H}_\alpha$ is normal. Since $\{D_\alpha\}$ is normal by Theorem 2-27,

[†] This notion is due to Dowker [3]. All results in this section are also due to him, except Assertions 2 and 3 in Section 7-6 which are due to Bing [1].

$\cup \mathcal{H}_\alpha$ is normal by Theorem 2-26. Clearly $\cup \mathcal{H}_\alpha$ refines $\{H_i\}$. Hence $\{H_i\}$ is normal and $\{G_i\}$ is also. Again by Theorem 2-14, Y is normal. $\square$

7-5. PROPOSITION. Every subset Y of a totally normal space X is also totally normal.

PROOF. Y is normal by the preceding proposition. Let G be an arbitrary relatively open set in Y . Let H be an open set in X with $H \cap Y = G$. Let $\{H_\alpha\}$ be a locally finite (in H) covering of H such that each H_α is cozero in X . Then $\{H_\alpha \cap G\}$ is a locally finite (in G) covering of G such that each $H_\alpha \cap G$ is cozero in Y . $\square$

7-6. REMARK. Bing's example F in Section 2-3 illustrates the status of totally normal spaces.

ASSERTION 1. F is totally normal.

PROOF. F is already normal. Let U be an arbitrary open set of F . Set

$$U_0 = U \cap F_0, \quad P_1 = \{p \in P, f_p \in U_0\}$$

Fix a function f_{p_0} in U_0 if $U_0 \neq \phi$. Let $V = (f_{p_0} : \{P_1\})$. Then $U_0 \subset V$. To see that $U \cap V$ is closed in F let f_p be an element of $F_0 - U_0$. Then $p \notin P_1$. Hence

$$(f_p : \{P_1\}) \cap (f_{p_0} : \{P_1\}) = \phi$$

Now

$$\{U \cap V\} \cup \{ \{f\} : f \in U - V \}$$

is a locally finite (in U) covering of U , each element of which is cozero in F . $\square$

ASSERTION 2. F is not perfectly normal.

PROOF. To prove that an f_p is not G_δ , assume the contrary: $f_p = \cap G_i$, where the G_i 's are open in F . Take $r_i \in Q_0$ with $(f_p : r_i) \subset G_i$. Since $\cup r_i$ consists of a countable number of elements of Q , say $q_1, q_2, \dots$, there is a $q \in Q$ such that $q \neq q_i$ for any i . Let f be an element of F such that

$$f(q) \neq f_p(q) \quad \text{and} \quad f(q_i) = f_p(q_i) \quad \text{for } i = 1, 2, \dots$$

Then $f \in \cap G_i$, which is a contradiction. $\square$

ASSERTION 3. F is not paracompact.

PROOF. We already know that F is not collectionwise normal. Since every paracompact Hausdorff space is collectionwise normal by Theorem 2-5, the assertion is true. $\square$

Chapter 2

DIMENSION OF NORMAL SPACES

In the following n denotes a nonnegative integer. The class of normal spaces is the weakest class of spaces where we define and study dimension functions. Although it is possible to define some kind of dimension for more general spaces, the result may be trivial and have little mathematical meaning. Our efforts should be not toward generalizing the theory beyond normal spaces but rather in finding good intermediate classes between normal spaces and metric spaces in which to develop the theory.

In this chapter we present three basic dimension functions for normal spaces, their fundamental properties, and relations between them. In Section 8, we give definitions of the three functions and study elementary properties of 0-dimensional normal spaces. The three functions are $\dim$ (the covering dimension), Ind (the large inductive dimension), and ind (the small inductive dimension). Among these $\dim$ is the most important and Ind is next. The dimension function ind is not a global concept but a local one; through ind we can realize an interesting gap between the global and the local. In Section 9 we study $\dim$ for normal spaces. The two important expressions equivalent to $\dim X \leq n$ will be given in Theorem 9-9. They are the characterization theorems: by S^n and by Eilenberg–Otto separation. The sum theorem for $\dim$ (Theorem 9-10) is then a trivial corollary. Section 10 is just an elementary observation of Ind for normal spaces, while Section 11 gives the sum and the subset theorems for totally normal spaces (Theorem 11-5). Local dimension is also studied in this section.

8. Dimension Functions and 0-Dimensional Spaces

8-1. DEFINITION. Let X be a normal space. The *covering dimension* of X , or simply the dimension of X , denoted by $\dim X$, is at most n if

every finite open covering of X can be refined by an open covering whose order is at most $n + 1$. If $\dim X \leq n$ and the statement $\dim X \leq n - 1$ is false, we say $\dim X = n$. If the statement $\dim X \leq n$ is false for all n , then we say $\dim X = \infty$. For the empty set, $\dim \phi = -1$.

The *large inductive dimension* of X , denoted by $\text{Ind } X$, is defined inductively as follows. $\text{Ind } \phi = -1$. If for any disjoint pair of closed sets F, H of X there exists a closed set K with $\text{Ind } K \leq n - 1$ separating F and H , then $\text{Ind } X \leq n$. If $\text{Ind } X \leq n$ and the statement $\text{Ind } X \leq n - 1$ is false, then we say $\text{Ind } X = n$. If the statement $\text{Ind } X \leq n$ is false for all n , then we say $\text{Ind } X = \infty$.

The *small inductive dimension* of X , denoted by $\text{ind } X$, is defined inductively as follows. $\text{ind } \phi = -1$. If for any point x and any closed set F with $x \in X - F$ there exists a closed set K of X with $\text{ind } K \leq n - 1$ separating x and F , then $\text{ind } X \leq n$. If $\text{ind } X \leq n$ and the statement $\text{ind } X \leq n - 1$ is false, then we say $\text{ind } X = n$. If the statement $\text{ind } X \leq n$ is false for all n , then we say $\text{ind } X = \infty$.†

It is evident that the statement $\text{ind } X \leq \text{Ind } X$ is always true. It is also evident that $\dim X \leq n$ if and only if the following condition is satisfied: Every finite open covering $\{G_i\}$ of X can be refined by an open covering $\{E_i\}$ in a one-one corresponding way, i.e., $G_i \supset E_i$ for each i , whose order is at most $n + 1$.

8-2. THEOREM. If X is a separable metric space, then

$$\dim X = \text{ind } X = \text{Ind } X$$

This is one of the main results in Hurewicz and Wallman's "Dimension Theory" [1]. Our Theorem 12-6 combined with Theorem 11-13 or Lemma 19-1 gives another proof of this theorem.

8-3. PROPOSITION. Let X be a normal space, then $\dim X = 0$ if and only if $\text{Ind } X = 0$.

PROOF. It is evident that if $\dim X = 0$, then $\text{Ind } X = 0$. Suppose $\text{Ind } X = 0$. Let $\{G_i\}$ be an arbitrary finite open covering of X . Since $\{G_i\}$ is shrinkable by Theorem 1-7, there exists a closed covering $\{F_i\}$ in a one-one correspondent way such that $G_i \supset F_i$ for each i . Take an open-closed set E_i with $G_i \supset E_i \supset F_i$. Then

$$\wedge \{E_i, X - E_i\}$$

† Sometimes we consider ind for merely regular spaces.

is a disjoint open covering of X which refines $\{G_i\}$. $\square$

8-4. PROPOSITION. Let X be a compact Hausdorff space, then the following three statements are equivalent:

$$(i) \dim X = 0, \quad (ii) \text{Ind } X = 0, \quad (iii) \text{ind } X = 0$$

This is evident.

8-5. PROPOSITION. A space X is a compact Hausdorff space with $\dim X \leq 0$ if and only if X is an inverse limit of finite discrete spaces.

PROOF. When X is an inverse limit of finite discrete spaces, then X is a compact Hausdorff space by Assertion 3 of Section 6-1. Since evidently $\text{ind } X \leq 0$, $\dim X \leq 0$ by Proposition 8-4.

Conversely, let X be a compact Hausdorff space with $\dim X \leq 0$. Let

$$\{\mathcal{G}_\lambda = \{G_\alpha : \alpha \in A_\lambda\} : \lambda \in \Lambda\}$$

be the family of all finite, disjoint, open coverings of X . When $\mathcal{G}_\lambda > \mathcal{G}_\mu$, we say $\lambda < \mu$. Then Λ is a directed set. When $\lambda < \mu$, define $f_\lambda^\mu : A_\mu \rightarrow A_\lambda$ by

$$f_\lambda^\mu(\alpha) = \beta \quad \text{implies} \quad G_\alpha \subset G_\beta$$

Then

$$\{A_\lambda \text{ with the discrete topology, } f_\lambda^\mu\}$$

constitutes an inverse limiting system. Let A be its inverse limit. Let $\pi_\lambda : A \rightarrow A_\lambda$ be the projection. Define $f : A \rightarrow X$ by

$$f(a) = \bigcap G_{\alpha_\lambda}, \quad \text{where} \quad a = \langle \alpha_\lambda \rangle$$

Then f is a homeomorphism of A onto X . $\square$

When X is a completely regular space with $\text{ind } X = 0$, we get a compact Hausdorff space A with $\dim A = 0$ and a homeomorphism f^{-1} of X into A in the same way as above. Therefore, we may decide: A completely regular space X has $\text{ind } X \leq 0$ if and only if X is a dense subset of a compact Hausdorff space Y with $\dim Y \leq 0$.

8-6. PROPOSITION. If a compact Hausdorff space X is totally disconnected, then $\dim X \leq 0$.

PROOF. Let x be a point of X and F a closed set of X which does not contain x . Then there is no compactum meeting x and F . Hence, by

Theorem 6-7, we have $\text{ind } X \leq 0$, which implies $\dim X \leq 0$ since X is now compact. $\square$

It is to be noted that there is a separable, metric, totally disconnected space X with $\dim X \neq 0$. An example will be given in Section 9-12.

As an immediate corollary of Proposition 8-6 we have the assertion: A compact Hausdorff space X has $\dim X \leq 0$ if and only if X is a closed subset of a product of finite discrete spaces.

9. Covering Dimension of Normal Spaces

9-1. DEFINITION. Let $\{S_\alpha: \alpha \in A\}$ and $\{T_\alpha: \alpha \in A\}$ be two collections of subsets of a fixed set X . $\{S_\alpha\}$ is *similar* to $\{T_\alpha\}$, denoted by $\{S_\alpha\} \sim \{T_\alpha\}$, if for any subset B of A , $\cap \{S_\alpha: \alpha \in B\} \neq \phi$ when and only when $\cap \{T_\alpha: \alpha \in B\} \neq \phi$.

9-2. PROPOSITION (Morita [3]). Let X be a normal space, $\{G_\alpha: \alpha \in A\}$ a locally finite collection of open sets of X and $\{F_\alpha: \alpha \in A\}$ a collection of closed sets of X such that $F_\alpha \subset G_\alpha$ for each α . Then there is an open collection $\{H_\alpha: \alpha \in A\}$ such that $\{H_\alpha\} \sim \{F_\alpha\}$ and $F_\alpha \subset H_\alpha \subset G_\alpha$ for each α .

PROOF (by transfinite induction). Well-order the index set A . Let F be the sum of all finite intersections of elements from $\{F_\alpha\}$ such that the intersections do not meet the first element, F_0 . Let H_0 be an open set with

$$F_0 \subset H_0 \subset \bar{H}_0 \subset (X - F) \cap G_0$$

Then $\{\bar{H}_0, F_1, F_2, \dots\}$ is similar to $\{F_\alpha\}$. In this fashion we can enlarge F_α to an open set H_α , step by step, transfinitely. Finally, we get a collection $\{H_\alpha\}$, which satisfies the required condition. $\square$

9-3. PROPOSITION (Morita [3]). Let X be a fully normal space and $\{F_\alpha\}$ a locally finite collection of closed sets of X , then there is a locally finite open collection $\{H_\alpha\}$ with $\{H_\alpha\} \sim \{F_\alpha\}$ such that $F_\alpha \subset H_\alpha$ for each α .

PROOF. For every point x in X let $U(x)$ be an open neighborhood of x which meets at most a finite number of elements of $\{F_\alpha\}$. Let $\mathcal{U}$ be an open covering of X which $*$ -refines $\{U(x): x \in X\}$. To see that

$$\{G_\alpha = S(F_\alpha, \mathcal{U})\}$$

is locally finite, let x be an arbitrary point of X . Take an element U from $\mathcal{U}$ with $x \in U$. Set

$$B = \{\alpha: U \cap G_\alpha \neq \phi\}$$

If $U \cap G_\alpha \neq \phi$, then there is an element of $\mathcal{U}$ meeting U and F_α at the same time. Hence, $S(U, \mathcal{U}) \cap F_\alpha \neq \phi$. Take $U(y)$ with $S(U, \mathcal{U}) \subset U(y)$. Then $U(y) \cap F_\alpha \neq \phi$. Since $U(y)$ meets at most a finite number of F_α 's, B is finite. Thus $\{G_\alpha\}$ is locally finite. Hence, by Proposition 9-2 we get the desired open collection $\{H_\alpha\}$. $\square$

9-4. PROPOSITION. The following are equivalent for a normal space X :

1. $\dim X \leq n$
2. Every finite open covering of X can be refined by a (locally) finite closed covering of order $\leq n + 1$
3. Every finite open covering $\{G_1, \dots, G_k\}$ can be refined by a closed covering $\{H_1, \dots, H_k\}$ of order $\leq n + 1$ such that $H_i \subset G_i$ for each i

PROOF. Since a finite open covering is shrinkable by Theorem 1-7, it is obvious that (1) implies (2). That (2) implies (3) is almost evident. By Proposition 9-2, it is also evident that (3) implies (1). $\square$

9-5. THEOREM (Vendennisoff [1]). For a normal space X , $\dim X = \dim \beta X$.

PROOF. Let $\dim X \leq n$. Let $\mathcal{G}$ be an arbitrary finite open covering of βX . Then there exists a finite open covering $\mathcal{H}$ of βX with $\bar{\mathcal{H}} < \mathcal{G}$. Take a finite closed covering $\{F_1, \dots, F_k\}$ and a finite open covering $\{G_1, \dots, G_k\}$ of X such that

$$F_i \subset G_i \text{ for each } i, \quad \{G_i\} \text{ refines } \mathcal{H} \mid X$$

$$\text{ord}\{G_i\} \leq n + 1$$

Let $f_i: X \rightarrow [0, 1]$ be a mapping such that

$$f_i(x) = 0, \quad x \in F_i$$

$$f_i(x) = 1, \quad x \in X - G_i$$

Let $g_i: \beta X \rightarrow [0, 1]$ be an extension of f_i . $\{\bar{F}_i\}$ covers βX . To see $\text{ord}\{\bar{F}_i\} \leq n + 1$, suppose $\bar{F}_{i_1} \cap \dots \cap \bar{F}_{i_s} \neq \phi$. Set

$$g = \max\{g_{i_j}: j = 1, \dots, s\}$$

$$V = \{z: g(z) < 1\}$$

Then

$$\bar{F}_{i_1} \cap \cdots \cap \bar{F}_{i_s} \subset V$$

Since V is open, $V \cap X \neq \emptyset$. Since $g_{i_j}(x) = 1$, if $x \in X - G_{i_j}$,

$$V \cap X \subset G_{i_1} \cap \cdots \cap G_{i_s}$$

Thus we have

$$\text{ord}\{\bar{F}_i\} \leq \text{ord}\{G_i\} \leq n + 1$$

Moreover, $\{\bar{F}_i\} < \bar{\mathcal{H}} < \mathcal{G}$. Hence, $\dim \beta X \leq n$. Thus we have $\dim X \geq \dim \beta X$, whether $\dim X$ is finite or infinite.

Conversely, let $\dim \beta X \leq n$. Let $\mathcal{U}$ be an arbitrary finite open covering of X . Since $\mathcal{U}$ is shrinkable, there exist finite coverings of X , $\{V_1, \dots, V_k\}$ and $\{H_1, \dots, H_k\}$, such that:

- i. $H_i \subset V_i$ for each i
- ii. Each V_i is a cozero set of X
- iii. Each H_i is a zero set of X
- iv. $\{V_i\}$ refines $\mathcal{U}$

Let $h_i: X \rightarrow [0, 1]$ be a mapping such that

$$\{x: h_i(x) = 0\} = H_i$$

$$\{x: h_i(x) < 1\} = V_i$$

Let $h_i^*: \beta X \rightarrow [0, 1]$ be an extension of h_i . Set

$$H_i^* = \{z: h_i^*(z) = 0\}$$

$$V_i^* = \{z: h_i^*(z) < 1\}$$

Then, $H_i^* \cap X = H_i$ and $V_i^* \cap X = V_i$. Since $\{\bar{H}_i\}$ covers βX and $\bar{H}_i \subset H_i^*$, $\{V_i^*\}$ covers βX . Let $\mathcal{W}$ be an open covering of βX with $\mathcal{W} < \{V_i^*\}$ and with $\text{ord } \mathcal{W} \leq n + 1$. Then we have

$$\mathcal{W} \mid X < \{V_i\} < \mathcal{U} \quad \text{and} \quad \text{ord } \mathcal{W} \mid X \leq n + 1$$

Hence $\dim X \leq n$. Thus we have $\dim X \leq \dim \beta X$ whether $\dim \beta X$ is finite or infinite. $\square$

9-6. PROPOSITION. Let X be a normal space and F a closed set of X . If $\mathcal{G} = \{G_i\}$ is a finite covering of F such that each G_i is a cozero set of F , then there exists a collection $\{G_i^*\}$ in a one-one corresponding way such that

- i. Each G_i^* is a cozero set of X
- ii. $G_i^* \cap F = G_i$ for each i
- iii. $\{G_i^*\}$ is similar to $\{G_i\}$

PROOF. Let N be the geometric nerve of $\mathcal{G}$ and f a canonical mapping of F to N (compare Theorem 1-5). Since N is an NES (normal) by Theorem 3-10, f has an extension g to an open neighborhood U of F . Let v_i be the vertex of N corresponding to G_i . Set $H_i = g^{-1}(\text{St}(v_i))$. Then H_i is cozero in U and $H_i \cap F = G_i$. Take a cozero set V in X with $F \subset V \subset U$. Then $\{G_i^* = H_i \cap V\}$ is the desired collection. $\square$

9-7. DEFINITION. Let d be any one of dimension functions. Let X be a normal space and F a closed set of X . Then we say

$$d(X \bmod F) \leq n$$

if for any closed set H with $H \cap F = \emptyset$, $dH \leq n$.

9-8. THEOREM. Let X be a normal space, F a closed set of X with $\dim(X \bmod F) \leq n$ and f a mapping of F to a finite n -full polyhedron Y , then f has an extension over X .

PROOF. Let $\{v_i\}$ be the vertex set of Y . Let g be an extension of f to some open neighborhood U of F by Theorem 3-10. For a point x of F let $f_i(x)$ be the weight of $f(x)$ on v_i . Then, $f_i: F \rightarrow [0, 1]$ is a mapping. For a point x of U let $g_i(x)$ be the weight of $g(x)$ on v_i . Then $g_i: U \rightarrow [0, 1]$ is an extension of f_i to U . Let D, D' be cozero sets in X and H, H' zero sets in X such that

$$F \subset D \subset H \subset D' \subset H' \subset U$$

Set

$$G_i = f^{-1}(\text{St}(v_i)), \quad E_i = g^{-1}(\text{St}(v_i))$$

Then, $E_i \cap F = G_i$ and $\cup E_i = U$. Since $\dim(X - D) \leq n$, the open covering

$$\{(E_i - D) \cup (X - H')\}$$

of $X - D$ can be refined by a covering $\{L_i\}$ of $X - D$ in a one-one corresponding way such that

- i. Each L_i is cozero in $X - D$
- ii. $L_i \subset (E_i - D) \cup (X - H')$ for each i
- iii. $\text{ord}\{L_i\} \leq n + 1$

Set

$$M_i = (E_i \cap D') \cup (L_i - H)$$

Then M_i is cozero in X and $\{M_i\}$ is a covering of X of order $\leq n+1$. Let $h: X \rightarrow [0, 1]$ be a mapping such that

$$\begin{aligned} h(x) &= 1, & x \in F \\ \{x: h(x) = 0\} &= X - D' \end{aligned}$$

Let $h_i: X \rightarrow [0, 1]$ be a function such that

$$\begin{aligned} h_i(x) &= 0 & \text{if } x \in X - D' \\ h_i(x) &= f_i(x) & \text{if } x \in F \\ h_i(x) &= g_i(x) \cdot h(x) & \text{otherwise} \end{aligned}$$

Then h_i is continuous and $h_i(x) > 0$ if and only if $x \in E_i \cap D'$. Since $L_i - H$ is cozero in X , there is a mapping $h'_i: X \rightarrow [0, 1]$ such that $h'_i(x) > 0$ if and only if $x \in L_i - H$. Set

$$k_i = h_i + h'_i$$

Then $k_i: X \rightarrow [0, 2]$ is a mapping such that $k_i(x) > 0$ if and only if $x \in M_i$. Moreover, k_i is an extension of f_i . Let $k: X \rightarrow Y$ be the canonical mapping of $\{M_i\}$ with respect to the k_i 's. Then k is an extension of f over X . $\square$

9-9. THEOREM.[†] In a normal space X the following three conditions are equivalent:

1. $\dim X \leq n$
2. The n -sphere S^n is an extensor for X
3. For any $n+1$ disjoint pairs of closed sets: $C_1, C_1'; \dots; C_{n+1}, C'_{n+1}$, there exist closed sets $B_1, \dots, B_{n+1}$ such that (i) each B_i separates C_i and C'_i , and (ii) $\bigcap B_i = \emptyset$

PROOF. That (1) implies (2): Since each closed set of X has dimension at most n , $\dim(X \text{ mod } F) \leq n$ for any closed set F . Since S^n is nothing but an n -full polyhedron spanned by $n+2$ vertices any mapping $f: F \rightarrow S^n$ has an extension over X by the preceding theorem.

That (2) implies (3): Let $f_i: X \rightarrow [0, 1] = I$ be a mapping which assumes 0 on C_i and 1 on C'_i . Let $F = \bigcup (C_i \cup C'_i)$. Define a function f on F to the $(n+1)$ -cube, I^{n+1} , as follows:

$$f(x) = \langle f_1(x), \dots, f_{n+1}(x) \rangle$$

[†] This was proved by Morita [2]. The equivalence of (1) and (3) was first proved for separable metric spaces by Eilenberg and Otto [1]. Condition (3) is called the Eilenberg–Otto characterization of dimension.

Then, $f(F)$ is on the surface of I^{n+1} which is essentially the n -sphere S^n . So f has an extension g over X whose values are still on the surface. Let B_i be the aggregate of points x such that the i -th coordinate of $g(x)$ is $\frac{1}{2}$. Then the B_i 's are the desired closed sets.

That (3) implies (1): Let $\mathcal{G} = \{G_1, \dots, G_m\}$ be an arbitrary finite open covering of X . Let $\{H_1, \dots, H_m\}$ be an open covering of X such that $\bar{H}_i \subset G_i$ for each i . Let

$$\{\lambda_1, \dots, \lambda_k\}, \quad \text{where } k = \binom{m}{n+1}$$

be the set of all subsets of the index set $\{1, \dots, m\}$ containing exactly $n+1$ elements. For each i in λ_1 there is an open set U_i' such that

$$\bar{H}_i \subset U_i' \subset \bar{U}_i' \subset G_i^\dagger$$

$$\cap \{B(U_i') : i \in \lambda_1\} = \phi$$

We now have a closed collection $\{B(U_i') : i \in \lambda_1\}$ and an open collection $\{G_i - \bar{H}_i : i \in \lambda_1\}$ such that $B(U_i') \subset G_i - \bar{H}_i$ for each i in λ_1 . Since λ_1 has exactly $n+1$ elements, there are, by Proposition 9-2, open sets U_{1i} , V_{1i} , $i \in \lambda_1$, such that

$$\bar{H}_i \subset U_{1i} \subset \bar{U}_{1i} \subset U_i' \subset \bar{U}_i' \subset V_{1i} \subset \bar{V}_{1i} \subset G_i, \quad i \in \lambda_1$$

$$\cap \{\bar{V}_{1i} - U_{1i} : i \in \lambda_1\} = \phi$$

The next step is to obtain open sets U_{2i} , V_{2i} , $i \in \lambda_2$, such that

$$\bar{H}_i \subset U_{2i} \subset \bar{U}_{2i} \subset V_{2i} \subset \bar{V}_{2i} \subset G_i, \quad i \in \lambda_2 - \lambda_1$$

$$\bar{U}_{1i} \subset U_{2i} \subset \bar{U}_{2i} \subset V_{2i} \subset \bar{V}_{2i} \subset V_{1i}, \quad i \in \lambda_1 \cap \lambda_2$$

$$\cap \{\bar{V}_{2i} - U_{2i} : i \in \lambda_2\} = \phi$$

The existence of these open sets is assured by the same argument as in the above. Continuing these processes step by step from λ_j to λ_{j+1} , we finally get open sets

$$U_{ji}, V_{ji}; \quad i \in \lambda_j, \quad j = 1, \dots, k$$

† It can easily be seen that the existence of a closed set separating $\bar{H}_i$ and $X - G_i$ is equivalent to the existence of an open set U_i' with this condition. Since U_i' is open, $B(U_i') = \bar{U}_i' - U_i'$.

which satisfy the following conditions:

$$\begin{aligned} \bar{H}_i &\subset U_{ji} \subset \bar{U}_{ji} \subset V_{ji} \subset \bar{V}_{ji} \subset G_i \quad \text{for } i \in \lambda_j \text{ and } j = 1, \dots, k \\ \bar{U}_{ji} &\subset U_{ri} \text{ and } \bar{V}_{ri} \subset V_{ji} \text{ if } j < r \text{ and } i \in \lambda_j \cap \lambda_r \\ \cap \{ \bar{V}_{ji} - U_{ji} : i \in \lambda_j \} &= \phi \quad \text{for } j = 1, \dots, k \end{aligned}$$

Set

$$U_i = \cup \{ U_{ji} : i \in \lambda_j \}, \quad V_i = \cap \{ V_{ji} : i \in \lambda_j \}$$

Then

$$\bar{H}_i \subset U_i \subset \bar{U}_i \subset V_i \subset \bar{V}_i \subset G_i \quad \text{for } i = 1, \dots, m$$

Moreover,

$$\text{ord} \{ \bar{V}_i - U_i : i = 1, \dots, m \} \leq n$$

Let W_i be open sets with

$$\bar{U}_i \subset W_i \subset \bar{W}_i \subset V_i, \quad i = 1, \dots, m$$

Let

$$\mathcal{W} = \wedge \{ W_i, X - \bar{W}_i \}$$

Then $\mathcal{W}$ is a disjoint open collection which refines $\mathcal{G}$. Now it can easily be seen that

$$\mathcal{U} = \mathcal{W} \cup \{ V_i - \bar{U}_i : i = 1, \dots, m \}$$

is an open covering of X such that $\text{ord } \mathcal{U} \leq n + 1$ and $\mathcal{U} < \mathcal{G}$. $\square$

9-10. THEOREM (sum theorem).[†] If a normal space X is a countable sum of closed sets F_i with $\dim F_i \leq n$ for each i , then $\dim X \leq n$.

PROOF. Let F be an arbitrary closed set of X and f a mapping of F to S^n . Since S^n is considered as an n -full polyhedron spanned by $n + 2$ vertices, f has an extension f_1 over $F \cup F_1$, by Theorem 9-8. Since S^n is an NES (normal) by Theorem 3-10, f_1 has an extension g_1 over some closed neighborhood U_1 of $F \cup F_1$. Repeating this process we get a sequence of closed neighborhoods $U_1, U_2, \dots$ of $F \cup F_1, F \cup F_1 \cup F_2, \dots$, respectively, with $U_i \subset U_{i+1}$, and a sequence of mappings $g_i: U_i \rightarrow S^n$ such that

[†] This was proved first by Čech [1]. The sum theorem for cohomological dimension of paracompact Hausdorff spaces was also shown to be true by Okuyama [1] and Kodama [6].

g_{i+1} is an extension of g_i for each i . Define $g: X \rightarrow S^n$ in such a way that

$$g(x) = g_i(x) \quad \text{when } x \in F_i$$

Then g is uniquely defined and continuous. Clearly g is an extension of f over X . Then, by the preceding theorem $\dim X \leq n$. $\square$

This proof essentially contains a proof of the following theorem.

9-11. THEOREM. Let X be a normal space and F a closed set of X . If $\dim F \leq n$ and $\dim(X \text{ mod } F) \leq n$, then $\dim X \leq n$.

9-12. EXAMPLE (Knaster and Kuratowski [1]). There exists a totally disconnected separable metric space X with $\dim X = 1$. Let C be the Cantor discontinuum on the unit interval $[0, 1]$ lying on the x axis. Let a be the point in the plane whose coordinates are $(1, 1)$. Let P be the set of all end points in C and $Q = C - P$. P is countable. For any point $c \in C$, let $L(c)$ be the closed segment joining c and a . When p is in P , let $L^*(p)$ be the set of all points in $L(p)$ whose heights are rational. Let $L^*(q)$, $q \in Q$, be the set of all points in $L(q)$ whose heights are irrational. Set

$$X^* = (\cup \{L^*(p): p \in P\}) \cup (\cup \{L^*(q): q \in Q\})$$

$$X = X^* - \{a\}$$

PROOF. To prove first that X^* is connected assume the contrary. Then there would be a separation D, E , where $a \in D$. For each nonnegative number r and each $q \in Q$ let $L_r^*(q)$ be the set of all points in $L^*(q)$ whose heights are not less than r . Let

$$h(q) = \inf\{r: L_r^*(q) \subset D\}$$

Then $h(q)$ is rational. Let $\{r_1, r_2, \dots\}$ be the set of all rationals in $[0, 1]$. Set

$$Q_i = \{q \in Q: h(q) = r_i\}$$

Then $C = P \cup (\cup Q_i)$. Since C is complete metric and P is countable, there exists an i , say i_0 , such that $\bar{Q}_{i_0}$ contains an interval T (in C), by Baire's theorem. Take a point p from $T \cap P$. Let p_0 be the point in $L^*(p)$ of height r_{i_0} . Then, p_0 is in $\bar{D} \cap \bar{E}$, which is a contradiction. Hence, X^* is connected.

X is clearly disconnected. To prove $\dim X > 0$, assume $\dim X = 0$. Let U_i , $i = 1, 2, \dots$, be a sequence of open neighborhoods of a whose intersection is a . Then $\dim(X - U_i) \leq 0$. Since

$$X^* = (\cup (X - U_i)) \cup \{a\}$$

and $\dim\{a\} = 0$, we have $\dim X^* = 0$ by the sum theorem Section 9-10, which contradicts the connectedness of X^* . Hence, $\dim X > 0$.

Actually we know $\dim X = 1$: As can easily be seen, every point in X has an open neighborhood whose boundary consists of countable points. This implies $\dim X \leq 1$, by Theorem 8-2 since X is separable metric. $\square$

9-13. THEOREM (Morita [2] and Dowker [1]). If X is a normal space with $\dim X \leq n$, then every normal open covering can be refined by an open covering of order $\leq n + 1$.

PROOF. Let $\mathcal{G}$ be an arbitrary normal open covering of X . Then by Theorem 2-7 and Theorem 2-27, $\mathcal{G}$ can be refined by an open covering $\cup \mathcal{G}_i$ such that:

- i. Every $\mathcal{G}_i$ is discrete
- ii. $G_i = \cup \{G : G \in \mathcal{G}_i\}$ is cozero in X for each i
- iii. $\{G_i\}$ is star-finite

Let $1 = n_1 < n_2 < \dots$ be a sequence of integers such that

$$G_i \cap G_j = \emptyset \quad \text{if } i \leq n_k \quad \text{and} \quad j \geq n_{k+2}, \quad k = 1, 2, \dots$$

Such a sequence does exist by the star-finiteness of $\{G_i\}$. Since $\cup \{G_i : i \leq n_2\}$ is F_σ in X , it is a normal subspace by Theorem 3-12. Moreover, its dimension is at most n by the sum theorem. Hence, it has a cozero covering

$$\mathcal{U}_1 = \{W_1, \dots, W_{n_1}, V_{n_1+1}, \dots, V_{n_2}\}$$

such that

$$\text{ord } \mathcal{U}_1 \leq n + 1$$

$$W_i \subset G_i \quad \text{for } i \leq n_1$$

$$V_i \subset G_i \quad \text{for } i \text{ with } n_1 < i \leq n_2$$

Similarly, there exists a cozero covering

$$\mathcal{U}_2 = \{W_{n_1+1}, \dots, W_{n_2}, V_{n_2+1}, \dots, V_{n_3}\}$$

of $V_{n_1+1} \cup \dots \cup V_{n_2} \cup G_{n_2+1} \cup \dots \cup G_{n_3}$ such that

$$\text{ord } \mathcal{U}_2 \leq n + 1$$

$$W_i \subset V_i \quad \text{for } i \text{ with } n_1 < i \leq n_2$$

$$V_i \subset G_i \quad \text{for } i \text{ with } n_2 < i \leq n_3$$

Continuing these procedures we get an open covering $\mathcal{W} = \{W_1, W_2, \dots\}$ of X such that

$$\text{ord } \mathcal{W} \leq n + 1, \quad W_i \subset G_i \quad \text{for every } i$$

Now $\cup(\mathcal{G} \mid W_i)$ is an open covering of order $\leq n + 1$ which refines $\mathcal{G}$. $\square$

9-14. COROLLARY. A fully normal space X has $\dim X \leq n$ if and only if every open covering is refined by an open covering of order $\leq n + 1$.

10. Large Inductive Dimension of Normal Spaces

10-1. THEOREM. If X is a normal space, $\dim X \leq \text{Ind } X$.

PROOF (by induction). When $\text{Ind } X = \infty$, the inequality is trivially true. When $\text{Ind } X = -1$, i.e. $X = \phi$, $\dim X = -1$ by definition. Take as the induction assumption that the theorem is true when Ind of spaces is at most $n - 1$. Suppose that $\text{Ind } X \leq n$, $n \geq 0$. Let $\mathcal{G} = \{G_i\}$ be an arbitrary finite open covering of X . Let $\{F_i\}$ be a closed covering of X in one-one correspondence with $\{G_i\}$ such that $F_i \subset G_i$ for each i . For each i , take an open set U_i with $F_i \subset U_i \subset \bar{U}_i \subset G_i$ such that $\text{Ind } B(U_i) \leq n - 1$. By the induction assumption, $\dim B(U_i) \leq n - 1$ for each i . Hence, $\dim \cup B(U_i) \leq n - 1$ by the sum theorem. $\mathcal{G} \mid \cup B(U_i)$ can be refined by an open covering $\mathcal{H}$ of $\cup B(U_i)$ of order $\leq n$. By the same process as in the proof of Proposition 9-6 we get an open collection $\mathcal{H}_1$ of X such that (i) $\mathcal{H}_1 \mid \cup B(U_i) < \mathcal{H}$, (ii) $\text{ord } \mathcal{H}_1 \leq n$, and (iii) $\mathcal{H}_1 < \mathcal{G}$. Set

$$\mathcal{H}_2 = \wedge \{U_i, X - \bar{U}_i\}$$

Then $\mathcal{H}_2$ is a disjoint open collection of X which refines $\mathcal{G}$. Now $\mathcal{H}_1 \cup \mathcal{H}_2$ is an open covering of X of order $\leq n + 1$ which refines $\mathcal{G}$. $\square$

10-2. PROPOSITION. If X is a normal space and F is a closed set of X , then $\text{Ind } F \leq \text{Ind } X$.

The proof by induction is easy.

10-3. THEOREM (Isbell [1]). If X is a normal space, then $\text{Ind } X = \text{Ind } \beta X$.

PROOF (by induction). $\text{Ind } X \leq \text{Ind } \beta X$: The proof is carried by induction on $n = \text{Ind } \beta X$. If $n = -1$, then $\text{Ind } X = \text{Ind } \phi = -1$. Suppose the inequality is true for n . Let $\text{Ind } \beta X = n + 1$, and let C_1, C_2 be a disjoint pair of closed sets of X . Then $\bar{C}_1$ and $\bar{C}_2$ are disjoint in $\beta X^\dagger$ and separated by a closed set B in βX such that $\text{Ind } B \leq n$. Then $B \cap X$ separates C_1 and C_2 , and $\ddagger$

$$\beta(B \cap X) = \overline{B \cap X} \subset B$$

By the preceding proposition, $\text{Ind } \beta(B \cap X) \leq n$. By the induction assumption, $\text{Ind } (B \cap X) \leq n$. Thus, $\text{Ind } X \leq n + 1$. When $\text{Ind } \beta X = \infty$, the inequality is also true.

$\text{Ind } \beta X \leq \text{Ind } X$: The proof will be carried by induction on $n = \text{Ind } X$. Again if $n = -1$, the inequality is trivially true. Assume that the inequality is true for n and let $\text{Ind } X = n + 1$. Let C_1, C_2 be a disjoint pair of closed sets in βX . There exist open sets O_1 and O_2 of βX such that

$$C_1 \subset O_1, \quad C_2 \subset O_2, \quad \bar{O}_1 \cap \bar{O}_2 = \phi$$

Let $C_i' = \bar{O}_i \cap X$. Then, C_1' and C_2' can be separated by a closed set B of X such that $\text{Ind } B \leq n$. Let U_1, U_2 be open sets of X such that

$$C_1' \subset U_1, \quad C_2' \subset U_2, \quad U_1 \cap U_2 = \phi$$

$$U_1 \cup B \cup U_2 = X, \quad C_1' \cap B = \phi, \quad C_2' \cap B = \phi$$

Claim: $\bar{B}$ separates U_1 and U_2 in βX .

To establish the claim, it suffices to show that $\bar{U}_1 - \bar{B}$ and $\bar{U}_2 - \bar{B}$ are disjoint. Suppose that there exists a point p in $(\bar{U}_1 \cap \bar{U}_2) - \bar{B}$. Let $f: \beta X \rightarrow [0, 1]$ be a mapping such that $f(p) = 0$ and $f(x) = 1$ when $x \in \bar{B}$. Define $f': X \rightarrow [0, 1]$ such that

$$f'(x) = f(x), \quad x \in U_1 \cup B$$

$$f'(x) = \max\{f(x), 1/2\}, \quad x \in U_2 \cup B$$

$\dagger$ In this argument the normality of X is used: Let $h: X \rightarrow [0, 1]$ be a mapping with (i) $h(x) = 0, x \in C_1$, (ii) $h(x) = 1, x \in C_2$ and h_1 the extension of h to βX . Then $\bar{C}_1 \subset \{x: h_1(x) = 0\}$ and $\bar{C}_2 \subset \{x: h_1(x) = 1\}$ and the right sides of both inequalities are of course closed in βX .

$\ddagger$ This equality comes from the normality of X . Notice Tietze's extension theorem.

Then f' is continuous and has an extension g to βX . Since $g(x) \geq \frac{1}{2}$ for all $x \in \overline{U_2 \cup B}$, then $g(p) \geq \frac{1}{2}$. But f and g agree on the dense subset U_1 of $\overline{U_1}$. Thus $g(p) = f(p) = 0$, which gives a contradiction, establishing the fact that $\bar{B}$ separates U_1 and U_2 in βX .

Finally observe that since C_1' and $U_2 \cup B$ are disjoint closed sets of X , then $\bar{C}_1'$ and $\overline{U_2 \cup B} = \overline{U_2} \cup \bar{B}$ are disjoint in βX . Thus, $\bar{C}_1' \subset \overline{U_1} - \bar{B}$. Likewise, $\bar{C}_2' \subset \overline{U_2} - \bar{B}$. So $\bar{B}$ separates $\bar{C}_1'$ and $\bar{C}_2'$ in βX . Thus

$$C_i \subset \bar{O}_i = \overline{O_i \cap X} \subset (\overline{\bar{O}_i \cap X}) = \bar{C}_i'$$

and therefore $\bar{B}$ separates C_1 and C_2 in βX . By the induction assumption, $\text{Ind } \bar{B} = \text{Ind } \beta B \leq n$ which implies that $\text{Ind } \beta X \leq n + 1$. When $\text{Ind } X = \infty$, the inequality is trivially true. $\square$

We already know that if X is a normal space, then $\dim X = \dim \beta X$ by Theorem 9-5 and $\text{Ind } X = \text{Ind } \beta X$ by the present theorem. So it is natural to ask about the validity of the equality $\text{ind } X = \text{ind } \beta X$. But the answer is negative as is shown by Roy's example R (Roy [1]). R is a complete metric space with $\text{ind } R = 0$ and $\dim R = \text{Ind } R = 1$. By Proposition 8-4, $\text{ind } \beta R = 0$ implies $\dim \beta R = \text{Ind } \beta R = 0$, which is impossible. Actually, $\text{ind } \beta R = 1$, since $0 < \text{ind } \beta R \leq \text{Ind } \beta R = 1$.

It can easily be seen that ind is monotone: If X is a subset of a (completely) regular space Y , then $\text{ind } X \leq \text{ind } Y$.[†] As a special case we have $\text{ind } X \leq \text{ind } \beta X$ for any completely regular space X .

10-4. THEOREM (Dowker [3]). Let X be a hereditarily normal space[‡] and $F_i, i = 1, 2, \dots$, a disjoint sequence of sets covering X such that:

- i. $\text{Ind } F_i \leq n$ for each i
- ii. $\cup_{j \leq i} F_j$ is closed for each i

then $\text{Ind } X \leq n$.

[†] Let d be any one of dimension functions. d satisfies the *subset theorem* for a space Y if $X \subset Y$ implies $dX \leq dY$. d satisfies the *open subset theorem* for a space Y if X being open in Y implies $dX \leq dY$.

[‡] A hereditarily normal space is sometimes called a *completely normal* space. A normal space is hereditarily normal if and only if the following condition is satisfied: For any sets F, H in X with $\bar{F} \cap H = F \cap \bar{H} = \emptyset$ there exist open sets D, E with $F \subset D$, $H \subset E$, and $D \cap E = \emptyset$. For the details, see Alexandroff and Hopf [1].

PROOF (by induction). The theorem is true for dimension -1 . Assume it is true for dimension $n-1$, $n \geq 0$. Let S and T be a disjoint pair of closed sets of X . Take a disjoint covering $\{G_1, H_1, C_1\}$ of F_1 such that

$$\begin{aligned} G_1 \text{ and } H_1 \text{ are open in } F_1, \quad C_1 \text{ is closed in } F_1 \\ S \cap F_1 \subset G_1, \quad T \cap F_1 \subset H_1, \quad \text{Ind } C_1 \leq n-1 \end{aligned}$$

Since $S \cup G_1$ and $T \cup H_1$ are disjoint and closed in $X - C_1$, there exist sets U_1 and V_1 such that

$$\begin{aligned} U_1 \text{ and } V_1 \text{ are open in } X - C_1 \\ S \cup G_1 \subset U_1, \quad T \cup H_1 \subset V_1 \\ (\bar{U}_1 \cap \bar{V}_1) - C_1 = \phi \end{aligned}$$

Since $\bar{U}_1 \cap F_2$ and $\bar{V}_1 \cap F_2$ are disjoint and closed in F_2 , there exists a disjoint covering $\{G_2, H_2, C_2\}$ of F_2 such that

$$\begin{aligned} G_2 \text{ and } H_2 \text{ are open in } F_2, \quad C_2 \text{ is closed in } F_2 \\ \bar{U}_1 \cap F_2 \subset G_2, \quad \bar{V}_1 \cap F_2 \subset H_2, \quad \text{Ind } C_2 \leq n-1 \end{aligned}$$

It is to be noticed that $C_1 \cup C_2$ is closed in X . Continuing recursively in this fashion we get three sequences of sets $U_1, U_2, \dots, V_1, V_2, \dots, C_1, C_2, \dots$ which satisfy the following conditions for each i :

- i. U_i, V_i , and C_i are disjoint
- ii. U_i and V_i are open in $X - \cup_{j \leq i} C_j$
- iii. C_i is a closed set of F_i with $\text{Ind } C_i \leq n-1$
- iv. $F_i \subset U_i \cup V_i \cup C_i$
- v. $\bar{U}_i \cap \bar{V}_i \subset \cup_{j \leq i} C_j$
- vi. $\bar{U}_i - \cup_{j \leq i} C_j \subset U_{i+1}$ and $\bar{V}_i - \cup_{j \leq i} C_j \subset V_{i+1}$
- vii. $S \subset U_1, T \subset V_1$

If we set $C = \cup C_i$, then C is a closed set of X separating S and T . Since every finite sum of C_i 's is closed in X , $\text{Ind } C \leq n-1$ by the induction assumption. $\square$

11. Subset Theorem and Local Dimension

11-1. TERMINOLOGY. Let X be an hereditarily normal space. We consider the following conditions which X may satisfy:

- (α_n) : If $Z \subset Y \subset X$ and $\text{Ind } Y \leq n$, then $\text{Ind } Z \leq n$
- (β_n) : If $G \subset Y \subset X$ with G open in Y and $\text{Ind } Y \leq n$, then $\text{Ind } G \leq n$
- (γ_n) : If $Y = \cup_{i=1}^{\infty} F_i \subset X$ with each F_i closed in Y and $\text{Ind } F_i \leq n$, then $\text{Ind } Y \leq n$

11-2. LEMMA (Dowker [3]). Let X be a hereditarily normal space. If X satisfies (β_n) , then X satisfies (γ_n) . If X satisfies (β_n) for all n , then X satisfies (α_n) for all n .

PROOF. That (β_n) implies (γ_n) is a direct consequence of Theorem 10-4.

To show that (α_{n-1}) and (β_n) imply (α_n) : Let $Z \subset Y \subset X$ and $\text{Ind } Y \leq n$. Let F be a closed set of Z and G an open set of Z with $F \subset G$. Take a closed set F_1 of Y and an open set G_1 of Y with $F_1 \cap Z = F$ and $G_1 \cap Z = G$. Set $H = (Y - F_1) \cup G_1$. Then H is an open set of Y . Take an open set U_1 of H with

$$F_1 \cap H \subset U_1 \subset \bar{U}_1 \cap H \subset G_1$$

$$\text{Ind}(\bar{U}_1 \cap H - U_1) \leq n - 1$$

Set $U = U_1 \cap Z$. Then $\bar{U} \cap Z - U$ separates F and G in Z and $\text{Ind}(\bar{U} \cap Z - U) \leq n - 1$ by (α_{n-1}) . Since (α_{n-1}) is trivially true, (α_n) is true for all n . $\square$

11-3. LEMMA (Dowker [3]). Let X be a normal space. Let F_i, C_i , $i = 1, 2, \dots$, be closed sets of X such that (i) $F_i \subset \text{Int } C_i^\dagger$ for each i , (ii) $X = \cup F_i$, (iii) $\text{Ind } C_i \leq n$. If X satisfies (γ_{n-1}) , then $\text{Ind } X \leq n$.

PROOF. Let F be a closed set and G an open set with $F \subset G$. Let K be a zero set with $F \subset K \subset G$. Take open sets W_i , $i = 1, 2, \dots$, such that

$$K \subset \bar{W}_{i+1} \subset W_i \subset G \quad \text{for each } i$$

$$K = \cap W_i$$

Consider the pair $K \cap F_i \subset \text{Int } C_i \cap W_i$. Take an open set U_i such that

$$K \cap F_i \subset U_i \subset \bar{U}_i \subset \text{Int } C_i \cap W_i$$

$$\text{Ind}(\bar{U}_i - U_i) \leq n - 1$$

Set $U = \cup_{i=1}^{\infty} U_i$. Since $\{U_i\}$ is locally finite in $X - K$, $B(U) \subset \cup B(U_i)$. By (γ_{n-1}) , $\text{Ind } B(U) \leq n - 1$. $F \subset U \subset \bar{U} \subset G$ is evident. $\square$

11-4. LEMMA (Dowker [3]). If X is a totally normal space, then (γ_{n-1}) implies (β_n) .

$\dagger \text{Int } C_i$ denotes the interior of C_i .

PROOF. Let Y be a subset of X with $\text{Ind } Y \leq n$ and G an open set in Y . Let $\{G_\alpha\}$ be a locally finite (in G) covering of G such that each G_α is cozero in Y . Since $\{G_\alpha\}$ is normal by Theorem 2-27, there exists a covering

$$\bigcup_{i=1}^{\infty} \mathcal{H}_i, \quad \text{where } \mathcal{H}_i = \{H_\alpha: \alpha \in A_i\}$$

of G such that:

- i. $\bigcup \mathcal{H}_i < \{G_\alpha\}$
- ii. Every element of $\bigcup \mathcal{H}_i$ is cozero in G
- iii. Each $\mathcal{H}_i$ is discrete in G

Since every element H of $\bigcup \mathcal{H}_i$ is contained in some G_α which is cozero in Y , H is cozero in Y . So we can express each element of $\bigcup \mathcal{H}_i$ as follows

$$H_\alpha = \bigcup_{i=1}^{\infty} F_{\alpha i}$$

$$F_{\alpha i} \text{ is closed in } Y, \quad F_{\alpha i} \subset \text{Int } F_{\alpha, i+1}$$

Set

$$F_{ij} = \bigcup \{F_{\alpha j}: \alpha \in A_i\}$$

$$C_{ij} = \bigcup \{F_{\alpha, j+1}: \alpha \in A_i\}$$

Then $F_{ij} \subset \text{Int } C_{ij}$ and F_{ij}, C_{ij} are closed in G . Moreover, $\text{Ind } C_{ij} \leq n$ since C_{ij} is the sum of the discrete collection $\{F_{\alpha, j+1}: \alpha \in A_i\}$ whose elements have $\text{Ind} \leq n$. Since $\{F_{ij}: i, j = 1, 2, \dots\}$ covers G and G is normal by Proposition 7-4, $\text{Ind } G \leq n$. Thus (β_n) is true. $\square$

11-5. THEOREM (Dowker [3]). A totally normal space X satisfies the subset theorem and the sum theorem for the large inductive dimension.

PROOF. (γ_{-1}) is evidently true for X . By repeated applications of Proposition 11-2 and Lemma 11-4 we have a sequence of implications:

$$(\gamma_{-1}) \rightarrow (\beta_1) \rightarrow (\gamma_1) \rightarrow (\beta_2) \rightarrow \dots$$

Thus we have (α_n) for all n , again by Lemma 11-2. $\square$

11-6. DEFINITION. Let X be a normal space. Then we say the *local dimension* of X is at most n , $\text{loc dim } X \leq n$, if each point x of X has a closed neighborhood U with $\text{dim } U \leq n$. $\text{loc dim } X$ is the least of such n . If there is no such n , $\text{loc dim } X = \infty$. $\text{loc Ind } X$ is defined similarly.

If X has a normal cozero covering $\mathcal{G}$ such that $\dim G \leq n$ for each element G of $\mathcal{G}$, then the *uniform local dimension* of X is at most n , and we write $\text{uloc dim } X \leq n$. $\text{uloc dim } X$ is the least of such n . $\text{uloc Ind } X$ is defined similarly.

The following implications are evident: $\dim X \leq n \rightarrow \text{uloc dim } X \leq n \rightarrow \text{loc dim } X \leq n$, and $\text{Ind } X \leq n \rightarrow \text{uloc Ind } X \leq n \rightarrow \text{loc Ind } X \leq n$.

11-7. THEOREM. If X is a normal space, $\text{uloc dim } X \leq n$ implies $\dim X \leq n$. If X is a totally normal space, $\text{uloc Ind } X \leq n$ implies $\text{Ind } X \leq n$.

PROOF. Let $\text{uloc dim } X \leq n$. Then there is a normal cozero covering $\mathcal{G}$ such that each element G of $\mathcal{G}$ has $\dim G \leq n$. Let $\cup_{i=1}^{\infty} \mathcal{H}_i$ be an open covering of X such that (i) $\cup \overline{\mathcal{H}_i} < \mathcal{G}$ and (ii) each $\mathcal{H}_i$ is discrete. Let H be an arbitrary element of $\cup \mathcal{H}_i$. Then $\overline{H}$ is contained in some $G \in \mathcal{G}$. Since G is normal by Theorem 3-12, $\dim \overline{H} \leq \dim G \leq n$. If we set

$$H_i = \{\overline{H} : H \in \mathcal{H}_i\}$$

then H_i is a closed set of X with $\dim H_i \leq n$. By the sum theorem $\dim X \leq \sup \dim H_i \leq n$.

The second statement is proved similarly. $\square$

11-8. COROLLARY. If X is a paracompact Hausdorff space, then $\text{loc dim } X \leq n$ implies $\dim X \leq n$. If X is a hereditarily paracompact Hausdorff space, then $\text{loc Ind } X \leq n$ implies $\text{Ind } X \leq n$.[†]

PROOF. Consider the first case. If $\text{loc dim } X \leq n$, every point x in X has a closed neighborhood $U(x)$ with $\dim U(x) \leq n$. Let $V(x)$ be a cozero set in X with $x \in V(x) \subset U(x)$. Then $\dim V(x) \leq n$ by the sum theorem. Since $\{V(x) : x \in X\}$ is normal, we get $\text{uloc dim } X \leq n$, which implies $\dim X \leq n$ by the theorem. The remaining case is proved similarly. $\square$

[†] We can prove the latter case under the assumption that X is paracompact and totally normal. This condition is at first sight weaker than the condition in the corollary. But as a matter of fact every paracompact, totally normal space is hereditarily paracompact. It is to be noted that a metric space is hereditarily paracompact but a hereditarily paracompact Hausdorff space need not be metrizable. The one point compactification of an uncountable discrete space is not perfectly normal, and hence not metrizable, but hereditarily paracompact Hausdorff.

11-9. COROLLARY. If X is a paracompact Hausdorff space and $\{F_\alpha\}$ is a locally countable collection of closed sets in X with $\dim F_\alpha \leq n$ for each α , then $\cup F_\alpha$ is a paracompact space with $\dim \cup F_\alpha \leq n$. If moreover X is hereditarily paracompact and $\text{Ind } F_\alpha \leq n$ for each α , then $\text{Ind } \cup F_\alpha \leq n$.

PROOF. For each point x in X let $U(x)$ be an open neighborhood of x meeting at most a countable number of elements of $\{F_\alpha\}$. $\{U(x): x \in X\}$ can be refined by a covering $\cup \mathcal{H}_i$ such that

- i. Each element of $\cup \mathcal{H}_i$ is cozero in X
- ii. Each $\mathcal{H}_i$ is discrete

Set

$$H_i = \cup \{H: H \in \mathcal{H}_i\}$$

Then $H_i \cap (\cup F_\alpha)$ is F_σ . Hence $\cup F_\alpha$ is F_σ . By Theorem 3-13, $\cup F_\alpha$ is paracompact. By the sum theorem $\text{loc dim}(\cup F_\alpha) \leq n$. Hence, by Corollary 11-8 $\dim(\cup F_\alpha) \leq n$.

The remaining claim is proved similarly. $\square$

11-10. COROLLARY. Let X be a totally normal space and F a closed set of X . If $\text{Ind } F \leq n$ and $\text{Ind}(X \text{ mod } F) \leq n$, then $\text{Ind } X \leq n$.

PROOF. Let $\{G_\alpha\}$ be a locally finite (in $X - F$) covering of $X - F$ such that each G_α is cozero in X . Since G_α is F_σ in X , $\text{Ind } G_\alpha \leq n$ by the sum theorem for Ind . Since $\{G_\alpha\}$ is normal by Theorem 2-27,

$$\text{uloc Ind}(X - F) \leq n$$

which implies $\text{Ind}(X - F) \leq n$ by Theorem 11-7 and Proposition 7-5. Hence, by Theorem 10-4 we get $\text{Ind } X \leq n$. $\square$

11-11. COROLLARY. A totally normal space X satisfies the subset theorem for covering dimension.

PROOF. If G is an open set of X , then $\text{uloc dim } G \leq \dim X$ by an argument similar to the proof above. Since $\text{uloc dim } G = \dim G$, $\dim G \leq \dim X$. Hence, X satisfies the open subset theorem which implies the subset theorem, as can easily be seen. $\square$

11-12. THEOREM.[†] Let X be a hereditarily paracompact Hausdorff space, then the following conditions are equivalent:

1. $\text{Ind } X \leq n$
2. Every open covering of X can be refined by a locally finite σ -discrete open covering $\{V\}$ such that $\text{Ind } B(V) \leq n - 1$ for each V
3. Every binary open covering of X can be refined by a σ -locally finite open covering $\{V\}$ such that $\text{Ind } B(V) \leq n - 1$ for each V

PROOF. Since it is evident that (1) implies (2) which implies (3), we prove only that (3) implies (1). Let F_1 and F_2 be an arbitrary disjoint pair of closed sets of X . Let H_1 and H_2 be a pair of open sets with $F_i \subset H_i$, $i = 1, 2$, and $\bar{H}_1 \cap \bar{H}_2 = \phi$. Let $\cup \mathcal{V}_i$ be an open covering of X such that

- i. $\cup \mathcal{V}_i$ refines $\{X - \bar{H}_1, X - \bar{H}_2\}$
- ii. Each $\mathcal{V}_i$ is locally finite
- iii. $\text{Ind } B(V) \leq n - 1$ for every element V of $\cup \mathcal{V}_i$

Set

$$V_i = \cup \{V \in \mathcal{V}_i : V \cap \bar{H}_1 \neq \phi\}$$

$$V'_i = \cup \{V \in \mathcal{V}_i : V \cap \bar{H}_1 = \phi\}$$

Then $\cup(V_i \cup V'_i)$ covers X . Set

$$U_i = V_i - \left(\bigcup_{j < i} \bar{V}_j \right) \cup \left(\bigcup_{j < i} \bar{V}'_j \right)$$

$$U'_i = V'_i - \left(\bigcup_{j \leq i} \bar{V}_j \right) \cup \left(\bigcup_{j < i} \bar{V}'_j \right)$$

$$B = \cup \{B(V) : V \in \cup \mathcal{V}_i\}$$

Then $\cup(U_i \cup U'_i)$ covers $X - B$. Moreover, the restricted collection

$$(\{U_i\} \cup \{U'_i\}) \upharpoonright (X - B)$$

is locally finite in $X - B$. Thus, $F_1 \cup (\cup U_i)$ and $F_2 \cup (\cup U'_i)$ are mutually separated. Hence, there is an open set D such that

$$F_1 \cup (\cup U_i) \subset D \subset \bar{D} \subset X - F_2 \cup (\cup U'_i)$$

Since $\{U_i, U'_i : i = 1, 2, \dots\}$ covers $X - B$,

$$B(D) = \bar{D} - D \subset B$$

Since $\text{Ind } B \leq n - 1$ by Corollary 11-8, $\text{Ind } B(D) \leq n - 1$ by Theorem 11-5. $\square$

[†] This is due to Nagami [5]. The equivalence of (1) and (3) will be proved for more general spaces in Lemma 25-4.

11-13. COROLLARY (Katsuta [1]). Let X be a hereditarily paracompact Hausdorff space with the star-finite property. Then $\text{Ind } X = \text{ind } X$.

PROOF (by induction). When $\text{ind } X = \infty$ or -1 , the assertion is trivially true. Suppose that the assertion is true for dimension $\leq n$. Let $\text{ind } X \leq n + 1$. Let $\mathcal{G}$ be an arbitrary binary open covering of X . For each point x in X let $U(x)$ be an open neighborhood of x such that $\text{ind } B(U(x)) \leq n$ and $U(x)$ refines $\mathcal{G}$. Then $\text{Ind } B(U(x)) \leq n$, by the assumption. Since X has the star-finite property, $\{U(x) : x \in X\}$ can be refined by a star-finite open covering $\mathcal{H}$. $\mathcal{H}$ can be indexed as follows

$$\mathcal{H} = \{H_{\alpha i} : \alpha \in A, \quad i = 1, 2, \dots\}$$

where $H_{\alpha i} \cap H_{\beta j} = \emptyset$ whenever $\alpha \neq \beta$.

Set $H_\alpha = \cup_i H_{\alpha i}$. Then H_α is open-closed. Each $H_{\alpha i}$ is contained in some $U(x)$, say $U(x_{\alpha i})$. Set $V_{\alpha i} = U(x_{\alpha i}) \cap H_\alpha$. Then

$$\mathcal{V}_i = \{V_{\alpha i} : \alpha \in A\}$$

is discrete. $\cup \mathcal{V}_i$ covers X and refines $\mathcal{G}$. Moreover

$$\text{Ind } B(V_{\alpha i}) \leq \text{Ind } B(U(x_{\alpha i})) \leq n$$

Thus $\text{Ind } X \leq n + 1$ by the theorem. The validity of the (trivial) inequality $\text{Ind } X \geq \text{ind } X$ was already mentioned. $\square$

11-14. PROBLEM. Is the subset theorem for dim or Ind true for every hereditarily normal space? Embed Roy's example R , Section 10-3, in a compact Hausdorff space S as stated in Section 8-5. Even if R is metric, $\text{dim } R = \text{Ind } R = 1$ and $\text{dim } S = \text{Ind } S = 0$.

11-15. THEOREM (Morita [3]). Let X be a normal space and S a subset of X with the star-finite property, then $\text{dim } S \leq \text{dim } X$.

PROOF. First, since S is a paracompact Hausdorff space, S is normal. When $\text{dim } X = \infty$, the assertion is trivially true. Let $\text{dim } X \leq n$. Let $\mathcal{G}$ be an arbitrary finite open covering of S . Let $\mathcal{G}_1$ be a finite open collection of X with $\mathcal{G}_1 \upharpoonright S < \mathcal{G}$. For each point x in S let $U(x)$ be a cozero set in X such that $x \in U(x)$ and $U(x)$ refines $\mathcal{G}_1$. $\{U(x) \cap S : x \in S\}$ is refined by a star-finite open covering $\mathcal{H}$ of S . $\mathcal{H}$ can be indexed as follows

$$\mathcal{H} = \{H_{\alpha i} : \alpha \in A, \quad i = 1, 2, \dots\}$$

$$H_{\alpha i} \cap H_{\beta j} \neq \emptyset \quad \text{implies} \quad \alpha = \beta$$

Set

$$H_\alpha = \cup \{H_{\alpha i} : i = 1, 2, \dots\}$$

Then H_α is open-closed in S and $\{H_\alpha : \alpha \in A\}$ is disjoint. Each $H_{\alpha i}$ is contained in some $U(x)$, say $U(x_{\alpha i})$. Set

$$U_\alpha = \cup \{U(x_{\alpha i}) : i = 1, 2, \dots\}$$

Then U_α is cozero in X . So U_α is normal by Theorem 3-12 and $\dim U_\alpha \leq n$ by the sum theorem. Since

$$\mathcal{U}_\alpha = \{U(x_{\alpha i}) : i = 1, 2, \dots\}$$

is normal by Theorem 2-27, $\mathcal{U}_\alpha$ is refined by an open covering $\mathcal{V}_\alpha$ of U_α such that $\text{ord } \mathcal{V}_\alpha \leq n+1$ by virtue of Theorem 9-13. Since $H_\alpha \subset U_\alpha$, $\mathcal{V}_\alpha \upharpoonright H_\alpha$ is an open covering of H_α . $\cup (\mathcal{V}_\alpha \upharpoonright H_\alpha)$ is an open covering of S of order $\leq n+1$ which refines $\mathcal{G}$. Thus, $\dim S \leq n$. $\square$

If S is separable metric, then S has the star-finite property by Theorem 2-7. Hence, $\dim S \leq \dim X$. On the other hand, we already know that if S is merely metric, there is a case with $\dim S > \dim X$.

11-16. EXAMPLE (Dowker [4]). A normal space M with $\text{ind } M = \text{loc dim } M = \text{loc Ind } M = 0$ and $\dim M = \text{Ind } M = 1$: Let J be the space given in Section 2-20: $J = [0, \omega_1)$. Let J^* be the closed interval $[0, \omega_1] = J \cup \{\omega_1\}$ with the interval topology. Let I be the unit interval $[0, 1]$. Let $\{Q_\alpha : \alpha \in A\}$ be the congruence classes of I modulo the rational numbers, where A is well-ordered. The ordinal type of A is not less than ω_1 . Set

$$I_\alpha = I - \cup \{Q_\beta : \alpha < \beta < \omega_1\}, \quad \alpha < \omega_1$$

Then (i) $\dim I = 1$, (ii) $\dim I_\alpha = 0$ for each $\alpha < \omega_1$, (iii) $I_\alpha \subset I_\beta$ whenever $\alpha < \beta < \omega_1$, (iv) $\cup \{I_\alpha : 0 \leq \alpha < \omega_1\} = I$. The set M is the sum of all subsets $\{\alpha\} \times I_\alpha$ in $J^* \times I$. The topology of M is the relative topology of $J^* \times I$. Let

$$\pi' : J^* \times I \rightarrow I \quad \text{and} \quad \rho' : J^* \times I \rightarrow J^*$$

be the projections. Set $\pi = \pi' \upharpoonright M$ and $\rho = \rho' \upharpoonright M$. If S is any set in I and T is any set in J , $[\pi^{-1}(S)]_T$ denotes the intersection $\pi^{-1}(S) \cap \rho^{-1}(T)$. M is clearly an Hausdorff space.

ASSERTION 1. M is a normal space with $\text{ind } M \leq 0$.

PROOF. Let $\mathcal{G}$ be an arbitrary finite open covering of M . For each x in I , $\mathcal{G}$ has an element equifinal on $\pi^{-1}(x)$, as can be seen by an argument analagous to that in Section 2-20, since $\pi^{-1}(x)$ is normal. Let G be such an element. Then there exist an $\varepsilon = \varepsilon(x) > 0$ and an $\alpha = \alpha(x)$ such that $[\pi^{-1}(S_\varepsilon(x))]_{(\alpha, \omega_1)} \subset G$. If not, there would exist sequences $\alpha < \alpha_1 < \alpha_2 < \dots < \omega_1$ and $x_1, x_2, \dots$ in I such that

$$\lim x_i = x$$

$$p_i = \langle \alpha_i, x_i \rangle \in M - G$$

Let $\lim \alpha_i = \alpha$. Then $\alpha < \omega_1$. Hence,

$$\lim p_i = \langle \alpha, x \rangle \in M - G$$

which gives a contradiction.

Now we have an open covering $\{S_{\varepsilon(x)}(x) : x \in I\}$ of I which has a finite subcovering $\mathcal{S} = \{S_{\varepsilon(x_i)}(x_i)\}$. Set $\beta = \max \alpha(x_i)$. Then

$$\{[\pi^{-1}(S)]_{(\beta, \omega_1)} : S \in \mathcal{S}\} \subset \mathcal{G}$$

Let $\mathcal{S} = \mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \dots$ be a sequence of open coverings of I . Set

$$\mathcal{S}'_i = \{[\pi^{-1}(S)]_{(\beta, \omega_1)} : S \in \mathcal{S}_i\}$$

Then $\mathcal{S}'_1 \supset \mathcal{S}'_2 \supset \dots$. Hence $\mathcal{S}'_1$ is a normal open covering of an open set $\rho^{-1}((\beta, \omega_1))$. Since $\rho^{-1}([0, \beta])$ is a separable metric space, $\mathcal{S}_1 = \mathcal{G} \mid \rho^{-1}([0, \beta])$ is normal. Thus, $\mathcal{G}$ is refined by an open covering $\mathcal{S}_1 \cup \mathcal{S}'_1$ which is normal. Hence $\mathcal{G}$ itself is normal, which proves the normality of M .[†]□

ASSERTION 2. $\dim M \geq 1$.

PROOF. Let $\mathcal{H}$ be an open covering of M consisting of two elements

$$\pi^{-1}(I - \{0\}), \quad \pi^{-1}(I - \{1\})$$

If we assume $\dim M < 1$, there would exist an open covering $\mathcal{G} = \{G_1, G_2\}$ such that $G_1 \cap G_2 = \emptyset$, $G_1 \subset \pi^{-1}(I - \{0\})$ and $G_2 \subset \pi^{-1}(I - \{1\})$. Let $\beta, \mathcal{S}_1$ and $\mathcal{S}$ be the same as in the proof of Assertion 1. Let

$$S_i = \cup \{S \in \mathcal{S} : [\pi^{-1}(S)]_{(\beta, \omega_1)} \subset G_i\}$$

Then S_1 and S_2 constitute a separation of I , which is a contradiction.□

[†] A space is normal if and only if every binary open covering is normal. A proof is quite easy.

ASSERTION 3. $\text{Ind } M \leq 1$.

PROOF. Let F_1 and F_2 be a disjoint pair of closed sets of M . Then $\mathcal{G} = \{M - F_1, M - F_2\}$ can be refined by $\mathcal{G}_1 \cup \mathcal{S}_1'$, as in the above. Since $\dim \rho^{-1}([0, \beta]) = 0$, $\mathcal{G}_1$ has an open refinement $\mathcal{G}_2$ with $\text{ord } \mathcal{G}_2 = 1$ such that $\mathcal{G}_2$ covers $\rho^{-1}([0, \beta])$. Let $S_1, \dots, S_k$ be all elements of $\mathcal{S}$ such that

$$[\pi^{-1}(S_i)]_{(\beta, \omega_1)} \cap F_1 \neq \emptyset.$$

$B = B(\cup S_i)$ consists of finite points. It is easy to see that

$$\text{Ind}[\pi^{-1}(B)]_{(\beta, \omega_1)} \leq 0$$

Let

$$D = \cup \{G \in \mathcal{G}_2 : F_1 \cap G \neq \emptyset\}$$

$$E = D \cup [\pi^{-1}(\cup S_i)]_{(\beta, \omega_1)}$$

Then $B(E) = [\pi^{-1}(B)]_{(\beta, \omega_1)}$. Hence $\text{Ind } B(E) \leq 0$. $B(E)$ separates F_1 and F_2 , which proves $\text{Ind } M \leq 1$. $\square$

ASSERTION 4. $\text{ind } M = \text{loc dim } M = \text{loc Ind } M = 0$, and $\dim M = \text{Ind } M = 1$.

PROOF. $\text{ind } M = \text{loc dim } M = \text{loc Ind } M = 0$ is clear. Since $1 \leq \dim M \leq \text{Ind } M \leq 1$ by Theorem 10-1, we have the desired equalities. $\square$

11-17. THEOREM (Dowker [4]). Let X be a normal space, then there exists a normal space X^* containing X as an open set such that (i) $\dim X^* = \text{loc dim } X$, (ii) if X is hereditarily normal, X^* is also hereditarily normal.

PROOF. When $X = \emptyset$ or $\text{loc dim } X = \infty$, set $X^* = X$. Then X^* is the desired space. Let $\text{loc dim } X = n \neq -1$. The point set X^* is the disjoint sum of X and one new point a . A set U of X^* is to be open if either (i) $U \subset X$ and U is open in X or (ii) $a \in U$ and $X^* - U$ is a closed set of X which is contained in an open set V of X such that $\dim \bar{V} \leq n$. Then it is easy to see that X^* is a Hausdorff space containing X as an open set without disturbing the topology of X . To see that X^* is normal, let E and F be disjoint closed sets of X^* . When $E \cup F \subset X$, then we can find open sets U, V of X with $E \subset U, F \subset V, U \cap V = \emptyset$. But U and V are open in X^* .

Consider the case when $a \in E$. Then $X^* - F$ is an open set of X^* of type (ii). There is an open set V of X with $F \subset V$ and $\dim \bar{V} \leq n$.[†] Then $V \cap (X^* - E)$ is an open set of X containing the closed set F . Take an open set W of X with

$$F \subset W \subset \bar{W} \subset V \cap (X^* - E)$$

Then $X^* - \bar{W}$ is an open set of X^* containing E . Since $E \cap (V \cap (X^* - E)) = \phi$, X^* is normal.

Let X be hereditarily normal. If U is an open set of type (ii) in X^* , then there is an open set V in X with $X^* - U \subset V$ and $\dim \bar{V} \leq n$. Take an open set W with

$$X^* - U \subset W \subset \bar{W} \subset V$$

Then $X^* - \bar{W}$ is open of type (ii) and $\bar{W}$ is closed in X^* . Since $X^* - W$ is closed in the normal space X^* , $X^* - W$ is normal. Since $\bar{W} \cap U$ is a subset of the hereditarily normal space X , $\bar{W} \cap U$ is normal. Then U is the union of two relatively closed normal subsets $X^* - W$ and $\bar{W} \cap U$. Hence U is normal.[‡] Every open set of type (i) is also normal. Therefore, X^* is hereditarily normal.[§]

Since $\dim\{a\} = 0$ and $\dim(X^* \text{ mod } a) \leq n$, then $\dim X^* \leq n$ by Theorem 9-11. Since $\dim X^* \geq \text{loc dim } X^* = n$, we have $\dim X^* = \text{loc dim } X^*$. $\square$

11-18. REMARK. For a hereditarily normal space X , does $\text{loc dim } X = \dim X$? Example 11-16 is not hereditarily normal. If there existed a hereditarily normal space X with $\text{loc dim } X < \dim X$, then by Theorem 11-17 there would exist a hereditarily normal space X^* for which the subset theorem is not true. Thus, a negative answer for the present problem

[†] The closure signs indicate closure in X hereafter in the proof.

[‡] Let F_1, F_2 be closed sets in a Hausdorff space $F_1 \cup F_2$. If F_1 and F_2 are normal, then $F_1 \cup F_2$ is normal. Proof: Let F be a closed set of $F_1 \cup F_2$ and $f: F \rightarrow [0, 1]$ be a mapping. Let $g': F_1 \rightarrow [0, 1]$ be an extension of $f|F_1 \cap F$. Let $g'': F_2 \rightarrow [0, 1]$ be an extension of $g'| (F_1 \cap F_2) \cup (F \cap F_2)$. Let $g: F_1 \cup F_2 \rightarrow [0, 1]$ be a function such that $g|F_1 = g'$ and $g|F_2 = g''$. Then g is an extension of f . $\square$

[§] Let Y be a normal space. If every open set of Y is normal, then Y is hereditarily normal. Proof: Let S be a subset of Y and $\mathcal{G}$ a binary open covering of S . Let $\mathcal{H} = \{H_1, H_2\}$ be an open collection with $\mathcal{H}|S = \mathcal{G}$. Then $\mathcal{H}$ is normal by the normality of $H_1 \cup H_2$. Hence $\mathcal{G}$ is normal. $\square$

implies a negative one for the problem concerning the subset theorem, Section 11-14. Conversely, if there existed a hereditarily normal space X^* for which the subset theorem is not true, then there would exist an open set X of X^* with $\dim X > \dim X^*$. Since $\text{loc dim } X \leq \dim X^*$, $\dim X > \text{loc dim } X$, which gives a negative answer for the present problem. Therefore, the present problem is another expression of Problem 11-14.†

† This was mentioned by Dowker [4].

Chapter 3

DIMENSION OF METRIC SPACES

In this chapter our concern is restricted to metric spaces, where we can get more information about dimension than in the nonmetric case. The results in this chapter will be used in all succeeding chapters. In Section 12 the fundamental theorem for metric spaces will be given (Theorem 12-6), showing the equivalence of the four conditions: (i) $\dim X \leq n$, (ii) $\text{Ind } X \leq n$, (iii) the decomposition theorem, and (iv) X is generated by 0-dimensional metric space under a closed mapping of order $\leq n + 1$. The last condition may be one of the keys to open the door to classes of nonmetric spaces where a harmonious dimension theory can be developed. The existence of an equidimensional G_δ -envelope will be given in Theorem 12-15. The rest of this section furnishes preliminary theorems concerning growing finer sequences of coverings. This material will be used later in Chapter 7, where metric-dependent dimensions will be studied. Section 13 provides a quick review of Nagata's metric and introduces two kinds of infinite dimensional spaces. One is a space which is the countable sum of finite dimensional closed sets, say a σ_f -space, and the other is a space which is the countable sum of 0-dimensional sets, say a σ_0 -space. Two decomposition theorems for metric spaces and for σ_0 -spaces will also be given, Theorems 13-14 and 13-15. In Section 14 we study a sequence of open coverings, the existence of which characterizes dimension. It is to be noted that we obtain a characterization of dimension in terms of a sequence of locally finite coverings in Theorem 12-6. As an application, the dimension-preserving completion (Theorem 14-4) is obtained. In Section 15 we study the relation between open mappings and dimension.

12. Fundamental Theorems for the Dimension of Metric Spaces

12-1. DEFINITION. An inverse limit of a sequence of discrete spaces is called a *Baire 0-dimensional space*.

12-2. THEOREM (Katětov [1] and Morita [6]). A space X is a metric space with $\dim X \leq 0$ if and only if X is a (dense) subset of a Baire 0-dimensional space.

PROOF. Let X be a metric space with $\dim X \leq 0$. Since every open covering of X can be refined by an open covering of order ≤ 1 by Theorem 9-13, there is a sequence $\mathcal{U}_i = \{U_\alpha : \alpha \in A_i\}$ of open coverings of X such that:

- i. $\text{ord } \mathcal{U}_i \leq 1$ for each i
- ii. mesh $\mathcal{U}_i$ converge to 0
- iii. $\mathcal{U}_{i+1} < \mathcal{U}_i$ for each i

Let $f_i^{i+1}: A_{i+1} \rightarrow A_i$ be a transformation defined by

$$f_i^{i+1}(\alpha) = \beta \quad \text{implies} \quad U_\alpha \subset U_\beta$$

For any ordered pair $i < j$ set

$$f_i^j = f_i^{i+1} \cdots f_{j-1}^j$$

If we consider each A_i as a discrete space, we get an inverse limiting system $\{A_i, f_i^j\}$. Let $\pi_i: \text{inv lim } A_i \rightarrow A$ be the projection. Let A be the aggregate of points x in $\text{inv lim } A_i$ such that

$$f(x) = \bigcap U_{\pi_i(x)} \neq \emptyset$$

Then $f(x)$ is a single point and $f: A \rightarrow X$ is a homeomorphism. It can easily be seen that A is dense in $\text{inv lim } A_i'$, where $A_i' = \pi_i(A)$.

Conversely, suppose that A is a subset of $A' = \text{inv lim } \{A_i, f_i^j\}$. Then A is metrizable since A' is metrizable. To prove $\dim A \leq 0$ it suffices to prove $\dim A' \leq 0$ since we already know the subset theorem is true in metric spaces by Theorem 11-10. We associate $\text{inv lim } A_i$ with *Baire's metric* as follows

$$\begin{aligned} \rho(x, y) &= 0 && \text{if } x = y \\ \rho(x, y) &= 1/n && \text{if } n \text{ is the first index with } \pi_n(x) \neq \pi_n(y) \end{aligned}$$

Then this metric agrees with the topology and (A', ρ) is a complete metric space. Let F and H be disjoint closed sets of A' . Let

$$\mathcal{U}_i = \{\pi_i^{-1}(\alpha) : \alpha \in \pi_i(A')\}$$

Then $\mathcal{U}_i$ is a disjoint open covering of A' . Set

$$D_i = \cup \{U \in \mathcal{U}_i : U \cap F \neq \phi, U \cap H = \phi\}$$

Then D_i is open-closed in A' and $\{D_i\}$ is locally finite in $A' - F \cup H$. Hence, if we set $D = \cup D_i$, then D is open-closed in A' . Obviously, $F \subset D \subset A' - H$. $\square$

12-3. COROLLARY. Every metric space X with $\dim X = 0$ is embedded densely in a complete metric space Y with $\dim Y = 0$.

The above proof contains a proof for this corollary.

12-4. COROLLARY. A space X is a metric space with $\dim X \leq 0$ if and only if X is embedded in a countable product of discrete spaces.

PROOF. Suppose X is a metric space with $\dim X \leq 0$. Then X is in some $\text{inv lim } A_i$. $\text{inv lim } A_i$ is a subspace of ΠA_i by Section 6-1.

Conversely, let a space X be embedded in ΠA_i where A_i is discrete for every i . ΠA_i is nothing but the inverse limit of $\Pi_{j=1}^i A_j$ with the natural projections. $\square$

12-5. LEMMA.[†] Let X and Y be metric spaces with $\dim X \leq n$ and f a closed mapping of X onto Y . If for each point y of Y , $f^{-1}(y)$ consists of exactly k ($k < \infty$) points, then $\dim Y \leq n$.

PROOF. Let $\mathcal{F}_i = \{F_\alpha : \alpha \in A_i\}$, $i = 1, 2, \dots$, be locally finite closed coverings of X such that $\text{mesh } \mathcal{F}_i \rightarrow 0$ as $i \rightarrow \infty$. Then $f(\mathcal{F}_i) = \{f(F_\alpha) : \alpha \in A_i\}$ is a point-finite closed covering of Y . Let y be an arbitrary point of Y and B_i a subset of A_i such that $\alpha \in B_i$ if and only if $y \notin f(F_\alpha)$. Since $\cup \{F_\alpha : \alpha \in B_i\}$ is closed,

$$V = Y - \cup \{f(F_\alpha) : \alpha \in B_i\}$$

is an open neighborhood of y . If $f(F_\alpha) \cap V \neq \phi$, then $\alpha \notin B_i$. Since $A_i - B_i$ is finite, V meets only a finite number of $f(F_\alpha)$, $\alpha \in A_i$. Thus, $f(\mathcal{F}_i)$ is locally finite.

[†] This is due to Suzuki [1]. Actually we can prove $\dim X = \dim Y$, since $\dim X \leq \dim Y$ by Corollary 21-2. The present lemma will be generalized thoroughly in Theorem 24-5.

Let y be an arbitrary point of Y and $\{x_1, \dots, x_k\}$ the inverse image of y . Then there exists an i such that when $x_s \in F_\alpha \in \mathcal{F}_i$, $x_t \in F_\beta \in \mathcal{F}_i$, $s \neq t$, then $F_\alpha \cap F_\beta = \emptyset$. Choose $F_{\alpha(j)} \in \mathcal{F}_i$, $j = 1, \dots, k$, such that $x_j \in F_{\alpha(j)}$ and put

$$F(y) = \bigcap_{j=1}^k f(F_{\alpha(j)})$$

Then $F(y)$ is a closed set of Y which contains y . Since $F_{\alpha(1)} \cap f^{-1}(F(y))$ is a closed set of X and $f|_{F_{\alpha(1)} \cap f^{-1}(F(y))}$ is a homeomorphism of $F_{\alpha(1)} \cap f^{-1}(F(y))$ onto $F(y)$,

$$\dim F(y) = \dim(F_{\alpha(1)} \cap f^{-1}(F(y))) \leq n$$

Since $\{F(y): y \in Y\}$ is a closed covering of Y which is a countable sum of locally finite subfamilies,

$$\dim Y = \sup \dim F(y) \leq n$$

by Corollary 11-9 $\square$

12-6. THEOREM.† For a space X the following conditions are equivalent:

1. X is a metric space with $\dim X \leq n$
2. There exists a sequence of locally finite coverings $\mathcal{F}_i = \{F_\alpha: \alpha \in A_i\}$, $i = 1, 2, \dots$, of X such that (i) $\overline{\mathcal{F}}_{i+1}$ refines $\mathcal{F}_i$ for each i , (ii) $\text{ord } \mathcal{F}_i \leq n + 1$ for each i , and (iii) for each neighborhood U of each point x in X there exists an i with $S(x, \mathcal{F}_i) \subset U$
3. There exist a metric space A with $\dim A \leq 0$ and a closed mapping f of A onto X with $\text{ord } f^\dagger \leq n + 1$
4. (decomposition theorem) X is a metric space which is the sum of $n + 1$ subspaces X_i , $i = 1, \dots, n + 1$, with $\dim X_i \leq 0$ for each i
5. X is a metric space with $\text{Ind } X \leq n$

PROOF. With the aid of Theorem 1-7 and Corollary 9-14, that (1) implies (2) is verified at once.

† Proof of the equivalence of (1), (4), and (5) is due to Katětov [1] and Morita [6]. The equivalence of (1) and (2) is due to Nagami [3]. The equivalence of (1) and (3) is due to Morita [7].

‡ $\text{ord } f$ indicates the *order* of f , i.e. the maximum of the cardinals of $f^{-1}(x)$, $x \in X$.

That (2) implies (3): For every i define $f_i^{i+1}: A_{i+1} \rightarrow A_i$ in such a way that

$$f_i^{i+1}(\alpha) = \beta \quad \text{yields} \quad \bar{F}_\alpha \subset F_\beta$$

For a pair $i > j$, let

$$f_j^i = f_j^{j+1} \dots f_{i-1}^i$$

Let A' be the inverse limit of $\{A_i, f_j^i\}$ where we consider the A_i 's as discrete spaces. Let A be a subspace of A' consisting of points $\langle \alpha_1, \alpha_2, \dots \rangle \in A'$ such that $\cap F_{\alpha_i} \neq \emptyset$. This intersection is a single point. Let f be a transformation of A to X such that $f(a) = \cap F_{\pi_i(a)}$, where $\pi_i: A \rightarrow A_i$ are the projections.

To prove f is onto, let x be an arbitrary point in X . Since $\mathcal{F}_i$ is locally finite,

$$B_i = \{\alpha: x \in F_\alpha \in \mathcal{F}_i\}$$

is a nonempty finite set of A_i . Since

$$f_i^{i+1}(B_{i+1}) \subset B_i \quad \text{for every } i$$

$\{B_i, f_i^j \mid B_j\}$ forms an inverse limiting system. Since every B_i is a nonempty compact space, the inverse limit B of this system is not empty by Section 6-1. Let $a = \langle \alpha_1, \alpha_2, \dots \rangle$ be a point of B . Then $f(a) = \cap F_{\alpha_i} = x$, which proves f is onto.

To prove f is continuous, let a be an arbitrary point of A and U an arbitrary neighborhood of $f(a)$. Then there exists an i with $S(f(a), \mathcal{F}_i) \subset U$. Since

$$f(a) \in F_{\pi_i(a)} \quad \text{and} \quad f(\pi_i^{-1}(\pi_i(a))) \subset F_{\pi_i(a)}$$

we get $f(\pi_i^{-1}(\pi_i(a))) \subset U$, which proves the continuity of f .

To prove $\text{ord } f \leq n+1$ assume that $\text{ord } f > n+1$. Then there would exist a point x in X such that $f^{-1}(x)$ would contain at least $n+2$ points, say $a_1, \dots, a_{n+2}$. Then there would exist a j such that $\pi_j(a_1), \dots, \pi_j(a_{n+2})$ are different points in A_j . Since

$$x = f(a_i) \in f(\pi_j^{-1}(\pi_j(a_i))) \subset F_{\pi_j(a_i)}, \quad i = 1, \dots, n+2$$

$\text{ord } \mathcal{F}_j \geq n+2$, which is a contradiction, proving $\text{ord } f \leq n+1$.

To show f is closed let F be an arbitrary closed set of A . Then

$$f(F) = \cup \{f(F \cap \pi_1^{-1}(\alpha)): \alpha \in \pi_1(A)\}$$

Since $f(\pi_1^{-1}(\alpha)) \subset F_\alpha$ for any $\alpha \in \pi_1(A)$ and $\mathcal{F}_1 = \{F_\alpha : \alpha \in A_1\}$ is locally finite, we get

$$\overline{f(F)} = \cup \{ \overline{f(F \cap \pi_1^{-1}(\alpha))} : \alpha \in \pi_1(A) \}$$

Let x_0 be an arbitrary point of $\overline{f(F)}$. Then there exists an $\alpha_1 \in \pi_1(A)$ such that

$$x_0 \in \overline{f(F \cap \pi_1^{-1}(\alpha_1))}$$

By an inductive process we can find indices $\alpha_i \in \pi_i(A)$ for $i = 1, 2, \dots$ such that

$$x_0 \in \overline{f(F \cap \pi_1^{-1}(\alpha_1) \cap \dots \cap \pi_i^{-1}(\alpha_i))} \quad \text{for each } i$$

Since $\pi_i^{-1}(\alpha_i) \cap \pi_{i+1}^{-1}(\alpha_{i+1}) \neq \emptyset$ for any i , $f_i^{i+1}(\alpha_{i+1}) = \alpha_i$ for any i . Therefore, $a = \langle \alpha_1, \alpha_2, \dots \rangle$ is a point of A' . Since

$$f(\pi_{i+1}^{-1}(\alpha_{i+1})) \subset F_{\alpha_{i+1}} \subset \bar{F}_{\alpha_{i+1}} \subset F_{\alpha_i}$$

for each i ,

$$\begin{aligned} x_0 &\in \bigcap_{i=1}^{\infty} \overline{f(F \cap \pi_1^{-1}(\alpha_1) \cap \dots \cap \pi_i^{-1}(\alpha_i))} \\ &\subset \bigcap_{i=1}^{\infty} \overline{f(\pi_i^{-1}(\alpha_i))} \subset \bigcap_{i=1}^{\infty} \bar{F}_{\alpha_i} \subset \bigcap_{i=2}^{\infty} \bar{F}_{\alpha_i} \subset \bigcap_{i=1}^{\infty} F_{\alpha_i} \end{aligned}$$

Since $\bigcap_{i=1}^{\infty} F_{\alpha_i}$ is either the empty set or a single point, we get $x_0 = \bigcap_{i=1}^{\infty} F_{\alpha_i}$. Thus a is a point of A and $f(a) = x_0$. On the other hand, $\{\pi_i^{-1}(\alpha_i) : i = 1, 2, \dots\}$ is a neighborhood base of a and $F \cap \pi_i^{-1}(\alpha_i) \neq \emptyset$ for any i , by the construction of α_i 's. Therefore, $a \in \bar{F} = F$ and $x_0 = f(a) \in f(F)$, which shows the closedness of f . We already know A is a metrizable space with $\dim A \leq 0$, by Theorem 12-2.

That (3) implies (4): X is metrizable by Corollary 5-11. Let X_i be the aggregate of all points x in X such that $f^{-1}(x)$ consists of exactly i points. Then by Lemma 12-5 $\dim X_i \leq 0$. Evidently, $X = \bigcup_{i=1}^{n+1} X_i$.

That (4) implies (5): We prove this by induction on n . When $\dim X \leq 0$, it is evident that $\text{Ind } X \leq 0$. Assume that the implication is true for dimension n . Let X be the sum of $n+2$ subspaces $X_i; i = 1, \dots, n+2$, with $\dim X_i \leq 0$. Let F be a closed set of X and G an open set of X with $F \subset G$. Let H_1 and H_2 be open sets such that

$$F \subset H_1 \subset \bar{H}_1 \subset H_2 \subset \bar{H}_2 \subset G$$

Since $\dim X_1 \leq 0$, there exists a relatively open-closed set U_1 of X_1 with

$$X_1 \cap \bar{H}_1 \subset U_1 \subset X_1 \cap H_2$$

Set

$$P_1 = F \cup U_1, \quad P_2 = (X - G) \cup (X_1 - U_1)$$

Then $\bar{P}_1 \cap P_2 = P_1 \cap \bar{P}_2 = \phi$. Hence, there exists an open set U of X such that

$$P_1 \subset U \subset \bar{U} \subset X - P_2$$

Then $B(U) \cap X_1 = \phi$ and $F \subset U \subset \bar{U} \subset G$. Since $\text{Ind } \cup_{i=2}^{n+2} X_i \leq n$ by the induction assumption and $B(U)$ is a closed subset of $\cup_{i=2}^{n+2} X_i$, we get $\text{Ind } B(U) \leq n$, which proves $\text{Ind } X \leq n + 1$.

That (5) implies (1) was proved in Theorem 10-1. $\square$

12-7. PROPOSITION. Let X be a metric space and S a subset of X with $\dim S \leq n$, then for any disjoint closed sets F_1, F_2 in X there exists a closed set B such that B separates F_1 and F_2 and $\dim B \cap \bar{S} \leq n - 1$.

This can be proved by a method similar to that of the proof for the implication of (5) by (4) in the preceding theorem.

12-8. LEMMA. Let X and Y be metric spaces with $\dim X \leq 0$ and f a closed mapping of X onto Y with $\text{ord } f < \infty$, then $\dim Y + 1 \leq$ the number of integers i with $\{y: |f^{-1}(y)| = i\} \neq \phi$.

PROOF. Set

$$\{y: |f^{-1}(y)| = i\} = Y_i$$

Then $\dim Y_i \leq 0$. Since Y is the sum of these Y_i 's, the assertion is true by the decomposition theorem. $\square$

12-9. THEOREM (Nagami [3]). Let X be a metric space, $C_i, i = 1, 2, \dots$, closed sets of X with $\dim C_i \leq n_i$ and $\mathcal{U}$ an arbitrary open covering of X , then there exist a sequence of locally finite closed coverings $\mathcal{H}_i = \{H_\alpha: \alpha \in A_i\}, i = 1, 2, \dots$, and a sequence of transformations $f_i^j: A_j \rightarrow A_i, i < j$, which satisfy the following conditions:

1. $\mathcal{H}_1$ refines $\mathcal{U}$
2. $\text{mesh } \mathcal{H}_i \rightarrow 0$ as $i \rightarrow \infty$
3. For any i and any j , $\text{ord } \mathcal{H}_i \mid C_j \leq n_j + 1$
4. For any i , any $j > i$, and any $\alpha \in A_i$:

$$H_\alpha = \cup \{H_\beta: f_i^j(\beta) = \alpha\}$$

5. For any i , any s with $s \leq n_i + 2$, and any t :

$$\dim \bigcap_{j=1}^s \{H_{\alpha_j} \cap C_i : \alpha_1, \dots, \alpha_s \text{ are different indices of } A_i\} \leq n_i - s + 1$$

PROOF. First we construct by induction a sequence of locally finite closed coverings $\mathcal{F}_i = \{F_\alpha : \alpha \in B_i\}$, $i = 1, 2, \dots$, and a sequence of locally finite open coverings $\mathcal{U}_i = \{U_\alpha : \alpha \in B_i\}$, $i = 1, 2, \dots$, which satisfy the following conditions:

- i. $\mathcal{U} > \mathcal{F}_1 > \mathcal{U}_1 > \mathcal{F}_2 > \mathcal{U}_2 > \dots$
- ii. $\text{mesh } \mathcal{F}_i < 1/i$
- iii. $\text{ord } \mathcal{F}_i \mid C_j \leq n_j + 1$ if $i \leq j$

Let $\mathcal{V}_1 = \{V_\alpha : \alpha \in B_1\}$ be a locally finite open covering of X with $\text{mesh } \mathcal{V}_1 < 1$ such that $\mathcal{V}_1$ refines $\mathcal{U}$. Let $\mathcal{W}' = \{W'_\alpha : \alpha \in B_1\}$ be an open covering of C_1 such that $\text{ord } \mathcal{W}' \leq n_1 + 1$ and $W'_\alpha \subset V_\alpha$ for every $\alpha \in B_1$. Let $W''_\alpha, \alpha \in B_1$, be an open set of X with $W''_\alpha \cap C_1 = W'_\alpha$. Set

$$W_\alpha = (W''_\alpha \cap V_\alpha) \cup (V_\alpha - C_1)$$

Then $\mathcal{W} = \{W_\alpha : \alpha \in B_1\}$ is a locally finite open covering of X and hence, there exist by Theorem 1-7 a closed covering $\mathcal{F}_1 = \{F_\alpha : \alpha \in B_1\}$ and an open covering $\mathcal{U}_1 = \{U_\alpha : \alpha \in B_1\}$ such that $U_\alpha \subset F_\alpha \subset W_\alpha (\subset V_\alpha)$, for every $\alpha \in B_1$. Since $\mathcal{W} \mid C_1 = \mathcal{W}'$,

$$\text{ord } \mathcal{F}_1 \mid C_1 \leq \text{ord } \mathcal{W} \mid C_1 = \text{ord } \mathcal{W}' \leq n_1 + 1$$

Now make the induction assumption that there exist locally finite closed coverings $\mathcal{F}_1, \dots, \mathcal{F}_{m-1}$ and locally finite open coverings $\mathcal{U}_1, \dots, \mathcal{U}_{m-1}$ which satisfy conditions (i), (ii), and (iii) for $i = 1, \dots, m-1$. Let $\mathcal{V}_m = \{V_\alpha : \alpha \in B_m\}$ be a locally finite open covering of X such that $\mathcal{V}_m$ refines $\mathcal{U}_{m-1}$ and $\text{mesh } \mathcal{V}_m < 1/m$. Let $D_1, \dots, D_m$ be a system with $\dim D_1 \leq \dim D_2 \leq \dots \leq \dim D_m$ obtained by changing the order of $C_1, \dots, C_m$. By the same argument as in the above, there exist open coverings $\mathcal{W}_i = \{W_{i\alpha} : \alpha \in B_m\}$, $i = 1, \dots, m$, such that

$$V_\alpha \supset W_{1\alpha} \supset \dots \supset W_{m\alpha} \quad \text{for every } \alpha \in B_m$$

$$\text{ord } \mathcal{W}_i \mid D_i \leq \dim D_i + 1 \quad \text{for } i = 1, \dots, m$$

Since $\text{ord } \mathcal{W}_i \mid D_j \leq \text{ord } \mathcal{W}_j \mid D_j$, if $j \leq i$, we get

$$\text{ord } \mathcal{W}_m \mid C_j \leq \text{ord } \mathcal{W}_{i_j} \mid C_i \leq n_i + 1 \quad \text{for } i = 1, \dots, m$$

where $C_i = D_{i_1}$. Since $\mathcal{W}_m = \{W_{m\alpha} : \alpha \in B_m\}$ is locally finite, there exist a closed covering $\mathcal{F}_m = \{F_\alpha : \alpha \in B_m\}$ and an open covering $\mathcal{U}_m = \{U_\alpha : \alpha \in B_m\}$ such that $U_\alpha \subset F_\alpha \subset W_\alpha (\subset V_\alpha)$ for every $\alpha \in B_m$. It is obvious that $\text{ord } \mathcal{F}_m \mid C_i \leq n_i + 1$ for $i \leq m$, $\mathcal{U}_{m-1} > \mathcal{F}_m > \mathcal{U}_m$, and $\text{mesh } \mathcal{F}_m < 1/m$, which completes the induction.

Now define $f_i^{i+1} : B_{i+1} \rightarrow B_i$ in such a way that $f_i^{i+1}(\alpha) = \beta$ yields $F_\alpha \subset F_\beta$. Set

$$f_i^j = f_i^{i+1} \dots f_{j-1}^j \quad \text{for } j > i$$

Let B be the aggregate of points $a = \langle \alpha_1, \alpha_2, \dots \rangle$ of $\text{inv } \lim \{B_i, f_i^j\}$ such that $\cap F_{\alpha_i} \neq \phi$. Let f be a transformation of B to X defined by: $f(a) = \cap F_{\pi_i(a)}$, where π_i 's are the projections of B to B_i . Then we know by an argument analogous to that in the proof for the implication of (3) by (2) of Theorem 12-6 that f is a closed mapping of B onto X . Let

$$\mathcal{H}_i = \{H_\alpha = f(\pi_i^{-1}(\alpha)) : \alpha \in A_i = \pi_i(B)\}, \quad i = 1, 2, \dots$$

Then $\mathcal{H}_1 < \mathcal{F}_i$; and these form a sequence of locally finite closed coverings of X which satisfy conditions (1), (2), (3), and (4).

Let us prove that the conditions (1), (2), (3), and (4) imply the last condition, (5). Set

$$H = \bigcap_{j=1}^s \{H_{\alpha_j} \cap C_i : \alpha_1, \dots, \alpha_s \text{ are different indices of } A_i\}$$

Consider the sequence

$$\mathcal{H}_1 \mid H, \quad \mathcal{H}_{i+1} \mid H, \quad \dots$$

of closed coverings of H . Then, $\text{ord } \mathcal{H}_j \mid H \geq s$ for any $j \geq i$. Corresponding to this sequence we can get, by the same argument as in the proof that (2) implies (3) in Theorem 12-6, a metric space Z with $\dim Z \leq 0$ and a closed mapping g of Z onto H such that

$$s \leq |f^{-1}(y)| \leq n_i + 1$$

Hence, by the lemma

$$\dim H \leq n_i + 1 - (s - 1) - 1 = n_i - s + 1 \quad \square$$

12-10. COROLLARY. Under the assumptions of Theorem 12-9, omitting the open covering $\mathcal{U}$, there exist a metric space A with $\dim A \leq 0$ and a closed mapping f of A onto X such that for any i and any $x \in C_i$,

$$|f^{-1}(x)| \leq n_i + 1$$

12-11. COROLLARY (Morita [6]). Let X be a metric space and C_i , $i = 1, 2, \dots$, closed sets of X such that $\cup C_i = X$ and $\dim C_i \leq n_i$. Then there exist subsets X_i , $i = 1, 2, \dots$, of X with $\dim X_i \leq 0$ such that

$$C_i \subset \cup \{X_j : j = 1, \dots, n_i + 1\} \quad \text{for each } i$$

These corollaries are almost evident by the theorem.

12-12. COROLLARY (Morita [6] and Katětov [2]). Let X be a metric space and C_i , $i = 1, 2, \dots$, closed sets of X with $\dim C_i \leq n_i$. If F is a closed set and G is an open set with $F \subset G$, then there exists an open set U with $F \subset U \subset \bar{U} \subset G$ such that

$$\dim((\bar{U} - U) \cap C_i) \leq n_i - 1 \quad \text{for every } i$$

PROOF. Let X_i , $i = 1, 2, \dots$, be subsets of X having the property in the preceding corollary. Setting $S = X_1$ in Proposition 12-7, we get the desired result. $\square$

12-13. THEOREM (Morita [6]). In a regular space X the following propositions are equivalent:

1. X is a metric space with $\dim X \leq n$
2. There exists a σ -discrete base $\mathcal{U}$ such that $\text{ord } B(\mathcal{U}) \leq n$, where $B(\mathcal{U}) = \{B(U) : U \in \mathcal{U}\}$
3. There exists a σ -locally finite base $\mathcal{U}$ such that $\text{ord } B(\mathcal{U}) \leq n$
4. There exists a σ -discrete base $\mathcal{U}$ such that $\dim B(U) \leq n - 1$ for any $U \in \mathcal{U}$
5. There exists a σ -locally finite base $\mathcal{U}$ such that $\dim B(U) \leq n - 1$ for any $U \in \mathcal{U}$

PROOF. That (1) implies (2) and (4): For positive integers i and j , let $\mathcal{U}_{ij} = \{U_\alpha : \alpha \in A_{ij}\}$ be a discrete open collection and $\mathcal{F}_{ij} = \{F_\alpha : \alpha \in A_{ij}\}$ a discrete closed collection such that

$$\text{mesh } \mathcal{U}_{ij} < 1/i, \quad F_\alpha \subset U_\alpha \text{ for each } \alpha \in \cup A_{ij}$$

$$\cup_j \mathcal{F}_{ij} \text{ covers } X$$

Set

$$U_{ij} = \cup \{U : U \in \mathcal{U}_{ij}\}, \quad F_{ij} = \cup \{F : F \in \mathcal{F}_{ij}\}$$

Then F_{ij} is a closed set contained in an open set U_{ij} . We count all pairs $F_{ij} \subset U_{ij}$ as $F_{\lambda_i} \subset U_{\lambda_i}$, $i = 1, 2, \dots$. First, there exists an open set G_1 , by Theorem 12-6, with

$$F_{\lambda_1} \subset G_1 \subset \bar{G}_1 \subset U_{\lambda_1} \quad \text{and} \quad \dim B(G_1) \leq n - 1$$

Make the induction assumption $[\xi_k]$:

$G_1, \dots, G_k$ are open sets such that:

- i. $F_{\lambda_j} \subset G_j \subset \bar{G}_j \subset U_{\lambda_j}$ for $j = 1, \dots, k$
- ii. If $\{i_1, \dots, i_t\}$, $t \leq n + 1$, are different indices from $\{1, \dots, k\}$, then $\dim \cap \{B(G_j) : j = i_1, \dots, i_t\} \leq n - t$

Now there is a finite sequence of closed sets $C_1 = X, C_2, \dots, C_s$ such that $C_2, \dots, C_s$ are all possible intersections of n elements from $\{B(G_j) : j = 1, \dots, k\}$. Applying Corollary 12-12 we can find an open set G_{k+1} which satisfies the proposition $[\xi_{k+1}]$.

Thus, by induction there exist open sets G_{ij} such that:

- i. $F_{ij} \subset G_{ij} \subset \bar{G}_{ij} \subset U_{ij}$ for each i, j
- ii. If B is the intersection of $t (\leq n + 1)$ elements from $\{B(G_{ij})\}$, then $\dim B \leq n - t$

Setting

$$\mathcal{U} = \{G_{ij} \cap U : U \in \mathcal{U}_{ij}, \quad i, j = 1, 2, \dots\}$$

we have a base $\mathcal{U}$ satisfying conditions (2) and (4) at the same time.

It is evident that (2) implies (3) and that (4) implies (5).

That (3) implies (5): Let $\mathcal{U}$ be a σ -locally finite base with $\text{ord } B(\mathcal{U}) \leq n$. If U is an arbitrary element of $\mathcal{U}$,

$$\text{ord}(\mathcal{U} - \{U\}) \mid B(U) \leq n - 1$$

When $n = 0$, (3) implies (5), evidently. So an easy application of induction on n proves the implication.

That (5) implies (1): X is metrizable by the Nagata–Smirnov metrization theorem, Section 5-3. Since every binary open covering of X can be refined by an open covering $\mathcal{U}'$ which is a subcollection of $\mathcal{U}$, we have $\dim X \leq n$ by Theorem 11-12. $\square$

12-14. THEOREM (product theorem) (Katětov [1] and Morita [6]). Let X and Y be nonempty metric spaces with $\dim X \leq m$ and $\dim Y \leq n$, then $\dim(X \times Y) \leq m + n$.

PROOF (by induction). When $m + n = 0$, $\dim X = 0$ and $\dim Y = 0$. Then, by Corollary 12-4, $X \times Y$ is a subset of a countable product of discrete spaces and hence $\dim(X \times Y) = 0$. Suppose that the theorem is true for X' , Y' with $\dim X' + \dim Y' \leq i$. Let $m + n = i + 1$. By Theorem 12-13 there exist σ -locally finite bases $\mathcal{U}_1$ of X and $\mathcal{U}_2$ of Y such that:

- i. $\dim B(U_1) \leq m - 1$ for each $U_1 \in \mathcal{U}_1$
- ii. $\dim B(U_2) \leq n - 1$ for each $U_2 \in \mathcal{U}_2$

Then

$$\{U_1 \times U_2 : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2\}$$

is a σ -locally finite base of $X \times Y$. Since

$$B(U_1 \times U_2) \subset (B(U_1) \times \bar{U}_2) \cup (\bar{U}_1 \times B(U_2))$$

we have

$$\begin{aligned} \dim B(U_1 \times U_2) &\leq \max\{\dim(B(U_1) \times \bar{U}_2), \dim(\bar{U}_1 \times B(U_2))\} \\ &\leq \max\{(m - 1) + n, m + (n - 1)\} = m + n - 1 \end{aligned}$$

by the induction assumption. Again by Theorem 12-13, $\dim(X \times Y) \leq m + n$. $\square$

12-15. THEOREM (Katětov [1] and Morita [6]). Let X be a metric space and S a subset of X with $\dim S \leq n$, then there exists a G_δ -set T in X such that $S \subset T$ and $\dim T \leq n$.

PROOF. First consider the case $n = 0$. Set $\mathcal{F} = \cup \mathcal{F}_{ij}$ and $\mathcal{U} = \cup \mathcal{U}_{ij}$ where $\mathcal{F}_{ij}$ and $\mathcal{U}_{ij}$ are the same as in the proof of Theorem 12-13. $\mathcal{F}$ and $\mathcal{U}$ are in one-one correspondence

$$\mathcal{F} = \{F_\alpha\}, \quad \mathcal{U} = \{U_\alpha\}, \quad F_\alpha \subset U_\alpha \quad \text{for each } \alpha$$

By Proposition 12-7, there exists an open set V_α for each α such that $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$ and $B(V_\alpha) \cap S = \emptyset$. Set $T = X - \cup B(V_\alpha)$. Then T is a G_δ -set in X containing S . $\{V_\alpha\} \mid T$ is a σ -locally finite base of T . Since $\dim(B(V_\alpha) \cap T) = -1$, $\dim T \leq 0$ by Theorem 12-13.

When $\dim S \leq n$, we can set $S = \cup_{i=1}^{n+1} S_i$, where $\dim S_i \leq 0$ for each i , by the decomposition theorem. Let T_i be a G_δ -set with $S_i \subset T_i$ and $\dim T_i \leq 0$. Then $T = \cup T_i$ contains S and $\dim T \leq n$, again by the decomposition theorem. Clearly T is a G_δ -set in X . $\square$

13. Topics on Nagata's Metric and Infinite Dimensional Spaces

13-1. THEOREM.[†] Let X be a metric space, then $\dim X \leq n$ if and only if X has a metric ρ agreeing with the topology such that for each $\varepsilon > 0$ and each $n + 3$ points $x, y_1, y_2, \dots, y_{n+2}$ with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon, i = 1, \dots, n + 2$, there exist i and j with $i \neq j$ and $\rho(y_i, y_j) < \varepsilon$.

13-2. PROBLEM. Let (X, ρ) be a metric space. If for each $\varepsilon > 0$ and each $n + 3$ points $x, y_1, \dots, y_{n+2}$ with $\rho(x, y_i) < \varepsilon, i = 1, \dots, n + 2$, there exist i and j with $i \neq j$ and $\rho(y_i, y_j) < \varepsilon$, then is it true that $\dim X \leq n$?

13-3. THEOREM (Nagata [5] and Ostrand [1]). A metric space X has $\dim X \leq n$ if and only if one can introduce a metric ρ in X satisfying the following condition: For every $n + 3$ points $x, y_1, \dots, y_{n+2}$ in X there are i and j with $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_j)$.

13-4. DEFINITION. Let X be a space. If for every sequence of disjoint pairs of closed sets of $X, C_1, C_1'; C_2, C_2'; \dots$, there are closed sets $B_1, B_2, \dots$ such that:

- i. B_i separates C_i and $C_i', i = 1, 2, \dots$
- ii. $\bigcap_{i=1}^k B_i = \emptyset$ for some number k

then X is called a *weakly infinite dimensional space* or a σ_w -space.

If X is the countable sum of sets X_i with $\dim X_i \leq 0$, then X is a *countable dimensional space* or a σ_0 -space. If not, X is an *uncountable dimensional space*. If X is the countable sum of closed sets X_i with $\dim X_i < \infty$, then X is a *strongly countable dimensional space* or a σ_f -space.

13-5. THEOREM (Nagata [4]). A space X is a metric σ_0 -space if and only if there exist a metric space A with $\dim A \leq 0$ and a closed mapping f of A onto X such that $f^{-1}(x)$ is finite for each point x in X .

13-6. THEOREM (Nagata [6]). A metric space X is σ_0 if and only if for every sequence of disjoint pairs of closed sets of $X, C_1, C_1'; C_2, C_2'; \dots$, there are closed sets $B_1, B_2, \dots$ such that:

- i. B_i separates C_i and C_i' for each i
- ii. $\{B_i\}$ is point-finite

[†] Metric-characterizing dimension was studied in detail by Nagata. In this book we merely list theorems without proof. For further information see "Modern Dimension Theory," Nagata [6]. Our Theorems 13-1, 13-3, 13-5, and 13-6 are stated without proof. The present theorem is due to Nagata [3].

13-7. PROBLEM.† Let X be a compact metric space. Is $X \sigma_0$ whenever X is σ_w ?

13-8. PROBLEM (Tumarkin's problem). Let X be a compact metric space with $\dim X = \infty$. Does X contain for every n a closed subset Y_n with $\dim X_n = n$?

It is to be noted that when X is σ_0 this problem is solved affirmatively.‡

13-9. EXAMPLE (Hurewicz). There is a separable metric space H with $\dim H = \infty$ containing no subset with positive dimension. Assume the continuum hypothesis. Let I^ω be the Hilbert cube. Let $\{X_\alpha\}$ be the aggregate of all nonempty G_δ -sets in I^ω with $\dim X_\alpha < \infty$. By our assumption $\{X_\alpha\}$ can be indexed as $\{X_\alpha: \alpha < \omega_1\}$, where ω_1 is the first uncountable ordinal. The construction of the space H is by transfinite induction. Let x_0 be a point of X_0 . Let x_α be a point of $X_\alpha - \bigcup_{\beta < \alpha} X_\beta$. When $X_\alpha - \bigcup_{\beta < \alpha} X_\beta = \phi$, we do not select a point from H_α . H is the aggregate of all such points x_α .

ASSERTION 1. $\dim H = \infty$.

PROOF. If $\dim H < \infty$, there is some X_α with $H \subset X_\alpha$ and $\dim H = \dim X_\alpha$ by Theorem 12-15. Since $\bigcup_{\beta \leq \alpha} X_\beta$ is σ_0 and I^ω is not σ_0 (see Hurewicz and Wallman [1]), there is a point p in

$$I^\omega - \bigcup_{\beta \leq \alpha} X_\beta$$

Since $p = X_\gamma$ for some γ , there is a first set X_δ such that

$$X_\delta - \bigcup_{\beta \leq \delta} X_\beta \neq \phi$$

Then a point x_δ from $X_\delta - \bigcup_{\beta \leq \alpha} X_\beta$ is in H , which is a contradiction. $\square$

ASSERTION 2. If a nonempty set S of H has $\dim S < \infty$, then $\dim S = 0$.

PROOF. By Theorem 12-15, there exists some X_α with $S \subset X_\alpha$ and $\dim S = \dim X_\alpha$. Since

$$S \subset H \cap X_\alpha \subset H \cap \left(\bigcup_{\beta \leq \alpha} X_\beta \right)$$

and $H \cap \left(\bigcup_{\beta \leq \alpha} X_\beta \right)$ consists of countable points, $\dim S = 0$. $\square$

† The importance of this problem was emphasized by Alexandroff [1].

‡ See Nagami and Roberts [1]. After this book had been completed, Henderson [1] constructed a wonderful space which gives a negative answer for Tumarkin's problem.

13-10. LEMMA (Morita and Hanai [1]). Let X and Y be metric spaces and f a closed mapping of X onto Y , then $B(f^{-1}(y))$ is compact for every point y in Y .

PROOF. Suppose that there existed a point y in Y such that $B(f^{-1}(y))$ were not compact. Then there would exist a sequence $x_1, x_2, \dots$ of points in $B(f^{-1}(y))$ such that $\{x_i\}$ would be discrete. Let $\{U_i\}$ be a discrete open collection of X with $x_i \in U_i$ for each i . Let $\{V_i\}$ be an open neighborhood base of y . Choose a point x'_i from $U_i \cap f^{-1}(V_i) - f^{-1}(y)$ for each i . Since $\{x'_i\}$ is discrete, $Y - f(\{x'_i\}) = V$ is an open neighborhood of y . Then there would exist a j such that $V_j \subset V$, which implies $x'_j \notin f^{-1}(V_j)$, which is a contradiction. $\square$

13-11. LEMMA. Let X and Y be metric spaces and f a closed mapping of X onto Y such that $f^{-1}(y)$ is compact and is not dense-in-itself[†] for any point y in Y . If X is σ_0 , then Y is.

PROOF. Let $\cup \mathcal{U}_i$ be a σ -discrete base of X , where $\mathcal{U}_i = \{U_\alpha : \alpha \in A_i\}$ and each $\mathcal{U}_i$ is discrete. By the condition every $f^{-1}(y)$ has an isolated point $x(y)$. There exists an $\alpha(y) \in \cup A_i$ such that

$$x(y) = \bar{U}_{\alpha(y)} \cap f^{-1}(y)$$

For each $\alpha \in \cup A_i$, let

$$Y_\alpha = \{y \in Y : \alpha(y) = \alpha\}, \quad Y_i = \cup \{Y_\alpha : \alpha \in A_i\}$$

Then $Y = \cup Y_i$. For each $\alpha \in \cup A_i$, let

$$X_\alpha = \{x(y) : y \in Y_\alpha\}, \quad X_i = \{x(y) : y \in Y_i\}$$

Then $f(X_\alpha) = Y_\alpha$ and $f(X_i) = Y_i$. Since $f|_{\bar{U}_\alpha}$ is closed and

$$\bar{U}_\alpha \cap f^{-1}(Y_\alpha) = X_\alpha$$

f maps X_α onto Y_α homeomorphically. Hence each Y_α is σ_0 .

Let us prove that $f(\bar{\mathcal{U}}_i)$ is locally finite in Y . For every point y in Y , $f^{-1}(y)$ meets at most finite elements of $\mathcal{U}_i$ by the compactness of $f^{-1}(y)$ and the discreteness of $\mathcal{U}_i$. Set

$$V = Y - \cup \{f(\bar{U}_\alpha) : \alpha \in A_i, \bar{U}_\alpha \cap f^{-1}(y) = \emptyset\}$$

[†] A set S is *dense-in-itself* if every point x in S is an accumulating point of $S - \{x\}$.

Then V is an open neighborhood of y since f is closed and $\mathcal{U}_i$ is discrete. Since V meets at most a finite number of elements of $f(\overline{\mathcal{U}_i})$, $f(\overline{\mathcal{U}_i})$ is locally finite in Y . Therefore $\{f(X_\alpha) = Y_\alpha : \alpha \in A_i\}$ is locally finite in Y . Hence Y_i is locally σ_0 . By an argument analogous to the proof of Corollary 11-8 we can show that Y_i is σ_0 . Hence Y itself is σ_0 . $\square$

13-12. THEOREM (Nagami [10], compare Arhangel'skii [1]). Let X be a metric σ_0 -space, Y a metric, uncountable dimensional space and f a closed mapping of X onto Y , then Y contains an uncountable dimensional subset Y_0 such that for every $y \in Y_0$, $B(f^{-1}(y))$ is dense-in-itself and nonempty.

PROOF. Let Y_1 be the aggregate of all points y in Y with $B(f^{-1}(y)) = \emptyset$. For each y in Y_1 select a point $x(y)$ from $f^{-1}(y)$. Let $X_1 = \{x(y) : y \in Y_1\}$. Then X_1 is closed in $f^{-1}(Y_1)$ since $\{x(y)\}$ is a discrete collection in $f^{-1}(Y_1)$. Therefore, $f|X_1$ is a one-one, closed mapping and hence a homeomorphism of X_1 onto Y_1 . Since $\dim X_1 \leq 0$, $\dim Y_1 \leq 0$.

Let Y_2 be the aggregate of all points y in $Y - Y_1$ such that $B(f^{-1}(y)) (\neq \emptyset)$ is not dense-in-itself. Set

$$X_2 = \cup \{B(f^{-1}(y)) : y \in Y_2\}$$

Since

$$X_2 = f^{-1}(Y_2) - \cup \{\text{Int } f^{-1}(y) : y \in Y_2\}$$

X_2 is closed in $f^{-1}(Y_2)$. Hence, $f_1 = f|X_2$ is a closed mapping of X_2 onto Y_2 . Since

$$f_1^{-1}(y) = B(f^{-1}(y)) \quad \text{for every } y \in Y_2$$

$f_1^{-1}(y)$ is compact by Lemma 13-10. By Lemma 13-11, Y_2 is σ_0 . If we set

$$Y_0 = Y - Y_1 \cup Y_2$$

Y_0 satisfies the condition. $\square$

13-13. LEMMA.[†] Let X be a metric space, $X_1, X_2, \dots$, a sequence of subsets of X with $\dim X_i \leq 0$ for each i and $C_1, C_1'; C_2, C_2'; \dots$ a sequence of disjoint pairs of closed sets of X . Let L be a countable subset

[†] This is a special case of a lemma of Nagata [4, Lemma 4.1].

of the unit interval. Then there exist open sets $U(i, r)$, $i = 1, 2, \dots$, $r \in L$ such that:

- i. $C_i \subset U(i, r) \subset \overline{U(i, r)} \subset X - C'_i$ for each i and r
- ii. $\overline{U(i, r)} \subset U(i, r')$ for each i , whenever $r < r'$
- iii. $\text{ord}_p\{B(U(i, r))\} \leq n - 1$ whenever $p \in X_n$

PROOF (by induction). Enumerate all (i, r) as

$$(i_1, r_1), (i_2, r_2), \dots$$

By Proposition 12-7, there exists an open set $U(i_1, r_1)$ such that

$$C_{i_1} \subset U(i_1, r_1) \subset \overline{U(i_1, r_1)} \subset X - C'_{i_1}$$

$$B(U(i_1, r_1)) \cap X_1 = \phi$$

Make the induction assumption that there exist open sets $U(i_j, r_j)$, $j < k$, satisfying the following conditions:

- i. $C_{i_j} \subset U(i_j, r_j) \subset \overline{U(i_j, r_j)} \subset X - C'_{i_j}$, $j < k$
- ii. $\overline{U(i_j, r_j)} \subset U(i_s, r_s)$ if $j, s < k$, $i_j = i_s$ and $r_j < r_s$
- iii. $\text{ord}_p\{B(U(i_j, r_j)): j < k\} \leq n - 1$ if $p \in X_n$, $n = 1, 2, \dots$

Set $U(i_j, r_j) = U_j$ for brevity. Set $B = \cup_{j < k} B(U_j)$,

$$H_n = \{x \in X_n \cap B: \text{ord}_x\{B(U_j): j < k\} = n - 1\}, \quad n = 1, 2, \dots$$

Since $\dim X_n \leq 0$, $\dim H_n \leq 0$ for each n . If we set

$$H = \bigcup_{n=1}^{\infty} H_n$$

then for every n , $\cup_{i \leq n} H_i$ is open in H . For let p be a point in $\cup_{i \leq n} H_i$, then $p \in \cup_{i \leq n} X_i$. Hence $\text{ord}_p\{B(U_j)\} \leq n - 1$. Set

$$B' = \{x \in B: \text{ord}_x\{B(U_j)\} \geq n\}$$

Then B' is closed and $p \notin B'$. Since $B' \subset \cup_{i > n} H_i$, $X - B'$ is an open neighborhood of p which does not meet $\cup_{i > n} H_i$. Thus, $\cup_{i \leq n} H_i$ is open in H . Therefore, for every n , $H_n - \cup_{i=1}^{n-1} H_i$ is a zero dimensional F_σ -set of H . By the sum theorem, $\dim H \leq 0$.

By Proposition 12-7, there exists an open set $U_k (= U(i_k, r_k))$ such that

$$B(U_k) \cap H = \phi, \quad C_{i_k} \subset U_k \subset \overline{U_k} \subset X - C'_{i_k}$$

Furthermore, we can assume that U_k satisfies the following conditions

- i. When $i_k = i_j, j < k$ and $r_j < r_k$, then $\bar{U}_j \subset U_k$
- ii. When $i_k = i_j, j < k$ and $r_j > r_k$, then $\bar{U}_k \subset U_j$

Thus the induction is completed. Consequently $\{U(i_j, r_j): j = 1, 2, \dots\}$ satisfies the required conditions. $\square$

13-14. THEOREM (Nagami [8]). Let X be a nonempty metric σ_0 -space. Then there exists a sequence $X_1, X_2, \dots$, of subsets of X such that: (i) $X_1 \subset X_2 \subset \dots$, (ii) $\dim X_i \leq 0$ for each i , and (iii) $\cup X_i = X$.

PROOF. By Theorems 1-7 and 2-22 and Corollary 2-8 there exist collections

$$\mathcal{U}_{ij} = \{U_\lambda: \lambda \in \Lambda_{ij}\}, \quad \mathcal{F}_{ij} = \{F_\lambda: \lambda \in \Lambda_{ij}\}, \quad i, j = 1, 2, \dots,$$

of subsets of X which satisfy the following conditions:

1. Each $\mathcal{U}_{ij}$ is a discrete open collection
2. Each element of each $\mathcal{F}_{ij}$ is a nonempty closed set of X
3. $F_\lambda \subset U_\lambda$ for each $\lambda \in \Lambda$, where $\Lambda = \cup_{i,j} \Lambda_{ij}$
4. $\mathcal{F}_i = \{F_\lambda: \lambda \in \Lambda_i\}$ covers X for each i , where $\Lambda_i = \cup_j \Lambda_{ij}$
5. Each $\mathcal{U}_i = \{U_\lambda: \lambda \in \Lambda_i\}$ is locally finite
6. $\text{mesh } \mathcal{U}_i < 1/i$ for each i

Set

$$U_{ij} = \cup \{U_\lambda: \lambda \in \Lambda_{ij}\}, \quad F_{ij} = \cup \{F_\lambda: \lambda \in \Lambda_{ij}\}$$

Let L be a countably infinite subset of the unit interval. By the lemma there exists a collection

$$\{U_{ijr}: i, j = 1, 2, \dots, r \in L\}$$

of open sets of X which satisfies the following conditions:

- i. $F_{ij} \subset U_{ijr} \subset \bar{U}_{ijr} \subset U_{ij}$ for each i, j and r
- ii. $\bar{U}_{ijr} \subset U_{ijs}$ for each i, j whenever $r < s$
- iii. $\{B(U_{ijr})\}$ is point-finite

Let $L = \{r_1, r_2, \dots\}$ and

$$X_i = X - \cup \{B(U_{jkr}): j, k = 1, 2, \dots, r \notin \{r_1, \dots, r_i\}\}$$

Then the restriction of

$$\{U_{jkr}: j, k = 1, 2, \dots, r \notin \{r_1, \dots, r_i\}\}$$

to X_i , say $\mathcal{V}_i$, is a σ -discrete base of X_i such that the relative boundary of each element of $\mathcal{V}_i$ is empty. Hence by Theorem 12-13, $\dim X_i \leq 0$. It is evident that $X_1 \subset X_2 \subset \dots$.

To see $\cup X_i = X$ let x be an arbitrary point of X . By condition (iii) there exists an i such that

$$x \notin B(U_{jkr}) \quad \text{for any } j, k \text{ and any } r \in L - \{r_1, \dots, r_i\}$$

Hence $x \in X_i$ for this i . $\square$

13-15. THEOREM (Nagami [8]). Let X be a nonempty metric space. Then there exists a transfinite sequence $\{X_\alpha: \alpha < \omega_1\}$ of subsets of X such that:

- i. $X_\alpha \subset X_\beta$ whenever $\alpha < \beta$
- ii. $\dim X_\alpha \leq 0$ for each α
- iii. $\cup X_\alpha = X$

PROOF. We continue to use the notions in the preceding theorem. Let I be the unit interval and I_α , $\alpha < \omega_1$, the subsets of I given in Example 11-16 such that $I_\alpha \subset I_\beta$ whenever $\alpha < \beta$, $\dim I_\alpha = 0$ for each α , and $\cup I_\alpha = I$. Let $f_{ij}: X \rightarrow I$ be a mapping such that

$$\begin{aligned} f_{ij}(x) &= 0, & x \in X - U_{ij} \\ f_{ij}(x) &= 1, & x \in F_{ij} \end{aligned}$$

Put

$$\sigma(x, y) = \rho(x, y) + \sum_{i,j} (|f_{ij}(x) - f_{ij}(y)|) / 2^{i+j}$$

where ρ is the preassigned metric on X . Then σ is a metric equivalent to ρ such that

$$\sigma(F_{ij}, X - U_{ij}) = d_{ij} > 0$$

for every i, j . For any t with $0 < t \leq 1$ set

$$H'(i, j, t) = \{x: \sigma(x, F_{ij}) = t\}$$

$$H(i, j, t) = H'(i, j, t) \cap U_{ij}$$

$$X_\alpha = X - \cup \{H(i, j, t): i, j = 1, 2, \dots, 0 < t \in I - I_\alpha\}$$

Then $\{X_\alpha: \alpha < \omega_1\}$ satisfies the required conditions. The inequalities $X_0 \subset X_1 \subset \dots \subset X_\alpha \subset \dots$ arise from the condition $I_0 \subset I_1 \subset \dots \subset I_\alpha \subset \dots$.

Let us prove $\dim X_\alpha \leq 0$. Since $I - I_\alpha$ is dense in I , we can pick a number t_{ij} from $I - I_\alpha$ with $0 < t_{ij} < d_{ij}$ for every i and j . Set

$$V_{ij} = \{x: \sigma(x, F_{ij}) < t_{ij}\}$$

$$\mathcal{V}_{ij} = \{V_\lambda = U_\lambda \cap V_{ij}: \lambda \in \Lambda_{ij}\}$$

By (4) and (5) of Theorem 13-14, $\mathcal{V}_i = \{V_\lambda: \lambda \in \Lambda_i\}$ is a locally finite open covering of X for each i . By (6) of the same theorem and by the fact that $\mathcal{V}_i$ refines $\mathcal{U}_i$, the mesh of $\mathcal{V}_i$ with respect to ρ is less than $1/i$. Hence, $\{V_\lambda: \lambda \in \Lambda\}$ is a σ -locally finite base of X . Let λ be an arbitrary index from Λ_{ij} . Since

$$B(V_\lambda) = \bar{V}_\lambda - V_\lambda \subset \bar{V}_{ij} - V_{ij} \subset H'(i, j, t_{ij}) \subset U_{ij}$$

$B(V_\lambda)$ does not meet X_α . Hence by Theorem 12-13, $\dim X_\alpha \leq 0$.

To prove $\cup X_\alpha = X$ let x be an arbitrary point of X . Set

$$\sigma_{ij} = \sigma(x, F_{ij})$$

$$L = \{\sigma_{ij}: 0 < \sigma_{ij} \leq 1, x \in U_{ij} - F_{ij}\}$$

Since L is countable, there exists an α with $L \subset I_\alpha$. If $0 < t \in I - I_\alpha$ and $\sigma_{ij} \in L$, then $x \notin H(i, j, t)$. If either $x \in X - U_{ij}$, $x \in F_{ij}$, or $\sigma_{ij} > 1$, then $x \notin H(i, j, t)$ for any t . Therefore $x \notin H(i, j, t)$ for any i, j and t with $0 < t \in I - I_\alpha$, which implies $x \in X_\alpha$. $\square$

When X is a nonempty separable metric space, we have a simple proof (by Proizvoloff in Smirnov [2]) of the theorem: Embed X into the Hilbert cube I^ω . I^ω is essentially the same space as a countable product $I \times I \times \dots$ of I . Set

$$X_\alpha = X \cap (I_\alpha \times I_\alpha \times \dots)$$

By Corollary 12-4, $\dim X_\alpha = 0$. $\square$

14. Sequences of Coverings Defining Dimension

14-1. THEOREM (Nagami and Roberts [1], [3]). A space X is a metric space with $\dim X \leq n$ if and only if there exists a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \dots$ of open coverings of X such that:

1. $\{S(x, \mathcal{U}_i^\Delta)\}$ is a neighborhood base of x for each $x \in X$
2. $\text{ord } \mathcal{U}_i \leq n + 1$ for each i

PROOF. The necessity is clear by Theorem 9-13. Suppose that the conditions are satisfied, then X is metrizable by Theorem 5-8. To prove

$\dim X \leq n$, let us start by taking an arbitrary finite open covering $\mathcal{G}$ of X . Let

$$\mathcal{U}_i = \{U(\alpha_i) : \alpha_i \in A_i\}, \quad i = 1, 2, \dots$$

Define a refine transformation $f_i^{i+1} : A_{i+1} \rightarrow A_i$ such that $f_i^{i+1}(\alpha_{i+1}) = \alpha_i$ yields $U(\alpha_{i+1}) \subset U(\alpha_i)$. For each pair $i < j$, let

$$f_i^j = f_i^{i+1} \dots f_{j-1}^j$$

and let f_i^i be the identity transformation. Set

$$X_i = \cup \{U(\alpha_i) : S(U(\alpha_i), \mathcal{U}_i) < \mathcal{G}\}$$

Then by condition (1), $\{X_i\}$ is an open covering of X . Set $X_0 = \phi$. Set

$$B_i = \{\alpha_i : U(\alpha_i) \cap X_i \neq \phi\}$$

$$C_i = \{\alpha_i \in B_i : U(\alpha_i) \cap (\bigcup_{j < i} X_j) = \phi\}$$

$$D_i = \{\alpha_i \in B_i : U(\alpha_i) \cap (\bigcup_{j < i} X_j) \neq \phi\}$$

Then $B_1 = C_1$ and B_i is the disjoint sum of C_i and D_i .

For every pair $i < j$ and every $\alpha_i \in C_i$ set

$$D_j(\alpha_i) = \{\alpha_j : f_i^j(\alpha_j) = \alpha_i, f_k^j(\alpha_j) \in D_k \text{ for } k = i+1, \dots, j\}$$

Then

$$f_k^j(D_j(\alpha_i)) \subset D_k(\alpha_i), \quad i < k \leq j$$

$$D_j = \cup \{D_j(\alpha_i) : \alpha_i \in C_i, \quad i < j\}$$

For every $\alpha_i \in C_i$ let

$$V(\alpha_i) = (U(\alpha_i) \cap X_i) \cup (\cup \{U(\alpha_j) \cap X_j : \alpha_j \in D_j(\alpha_i), \quad i < j\})$$

Let us show that

$$\mathcal{V} = \{V(\alpha_i) : \alpha_i \in C_i, \quad i = 1, 2, \dots\}$$

is an open covering of X with $\text{ord } \mathcal{V} \leq n+1$ and with $\mathcal{V} < \mathcal{G}$, which will prove $\dim X \leq n$.

Let x be an arbitrary point of X . Since $X_0 = \phi$, there exists a j with $x \in X_j - \cup_{i < j} X_i$. Pick an $\alpha_j \in B_j$ with $x \in U(\alpha_j)$. When $\alpha_j \in C_j$,

$$x \in U(\alpha_j) \cap X_j \subset V(\alpha_j)$$

When $\alpha_j \in B_j - C_j = D_j$, there exists an i with $i < j$ such that $\alpha_j \in D_j(\alpha_i)$ for some $\alpha_i \in C_i$. Then

$$x \in U(\alpha_j) \cap X_j \subset V(\alpha_i)$$

Thus $\mathcal{V}$ is an open covering of X .

Let $V(\alpha_i)$ be an arbitrary element of $\mathcal{V}$, where $\alpha_i \in C_i$. Since $U(\alpha_i) \supset V(\alpha_i) \supset U(\alpha_i) \cap X_i \neq \phi$, there exists a $\beta_i \in A_i$ such that

$$U(\beta_i) \cap U(\alpha_i) \neq \phi \quad \text{and} \quad S(U(\beta_i), \mathcal{U}_i) < \mathcal{G}$$

Thus $U(\alpha_i)$ refines $\mathcal{G}$ and hence $V(\alpha_i)$ does. Therefore $\mathcal{V}$ refines $\mathcal{G}$.

To prove $\text{ord } \mathcal{V} \leq n+1$ assume the contrary. Then there would exist a point x in X and $n+2$ indices $\alpha^1, \dots, \alpha^{n+2}$ such that

$$\begin{aligned} \alpha^i &\in C_{m_i}, & i &= 1, \dots, n+2 \\ x &\in V(\alpha^i), & i &= 1, \dots, n+2 \end{aligned}$$

Let k be the integer such that

$$x \in X_k - \bigcup_{j < k} X_j$$

Since $V(\alpha^i) \subset U(\alpha^i)$ and

$$U(\alpha^i) \cap (\bigcup \{X_j : j < m_i\}) = \phi$$

every m_i cannot be greater than k . When $m_i < k$, there exist a $j(i) \geq k$ and a $\beta^i \in D_{j(i)}(\alpha^i)$ such that $x \in U(\beta^i)$. If we set $\gamma^i = f_k^{j(i)}(\beta^i)$, then $x \in U(\gamma^i)$. These γ^i 's are different from each other since $m_i < k$.

When $m_i = k$, $x \in U(\alpha^i)$ because of the inequality $V(\alpha^i) \subset U(\alpha^i)$. We now have $U(\alpha^i)$'s for i with $m_i = k$ and $U(\gamma^i)$'s for i with $m_i < k$, which contain x at the same time. Hence

$$\text{ord}_x \mathcal{U}_k \geq n+2$$

which is a contradiction. $\square$

14-2. COROLLARY (Vopenka [2]). A metric space X has $\dim X \leq n$ if and only if X has a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \dots$ of open coverings of X such that

- i. $\text{ord } \mathcal{U}_i \leq n+1$ for each i
- ii. $\text{mesh } \mathcal{U}_i \rightarrow 0$ as $i \rightarrow \infty$

This is a direct consequence of the theorem.

14-3. REMARK. In Theorem 12-6 we have a sequence of coverings characterizing dimension. The sequence stated in this theorem is not assumed to be a sequence of open coverings. In this direction, the following is the most general characterization (Nagami [4]):

A space X is a metric space with $\dim X \leq n$ if and only if X has a sequence of coverings $\mathcal{F}_1, \mathcal{F}_2, \dots$ such that:

- i. For each point $x \in X$ and each neighborhood U of x there is an i with $S(x, \mathcal{F}_i) \subset U$
- ii. $\mathcal{F}_{i+1}$ is a cushioned refinement of $\mathcal{F}_i$ for each i
- iii. $\liminf \text{ord}_x \mathcal{F}_i \leq n + 1$ for each $x \in X$

Here $\mathcal{F}_{i+1} = \{F_\alpha : \alpha \in A_{i+1}\}$ is a *cushioned refinement*[†] of $\mathcal{F}_i = \{F_\alpha : \alpha \in A_i\}$ if there is a refine transformation $f: A_{i+1} \rightarrow A_i$ such that for each subset B of A_{i+1} , the closure of the sum of U_α 's, $\alpha \in B$, is contained in the sum of U_β 's, $\beta \in f(B)$.

14-4. THEOREM (Nagami [3]). Let (X, d) be a metric space with $\dim X \leq n$ whose metric d is generated, as in Theorem 2-16, by a sequence $\mathcal{U}_1^* > \mathcal{U}_2^* > \dots$ of open coverings of X such that

$$\text{ord } \mathcal{U}_i \leq n + 1 \text{ for each } i.$$

Then the completion (X^*, d^*) of (X, d) has $\dim X^* \leq n$.

PROOF. Set

$$\mathcal{U}_i = \{U_\alpha : \alpha \in A_i\}, \quad i = 1, 2, \dots$$

For each $U_\alpha, \alpha \in \cup A_i$, let

$$U_\alpha^* = X^* - \overline{X - U_\alpha}$$

It is evident that $U_\alpha^* \cap X \supset U_\alpha$. To prove $U_\alpha^* \cap X = U_\alpha$ assume that there exists a point x in $U_\alpha^* \cap X - U_\alpha$. Then

$$\begin{aligned} x \in U_\alpha^* \cap (X - U_\alpha) &= (X^* - \overline{X - U_\alpha}) \cap (X - U_\alpha) \\ &\subset (X^* - (X - U_\alpha)) \cap (X - U_\alpha) \end{aligned}$$

which is a contradiction. Hence $U_\alpha^* \cap X = U_\alpha$.

Let us show that $\{U_\alpha^* : \alpha \in A_i\}$ covers X^* . For each $\beta \in A_{i+1}$ there is an $\alpha(\beta) \in A_i$ such that

$$S(U_\beta, \mathcal{U}_{i+1}) \subset U_{\alpha(\beta)}$$

By the inequality (14) in 2-16

$$d(U_\beta, X - U_{\alpha(\beta)}) > 0$$

[†] This notion is due to Michael [4]. Compare Lemma 22-6.

Hence $\bar{U}_\beta \cap \overline{X - U_{\alpha(\beta)}} = \phi$, which implies $\bar{U}_\beta \subset U_{\alpha(\beta)}^*$. Since $\{\bar{U}_\beta: \beta \in A_{i+1}\}$ covers X^* , $\{U_{\alpha(\beta)}^*: \beta \in A_{i+1}\}$ covers X^* . Therefore, $\{U_\alpha^*: \alpha \in A_i\}$ is an open covering of X^* .

To see that $\text{ord}\{U_\alpha^*: \alpha \in A_i\} \leq n+1$, assume the contrary. Then there would exist $n+2$ indices $\alpha_1, \dots, \alpha_{n+2}$ in A_i such that

$$U^* = U_{\alpha_1}^* \cap \dots \cap U_{\alpha_{n+2}}^* \neq \phi$$

Then

$$U^* \cap X = U_{\alpha_1} \cap \dots \cap U_{\alpha_{n+2}} \neq \phi$$

which is a contradiction. By Theorem 14-1, we have $\dim X^* \leq n$. $\square$

When X is separable, we can start with a sequence $\mathcal{U}_i$ of finite open coverings. Then (X^*, d^*) is a totally bounded, complete metric space and hence is a compact metric space. Thus we have Hurewicz's dimension-preserving compactification.

There is another way of providing a dimension-preserving completion. If (X, ρ) is a metric space with $\dim X \leq n$ accompanied by a Nagata metric ρ , as presented in Theorem 13-1. Then it is easy to see that the metric ρ^* in the completion (X^*, ρ^*) of (X, ρ) is also a Nagata metric on X^* . Hence, $\dim X^* \leq n$. It can be proved that Nagata's metric satisfies the condition in this theorem.

15. Open Mappings and Dimension

15-1. THEOREM.[†] A nonempty metric space X is always the image of a metric space A with $\dim A = 0$ under an open mapping f such that $f^{-1}(x)$ is compact for every point x in X .

PROOF. Let

$$\mathcal{U}_i = \{U(\alpha): \alpha \in A_i\}$$

be a locally finite open covering of X with $\text{mesh } \mathcal{U}_i < 1/i$. Let A be the aggregate of points $a = \langle \alpha_1, \alpha_2, \dots \rangle$ in the product space ΠA_i , considering A_i 's as discrete spaces, such that $\cap U(\alpha_i) \neq \phi$. Here $\cap U(\alpha_i)$ is a single

[†] This is due to Nagami [6]. It is to be noted that when X is merely a paracompact Hausdorff space, X is the image of a paracompact Hausdorff space A with $\dim A = 0$ under f with the same property as in the theorem. This was also proved by Nagami [6].

point and A is a metric space with $\dim A = 0$ by Corollary 12-4. Define $f: A \rightarrow X$ by

$$f(a) = \cap U(\pi_i(a))$$

where $\pi_i: A \rightarrow A_i$ are the projections. Clearly f is onto. Since for each i and each $\alpha \in A_i$ $f(\pi_i^{-1}(\alpha)) = U(\alpha)$, f is open and continuous. Let x be an arbitrary point of X . Set

$$B_i = \{\alpha: x \in U(\alpha) \in \mathcal{U}_i\}, \quad i = 1, 2, \dots$$

Then every B_i is a nonempty finite set of indices. Since $f^{-1}(x) = \Pi B_i$, $f^{-1}(x)$ is compact. $\square$

15-2. COROLLARY. A nonempty separable metric space X is always the image of a separable metric space A with $\dim A = 0$ under an open mapping f such that $f^{-1}(x)$ is compact for every point x in X .

PROOF. For this case we can impose an additional condition upon the sequence $\mathcal{U}_i$ in the above proof: Every $\mathcal{U}_i$ consists of countable elements. Then A has to be separable. $\square$

15-3. THEOREM. If f is an open transformation of a metric space X onto a metric space Y , then there exists a dense subset X_1 of X such that:

- i. $f_1 = f|X_1$ is open
- ii. $f_1(X_1) = Y$
- iii. For every $y \in Y$, $f_1^{-1}(y)$ is a σ -discrete family of points (which implies that $\dim f_1^{-1}(y) = 0$)

PROOF. Let $\mathcal{U}_i = \{U_\alpha: \alpha \in A_i\}$, $i = 1, 2, \dots$, be a sequence of locally finite open coverings of X with mesh $\mathcal{U}_i < 1/i$. For each $y \in Y$ set

$$A_i(y) = \{\alpha \in A_i: U_\alpha \cap f^{-1}(y) \neq \emptyset\}$$

For each $\alpha \in A_i(y)$ pick a point $x(y, \alpha)$ from $U_\alpha \cap f^{-1}(y)$. Set

$$X_1 = \{x(y, \alpha): \alpha \in \bigcup_{i=1}^{\infty} A_i(y), \quad y \in Y\}$$

Then we can show X_1 is the desired subset.

Clearly f_1 maps X_1 onto Y and X_1 is dense in X . For every point $y \in Y$,

$$f_1^{-1}(y) = \bigcup_i \{x(y, \alpha): \alpha \in A_i(y)\}$$

where each $\{x(y, \alpha): \alpha \in A_i(y)\}$ is a discrete family of points. Hence $f_1^{-1}(y)$ is a σ -discrete family of points.

To prove f_1 is open, assume the contrary. Then there exists a point x in X and a positive number ε such that $f_1(S_\varepsilon(x) \cap X_1)$ would not be a neighborhood of $f_1(x)$. Pick a sequence of points $y_1, y_2, \dots$ in $Y - f_1(S_\varepsilon(x) \cap X_1)$ with $\lim y_i = f_1(x)$. Let j be a positive integer with $1/j \leq \varepsilon/2$. Since $f(S_{\varepsilon/2}(x))$ is an open neighborhood of $f_1(x)$, there exists a positive integer $k \geq j$ such that $y_k \in f(S_{\varepsilon/2}(x))$. Hence

$$f^{-1}(y_k) \cap S_{\varepsilon/2}(x) \neq \emptyset$$

There exists an index β in $A_k(y_k)$ such that

$$U_\beta \cap f^{-1}(y_k) \cap S_{\varepsilon/2}(x) \neq \emptyset$$

Then

$$\begin{aligned} d(x, x(y_k, \beta)) &< \varepsilon/2 + 1/k \\ &\leq \varepsilon/2 + 1/j \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

where d is the distance in X . Hence $x(y_k, \beta) \in S_\varepsilon(x)$. Therefore

$$y_k = f_1(x(y_k, \beta)) \in f_1(S_\varepsilon(x) \cap X_1)$$

which is a contradiction. $\square$

15-4. COROLLARY. If f is an open transformation of a separable metric space X onto a separable metric space Y , then there exists a dense subset X_1 of X such that

- i. $f_1 = f|X_1$ is open
- ii. $f_1(X_1) = Y$
- iii. $f_1^{-1}(y)$ is countable for each point y in Y

This is evident from the above argument because of the fact that a discrete family of points in a separable metric space consists of countable points.

15-5. LEMMA (Nagami [2]). Let X be a metric space and $\{F_\alpha: \alpha < \eta\}$ a well-ordered covering of X which satisfies the following conditions:

- i. For each $\alpha < \eta$, $H_\alpha = \cup_{\beta < \alpha} F_\beta$ is closed
- ii. For each $\alpha < \eta$, $\dim F_\alpha \leq n$

Then $\dim X \leq n$.

PROOF. Set

$$F_{\alpha i} = H_{\alpha+1} - S_{1/i}(H_{\alpha})$$

Then $F_{\alpha i}$ is a closed set with $\dim F_{\alpha i} \leq n$. Since $\{F_{\alpha i} : \alpha < \eta\}$ is a discrete collection for each i ,

$$\dim \cup \{F_{\alpha i} : \alpha < \eta\} \leq n$$

Set

$$F_i = \cup \{F_{\alpha i} : \alpha < \eta\}$$

Since $\cup F_i = X$, $\dim X \leq n$ by the sum theorem. $\square$

15-6. THEOREM. Let X and Y be metric spaces and f an open-closed mapping of X onto Y such that $B(f^{-1}(y))$ is not dense-in-itself for any point y in Y , then $\dim Y \leq \dim X$.

PROOF. Let Y_0 be the aggregate of all points y in Y such that $\text{Int } f^{-1}(y) \neq \phi$. Then every point y in Y_0 is open. Hence $\dim Y_0 \leq 0$. Set

$$Y_1 = Y - Y_0, \quad X_1 = f^{-1}(Y_1), \quad f_1 = f|X_1$$

Then $B(f_1^{-1}(y)) = f_1^{-1}(y)$ for every point y in Y_1 and f_1 is an open-closed mapping of X_1 onto Y_1 . Since Y_0 is open in Y , X_1 is closed in X . In the following we confine ourselves to the spaces X_1 and Y_1 and the mapping f_1 .

Let

$$\mathcal{U}_i = \{U_{\alpha} : \alpha \in A_i\}, \quad i = 1, 2, \dots$$

be a sequence of locally finite (relatively) open coverings of X_1 with mesh $\mathcal{U}_i \rightarrow 0$. Set for each $\alpha \in \cup A_i$

$$V_{\alpha} = \{x \in U_{\alpha} : f_1^{-1}(f_1(x)) \cap U_{\alpha} = x\}$$

To see V_{α} is closed in U_{α} , pick two points x and x' from U_{α} with $x \neq x'$ and $f_1(x) = f_1(x')$. Let $W(W')$ be an open neighborhood of x (respectively, of x') such that $W \cap W' = \phi$ and $W \cup W' \subset U_{\alpha}$. Set

$$U = W \cap f_1^{-1}(f_1(W) \cap f_1(W'))$$

Then U is an open neighborhood of x with $U \cap V_{\alpha} = \phi$, which shows V_{α} is closed in U_{α} . Since U_{α} is F_{σ} , we can let $V_{\alpha} = \cup_{i=1}^{\infty} F_{\alpha i}$, where $F_{\alpha i}$'s are closed sets in X_1 . Since $f_1|F_{\alpha i}$ is a homeomorphism, $f_1(F_{\alpha i})$ is a closed set with

$$\dim f_1(F_{\alpha i}) \leq \dim X$$

Since $\{f_1(F_{\alpha i}): \alpha \in A_j\}$ satisfies the condition of the lemma, its sum H_{ji} satisfies the inequality.

$$\dim H_{ji} \leq \dim X$$

It is to be noted that H_{ji} is closed in Y_1 , since

$$H_{ji} = f_1(\cup \{F_{\alpha i}: \alpha \in A_j\})$$

and $\cup \{F_{\alpha i}: \alpha \in A_j\}$ is closed. Since $\{f_1(V_\alpha): \alpha \in \cup A_i\}$ covers Y_1 ,

$$\cup \{H_{ji}: j, i = 1, 2, \dots\} = Y_1$$

By the sum theorem $\dim Y_1 \leq \dim X$. Since Y_0 is open and Y_1 is closed in Y , $\dim Y \leq \dim X$ again by the sum theorem. $\square$

If, moreover, $\dim f^{-1}(y) \leq 0$ for each $y \in Y$, we have $\dim X = \dim Y$. The reason is the inequality

$$\dim X \leq \dim Y$$

which will be proved in Corollary 21-2. In this connection the existence of metric spaces X, Y with $\dim X > \dim Y$ and an open mapping f of X onto Y with $\dim f^{-1}(y) = 0$ for each $y \in Y$ is interesting. Such a counterexample is given by using Example 9-12 as in the following paragraph.

15-7. EXAMPLE. There exists a dimension-lowering open mapping whose point inverses have dimension 0. Let X be a metric space and a the apex given in Example 9-12. Let $f: X \rightarrow C$ be the projection from a to the Cantor discontinuum C . It is evident that f is a dimension-lowering open mapping such that $\dim f^{-1}(c) = 0$ for each point c in C .

Chapter 4

GAPS BETWEEN DIMENSION FUNCTIONS

A summary of the relations between the three dimensions for normal spaces is as follows: (i) $\text{ind} \leq \text{Ind}$, (ii) $\text{dim} \leq \text{Ind}$, and (iii) there is a normal space with $\text{ind} < \text{dim}$ (See Chapter 2). The present chapter gives several counterexamples showing the gaps between dimension functions. We saw in Theorem 9-10 that the sum theorem for dim holds in normal spaces. In Section 16 we show that the sum theorems for ind and Ind are not true even for compact Hausdorff spaces. In Section 17 we construct for each n a normal space with $\text{ind} = n$ having arbitrarily large dim . In Section 18 we give for each n a compact Hausdorff space with $\text{dim} = n$ having arbitrarily large ind and Ind . Section 19 gives a normal space with $\text{ind} = 0$, $\text{dim} = 1$, and $\text{Ind} = 2$. Thus we see that normality and even compactness do not affect the equalities between dimensions.

16. A Compact Space Missing the Sum Theorem for ind and Ind

16-1. CONSTRUCTION. To each ordinal α less than the first uncountable ordinal ω_1 we correspond a copy I_α of the unit interval $I = [0, 1]$ identifying its left end with α and the right with $\alpha + 1$. Let P be the sum of all such I_α 's and ω_1 . Then the long line P with the usual order topology is a compact Hausdorff space. Let R be the product of P and the Cantor discontinuum C . Let G be an open set of R such that:

- i. $G_1 = G \cap (\omega_1 \times C) \neq \emptyset$
- ii. The least upper bound of G_1 , say $\langle \omega_1, x_0 \rangle$, is a two-sided point

ASSERTION. If $[\alpha, \omega_1] \times x_0$ is not contained in $B(G)$ for any $\alpha < \omega_1$, then there exists some $x_1 \in C$ with $x_1 > x_0$ such that

$$\omega_1 \times [x_0, x_1] \subset B(G)$$

PROOF. There exists a sequence $\langle \omega_1, p_1 \rangle, \langle \omega_1, p_2 \rangle, \dots$ of points in G_1 whose limit is $\langle \omega_1, x_0 \rangle$. Since G is open, for each i there exists a number $\alpha_i < \omega_1$ such that

$$[\alpha_i, \omega_1] \times p_i \subset G$$

Let $\beta = \sup \alpha_i$. Then

$$[\beta, \omega_1] \times x_0 \subset \bar{G}$$

Hence by the given condition there exists a cofinal (in $[0, \omega_1)$) subset T of $[\beta, \omega_1]$ such that

$$T \times x_0 \subset G$$

For each γ in T we can find a number $x_\gamma > x_0$ such that

$$\gamma \times [x_0, x_\gamma] \subset G$$

Since T is cofinal in $[0, \omega_1)$, there exist a subset T_1 of T and a number x_1 in C with $x_1 > x_0$ such that:

- i. T_1 is cofinal in $[0, \omega_1)$
- ii. $x_\gamma > x_1$ for every γ in T_1

Then it is easy to see that $\omega_1 \times [x_0, x_1]$ is contained in the boundary of G . $\square$

16-2. CONSTRUCTION. Let R' be a compact Hausdorff space obtained by the pairwise identification of the terminals of each dropped interval of $\omega_1 \times C$. By this identification $\omega_1 \times C$ passes into the unit interval F , while the set of one-sided points of $\omega_1 \times C$ passes into some set M . Since M is dense in F , it can easily be seen that $\text{ind } R' = 1$.

16-3. ASSERTION. $\text{Ind } R' = 1$.

PROOF. Let H be a closed set of R' and U an open set of R' with $H \subset U$. Since every point in R' has arbitrarily small neighborhoods whose boundaries are compact metric and has $\text{ind} \leq 0$, then every point x in H has an open neighborhood $U(x)$ such that $B(U(x))$ is compact metric and

$$\text{ind } B(U(x)) \leq 0, \quad \overline{U(x)} \subset U$$

Select a finite number of $U(x_i)$'s with $H \subset \cup U(x_i) \subset U$. Since $\text{ind } B(U(x_i)) = \text{Ind } B(U(x_i))$ for each i , we have

$$\text{Ind } B(\cup U(x_i)) \leq 0$$

by the sum theorem because $B(\cup U(x_i))$ is compact metric by Lemma 5-4. $\square$

16-4. ASSERTION. $\dim R' = 1$.

PROOF. Since $\dim R' = 0$ implies $\text{Ind } R' = 0$ by Proposition 8-4, it is true that $\dim R' \geq 1$. Since $\dim R' \leq \text{Ind } R'$ by Theorem 10-1, we have $\dim R' = 1$. $\square$

16-5. ASSERTION. If G' is an open set of R' such that

- i. $G_1' = G' \cap F \neq \phi$
- ii. The least upper bound of G_1' is not in M

Then $\text{ind } B(G') \geq 1$.

This is an immediate consequence of Construction 16-1.

16-6. EXAMPLE (Lokucievskii [1]). There exists a compact Hausdorff space S with the following properties:

- 1. $\dim S = 1$
- 2. $\text{ind } S = \text{Ind } S = 2$
- 3. S is the sum of closed subsets S_1 and S_2 such that $\text{ind } S_i = \text{Ind } S_i = 1$ for $i = 1, 2$

Let us consider the compact Hausdorff space $Q = R_1' \cup R_2'$, where R_1' and R_2' are nonintersecting copies of the space R' . Let F_i or M_i be the set lying in R_i' and corresponding respectively to the set F or M in R' , where $i = 1, 2$. Let us select from F_2 a dense subset N of the same ordinal type as M_1 having no common point with M_2 , except the terminals of F_2 . As is well known, by a simple argument, there exists an order preserving mapping g of F_1 onto F_2 , transferring M_1 onto N . Let S be the identification space

$$Z(R_1', F_1, g, R_2')$$

in Definition 4-1. Then S is a compact Hausdorff space. Let h be the natural mapping of Q onto S . Let

$$h(R_1') = S_1 \quad \text{and} \quad h(R_2') = S_2$$

Then $S = S_1 \cup S_2$ and $S_1 \cap S_2 = F_2$. Each S_i is of course closed in S .

ASSERTION 1. $\dim S = 1$.

PROOF. Since S_1 and S_2 are homeomorphic to R' , $\dim S_1 = \dim S_2 = 1$. By the sum theorem $\dim S = 1$. $\square$

ASSERTION 2. $\text{ind } S_i = \text{Ind } S_i = 1$ for $i = 1, 2$.

ASSERTION 3. $\text{ind } S = \text{Ind } S = 2$.

PROOF. Let y be an arbitrary nonterminal point of F_2 and U be an open neighborhood of y in S such that the least upper bound y_0 of $U \cap F_2$ is nonterminal. Then y_0 cannot belong to both $g(M_1)$ and M_2 . Let, for example, $y_0 \notin g(M_1)$. If $O = U \cap S_1$, then by 16-5, $B(O)$, the relative boundary of O in S_1 , has $\text{ind} \geq 1$. Since $B(O) \subset B(U)$, $\text{ind } B(U) \geq \text{ind } B(O) \geq 1$. Therefore $\text{ind } S \geq 2$. Since it is almost evident that $\text{ind } S \leq \text{Ind } S \leq 2$, we have $\text{ind } S = \text{Ind } S = 2$. $\square$

17. A Normal Space with $\text{ind} = n$ Having Large $\dim$

17-1. EXAMPLE. There exists a normal space with $\text{ind} = 0$ having large $\dim$. $\dagger$ Let X be a nonempty metric space and X_α , $\alpha < \omega_1$, be subsets of X such that

$$\begin{aligned} \dim X_\alpha &= 0 && \text{for each } \alpha \\ X_\alpha &\subset X_\beta && \text{whenever } \alpha < \beta \\ \bigcup X_\alpha &= X \end{aligned}$$

The existence of such a transfinite sequence is assured by Theorem 13-15. Let J be the space given in Section 2-20: $J = [0, \omega_1)$. Let J^* be $[0, \omega_1] = J \cup \{\omega_1\}$ with the interval topology. The set M_X is the sum of all subsets $\{\alpha\} \times X_\alpha$ in $J^* \times X$. The topology of M_X is the relative one of the product space $J^* \times X$. Let $\pi': J^* \times X \rightarrow X$ and $\rho': J^* \times X \rightarrow J^*$ be the projections. Set

$$\pi = \pi' \mid M_X \quad \text{and} \quad \rho = \rho' \mid M_X$$

If S is any set in X and T is any set in J , $[\pi^{-1}(S)]_T$ denotes the intersection $\pi^{-1}(S) \cap \rho^{-1}(T)$. M_X is clearly a Hausdorff space with $\text{ind } M_X = 0$. The space M given in Example 11-16 is M_I in the present expression.

$\dagger$ This modification of Dowker's space M in Example 11-16 was considered by Smirnov [2] for separable spaces X .

ASSERTION 1. M_X is a normal space with $\dim M_X \leq \dim X$.

PROOF. Let $\mathcal{G}$ be an arbitrary finite open covering of M_X . We construct an open covering $\mathcal{H}$ of M_X with (i) $\mathcal{H} \triangleleft \mathcal{G}$, (ii) $\text{ord } \mathcal{H} \leq \dim X + 1$, which will prove the assertion. By an argument analogous to that in Example 11-16, for each $x \in X$ there exist an open neighborhood $U(x)$ and an ordinal $\alpha(x)$ such that

$$[\pi^{-1}(U(x))]_{(\alpha(x), \omega_1)} < \mathcal{G}$$

Consider an open covering $\{U(x): x \in X\}$ of X . Let $\mathcal{V} = \{V_\lambda: \lambda \in \Lambda\}$ be a locally finite open covering of X with $\mathcal{V} < \{U(x)\}$. For each λ in Λ let $\alpha(\lambda)$ be an ordinal $\alpha(x)$ such that $V_\lambda \subset U(x)$. Then for every $\lambda \in \Lambda$,

$$[\pi^{-1}(V_\lambda)]_{(\alpha(\lambda), \omega_1)} < \mathcal{G}$$

Let $\cup_{i=1}^{\infty} \mathcal{U}_i$ be a locally finite open covering of X and $\cup_{i=1}^{\infty} \mathcal{F}_i$ a closed covering of X , where $\mathcal{U}_i = \{U_\xi: \xi \in \Xi_i\}$ and $\mathcal{F}_i = \{F_\xi: \xi \in \Xi_i\}$, such that:

- i. $\cup \mathcal{U}_i$ Δ -refines $\mathcal{V}$
- ii. $F_\xi \subset U_\xi$ for each $\xi \in \cup \Xi_i = \Xi$
- iii. Each element of $\cup \mathcal{U}_i$ meets at most a finite number of elements of $\mathcal{V}$
- iv. Each $\mathcal{U}_i$ is discrete
- v. $\text{ord } \cup \mathcal{U}_i \leq \dim X + 1$

Set

$$U_i = \cup \{U_\xi: \xi \in \Xi_i\}, \quad F_i = \cup \{F_\xi: \xi \in \Xi_i\}$$

Then U_i is an open set of X and F_i is a closed set of X with $F_i \subset U_i$ for each i . Since $\{\alpha(\lambda): U_\xi \subset V_\lambda\}$ is a nonempty finite set of ordinals for each $U_\xi \in \cup \mathcal{U}_i$, it has maximum, say $\alpha(\xi)$.

Set

$$\mathcal{W}_0 = \{[\pi^{-1}(U_\xi)]_{(\alpha(\xi), \omega_1)} = W_\xi: \xi \in \Xi\}$$

Then

$$\text{ord } \mathcal{W}_0 \leq \text{ord } \cup \mathcal{U}_i \leq \dim X + 1$$

Let us show that $\mathcal{W}_0$ Δ -refines $\mathcal{G}$. Let p be an arbitrary point of M_X . Set

$$\Xi' = \{\xi: p \in W_\xi\}$$

Then for each $\xi \in \Xi'$, $\pi(W_\xi)$, which is U_ξ , contains $\pi(p)$. Hence

$$\cup \{U_\xi: \xi \in \Xi'\} \subset S(\pi(p), \cup \mathcal{U}_i)$$

Since $\cup \mathcal{U}_i \triangleleft \mathcal{V}$, there exists some V_μ in $\mathcal{V}$ such that

$$\cup \{U_\xi: \xi \in \Xi'\} \subset V_\mu$$

Hence for this μ and every $\xi \in \Xi'$ $\alpha(\xi) \geq \alpha(\mu)$. Therefore

$$\cup \{W_\xi = [\pi^{-1}(U_\xi)]_{(\alpha(\xi), \omega_1)} : \xi \in \Xi'\} \subset [\pi^{-1}(V_\mu)]_{(\alpha(\mu), \omega_1)}$$

which shows that

$$S(p, \mathcal{W}_0) \triangleleft \{[\pi^{-1}(V_\lambda)]_{(\alpha(\lambda), \omega_1)} : \lambda \in \Lambda\}$$

Since the latter refines $\mathcal{G}$, $\mathcal{W}_0$ Δ -refines $\mathcal{G}$.

For each $\xi \in \Xi$, let D_ξ be an open set of X such that

$$F_\xi \subset D_\xi \subset \bar{D}_\xi \subset U_\xi$$

$$B(D_\xi) \cap X_{\alpha(\xi)} = \phi$$

Set

$$P_\xi = [\pi^{-1}(D_\xi)]_{[0, \alpha(\xi)]}, \quad \xi \in \Xi$$

Then every P_ξ is metrizable and is open-closed in M_X . By the product theorem 12-14 and the subset theorem 11-11, $\dim P_\xi \leq 0$ for each $\xi \in \Xi$. Set

$$P = \cup \{P_\xi : \xi \in \Xi\}$$

Since $\cup \mathcal{U}_i$ is locally finite in X , $\{P_\xi : \xi \in \Xi\}$ is locally finite in M_X . Hence P is open-closed in M_X . Since P is evidently paracompact (actually metrizable) and

$$\dim \cup \{P_\xi : \xi \in \Xi_i\} \leq 0 \text{ for each } i$$

$\dim P \leq 0$ by the sum theorem. Let $\mathcal{W}_1$ be an open covering of P such that

$$\mathcal{W}_1 \triangleleft \mathcal{G} \mid P \quad \text{and} \quad \text{ord } \mathcal{W}_1 \leq 1$$

Set

$$\mathcal{W} = (\mathcal{W}_0 \mid (M_X - P)) \cup \mathcal{W}_1$$

Then $\mathcal{W}$ is an open covering of M_X such that

$$\mathcal{W} \triangleleft \mathcal{G} \quad \text{and} \quad \text{ord } \mathcal{W} \leq \dim X + 1 \quad \square$$

ASSERTION 2. $\dim M_X \geq \dim X$.

PROOF. Suppose the contrary, then there would exist an n with $\dim M_X < n \leq \dim X$. Let $\mathcal{G}_X$ be a finite open covering of X such that whenever $\mathcal{H}_X$ is an open covering of X with $\mathcal{H}_X < \mathcal{G}_X$, then $\text{ord } \mathcal{H}_X \geq n+1$. By the inequality $\dim M_X < n$, $\pi^{-1}(\mathcal{G}_X)$ can be refined by a finite open covering $\mathcal{G} = \{G_1, \dots, G_k\}$ of M_X such that

$$\mathcal{G} < \pi^{-1}(\mathcal{G}_X) \quad \text{and} \quad \text{ord } \mathcal{G} \leq n$$

As in the proof of Assertion 1, for each $x \in X$, there exist an open neighborhood $U(x)$ and an ordinal $\alpha(x)$ such that

$$Q_x = [\pi^{-1}(U(x))]_{(\alpha(x), \omega_1)} < \mathcal{G}$$

Let Q_1 be the sum of all Q_x 's which are contained in G_1 . Let Q_i , $2 \leq i \leq k$, be the sum of all Q_x 's which are not contained in any $G_1, \dots, G_{i-1}$ but contained in G_i . Then $\{Q_1, \dots, Q_k\}$ is an open collection of M_X such that

$$\{Q_i\} \text{ refines } \{G_i\}, \quad \text{and} \quad \text{ord}\{Q_i\} \leq n$$

Now

$$\{\pi(Q_i): i = 1, \dots, k\}$$

is an open covering of X of order $\leq n$ which refines $\mathcal{G}_X$, which is a contradiction. $\square$

ASSERTION 3. $\text{Ind } M_X = \text{Ind } X$.

PROOF (by induction). Since $\text{Ind } M_X \geq \dim M_X$ by Theorem 10-1 and $\dim M_X = \dim X = \text{Ind } X$, we get $\text{Ind } M_X \geq \text{Ind } X$. Let us show $\text{Ind } M_X \leq \text{Ind } X$. When $\text{Ind } X = \infty$, this inequality is trivially true. Consider the case when X is of finite dimension. When $\text{Ind } X = 0$, we have $\text{Ind } M_X = 0$ since (i) $\text{Ind } M_X = 0$ if and only if $\dim M_X = 0$ by Proposition 8-3, and (ii) $\dim M_X = \dim X = \text{Ind } X = 0$. Take as the induction assumption that the inequality is true for every metric space X having $\text{Ind} \leq n$. Suppose now $\text{Ind } X \leq n + 1$.

Let F and H be disjoint closed sets of M_X . For each point x in X there exist an open neighborhood $U(x)$ and an ordinal $\alpha(x) \in J$ such that

$$[\pi^{-1}(U(x))]_{(\alpha(x), \omega_1)} < \{M_X - F, M_X - H\}$$

Let $\{U_x: x \in X\}$ be a locally finite open covering of X such that

$$U_x \subset U(x) \text{ for each } x \in X$$

Let $\{F_x: x \in X\}$ be a closed covering of X such that $F_x \subset U_x$ for each $x \in X$. Let V_x be an open set in X such that

$$F_x \subset V_x \subset \bar{V}_x \subset U_x \quad \text{and} \quad \text{Ind } B(V_x) \leq n$$

Let D_x be an open set of X such that

$$F_x \subset D_x \subset \bar{D}_x \subset V_x \quad \text{and} \quad B(D_x) \cap X_{\alpha(x)} = \phi$$

Let

$$P = \cup \{[\pi^{-1}(D_x)]_{[0, \alpha(x)]}: x \in X\}$$

Then P is an open-closed set in M_X with $\text{Ind } P \leq 0$ by the same reasoning as in the proof of Assertion 1. Set

$$\mathcal{V}' = \{[\pi^{-1}(V_x)]_{(\alpha(x), \omega_1)} : x \in X\}$$

Then $\mathcal{V}'$ covers $M_X - P$. Set

$$\mathcal{V} = \mathcal{V}' \mid (M_X - P) \quad \text{and} \quad S = S(F - P, \mathcal{V})$$

Then

$$\begin{aligned} B(S) &\subset \cup \{[\pi^{-1}(B(V_x))]_{(\alpha(x), \omega_1)} : x \in X\} \\ &\subset \pi^{-1}(\cup \{B(V_x) : x \in X\}) \end{aligned}$$

where the last term has $\text{Ind} \leq n$ by the assumption and since

$$\text{Ind}(\cup \{B(V_x) : x \in X\}) \leq n$$

by Corollary 11-9. Hence, $\text{Ind } B(S) \leq n$.

Let S' be an open-closed set of P such that

$$P \cap F \subset S' \subset P - H$$

Set $W = S \cup S'$. Then W is an open set of M_X such that

$$F \subset W \subset \bar{W} \subset M_X - H \quad \text{and} \quad \text{Ind } B(W) \leq n$$

Thus $\text{Ind } M_X \leq n + 1$ and the induction is completed. $\square$

17-2. REMARK. Let L be a closed, cofinal subset of J . Then by an analogous argument $[M_X]_L$ is a normal space with

$$\text{ind}[M_X]_L = 0 \quad \text{and} \quad \dim[M_X]_L = \text{Ind}[M_X]_L = \dim X$$

17-3. EXAMPLE. There exists a normal space with $\text{ind} = n$ having large $\dim$. Let $1 \leq n < m$. Let M_{nm} be the disjoint sum of I^n and M_{Im} topologized in such a way that I^n and M_{Im} are embedded as open sets. Then M_{nm} is a normal space.

ASSERTION. $\text{ind } M_{nm} = n$, $\dim M_{nm} = \text{Ind } M_{nm} = m$.

PROOF. Since $\text{ind } I^n = \dim I^n = n$ (see Hurewicz and Wallman [1, Theorem IV, 1]),

$$\text{ind } M_{nm} = \max\{\text{ind } I^n, \text{ind } M_{Im}\} = n$$

$$\dim M_{nm} = \max\{\dim I^n, \dim M_{Im}\} = m$$

$$\text{Ind } M_{nm} = \max\{\text{Ind } I^n, \text{Ind } M_{Im}\} = m$$

18. Compact Spaces with $\dim = n$ Having Large ind and Ind

18-1. LEMMA (Vopenka [1]). Let X be a normal space and Y a non-empty closed set of X which satisfy the following conditions:

- i. $\text{Ind}(X \text{ mod } Y) \leq n$
- ii. $\text{Ind } Y \leq m$
- iii. Y is a retract of X
- iv. For every neighborhood U of Y there is an open-closed neighborhood V of Y with $V \subset U$

Then, $\text{Ind } X \leq m + n$.

PROOF (by induction on $\text{Ind } Y$). Since the proof for the starting case when $\text{Ind } Y = 0$ is completely similar to the general case, we merely prove the lemma under the assumption that the lemma is true when $\text{Ind } Y < m$.

Let $\text{Ind } Y \leq m$. Let H be a closed set of X and W be an open set of X with $H \subset W$. Let G be a relatively open set of Y with

$$H \cap Y \subset G \subset \bar{G} \subset W \cap Y$$

$$\text{Ind}(\bar{G} - G) \leq m - 1$$

Let ϕ be a retraction of X onto Y . Set

$$X' = \phi^{-1}(\bar{G} - G) \quad \text{and} \quad Y' = \bar{G} - G$$

Then all conditions of the lemma are satisfied if X , Y , and m are replaced by X' , Y' , and $m - 1$ respectively. Hence by the induction assumption

$$\text{Ind } X' \leq m + n - 1$$

Let V be an open-closed neighborhood of Y with

$$V \cap ((\phi^{-1}(G) - W) \cup (H - \phi^{-1}(G))) = \phi$$

Let D_1 be an open set of $X - V$ such that

$$H - V \subset D_1 \subset \bar{D}_1 \subset W - V$$

$$\text{Ind}(\bar{D}_1 - D_1) \leq n - 1$$

Set

$$D_2 = V \cap \phi^{-1}(G)$$

Then

$$H \cap V \subset D_2 \subset \bar{D}_2 \subset W \cap V$$

Since $\bar{D}_2 - D_2 \subset X'$

$$\text{Ind}(\bar{D}_2 - D_2) \leq m + n - 1$$

Set $D = D_1 \cup D_2$. Then $B(D) = \bar{D} - D$ is the disjoint union of $\bar{D}_1 - D_1$ and $\bar{D}_2 - D_2$. Hence

$$\text{Ind } B(D) \leq \max\{n - 1, m + n - 1\} = m + n - 1$$

Moreover, $H \subset D \subset \bar{D} \subset W$. Therefore, $\text{Ind } X \leq m + n$. $\square$

18-2. LEMMA. Let X be a paracompact Hausdorff space with $\dim X = 0$ and Y a compact Hausdorff space, then

$$\dim(X \times Y) = \dim Y \quad \text{and} \quad \text{Ind}(X \times Y) = \text{Ind } Y$$

PROOF. Since it is evident that $\dim(X \times Y) \geq \dim Y$ and $\text{Ind}(X \times Y) \geq \text{Ind } Y$, let us prove, under the assumptions $\dim Y \leq m$ and $\text{Ind } Y \leq n$, that $\dim(X \times Y) \leq m$ and $\text{Ind}(X \times Y) \leq n$.

To prove the first inequality, let $\mathcal{G}$ be an arbitrary finite open covering of $X \times Y$. For every point x in X there exist an open neighborhood $U(x)$ of x and a finite open covering $\mathcal{G}_x$ of Y such that

$$\text{ord } \mathcal{G}_x \leq m + 1 \quad \text{and} \quad \{U(x) \times G : G \in \mathcal{G}_x\} \text{ refines } \mathcal{G}$$

Let $\{U_x : x \in X\}$ be an open covering of X such that

$$\text{ord}\{U_x\} = 1 \quad \text{and} \quad U_x \subset U(x) \text{ for each } x \in X$$

Then

$$\{U_x \times G : G \in \mathcal{G}_x, \quad x \in X\}$$

is an open covering of $X \times Y$ with order $\leq m + 1$ which refines $\mathcal{G}$. Hence, $\dim(X \times Y) \leq m$.

We prove the second inequality by induction on n . For this we make only the transition from $n - 1$ to n , since the proof for $n = 0$ is completely analogous. Let $\text{Ind } Y \leq n$ under the induction assumption that the inequality is true for spaces Y with $\text{Ind } Y \leq n - 1$. Let F be a closed set of $X \times Y$ and U an open set with $F \subset U$. For every x in X , there exist an open neighborhood $U(x)$ of x and an open set W_x in Y such that

$$F \cap (U(x) \times Y) \subset U(x) \times W_x \subset U(x) \times \bar{W}_x \subset U$$

$$\text{Ind } B(W_x) \leq n - 1$$

Let $\{U_x : x \in X\}$ be an open covering of X with $\text{ord}\{U_x\} = 1$ such that $U_x \subset U(x)$ for every $x \in X$. Set

$$W = \cup (U_x \times W_x)$$

Then

$$F \subset W \subset \overline{W} \subset U \quad \text{and} \quad B(W) = \cup (U_x \times B(W_x))$$

For each $x \in X$

$$\text{Ind}(U_x \times B(W_x)) \leq n - 1$$

by the induction assumption. Since $\{U_x \times B(W_x)\}$ is a discrete collection of closed sets in $X \times Y$, $\text{Ind } B(W) \leq n - 1$. Hence $\text{Ind}(X \times Y) \leq n$. $\square$

18-3. DEFINITION. A transformation f of a space Y to X is *nearly open* if for each point x in X there exists a point y in $f^{-1}(x)$ such that whenever U is a neighborhood of y , $f(U)$ is a neighborhood of x .

18-4. LEMMA. Let X be a nonempty compact Hausdorff space, then there exist a compact Hausdorff space Y with $\dim Y = 0$ and a nearly open mapping of Y onto X .

PROOF. Let

$$\{\mathcal{G}_\lambda = \{G(\alpha) : \alpha \in A_\lambda\} : \lambda \in \Lambda\}$$

be the set of all finite open coverings of X . Let $Y_0 = \Pi A_\lambda$, where A_λ 's are considered as discrete spaces. Let π_λ be the projection of Y_0 to A_λ . Let Y be the aggregate of all points y in Y_0 such that

$$\cap \{\overline{G(\pi_\lambda(y))} : \lambda \in \Lambda\} \neq \phi$$

When the intersection is nonempty, it is a single point, which is denoted by $f(y)$. Then f is a mapping of Y onto X .

To prove that Y is closed in Y_0 let $y = \langle \alpha_\lambda \rangle$ be an arbitrary point of the closure of Y in Y_0 . Let M be an arbitrary, finite, nonempty subset of Λ . Since

$$\cap \{\pi_\lambda^{-1}(\alpha_\lambda) : \lambda \in M\}$$

is a neighborhood of y in Y_0 , it contains a point $z = \langle \beta_\lambda \rangle$ in Y . Then $\alpha_\lambda = \beta_\lambda$ for every $\lambda \in M$. Hence

$$\cap \{\overline{G(\alpha_\lambda)} : \lambda \in M\} \neq \phi$$

Therefore $\{\overline{G(\alpha_\lambda)} : \lambda \in \Lambda\}$ has the finite intersection property. By the compactness of X , $\cap \{\overline{G(\alpha_\lambda)} : \lambda \in \Lambda\} \neq \phi$, which implies that y is in Y . Thus Y is closed in Y_0 . Since $\dim Y_0 = 0$, Y is a compact Hausdorff space with $\dim Y = 0$.

To prove f is nearly open let x be an arbitrary point of X . Let $y = \langle \alpha_\lambda \rangle$ be a point of $f^{-1}(x)$ such that $x \in G(\alpha_\lambda)$ for every $\lambda \in \Lambda$. Since

$$f(\pi_\lambda^{-1}(\alpha_\lambda)) = \overline{G(\alpha_\lambda)} \supset G(\alpha_\lambda)$$

we have,

$$f(\cap \{\pi_\lambda^{-1}(\alpha_\lambda) : \lambda \in M\}) \supset \cap \{G(\alpha_\lambda) : \lambda \in M\}$$

for each finite subset M of Λ . Thus every cubic neighborhood of y is mapped by f onto a neighborhood of x . $\square$

18-5. LEMMA. Let X be a space, X_1 a subset of X and X_2 a nonempty connected subset of X_1 . Let G be an open set of X such that $G \cap X_2 \neq \emptyset$ and $X_2 - \overline{G} \neq \emptyset$. Then either

1. There exists a relatively open set U of X_1 such that $U \cap X_2 \neq \emptyset$ and $U \subset B(G)$, or
2. $\overline{G \cap X_1} \cap \overline{X_1 - \overline{G}} \cap X_2 \neq \emptyset$

PROOF. Set

$$U = X_1 - (\overline{G \cap X_1} \cup \overline{X_1 - \overline{G}})$$

Then U is a relatively open set of X_1 with $U \subset B(G)$. If $U \cap X_2 = \emptyset$ and $\overline{G \cap X_1} \cap \overline{X_1 - \overline{G}} \cap X_2 \neq \emptyset$, then X_2 would have a separation $\overline{G \cap X_1} \cap X_2$ and $\overline{X_1 - \overline{G}} \cap X_2$, which is a contradiction. $\square$

18-6. DEFINITION. Let X, Y, Z be spaces, f a mapping of Y onto X and N an infinite discrete space. Let T be the disjoint sum of X and $Y \times N \times Z$. A base of T is the family of all open sets of the product space $Y \times N \times Z$ plus the family of all sets of the form

$$U \cup (f^{-1}(U) \times (N - K) \times Z)$$

where U is open in X and K is a finite set of N . Then

$$T = T(X, f, Y, N, Z)$$

is a Hausdorff space if X, Y , and Z are Hausdorff. This space is *Vopenka's space*. X is a closed subset in T . Let $\varphi: T \rightarrow X$ be a transformation defined by

$$\varphi(t) = t, \quad t \in X$$

$$\varphi(t) = f(y), \quad t = \langle y, v, z \rangle \in Y \times N \times Z$$

Then it is easy to see that φ is a retraction of T onto X .

18-7. THEOREM (Vopenka [1]). If X , Y , and Z are nonempty compact Hausdorff spaces, then $T = T(X, f, Y, N, Z)$ is also compact Hausdorff and for every neighborhood U of the set $X \subset T$ there exists an open-closed neighborhood V with $X \subset V \subset U$. If, furthermore, $\dim Y = 0$, then

$$\dim T = \max\{\dim X, \dim Z\}$$

$$\text{Ind } T \leq \text{Ind } X + \text{Ind } Z$$

PROOF. Let $\mathcal{G}$ be an arbitrary open covering of T . By the compactness of X there exist a finite number of open sets $V_1, \dots, V_k$ of the form

$$V_i = U_i \cup (f^{-1}(U_i) \times (N - K_i) \times Z), \quad i = 1, \dots, k$$

where U_i are open in X and K_i are finite sets of N , such that $\cup U_i = X$ and $\{V_i\}$ refines $\mathcal{G}$. Since

$$T - \cup V_i \subset Y \times (\cup K_i) \times Z$$

$T - \cup V_i$ is compact. Hence there exists a finite subfamily $\mathcal{G}_1$ of $\mathcal{G}$ such that $\mathcal{G}_1$ covers $T - \cup V_i$. For each i let G_i be an element of $\mathcal{G}$ with $V_i \subset G_i$. Then

$$\{G_1, \dots, G_k\} \cup \mathcal{G}_1$$

is a finite subcovering of $\mathcal{G}$. Therefore T is compact. By this argument we can easily notice that the sets of the form $X \cup (Y \times (N - K) \times Z)$, where K is finite, form a neighborhood base of X in T . These sets are clearly open-closed in T .

When $\dim Y = 0$, $\dim(Y \times N \times Z) = \dim Z$ by Lemma 18-2. Hence $\dim(T \text{ mod } X) = \dim Z$. By Theorem 9-11

$$\dim T \leq \max\{\dim X, \dim Z\}$$

Since T contains a closed copy of Z , $\dim T \geq \dim Z$. Since X is closed in T , $\dim T \geq \dim X$. Hence

$$\dim T = \max\{\dim X, \dim Z\}$$

By Lemma 18-2, $\text{Ind}(T \text{ mod } X) = \text{Ind } Z$. Hence, by Lemma 18-1

$$\text{Ind } T \leq \text{Ind } X + \text{Ind } Z \quad \square$$

18-8. LEMMA (Vopenka [1]). Let X , Y , and Z be nonempty compact Hausdorff spaces, f a nearly open mapping of Y onto X and N a discrete space the power of which is infinite and is greater than the density[†] of

[†] The *density* of a space is the minimum of powers of bases of a space.

X . Let, finally, $T = T(X, f, Y, N, Z)$, then the following assertion is true: If U and V are open in T and $\overline{U \cap X} \cap \overline{V \cap X} \neq \emptyset$, then $\overline{U \cap V}$ contains a copy of Z .

PROOF. Let us put $U_1 = U \cap X$ and $V_1 = V \cap X$. Let x_0 be a point of $\overline{U_1 \cap V_1}$ and y_0 be a point of $f^{-1}(x_0)$ such that each neighborhood of y_0 in Y is mapped by f onto a neighborhood of x_0 in X . Then

$$y_0 \in \overline{f^{-1}(U_1)} \cap \overline{f^{-1}(V_1)}$$

Let $\mathcal{G}$ be a base of X the power of which gives the density of X . Let $\mathcal{G}_U = \{G_\alpha : \alpha \in A_U\}$ be a subfamily of $\mathcal{G}$ such that

- i. $\cup \{G_\alpha : \alpha \in A_U\} = U_1$
- ii. $G_\alpha \cup (f^{-1}(G_\alpha) \times (N - K_\alpha) \times Z) \subset U$ for suitable finite sets $K_\alpha, \alpha \in A_U$

Analogously, let $\mathcal{G}_V = \{G_\alpha : \alpha \in A_V\}$ be a subfamily of $\mathcal{G}$ such that

- i. $\cup \{G_\alpha : \alpha \in A_V\} = V_1$
- ii. $G_\alpha \cup (f^{-1}(G_\alpha) \times (N - K'_\alpha) \times Z) \subset V$ for suitable finite sets $K'_\alpha, \alpha \in A_V$.

Let us put

$$M = N - ((\cup \{K_\alpha : \alpha \in A_U\}) \cup (\cup \{K'_\alpha : \alpha \in A_V\}))$$

Then $M \neq \emptyset$, $f^{-1}(U_1) \times M \times Z \subset U$ and $f^{-1}(V_1) \times M \times Z \subset V$. Therefore

$$\{y_0\} \times M \times Z \subset \overline{U \cap V}$$

which implies that $\overline{U \cap V}$ contains a copy of Z . $\square$

18-9. THEOREM (Vopenka [1]). If X is a compact Hausdorff space with $\dim X > 0$, then there exists a compact Hausdorff space T having the following properties: (1) $\dim T = \dim X$, (2) $\text{Ind } T \leq 2 \text{ Ind } X + 1$, and (3) There exist a point $t \in T$ and a neighborhood U in T such that for every open neighborhood $G \subset U$ of the point t the set $B(G)$ contains a copy of X .

PROOF. Let g be a mapping of a Cantor discontinuum C onto the unit interval $I = [0, 1]$ given in 16-2. Let M be a countably infinite discrete space. Set

$$T' = T(I, g, C, M, X)$$

By Theorem 18-7

$$\dim T' = \max\{\dim I, \dim X\} = \dim X$$

$$\text{Ind } T' \leq \text{Ind } I + \text{Ind } X = \text{Ind } X + 1$$

By Lemma 18-4 there exist a compact Hausdorff space Y with $\dim Y = 0$ and a nearly open mapping f of Y onto T' . Let N be a discrete space the power of which is greater than the density of T' . Set

$$T = T(T', f, Y, N, X)$$

Let us show that T has the required properties. By Theorem 18-7, T is a compact Hausdorff space accompanied by the inequalities

$$\dim T = \max\{\dim T', \dim X\} = \dim X$$

$$\text{Ind } T \leq \text{Ind } T' + \text{Ind } X \leq 2 \text{Ind } X + 1$$

Let t be the terminal point 0 in I and U its neighborhood in T such that $\bar{U}$ does not contain the other terminal point 1 in I . Let now G be an arbitrary open neighborhood of t in T with $G \subset U$. Then by Lemma 18-5 (setting $X = T$, $X_1 = T'$ and $X_2 = I$) either:

i. There exists a relatively open set V of T' such that $V \cap I \neq \emptyset$ and $V \subset B(G)$, or

ii. $\overline{G \cap T'} \cap \overline{T'} - \bar{G} \cap I \neq \emptyset$

In case (i), according to the construction of T' , V contains a copy of X and hence $B(G)$ contains a copy of X . In case (ii), it follows from Lemma 18-8 that $\overline{G \cap T'} - \bar{G}$ contains a copy of X . Since $\overline{G \cap T'} - \bar{G} \subset B(G)$, $B(G)$ contains a copy of X . $\square$

18-10. PROPOSITION (Vopenka [1]). Let n be an arbitrary natural number, then there exist compact Hausdorff spaces X and Y such that

$$\dim X = 1, \quad \text{ind } X = n, \quad \text{Ind } X < \infty$$

$$\dim Y = 1, \quad \text{Ind } Y = n$$

PROOF (by induction). Since $\dim I = \text{ind } I = \text{Ind } I = 1$, the existence of such an X is assured for $n = 1$. For $n \geq 1$ suppose that there exists a compact Hausdorff space X with $\dim X = 1$, $\text{ind } X = n$ and $\text{Ind } X < \infty$. By Theorem 18-9 there exists a compact Hausdorff space T such that

$$\dim T = \dim X = 1$$

$$\text{ind } X < \text{ind } T \leq \text{Ind } T \leq 2 \text{Ind } X + 1 < \infty$$

Set $k = \text{ind } T$. Then $n < k$. By inductive definition of ind , for each m with $n < m \leq k$, there exists a closed subset T_m of T with $\text{ind } T_m = m$. Then $\dim T_{n+1} \neq 0$, since $\text{ind } T_{n+1} \neq 0$. Hence $\dim T_{n+1} = 1$. Thus the induction is completed. The existence of Y with the properties mentioned can be proved quite analogously. $\square$

18-11. THEOREM (Vopenka [1]). Let $m \leq n$ be an arbitrary ordered pair of natural numbers, then there exist compact Hausdorff spaces X , Y , and Z such that

$$\begin{aligned} \dim X &= m, & \text{ind } X &= n, & \text{Ind } X &< \infty \\ \dim Y &= m, & \text{Ind } Y &= n \\ \dim Z &= m, & \text{ind } Z &= \infty \end{aligned}$$

PROOF. For $k = 1, 2, \dots$, let X_k be a compact Hausdorff space with

$$\dim X_k = 1, \quad \text{ind } X_k = k, \quad \text{Ind } X_k < \infty$$

Let X be the topologically disjoint sum[†] of X_n and an m -cube I^m .[‡] Then X satisfies the required properties.

The existence of a space Y with the properties mentioned is proved similarly. Let Z' be the disjoint sum of X_k , $k = 1, 2, \dots$, and an ideal point ξ . The topology of Z' is defined as follows:

- i. Each X_k is open in Z
- ii. $\{\{\xi\} \cup (\cup_{k \geq s} X_k), s = 1, 2, \dots\}$ is a neighborhood base of ξ

Then Z' is a compact Hausdorff space such that

$$\dim Z' = 1, \quad \text{ind } Z' = \infty$$

The first equality is assured by the sum theorem. Let Z be the topologically disjoint sum of Z' and I^m . Then Z is the required space. $\square$

19. A Normal Space with $\text{ind} = 0$, $\dim = 1$, $\text{Ind} = 2$

19-1. LEMMA. If X is a regular Lindelöf space, then $\dim X \leq \text{ind } X$.

[†] The *topologically disjoint sum* of spaces P_λ , $\lambda \in \Lambda$ is the disjoint sum of P_λ 's topologized in such a way that each P_λ is open in the sum without disturbing the topology of P_λ .

[‡] That $\dim I^m = m$ is one of the main results in Hurewicz and Wallman [1].

PROOF (by induction). When $\text{ind } X = \infty$, the inequality is trivially true. Let $\text{ind } X \leq n$. We are going to prove $\text{dim } X \leq n$. Since the proof for $n = 0$ is completely similar to the general case, we merely prove $\text{dim } X \leq n$ under the induction assumption that $\text{dim } X < n$ if $\text{ind } X < n$. Let F_1 and F_2 be disjoint closed sets of X . Let $\{U_i\}$ be an open covering of X such that $\{\bar{U}_i\}$ refines $\{X - F_1, X - F_2\}$ and such that $\text{ind } B(U_i) \leq n - 1$ for each i . Let $\{U_{i'}\}$ ($\{U_{i''}\}$) be the set of all U_i 's with $\bar{U}_i \cap F_1 \neq \phi$ ($\bar{U}_i \cap F_1 = \phi$, respectively). Using these subfamilies of $\{U_i\}$ let us define an open set U with $F_1 \subset U \subset \bar{U} \subset X - F_2$ such that $B(U)$ is a subset of the union of all $B(U_i)$, $i = 1, 2, \dots$. When $\{U_{i'}\}$ ($\{U_{i''}\}$) is finite, set

$$U = \cup U_{i'}, \quad (U = X - \cup \bar{U}_{i''), \quad \text{respectively})$$

Then U has the desired properties. Let us consider the case when both $\{U_{i'}\}$ and $\{U_{i''}\}$ are infinite. In this case set

$$V_i = U_{i'} - \cup_{j \leq i} \bar{U}_{j''} \quad \text{and} \quad U = \cup V_i$$

Then it can easily be seen that $F_1 \subset U \subset \bar{U} \subset X - F_2$. To prove $B(U) \subset \cup B(U_i)$ let x be an arbitrary point of $X - \cup B(U_i)$. If $x \notin U_{i''}$ for any i , then $x \notin \bar{U}_{i''}$ for any i . Hence $x \in V_j$ for j with $x \in U_{j'}$, which implies $x \notin B(U)$. If $x \notin U_{i'}$ for any i , then $x \notin \bar{U}_{i'}$ for any i . Let j be a number with $x \in U_{j''}$. Then

$$x \notin \bar{V}_i \quad \text{for } i < j$$

$$U_{j''} \cap V_i = \phi \quad \text{for } i \geq j$$

Hence

$$x \notin \cup_{i < j} \bar{V}_i \cup (\overline{\cup_{j \leq i} V_i}) = \bar{U}$$

If $x \in U_{i'} \cap U_{j''}$ for some i and j , let s and t be the smallest numbers with $x \in U_{s'} \cap U_{t''}$. When $s \leq t$, $x \in V_s$. Hence $x \notin B(U)$. When $s > t$, $V_i \cap U_{t''} = \phi$ for $i \geq t$ and $x \notin \bar{V}_i$ for $i < t$. Hence $x \notin \bar{U}$. Thus $B(U)$ is a subset of $\cup B(U_i)$. This argument contains a proof for the normality of X .

Since $\cup B(U_i)$ is a normal space by Theorem 3-12, $\text{dim } \cup B(U_i) \leq n - 1$ by the sum theorem and the induction assumption. Hence $\text{dim } B(U) \leq n - 1$. By a quite analogous argument to the proof of Theorem 10-1, we get $\text{dim } X \leq n$. $\square$

19-2. EXAMPLE (Nagami [9]). A normal space Z with $\text{ind } Z = 0$, $\text{dim } Z = 1$, $\text{Ind } Z = 2$. Let $J = \{\alpha: 0 \leq \alpha < \omega_1\}$ with the interval topology

stated in Example 11-16. Let I be the unit closed interval and I_α , $\alpha < \omega_1$, the subsets of I stated in Example 11-16 such that

$$I_\alpha \subset I_\beta \quad \text{whenever} \quad \alpha < \beta, \quad \dim I_\alpha = 0 \quad \text{for each} \quad \alpha \\ \cup I_\alpha = I, \quad \text{each } I_\alpha \text{ is dense in } I$$

By Corollary 15-2 there exist a separable metric space C_0 with $\dim C_0 = 0$ and an open mapping f of C_0 onto I . Let M be a discrete space whose power $|M|$ is $\mathfrak{f}$, the power of the function. Let

$$T = T(I, f, C_0, M, I)$$

given in Definition 18-6. Set

$$T_\alpha = I_\alpha \cup (f^{-1}(I_\alpha) \times M \times I_\alpha) \quad \alpha < \omega_1$$

The set Z is the sum of all $\{\alpha\} \times T_\alpha$, $\alpha < \omega_1$. The topology of Z is the relative topology of the product space $J \times T$. Let π' be the projection of $J \times T$ to T . Set

$$\pi = \pi' \mid Z$$

Let ρ' be the projection of $J \times T$ to J . Set

$$\rho = \rho' \mid Z$$

If S is a subset of Z and J' is a subset of J , then $[S]_{J'}$ denotes the intersection of S and $\rho^{-1}(J')$. If x is a point of I and ε is a positive number, $S_\varepsilon(x)$ denotes the open ε -sphere in I .

ASSERTION 1. Z is normal.

PROOF. (a). Z is evidently a Hausdorff space. To prove the normality of Z let F and H be a disjoint pair of closed sets of Z . Take an arbitrary point x of I . Both F and H cannot be cofinal on $\pi^{-1}(x)$ at the same time. Suppose that H is not cofinal on $\pi^{-1}(x)$. Then there exists an $\alpha(x)$ such that $\langle \alpha(x), x \rangle \in Z$ and

$$[\pi^{-1}(x)]_{[\alpha(x), \omega_1)} \cap H = \phi$$

For every β with $\alpha(x) < \beta < \omega_1$ let $\varepsilon(\beta)$ be the largest positive number for which there exist a γ with $\alpha(x) \leq \gamma < \beta$ and a finite subset K_β of M such that

$$[\pi^{-1}(S_{\varepsilon(\beta)}(x) \cup (f^{-1}(S_{\varepsilon(\beta)}(x)) \times (M - K_\beta) \times I))]_{(\gamma, \beta]} \cap H = \phi$$

Then it is easy to see that

$$\varepsilon(x) = \inf\{\varepsilon(\beta) : \alpha(x) < \beta < \omega_1\}$$

is positive. Set

$$K_x = \{\lambda \in M: [\pi^{-1}(f^{-1}(S_{\varepsilon(x)}(x)) \times \{\lambda\} \times I)_{(\alpha(x), \omega_1)} \cap H \neq \emptyset\}$$

To prove K_x is a finite set, assume the contrary. Then there would exist a countably infinite subset $\{\lambda_1, \lambda_2, \dots\}$ of K_x and a sequence $\alpha(x) < \alpha_1 \leq \alpha_2 \leq \dots$ such that

$$[\pi^{-1}(f^{-1}(S_{\varepsilon(x)}(x)) \times \{\lambda_i\} \times I)_{\alpha_i} \cap H \neq \emptyset$$

Let $\alpha_0 = \lim \alpha_i$. Then for any δ with $\alpha(x) \leq \delta < \alpha_0$ and for any finite set K of M ,

$$[\pi^{-1}(f^{-1}(S_{\varepsilon(x)}(x)) \times (M - K) \times I)_{(\delta, \alpha_0)} \cap H \neq \emptyset$$

which is a contradiction. Thus, K_x is finite and

$$[\pi^{-1}(S_{\varepsilon(x)}(x) \cup (f^{-1}(S_{\varepsilon(x)}(x)) \times (M - K_x) \times I))_{(\alpha(x), \omega_1)}$$

does not meet H .

(b). By (a), for every $x \in I$ we have a positive number $\varepsilon(x)$, an ordinal $\alpha(x) < \omega_1$ and a finite subset K_x of M such that

$$[\pi^{-1}(x)]_{\alpha(x)} \neq \emptyset$$

and

$$\{[\pi^{-1}(S_{\varepsilon(x)}(x) \cup (f^{-1}(S_{\varepsilon(x)}(x)) \times (M - K_x) \times I))_{(\alpha(x), \omega_1)} : x \in I\}$$

refines $\{Z - F, Z - H\}$. Pick a finite subset $\{x_1, \dots, x_n\}$ of I such that

$$\mathcal{U} = \{S_{\varepsilon(x_i)}(x_i) : i = 1, \dots, n\}$$

covers I . Set

$$K = \cup \{K_{x_i} : i = 1, \dots, n\}$$

Then K is a finite set of M . Set

$$\beta_0 = \sup\{\alpha(x_i) : i = 1, \dots, n\}$$

Let $\mathcal{V} = \{V_1, \dots, V_m\}$ be a finite, open (in I) covering of I which is a Δ -refinement of $\mathcal{U}$. We divide Z into three disjoint parts Z_1, Z_2 , and Z_3 each of which is open in Z as follows:

$$Z_1 = [\pi^{-1}(I \cup (C_0 \times (M - K) \times I))_{(\beta_0, \omega_1)}$$

$$Z_2 = [\pi^{-1}(C_0 \times K \times I)_{(\beta_0, \omega_1)}$$

$$Z_3 = [Z]_{[0, \beta_0]}$$

$$Z = Z_1 \cup Z_2 \cup Z_3$$

By construction

$$\tilde{\mathcal{V}} = \{[\pi^{-1}(V_i \cup (f^{-1}(V_i) \times (M - K) \times I))]_{(\beta_0, \omega_1)} : i = 1, \dots, m\}$$

Δ -refines $\{Z - F, Z - H\}$. Let D_1 and G_1 be respectively stars of F and H with respect to $\tilde{\mathcal{V}}$ (for example, $D_1 = S(F, \tilde{\mathcal{V}})$). Then

$$D_1 \cap G_1 = \phi, \quad D_1 \supset F \cap Z_1, \quad \text{and} \quad G_1 \supset H \cap Z_1$$

Since Z_2 is essentially the same as $M_{C_0 \times K \times I}$ in Example 17-1, Z_2 is normal. Hence, there exist open sets D_2 and G_2 of Z_2 such that

$$D_2 \cap G_2 = \phi, \quad D_2 \supset F \cap Z_2, \quad G_2 \supset H \cap Z_2$$

(c). Let us prove the normality of Z_3 . Let $\mathcal{W}$ be an arbitrary open covering of Z_3 . Consider an arbitrary ordinal α with $0 \leq \alpha \leq \beta_0$. By perfect separability of I there exist a sequence of open sets $A_1, A_2, \dots$ of I , a sequence of ordinals $\beta_1, \beta_2, \dots$ with $\beta_i < \alpha$, $i = 1, 2, \dots$, and a sequence of finite sets $K_1, K_2, \dots$ of M such that $\cup A_i = I$ and

$$\mathcal{W}_1 = \{[\pi^{-1}(A_i \cup (f^{-1}(A_i) \times (M - K_i) \times I))]_{(\beta_i, \alpha]} : i = 1, 2, \dots\}$$

refines $\mathcal{W}$, where we set $(\beta_i, \alpha] = \{0\}$ when $\alpha = 0$. Set $M_1 = \cup K_i$. Then M_1 is countable. Since $C_0 \times M_1 \times I$ is perfectly separable, we can find a countable open collection $\mathcal{W}_2$ of Z_3 such that

$$\mathcal{W}_2 < \mathcal{W} \quad \text{and} \quad \mathcal{W}_2 \text{ covers } [\pi^{-1}(C_0 \times M_1 \times I)]_\alpha$$

Thus we have a countable open collection $\mathcal{W}_1 \cup \mathcal{W}_2$ of Z_3 which covers $[Z]_\alpha$ and refines $\mathcal{W}$. Since $[0, \beta_0]$ contains only a countable number of ordinals, $\mathcal{W}$ can be refined by a countable open covering of Z_3 , which shows that Z_3 is a Lindelöf space. Since Z_3 is evidently regular, Z_3 is normal by Corollary 2-10. Hence there exist open sets D_3 and G_3 of Z_3 such that

$$D_3 \cap G_3 = \phi, \quad D_3 \supset F \cap Z_3, \quad \text{and} \quad G_3 \supset H \cap Z_3$$

Set

$$D = D_1 \cup D_2 \cup D_3, \quad G = G_1 \cup G_2 \cup G_3$$

Then D and G are open sets of Z such that $D \cap G = \phi$, $D \supset F$ and $G \supset H$. $\square$

ASSERTION 2. $\text{ind } Z = 0$.

ASSERTION 3. $\dim Z = 1$.

PROOF. Assertion 2 is evident. Since $\pi^{-1}(I)$ is nothing but the normal space M given in Example 11-16, $\dim \pi^{-1}(I) = 1$. Since $\pi^{-1}(I)$ is a closed set of Z , $\dim Z \geq 1$. Since $T - I$ is the sum of a disjoint collection of open metric subsets, $Z - \pi^{-1}(I)$ is a normal space with $\dim(Z - \pi^{-1}(I)) = 1$ by Assertions 1 and 2 of Example 17-1. Hence, by Theorem 9-11

$$\dim Z \leq \max\{\dim \pi^{-1}(I), \dim(Z - \pi^{-1}(I))\} = 1 \quad \square$$

ASSERTION 4. $\text{Ind } Z \leq 2$.

PROOF. For each $\lambda \in M$, $\pi^{-1}(C_0 \times \{\lambda\} \times I)$ is essentially the same as $M_{C_0 \times \{\lambda\} \times I}$ in Example 17-1. Hence,

$$\text{Ind } \pi^{-1}(C_0 \times \{\lambda\} \times I) = 1$$

by Assertion 3 in Example 17-1. Therefore we have

$$\text{Ind}(Z \bmod \pi^{-1}(I)) = 1$$

If we can show that the condition of Lemma 18-1 is satisfied, we will have $\text{Ind } Z \leq 2$. Let U be an arbitrary open set of Z with $U \supset \pi^{-1}(I)$. Set $H = Z - U$. By the same argument for H as in (a) of the proof of Assertion 1 there exist a finite subset K of M and an ordinal $\xi < \omega_1$ such that

$$V_1 = [\pi^{-1}(I \cup (C_0 \times (M - K) \times I))]_{(\xi, \omega_1)} \subset U$$

V_1 is open-closed in Z . Since we already knew in (c) of the proof of Assertion 1 that $[Z]_{[0, \xi]}$ is a Lindelöf space,

$$\dim [Z]_{[0, \xi]} \leq \text{ind } [Z]_{[0, \xi]} = 0$$

by Lemma 19-1. Hence there exists an open-closed set V_2 of $[Z]_{[0, \xi]}$ such that

$$[\pi^{-1}(I)]_{[0, \xi]} \subset V_2 \subset [U]_{[0, \xi]}$$

If we set $V = V_1 \cup V_2$, then V is an open-closed set of Z with $\pi^{-1}(I) \subset V \subset U$.

We define $\psi: T \rightarrow I$ as follows

$$\psi(x) = x, \quad x \in I$$

$$\psi(\langle c, \lambda, x \rangle) = f(c), \quad \langle c, \lambda, x \rangle \in C_0 \times M \times I$$

Then ψ is a retraction of T onto I . Define $\varphi: Z \rightarrow \pi^{-1}(I)$ as follows

$$\varphi(\langle \alpha, t \rangle) = \langle \alpha, \psi(t) \rangle, \quad \text{where } \alpha \in J \text{ and } t \in T_\alpha$$

Then φ is a retraction of Z onto $\pi^{-1}(I)$. By Lemma 18-1 $\text{Ind } Z \leq 2$. $\square$

ASSERTION 5. $\text{Ind } Z \geq 2$.

PROOF. Let 0 and 1 be the terminal points of I . It is to be noticed that 0 and 1 also represent the first and the second ordinals of J , but there ought not to be serious confusion. $\pi^{-1}(0)$ and $\pi^{-1}(1)$ are disjoint closed sets of Z . We prove that any closed set separating these two sets has to have $\text{Ind} \geq 1$, which will imply $\text{Ind } Z \geq 2$. Let P be an open set of Z with

$$\pi^{-1}(0) \subset P \subset \bar{P} \subset Z - \pi^{-1}(1)$$

Set

$$B = \bar{P} - P, \quad Z - \bar{P} = Q$$

We want to show $\text{Ind } B \geq 1$. Set

$$C_P = \{x \in I: P \text{ is cofinal on } \pi^{-1}(x)\}$$

$$C_Q = \{x \in I: Q \text{ is cofinal on } \pi^{-1}(x)\}$$

$$E_B = \{x \in I: B \text{ is equifinal on } \pi^{-1}(x)\}$$

Suppose that $C_P \cap C_Q \neq \emptyset$. Let h be a number of $C_P \cap C_Q$. Since $0 \notin C_Q$ and $1 \notin C_P$, $0 < h < 1$. For each point $p \in \pi^{-1}(h) \cap P$ there exists a positive integer $i(p)$ such that

$$1/i(p) < \min\{h, 1-h\}$$

$$[\pi^{-1}(S_{1/i(p)}(h))]_{\rho(p)} \subset P$$

Then there exists an i such that

$$P_1 = \{p \in \pi^{-1}(h) \cap P: i(p) = i\}$$

is cofinal. Analogously, for each point $q \in \pi^{-1}(h) \cap Q$, there exists a positive integer $i(q)$ such that

$$1/i(q) < \min\{h, 1-h\}$$

$$[\pi^{-1}(S_{1/i(q)}(h))]_{\rho(q)} \subset Q$$

Then there exists a j such that

$$Q_1 = \{q \in \pi^{-1}(h) \cap Q: i(q) = j\}$$

is cofinal. Let $k = \max\{i, j\}$. Then

$$B_1 = \{z \in \pi^{-1}(h): [\pi^{-1}(S_{1/k}(h))]_{\rho(z)} \subset B\}$$

is cofinal. Moreover, by the closedness of B , $\rho(B_1)$ is closed in J . Hence,

$$B_2 = \cup \{ [\pi^{-1}([h - 1/(2k), h + 1/(2k)])]_{\rho(z)} : z \in B_1 \}$$

is a closed subset of B . By Remark 17-2, $\text{Ind } B_2 = 1$. Hence,

$$\text{Ind } B \geq \text{Ind } B_2 = 1$$

It is to be noticed that the above observation contains the assertion: Both C_P and C_Q are open in I . Since

$$E_B = I - C_P \cup C_Q$$

E_B is closed in I .

Suppose that E_B is not nowhere-dense in I . Then by the closedness of E_B , E_B contains a closed interval $I' \subset I$. To prove $\rho(\pi^{-1}(I') \cap P)$ is not cofinal, assume the contrary. Then there would exist a positive number ε such that

$$\rho(\{p \in \pi^{-1}(I') \cap P : [\pi^{-1}(S_\varepsilon(\pi(p)))]_{\rho(p)} \subset P\})$$

would be cofinal. Then there would exist a closed subinterval I'' of I' whose length would be less than $\varepsilon/4$ such that

$$\{\alpha : [\pi^{-1}(I'')]_\alpha \subset P\}$$

would be cofinal. We have now $I'' \subset C_P$ and hence $I'' \cap E_B = \emptyset$, which is a contradiction. Thus, $\rho(\pi^{-1}(I') \cap P)$ is not cofinal. For the same reason $\rho(\pi^{-1}(I') \cap Q)$ is not cofinal. Hence, there exists an $\eta \in J$ such that

$$B_3 = [\pi^{-1}(I')]_{(\eta, \omega_1)} \subset B$$

Since B_3 is closed and $\text{Ind } B_3 = 1$ by Remark 17-2, $\text{Ind } B \geq \text{Ind } B_3 = 1$.

Let us consider the last case when $C_P \cap C_Q = \emptyset$ and E_B is nowhere-dense in I . Set

$$a = \sup C_P, \quad b = \inf C_Q$$

Since C_P and C_Q are disjoint open sets in I , $0 < a < 1$ and $0 < b < 1$. If $a < b$, then E_B would contain the interval $[a, b]$, which is a contradiction. Hence, $b \leq a$ and $a \in E_B$. Let $a_1, a_2, \dots$ be a monotonically increasing sequence of I such that (i) $\sup a_i = a$ and (ii) every $a_i \in C_P$. Let $b_1, b_2, \dots$ be a monotonically decreasing sequence of I such that (i) $\inf b_i = a$ and (ii) every $b_i \in C_Q$. Such a sequence does exist because we are now considering the case when E_B is nowhere-dense. Let c be a point of $f^{-1}(a)$. Let η_1 be an ordinal such that $[\pi^{-1}(a)]_{\eta_1} \neq \emptyset$. Set

$$J_1 = \{\alpha : \eta_1 < \alpha < \omega_1, \quad [\pi^{-1}(a)]_\alpha \in \overline{\pi^{-1}(I) \cap P} \cap \overline{\pi^{-1}(I) \cap Q}\}$$

To see J_1 is cofinal in J let α_0 be an arbitrary ordinal with $\eta_1 < \alpha_0$. Then there exist a sequence of points $p_i \in \pi^{-1}(a_i) \cap P$ and a sequence of points $q_i \in \pi^{-1}(b_i) \cap Q$ such that

$$\alpha_0 < \rho(p_1) < \rho(q_1) < \rho(p_2) < \rho(q_2) < \dots$$

Since

$$\langle \sup \rho(p_i), a \rangle \in \overline{\pi^{-1}(I) \cap P} \cap \overline{\pi^{-1}(I) \cap Q}$$

$\sup \rho(p_i) \in J_1$. Thus J_1 is cofinal. It is almost evident that J_1 is closed in J .

For every point $p \in \pi^{-1}(I) \cap P$ there exists a finite set K_p of M such that

$$[\pi^{-1}(f^{-1}(\pi(p)) \times (M - K_p) \times I)]_{\rho(p)} \subset P$$

For every point $q \in \pi^{-1}(I) \cap Q$ there exists a finite set K_q of M such that

$$[\pi^{-1}(f^{-1}(\pi(q)) \times (M - K_q) \times I)]_{\rho(q)} \subset Q$$

Set

$$M_1 = \cup \{K_p : p \in \pi^{-1}(I) \cap P\}$$

$$M_2 = \cup \{K_q : q \in \pi^{-1}(I) \cap Q\}$$

Since $|\pi^{-1}(I)| = \mathfrak{c}$, $|M_1| \leq \mathfrak{c}$ and $|M_2| \leq \mathfrak{c}$. Hence,

$$M - M_1 \cup M_2 \neq \emptyset$$

Take an arbitrary element μ from $M - M_1 \cup M_2$, an arbitrary ordinal γ from J_1 and an arbitrary point x from I_γ . Let c be an arbitrary point of $f^{-1}(a)$. Then $t = \langle c, \mu, x \rangle$ is a point of $T - I$.

We want to show that $\langle \gamma, t \rangle \in \bar{P} \cap \bar{Q}$. Let U be an arbitrary open neighborhood of c in C_0 , δ an arbitrary positive number and ξ an arbitrary ordinal with $\eta_1 \leq \xi < \gamma$. Consider a basic neighborhood

$$V = [\pi^{-1}(U \times \{\mu\} \times S_\delta(x))]_{(\xi, \gamma)}$$

of the point $\langle \gamma, t \rangle$ in Z and let us prove that V meets both P and Q . Since $f(U)$ is an open neighborhood of a ,

$$W = [\pi^{-1}(f(U))]_{(\xi, \gamma)}$$

is a relatively open neighborhood of $\langle \gamma, a \rangle$ in $\pi^{-1}(I)$. Hence W meets both P and Q by the definition of J_1 . Let p_0 be a point of $W \cap P$ and q_0 a point of $W \cap Q$. Then $f^{-1}(\pi(p_0)) \cap U \neq \emptyset$ and $f^{-1}(\pi(q_0)) \cap U \neq \emptyset$. Since $I_{\rho(p_0)}$ and $I_{\rho(q_0)}$ are dense in I , V meets both P and Q . Hence $\langle \gamma, t \rangle \in \bar{P} \cap \bar{Q} = B$. Since x was an arbitrary point of I_γ ,

$$[\pi^{-1}(\{c\} \times \{\mu\} \times I)]_\gamma \subset B$$

Therefore,

$$B_4 = [\pi^{-1}(\{c\} \times \{\mu\} \times I)]_{J_1} \subset B$$

Since B_4 is closed in Z and $\text{Ind } B_4 = 1$ by Remark 17-2, we get

$$\text{Ind } B \geq \text{Ind } B_4 = 1 \square$$

19-3. PROBLEM. Is there a compact Hausdorff space X with $\text{ind } X < \text{Ind } X$? This is almost the last, and a very exciting, problem concerning the gaps between the three basic dimension functions.

Chapter 5

DIMENSION-CHANGING CLOSED MAPPINGS

In this chapter we study dimension-lowering and dimension-raising closed mappings for metric and nonmetric spaces. Section 20 is a study of the relative dimension. As can be seen in Section 21, this idea is useful in the treatment of dimension-lowering closed mappings. Recall that in Section 12 we considered dimension-raising closed mappings for the metric case. Since dimension-raising closed mappings are very important, the remaining part of the chapter is devoted to a deeper study of such mappings. In Section 22 we give a necessary and sufficient condition for a nonmetric space to be generated by a 0-dimensional space under a closed mapping of order $\leq n + 1$. The concept of a fine directed family of coverings is used in proving this result. Section 23 gives a conventional dimension-raising theorem for nonmetric spaces. Section 24 is a deeper study of dimension-raising closed mappings for the metric case than was given in Section 12. These results, originally proved for separable metric spaces, are modernized to general metric spaces.

20. Relative Dimension

20-1. DEFINITION. Let X be a space and Y a subset of X . Let $\mathcal{G}$ be an open collection of X which covers Y . Then the *relative dimension* of Y with respect to $\mathcal{G}$, denoted by $\mathcal{G}\text{-dim } Y$, is at most n if $\mathcal{G} \upharpoonright Y$ is refined by a relatively open covering of Y whose order is at most $n + 1$.

20-2. LEMMA (Morita [2]). Let X be a normal space and F_1, F_2 be a disjoint pair of closed sets of X . If $\mathcal{G}$ is a locally finite open covering of X

refining the binary open covering $\{X - F_1, X - F_2\}$, then there exist a closed set B of X and a locally finite open covering $\mathcal{H}$ of X satisfying the following conditions:

1. B separates F_1 and F_2
2. $\mathcal{H}$ refines $\mathcal{G}$ in a one-one corresponding way
3. $\text{ord}_x \mathcal{H} < \text{ord}_x \mathcal{G}$ for any point x in B

PROOF. Since $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ is shrinkable by Theorem 1-7, there exists an open covering $\mathcal{U} = \{U_\alpha\}$ of X in a one-one corresponding way such that $\bar{U}_\alpha \subset G_\alpha$ for every α . Let $\mathcal{U}_1 = \{U_\alpha: \alpha \in A_1\}$ be the family of all elements of $\mathcal{U}$ which meets F_1 . Set

$$B = B(S(F_1, \mathcal{U}_1))$$

Then B is a closed subset of X separating F_1 and F_2 . Let x be an arbitrary point of B . Then there exists a U_α in $\mathcal{U}_1$ with $x \in \bar{U}_\alpha$ since $\mathcal{G}$ and hence $\mathcal{U}$ is locally finite. Since $\mathcal{U}$ covers X there exists an element U_β of $\mathcal{U} - \mathcal{U}_1 = \{U_\alpha: \alpha \in A_2\}$ with $x \in U_\beta$. By this observation it is now easy to see that

$$\mathcal{H} = \{G_\alpha: \alpha \in A_1\} \cup \{G_\alpha - B: \alpha \in A_2\}$$

is an open covering of X , actually in a one-one corresponding way, which satisfies conditions (2) and (3). $\square$

20-3. LEMMA. Let X be a normal space, $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ a locally finite open covering of X and B a closed set of X with $\mathcal{G}\text{-dim } B \leq n - 1$, then there exist an open covering $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ of X and an open set $V \supset B$ such that $H_\alpha \subset G_\alpha$ for each α and $\text{ord } \mathcal{H} \upharpoonright V \leq n$.

PROOF. Since $\mathcal{G}\text{-dim } B \leq n - 1$, it is easy to get a closed covering $\{K_\alpha\}$ of B such that $\text{ord}\{K_\alpha\} \leq n$ and $K_\alpha \subset G_\alpha$ for each α . By Proposition 9-2 there exists for each α an open set H'_α of X such that $\text{ord}\{H'_\alpha\} \leq n$ and $K_\alpha \subset H'_\alpha \subset G_\alpha$. Let V be an open set of X with $B \subset V \subset \bar{V} \subset \cup H'_\alpha$. Set

$$H_\alpha = H'_\alpha \cup (G_\alpha - \bar{V})$$

Then V and $\mathcal{H} = \{H_\alpha\}$ satisfy the desired condition. $\square$

20-4. THEOREM (Morita [2]). A normal space X has $\dim X \leq n$ if and only if, for each disjoint pair of closed sets F_1, F_2 of X and for each finite open covering $\mathcal{G}$ of X , there exists a closed set B separating F_1 and F_2 such that $\mathcal{G}\text{-dim } B \leq n - 1$.

PROOF. Suppose that $\dim X \leq n$. Let

$$\mathcal{G}_1 = \mathcal{G} \wedge \{X - F_1, X - F_2\}$$

$\mathcal{G}_1$ can be refined by a finite open covering $\mathcal{G}_2$ with $\text{ord } \mathcal{G}_2 \leq n + 1$. By Lemma 20-2 there exist a closed set B separating F_1 and F_2 and a finite open covering $\mathcal{H}$ of X such that $\text{ord } \mathcal{H} \mid B \leq n$, which implies $\mathcal{G}\text{-dim } B \leq n - 1$.

Conversely, suppose that the condition is satisfied. Let $\mathcal{G} = \{G_1, \dots, G_k\}$ be an arbitrary finite open covering of X . Let $\{F_i\}$ be a closed covering of X such that $F_i \subset G_i$ for each i . Let U_1 be an open set of X such that

$$F_1 \subset U_1 \subset \bar{U}_1 \subset G_1, \quad \text{and} \quad \mathcal{G}\text{-dim } B(U_1) \leq n - 1$$

By Lemma 20-3, there exist an open set V_1 of X with $V_1 \supset B(U_1)$ and an open covering $\mathcal{G}_1 = \{G_{1i}\}$ of X such that

$$G_{1i} \subset G_i \quad \text{for each } i, \quad \text{and} \quad \text{ord } \mathcal{G}_1 \mid V_1 \leq n$$

Analogously, by repeated application of Lemma 20-3, we can get for $i = 2, \dots, k$, open sets U_i with $F_i \subset U_i \subset \bar{U}_i \subset G_i$, open sets V_i with $V_i \supset B(U_i)$ and open coverings $\mathcal{G}_i = \{G_{ij}\}$ of X with $G_{ij} \subset G_{i-1,j}$ and with $\text{ord } \mathcal{G}_i \mid V_i \leq n$. Then,

$$\text{ord } \mathcal{G}_k \mid \bigcup V_i \leq n$$

Let

$$\mathcal{H}_i = \{U_i, U'_i = X - \bar{U}_i\} \quad \text{for } i = 1, \dots, k$$

Set $\mathcal{H} = \bigwedge \mathcal{H}_i$. Then $\mathcal{H}$ covers $X - \bigcup B(U_i)$ and $\text{ord } \mathcal{H} \leq 1$. Moreover, $\mathcal{H}$ refines $\mathcal{G}$ because $\bigcap U'_i = \phi$. Set

$$\mathcal{U} = \mathcal{H} \cup (\mathcal{G}_k \mid \bigcup V_i)$$

Then $\mathcal{U}$ is an open covering of X with $\mathcal{U} < \mathcal{G}$ and with $\text{ord } \mathcal{U} \leq n + 1$, which implies $\dim X \leq n$. $\square$

20-5. PROPOSITION. Let X be a normal space, $\mathcal{G}$ a locally finite open covering of X and F a closed set of X such that (i) $\mathcal{G}\text{-dim } F \leq n$ and (ii) $\dim(X \bmod F) \leq n$, then $\mathcal{G}\text{-dim } X \leq n$.

This is proved in a way similar to the proof of Theorem 9-8, if we use Proposition 9-2 as a lemma.

20-6. THEOREM.[†] Let X be a normal space, $\mathcal{G}$ a locally finite open covering of X , $\{F_\alpha: \alpha \in A\}$ a closed covering of X and $U_\alpha, \alpha \in A$, open sets which satisfy the following conditions:

1. $F_\alpha \subset U_\alpha$ for each $\alpha \in A$
2. $\{U_\alpha: \alpha \in A\}$ is locally finite in X
3. $\mathcal{G}\text{-dim } F_\alpha \leq n$ for each $\alpha \in A$
4. $\dim F_\alpha \cap F_\beta \leq n - 1$ whenever $\alpha \neq \beta$

Then $\mathcal{G}\text{-dim } X \leq n$.

PROOF. Set

$$F = \cup \{F_\alpha \cap F_\beta: \alpha \neq \beta\}$$

Then $\dim F \leq n - 1$ by Theorem 11-7, since $\text{uloc dim } F \leq n - 1$ as can easily be seen. Let $\{C_{\alpha\lambda}: \lambda \in \Lambda_\alpha\}$ be a closed covering of F_α and $\{V_{\alpha\lambda}: \lambda \in \Lambda_\alpha\}$ a locally finite open collection of X such that

$$V_{\alpha\lambda} \subset U_\alpha \quad \text{for each } \lambda \in \Lambda_\alpha$$

$$\text{ord}\{V_{\alpha\lambda}: \lambda \in \Lambda_\alpha\} \leq n + 1 \quad \text{for each } \alpha \in A$$

$$\{V_{\alpha\lambda}: \lambda \in \Lambda_\alpha, \alpha \in A\} \text{ refines } \mathcal{G}$$

Let

$$\mathcal{V} = \wedge \{ \{V_{\alpha\lambda}, X - C_{\alpha\lambda}\}: \lambda \in \Lambda_\alpha, \alpha \in A \}$$

Let $\mathcal{V}_1$ be the subcollection of all elements of $\mathcal{V}$ which meet $\cup F_\alpha$. Then $\mathcal{V}_1$ is locally finite in X . Let $\mathcal{W} = \{\mathcal{W}_\xi: \xi \in \Xi\}$ be an open collection of X such that

$$\mathcal{W} \text{ is locally finite in } X, \quad \mathcal{W} \text{ covers } F$$

$$\text{ord } \overline{\mathcal{W}} \leq n, \quad \text{and} \quad \overline{\mathcal{W}} \text{ refines } \mathcal{G} \wedge \mathcal{V}_1$$

Let W be the sum of all elements of $\mathcal{W}$. Fix an arbitrary α . Set $H_\alpha = F_\alpha - W$. Applying an easy transfinite induction on Ξ , considering it well-ordered, each $\overline{W}_\xi$ can be the sum of two closed sets $P_{\alpha\xi}$ and $Q_{\alpha\xi}$ such that

$$P_{\alpha\xi} \cap F_\alpha \subset W, \quad Q_{\alpha\xi} \cap F = \phi$$

$$P_{\alpha\xi} \cap Q_{\alpha\xi} \cap P_{\alpha\eta} \cap Q_{\alpha\eta} = \phi \quad \text{whenever } \xi \neq \eta$$

[†] This is called the addition theorem of Hurewicz who proved it for the case when X is separable metric and A consists of two indices. The present expression was given by Morita [5].

Consider Λ_α well-ordered. Take $\lambda \in \Lambda_\alpha$. Set

$$\Xi_{\alpha\lambda} = \{\xi \in \Xi: Q_{\alpha\xi} \cap (\bigcup_{\mu < \lambda} C_{\alpha\mu}) = \phi, \quad Q_{\alpha\xi} \cap C_{\alpha\lambda} \neq \phi\}$$

$$D_{\alpha\lambda} = (\bigcup \{Q_{\alpha\xi}: \xi \in \Xi_{\alpha\lambda}\}) \cup (C_{\alpha\lambda} \cap H_\alpha)$$

$$P_\xi = \bigcap \{P_{\alpha\xi}: \alpha \in A\}$$

Let us show that

$$\mathcal{F} = \{P_\xi: \xi \in \Xi\} \cup \{D_{\alpha\lambda}: \lambda \in \Lambda_\alpha, \quad \alpha \in A\}$$

is a locally finite closed covering of X with $\text{ord } \mathcal{F} \leq n+1$ and with $\mathcal{F} < \mathcal{G}$, which will imply $\mathcal{G}\text{-dim } X \leq n$ by Proposition 9-2. Evidently $\mathcal{F}$ is a closed covering of X . Since $\{P_\xi: \xi \in \Xi\} < \overline{\mathcal{W}}$, $\{P_\xi: \xi \in \Xi\} < \mathcal{G}$. Since

$$D_{\alpha\lambda} \subset S(C_{\alpha\lambda}, \overline{\mathcal{W}}) \subset V_{\alpha\lambda}$$

we have

$$\{D_{\alpha\lambda}: \lambda \in \Lambda_\alpha, \quad \alpha \in A\} < \mathcal{G}$$

Hence, $\mathcal{F} < \mathcal{G}$.

Let x be a point of X . When x is not contained in any $D_{\alpha\lambda}$,

$$\text{ord}_x \mathcal{F} = \text{ord}_x \{P_\xi: \xi \in \Xi\} \leq \text{ord}_x \overline{\mathcal{W}} \leq n$$

When x is not contained in any P_ξ , there is one and only one β with $x \in F_\beta - F$. Then

$$\begin{aligned} \text{ord}_x \mathcal{F} &= \text{ord}_x \{D_{\alpha\lambda}: \lambda \in \Lambda_\alpha, \quad \alpha \in A\} \\ &= \text{ord}_x \{D_{\beta\lambda}: \lambda \in \Lambda_\beta\} \\ &\leq \text{ord}_x \{V_{\beta\lambda}: \lambda \in \Lambda_\beta\} \leq n+1 \end{aligned}$$

Let us consider the last case. Let x be in $F_\beta - F$. Suppose that $\text{ord}_x \mathcal{F} \geq n+2$. Then there would be $n+2$ elements

$$P_{\xi_1}, \dots, P_{\xi_k}, \quad D_{\beta\lambda_{k+1}}, \dots, D_{\beta\lambda_{n+2}}$$

which would contain x at the same time. Since $x \notin H_\beta$ for this case, there would be $Q_{\beta\eta_i}$, $i = k+1, \dots, n+2$, such that

$$x \in Q_{\beta\eta_i} \subset D_{\beta\lambda_i}, \quad i = k+1, \dots, n+2$$

where η_i 's would be different from each other. Now x would be contained in each of

$$P_{\beta\xi_1}, \dots, P_{\beta\xi_k}, \quad Q_{\beta\eta_{k+1}}, \dots, Q_{\beta\eta_{n+2}}$$

Since $2 \leq k \leq n$,

$$P_{\beta\xi_i} \subset \overline{W}_{\xi_i} \quad \text{and} \quad Q_{\beta\eta_j} \subset \overline{W}_{\eta_j}$$

and $\text{ord } \overline{\mathcal{W}} \leq n$, then two of ξ_i 's would coincide with two η_j 's. Without loss of generality we could set

$$\xi_1 = \eta_{k+1}, \quad \xi_2 = \eta_{k+2}$$

Then

$$x \in P_{\beta\xi_1} \cap P_{\beta\xi_2} \cap Q_{\beta\xi_1} \cap Q_{\beta\xi_2}$$

which gives a contradiction. Therefore, $\text{ord } \mathcal{F} \leq n + 1$. $\square$

21. Dimension-Lowering Closed Mappings

21-1. THEOREM (Morita [8]). Let f be a closed mapping of a normal space X onto a nonempty paracompact Hausdorff space Y , then

$$\dim X \leq \sup_{y \in Y} \dim f^{-1}(y) + \text{Ind } Y$$

PROOF (by induction on $\text{Ind } Y$). Since the proof for the case when $\text{Ind } Y = 0$ is completely analogous to the general case, we omit it. Assume that the theorem is true for $\text{Ind } Y \leq n - 1$, $n \geq 2$. Let $\text{Ind } Y = n$ and $\sup \dim f^{-1}(y) = m$. Let $\mathcal{G}$ be an arbitrary finite open covering of X . For each point $y \in Y$, let H_y be an open set of X such that $\mathcal{G}\text{-dim } H_y \leq m$ and $f^{-1}(y) \subset H_y$. Set

$$V_y = Y - f(X - H_y)$$

Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be a locally finite open covering of Y refining $\{V_y : y \in Y\}$, where A is well-ordered. Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a closed covering of Y such that $F_\alpha \subset U_\alpha$ for each $\alpha \in A$. For each $\alpha \in A$ let W_α be an open set of Y such that

$$F_\alpha \subset W_\alpha \subset \overline{W}_\alpha \subset U_\alpha, \quad \text{Ind } B(W_\alpha) \leq n - 1$$

Set

$$H_0 = W_0$$

$$H_\alpha = W_\alpha - \bigcup_{\beta < \alpha} \overline{W}_\beta, \quad \alpha > 0$$

Then we have $Y = \bigcup \{\tilde{H}_\alpha : \alpha \in A\}$ and $\text{Ind } \tilde{H}_\alpha \cap \tilde{H}_\beta \leq n - 1$ whenever $\alpha \neq \beta$. By the induction assumption we have:

$$\dim f^{-1}(\tilde{H}_\alpha) \cap f^{-1}(\tilde{H}_\beta) \leq m + n - 1, \quad \alpha \neq \beta$$

Since $\mathcal{G}\text{-dim } f^{-1}(\tilde{H}_\alpha) \leq m$, we have

$$\mathcal{G}\text{-dim } X \leq m + n$$

by Theorem 20-6. Hence, $\dim X \leq m + n$.

When $\text{Ind } Y = \infty$ or $\sup \dim f^{-1}(y) = \infty$, the theorem is trivially true. $\square$

21-2. COROLLARY. Let f be a closed mapping of a normal space X onto a nonempty metric space Y . Then

$$\dim X \leq \sup \dim f^{-1}(y) + \dim Y$$

This is evident since $\dim Y = \text{Ind } Y$, by Theorem 12-6.

21-3. REMARK. It is to be noted that $\dim X \leq \dim Y$ under the same assumption as in Theorem 21-1 and with an additional condition: $\sup \dim f^{-1}(y) = 0$. Let $\mathcal{G}, f, H_y$ and V_y be those in the proof of Theorem 21-1. Let $\mathcal{U}_y$ be an open covering of H_y with $\text{ord } \mathcal{U}_y = 1$ and $\mathcal{U}_y < \mathcal{G}$. Let $\mathcal{V} = \{V(y)\}$ be an open covering of Y with $\text{ord } \mathcal{V} \leq \dim Y + 1$ such that $V(y) \subset V_y$ for every $y \in Y$. Set

$$\mathcal{U} = \cup \{\mathcal{U}_y \mid f^{-1}(V(y))\}; y \in Y\}$$

Then $\mathcal{U}$ is an open covering of X such that $\mathcal{U} < \mathcal{G}$ and $\text{ord } \mathcal{U} \leq \dim Y + 1$, which implies $\dim X \leq \dim Y$.

22. Directed Families of Closed Coverings

22-1. DEFINITION. Let X be a space. Let $\{A_\lambda, \varphi_\mu^\lambda: \lambda \in \Lambda\}$ be an inverse limiting system with discrete spaces of indices A_λ 's. Let

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha: \alpha \in A_\lambda\}: \lambda \in \Lambda\}$$

be a family of closed coverings of X such that for every ordered pair $\mu < \lambda$ and every $\alpha \in A_\mu$

$$F_\alpha = \cup \{F_\beta: \varphi_\mu^\lambda(\beta) = \alpha\}$$

Then $\mathfrak{F}$ is called a *directed family* of X . The *order* of $\mathfrak{F}$, $\text{ord } \mathfrak{F}$, is the supremum of $\{\text{ord } \mathcal{F}_\lambda: \lambda \in \Lambda\}$. A directed family $\mathfrak{F}$ is called *locally fine*, *globally fine*, or *fully fine*, if the following conditions are respectively satisfied:

1. For each point x of X and each neighborhood U of x , there exists a $\lambda \in \Lambda$ with $S(x, \mathcal{F}_\lambda) \subset U$
2. For each pair $F \subset U$ with F closed and U open, there exists a $\lambda \in \Lambda$ with $S(F, \mathcal{F}_\lambda) \subset U$
3. For each open covering $\mathcal{G}$ of X , there exists a $\lambda \in \Lambda$ with $\mathcal{F}_\lambda < \mathcal{G}$

22-2. THEOREM (Nagami [7] and Pasynkov [1]). For a nonempty completely regular space X the following conditions are equivalent:

1. There exists a locally fine, directed family

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha : \alpha \in A_\lambda\} : \lambda \in \Lambda\}$$

of locally finite closed coverings of X with $\text{ord } \mathfrak{F} \leq n + 1$, accompanied by an inverse limiting system $\{A_\lambda, \varphi_\mu^\lambda\}$

2. X_1 is the image of a completely regular space A with $\text{ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$

PROOF. That (1) implies (2): Let B be the limit space of $\{A_\lambda, \varphi_\mu^\lambda\}$. Let A be the aggregate of all points $a = \langle \alpha_\lambda : \lambda \in \Lambda \rangle \in B$ such that $\cap \{F_{\alpha_\lambda} : \lambda \in \Lambda\} \neq \emptyset$. Let x be an arbitrary point of X . Set

$$B_\lambda = \{\alpha : x \in F_\alpha \in \mathcal{F}_\lambda\}, \quad \lambda \in \Lambda$$

Then for any $\lambda \in \Lambda$, $B_\lambda \neq \emptyset$. Moreover, $\varphi_\mu^\lambda(B_\lambda) \subset B_\mu$ whenever $\mu < \lambda$. Hence, by Assertion 3 of Definition 6-1,

$$A(x) = \text{inv } \lim \{B_\lambda\} \neq \emptyset$$

Let $f: A \rightarrow X$ be a transformation in such a way that $f(A(x)) = x$ for every $x \in X$. Then it is easy to see that f is continuous and onto. A is evidently a completely regular space with $\text{ind } A = 0$. If we assume there exist $n + 2$ points $a_1, \dots, a_{n+2}$ of A with

$$f(a_1) = \dots = f(a_{n+2})$$

the a_i 's would have different coordinates $\alpha_1, \dots, \alpha_{n+2}$ respectively on a sufficiently large A_λ . Since $f(a_i) \in F_{\alpha_i}$,

$$\text{ord}_{f(a_1)} \mathcal{F}_\lambda > n + 1$$

which is a contradiction. Hence, $\text{ord } f \leq n + 1$.

To prove the closedness of f let H be an arbitrary nonempty closed set of A and x an arbitrary point of $\overline{f(H)}$. Let

$$D_\lambda = \{\alpha : x \in F_\alpha \in \mathcal{F}_\lambda, \quad f(H) \cap F_\alpha \neq \emptyset\}, \quad \lambda \in \Lambda$$

Since $S(x, \mathcal{F}_\lambda)$ contains $X - \cup \{F_\alpha : x \notin F_\alpha \in \mathcal{F}_\lambda\}$, $S(x, \mathcal{F}_\lambda)$ is a neighborhood of x . Hence, $D_\lambda \neq \emptyset$ for any $\lambda \in \Lambda$. It is evident that

$$\varphi_\lambda^\mu(D_\mu) \subset D_\lambda, \quad \lambda < \mu$$

Since $f(H) \cap S(x, \mathcal{F}_\lambda) \neq \emptyset$, $H \cap f^{-1}(S(x, \mathcal{F}_\lambda)) \neq \emptyset$. Since

$$f^{-1}(X - \cup \{F_\alpha : x \notin F_\alpha \in \mathcal{F}_\lambda\}) \subset \cup \{\pi_\lambda^{-1}(\alpha) : \alpha \in D_\lambda\}$$

where $\pi_\lambda : A \rightarrow A_\lambda$ is the projection, then

$$E_\lambda = \{\alpha \in D_\lambda : H \cap \pi_\lambda^{-1}(\alpha) \neq \emptyset\} \neq \emptyset$$

for each $\lambda \in \Lambda$. Since $\varphi_\mu^\lambda(E_\lambda) \subset E_\mu$ for each pair $\mu < \lambda$, $\{E_\lambda, \varphi_\mu^\lambda\}$ forms an inverse limiting system. Set

$$E = \text{inv lim } E_\lambda$$

Then $E \neq \emptyset$ by Assertion 3 of Section 6-1. Let $\langle \alpha_\lambda : \lambda \in \Lambda \rangle$ be an arbitrary point of E . Then $\cap \{F_{\alpha_\lambda} : \lambda \in \Lambda\} = x$. Hence we have $E \subset A$. Let a be an arbitrary point of E . Then $\pi_\lambda^{-1}(\pi_\lambda(a)) \cap H \neq \emptyset$ for any $\lambda \in \Lambda$. Hence we have $a \in \bar{H} = H$. On the other hand $f(a) \in f(E) = x$. Hence $x \in f(H)$, and the closedness of f is proved.

That (2) implies (1): By the remark at the end of Proposition 8-5, A is a dense subset of an inverse limit of the system $\{A_\lambda, \varphi_\mu^\lambda : \lambda \in \Lambda\}$ with finite nonempty discrete spaces A_λ . Let $\pi_\lambda : A \rightarrow A_\lambda$, $\lambda \in \Lambda$, be the projections. Set

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha = f(\pi_\lambda^{-1}(\alpha)) : \alpha \in A_\lambda\} : \lambda \in \Lambda\}$$

Then $\mathfrak{F}$ is a directed family of finite closed coverings with $\text{ord } \mathfrak{F} \leq n + 1$.

To prove the $\mathfrak{F}$ is actually locally fine let x be an arbitrary point of X and U an arbitrary open neighborhood of x . Let

$$|f^{-1}(x)| = j \quad \text{and} \quad f^{-1}(x) = \{a_1, \dots, a_j\}$$

For a sufficiently large $\mu \in \Lambda$

$$\{\pi_\mu(a_i) : i = 1, \dots, j\}$$

consists of mutually different representatives and

$$f(\pi_\mu^{-1}(\pi_\mu(a_i))) \subset U, \quad i = 1, \dots, j$$

Then we have

$$f(\pi_\mu^{-1}(\pi_\mu(f^{-1}(x)))) \subset U$$

$$|\pi_\mu(f^{-1}(x))| = j$$

Suppose that there exists an $\alpha \in A_\mu$ such that $\alpha \notin \pi_\mu(f^{-1}(x))$ and $x \in f(\pi_\mu^{-1}(\alpha)) = F_\alpha$. Then

$$|f^{-1}(x)| \geq |\pi_\mu(f^{-1}(x))| \geq j + 1$$

which is a contradiction. Therefore, $S(x, \mathcal{F}_\mu) \subset U$. $\square$

22-3. THEOREM (Nagami [7]). If a nonempty space X is the image of a normal space A with $\text{Ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$, then X is a normal space having a globally fine, directed family $\mathfrak{F}$, with $\text{ord } \mathfrak{F} \leq n + 1$, of finite closed coverings of X .

PROOF. To prove X is normal let H_1 and H_2 be a disjoint pair of closed sets of X . Let U_1 and U_2 be open sets of A such that

$$U_i \supset f^{-1}(H_i), \quad i = 1, 2 \quad \text{and} \quad U_1 \cap U_2 = \emptyset$$

Set

$$V_i = X - f(A - U_i)$$

Then the V_i 's are open, $V_1 \cap V_2 = \emptyset$ and $V_i \supset H_i$, $i = 1, 2$. Thus, X is normal.

Let βA be the Stone-Čech compactification of A . Then $\text{Ind } \beta A = 0$ by Theorem 10-3. By Proposition 8-5, we can consider βA as the inverse limit of an inverse limiting system

$$\{A_\lambda, \varphi_\mu^\lambda: \lambda \in \Lambda\}$$

of finite discrete spaces A_λ . Let $\pi_\lambda: A \rightarrow A_\lambda$ be the restriction of the projection $\tilde{\pi}_\lambda: \beta A \rightarrow A_\lambda$. Then

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha = f(\pi_\lambda^{-1}(\alpha)): \alpha \in A_\lambda\}: \lambda \in \Lambda\}$$

is a directed family of finite closed coverings of X . Moreover it is evident that $\text{ord } \mathfrak{F} \leq n + 1$.

To prove that $\mathfrak{F}$ is globally fine let $F \subset G$ be a pair consisting of a closed set F and an open set G . Let φ be a bounded real valued continuous function of A such that

$$\begin{aligned} \varphi(a) &= 0, & a \in f^{-1}(F) \\ \varphi(a) &= 1, & a \in A - f^{-1}(G) \end{aligned}$$

Let ψ be an extension of φ to βA . If we set

$$P_i = \{a: \psi(a) = i\}, \quad i = 0, 1$$

then we have an open covering

$$\mathcal{G} = \{\beta A - P_0, \beta A - P_1\}$$

of βA . Since βA is compact, there exists an index μ of Λ such that

$$\{\tilde{\pi}_\lambda^{-1}(\alpha): \alpha \in A_\mu\} \in \mathcal{G}$$

Let

$$D_0 = S(P_0, \{\tilde{\pi}_\lambda^{-1}(\alpha) : \alpha \in A_\mu\})$$

Then

$$P_0 \subset D_0 \subset \beta A - P_1$$

Hence we have

$$\begin{aligned} f^{-1}(F) &\subset A \cap P_0 \subset A \cap D_0 \subset A \cap (\beta A - P_1) \\ &= A \cap \beta A - A \cap P_1 = A - A \cap P_1 \\ &\subset A - (A - f^{-1}(G)) = f^{-1}(G) \end{aligned}$$

On the other hand

$$S(f^{-1}(F), \{\pi_\mu^{-1}(\alpha) : \alpha \in A_\mu\}) \subset A \cap D_0$$

Hence

$$f^{-1}(F) \subset S(f^{-1}(F), \{\pi_\mu^{-1}(\alpha) : \alpha \in A_\mu\}) \subset f^{-1}(G)$$

Therefore $F \subset S(F, \mathcal{F}_\mu) \subset G$ and we know $\mathfrak{F}$ is globally fine. $\square$

22-4. DEFINITION. Let $\{A_\lambda, \varphi_\mu^\lambda : \lambda \in \Lambda\}$ be an inverse limiting system of discrete spaces A_λ . Let π_λ be the projection of $A = \text{inv lim}\{A_\lambda\}$ to A_λ . The system $\{A_\lambda\}$ is *full* if every open covering of A is refined by $\{\pi_\lambda^{-1}(\alpha) : \alpha \in A_\lambda\}$ for some $\lambda \in \Lambda$.

22-5. LEMMA (Nagami [6]). In order that a space X be a paracompact Hausdorff space with $\dim X = 0$ it is necessary and sufficient that X be homeomorphic to the nonempty limit space of an inverse limiting full system of discrete spaces.

PROOF. It is evident that the condition is sufficient. To prove the necessity let

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha : \alpha \in A_\lambda\} : \lambda \in \Lambda\}$$

be the family of all closed coverings of X of order 1. Introduce the semi-order $<$ into Λ such that $\lambda < \mu$ implies $\mathcal{F}_\mu$ refines $\mathcal{F}_\lambda$. Then Λ is a directed set. Define $\varphi_\lambda^\mu : A_\mu \rightarrow A_\lambda$ in such a way that

$$\varphi_\lambda^\mu(\alpha) = \beta \quad \text{yields} \quad F_\alpha \subset F_\beta$$

Then $\mathfrak{F}$ is a directed family of locally finite closed coverings with $\text{ord } \mathfrak{F} = 1$ which is fully fine. In an argument analogous to the proof of Theorem 22-2, X is the image of a completely regular space A with $\text{ind } A = 0$ under a closed mapping f with $\text{ord } f = 1$. Since $\text{ord } f = 1$, f is an homeomor-

phism. Hence, A is paracompact. A is actually a dense subset of $\text{inv} \lim \{A_\lambda\} = B$. Let $\mathcal{G}$ be an arbitrary open covering of A . Then there exists a $\lambda \in \Lambda$ with $\mathcal{F}_\lambda < f(\mathcal{G})$. Since

$$f^{-1}(\mathcal{F}_\lambda) = \{\pi_\lambda^{-1}(\alpha) : \alpha \in A_\lambda\}$$

and $f^{-1}(\mathcal{F}_\lambda) < \mathcal{G}$, A is full. As a matter of fact $A = B$, because a paracompact Hausdorff space is, as can easily be seen, complete with respect to the finest uniformity and $\mathfrak{F}$ is a base of the finest uniformity of X . $\square$

22-6. LEMMA (Michael [4]). A space X is paracompact Hausdorff if for every open covering $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ there exists a covering $\mathcal{V}$ which is a cushioned refinement of $\mathcal{U}$.

PROOF. If a covering $\mathcal{V} = \{V_\beta : \beta \in B\}$ is a cushioned refinement, then there exists by definition a refine transformation $f: B \rightarrow A$ such that for any $B_1 \subset B$ the closure of the sum of all V_β , $\beta \in B_1$, is contained in the sum of all U_α , $\alpha \in f(B_1)$. Set

$$W_\alpha = \cup \{V_\beta : f(\beta) = \alpha\}, \quad \alpha \in A$$

Then $\{W_\alpha : \alpha \in A\}$ is a covering of X such that for any $A_1 \subset A$ the closure of the sum of all W_α , $\alpha \in A_1$, is contained in the sum of all U_α , $\alpha \in A_1$. This remark will be used often in the proof.

To prove first that X is normal, let $\{G_1, G_2\}$ be an arbitrary binary open covering of X . Let $\{W_1, W_2\}$ be a covering of X which is a cushioned refinement of $\{G_1, G_2\}$. Then $\overline{W_i} \subset G_i$, $i = 1, 2$. That is to say, a binary open covering of X is shrinkable. Thus, X is normal and hence, X is regular. Therefore, if we can get a σ -discrete open covering refining $\mathcal{U}$, then X is paracompact by Theorem 2-22. We begin by constructing for each positive integer i a covering $\{C_{\alpha i} : \alpha \in A\}$ of X which is a cushioned refinement of $\mathcal{U}$ such that for each α and i

$$\overline{\bigcup_{\beta < \alpha} C_{\beta i}} \cap C_{\alpha, i+1} = \phi \quad (1)$$

$$C_{\alpha i} \cap \overline{\bigcup_{\beta > \alpha} C_{\beta, i+1}} = \phi \quad (2)$$

Consider A as a well-ordered set. Let $\{C_{\alpha 1}\}$ be a covering of X which is a cushioned refinement of $\{U_\alpha\}$ in a one-one corresponding way as noted at the beginning of the proof. In the following this same kind of refinement is used without special notice, but with the same indices. Suppose that

suitable refinements $\{C_{\alpha i}\}$ have been found for $i = 1, \dots, n$, and let us construct $\{C_{\alpha, n+1}\}$. For each α let

$$U_{\alpha, n+1} = U_\alpha - \overline{\bigcup_{\beta < \alpha} C_{\beta n}} \quad (3)$$

Then $\{U_{\alpha, n+1}\}$ is an open covering of X , because $x \in X$ implies $x \in U_{\alpha, n+1}$ for the first α for which $x \in U_\alpha$ (since, by assumption, $\overline{\bigcup_{\beta < \alpha} C_{\beta n}} \subset \bigcup_{\beta < \alpha} U_\beta$). We now pick a cushioned refinement $\{C_{\alpha, n+1}\}$ of $\{U_{\alpha, n+1}\}$. Then (1) follows at once from (3) and the fact that $C_{\alpha, n+1} \subset U_{\alpha, n+1}$. To see (2), note that, by (3), $C_{\alpha n}$ is disjoint from $U_{\beta, n+1}$ for all $\beta > \alpha$, and hence from

$$\overline{\bigcup_{\beta > \alpha} C_{\beta, n+1}} \subset \bigcup_{\beta > \alpha} U_{\beta, n+1}$$

Let us construct an open covering $\{V_{\alpha i} : \alpha \in A, i = 1, 2, \dots\}$ of X such that, for all i

$$V_{\alpha i} \subset U_\alpha \quad \text{for each } \alpha \quad (4)$$

$$V_{\alpha i} \cap V_{\beta i} = \emptyset \quad \text{whenever } \alpha \neq \beta \quad (5)$$

For each α and i , let

$$V_{\alpha i} = X - \overline{\bigcup_{\beta \neq \alpha} C_{\beta i}}$$

Since $\{C_{\alpha i}\}$ covers X for each i , we have

$$V_{\alpha i} \subset C_{\alpha i} \subset U_\alpha$$

for all α and i , which proves (4) and (5). Since each $V_{\alpha i}$ is clearly open, it remains to show that the sets $V_{\alpha i}$ cover X .

Pick an $x \in X$, and let us find a $V_{\alpha i}$ containing x . Let

$$\alpha_i = \min\{\alpha \in A : x \in C_{\alpha i}\}, \quad i = 1, 2, \dots$$

and then pick a positive integer k such that

$$\alpha_k = \min\{\alpha_i : i = 1, 2, \dots\}$$

using the well-ordering of A . Let us show that

$$x \in V_{\alpha_k k+1}$$

Note first that $x \in C_{\alpha_k k}$ by definition of α_k , and hence, by (2) (with $i = k$)

$$x \notin \overline{\bigcup_{\alpha > \alpha_k} C_{\alpha, k+1}} \quad (6)$$

Again from the definition of α_k , we have $x \in C_{\alpha, k+2}$ for some $\alpha \geq \alpha_k$, and hence

$$x \notin \overline{\bigcup_{\beta < \alpha_k} C_{\beta, k+1}} \quad (7)$$

by (1) with $i = k + 1$. It follows from (6) and (7) that $x \in V_{\alpha_k k+1}$.

To complete the proof we obtain once more a covering

$$\mathcal{D} = \{D_{\alpha i} : \alpha \in A, \quad i = 1, 2, \dots\}$$

of X which is a cushioned refinement of $\{V_{\alpha i} : \alpha \in A, i = 1, 2, \dots\}$. Set

$$D_i = \bigcup \{D_{\alpha i} : \alpha \in A\}, \quad V_i = \bigcup \{V_{\alpha i} : \alpha \in A\}$$

Let W_i be an open set with $\bar{D}_i \subset W_i \subset \bar{W}_i \subset V_i$. Set

$$W_{\alpha i} = W_i \cap V_{\alpha i}$$

Then

$$\{W_{\alpha i} : \alpha \in A, \quad i = 1, 2, \dots\}$$

is a σ -discrete open covering of X refining $\mathcal{U}$. $\square$

22-7. COROLLARY (Michael [3]). If a space Y is the image of a paracompact Hausdorff space X under a closed mapping f , then Y is a paracompact Hausdorff space.

PROOF. Let $\mathcal{U}$ be an arbitrary open covering of Y . Let $\mathcal{V}$ be a covering of X which is a cushioned refinement of $f^{-1}(\mathcal{U})$. Then by the closedness of f , $f(\mathcal{V})$ is a cushioned refinement of $\mathcal{U}$, which implies that Y is a paracompact Hausdorff space by lemma 22-6. $\square$

22-8. THEOREM (Nagami [7]). For a nonempty space X the following conditions are equivalent:

1. X is a paracompact Hausdorff space having a fully fine, directed family

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha : \alpha \in A_\lambda\} : \lambda \in \Lambda\}$$

of locally finite closed coverings of X with $\text{ord } \mathfrak{F} \leq n + 1$, accompanied by an inverse limiting system $\{A_\lambda, \varphi_\mu^\lambda\}$

2. X is the image of a paracompact Hausdorff space A with $\text{Ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$

PROOF. That (1) implies (2): Let $A, f: A \rightarrow X$ and $\pi_\lambda: A \rightarrow A_\lambda$ be the same as constructed in the proof of Theorem 22-2. Since $\mathfrak{F}$ is fully fine, it is locally fine. Hence f is a closed mapping of A onto X with $\text{ord } f \leq n + 1$. Thus, what we have to do is to prove that A is a paracompact space with $\text{Ind } A = 0$.

Let $\mathcal{G}$ be an arbitrary open covering of A . For every point x in X let

$$f^{-1}(x) = \{a(x, 1), \dots, a(x, m(x))\}$$

where $m(x) = |f^{-1}(x)|$. Let $V(x, i)$ be an open neighborhood of $a(x, i)$ such that

$$\{V(x, i): i = 1, \dots, m(x)\}$$

is a disjoint collection which refines $\mathcal{G}$. Let

$$W_x = \cup \{V(x, i): i = 1, \dots, m(x)\}$$

$$V(x) = X - f(A - W_x)$$

Then $V(x)$ is an open neighborhood of x .

Let $\mathcal{F}_\lambda$ be an element of $\mathfrak{F}$ such that $\mathcal{F}_\lambda$ refines $\{V(x): x \in X\}$. Then

$$\begin{aligned} \{W_x: x \in X\} &> f^{-1}\{V(x): x \in X\} \\ &> f^{-1}(\mathcal{F}_\lambda) \\ &> \{\pi_\lambda^{-1}(\alpha): \alpha \in A_\lambda\} \end{aligned}$$

Since $\{\pi_\lambda^{-1}(\alpha): \alpha \in A_\lambda\}$ is mutually disjoint, we can get a disjoint open covering $\{U_x: x \in X\}$ of A such that $U_x \subset W_x$ for every $x \in X$. Since

$$U_x \cap (\cup \{V(x, i): i = 1, \dots, m(x)\}) = U_x \cap W_x = U_x$$

then

$$\{U_x \cap V(x, i): i = 1, \dots, m(x), \quad x \in X\}$$

is an open covering of order 1 which refines $\mathcal{G}$. Thus, A is a paracompact space with $\text{Ind } A = 0$.

That (2) implies (1): By Corollary 22-7, X is a paracompact Hausdorff space. By Lemma 22-5, there exists an inverse limiting full system $\{A_\lambda, \varphi_\mu^\lambda: \lambda \in \Lambda\}$ of discrete spaces A_λ such that $A = \text{inv lim}(A_\lambda)$. Let $\pi_\lambda: A \rightarrow A_\lambda$ be the projections. Then

$$\mathfrak{F} = \{\mathcal{F}_\lambda = \{F_\alpha = f(\pi_\lambda^{-1}(\alpha)): \alpha \in A_\lambda\}: \lambda \in \Lambda\}$$

is a directed family of locally finite closed coverings of X with $\text{ord } \mathfrak{F} \leq n + 1$.

To prove that $\mathfrak{F}$ is fully fine let $\mathcal{G}$ be an arbitrary open covering of X . Then by the fullness of $\{A_\lambda\}$, there exists an index $\lambda \in \Lambda$ such that $\{\pi_\lambda^{-1}(\alpha) : \alpha \in A_\lambda\}$ refines $f^{-1}(\mathcal{G})$. It is now evident that $\mathcal{F}_\lambda$ refines $\mathcal{G}$. $\square$

22-9. THEOREM (Morita [8] and Nagami [8]). Let X be a space satisfying condition (1) of Theorem 22-2 with $n \geq 0$, then X is a regular space with $\text{ind } X \leq n$.

PROOF (by induction on n). Let (P_i) be the theorem for $n = i$. Then (P_0) is evidently true. Make the induction assumption that (P_i) is true for $i < n$. Let x be an arbitrary point in X and G an arbitrary open set containing x . Let λ be an index from Λ with $S(x, \mathcal{F}_\lambda) \subset G$. Let H be the open kernel of $S(x, \mathcal{F}_\lambda)$. Then it is easy to see that $x \in H \subset \bar{H} \subset G$. Thus, X is a regular space. Let

$$H_1 = X - \cup \{F_\alpha : x \notin F_\alpha \in \mathcal{F}_\lambda\}$$

Then $x \in H_1 \subset H \subset S(x, \mathcal{F}_\lambda)$. Since

$$B(H) = \bar{H} - H \subset X - H \subset X - H_1$$

$B(H)$ is covered by

$$\mathcal{F}_\lambda' = \{F_\alpha : \alpha \in B_\lambda\}$$

where

$$B_\lambda = \{\alpha : x \notin F_\alpha \in \mathcal{F}_\lambda\}$$

Let for every $\mu > \lambda$,

$$B_\mu = \{\beta : \varphi_\lambda^\mu(\beta) \in B_\lambda\}$$

Then for every $\mu \in M = \{\nu : \nu > \lambda\}$, $\mathcal{F}_\mu' = \{F_\alpha : \alpha \in B_\mu\}$ covers $B(H)$. Let $\mathcal{H}_\mu$ be the restriction of $\mathcal{F}_\mu'$ to $B(H)$, $\mu \in M$. Since

$$\bar{H} \subset \cup \{F_\alpha : \alpha \in A_\lambda - B_\lambda\}$$

ord $\mathcal{H}_\mu \leq n - 1$ for every $\mu \in M$. It can easily be seen that

$$\mathfrak{H} = \{\mathcal{H}_\mu : \mu \in M\}$$

is a locally fine, directed family of finite closed coverings of $B(H)$. Thus we have $\text{ind } B(H) \leq n - 1$ by the assumption. Hence we have $\text{ind } X \leq n$, and the induction is completed. $\square$

22-10. COROLLARY. If a space X is the image of a completely regular space A with $\text{ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$, then X is a regular space with $\text{ind } X \leq n$.

This is an immediate consequence of Theorems 22-2 and 22-9.

22-11. THEOREM. If a space X has a directed family with the condition in Theorem 22-3 ($n \geq 1$), then X is a normal space with $\text{Ind } X \leq n$.

This is proved in a way quite analogous to that of the proof for Theorem 22-9.

22-12. COROLLARY. If a space X is the image of a normal space A with $\text{Ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$, then X is a normal space with $\text{Ind } X \leq n$.

This is a direct consequence of Theorems 22-3 and 22-11.

22-13. COROLLARY. If a space X is the image of a paracompact Hausdorff space A with $\text{Ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$, then X is a paracompact Hausdorff space with $\text{Ind } X \leq n$.

This is a direct consequence of Corollaries 22-7 and 22-12.

22-14. REMARK. We already know by Theorems 12-6 and 22-8 that a metric space X with $\dim X \leq n$ has a fully fine, directed family of locally finite closed coverings of order $\leq n + 1$. There is a simple example of a nonmetric but compact Hausdorff space having such a family: Let D be an uncountable product of two points sets. D is called a *generalized Cantor discontinuum*. Let

$$X = I^n \times D$$

Then X is the desired space. The identity element of an arbitrary locally compact group G with $\dim G = n$ has a neighborhood homeomorphic to this X . There is another nonmetric example: Any CW-complex K with $\dim K = n$, introduced by Whitehead [1], has such a family. This fact is proved in Nagami [7].

For a metric space the following is true, again by Theorem 12-6: A metric space X has $\dim X \leq n$ if and only if X is the image of a metric space A with $\dim A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$. This theorem of Hurewicz and Kuratowski is not valid even for compact Hausdorff spaces. Let S be the compact Hausdorff space given in Example 16-6. Then $\dim S = 1$. If there existed a normal space A with $\dim A = 0$ and a closed mapping f of A onto S with $\text{ord } f \leq 2$, then $\text{Ind } S \leq 1$ by Corollary 22-12. But $\text{Ind } S = 2$, a contradiction. From these observations it is natural to raise the following problem.

22-15. PROBLEM. Is a compact Hausdorff space X with $\text{Ind } X = n$ always the image of a compact Hausdorff space A with $\text{Ind } A = 0$ under a closed mapping f with $\text{ord } f \leq n + 1$? If this question had an affirmative answer, the theorem of Hurewicz and Kuratowski would be proved not to be combined with the covering dimension but with the large inductive dimension. But the author's conjecture is that the problem may be solved negatively.

23. Dimension-Raising Closed Mappings for Nonmetric Spaces

23-1. THEOREM (Morita [8]). Let f be a closed mapping of a nonempty normal space X onto a normal space Y such that $f^{-1}(y)$ consists of at most $k + 1$ points for each point y in Y , then

$$\dim Y \leq \text{Ind } X + k$$

PROOF (by double induction). When $\text{Ind } X = 0$, $\text{Ind } Y \leq k$ by Corollary 22-12. Since $\dim Y \leq \text{Ind } Y$ by Theorem 10-1, we know the theorem is valid for this case. Assume that the theorem is true for $\text{Ind } X \leq n - 1$.

Let $\text{Ind } X \leq n$. When $k = 0$, f is a homeomorphism. Hence, $\dim Y = \dim X \leq \text{Ind } X \leq n$, and the theorem is true. We prove the theorem for $k = k_0$ assuming it for $k \leq k_0 - 1$. Let $F \subset G$ be a pair consisting of a closed set F of Y and an open set G of Y . Let H be an open set of X with

$$f^{-1}(F) \subset H \subset \bar{H} \subset f^{-1}(G) \quad \text{and} \quad \text{Ind } B(H) \leq n - 1$$

Set

$$V = Y - f(X - H)$$

Then

$$B(V) = \bar{V} - V \subset f(\bar{H}) - V \quad \text{and} \quad F \subset V \subset \bar{V} \subset G$$

If we put

$$K = f(\bar{H}) - V, \quad K_1 = f(B(H))$$

then $\dim K_1 \leq n - 1 + k_0$, since $\text{Ind } B(H) \leq n - 1$ and the partial mapping $f|_{B(H)}$ is closed.

Let M be any closed set of K (and hence of Y) contained in $K - K_1$. If we denote by f_1 the partial mapping of f whose domain is $(X - H) \cap f^{-1}(M)$ and whose range is M , then f_1 is a closed onto mapping such that $f^{-1}(y)$ consists of at most k_0 points for each point y in M , since

$$M \subset K - K_1 \subset f(H) - V \subset f(H) \cap f(X - H)$$

Hence, by the induction assumption on k we have

$$\dim M \leq n + k_0 - 1$$

since $\text{Ind}((X - H) \cap f^{-1}(M)) \leq \text{Ind } X \leq n$. Therefore

$$\dim(K \bmod K_1) \leq n + k_0 - 1$$

Since $\dim K \leq n + k_0 - 1$ by Theorem 9-11, we have

$$\dim B(V) \leq n + k_0 - 1$$

If $\mathcal{G}$ is an arbitrary finite open covering of X , it is trivially true that

$$\mathcal{G}\text{-dim } B(V) \leq n + k_0 - 1$$

Hence, by Theorem 20-4

$$\dim Y \leq n + k_0$$

which completes the induction. $\square$

23-2. LEMMA. Let f be a closed mapping of a normal space X onto a normal space Y . Let

$$X_1 = \cup \{B(f^{-1}(y)) : y \in Y\}$$

$$Y_1 = f(X_1)$$

then

1. X_1 is closed,
2. $\dim Y = \max\{\dim Y_1, 0\}$.

Moreover, if Y is a hereditarily normal space,

3. $\text{Ind } Y = \max\{\text{Ind } Y_1, 0\}$.

PROOF. If a point y is in $Y - Y_1$, $f^{-1}(y)$ is open. Hence, each point in $Y - Y_1$ is isolated. Therefore, $\dim(Y - Y_1) \leq 0$ and $Y - Y_1$ is open. Since

$$X_1 = f^{-1}(Y_1) - \cup \{\text{the open kernel of } f^{-1}(y) : y \in Y_1\}$$

X_1 is closed. Hence Y_1 is closed. Statements (2) and (3) are now immediate consequences of Theorem 9-11 and Theorem 10-4, respectively. $\square$

23-3. COROLLARY. Let f be a closed mapping of a nonempty normal space X onto a normal space Y such that $B(f^{-1}(y))$ consists of at most $k + 1$ points for each point y in Y , then

$$\dim Y \leq \text{Ind } X + k$$

PROOF. When $Y_1 = \phi$, $\dim Y = 0$ by the lemma. Hence, the theorem is true. When $Y_1 \neq \phi$, $\dim Y = \dim Y_1$ and $\text{Ind } X_1 \leq \text{Ind } X$ by the lemma. Hence by Theorem 23-1

$$\dim Y \leq \text{Ind } X_1 + k \leq \text{Ind } X + k \quad \square$$

23-4. THEOREM (Nagata [6]). Let f be a closed mapping of a nonempty normal space X onto a hereditarily normal space Y such that $B(f^{-1}(y))$ consists of at most $k + 1$ points for each point y in Y , then

$$\text{Ind } Y \leq \text{Ind } X + k$$

PROOF. If we apply Theorem 10-4 instead of Theorem 9-11 in the proof of Theorem 23-1 we get the desired inequality by induction on $\text{Ind } X + \text{ord } f$. $\square$

24. Dimension-Raising Closed Mappings for Metric Spaces[†]

24-1. LEMMA. Let X be a metric space with $\dim X \leq n$ and X_1 an F_σ set in X with $\dim X_1 \leq m$ ($\leq n$), then there exists an F_σ set N of X such that:

- i. $N \cap X_1 = \phi$
- ii. $\dim N \leq n - m - 1$
- iii. $\dim(X - N) \leq m$

PROOF. By the decomposition theorem, X is the sum of two sets X_2 and X_3 with

$$\dim X_2 \leq m, \quad \dim X_3 \leq n - m - 1$$

By Theorem 12-15 we suppose without loss of generality that X_2 is F_σ in X . Since $\dim X_1 \cup X_2 \leq m$ by the sum theorem, there exists a G_δ set X_4 of X with $X_1 \cup X_2 \subset X_4$ and $\dim X_4 \leq m$, again by Theorem 12-15. Set $N = X - X_4$. Then N satisfies the desired conditions. $\square$

24-2. THEOREM. Let f be a closed mapping of a metric space X onto a metric space Y such that $\dim f^{-1}(y) \leq 0$ for each point y in Y . Set

$$Y_2 = \{y \in Y: |f^{-1}(y)| \geq 2\}$$

If $\dim X \leq n$ and $\dim Y_2 \leq n - 1$, then $\dim Y \leq n$.

[†] All of the results in this section are essentially due to Freudenthal [1] who proved them for separable metric spaces.

PROOF (by induction). When $n = 0$, Y is homeomorphic to X . Hence the theorem is trivially true. Assume the theorem is true for the case when n is replaced by $n - 1$.

Let $\dim X \leq n$ and $\dim Y_2 \leq n - 1$. We prove first that Y_2 is F_σ . Set

$$F_i = \{y: \text{dia } f^{-1}(y) = \text{the diameter of } f^{-1}(y) \geq 1/i\}, \quad i = 1, 2, \dots$$

Then $Y_2 = \cup F_i$. Take a point $y \notin F_i$. Then $\text{dia } f^{-1}(y) < 1/i$. There exists an $\varepsilon > 0$ with $\text{dia } S_{2\varepsilon}(f^{-1}(y)) < 1/i$. Set

$$V = Y - f(X - S_\varepsilon(f^{-1}(y)))$$

Then V is an open neighborhood of y such that $\text{dia } f^{-1}(z) < 1/i$ for each $z \in V$. Hence, $V \cap F_i = \emptyset$, which shows F_i is closed.

Since Y_2 is F_σ , $f^{-1}(Y_2)$ is F_σ . By Corollary 21-2,

$$\dim f^{-1}(Y_2) \leq \dim Y_2 \leq n - 1$$

Hence, by the lemma there exists an F_σ set N of X such that

$$N \cap f^{-1}(Y_2) = \emptyset$$

$$\dim N \leq 0, \quad \dim(X - N) \leq n - 1$$

By the lemma there exists a subset M of Y_2 such that

$$\dim M \leq 0, \quad \dim(Y_2 - M) \leq n - 2$$

and M is F_σ in Y_2 (and hence also in Y). Set

$$X' = X - N \cup f^{-1}(M)$$

$$Y' = Y - f(N) \cup M$$

Then

$$\dim X' \leq n - 1$$

$$\dim\{y \in Y': |f^{-1}(y)| \geq 2\} \leq n - 2$$

Hence,

$$\dim Y' \leq n - 1$$

by the induction assumption. Since $f(N)$ and M are F_σ in Y , $\dim(f(N) \cup M) \leq 0$. Hence, $\dim Y \leq n$ and the induction is completed. $\square$

24-3. COROLLARY. Let f be a closed mapping of a metric space X onto a metric space Y such that $\dim f^{-1}(y) \leq 0$ for each point y in Y . Set

$$Y_2 = \{y \in Y: |f^{-1}(y)| \geq 2\}$$

If $\dim Y > \dim X$, then $\dim Y_2 \geq \dim Y - 1$.

PROOF. When $\dim Y = \infty$, $\dim X < \infty$. Hence, $\dim(Y - Y_2) \leq \dim X < \infty$, which implies $\dim Y_2 = \infty$. Thus, $\dim Y_2 \geq \dim Y - 1$.

When $\dim Y = n$, $\dim X \leq n - 1$. If $\dim Y_2 < \dim Y - 1 = n - 1$, then $\dim Y \leq n - 1$, which is a contradiction. $\square$

24-4. THEOREM. Let f be a closed mapping of a metric space X onto a nonempty metric space Y such that, for every point y in Y , $\dim f^{-1}(y) \leq 0$. Set

$$Y_i = \{y: |f^{-1}(y)| \geq i\}, \quad i = 1, 2, \dots$$

If k satisfies $Y_k = Y$ and $\dim Y > \dim X$, then

$$\dim Y_{k+1} \geq \dim Y_k - 1$$

PROOF. Set

$$P = \cup \{B(f^{-1}(y)): y \in Y\}$$

$$Q = f(P)$$

Then P is closed in Y and $\dim Y = \max\{\dim Q, 0\}$ as was shown in Lemma 23-2. Let us consider the case when $Q \neq \phi$. Let

$$\mathcal{F}_i = \{F_\alpha: \alpha \in A_i\}, \quad i = 1, 2, \dots$$

be a sequence of locally finite closed coverings of P such that $\text{mesh } \mathcal{F}_i < 1/i$ for every i . Let B_i be the family of all subsets τ of A_i such that:

- i. $|\tau| = k$
- ii. $F_\alpha \cap F_\beta = \phi$ whenever $\{\alpha, \beta\} \subset \tau$ and $\alpha \neq \beta$
- iii. $H_\tau = \cap \{f(F_\alpha): \alpha \in \tau\} \neq \phi$

Set $f_1 = f|P$. Then $f_1^{-1}(y)$ is compact for each point y in Q by Lemma 13-10. Assume that $\mathcal{F}_i|f_1^{-1}(y)$ were not finite. Then there would exist a sequence of points $\{x_i\}$ in $f_1^{-1}(y)$ and a sequence $\alpha_1, \alpha_2, \dots$ in A_i such that

$$x_1 \in F_{\alpha_1}, \quad x_i \in F_{\alpha_i} - \bigcup_{j < i} F_{\alpha_j} \quad \text{for } i > 1$$

Then the sequence $\{x_i\}$ would be a closed discrete subset of $f_1^{-1}(y)$, which would give a contradiction. Hence, $\mathcal{F}_i|f_1^{-1}(y)$ is finite, and hence $f(\mathcal{F}_i) = f_1(\mathcal{F}_i)$ is point-finite. By the closedness of f_1 , $f_1(\mathcal{F}_i)$ is closure preserving.[†] Hence, $f_1(\mathcal{F}_i)$ is locally finite in Q . Therefore, $\{H_\tau: \tau \in B_i\}$

[†] A collection $\mathcal{S}$ of subsets of a space is *closure preserving* if $\overline{\{S: S \in \mathcal{S}\}} = \{\bar{S}: S \in \mathcal{S}\}$ for any $\mathcal{T} \supset \mathcal{S}$.

is a locally finite closed collection of Q . Thus,

$$\{H_\tau: \tau \in \cup B_i\}$$

is a σ -locally finite closed covering of Q . Hence, by Corollary 11-9, there exist an i and a τ_0 in B_i such that

$$\dim H_{\tau_0} = \dim Q$$

Fix an index α in τ_0 . Set

$$X' = F_\alpha \cap f_1^{-1}(H_{\tau_0}), \quad Y' = H_{\tau_0}, \quad f' = f_1 | X'$$

Then f' is a closed mapping of X' onto Y' and

$$\dim Y' = \dim Q = \dim Y > \dim X \geq \dim X'$$

Set

$$Y_2' = \{y \in Y': |(f')^{-1}(y)| \geq 2\}$$

Since $Y_2' \subset Y_{k+1}$,

$$\dim Y_{k+1} \geq \dim Y_2' \geq \dim Y' - 1 = \dim Y_k - 1$$

by Corollary 24-3.

When $Q = \phi$, $\dim Y_k = 0$. Hence, it is trivially true that

$$\dim Y_{k+1} \geq -1 = \dim Y_k - 1. \square$$

24-5. THEOREM. Let f be a closed mapping of a nonempty metric space X onto a metric space Y . Set

$$Y_i = \{y \in Y: |f^{-1}(y)| \geq i\}, \quad i = 1, 2, \dots$$

$$Z_i = Y_i - Y_{i+1}$$

If there exists a sequence $k_1 < k_2 < \dots < k_{m+1}$ such that $Z_{k_i} \neq \phi$ if and only if $i \in \{1, 2, \dots, m+1\}$ then

$$\dim Y \leq \dim X + m$$

PROOF (by induction on m). When $m = 0$, the theorem is true by Lemma 12-5. Make the induction assumption that the theorem is true for the case when the number of nonempty Z_i is less than $m+1$. If $\dim Y \leq \dim X$,

the theorem is true. We treat the case when $\dim Y > \dim X$. Then by Theorem 24-4,

$$\begin{aligned}\dim Y - 1 &= \dim Y_{k_1} - 1 \\ &\leq \dim Y_{k_2} \\ &\leq \dim f^{-1}(Y_{k_2}) + m - 1 \\ &\leq \dim X + m - 1\end{aligned}$$

Hence, $\dim Y \leq \dim X + m$. $\square$

24-6. REMARK. The converse of Theorem 24-5 is true as follows: Let Y be a nonempty metric space with $\dim Y = n$. Then for any m with $0 < m < n$ there exist a metric space X and a closed mapping f of X onto Y such that

$$\text{ord } f = m + 1, \quad \dim X = n - m$$

This was proved by Roberts [1] for separable metric spaces and later by Nagami [3] for general metric spaces.

24-7. REMARK. It is also to be noted that Hurewicz proved the following interesting proposition: Let (α) be the property: "If F is closed in X and $\dim F = \dim X = n$, then F contains an open set of X ." Thus, if a metric space X has property (α) , Y is a metric space and f is a closed mapping of X onto Y , then

$$\dim Y \leq \dim X + \text{ord } f - 2$$

By Brouwer's invariance theorem of domains, I^n has property (α) . Any n -dimensional polyhedron has property (α) .

Chapter 6

PRODUCT THEOREM AND EXPANSION THEOREM

We have already obtained the product theorem for metric spaces in Theorem 12-14. In this chapter we prove the product theorem for non-metric spaces. The expansion theorem into an inverse limit of polyhedra is also studied. In Sections 25 and 26 the product theorem for Ind and dim will be proved under several conditions on nonmetric spaces. It is desirable that the product theorem for the nonmetric case be discussed by a unified method. Unfortunately, the present situation is far from being satisfactory. The difficulty lies in the product topology. Section 27 gives the expansion theorem under the dimensional restriction. It is interesting to see how the gap between dim and Ind plays a role in the expansion theory (see Theorem 27-12).

25. Product Theorem for Ind

25-1. LEMMA. Let X and Y be regular spaces at least one of which is non-empty, then

$$\text{ind}(X \times Y) \leq \text{ind } X + \text{ind } Y.$$

This is proved by an easy induction on $\text{ind } X + \text{ind } Y$.

25-2. THEOREM (Katsuta [1]). Let X and Y be spaces at least one of which is nonempty such that $X \times Y$ is a hereditarily paracompact Hausdorff space with the star-finite property, then

$$\text{Ind}(X \times Y) \leq \text{Ind } X + \text{Ind } Y.$$

PROOF. By Corollary 11-13,

$$\text{Ind}(X \times Y) = \text{ind}(X \times Y).$$

Since both X and Y are hereditarily paracompact Hausdorff spaces with the star-finite property,

$$\text{Ind } X = \text{ind } X \quad \text{and} \quad \text{Ind } Y = \text{ind } Y.$$

Hence, the desired inequality is a direct consequence of Lemma 25-1. $\square$

25-3. LEMMA (Kimura [1]). Let X be a totally normal space and $\{F_\alpha: \alpha \in A\}$ a locally finite closed covering of X such that $\text{Ind } F_\alpha \leq n$ for each α , then $\text{Ind } X \leq n$.

PROOF. Well-order the index set A . Set

$$H_\alpha = F_\alpha - \cup \{F_\beta: \beta < \alpha\}$$

Then, $\text{Ind } H_\alpha \leq n$ and $\cup_{\beta < \alpha} H_\beta$ is closed for each α . If we apply a transfinite argument analogous to the proof of Theorem 10-4, we obtain $\text{Ind } X \leq n$. $\square$

25-4. LEMMA. Let X be a totally normal space, then $\text{Ind } X \leq n$ if the following condition is satisfied:

For every binary open covering $\mathcal{G}$ of X there exists a σ -locally finite open covering $\mathcal{V}$ of X such that $\text{Ind } B(V) \leq n - 1$ for each $V \in \mathcal{V}$.

This is proved by the same method as in the proof of Theorem 11-12, with the aid of Lemma 25-3.

25-5. LEMMA (Morita [9]). Let X be a perfectly normal space and Y a metric space, then $X \times Y$ is perfectly normal.

PROOF. Let G be an arbitrary open set of $X \times Y$. Let $\mathcal{V} = \cup \mathcal{V}_i$ be a σ -discrete base of Y , where each $\mathcal{V}_i$ is discrete in Y . G is covered by sets of type $U_\nu \times V$ with U_ν open in X and $V \in \mathcal{V}$. Set

$$W_i = \cup \{U_\nu \times V: V \in \mathcal{V}_i\}$$

Then W_i is cozero in $X \times Y$. Since $G = \cup W_i$, G is cozero in $X \times Y$.

To prove the normality of $X \times Y$ let $\mathcal{G} = \{G_1, \dots, G_r\}$ be an arbitrary finite open covering of $X \times Y$. Since $\mathcal{G}$ is a finite cozero covering of $X \times Y$, $\mathcal{G}$ is normal by Theorem 2-7. Therefore, X is normal. $\square$

25-6. THEOREM.[†] Let X be a perfectly normal space and Y a metric space. If at least one of X and Y is nonempty, then

$$\text{Ind}(X \times Y) \leq \text{Ind } X + \text{Ind } Y.$$

PROOF (by induction). First note that $X \times Y$ is perfectly normal by the lemma. When $\text{Ind } X = \infty$ or $\text{Ind } Y = \infty$, the theorem is evidently true. Let us consider the case when $\text{Ind } X + \text{Ind } Y < \infty$. When $\text{Ind } X + \text{Ind } Y = -1$, either X or Y is empty. Hence the theorem is also evidently true. Take as the induction assumption that the theorem is true for $\text{Ind } X + \text{Ind } Y < k$.

Let $\text{Ind } X = m$, $\text{Ind } Y = n$ and $m + n = k$. By Theorem 12-13 there exists a σ -discrete base $\mathcal{V} = \cup \mathcal{V}_i$ of Y such that $\text{Ind } B(V) \leq n - 1$ for every V in $\mathcal{V}$, where each $\mathcal{V}_i$ is discrete in Y . Let $\mathcal{G}$ be an arbitrary binary open covering of $X \times Y$. We construct a refinement of $\mathcal{G}$ satisfying the condition of Lemma 25-4, with n replaced by k , which will imply $\text{Ind}(X \times Y) \leq k$, because $X \times Y$ is totally normal. $\mathcal{G}$ can be refined by an open covering of $X \times Y$ whose elements are of type

$$U_V \times V, \quad V \in \mathcal{V}$$

Let $U_{Vi}, i = 1, 2, \dots$, be open sets of X such that

$$\begin{aligned} \bar{U}_{Vi} &\subset U_{V,i+1}, \quad i = 1, 2, \dots \\ \bigcup_i U_{Vi} &= U_V \end{aligned}$$

$$\text{Ind } B(U_{Vi}) \leq m - 1, \quad i = 1, 2, \dots$$

Set

$$\mathcal{W} = \{U_{Vi} \times V: V \in \mathcal{V}, \quad i = 1, 2, \dots\}$$

Then $\mathcal{W}$ is a σ -locally finite open covering of $X \times Y$ which refines $\mathcal{G}$. Moreover, it is easy to see that

$$\text{Ind } B(W) \leq m + n - 1 = k - 1$$

for each $W \in \mathcal{W}$ by the induction assumption. $\square$

26. Product Theorem for dim

26-1. LEMMA. Let X be a paracompact Hausdorff space and Y a locally compact, paracompact Hausdorff space, then $X \times Y$ is paracompact.

[†] The proof given here is essentially the same as in Nagami [5].

PROOF. First we consider the case when Y is a compact Hausdorff space. Let $\mathcal{G}$ be an arbitrary open covering of $X \times Y$. By the compactness of Y each point x in X has an open neighborhood $U(x)$ such that for a suitable finite open covering $\mathcal{U}_x$ of Y

$$U(x) \times \mathcal{U}_x = \{U(x) \times U : U \in \mathcal{U}_x\} < \mathcal{G}$$

Let $\mathcal{V} = \{V_x : x \in X\}$ be a locally finite open covering of X such that $V_x \subset U(x)$ for each point x in X . Then

$$\{V_x \times \mathcal{U}_x : x \in X\}$$

is a locally finite open covering of $X \times Y$ which refines $\mathcal{G}$. Therefore, $X \times Y$ is paracompact.

Next let us consider a general Y . For each point y in Y let $W(y)$ be an open neighborhood of y whose closure is compact. Let $\mathcal{W}$ be a locally finite open covering of Y which refines $\{W(y) : y \in Y\}$. Let $\mathcal{G}$ be an arbitrary open covering of $X \times Y$. Then for each element W in $\mathcal{W}$, $\mathcal{G} \upharpoonright X \times \overline{W}$ can be refined by a locally finite closed covering $\mathcal{F}_W$ of $X \times \overline{W}$ since $X \times \overline{W}$ is already paracompact. Since $\cup \{\mathcal{F}_W : W \in \mathcal{W}\}$ is a locally finite closed covering of $X \times Y$ refining $\mathcal{G}$, $X \times Y$ is paracompact by Lemma 22-6. $\square$

26-2. LEMMA. Let T be a simplicial polyhedron with the weak topology whose combinatorial dimension is n , then

$$\dim T \leq n \quad \text{and} \quad \dim(T \times I) \leq n + 1$$

PROOF (by induction). When the combinatorial dimension of T is 0, T is discrete. Hence, $\dim T \leq 0$. Assume that the combinatorial dimension of T is less than n , then $\dim T < n$.

Let the combinatorial dimension of T be n . Let T' be the $(n-1)$ -skeleton of T . Then, $\dim T' \leq n-1$ by the assumption. Let $|s_\alpha|$, $\alpha \in A$, be all n -simplices of T . Let $K_{\alpha i}$, $\alpha \in A$, $i = 1, 2, \dots$, be compact sets such that

$$|s_\alpha| - T' = \cup_i K_{\alpha i}$$

Set $K_i = \cup_\alpha K_{\alpha i}$. Then K_i is closed in T . Since $\dim K_{\alpha i} \leq n$ and $\{K_{\alpha i} : \alpha \in A\}$ is discrete, $\dim K_i \leq n$. Since T' is closed and

$$T = T' \cup (\cup K_i)$$

$\dim T \leq n$ by the normality of T , which was noted in Section 3-11.

By Theorem 12-14,

$$\dim(K_{\alpha i} \times I) \leq n + 1$$

Hence, $\dim(K_i \times I) \leq n + 1$. Since

$$T \times I = (T' \times I) \cup (\cup (K_i \times I))$$

and $T \times I$ is normal by Lemma 26-1, $\dim(T \times I) \leq n + 1$ by an easy induction. $\square$

When T has the metric topology, we get $\dim T \leq n$ by the same argument as in the above. Actually, $\dim T = n$, since $\dim T \geq \dim I^n = n$ (see Hurewicz and Wallman [1]).

26-3. LEMMA. Let X be a nonempty paracompact Hausdorff space with $\dim X = n$, then

$$\dim(X \times I) \leq n + 1^\dagger$$

PROOF. Let $\mathcal{G}$ be an arbitrary finite open covering of $X \times I$. By the same technique as in the proof of Lemma 26-1 and by the equality $\dim X = n$, there exist a locally finite open covering $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of X with $\text{ord } \mathcal{U} \leq n + 1$ and finite open coverings $\mathcal{V}_\alpha$, $\alpha \in A$, of I such that

$$\{U_\alpha \times V : V \in \mathcal{V}_\alpha, \alpha \in A\}$$

refines $\mathcal{G}$. By Corollary 1-8, there exists a polyhedron T with the weak topology which approximates $\mathcal{U}$. Notice that the combinatorial dimension of T is n . Let f be a mapping of X to T and $\mathcal{W} = \{W_\alpha : \alpha \in A\}$ an open covering of T such that

$$f^{-1}(W_\alpha) \subset U_\alpha \quad \text{for each } \alpha$$

Let $g: X \times I \rightarrow T \times I$ be a mapping defined by:

$$g(\langle x, t \rangle) = \langle f(x), t \rangle$$

Set

$$\mathcal{H} = \{W_\alpha \times V : V \in \mathcal{V}_\alpha, \alpha \in A\}$$

† Actually it is true that $\dim(X \times I) = n + 1$, which will be proved in the appendix. The last section of the appendix will be devoted to considering the logarithmic law for cohomological dimension, which was studied primarily by Boltyanskii, Bockstein, Dyer, and Kodama.

Then $\mathcal{H}$ is an open covering of $T \times I$ such that $g^{-1}(\mathcal{H})$ refines $\mathcal{G}$. Since T is paracompact as noted in Section 3-11, $T \times I$ is paracompact by Lemma 26-1. Hence, by the inequality $\dim(T \times I) \leq n + 1$ as in Lemma 26-2, and by Corollary 9-14 there exists an open covering $\mathcal{D}$ of $T \times I$ with $\mathcal{D} < \mathcal{H}$ and with $\text{ord } \mathcal{D} \leq n + 2$. Since $g^{-1}(\mathcal{D})$ is an open covering of $X \times I$ with $g^{-1}(\mathcal{D}) < \mathcal{G}$ and with $\text{ord } g^{-1}(\mathcal{D}) \leq n + 2$, we get $\dim(X \times I) \leq n + 1$. $\square$

26-4. THEOREM. Let X be a paracompact Hausdorff space with $\dim X = m$ and Y a compact Hausdorff space with $\dim Y = n$. If either X or Y is not empty,

$$\dim(X \times Y) \leq m + n$$

PROOF. By the method in the preceding proof the theorem is reduced to applying the inequality $\dim(T \times Y) \leq m + n$ for a polyhedron T with the weak topology such that $\dim T \leq m$. As in the proof of Lemma 26-2, T is the sum $T' \cup (\cup K_i)$. Hence, the validity of the inequality $\dim(T \times Y) \leq m + n$ is reduced to showing the validity of the inequality $\dim(I^m \times Y) \leq m + n$ because every $K_{\alpha i}$ in the proof of Lemma 26-3 can be assumed to be an homeomorph of I^m . The last inequality is proved by repeated applications of Lemma 26-3 with the aid of Lemma 26-1. $\square$

26-5. COROLLARY.[†] Let X be a paracompact Hausdorff space and Y a locally compact, paracompact Hausdorff space. If either X or Y is not empty, then

$$\dim(X \times Y) \leq \dim X + \dim Y$$

This is evident by Corollary 11-8 and Theorem 26-4 since $X \times Y$ is paracompact by Lemma 26-1.

26-6. THEOREM (Morita [5]). Let X and Y be Hausdorff spaces such that $X \times Y$ has the star-finite property. If either X or Y is not empty, then

$$\dim(X \times Y) \leq \dim X + \dim Y$$

PROOF. By Theorem 26-4,

$$\dim(\beta X \times \beta Y) \leq \dim \beta X + \dim \beta Y$$

[†] This is due to Morita [5]. We owe the present method of proof to Begle [1].

By Theorem 9-5, $\dim \beta X = \dim X$ and $\dim \beta Y = \dim Y$. By Theorem 11-15, $\dim(X \times Y) \leq \dim(\beta X \times \beta Y)$. Hence, $\dim(X \times Y) \leq \dim X + \dim Y$. $\square$

26-7. REMARK. It is to be noted that even if X is a hereditarily paracompact Hausdorff space and Y is a separable metric space, $X \times Y$ need not be normal. This remarkable counter example was given by Michael [5]. Morita [10] obtained a necessary and sufficient condition on X in order that $X \times Y$ be normal for any metric space Y .

26-8. PROBLEM. Is it true that $\dim(X \times Y) \leq \dim X + \dim Y$ for a nonempty perfectly normal space X and a nonempty metric space Y ?

27. Expansion into Inverse Limits[†]

27-1. LEMMA. Let X be a compact Hausdorff space, P a finite polyhedron with a metric d , $r > 0$ a real number, and $f: X \rightarrow P$ a mapping, then there exist a finite polyhedron Q with $\dim Q \leq \dim X$, a mapping $g: X \rightarrow Q$, which is onto, and a mapping $p: Q \rightarrow P$ such that

$$d(f, pg) = \sup\{d(f(x), pg(x)): x \in X\} \leq r$$

PROOF. Let P^* be a simplicial complex whose geometric realization is P . Let K^* be a finite simplicial complex with vertex set $\{v\}$ and K a realization of K^* such that

- i. $K = P$
- ii. The vertex set of P is contained in $\{v\}$
- iii. Any simplex $|s|$ of K is on a simplex of P
- iv. $\text{mesh}\{\text{St}(v)\} \leq r$

A realization K with properties (i), (ii), and (iii) is called a *subdivision* of P . A subdivision K with property (iv) is assured by taking the m -th barycentric subdivision (see Hocking and Young [1, Theorem 5-19]). Set $\mathcal{V} = \{\text{St}(v)\}$. Then $\mathcal{V}$ is an open covering of K . Let $\mathcal{U} = \{U_\alpha \neq \phi: \alpha \in A\}$ be a finite open covering of X such that

$$\mathcal{U} < f^{-1}(\mathcal{V}), \quad \text{ord } \mathcal{U} \leq \dim X + 1$$

[†] All of the results in this section, except Theorem 27-7, are due to Mardešić [1] whose work stems from that of Freudenthal [2].

Let $N(\mathcal{U})$ be the nerve of $\mathcal{U}$ and Q' its realization. Let $g': X \rightarrow Q'$ be the canonical mapping introduced in Section 1-5 such that

$$U_\alpha \supset (g')^{-1}(\text{St}(\alpha)), \quad \alpha \in A$$

Let $|s|$ be a simplex of K , of the highest dimension k , such that $\sigma = |s| - |K_{k-1}^*|$ is not covered by $g'(X)$, where K_{k-1}^* is the $(k-1)$ -skeleton of the complex K^* with $|K^*| = K$. Take a point of $\sigma - g'(X)$ and compose g' with a mapping, which is the identity outside σ , and while in σ it is the projection from the selected point to the boundary of σ . Repeating this procedure, we arrive at a mapping $g: X \rightarrow Q'$, for which $g(X)$ is a subpolyhedron Q of Q' . The vertex-set of Q is still A . By the construction

$$g^{-1}(\text{St}(\alpha)) \subset U_\alpha, \quad \alpha \in A$$

Let $p': A \rightarrow \{v\}$ be a transformation such that

$$p'(\alpha) = v \quad \text{implies} \quad U_\alpha \subset f^{-1}(\text{St}(v))$$

Let $p: Q \rightarrow K = P$ be a mapping defined by:

$$p(\Sigma \lambda_\alpha \cdot \alpha) = \Sigma \lambda_\alpha \cdot p'(\alpha)$$

where λ_α are barycentric coordinates. It is evident that $\dim Q \leq \dim X$.

To prove the remaining claim let x be an arbitrary point of X . Pick an α_0 with $g(x) \in \text{St}(\alpha_0)$. Then $pg(x)$ has positive weight on $p'(\alpha_0) = v_0$, which implies that $pg(x) \in \text{St}(v_0)$. Since

$$x \in g^{-1}(\text{St}(\alpha_0)) \subset U_{\alpha_0} \subset f^{-1}(\text{St}(v_0))$$

$f(x) \in \text{St}(v_0)$. Since $d(\text{St}(v_0)) \leq r$, $d(f(x), pg(x)) \leq r$. Thus $d(f, pg) \leq r$. $\square$

27-2. LEMMA. Let X be a compact Hausdorff space, P_i , $i = 1, \dots, k$, a finite collection of finite polyhedra with metrics d_i , $r > 0$ a real number, and $f_i: X \rightarrow P_i$, $i = 1, \dots, k$, mappings, then there exist a finite polyhedron Q with $\dim Q \leq \dim X$, a mapping $g: X \rightarrow Q$, which is onto, and mappings $p_i: Q \rightarrow P_i$, $i = 1, \dots, k$, such that

$$d_i(f_i, p_i g) \leq r, \quad i = 1, \dots, k$$

PROOF. Let K_i be a subdivision corresponding to P_i constructed as in the above and $\{v_i\}$ its vertex set. Starting from a finite open covering $\mathcal{U}$ of X such that

$$\mathcal{U} < \bigwedge_i f^{-1}(\{\text{St}(v_i)\}), \quad \text{ord } \mathcal{U} \leq \dim X + 1$$

we can get the desired result in the same way as in the preceding lemma. $\square$

27-3. THEOREM. Let X be a compact Hausdorff space and f a mapping of X to the Hilbert cube I^ω , then there exist an inverse limiting system $\{Q_i, q_i^j: i=1, 2, \dots\}$ of finite polyhedra Q_i with $\dim Q_i \leq \dim X$, a mapping $g: X \rightarrow Q = \text{inv lim } Q_i$, which is onto, and a mapping $p: Q \rightarrow I^\omega$ such that $f = pg$.

PROOF.[†] Consider I^ω as the inverse limit of the system $\{I^i, \pi_j^i\}$ of Euclidean cubes I^i with the natural projections π_j^i . Let d be a metric on I^ω . Since we can assume $I^1 \subset I^2 \subset \dots \subset I^\omega$, every I^i can be considered to have the metric d . Choose a sequence of real numbers r_i such that (i) $\lim r_i = 0$ and (ii) $M_j \subset I^j$ and $\text{dia } M_j \leq 2r_j$ implies $\text{dia } \pi_i^j(M_j) \leq 2^{i-j}r_i$ for $i < j$. The latter condition is written in symbols as follows:

$$\text{dia } M_j \leq 2r_j \quad \text{implies} \quad \text{dia } \pi_i^j(M_j) \leq 2^{i-j}r_i, \quad i < j \quad (1)$$

By induction we can construct a sequence of real numbers $s_i > 0$, an inverse limiting system $\{Q_i, q_i^j\}$ of compact (or equivalently finite) polyhedra Q_i with metrics d_i and with $\dim Q_i \leq \dim X$ for each i , a sequence $g_i: X \rightarrow Q_i$ of mappings, which are onto, and a sequence $p_i: Q_i \rightarrow I^i$ of mappings satisfying the following conditions:

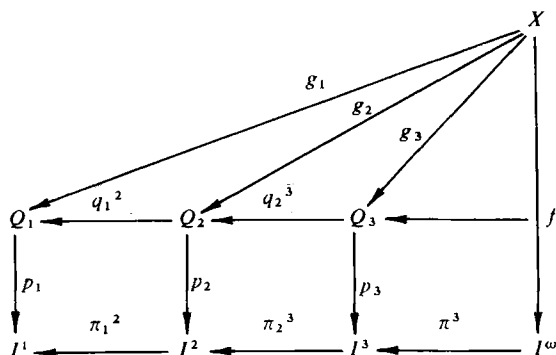
$$d_i(g_i, q_i^{i+1}g_{i+1}) \leq \frac{1}{2}s_i \quad (2)$$

$$d(\pi_i^j f, p_i g_i) \leq \frac{1}{2}r_i \quad (3)$$

$$N_i \subset Q_i, \quad \text{dia } N_i \leq s_i \quad \text{implies} \quad \text{dia } p_i(N_i) \leq \frac{1}{2}r_i \quad (4)$$

$$N_j \subset Q_j, \quad \text{dia } N_j \leq s_j \quad \text{implies} \quad \text{dia } q_i^j(N_j) \leq 2^{i-j}s_i, \quad i < j \quad (5)$$

Here $\pi^i: I^\omega \rightarrow I^i$ is the projection. The situation will be clarified by the following diagram



[†] An improvement of this proof due to Mardešić [1] was attempted by Isbell [1].

The induction can be completed easily by repeated applications of Lemma 27-2, since I^i can be considered as a finite polyhedron, and by the uniform continuity of mappings defined on compact metric spaces. Here q_i^j , $i < j$, is the composite $q_i^{i+1} \cdots q_{j-1}^j$ and q_i^i is the identity mapping. (2) and (5) imply that

$$d_i(g_i, q_i^j g_j) \leq s_i, \quad i \leq j \quad (6)$$

If $j \leq k$, $d_j(g_j, q_j^k g_k) \leq s_j$ by (6). Hence, by (5) we get

$$d_i(q_i^j g_j, q_i^k g_k) \leq 2^{i-j} s_i, \quad i < j \leq k \quad (7)$$

Consider now the transformation $g^i: X \rightarrow Q_i$ defined by

$$g^i = \lim_j q_i^j g_j \quad (8)$$

Relation (7) guarantees the existence and the continuity of g^i . Notice that, as $j \rightarrow \infty$, (6) becomes

$$d_i(g_i, g^i) \leq s_i \quad (9)$$

Let Q be the inverse limit of $\{Q_i\}$ and $q^i: Q \rightarrow Q_i$ the projection. It follows from (8) that

$$g^i = q_i^j g^j \quad (10)$$

so that g^i induces a mapping $g: X \rightarrow Q$, defined by

$$g^i = q^i g \quad (11)$$

In order to prove that $g(X) = Q$, it suffices to show that $g(X)$ is dense in Q , because X is compact. Let $y \in Q$, and let U be an open set of Q containing y . Let U_i be an ε -neighborhood of $q^i(y)$ with $(q^i)^{-1}(U_i) \subset U$. Choose j so large that $2^{i-j} s_i < \varepsilon$, and consider the point $q^j(y) \in Q_j$. Since g_j is a mapping onto, there is an $x \in X$ such that $g_j(x) = q^j(y)$. By (9) $d_j(g_j(x), g^j(x)) \leq s_j$. Therefore, by (5)

$$d_i(q_i^j g_j(x), q_i^j g^j(x)) \leq 2^{i-j} s_i < \varepsilon$$

Since $q_i^j g_j(x) = q^i(y)$, while $q_i^j g^j(x) = q^i g(x)$, we conclude that $q^i g(x) \in U_i$ and $g(x) \in U$.

Observe that (9) and (4) imply $d(p_i g_i, p_i g^i) \leq \frac{1}{2} r_i$. This relation combined with (3) yields

$$d(\pi_i^! f, p_i g^i) \leq r_i \quad (12)$$

Applying (1) to (12) we obtain

$$d(\pi^{i-1} f, \pi_{i-1}^! p_i g^i) \leq \frac{1}{2} r_{i-1} \quad (13)$$

Combining (12) and (13) and replacing i in (13) by $i+1$ we obtain

$$d(p_i g^i, \pi_i^{i+1} p_{i+1} g^{i+1}) \leq \frac{3}{2} r_i \quad (14)$$

Hence by (1) we obtain

$$d(p_i g^i, \pi_i^j p_j g^j) \leq 2r_i, \quad i \leq j \quad (15)$$

Applying (1) to (15) we have

$$d(\pi_i^j p_j g^j, \pi_i^k p_k g^k) \leq 2^{i-j} r_i, \quad i < j \leq k \quad (16)$$

This relation and (11) guarantee the existence and the continuity of transformations $p^i: Q \rightarrow I^i$ defined by

$$p^i = \lim_j \pi_i^j p_j q^j \quad (17)$$

Clearly $p^i = \pi_i^j p^j$, so that the p^i induce a mapping $p: Q \rightarrow I^\omega$ defined by

$$p^i = \pi_i^i p \quad (18)$$

Notice that, as $j \rightarrow \infty$, (15) becomes

$$d(p_i q^i, p^i) \leq 2r_i \quad (19)$$

In order to show that $f = pg$, choose a fixed $x \in X$ and a fixed $\varepsilon > 0$. Since $I^1 \subset I^2 \subset \dots \subset I^\omega$, then $f(x) = \lim p^i f(x)$ and $pg(x) = \lim p^i g(x)$. Hence, there is an i such that each of the numbers

$$d(f(x), \pi^i f(x)), \quad d(pg(x), p^i g(x)), \quad 3r_i$$

is not greater than $\varepsilon/3$. Considering the points $f(x)$, $\pi^i f(x)$, $p_i g^i(x) [= p_i q^i g(x)]$, $p^i g(x)$, $pg(x)$, and taking into account (12) and (19), we conclude that $d(f(x), pg(x)) \leq \varepsilon$, which implies $f = pg$. $\square$

27-4. COROLLARY (Freudenthal's expansion theorem, Freudenthal [1]). Let X be a compact metric space, then X is an inverse limit of a sequence of compact polyhedra Q_i with $\dim Q_i \leq \dim X$.[†]

PROOF. Let $f: X \rightarrow I^\omega$ be an embedding. By the theorem f can be factored through Q , which is an inverse limit of a sequence of compact polyhedra Q_i with $\dim Q_i \leq \dim X$. However, g is a homeomorphism between X and Q , because $g(X) = Q$ and $pg = f$. $\square$

[†] It is to be noted that for any $n > 0$ it is easy to construct an inverse limiting system $\{Q_i, g_j^i\}$ such that (i) every Q_i is a compact polyhedron with $\dim Q_i = n$, (ii) every g_j^i is onto, and (iii) $\dim Q = 0$, where $Q = \text{inv lim } Q_i$.

27-5. COROLLARY. Let X be a compact Hausdorff space, P_i , $i = 1, \dots, k$, a finite collection of compact metric spaces, and $f_i: X \rightarrow P_i$, $i = 1, \dots, k$, mappings, then there exist a compact metric space Q and mappings $g: X \rightarrow Q$, $p_i: Q \rightarrow P_i$, $i = 1, \dots, k$, such that g is onto and

$$\dim Q \leq \dim X, \quad f_i = p_i g \quad \text{for } i = 1, \dots, k$$

PROOF. We consider the product space $P = P_1 \times \dots \times P_k$ as a subset of I^ω . With the mapping

$$f = f_1 \times \dots \times f_k: X \rightarrow P$$

we can associate, by Theorem 27-3, a compact metric space Q with $\dim Q \leq \dim X$ and mappings $g: X \rightarrow Q$, $p: Q \rightarrow P$ such that g is onto and $pg = f$. The mapping p splits into mappings $p_i: Q \rightarrow P_i$ and $f_i = p_i g$, for each i . $\square$

27-6. LEMMA. Let P be a polyhedron with the metric topology and with $\dim P \leq n$ and P' the barycentric subdivision of P . Let $\{S\}$ or $\{S'\}$ be respectively the open covering of all stars with respect to P or to P' , then

1. P is complete
2. $\text{mesh } \{S\} \leq \sqrt{2}$
3. $\text{mesh } \{S'\} \leq (n/(n+1)) \text{mesh } \{S\}$
4. Every S' is open in P

PROOF. Since the inequality $\dim P \leq n$ implies that the combinatorial dimension of P is at most n by the remark at the end of Section 26-2, (2), (3) and (4) are well known (see Hurewicz and Wallman [1]). To prove the first assertion, let $\{x_i\}$ be a Cauchy sequence of P and $\{v\}$ the vertex set of P . Let

$$x_i = \sum \lambda_v^i \cdot v$$

Then $\{\lambda_v^i\}$ is a Cauchy sequence for every v . Hence there exists a non-negative number λ_v such that $\lim \lambda_v^i = \lambda_v$. Consider the form $\sum \lambda_v \cdot v = x$. Since $\sum \lambda_v^i = 1$ for every i , $\sum \lambda_v = 1$. If we assume that $\lambda_{v_i} \neq 0$ for $i = 0, 1, \dots, m$, then there exists a j such that x_j has positive weight on each of $v_0, \dots, v_m$. Hence, the simplex spanned by $v_0, \dots, v_m$ is contained in P . From this observation we conclude that x is a point of P . Since it is evident that $\lim x_i = x$, P is complete. $\square$

27-7. THEOREM. Let X be a complete metric space with $\dim X \leq n$, then X is an inverse limit of a sequence of metric polyhedra Q_i with $\dim Q_i \leq n$.

PROOF. Let $\mathcal{U}_i, i = 1, 2, \dots$, be a sequence of locally finite open coverings of X with $\text{ord } \mathcal{U}_i \leq n + 1$ and with $\text{mesh } \mathcal{U}_i < 1/i$. By Theorem 1-5 and Remark 1-3, there exists a metric polyhedron P_i such that P_i approximates $\mathcal{U}_i$ and $\dim P_i \leq n$. Let $f_i: X \rightarrow P_i$ be a canonical mapping and d_i the metric P_i given in Definition 1-2. Then, by Lemma 27-6, (P_i, d_i) is complete. Set

$$P^\omega = \prod P_i, \quad d = \sum d_i / 2^i, \quad f = \prod f_i$$

Then (P^ω, d) is a complete metric space and f is a homeomorphism. Notice that Lemma 27-1 can be restated as follows: Let X be a metric space, P a metric polyhedron with $\dim P < \infty$, k a natural number and $f: X \rightarrow P$ a mapping. Then there exist a metric polyhedron Q with $\dim Q \leq \dim X$, a mapping $g: X \rightarrow Q$ which is onto and a mapping $p: Q \rightarrow P$ such that for each point x in X $\{f(x), pg(x)\}$ is contained in a star of the k -th barycentric subdivision of P . Combining this assertion with Lemma 27-6, we obtain the theorem by an argument analogous to that in the proof of Theorem 27-3. $\square$

27-8. THEOREM. A compact Hausdorff space X is an inverse limit of compact metric spaces Q_β with $\dim Q_\beta \leq \dim X$.

PROOF (by induction). When $X = \phi$, the theorem is trivially true. When $\dim X = 0$, X is an inverse limit of finite discrete spaces by Proposition 8-5. Hence, the theorem is also true for this case. We consider the case when $\dim X \geq 1$. By 6-2, X is embedded in

$$\prod \{I_\alpha: \alpha \in A\}$$

where each I_α is a copy of the unit interval I . Let $f: X \rightarrow \prod I_\alpha$ be an embedding. Let $f_\alpha: X \rightarrow I_\alpha$ be the composite of f and the projection: $\prod I_\alpha \rightarrow I_\alpha$. Let $(B, <)$ be the set of all nonempty finite subsets of A , ordered by inclusion $\subset$, and let $B_i \subset B$ consist of all subsets of A having precisely i elements. We can identify A with B_1 . Clearly $B = \cup B_i$.

Now we shall define, by induction on i , for every $\beta \in B$, a compact metric space Q_β with $\dim Q_\beta \leq \dim X$, mappings $g_\beta: X \rightarrow Q_\beta$, which are

onto, and mappings $p_{\beta}^{\beta'}: Q_{\beta'} \rightarrow Q_{\beta}$, $\beta < \beta'$, in such a way that $\{Q_{\beta}, p_{\beta}^{\beta'}\}$ will be an inverse system and that

$$g_{\beta} = p_{\beta}^{\beta'} g_{\beta'}, \quad \beta < \beta' \quad (1)$$

We start by setting $Q_{\alpha} = f_{\alpha}(X)$ and $g_{\alpha} = f_{\alpha}$ for $\alpha \in A = B_1$. Assume that $Q_{\beta}, g_{\beta}, p_{\beta}^{\beta'}$ have already been defined in accordance with our requirements for all $\beta, \beta' \in B_i$ with $\beta < \beta'$, $i < k$, $k > 1$. Take any $\beta \in B_k$ and consider all its immediate predecessors $\beta_1, \dots, \beta_k$ which are of course in B_{k-1} . Apply Corollary 27-5 to X , all Q_{β_j} , and all $g_{\beta_j}: X \rightarrow Q_{\beta_j}$. We obtain a compact metric space Q_{β} with $\dim Q_{\beta} \leq \dim X$ and a mapping $g_{\beta}: X \rightarrow Q_{\beta}$, which is onto. Moreover, we obtain mappings

$$p_{\beta_j}^{\beta}: Q_{\beta} \rightarrow Q_{\beta_j}$$

satisfying

$$g_{\beta_j} = p_{\beta_j}^{\beta} g_{\beta} \quad (2)$$

If $\gamma \in B_i$ and $i < k-1$, we choose a β_j for some $\beta \in B_k$ such that $\gamma < \beta_j$ and define p_{γ}^{β} by composing $p_{\beta_j}^{\beta}$ with $p_{\gamma}^{\beta_j}$. This last mapping p_{γ}^{β} is independent of the choice of B_j , because g_{β} is a mapping onto and we have (1) and (2).

Let Q be the inverse limit of the system

$$\{Q_{\beta}, p_{\gamma}^{\beta}: \beta \in B\}$$

obtained in this way and let $p^{\beta}: Q \rightarrow Q_{\beta}$ be the corresponding projections. By (1), the mappings $g_{\beta}: X \rightarrow Q_{\beta}$ induce a mapping $g: X \rightarrow Q$, defined by

$$g_{\beta} = p^{\beta} g, \quad \beta \in B \quad (3)$$

Since all g_{β} are mappings onto, so is g . Finally, if $\beta = \alpha \in B_1$, then (3) becomes

$$f_{\alpha} = p^{\alpha} g \quad (4)$$

proving that $p^{\alpha}, \alpha \in A$, defines a mapping

$$p: Q \rightarrow \Pi I_{\alpha}$$

satisfying $pg = \Pi f_{\alpha} = f$. Since f is a homeomorphism into and g is a mapping onto, g is a homeomorphism onto. $\square$

27-9. REMARK. Let a metric space X be an inverse limit of a sequence of metric spaces X_i with $\dim X_i \leq n$ accompanied by mappings $p_j^i: X_i \rightarrow X_j$. Then it is true that $\dim X \leq n$ as follows: Let $\mathcal{U}_{ij}, j = 1, 2, \dots$,

be a sequence of open coverings of X_i with mesh $\mathcal{U}_{ij} < 1/j$. Let $\mathcal{V}_1$ be an open covering of X_1 such that

$$\text{ord } \mathcal{V}_1 \leq n+1 \quad \text{and} \quad \mathcal{V}_1 < \mathcal{U}_{11}$$

Let $\mathcal{V}_2$ be an open covering of X_2 such that

$$\begin{aligned} \text{ord } \mathcal{V}_2 &\leq n+1 \\ \mathcal{V}_2 &< (p_1^2)^{-1}(\mathcal{V}_1 \wedge \mathcal{U}_{11} \wedge \mathcal{U}_{12}) \end{aligned}$$

Repeating this procedure, we obtain for each i an open covering $\mathcal{V}_i$ of X_i such that

$$\text{ord } \mathcal{V}_i \leq n+1$$

$$\mathcal{V}_i < (\wedge \{(p_j^i)^{-1}(\wedge_{k=1}^j \mathcal{U}_{jk}) : j=1, \dots, i-1\}) \wedge (p_{i-1}^i)^{-1}(\mathcal{V}_{i-1})$$

Let $p_i: X \rightarrow X_i$ be the projections. Then it is easy to see that the sequence

$$\mathcal{W}_i = p_i^{-1}(\mathcal{V}_i), \quad i=1, 2, \dots$$

of open coverings of X satisfies the following conditions:

- i. $\mathcal{W}_1 > \mathcal{W}_2 > \dots$
- ii. $\text{ord } \mathcal{W}_i \leq n+1$ for each i
- iii. For each $x \in X$, $\{S(x, \mathcal{W}_i^A)\}$ is a neighborhood base of x

Therefore, we have $\dim X \leq n$ by Theorem 14-1. $\square$

When a compact Hausdorff space X is an inverse limit of compact Hausdorff spaces X_α with $\dim X_\alpha \leq n$, then $\dim X \leq n$. This is almost evident.

27-10. LEMMA. Let P be a polyhedron of dimension n , F a closed subset of P , and U an open set of F , then $\dim B(U) \leq n-1$.

This statement is easily derived from the fact that the boundary of an open set in Euclidean n -space has dimension not greater than $n-1$. See Hurewicz and Wallman [1, Theorem IV 3].

27-11. LEMMA. Let $\{X_\alpha, p_\beta^\alpha: \alpha \in A\}$ be an inverse limiting system of compact metric spaces X_α , having the property that each X_α can be embedded in a finite polyhedron P_α with $\dim P_\alpha \leq 1$, then $X = \text{inv lim } X_\alpha$ satisfies $\dim X \leq 1$.

PROOF. Let x be an arbitrary point of X and U an arbitrary open set with $x \in U$. Let $p_\alpha: X \rightarrow X_\alpha$ be projections. Choose an index α and an open neighborhood V_α of $p_\alpha(x)$ such that

$$V = p_\alpha^{-1}(V_\alpha) \subset U$$

Set

$$V_\beta = (p_\alpha^\beta)^{-1}(V_\alpha) \quad \text{and} \quad F_\beta = X_\beta - V_\beta$$

Clearly p_β^γ maps V_γ , $\bar{V}_\gamma$ or F_γ into V_β , $\bar{V}_\beta$ or F_β , respectively, while p_β maps V , $\bar{V}$ or $F = X - V$ into V_β , $\bar{V}_\beta$ or F_β , respectively. Since $\bar{V}_\beta \cap F_\beta = B(V_\beta)$ and $\bar{V} \cap F = B(V)$, we conclude that

$$\{B(V_\beta), p_\beta^\gamma: \alpha < \beta \in A\}$$

forms an inverse limiting system whose limit $\text{inv lim } B(V_\beta)$ contains $B(V)$. By the assumption of the lemma and by the preceding lemma, we know that $\dim B(V_\beta) \leq 0$ and thus

$$\dim B(V) \leq \dim(\text{inv lim } B(V_\beta)) \leq 0$$

This implies $\text{ind } B(V) \leq 0$, proving that $\text{ind } X \leq 1$. $\square$

27-12. THEOREM. There exists a compact Hausdorff space X with $\dim X = 1$, which cannot be obtained as any inverse limit of compact polyhedra P_α with $\dim P_\alpha \leq 1$.

PROOF. Let S be the compact Hausdorff space given in Example 16-6 such that $\dim S = 1$ and $\text{ind } S = 2$. If we assume that S had an expansion as in the theorem, then $\text{ind } S \leq 1$ by Lemma 27-11, which would give a contradiction. $\square$

If we combine Theorem 27-8 with Corollary 27-4, then we obtain the assertion: Every compact Hausdorff space X is homeomorphic to a double iterated inverse limit

$$\text{inv lim}_\alpha (\text{inv lim}_i P_{\alpha i})$$

of compact polyhedra $P_{\alpha i}$ satisfying $\dim P_{\alpha i} \leq \dim X$, where i ranges through the positive integers.

Chapter 7

METRIC-DEPENDENT DIMENSION FUNCTIONS

The aim of this chapter is to give in full detail the known behavior of the metric-dependent dimension functions d_2 , d_3 , and $\mu \dim$. Up to now we have considered the three dimension functions $\dim$, Ind , and ind . They are, of course, topological invariants, while d_2 , d_3 , and $\mu \dim$ depend on the metric and are not topological invariants. Yet they have deep topological meaning as follows. Let X be a subset of the Hilbert cube with $\dim X \leq n$. Then for each $\varepsilon > 0$, X can be ε -transformed to a polyhedron X_ε in the cube with $\dim X_\varepsilon \leq n$. This approximation theorem was one of the fundamental theorems from which homology theory has developed. The condition is sufficient for X to have $\dim \leq n$ when X is compact. Thus the question occurs: Is the condition sufficient for the statement $\dim X \leq n$ for a noncompact X ? This problem had been unanswered for a long time. Actually, as can be seen in this chapter, there is a gap between the condition and the statement $\dim X \leq n$. This gap played an important role implicitly when the study of dimension theory moved to general metric spaces from separable metric spaces. The gap is now an interesting object to study. We study this gap in the present chapter and our functions d_2 , d_3 , and $\mu \dim$ may be said to be skillful guides to our sightseeing trip around the gap. Although we now have a fair amount of knowledge about metric-dependent dimension functions, several interesting problems still remain unsolved. The reader will see some of them in this chapter. It is to be noticed at this occasion that our study, in Section 14, of sequences of coverings defining dimension was singing a sufficient condition for the equality $\mu \dim = \dim$.

In Section 28 we will estimate the dimension of the complement of a disjoint sum of a countable number of closed sets (Theorem 28-7). This estimation is not only useful for the remaining part of the chapter but also

interesting in itself. Section 29 gives the definitions of several metric-dependent dimension functions and their relations: $d_2 \leq d_3 \leq \mu \dim \leq \dim = d_1 = d_4 \leq 2d_3$. In Section 30 we will construct for each positive n a space X_n with $\dim \geq n-1$ and with $d_2 = [n/2]$. In Section 31 we will construct for each positive n a space Y_n with $\dim \geq n-1$ and with $\mu \dim = [n/2]$. In Section 32 we will construct a space Z_n where $\mu \dim$ and d_2 have many values when the metric on Z_n changes while giving the same topology. The final Section 33 offers a space R with $d_2 = 2$, $\mu \dim = 3$ and $\dim = 4$.

28. Dimension of the Complement of a Disjoint Sum

28-1. LEMMA. Let X be a normal space and Y a subset of X such that (i) $X - Y$ is normal, (ii) $\dim(X - Y) < n$, and (iii) $X - Y$ is dense in X , then for any n pairs of disjoint closed sets of X , $C_1, C_1'; \dots; C_n, C_n'$, there exist closed sets of X , $B_1, \dots, B_n$, such that $\cap B_i \subset Y$ and B_i separates C_i and C_i' for each i .

PROOF. Let D_i, D_i' be open sets of X such that

$$C_i \subset D_i, \quad C_i' \subset D_i', \quad \text{and} \quad \bar{D}_i \cap \bar{D}_i' = \phi$$

We apply the Eilenberg-Otto characterization of dimension, Theorem 9-9, to the space $X - Y$ and the n pairs of the sets $\bar{D}_i - Y, \bar{D}_i' - Y$. We thus obtain open (in $X - Y$) sets $T_1, \dots, T_n$ such that

$$\bar{D}_i - Y \subset T_i, \quad (\bar{D}_i' - Y) \cap \bar{T}_i = \phi, \quad \cap B_i' = \phi$$

where B_i' is the relative boundary of T_i for each i .

Let U_i be the interior of $\bar{T}_i$ in the space X . Since $T_i \cap D_i' = \phi$, $\bar{T}_i \cap D_i' = \phi$ and hence $\bar{T}_i \cap C_i' = \phi$. If $D_i - \bar{T}_i \neq \phi$, this would be a nonempty open set of Y , contradicting the assumption that $X - Y$ is dense in X . Thus $D_i - \bar{T}_i = \phi$ and hence $D_i \subset U_i$. Therefore, $B_i = B(U_i)$ separates C_i and C_i' .

Let x be an arbitrary point of $B_i - Y$ and V an arbitrary open neighborhood of x . Since $V \cap U_i \neq \phi$, $V \cap (U_i - Y) \neq \phi$. Since $U_i - Y \subset \bar{T}_i$, $V \cap T_i \neq \phi$. If $V \subset \bar{T}_i$, $\bar{T}_i$ would contain a nonempty open set $V - \bar{U}_i$ having no point in common with U_i , which is a contradiction. Hence, $V - \bar{T}_i \neq \phi$, which implies $(V - \bar{T}_i) - Y \neq \phi$. Therefore, x is a point of B_i' . We now have the inequality $B_i - Y \subset B_i'$. Since $\cap B_i' = \phi$, $\cap B_i \subset Y$. $\square$

28-2. LEMMA (Borsuk's extension theorem). Let X be a space such that $X \times I$ is normal, F a closed set of X , and f and g mappings of F to S^n . If f is homotopic to g and g can be extended over X , then f can also be extended over X .

See Hurewicz and Wallman [1, Theorem VI 5]. Note that $X \times I$ is normal if and only if X is normal and countably paracompact. This was proved by Dowker [2]. Since S^n is an NES (normal) by Theorem 3-10, the proof given by Hurewicz and Wallman is valid for the present case.

28-3. DEFINITION. Let X be a normal space. A system of pairs $C_1, C_1'; \dots; C_n, C_n'$ is an n -defining system of X if (i) C_i and C_i' are closed sets of X for each i , (ii) $C_i \cap C_i' = \emptyset$ for each i , and (iii) for any closed sets B_i , $i = 1, \dots, n$, separating C_i and C_i' , $\bigcap B_i$ is nonempty.

28-4. LEMMA.[†] Let X be a compact Hausdorff space, F a closed set of X and f a mapping of F to S^{n-1} . Consider S^{n-1} as the surface of $I^n = \{\langle x_1, \dots, x_n \rangle : -1 \leq x_i \leq 1\}$. Let $C_1, C_1'; \dots; C_n, C_n'$ be the n pairs of opposite faces of I^n defined by

$$C_i = \{\langle x_1, \dots, x_n \rangle : x_i = -1\}$$

$$C_i' = \{\langle x_1, \dots, x_n \rangle : x_i = 1\}$$

If the system $f^{-1}(C_i), f^{-1}(C_i'), i = 1, \dots, n$, is not n -defining, then f has an extension $f^*: X \rightarrow S^{n-1}$.

PROOF. Let $B_1, \dots, B_n$ be closed sets of X with $\bigcap B_i = \emptyset$ such that B_i separates $f^{-1}(C_i)$ and $f^{-1}(C_i')$ for each i . By Proposition 9-2 we can assume without loss of generality that every B_i is G_δ . Let

$$f = f_1 \times \dots \times f_n$$

Let $g_i: X \rightarrow [-1, 1]$ be an extension of $f_i|_{f^{-1}(C_i) \cup f^{-1}(C_i')}$ such that $g_i(x) = 0$ if and only if $x \in B_i$ and such that $|g_i(x)| = 1$ if and only if $x \in f^{-1}(C_i) \cup f^{-1}(C_i')$. Set

$$g = g_1 \times \dots \times g_n$$

[†] One of my students, T. J. Grilliot, pointed out that this lemma remains valid even if X is merely normal.

Then g is a mapping of X into I^n and $g(F) \subset S^{n-1}$. If $x \in F$, then $f(x)$ and $g(x)$ cannot be a pair of opposite points on S^{n-1} . Hence, f is homotopic to $g|F$. Let p be the origin, $\langle 0, \dots, 0 \rangle$, of I^n . Then $p \notin g(X)$. Let $r: I^n - \{p\} \rightarrow S^{n-1}$ be a retraction. Then rg maps X into S^{n-1} . Since $rg|F = g|F$ and $X \times I$ is normal by Lemma 26-1, f has an extension $f^*: X \rightarrow S^{n-1}$ by Lemma 28-2. $\square$

28-5. DEFINITION. A compact Hausdorff space X with $\dim X = n$, $n \geq 1$, is called a *Cantor n -manifold* if it cannot be separated by a closed set S with $\dim S \leq n - 2$.

28-6. PROPOSITION. A compact Hausdorff space X with $\dim X = n$, $n \geq 1$, contains a Cantor n -manifold.

Since the proof by Hurewicz and Wallman [1, Theorem VI 8] can be applied to this case with a trivial change, we omit the proof.

28-7. THEOREM (Nagami and Roberts [3]). Let X be a compact Hausdorff space with $\dim X = n$, $n \geq 1$, and $A_1, A_2, \dots$ be a sequence of disjoint closed sets of X such that (i) $\dim A_i \leq n - 1$ for each i and (ii) $X - \cup A_i$ is normal, then

$$\dim(X - \cup A_i) \geq n - 1$$

PROOF. Since $\dim X = n$ there exist, by Theorem 9-9, a closed set F of X and a mapping $f: F \rightarrow S^{n-1}$ such that f cannot be extended over X . Then there exists a closed set Y of X such that (i) $f|F \cap Y$ cannot be extended over Y and (ii) if Z is any proper closed subset of Y , then $f|F \cap Z$ is extendable over Z . Then Y is actually a Cantor n -manifold (see Hurewicz and Wallman [1, Theorem VI 8]). Since $\dim(Y - \cup A_i) \geq n - 1$ implies $\dim(X - \cup A_i) \geq n - 1$, we assume hereafter X is Y itself with the above minimal property. Since any nonempty cozero set of X has dimension n and $\dim(\cup A_i) \leq n - 1$ by the normality of $\cup A_i$ by Theorem 3-12 and the sum theorem, $X - \cup A_i$ is dense in X .

Since $\dim A_1 \leq n - 1$, there exists a mapping

$$f': F \cup A_1 \rightarrow S^{n-1} \quad \text{with } f'|F = f$$

Since S^{n-1} is an NES (normal) by Theorem 3-10, there exist an open set U_1 with $U_1 \supset F \cup A_1$ and a mapping

$$f_1: \bar{U}_1 \rightarrow S^{n-1} \quad \text{with } f_1|F \cup A_1 = f'$$

Continuing this procedure, we have a sequence of open sets $U_1, U_2, \dots$ and a sequence of mappings $f_i: \bar{U}_i \rightarrow S^{n-1}$ such that, for each i ,

- i. $\bar{U}_i \subset U_{i+1}$
- ii. $F \cup (\cup_{i \leq j} A_i) \subset U_i$
- iii. f_i is an extension of f_{i-1} , where $f_0 = f$

Define $g: \cup U_i \rightarrow S^{n-1}$ in such a way that $g|_{\bar{U}_i} = f_i$ for each i . Then g is an extension of f over $\cup U_i$. Let $\varphi: X \rightarrow [0, 1]$ be a mapping such that

$$\varphi(x) = 0 \quad \text{if and only if } x \notin \cup U_i$$

$$\varphi(x) = 1 \quad \text{if } x \in F$$

$$\bar{U}_i \subset \{x: \varphi(x) > 2^{-i}\} \subset U_{i+1} \quad \text{for each } i$$

Consider S^{n-1} as the surface of the solid n -ball I^n of radius 1 whose center is the origin p . We define $h: X \rightarrow I^n$ by

$$h(x) = p \quad \text{if } x \notin \cup U_i$$

$$h(x) = \varphi(x)g(x) \quad \text{if } x \in \cup U_i$$

where $g(x)$ is considered as a vector from p to $g(x)$. Then h is continuous and $h|_F = f$. Moreover,

$$h^{-1}(p) \cap (\cup A_i) = \emptyset$$

which will be a meaningful fact later.

Here we reconsider that I^n is the n -cube expressed by

$$\{\langle x_1, \dots, x_n \rangle: -1 \leq x_i \leq 1\}$$

whose surface is S^{n-1} and whose origin $\langle 0, \dots, 0 \rangle$ is p . Consider the solid pyramid P in I^n whose base is

$$B = \{\langle x_1, \dots, x_n \rangle: x_n = -1\}$$

and whose apex is p . The $n-1$ pairs of opposite sides of P are denoted by (S_i, T_i) , $i = 1, \dots, n-1$, where S_i is spanned by

$$S'_i = \{\langle x_1, \dots, x_n \rangle: x_i = x_n = -1\}$$

and p , and T_i is spanned by

$$T'_i = \{\langle x_1, \dots, x_n \rangle: x_i = 1, x_n = -1\}$$

and p . Then

$$C_i = h^{-1}(S_i) - h^{-1}(p), \quad C'_i = h^{-1}(T_i) - h^{-1}(p), \quad i = 1, \dots, n-1$$

are $n - 1$ pairs of disjoint closed sets of $X' = X - h^{-1}(p)$. Figure 1 will help illustrate the situation.

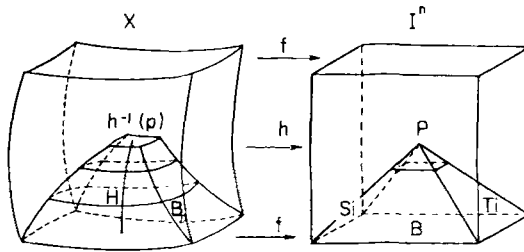


FIGURE 1

Assume that $\dim(X - \cup A_i) < n - 1$. Then

$$\dim(X - ((\cup A_i) \cup h^{-1}(p))) < n - 1$$

since $h^{-1}(p)$ is G_δ . By Lemma 28-1 there exist closed sets B_i , $i = 1, \dots, n - 1$, of $X - h^{-1}(p)$ such that

- i. B_i separates C_i and C_i' in $X - h^{-1}(p)$ for each i
- ii. $\bigcap_{i=1}^{n-1} B_i \subset \cup A_i$

Setting $H = \cap B_i$, consider the compact set $H \cup h^{-1}(p)$ and its disjoint subsets $H \cap h^{-1}(B)$ and $h^{-1}(p)$. Suppose that $H \cap h^{-1}(B) \neq \emptyset$ and that there existed a connected closed set $K \subset H \cup h^{-1}(p)$ such that

$$K \cap H \cap h^{-1}(B) \neq \emptyset \quad \text{and} \quad K \cap h^{-1}(p) \neq \emptyset.$$

Then for some i , $K \cap A_i \neq \emptyset$. Since $h^{-1}(p) \neq \emptyset$ by the minimal property of X and $h^{-1}(p)$ does not meet any A_i , we have a contradiction by Theorem 6-10 and by the statement

$$K \subset h^{-1}(p) \cup A_1 \cup A_2 \cup \dots$$

By Corollary 6-8 we can now conclude that there exist disjoint compact sets H_1 and H_2 such that

$$\begin{aligned} H_1 \cup H_2 &= h^{-1}(B) \cup H \cup h^{-1}(p) \\ H_1 &\supset h^{-1}(p), \quad H_2 \supset h^{-1}(B) \end{aligned}$$

whether or not $H \cap h^{-1}(B) = \phi$. Hence, there exists a closed set B_n of X separating $h^{-1}(p)$ and $h^{-1}(B)$ without touching H . Let c be a number with $0 < c < 1$, Q_c the intersection of P and the hyperplane $\{\langle x_1, \dots, x_n \rangle: x_n = -c\}$, P_c the intersection of P and $\{\langle x_1, \dots, x_n \rangle: x_n \leq -c\}$ and R_c the surface of P_c . Then there exists a number b with $0 < b < 1$ such that

$$h^{-1}(\overline{P - P_b}) \cap B_n = \phi$$

If we confine our attention to the set $h^{-1}(P_b)$, there are closed sets

$$B_1 \cap h^{-1}(P_b), \dots, B_n \cap h^{-1}(P_b)$$

which separate pairs

$$h^{-1}(S_1 \cap P_b), \quad h^{-1}(T_1 \cap P_b); \quad \dots$$

$$h^{-1}(S_{n-1} \cap P_b), \quad h^{-1}(T_{n-1} \cap P_b); \quad h^{-1}(B), \quad h^{-1}(Q_b)$$

respectively. Denote this system of pairs by α . Since

$$\bigcap_{i=1}^n (B_i \cap h^{-1}(P_b)) \subset \bigcap_{i=1}^n B_i = \phi$$

α is not defining. Then by Lemma 28-4, there exists a mapping

$$k_1: h^{-1}(P_b) \rightarrow R_b \quad \text{with} \quad k_1 \mid h^{-1}(R_b) = h \mid h^{-1}(R_b)$$

Let $k: X \rightarrow I^n$ be a mapping defined by

$$k \mid I^n - h^{-1}(P_b) = h \mid I^n - h^{-1}(P_b)$$

$$k \mid h^{-1}(P_b) = k_1$$

Let s be an inner point of P_b and r a retraction of $I^n - \{s\}$ onto S^{n-1} . Then $rk: X \rightarrow S^{n-1}$ is an extension of f , which is a contradiction. Thus we have

$$\dim(X - \cup A_i) \geq n - 1 \quad \square$$

28-8. COROLLARY (Sitnikov [1]). Let $n \geq 1$. Let A_i , $i = 1, 2, \dots$, be a disjoint sequence of closed sets of I^n at least two of which are not empty, then

$$\dim(I^n - \cup A_i) \geq n - 1$$

PROOF. Let

$$I^n = \{\langle x_1, \dots, x_n \rangle: -1 \leq x_i \leq 1\}$$

and S^{n-1} the surface of I^n . Let $f: S^{n-1} \rightarrow S^{n-1}$ be the identity mapping. Since it is impossible that $I^n - S^{n-1}$ is contained in only one A_i , we have one of the following two cases:

- i. There exists an i such that $(I^n - S^{n-1}) - A_i \neq \emptyset$ and $A_j \subset S^{n-1}$ for any $j \neq i$.
- ii. There exist i and j with $i \neq j$ such that $(I^n - S^{n-1}) \cap A_i \neq \emptyset$ and $(I^n - S^{n-1}) \cap A_j \neq \emptyset$.

The first case yields $\dim(I^n - \cup A_i) = n$. If the second case happens, then there exists a number ε with $0 < \varepsilon < 1$ such that

$$I_\varepsilon^n = \{\langle x_1, \dots, x_n \rangle : |x_i| \leq \varepsilon, \quad i = 1, \dots, n\}$$

meets A_i and A_j . Then by Corollary 6-10 there exists a point q in $I_\varepsilon^n - \cup A_i$. Considering q as $h^{-1}(p)$ in the preceding proof we can apply a quite analogous argument on f and q and obtain $\dim(I^n - \cup A_i) \geq n-1$. $\square$

28-9. EXAMPLE (Nagami and Roberts [3]). Figure 2 gives a Cantor 2-manifold X such that a proposition analogous to Corollary 28-8 fails for the space X . In fact, $\dim X = 2$, yet $\dim(X - \cup A_{ij}) = 0$, since $X - \cup A_{ij}$ is a subset of the Cantor discontinuum.

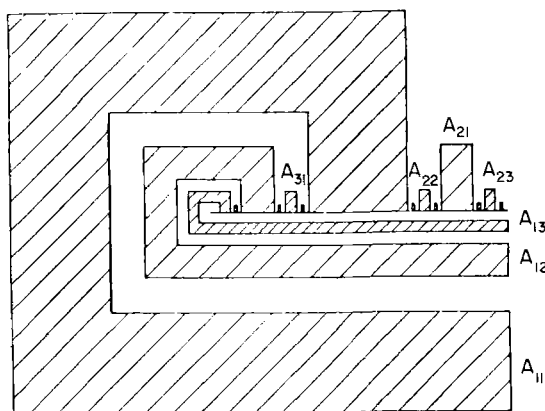


FIGURE 2

29. Relations between Metric-Dependent Dimension Functions

29-1. DEFINITION. Let (X, ρ) be a metric space. If there exists a sequence $\mathcal{U}_i$ of open coverings of X such that

$$\begin{aligned} \text{ord } \mathcal{U}_i &\leq n + 1 \quad \text{for each } i \\ \lim \text{mesh } \mathcal{U}_i &= 0 \end{aligned}$$

then the *metric dimension* of (X, ρ) , $\mu \dim(X, \rho)$, is at most n . When $X = \phi$, set $\mu \dim(X, \rho) = -1$. If $\mu \dim(X, \rho) \leq n$ and the statement $\mu \dim(X, \rho) \leq n - 1$ is false, then $\mu \dim(X, \rho) = n$. If $\mu \dim(X, \rho)$ is not less than n for any n , then $\mu \dim(X, \rho) = \infty$.

The original definition of metric dimension is as follows: If for any $\varepsilon > 0$ the ε -covering of X is approximated by polyhedra of dimension $\leq n$, then the metric dimension of (X, ρ) is at most n . By Corollary 1-8 it is easy to see that the two definitions are equivalent.

29-2. PROPOSITION. For a metric space (X, ρ) the following conditions are equivalent:

1. $\mu \dim(X, \rho) \leq n$
2. There exists a sequence $\mathcal{V}_i, i = 1, 2, \dots$, of locally finite open coverings of X such that

$$\text{ord } \mathcal{V}_i \leq n + 1, \quad \lim \text{mesh } \mathcal{V}_i = 0$$

3. There exists a sequence $\mathcal{F}_i, i = 1, 2, \dots$, of locally finite closed coverings of X such that

$$\text{ord } \mathcal{F}_i \leq n + 1, \quad \lim \text{mesh } \mathcal{F}_i = 0$$

PROOF. Let $\{\mathcal{U}_i\}$ be a sequence of open coverings as in Definition 29-1. Let $\mathcal{V}_i$ be a locally finite open covering of X refining $\mathcal{U}_i$ in a one-one corresponding way. Then the sequence $\mathcal{V}_i, i = 1, 2, \dots$, satisfies condition (2). Thus (1) implies (2). The implication by (2) of (3) follows from the fact that $\mathcal{V}_i$ is shrinkable. By Proposition 9-3, (3) implies (2). The implication (2) $\rightarrow$ (1) is trivially true. $\square$

29-3. DEFINITION. Let (X, ρ) be a metric space. We define $d_1(X, \rho)$ as follows: For the empty set ϕ , $d_1(\phi, \rho) = -1$. If for every pair of closed sets F, H of X with $\rho(F, H) > 0$ there exists a closed set B , separating F and H , with $d_1(B, \rho) \leq n - 1$, then we say $d_1(X, \rho) \leq n$. If there is no such

integer n , then we say $d_1(X, \rho) = \infty$. The meaning of the equality $d_1(X, \rho) = n$ is clear.

29-4. THEOREM (Nagami and Roberts [2]). For any metric space (X, ρ) , $d_1(X, \rho) = \text{Ind } X$.

PROOF (by induction). It is evident that $d_1(X, \rho) \leq \text{Ind } X$. When $d_1(X, \rho) = \infty$, $d_1(X, \rho) \geq \text{Ind } X$. When $d_1(X, \rho) = -1$, $\text{Ind } X = -1$. We suppose that $d_1(X, \rho) \leq n$ and make the induction assumption that $d_1(X', \rho') \geq \text{Ind } X'$ for any metric space (X', ρ') with $d_1(X', \rho') \leq n - 1$.

Let H and F be disjoint closed sets of X . Put, for $i = 1, 2, \dots$,

$$D_i = \{x: \rho(x, H) < 1/i\}$$

$$E_i = \{x: \rho(x, F) < 1/i\}$$

Then there exist open sets $M_i, i = 1, 2, \dots$, with

$$H \subset M_i \subset D_i, \quad d_1(B(M_i), \rho) \leq n - 1$$

and open sets $N_i, i = 1, 2, \dots$, with

$$F \subset N_i \subset E_i, \quad d_1(B(N_i), \rho) \leq n - 1$$

Put

$$G_i = M_i - \bar{N}_i, \quad i = 1, 2, \dots$$

$$G = \cup G_i$$

Then $H \subset G \subset \bar{G} \subset X - F$. Since $B(G_i) \subset B(M_i) \cup B(N_i)$, we have $\text{Ind } B(G_i) \leq n - 1$ by the induction assumption. Since $\{G_i: i = 1, 2, \dots\}$ is locally finite in $X - H \cup F$, we have

$$B(G) \subset \cup B(G_i)$$

Hence, $\text{Ind } B(G) \leq n - 1$, which implies $\text{Ind } X \leq n$. $\square$

29-5. COROLLARY. If (X, ρ) is a metric space with $\dim X = 1$, then there exists a pair of closed sets H and F with $\rho(H, F) > 0$ such that the empty set cannot separate H and F .

29-6. DEFINITION. The empty set ϕ has $d_2\phi = -1$. $d_2(X, \rho) \leq n$ if a metric space (X, ρ) satisfies the condition:

For any $n + 1$ pairs of closed sets $C_1, C_1'; \dots; C_{n+1}, C_{n+1}'$ with $\rho(C_i, C_i') > 0$ for every i , there exist closed sets $B_1, \dots, B_{n+1}$ such that $\cap B_i = \phi$ and such that B_i separates C_i and C_i' for every i .

If $d_2(X, \rho) \leq n$ and the statement $d_2(X, \rho) \leq n - 1$ is false, we say $d_2(X, \rho) = n$. If $d_2(X, \rho)$ is not less than n for any n , then we say $d_2(X, \rho) = \infty$.

By Theorem 9-9 it is evident that $d_2(X, \rho) \leq \dim X$.

29-7. PROPOSITION. Let (X, ρ) be a metric space. If $\text{Ind } X = 1$, then $d_2(X, \rho) = 1$.

This is clear from Corollary 29-5.

29-8. DEFINITION. The empty set ϕ has $d_3\phi = -1$. $d_3(X, \rho) \leq n$ if a metric space (X, ρ) satisfies the condition:

For any finite number of pairs of closed sets $C_1, C_1'; \dots; C_m, C_m'$ with $\rho(C_i, C_i') > 0$ for each i , there exist closed sets $B_1, \dots, B_m$ such that (i) B_i separates C_i and C_i' for each i and (ii) $\text{ord}\{B_i\} \leq n$

The empty set ϕ has $d_4\phi = -1$. $d_4(X, \rho) \leq n$ if a metric space (X, ρ) satisfies the condition:

For any sequence of pairs of closed sets $C_1, C_1'; C_2, C_2'; \dots$ with $\rho(C_i, C_i') > 0$ for each i , there exist closed sets $B_1, B_2, \dots$ such that (i) B_i separates C_i and C_i' for each i and (ii) $\text{ord}\{B_i\} \leq n$.

The definition for $d_i(X, \rho) = n$ or $= \infty$, $i = 3, 4$, is analogous to the corresponding one for d_2 .

It is evident that $d_2(X, \rho) \leq d_3(X, \rho) \leq d_4(X, \rho)$.

29-9. THEOREM (Nagami and Roberts [3]). $d_4(X, \rho) = \dim X$ for any metric space (X, ρ) .

PROOF. First let us prove the inequality $d_4(X, \rho) \leq \dim X$. When $\dim X = \infty$, the inequality is trivially true. Consider the case when $\dim X \leq n$. Let $C_1, C_1'; C_2, C_2'; \dots$ be a sequence of pairs of closed sets with $\rho(C_i, C_i') > 0$ for each i . Let B_1 be a closed set with $\dim B_1 \leq n - 1$, separating C_1 and C_1' . By Corollary 12-12 there exists a closed set B_2 , separating C_2 and C_2' , such that

$$\dim B_2 \leq n - 1, \quad \dim(B_1 \cap B_2) \leq n - 2$$

By repeated applications of Corollary 12-12 we easily obtain a sequence of closed sets $B_1, B_2, \dots$ such that B_i separates C_i and C_i' for each i and $\text{ord}\{B_i\} \leq n$. Hence, $d_4(X, \rho) \leq n$.

To prove $\dim X \leq d_4(X, \rho)$ consider the case when $d_4(X, \rho) \leq n$ since the inequality is true for the case when $d_4(X, \rho) = \infty$. Let us show the existence of a σ -discrete base $\mathcal{U}$ of X such that $\text{ord } B(\mathcal{U}) \leq n$, which will imply $\dim X \leq n$ by Theorem 12-13. Let $\mathcal{V} = \cup \mathcal{V}_i$ be a σ -discrete base of X , where each $\mathcal{V}_i$ is discrete. Set

$$V_i = \cup \{V : V \in \mathcal{V}_i\}$$

$$F_{ij} = \{x : \rho(x, X - V_i) \geq 1/j\}, \quad j = 1, 2, \dots$$

Then by $d_4(X, \rho) \leq n$ there exist open sets U_{ij} , $i, j = 1, 2, \dots$, such that

$$V_i \supset \bar{U}_{ij} \supset U_{ij} \supset F_{ij}$$

$$\text{ord}\{B(U_{ij}) : i, j = 1, 2, \dots\} \leq n$$

Set

$$\mathcal{U}_{ij} = \{V \cap U_{ij} : V \in \mathcal{V}_i, \quad j = 1, 2, \dots\}$$

Then $\mathcal{U}_{ij}$ is discrete. Set

$$\mathcal{U} = \cup_{i,j} \mathcal{U}_{ij}$$

Then $\mathcal{U}$ is a σ -discrete base of X with $\text{ord } B(\mathcal{U}) \leq n$. $\square$

29-10. THEOREM (Nagami and Roberts [3]). If (X, ρ) is a metric space, then $d_3(X, \rho) \leq \mu \dim(X, \rho)$.

PROOF. Suppose $\mu \dim(X, \rho) \leq n$. Let $C_1, C_1'; \dots; C_m, C_m'$ be a finite number of pairs of closed sets of X such that $m \geq n+1$ and $\rho(C_i, C_i') > 0$ for every i . Let ε be a positive number such that $\rho(C_i, C_i') > \varepsilon$ for every i . By Proposition 29-2 there exists a locally finite open covering $\mathcal{U}_0$ of X such that

$$\text{ord } \mathcal{U}_0 \leq n+1 \quad \text{and} \quad \text{mesh } \mathcal{U}_0 < \varepsilon/2$$

By repeated applications of Lemma 20-2 we obtain closed sets $B_1, \dots, B_m$ and locally finite open coverings $\mathcal{U}_1, \dots, \mathcal{U}_m$ of X which satisfy the following conditions:

- i. Each B_i separates C_i and C_i'
- ii. $\mathcal{U}_i$ refines $\mathcal{U}_{i-1}$ in a one-one corresponding way
- iii. $\text{ord}_x \mathcal{U}_i < \text{ord}_x \mathcal{U}_{i-1}$ for any point x in B_i

To prove $\text{ord}\{B_i\} \leq n$, assume the contrary. Then there would be a point x and a sequence $1 \leq i_1 < i_2 < \dots < i_{n+1} \leq m$ such that

$$x \in B_{i_j}, \quad j = 1, \dots, n+1$$

By condition (iii)

$$\begin{aligned} n+1 &\geq \text{ord}_x \mathcal{U}_0 \geq \text{ord}_x \mathcal{U}_1 \geq \dots > \text{ord}_x \mathcal{U}_{i_1} \geq \dots \\ &> \text{ord}_x \mathcal{U}_{i_2} \geq \dots > \text{ord}_x \mathcal{U}_{i_{n+1}} \end{aligned}$$

Hence

$$\text{ord}_x \mathcal{U}_{i_{n+1}} < n+1 - (n+1) = 0$$

a contradiction. Thus we obtain $d_3(X, \rho) \leq n$. $\square$

29-11. THEOREM (Nagami and Roberts [3]). If (X, ρ) is a totally bounded metric space, then

$$d_3(X, \rho) = \mu \dim(X, \rho)$$

PROOF. By the preceding theorem we already have $d_3(X, \rho) \leq \mu \dim(X, \rho)$. So we merely need to prove $d_3(X, \rho) \geq \mu \dim(X, \rho)$. Let $d_3(X, \rho) \leq n$ and ε an arbitrary positive number. By the total boundedness of (X, ρ) there exist finite points $x_1, \dots, x_m$ such that

$$\{U_i = S_{\varepsilon/4}(x_i); i = 1, \dots, m\}$$

covers X . Set

$$V_i = S_{\varepsilon/2}(x_i), \quad i = 1, \dots, m$$

Then by $d_3(X, \rho) \leq n$ there exist open sets $W_1, \dots, W_m$ such that

$$\begin{aligned} \bar{U}_i &\subset W_i \subset \bar{W}_i \subset V_i \quad \text{for each } i \\ \text{ord}\{B(W_i); i = 1, \dots, m\} &\leq n \end{aligned}$$

By Proposition 9-2 there exist open sets $G_1, \dots, G_m$ of X such that

$$\begin{aligned} B(W_i) &\subset G_i \subset V_i \quad \text{for each } i \\ \text{ord}\{G_1, \dots, G_m\} &\leq n \end{aligned}$$

Set

$$\mathcal{W}_i = \{W_{i1} = W_i, \quad W_{i2} = X - \bar{W}_i\}$$

$$\mathcal{W} = \bigwedge_{i=1}^m \mathcal{W}_i$$

Since

$$\bigcap_{i=1}^m W_{i2} = \bigcap_{i=1}^m (X - \bar{W}_i) = X - \bigcap_{i=1}^m \bar{W}_i = X - X = \phi$$

$\mathcal{W}$ refines $\{W_1, \dots, W_m\}$ and hence also $\{V_1, \dots, V_m\}$. Moreover, $\text{ord } \mathcal{W} \leq 1$ and $\mathcal{W}$ covers $X - \cup_{i=1}^m B(W_i)$. If we set

$$\mathcal{G} = \mathcal{W} \cup \{G_1, \dots, G_m\}$$

then $\mathcal{G}$ is an open covering of X such that

$$\text{mesh } \mathcal{G} < \varepsilon \quad \text{and} \quad \text{ord } \mathcal{G} \leq n + 1$$

Thus we have $\mu \dim(X, \rho) \leq n$. $\square$

29-12. PROBLEM. Is it true that $d_3(X, \rho) = \mu \dim(X, \rho)$ for any metric space (X, ρ) ?

29-13. REMARK. When (X, ρ) is locally compact, it is easy to see that $d_2(X, \rho) = d_3(X, \rho) = \mu \dim(X, \rho) = \dim X$. On the other hand, there exists a space (X, ρ) which is not locally compact at any point but yet $d_2(X, \rho) = d_3(X, \rho) = \mu \dim(X, \rho) = \dim X$. Consider

$$I^3 = \{\langle x_1, x_2, x_3 \rangle : 0 \leq x_i \leq 1\}$$

Let ρ be an arbitrary metric of I^3 which is equivalent to the Euclidean one. Set

$$B = \{\langle x_1, x_2, x_3 \rangle \in I^3 : x_1 = 0\}$$

Let D be the set of all points of I^3 all of whose coordinates are rational. If we set $X = B \cup D$, then (X, ρ) satisfies the condition:

$$\begin{aligned} 2 &= d_2(B, \rho) \leq d_2(X, \rho) \leq d_3(X, \rho) \\ &\leq \mu \dim(X, \rho) \leq \dim X = 2 \end{aligned}$$

29-14. LEMMA. Let $\mathcal{G} = \{G_i\}$ be a countable open covering of a metric space (X, ρ) , then there exist a countable closed covering $\{F_i\}$ of X and a sequence of positive numbers $\varepsilon_1, \varepsilon_2, \dots$ such that

$$\{S(F_i : \varepsilon_i) = S_{\varepsilon_i}(F_i) : i = 1, 2, \dots\}$$

is star-finite and refines $\mathcal{G}$.

PROOF. Set

$$\begin{aligned} E_{ij} &= X - S(X - G_i : 1/2^j), \quad j = 1, 2, \dots \\ E_i &= \cup \{E_{ji} : j \leq i\} \end{aligned}$$

Then $G_i = \cup_j E_{ij}$ and $X = \cup E_i$. Let D_{ij} be the open kernel of E_{ij} . Set

$$D_i = \cup \{D_{ji} : j \leq i\}$$

Then

$$D_{i1} \subset E_{i1} \subset D_{i2} \subset E_{i2} \subset \dots$$

$$D_1 \subset E_1 \subset D_2 \subset E_2 \subset \dots$$

Set

$$F_{ij} = (E_i - D_{i-1}) \cap E_{ji}, \quad j \leq i, \quad i = 1, 2, \dots$$

where $D_0 = \phi$. Then

$$\{F_{ij} : j \leq i, \quad i = 1, 2, \dots\}$$

is a closed covering of X . Let us prove that

$$\mathcal{S} = \{S_{ij} = S(F_{ij} : 1/2^{i+1}) : j \leq i, \quad i = 1, 2, \dots\}$$

is star-finite and refines $\mathcal{G}$. Let i and j be arbitrary positive integers with $j \leq i$. Since $E_{ji} \cap S(X - G_j : 1/2^i) = \phi$,

$$\begin{aligned} S_{ij} \cap (X - G_j) &\subset S_{ij} \cap S(X - G_j : 1/2^{i+1}) \\ &\subset S(E_{ji} : 1/2^{i+1}) \cap S(X - G_j : 1/2^{i+1}) = \phi \end{aligned}$$

Hence $\mathcal{S}$ refines $\mathcal{G}$ and

$$S_{ij} \subset S(E_{ji} : 1/2^{i+1}) \subset E_{j,i+1}$$

The latter inequality yields that

$$S(E_{j,i+1} : 1/2^{i+2}) \subset E_{j,i+2} \subset E_{i+2} \subset D_{i+3}$$

Hence

$$E_{j,i+1} \cap S(X - D_{i+3} : 1/2^{i+2}) = \phi$$

Let m and n be arbitrary positive integers with $i + 4 \leq m$ and with $n \leq m$. Since $F_{mn} \cap D_{m-1} = \phi$, $F_{mn} \cap D_{i+3} = \phi$. Hence

$$\begin{aligned} S_{ij} \cap S_{mn} &\subset S(E_{ji} : 1/2^{i+1}) \cap S(F_{mn} : 1/2^{m+1}) \\ &\subset E_{j,i+1} \cap S(X - D_{i+3} : 1/2^{m+1}) \\ &\subset E_{j,i+1} \cap S(X - D_{i+3} : 1/2^{i+2}) = \phi \end{aligned}$$

Therefore S_{ij} meets at most a finite number of elements of $\mathcal{S}$. $\square$

29-15. THEOREM (Hodel [1]). If (X, ρ) is a metric space, then $\dim X \leq 2d_3(X, \rho)$.

PROOF. When $d_3(X, \rho) = \infty$, the inequality is trivially true. Let $d_3(X, \rho) = n$. Let $\mathcal{G}$ be an arbitrary finite open covering of X . Let $\{F_i\}$ be a closed covering of X and $\varepsilon_i, i = 1, 2, \dots$, a sequence of positive numbers which satisfy the condition of Lemma 29-14. Let $i_1 < i_2 < \dots$ be a sequence such that

$$\bigcup_{i \leq i_j} S(F_i; \varepsilon_i) \subset \bigcup_{i \leq i_{j+1}} F_i, \quad j = 1, 2, \dots$$

Such a sequence can be found by the star-finiteness of $\{S(F_i; \varepsilon_i)\}$. For each j , consider i_j pairs with positive distances

$$F_1, X - S(F_1; \varepsilon_1); \dots; F_{i_j}, X - S(F_{i_j}; \varepsilon_{i_j})$$

By the equality $d_3(X, \rho) = n$ there exists open sets $U(j, 1), \dots, U(j, i_j)$ such that

$$F_k \subset U(j, k) \subset \overline{U(j, k)} \subset X - S(F_k; \varepsilon_k), \quad k = 1, \dots, i_j$$

$$\text{ord}\{B(j, k) = B(U(j, k)): k = 1, \dots, i_j\} \leq n$$

By Proposition 9-2 there exist open sets $V(j, 1), \dots, V(j, i_j)$ such that

$$B(j, k) \subset V(j, k) \subset S(F_k; \varepsilon_k) - F_k, \quad k = 1, \dots, i_j$$

$$\text{ord}\{V(j, k): k = 1, \dots, i_j\} \leq n$$

Set

$$\mathcal{W}_{jk} = \{U(j, k), X - \overline{U(j, k)}, k = 1, \dots, i_j\}$$

$$\mathcal{W}_j' = \bigwedge \{\mathcal{W}_{jk}: k = 1, \dots, i_j\}$$

Then it is easy to see that

$$\text{ord}_x \mathcal{W}_j' \leq 1$$

at every point x in $\bigcup \{U(j, k): k = 1, \dots, i_j\}$. Set

$$U_j = \bigcup \{U(j, k): k = 1, \dots, i_j\}$$

$$B_j = \bigcup \{B(j, k): k = 1, \dots, i_j\}$$

$$\mathcal{W}_j = \mathcal{W}_j' \mid (U_j - B_j)$$

Then $\mathcal{W}_j$ is an open covering of $U_j - B_j$ with $\text{ord } \mathcal{W}_j \leq 1$.

Set

$$\mathcal{H}_j = \{V(j, k): k = 1, \dots, i_j\} \cup \mathcal{W}_j$$

Then $\mathcal{H}_j$ is a collection of open sets, covering $\overline{U}_j$, such that

$$\text{ord } \mathcal{H}_j \leq n + 1, \quad \mathcal{H}_j < \mathcal{G}$$

Set

$$H_j = \cup \{H: H \in \mathcal{H}_j\}$$

Then $\bar{U}_j \subset H_j \subset U_{j+1}$ for each j . Set

$$\mathcal{H} = \cup \{\mathcal{H}'_j = \mathcal{H}_j \mid (H_j - \bar{U}_{j-1}): j = 1, 2, \dots\}$$

where $U_0 = \phi$. Then $\mathcal{H}$ is an open covering of X which refines $\mathcal{G}$. To prove $\text{ord } \mathcal{H} \leq 2n + 1$ let x be an arbitrary point of X . Then there exists a j with $x \in H_j - \bar{U}_{j-1}$. There are two cases: (1) $x \in H_j - \bar{U}_j$, (2) $x \in \bar{U}_j - \bar{U}_{j-1}$. In case (1), $\text{ord}_x \mathcal{H}'_s = 0$ with either $s \leq j - 1$ or $s \geq j + 2$. Hence,

$$\begin{aligned} \text{ord}_x \mathcal{H} &= \text{ord}_x \mathcal{H}'_j + \text{ord}_x \mathcal{H}'_{j+1} \\ &\leq \text{ord}_x \{V(j, k): k = 1, \dots, i_j\} + (n + 1) \\ &\leq n + (n + 1) = 2n + 1 \end{aligned}$$

In case (2), $\text{ord}_x \mathcal{H}'_s = 0$ with either $s \leq j - 2$ or $s \geq j + 1$. Hence,

$$\begin{aligned} \text{ord}_x \mathcal{H} &= \text{ord}_x \mathcal{H}'_{j-1} + \text{ord}_x \mathcal{H}'_j \\ &\leq \text{ord}_x \{V(j - 1, k): k = 1, \dots, i_{j-1}\} + (n + 1) \\ &\leq n + (n + 1) = 2n + 1 \end{aligned}$$

Thus we obtain $\text{ord } \mathcal{H} \leq 2n + 1$, which proves $\dim X \leq 2n$. $\square$

Since $d_3(X, \rho) \leq \mu \dim(X, \rho)$ by Theorem 29-10 we get at once the following corollary.

29-16. COROLLARY (Katětov [3]). If (X, ρ) is a metric space, then $\dim X \leq 2\mu \dim(X, \rho)$

Later we shall construct a metric space (X, ρ) for which $2d_3(X, \rho) = 2\mu \dim(X, \rho) = \dim X$. Thus the inequalities $\dim X \leq 2d_2(X, \rho)$ and $\dim X \leq 2\mu \dim(X, \rho)$ are the best possible.

29-17. PROBLEM. Let (X, ρ) be a metric space. Is it true that $\dim X \leq 2d_2(X, \rho)$? At the present stage even the following problem has not been solved: If $d_2(X, \rho)$ is finite, is $\dim X$ finite?

30. Spaces with $d_2 = [n/2]$ and $\dim \geq n - 1$

30-1. LEMMA. Let $F_1, F_2, \dots$ be a sequence of closed sets of a metric

space X with $\dim F_i = n_i$. Let $C_1, C_1'; \dots; C_m, C_m'$ be m disjoint pairs of closed sets of X , then there exist closed sets $B_1, \dots, B_m$ such that:

- i. B_i separates C_i and C_i' for each i
- ii. For each j and for each sequence $1 \leq i_1 < i_2 < \dots < i_t \leq \min\{n_j + 1, m\}$:

$$\dim(B_{i_1} \cap \dots \cap B_{i_t} \cap F_j) \leq n_j - t$$

By repeated applications of Corollary 12-12 we can obtain the lemma at once.

30-2. EXAMPLE (Nagami and Roberts [3]). There exist spaces (X_n, ρ) with $d_2(X_n, \rho) = [n/2]$ and $\dim X_n \geq n - 1$. Let (K_n, ρ) be a metric Cantor n -manifold with $n \geq 3$. Put

$$m = [n/2] + 1$$

where $[n/2]$ denotes the integral part of $n/2$. By the compactness of K_n there exists a sequence of m disjoint pairs of closed sets of K_n , say

$$C_{i1}, C'_{i1}; \dots; C_{im}, C'_{im}, \quad i = 1, 2, \dots$$

such that for any m disjoint pairs of closed sets $C_1, C_1'; \dots; C_m, C_m'$ there exists an i with

$$C_j \subset C_{ij} \quad \text{and} \quad C'_j \subset C'_{ij} \quad \text{for } j = 1, \dots, m$$

By Lemma 30-1 there exist closed sets $B_{11}, \dots, B_{1m}$ such that

- i. B_{1i} separates C_{1i} and C'_{1i} for each i
- ii. $\dim B_1 \leq n - m$ where $B_1 = \bigcap_{j=1}^m B_{1j}$

Again by Lemma 30-1 there exist closed sets $B_{21}, \dots, B_{2m}$ such that

- i. B_{2i} separates C_{2i} and C'_{2i} for each i
- ii. $\dim B_2 \leq n - m$ where $B_2 = \bigcap_{j=1}^m B_{2j}$
- iii. $B_1 \cap B_2 = \phi$

Continuing in this fashion we obtain finally a sequence of closed sets B_{ij} , $i = 1, 2, \dots, j = 1, \dots, m$, satisfying the following conditions:

- i. B_{ij} separates C_{ij} and C'_{ij} for each i and j
- ii. $\dim B_i \leq n - m$ for $i = 1, 2, \dots$, where $B_i = \bigcap_{j=1}^m B_{ij}$
- iii. $B_i \cap B_j = \phi$ if $i \neq j$

If we set

$$X_n = K_n - \bigcup B_i$$

then we have the space (X_n, ρ) .

ASSERTION 1. $d_2(X_n, \rho) \leq [n/2]$.

PROOF. Let $C_1, C_1'; \dots; C_m, C_m'$ be m pairs of closed sets of X_n such that $\rho(C_i, C_i') > 0$ for $i = 1, \dots, m$. Since their closures $\bar{C}_1, \bar{C}_1'; \dots; \bar{C}_m, \bar{C}_m'$ in K_n constitute m disjoint pairs of closed sets of K_n , there exists an i such that

$$\bar{C}_j \subset C_{ij} \quad \text{and} \quad \bar{C}_j' \subset C_{ij}' \quad \text{for } j = 1, \dots, m$$

Then

$$B_{i1} \cap X_n, \dots, B_{im} \cap X_n$$

are closed sets of X_n such that $B_{ij} \cap X_n$ separates C_j and C_j' in X_n for $j = 1, \dots, m$. Since

$$\bigcap_{j=1}^m (B_{ij} \cap X_n) = B_i \cap X_n = \phi$$

we have

$$d_2(X_n, \rho) \leq m - 1 = [n/2] \quad \square$$

ASSERTION 2. $d_2(X_n, \rho) \geq [n/2]$.

PROOF. If G is a nonempty open set of K_n , then $\dim G = n$. Since $\dim(\cup B_i) \leq n - m < n$, $G - (\cup B_i) \neq \phi$ and hence $G \cap X_n \neq \phi$. Thus, X_n is dense in K_n . Assume $d_2(X_n, \rho) = t < [n/2]$. Take a defining system $\bar{D}_1, \bar{D}_1'; \dots; \bar{D}_{t+1}, \bar{D}_{t+1}'$ of K_n such that: (1) each D_i and D_i' are open in K_n , (2) for any closed sets $A_i, i = 1, \dots, t + 1$, separating $\bar{D}_i$ and $\bar{D}_i'$

$$\dim\left(\bigcap_{i=1}^{t+1} A_i\right) \geq n - (t + 1)$$

Set

$$C_i = \bar{D}_i \cap X_n, \quad C_i' = \bar{D}_i' \cap X_n$$

Then it is easy to see that $\bar{C}_i = \bar{D}_i$ and $\bar{C}_i' = \bar{D}_i'$, since X_n is dense in K_n .

Set

$$\varepsilon = \min\{\rho(C_i, C_i') : i = 1, \dots, t + 1\}$$

By the assumption $d_2(X_n, \rho) = t$, there exist open sets $U_i, i = 1, \dots, t + 1$, of K_n such that (3)

$$\begin{aligned} \{x \in X_n : \rho(x, C_i) < \varepsilon/4\} &\subset U_i \cap X_n \subset \bar{U}_i \cap X_n \\ &\subset X_n - \{x \in X_n : \rho(x, C_i') < \varepsilon/4\} \end{aligned}$$

for each i , and (4):

$$\bigcap_{i=1}^{t+1} (B(U_i) \cap X_n) = \phi$$

It is easy to see that $B(U_i)$ thus chosen separates $\bar{C}_i = \bar{D}_i$ and $\bar{C}'_i = \bar{D}'_i$ for each i . Hence setting

$$B = \bigcap_{i=1}^{t+1} B(U_i)$$

we obtain

$$\dim B \geq n - (t + 1) \geq n - [n/2]$$

On the other hand

$$\dim B \leq \dim(\cup B_i) \leq n - m = n - [n/2] - 1$$

because $B \cap X_n = \phi$, which is a contradiction $\square$

ASSERTION 3. $\dim X_n \geq n - 1$.

PROOF. Since $\dim B_i \leq n - m = n - [n/2] - 1 \leq n - 1$, we have $\dim X_n \geq n - 1$ by Theorem 28-7. $\square$

Thus we obtain

$$d_2(X_n, \rho) = [n/2], \quad \dim X_n \geq n - 1$$

31. Spaces with $\mu \dim = [n/2]$ and $\dim \geq n - 1$

31-1. EXAMPLE (Nagami and Roberts [3]). There exist spaces (Y_n, ρ) with $\mu \dim(Y_n, \rho) = [n/2]$ and $\dim Y_n \geq n - 1$. Let (K_n, ρ) be a metric Cantor n -manifold, $n \geq 3$. Set $m = [n/2] + 2$. By Theorem 12-9 there exists a locally finite closed covering $\mathcal{F}_1 = \{F_\alpha : \alpha \in \Lambda_1\}$ of K_n such that

$$\text{mesh } \mathcal{F}_1 < 1 \tag{1}$$

$$\text{ord } \mathcal{F}_1 \leq n + 1 \tag{2}$$

$$\dim B_1 \leq n - m + 1, \quad \text{where } B_1 = \{x : \text{ord}_x \mathcal{F}_1 \geq m\} \tag{3}$$

Since $\mathcal{F}_1$ is locally finite, B_1 is closed. It follows that $\text{ord } \mathcal{F}_1 \mid (K_n - B_1) < m$. Again by Theorem 12-9, there exists a locally finite closed covering $\mathcal{F}_2 = \{F_\alpha : \alpha \in \Lambda_2\}$ of K_n such that

$$\text{mesh } \mathcal{F}_2 < 1/2 \tag{4}$$

$$\text{ord } \mathcal{F}_2 \leq n + 1 \tag{5}$$

$$\dim B_2 \leq n - m + 1, \quad \text{where } B_2 = \{x : \text{ord}_x \mathcal{F}_2 \geq m\} \tag{6}$$

$$\dim \bigcap_{k=1}^j (F_{\alpha(k)} \cap B_1) \leq \dim B_1 - j + 1 \tag{7}$$

for any $j \leq \dim B_1 + 2$ and any different indices $\alpha(1), \dots, \alpha(j)$ of Λ_2 ,

To show that condition (7) implies $B_1 \cap B_2 = \phi$, let $\dim B_1 = n_1$. Take $n_1 + 2$ different indices $\alpha(1), \dots, \alpha(n_1 + 2)$ of Λ_2 . Then

$$\dim \bigcap_{k=1}^{n_1+2} (F_{\alpha(k)} \cap B_1) \leq n_1 - (n_1 + 2) + 1 = -1$$

Hence we have

$$B_1 \cap \{x: \text{ord}_x \mathcal{F}_2 \geq n_1 + 2\} = \phi$$

Since, by (3), (6) and (7)

$$\begin{aligned} n_1 + 2 &\leq (n - m + 1) + 2 = n - m + 3 \\ &= n - ([n/2] + 2) + 3 \\ &= n - [n/2] + 1 \\ &\leq (2[n/2] + 1) - [n/2] + 1 \\ &= [n/2] + 2 = m \end{aligned}$$

we have $B_1 \cap B_2 = \phi$.

Repeating these procedures we have a sequence of locally finite closed coverings $\mathcal{F}_i$, $i = 1, 2, \dots$, which satisfies the following conditions:

- i. For each i , $\text{mesh } \mathcal{F}_i < 1/i$
- ii. For each i , $\dim B_i \leq n - m + 1$, where B_i is a closed set defined by $B_i = \{x: \text{ord}_x \mathcal{F}_i \geq m\}$
- iii. B_i , $i = 1, 2, \dots$, are mutually disjoint

We set $Y_n = K_n - \cup B_i$. Then (Y_n, ρ) is what we want to have.

ASSERTION 1. $\dim Y_n \geq n - 1$.

PROOF. Since

$$\dim B_i \leq n - m + 1 = n - [n/2] - 1 \leq n - 1$$

the assertion is true by Theorem 28-7. $\square$

ASSERTION 2. $\mu \dim(Y_n, \rho) \leq [n/2]$.

PROOF. Since

$$\text{ord } \mathcal{F}_i | Y_n \leq \text{ord } \mathcal{F}_i | (K_n - B_i) \leq m - 1 = [n/2] + 1$$

the assertion is true by Proposition 29-2. $\square$

ASSERTION 3. $\dim Y_n \leq n - 1$ when n is odd.

PROOF. Since $\dim Y_n \leq 2\mu \dim(Y_n, \rho)$ by Theorem 29-16, we have

$$\dim Y_n \leq 2[n/2] = 2((n-1)/2) = n-1 \quad \square$$

ASSERTION 4. $\mu \dim(Y_n, \rho) \geq [n/2]$.

PROOF. Assume the contrary. Then

$$\dim Y_n \leq 2\mu \dim(Y_n, \rho) \leq 2([n/2] - 1) \leq n-2$$

which is a contradiction. $\square$

Thus (Y_n, ρ) satisfies

$$\dim Y_n \geq n-1, \quad \mu \dim(Y_n, \rho) = [n/2]$$

Especially when n is odd,

$$\dim Y_n = n-1$$

31-2. REMARK. For X_n and Y_n obtained by replacing K_n with I^n , $n \geq 1$,

$$\dim X_n = \dim Y_n = n-1$$

because of the fact that $I^n - X_n$ and $I^n - Y_n$ are dense in I^n and of the invariance of domain theorem (see Hurewicz and Wallman [1, Theorem VI 9]).

Now that the existence of a sequence of open coverings $\mathcal{U}_i$, $i = 1, 2, \dots$, with $\text{ord } \mathcal{U}_i \leq n+1$ and $\lim \text{mesh } \mathcal{U}_i = 0$ is seen not to characterize dimension, the reader will recognize the *raison d'être* of Theorem 14-1 or Vopenka's theorem, Section 14-2. Theorems of this type, which characterized dimension, are one of the main foundations on which modern dimension theory has built up. Nagata is the first man who found such a characterization theorem and utilized it (Nagata [3], [6]).

32. Spaces Illustrating the Dependence of d_2 , d_3 and $\mu \dim$ on the Metric

32-1. LEMMA. If (X, ρ) is a metric space with $\dim X = n$, then there exists an equivalent metric ρ' to ρ such that $d_2(X, \rho') = n$.

PROOF. Since $\dim X = n$, there exists by Theorem 9-9, a defining system of n pairs $C_1, C_1'; \dots; C_n, C_n'$. Let $f_i: X \rightarrow [0, 1]$, $i = 1, \dots, n$, be mappings such that

$$\begin{aligned} f_i(x) &= 0, & x &\in C_i \\ f_i(x) &= 1, & x &\in C_i' \end{aligned}$$

Set

$$\rho'(x, y) = \rho(x, y) + \sum_{i=1}^n |f_i(x) - f_i(y)|$$

Then ρ' is an equivalent metric to ρ and $\rho'(C_i, C_i') > 0$ for each i . Thus we have $d_2(X, \rho') \geq n$ and hence $d_2(X, \rho') = n$. $\square$

32-2. EXAMPLE (Nagami and Roberts [3]). There exist spaces (Z_n, σ_i) illustrating the dependence of $\mu \dim$ and d_2 on the metric. Let $n \geq 2$. Set

$$m = [(n+1)/2] + 1$$

In every (I^i, ρ_i) , $i = m, m+1, \dots, n+1$, we construct (Y_i, ρ_i) as in the preceding section. Then

$$\mu \dim(Y_i, \rho_i) = [i/2] \leq [(n+1)/2]$$

$$\dim Y_i = i - 1$$

We let $\rho_i(I^i) \leq 1$ for each i . Choose an equivalent metric ρ_i' to ρ_i as in the proof of Lemma 32-1 such that

$$d_2(Y_i, \rho_i') = i - 1$$

$$\rho_i'(Y_i) \leq i + 1$$

Since

$$d_2(Y_i, \rho_i') \leq d_3(Y_i, \rho_i') \leq \mu \dim(Y_i, \rho_i') \leq \dim Y_i$$

we have

$$d_3(Y_i, \rho_i') = \mu \dim(Y_i, \rho_i') = i - 1$$

Z_n is the topologically disjoint sum of $Y_m, Y_{m+1}, \dots, Y_{n+1}$. Then it is a metric space. Define for $i = m, \dots, n+1$ the metrics σ_i of Z_n by:

- i. $\sigma_i | Y_j = \rho_j$ if $i \neq j$
- ii. $\sigma_i | Y_i = \rho_i'$
- iii. $\sigma_i(x, y) = n + 2$ if x and y are not in the same Y_j , $j = m, \dots, n + 1$

Then $\sigma_m, \dots, \sigma_{n+1}$ are equivalent metrics all of which give the preassigned topology of Z_n .

ASSERTION 1. $\dim Z_n = n$.

PROOF. $\dim Z_n = \max \dim Y_i = n$. $\square$

ASSERTION 2. $d_2(Z_n, \sigma_i) = d_3(Z_n, \sigma_i) = \mu \dim(Z_n, \sigma_i) = i - 1$ for $i = m, m + 1, \dots, n + 1$.

PROOF. If $j \neq i$, then

$$\mu \dim(Y_j, \sigma_i) = \mu \dim(Y_j, \rho_j) \leq [(n+1)/2]$$

Since

$$\begin{aligned} d_2(Y_i, \sigma_i) &= d_3(Y_i, \sigma_i) \\ &= \mu \dim(Y_i, \sigma_i) \\ &= \mu \dim(Y_i, \rho_i') \\ &= i-1 \geq [(n+1)/2] \end{aligned}$$

then

$$\begin{aligned} i-1 &= d_2(Z_n, \sigma_i) \leq d_3(Z_n, \sigma_i) \leq \mu \dim(Z_n, \sigma_i) \\ &= \max\{\mu \dim(Y_i, \rho_i'), \mu \dim(Y_j, \rho_j): j \neq i\} \\ &= i-1 \quad \square \end{aligned}$$

32-3. LEMMA. Let (X, ρ) be a metric space with $\mu \dim(X, \rho) = k$, then for every positive number ε there exists a locally finite, σ -discrete closed covering $\mathcal{H}$ of X such that

$$\text{ord } \mathcal{H} \leq k+1 \quad \text{and} \quad \text{mesh } \mathcal{H} < \varepsilon$$

PROOF. Let ε be an arbitrary positive number. Then by Proposition 29-2 there exists a locally finite open covering $\mathcal{V}$ of X such that

$$\text{mesh } \mathcal{V} < \varepsilon, \quad \text{ord } \mathcal{V} \leq k+1$$

As can be seen in the proof of Theorem 1-5 there exists a polyhedron P with the weak topology such that

- i. P approximates $\mathcal{V}$
- ii. The combinatorial dimension of P is at most k

Let $\{v\}$ be the vertex set of P . Then condition (i) implies that there exists a mapping $f: X \rightarrow P$ such that $\mathcal{V}$ is refined by $f^{-1}(\{\text{St}(v)\})$. Condition (ii) implies $\dim P \leq k$ by Lemma 26-2. Since P is a paracompact Hausdorff space by Section 3-11, the open covering $\{\text{St}(v)\}$ of P is refined by a locally finite open covering $\mathcal{U} = \cup \mathcal{U}_i$ such that each $\mathcal{U}_i$ is discrete. By Theorem 9-13, there exists an open covering $\mathcal{G}$ of P such that

$$\text{ord } \mathcal{G} \leq k+1, \quad \mathcal{G} < \mathcal{U}$$

Well-order the elements U of $\mathcal{U}$. Let G_U be the sum of all elements G of $\mathcal{G}$ such that $G \subset U$ and $G \not\subset U'$ for any U' preceding U . Set

$$\mathcal{G}_i = \{G_U: U \in \mathcal{U}_i\}, \quad i = 1, 2, \dots$$

Then each $\mathcal{G}_i$ is discrete and $\cup \mathcal{G}_i$ is locally finite. Moreover,

$$\cup \mathcal{G}_i \text{ covers } P, \quad \cup \mathcal{G}_i < \mathcal{U}, \quad \text{ord}(\cup \mathcal{G}_i) \leq k + 1$$

By Theorem 1-7, $\cup \mathcal{G}_i$ is shrinkable. Hence there exists a closed covering $\cup \mathcal{F}_i$ of P refining $\cup \mathcal{G}_i$ in a one-one corresponding way, where each $\mathcal{F}_i$ is correspondent to $\mathcal{G}_i$. Then

- i. Each $\mathcal{F}_i$ is discrete
- ii. $\cup \mathcal{F}_i$ is locally finite and refines $\mathcal{U}$
- iii. $\text{ord}(\cup \mathcal{F}_i) \leq k + 1$

Set

$$\mathcal{H}_i = f^{-1}(\mathcal{F}_i)$$

Then $\mathcal{H} = \cup \mathcal{H}_i$ is a locally finite closed covering of X such that:

- i. Each $\mathcal{H}_i$ is discrete
- ii. $\mathcal{H} < \mathcal{V}$
- iii. $\text{ord } \mathcal{H} \leq k + 1$

and (ii) implies $\text{mesh } \mathcal{H} < \varepsilon$. $\square$

32-4. LEMMA (Roberts and Slaughter [1]). Let (X, ρ) be a metric space with $\mu \dim(X, \rho) = k$. Let $f: X \rightarrow [0, 1]$ be a mapping. Define an equivalent metric ρ^* by

$$\rho^*(x, y) = \rho(x, y) + |f(x) - f(y)|$$

then,

$$k \leq \mu \dim(X, \rho^*) \leq k + 1$$

PROOF. Since ρ^* enlarges ρ , it is evident that $k \leq \mu \dim(X, \rho^*)$. Let us prove that $\mu \dim(X, \rho^*) \leq k + 1$. Let ε be an arbitrary positive number. Then by Lemma 32-3, there exists a locally finite closed covering $\mathcal{H} = \cup \mathcal{H}_i$, where each $\mathcal{H}_i$ is discrete, such that

$$\rho\text{-mesh } \mathcal{H} < \varepsilon/2, \quad \text{ord } \mathcal{H} \leq k + 1$$

Set

$$H_i = \cup \{H: H \in \mathcal{H}_i\}$$

Then $\{H_i\}$ is a closed covering of X of order at most $k + 1$. Now pick an infinite sequence of finite sequences

$$a_{i0} = 0 < a_{i1} < a_{i2} < \cdots < a_{im_i} = 1$$

such that:

- i. $a_{ij} - a_{i,j-1} < \varepsilon/2$
- ii. For different i 's, say i' and i'' , $a_{i',j} \neq a_{i'',s}$, for all j and s , except, of course, all $a_{i0} = 0$, all $a_{im_i} = 1$

For each i , break the closed set H_i into m_i closed sets

$$H_i \cap \{x: a_{i,j-1} \leq f(x) \leq a_{ij}\}, \quad j = 1, 2, \dots, m_i$$

Let $\mathcal{F}_i$ be the collection of all such pieces. Then

$$\text{ord}(\cup \mathcal{F}_i) \leq k + 2$$

because any x which is on a dividing line

$$H_i \cap \{x: f(x) = a_{ij}\} \quad (j \neq 0, j \neq m_i)$$

for some i , will not be on a dividing line for any other i . Set

$$\mathcal{K} = \cup_i (\mathcal{H}_i \wedge \mathcal{F}_i)$$

Then $\mathcal{K}$ is a locally finite closed covering of X satisfying

$$\rho^*\text{-mesh } \mathcal{K} < \varepsilon \quad \text{and} \quad \text{ord } \mathcal{K} \leq k + 2$$

By Proposition 29-2, we conclude that $\mu \dim(X, \rho^*) \leq k + 1$. $\square$

32-5. THEOREM (Roberts and Slaughter [1]). Let (X, ρ) be a metric space with $\mu \dim(X, \rho) = k < \dim X = n$, then there exist equivalent metrics $\rho_1, \dots, \rho_{n-k}$ to ρ such that

$$\mu \dim(X, \rho_i) = k + i, \quad i = 1, \dots, n - k$$

PROOF. Let $C_1, C_1'; \dots; C_n, C_n'$ be a defining system of n pairs of closed sets. Let $f_i: X \rightarrow [0, 1]$, $i = 1, \dots, n$, be mappings such that

$$f_i(x) = 0, \quad x \in C_i$$

$$f_i(x) = 1, \quad x \in C_i'$$

Set

$$\sigma_0(x, y) = \rho(x, y)$$

$$\sigma_i(x, y) = \rho(x, y) + \sum_{j=1}^i |f_j(x) - f_j(y)|, \quad i = 1, \dots, n$$

Then by Lemma 32-4

$$\mu \dim(X, \sigma_{i-1}) \leq \mu \dim(X, \sigma_i) \leq \mu \dim(X, \sigma_{i-1}) + 1 \text{ for } i = 1, \dots, n.$$

Since

$$\begin{aligned} d_2(X, \sigma_i) &\leq \mu \dim(X, \sigma_i), & i = 0, \dots, n \\ d_2(X, \sigma_{i-1}) &\leq d_2(X, \sigma_i), & i = 1, \dots, n \end{aligned}$$

and moreover

$$d_2(X, \sigma_n) = n$$

by Lemma 32-1, then the theorem follows. $\square$

32-6. PROBLEM. If for a metric space (X, ρ) there is a gap between $d_i(X, \rho)$ ($i = 2, 3$) and $\dim X$, can we get equivalent metrics for which a statement analogous to the above theorem holds?

32-7. PROBLEM. Find conditions on (X, ρ) which have influence on the difference between $\dim X$ and the metric-dependent dimension functions. Especially it is desirable to obtain various sufficient conditions with which $\dim X = \mu \dim(X, \rho)$ (or $= d_i(X, \rho)$).

33. A Space with $d_2 = 2$, $\mu \dim = 3$, $\dim = 4$

33-1. EXAMPLE (Nagami and Roberts [3]). There exists a metric space (S, σ) with $d_2(S, \sigma) = 2$ and $\mu \dim(S, \sigma) = \dim S = 3$. (S, σ) will be a subset of $I^4 = \{\langle x_1, \dots, x_4 \rangle : 0 \leq x_i \leq 1\}$ with the Euclidean metric σ . Let C_{ij}, C'_{ij} , $i = 1, 2, \dots, j = 1, 2, 3$, be disjoint pairs of closed sets of I^4 such that for any three disjoint pairs of closed sets C_j, C'_j , $j = 1, 2, 3$, there exists an i with $C_i \subset C_{ij}$ and $C'_i \subset C'_{ij}$ for $j = 1, 2, 3$. Let π be a prime number with $5 \leq \pi$. Consider an open covering $\mathcal{D}(\pi)$ of the unit interval $[0, 1]$ consisting of overlapping intervals

$$\begin{aligned} [0, 2/\pi], ((\pi-2)/\pi, 1], & \quad ((2k-1)/\pi, (2k+2)/\pi) \\ k = 1, \dots, (\pi-3)/2 \end{aligned}$$

Define an open covering $\mathcal{E}(\pi)$ of I^4 by

$$\begin{aligned} \mathcal{E}(\pi) &= \{D_1 \times D_2 \times D_3 \times D_4 : D_i \in \mathcal{D}(\pi)\} \\ &= \{E_\lambda : \lambda \in \Lambda(\pi)\} \end{aligned}$$

Let π_{ij} , $i = 1, 2, \dots, j = 1, 2, 3$, be prime numbers which are different from each other and satisfy the following conditions:

- i. $5 \leq \pi_{ij}$ for each i and j
- ii. $\max\{\text{mesh } \mathcal{E}(\pi_{ij}) : j = 1, 2, 3\} < \min\{\sigma(C_{ij}, C'_{ij}) : j = 1, 2, 3\}$ for each i

Let U_{ij} be the sum of all elements of $\mathcal{E}(\pi_{ij})$ which meet C_{ij} . Set

$$B_{ij} = B(U_{ij}) \quad \text{and} \quad B_i = \bigcap_{j=1}^3 B_{ij}$$

Then B_{ij} separates C_{ij} and C'_{ij} . Set

$$S = I^4 - \bigcup B_i$$

Then (S, σ) satisfies the required equalities.

ASSERTION 1. $B_i \cap B_k = \emptyset$ whenever $i \neq k$.

PROOF. Set

$$L_{ij} = \{a/\pi_{ij} : a = 1, \dots, \pi_{ij} - 1\}$$

Then $L_{ij} \cap L_{kt} \neq \emptyset$ if and only if $i = k$ and $j = t$. If $x = \langle x_1, \dots, x_4 \rangle$ is a point of B_{ij} , then for some s , $x_s \in L_{ij}$. Hence, if x is a point of B_i , three coordinates are restricted by L_{ij} , $j = 1, 2, 3$. The inequality $B_i \cap B_k \neq \emptyset$ for $i \neq k$ would imply the existence of a point having at least 6 coordinates, which is a contradiction. $\square$

ASSERTION 2. B_i does not meet any 2-dimensional edge of I^4 . B_i meets the surface of I^4 at only a finite number of points. B_i is the sum of a finite number of segments.

This is evident from the above argument.

ASSERTION 3. B_i is the disjoint sum of a finite number of simple closed curves and a finite number of simple arcs.

PROOF. If three different lines q_1, q_2 , and q_3 lying on B_i had a common point, then they would lie on some hyperplane $H: x_j = \text{constant}$. We can easily visualize H , because H is 3-dimensional. It is now easy to see that $H \cap B_i$ could not contain q_1, q_2 , and q_3 at the same time, because

- i. $\mathcal{E}(\pi_{ij}) \mid H, j = 1, 2, 3$, are collections of bordered blocks
- ii. $\pi_{ij}, j = 1, 2, 3$, are different from each other $\square$

ASSERTION 4. $d_2(S, \sigma) = 2$ and $\dim S = 3$.

The first equality was proved in Section 30. As for the second, see Remark 31-2.

ASSERTION 5. $\mu \dim(S, \sigma) = 3$.

PROOF. To show $\mu \dim(S, \sigma) > 2$ assume that $\mu \dim(S, \sigma) \leq 2$. Then there would exist a finite, closed (in S) covering $\mathcal{F} = \{F\}$ of S satisfying the following conditions:

- i. $\{G(F) = \text{the interior of } F \text{ with respect to } S: F \in \mathcal{F}\}$ covers S
- ii. $\text{mesh } \mathcal{F} < 1$
- iii. $\text{ord } \mathcal{F} \leq 3$

The proof for the existence of such an $\mathcal{F}$ is left to the reader. Use the total boundedness of (S, σ) . Set

$$\mathcal{F}_1 = \{F \in \mathcal{F}: \bar{F} \cap \{x: x_1 = 0\} \neq \emptyset\}$$

$$M_1 = \text{the boundary of } \cup \{\bar{F}: F \in \mathcal{F}_1\} \text{ in } I^4$$

Let F be an arbitrary element of $\mathcal{F}_1$. Let G' be an open set of I^4 with $G' \cap S = G(F)$. Since $\dim(\cup B_i) = 1$, S is dense in I^4 . Hence $G' - \bar{F} \neq \emptyset$ yields $(G' - \bar{F}) \cap S \neq \emptyset$, a contradiction. Thus $G' \subset \bar{F}$ and $G(F) \cap M_1 = \emptyset$. Pick an arbitrary point x from $M_1 \cap S$. Since $x \notin G(F)$ for any F in $\mathcal{F}_1$, there exists an element $F_0 \in \mathcal{F} - \mathcal{F}_1$ such that $x \in G(F_0)$ by condition (i) imposed upon $\mathcal{F}$. Hence

$$\text{ord } \mathcal{F}_1 \mid (M_1 \cap S) \leq \text{ord } \mathcal{F} - 1 \leq 2$$

Set

$$\mathcal{F}_2 = \{F \in \mathcal{F}_1: \bar{F} \cap \{x: x_2 = 0\} \neq \emptyset\}$$

$$M_2 = \text{the boundary of } \cup \{\bar{F} \cap M_1: F \in \mathcal{F}_2\} \text{ in } M_1$$

Pick an arbitrary point x' from $M_2 \cap S$. Let $y^1, y^2, \dots$ be a sequence of points in

$$M_1 - \cup \{\bar{F} \cap M_1: F \in \mathcal{F}_2\}$$

with

$$\lim y^i = x'$$

Since $\mathcal{F}_1$ is finite and $\bar{\mathcal{F}}_1$ covers M_1 , we can assume without loss of generality that the sequence $\{y^i\}$ is contained in one $\bar{F}_1$ with $F_1 \in \mathcal{F}_1 - \mathcal{F}_2$. For each i , let z^i be a point of F_1 with $\sigma(y^i, z^i) < \sigma(y^i, x')$. Since $\lim z^i = x'$ and $x' \in S, x' \in F_1$. Therefore,

$$\text{ord } \mathcal{F}_2 \mid (M_2 \cap S) \leq \text{ord } \mathcal{F}_1 \mid (M_1 \cap S) - 1 \leq 1$$

Set

$$\mathcal{F}_3 = \{F \in \mathcal{F}_2: \bar{F} \cap \{x: x_3 = 0\} \neq \emptyset\}$$

$$M_3 = \text{the boundary of } \cup \{\bar{F} \cap M_2: F \in \mathcal{F}_3\} \text{ in } M_2$$

Since $\text{ord } \overline{\mathcal{F}}_2 \mid (M_2 \cap S) = \text{ord } \mathcal{F}_2 \mid (M_2 \cap S) \leq 1$

$$M_3 \cap S = \phi \quad (1)$$

Set

$$T = \{x \in M_2 : \text{ord}_x \overline{\mathcal{F}}_2 \geq 2\}$$

Then T is a closed set of I^4 such that

$$M_3 \subset T \subset M_2 \cap (\cup B_i) \quad (2)$$

Let K_1 and K_2 be mutually separated relatively open sets of M_2 such that:

$$M_2 - M_3 = K_1 \cup K_2$$

$$K_1 \supset M_2 \cap \{x : x_3 = 0\} \quad \text{and} \quad K_2 \supset M_2 \cap \{x : x_3 = 1\}$$

Let P , P' , Q or Q' be the union of all components of T which meet $\{x : x_3 = 0\}$, $\{x : x_3 = 1\}$, $\{x : x_4 = 0\}$, or $\{x : x_4 = 1\}$, respectively. Then these four sets are closed. Let us show for instance that P is closed.

Let x^0 be an arbitrary point of the closure of P . Then there exists a sequence of components of T , $C_1, C_2, \dots$, such that

- i. Each C_i intersects $\{x : x_3 = 0\}$
- ii. Each C_i contains a point z^i with $\lim z^i = x^0$

Since $x^0 \in \liminf C_i$, $\limsup C_i$ is connected by Theorem 6-6. Since $\limsup C_i$ intersects $\{x : x_3 = 0\}$ and $\limsup C_i \subset T$, then $\limsup C_i \subset P$. In particular, $x^0 \in P$ and hence P is closed.

To show that $P \cup P'$ and $Q \cap Q'$ are disjoint closed sets of T such that there is no continuum in T between them, assume the contrary. Then there would exist a continuum L in T touching three sides $\{x : x_3 = 0 \text{ (or } = 1)\}$, $\{x : x_4 = 0\}$, and $\{x : x_4 = 1\}$ at the same time. Since $L \subset \cup B_i$ by (2) and $\{B_i\}$ is disjoint by Assertion 1, L would be a subset of only one B_i by Sierpinski's theorem, Section 6-10. By Assertion 3, L would be a subset of a simple closed curve or a simple arc and thus L itself would be a simple closed curve or a simple arc. Then by Assertion 2, L would have three end points, a contradiction.

Hence by Corollary 6-8 there exists a subset V of T such that

$$V \text{ is open-closed in } T \quad (3)$$

$$Q \cap Q' \subset V \quad (4)$$

$$V \cap (P \cup P') = \phi \quad (5)$$

Since $Q \cap Q' \cap \{x: x_3 = 0, 1\} = \phi$, there exists a subset W of M_2 such that

$$W \text{ is open in } M_2 \quad (6)$$

$$W \cap T = V \quad (7)$$

$$\bar{W} \cap \{x: x_3 = 0, 1\} = \phi \quad (8)$$

Then

$$(\bar{W} - W) \cap T = \phi \quad (9)$$

$$Q \cap Q' \subset W \quad (10)$$

$$\bar{W} \cap (P \cup P' \cup \{x: x_3 = 0, 1\}) = \phi \quad (11)$$

Set

$$M = (M_3 - W) \cup (\bar{W} - W)$$

$$G_1 = K_1 - \bar{W}, \quad G_2 = (K_2 \cup W) - (\bar{W} - W)$$

Then

$$M_2 - M = G_1 \cup G_2, \quad G_1 \cap G_2 = \phi$$

$$G_1 \supset M_2 \cap \{x: x_3 = 0\}, \quad G_2 \supset M_2 \cap \{x: x_3 = 1\}$$

Since G_1 and G_2 are open in M_2 , M separates $M_2 \cap \{x: x_3 = 0\}$ and $M_2 \cap \{x: x_3 = 1\}$ in M_2 .

Let us show that no component of M meets both $\{x: x_4 = 0\}$ and $\{x: x_4 = 1\}$. Take an arbitrary element F from $\mathcal{F}_2$. Set

$$U(F) = M_2 - \cup \{\bar{F}': F' \in \mathcal{F}_2, F' \neq F\}$$

Then $\{U(F): F \in \mathcal{F}_2\}$ is a disjoint collection of open sets of M_2 . Since

$$M_2 - T = \cup \{U(F): F \in \mathcal{F}_2\}$$

and $(\bar{W} - W) \cap T = \phi$, $\bar{W} - W$ is the sum of the disjoint collection

$$\mathcal{H} = \{(\bar{W} - W) \cap U(F) = H(F): F \in \mathcal{F}_2\}$$

Since

$$H(F) = (\bar{W} - W) - \cup \{U(F'): F' \neq F, F' \in \mathcal{F}_2\}$$

$H(F)$ is closed and hence $\mathcal{H}$ is a disjoint collection of closed sets. Since

$$\text{mesh } \mathcal{H} \leq \text{mesh } \mathcal{F}_2 < 1$$

no $H(F)$ meets both $\{x: x_4 = 0\}$ and $\{x: x_4 = 1\}$. Now M is the sum of the disjoint closed sets $M_3 - W$ and $\bar{W} - W$ by (2) and (9). By (1) $M_3 - W$ is the sum of disjoint closed sets

$$(M_3 - W) \cap B_i, \quad i = 1, 2, \dots$$

By our construction no $(M_3 - W) \cap B_i$ meets both $\{x: x_4 = 0\}$ and $\{x: x_4 = 1\}$, since $Q \cap Q' \cap M = \phi$. Therefore, no component of M meets both $\{x: x_4 = 0\}$ and $\{x: x_4 = 1\}$, by Theorem 6-10.

Consider the closed set

$$X = \{x: x_4 = 0, 1\} \cup M$$

Let X_1 be the sum of $\{x: x_4 = 0\}$ and all components of M which meet $\{x: x_4 = 0\}$. Let X_2 be the sum of $\{x: x_4 = 1\}$ and all components of M which meet $\{x: x_4 = 1\}$. Then X_1 and X_2 are closed by an argument analogous to the proof of the closedness of P . With the aid of Corollary 6-8, we can find a closed set N of M_2 which separates $\{x: x_4 = 0\}$ and $\{x: x_4 = 1\}$ such that $N \cap M = \phi$. Thus, two pairs of opposite sides of M_2 are not defining, which implies in turn three pairs of opposite sides of M_1 are not defining, as can easily be seen. At last four pairs of opposite faces of I^4 are not defining, which is a contradiction (see Hurewicz and Wallman [1, VI]). Hence $2 < \mu \dim(S, \sigma)$. Since $\mu \dim(S, \sigma) \leq \dim S = 3$, $\mu \dim(S, \sigma) = 3$. $\square$

ASSERTION 6. $d_3(S, \sigma) = 3$.

PROOF. Since σ is totally bounded, $d_3(S, \sigma) = \mu \dim(S, \sigma) = 3$ by Theorem 29-11. $\square$

33-2. EXAMPLE (Nagami and Roberts [3]). A metric space (R, ρ) with $d_2(R, \rho) = 2$, $d_3(R, \rho) = \mu \dim(R, \rho) = 3$ and $\dim R = 4$. Take the space (Z_4, σ_3) constructed in the preceding section. Then

$$\dim Z_4 = 4$$

$$d_2(Z_4, \sigma_3) = d_3(Z_4, \sigma_3) = \mu \dim(Z_4, \sigma_3) = 2$$

R is the topologically disjoint sum of Z_4 and (S, σ) just constructed. The metric ρ on R is defined as follows

$$\rho \mid Z_4 = \sigma_3$$

$$\rho \mid S = \sigma$$

$$\rho(x, y) = \max\{\sigma_3(Z_4), \sigma(S)\} \quad (\leq 6) \quad \text{if } \{x, y\} \text{ is contained in neither } Z_4 \text{ nor } S$$

Then it is evident that $d_2(R, \rho) = 2$, $d_3(R, \rho) = \mu \dim(R, \rho) = 3$ and $\dim R = 4$.

Appendix

COHOMOLOGICAL DIMENSION THEORY†

The appendix is devoted to investigating cohomological dimension theory. We assume familiarity with the Čech cohomology theory and some homotopy theory. In particular, Chapters IX and X of Eilenberg and Steenrod [1] and Chapter III of Hu [1] form a good introduction to the appendix. Throughout the appendix we mean by a *space* a normal Hausdorff space with finite covering dimension and by a *mapping* a continuous transformation.

34. Preliminaries: Čech Cohomology Group

In this section we shall give a quick review, without proofs, of the Čech cohomology theory. For the proofs, we refer the reader to Chapters IX and X of the book by Eilenberg and Steenrod [1].

34-1. DEFINITION. Let X be a space and let $\mathcal{A}$ be the collection of all open coverings of X . If $\mathcal{V}$ is a refinement of $\mathcal{U}$ for $\mathcal{V} \in \mathcal{A}$ and $\mathcal{U} \in \mathcal{A}$, then we write $\mathcal{U} < \mathcal{V}$. If A is a closed set of X and $\mathcal{U} \in \mathcal{A}$, then we denote by $\mathcal{U} \mid A$ the open covering of A consisting of all intersections of A and members of $\mathcal{U}$. For a pair, a space X and its closed set A , (X, A) of spaces and $\mathcal{U} \in \mathcal{A}$, denote by $(M_{\mathcal{U}}, N_{\mathcal{U}})$ the pair of the nerves of $\mathcal{U}$ and $\mathcal{U} \mid A$ with the weak topology (see Section 1.1). If $\mathcal{U} < \mathcal{V}$, then a projection

† The appendix was written by Yukihiro Kodama, Tokyo University of Education.

$\pi_{\mathcal{U}}^{\mathcal{V}}: (M_{\mathcal{V}}, N_{\mathcal{V}}) \rightarrow (M_{\mathcal{U}}, N_{\mathcal{U}})$ is defined. Let G be an abelian group. A projection $\pi_{\mathcal{U}}^{\mathcal{V}}$ induces the homomorphism

$$\pi_{\mathcal{U}}^{\mathcal{V}*}: H^n(M_{\mathcal{U}}, N_{\mathcal{U}}; G) \rightarrow H^n(M_{\mathcal{V}}, N_{\mathcal{V}}; G)$$

where H^n is the usual n -dimensional cohomology group. The limit group of the directed system $\{H^n(M_{\mathcal{U}}, N_{\mathcal{U}}; G), \pi_{\mathcal{U}}^{\mathcal{V}*}; \mathcal{U} \in \mathcal{A} \text{ and } \mathcal{V} \in \mathcal{A}\}$ is called the n -dimensional unrestricted Čech cohomology group of (X, A) with coefficients in G and it is denoted by $H^n(X, A; G)$. If $A = \phi$ (the void set), then we simply denote this group by $H^n(X; G)$.

Let (X, A) and (Y, B) be pairs of spaces and let $f: (X, A) \rightarrow (Y, B)$ be a mapping. Then f induces the homomorphism $f^*: H^n(Y, B; G) \rightarrow H^n(X, A; G)$. If (X, A, B) is a triple of spaces (i.e. a space X , closed subsets A and B with $A \supset B$), then the coboundary homomorphism $\delta: H^{n-1}(A, B; G) \rightarrow H^n(X, A; G)$ is defined. It is known that the unrestricted Čech cohomology theory satisfies Eilenberg–Steenrod axioms. (See Eilenberg and Steenrod [1, Chapters I and IX].)

34-2. LEMMA. If $f: (X, A) \rightarrow (X, A)$ is the identity mapping, then $f^*: H^n(X, A; G) \rightarrow H^n(X, A; G)$ is the identity homomorphism. If $f: (X, A) \rightarrow (Y, B)$ and $g: (Y, B) \rightarrow (Z, C)$ are mappings, then $(gf)^* = f^*g^*: H^n(Z, C; G) \rightarrow H^n(X, A; G)$.

34-3. LEMMA. Let (X, A, B) be a triple of spaces. The sequence

$$\cdots \xrightarrow{i^*} H^{n-1}(A, B; G) \xrightarrow{\delta} H^n(X, A; G) \xrightarrow{i^*} H^n(X, B; G) \xrightarrow{j^*} H^n(A, B; G) \xrightarrow{\delta} \cdots$$

is exact, where i and j are the inclusions. Moreover, this exact sequence is natural with respect to a mapping $f: (X, A, B) \rightarrow (X', A', B')$.

34-4. LEMMA. If $f_i: (X, A) \rightarrow (Y, B)$, $i = 0, 1$, are mappings and $f_0 \sim f_1: (X, A) \rightarrow (Y, B)$, that is, there is a homotopy $F: (X \times I, A \times I) \rightarrow (Y, B)$ connecting f_0 and f_1 , then $f_0^* = f_1^*: H^n(Y, B; G) \rightarrow H^n(X, A; G)$, where I is the closed interval $[0, 1]$.

34-5. LEMMA. Let (X, A) be a pair of spaces and let U be an open set of X whose closure $\bar{U}$ is contained in the interior of A , then the inclusion mapping $i: (X - U, A - U) \subset (X, A)$ induces an isomorphism $i^*: H^n(X, A; G) \rightarrow H^n(X - U, A - U; G)$.

34-6. DEFINITION. Let $\mathcal{F}$ be the collection of all finite open coverings of a space X . For a pair (X, A) , if we use only members of $\mathcal{F}$ in the definition of the unrestricted group $H^n(X, A; G)$, then the *n-dimensional Čech group of (X, A) with coefficients in G based on $\mathcal{F}$* is defined. We denote this group by $H_{\mathcal{F}}^n(X, A; G)$.

34-7. LEMMA. The lemmas 34-2, 34-3, and 34-5 in which the groups H^* are replaced by the groups $H_{\mathcal{F}}^*$ are true.

This is evident. The groups $H_{\mathcal{F}}^n(X, A; G)$ and $H^n(X, A; G)$ are not generally isomorphic.† If X is compact, then these groups are isomorphic.

Let X be paracompact. Any open covering of X has a locally finite refinement consisting of cozero sets (see Definition 1-4). Thus, if (X, A) is a pair of paracompact spaces, then the group $H^n(X, A; G)$ is defined by using only locally finite open coverings consisting of cozero sets.

34-8. DEFINITION. An open covering of a space X is called a *locally finite cozero covering* (abbreviated: lfc covering) if it is locally finite and consists of cozero sets.

34-9. LEMMA. If (M, N) is a pair of simplicial polyhedra, then the usual cohomology group of (M, N) is isomorphic to the unrestricted Čech cohomology group of (M, N) .

PROOF. Denote by $\check{H}^n(M, N; G)$ and $H^n(M, N; G)$ the usual group and the Čech group, respectively. For any open covering $\mathcal{U}$ of M there is a subdivision M' of M such that the open covering of M consisting of all open stars (see Definition 1-2) refines $\mathcal{U}$. Let N' be the corresponding subdivision of N and let $\pi: (M', N') \rightarrow (M, N)$ be a projection. It is well known that $\pi^*: H^n(M, N; G) \rightarrow H^n(M', N'; G)$ is an isomorphism. This completes the proof. $\square$

† Dowker [5, Theorem 9.6] showed that: If X is a real line and Z is the additive group of integers, then $H_{\mathcal{F}}^1(X; Z)$ has an infinite number of elements. This shows that the theory $H_{\mathcal{F}}^*$ does not satisfy the homotopy axiom (see Lemma 34-4). For a pair (X, A) of spaces a natural homomorphism $\mu: H_{\mathcal{F}}^n(X, A; G) \rightarrow H^n(X, A; G)$ is defined. If (X, A) is a pair of paracompact spaces and G is finitely generated, then μ is onto. See Kodama [7].

We identify the usual cohomology group and the unrestricted Čech cohomology group for a pair of simplicial polyhedra with the weak topology.

34-10. LEMMA. Let (X, A) be a pair of paracompact spaces and let $\mathcal{U}$ be an lfc covering of X (see Definition 34-8). Let $\varphi: (X, A) \rightarrow (M_{\mathcal{U}}, N_{\mathcal{U}})$ be a canonical mapping (see Section 1-5), then $\varphi^* = p_{\mathcal{U}}: H^n(M_{\mathcal{U}}, N_{\mathcal{U}}; G) \rightarrow H^n(X, A; G)$, where $p_{\mathcal{U}}$ is the projection.

The simple proof is omitted.

34-11. LEMMA. Let X be paracompact and let A and B be closed sets of X whose union is X , then the sequence

$$\begin{aligned} \cdots \rightarrow H^{n-1}(A \cap B; G) \rightarrow H^n(X; G) \rightarrow H^n(A; G) \\ \oplus H^n(B; G) \rightarrow H^n(A \cup B; G) \rightarrow \cdots \end{aligned}$$

is exact.

The proof is found in Eilenberg and Steenrod [1, Chapter I]. This exact sequence is called the *Mayer-Vietoris sequence*.

Let $\{(X_{\alpha}, A_{\alpha}), f_{\alpha}^{\beta}\}$ be an inverse limiting system of pairs of compact spaces (see Section 6-1), $(X, A) = (\text{inv lim } X_{\alpha}, \text{inv lim } A_{\alpha})$ and f_{α} the projection of (X, A) into (X_{α}, A_{α}) . Since $f_{\alpha}^{\beta} f_{\beta} = f_{\alpha}$ for $\alpha < \beta$, a unique homomorphism

$$\pi: \varinjlim \{H^n(X_{\alpha}, A_{\alpha}; G), f_{\alpha}^{\beta*}\} \rightarrow H^n(X, A; G)$$

is induced by f_{α}^* . The proof of the following is found in Eilenberg and Steenrod [1, Chapter X].

34-12. THEOREM. The homomorphism π is an isomorphism.

35. Cohomological Dimension

35-1. DEFINITION. The *large cohomological dimension* $D(X; G)$ of a space X with respect to an abelian group G is defined to be the largest integer n such that $H^n(X, A; G) \neq 0$ for some closed set A of X . The *small cohomological dimension* $d(X; G)$ is defined to be the smallest integer n such that, whenever $m \geq n$ and A is closed in X , the homomorphism $i^*: H^m(X; G) \rightarrow H^m(A; G)$ induced by the inclusion $i: A \subset X$ is onto. If we use the groups $H_{\mathcal{F}}^*$ in place of H^* in the definitions of

$D(X: G)$ and $d(X: G)$, then the *cohomological dimensions* $D_{\mathcal{F}}(X: G)$ and $d_{\mathcal{F}}(X: G)$ based on $\mathcal{F}$ (where $\mathcal{F}$ is the collection of all finite open coverings) are defined.

35-2. LEMMA. Let X be a space and A a closed set, then the following relations hold:

$$d(X: G) \leq D(X: G), \quad D(A: G) \leq D(X: G), \quad d(A: G) \leq d(X: G)$$

$$d_{\mathcal{F}}(X: G) \leq D_{\mathcal{F}}(X: G), \quad D_{\mathcal{F}}(A: G) \leq D_{\mathcal{F}}(X: G), \quad d_{\mathcal{F}}(A: G) \leq d_{\mathcal{F}}(X: G)$$

PROOF. Let B be a closed set of A . Consider the following exact sequence (see Lemma 34-3):

$$\cdots \rightarrow H^{m-1}(A, B: G) \xrightarrow{\delta} H^m(X, A: G) \xrightarrow{i^*} H^m(X, B: G) \xrightarrow{j^*} H^m(A, B: G) \xrightarrow{\delta} \cdots \quad (1)$$

If $m > D(X: G)$, then $H^m(X, A: G) = H^m(X, B: G) = 0$ and hence $H^m(A, B: G) = 0$. This shows $D(A: G) \leq D(X: G)$. If we put $B = \phi$ in sequence (1), then $j^*: H^{m-1}(X: G) \rightarrow H^{m-1}(A: G)$ is onto. Thus, $d(X: G) \leq D(X: G)$. Let $j_1: B \subset A$ and $j_2: B \subset X$. Since $j_1 j_2 = j_2$, $j_1^* j_2^* = j_2^*$: $H^m(X: G) \rightarrow H^m(B: G)$. Thus, if $m \leq d(X: G)$, then $j_1^*: H^m(A: G) \rightarrow H^m(B: G)$ is onto, because j_2^* is onto. This shows $d(A: G) \leq d(X: G)$. The other relations are similarly proved by using the exact sequence for the groups $H_{\mathcal{F}}^*$. (See Lemma 34-8.) $\square$

The following lemma is evident.

35-3. LEMMA. The following conditions are equivalent: (i) $\dim X = 0$, (ii) $D_{\mathcal{F}}(X: G) = 0$, (iii) $d_{\mathcal{F}}(X: G) = 0$. If X is paracompact, then (i), (ii), (iii) and the following are equivalent: (iv) $D(X: G) = 0$, (v) $d(X: G) = 0$.

Denote by βX the Stone-Čech compactification of a space X . Set $\eta U = \beta X - \overline{X - U}$ for an open set U of X and $\eta A = \bar{A}$ for a closed set A of X , where the bar indicates the closure operation in βX . If $U_i, i = 1, 2, \dots, k$, are open and A is closed in X , then $U_1 \cap \cdots \cap U_k \cap A \neq \phi$ if and only if $\eta U_1 \cap \cdots \cap \eta U_k \cap \eta A \neq \phi$. For a finite open covering $\mathcal{U}$ of X put $\eta \mathcal{U} = \{\eta U: U \in \mathcal{U}\}$. If $(M_{\mathcal{U}}, N_{\mathcal{U}})$ and $(M_{\eta \mathcal{U}}, N_{\eta \mathcal{U}})$ are the pair of nerves of $\mathcal{U}$ and $\eta \mathcal{U}$ for (X, A) and $(\beta X, \eta A)$, then $(M_{\mathcal{U}}, N_{\mathcal{U}})$ and $(M_{\eta \mathcal{U}}, N_{\eta \mathcal{U}})$ are naturally isomorphic. Hence we have the following lemma.

35-4. LEMMA (Morita [11]). $D_{\mathcal{F}}(X: G) = D(\beta X: G)$ and $d_{\mathcal{F}}(X: G) = d(\beta X: G)$.

36. A Characteristic Property of $d(X: G)$

36-1. DEFINITION. For an abelian group G and a nonnegative integer n , an *Eilenberg–MacLane space* is a topological space X such that $\pi_i(X) = 0$ for $i \neq n$ and $\pi_n(X) = G$, where π_* is the homotopy group. Such a space is denoted by $K(G, n)$.

36-2. LEMMA (Whitehead [2]). For any abelian group G and any non-negative integer n there is an Eilenberg–MacLane space $K(G, n)$ which is a simplicial polytope with metric topology (see Section 1-2).

PROOF. For $n = 0$, $K(G, 0)$ is a set G with discrete topology. Let $n \geq 1$. Let H be any subset of elements of G which generate G . Denote by M_n the one point union of n -spheres S_h^n , $h \in H$, that is, $S_h^n \cap S_{h'}^n = p_0$ (where p_0 is the one point) for $h \in H$, $h' \in H$ and $h \neq h'$. Then M_n is a simplicial polytope. Give the weak topology to M_n . Let $\alpha_h \in \pi_n(M_n)$ be the element which is represented by an homeomorphism $: S^n \rightarrow S_h^n$, $h \in H$. Define a homomorphism $\xi: \pi_n(M_n) \rightarrow G$ by $\xi(\alpha_h) = h$ for $h \in H$. Since $\pi_n(M_n)$ is the direct sum or the free product of copies of $\mathbb{Z}$, ξ is well defined. Since H generates G , ξ is onto. Let A be a subset of elements of $\xi^{-1}(0)$ ($\xi^{-1}(1)$ for $n = 1$) which generate $\xi^{-1}(0)$ ($\xi^{-1}(1)$). For each a of A , take an $(n+1)$ -simplex E_a and let S_a be the boundary of E_a . By subdividing E_a we can find a simplicial mapping $f_a: S_a \rightarrow M_n$ which represents the element a . Attach E_a to M_n under the mapping f_a for each $a \in A$, thus forming a CW-complex $K_n = M_n \cup (\cup_{a \in A} E_a)$. There is a simplicial subdivision of K_n under which M_n is a subcomplex. Thus we can consider K_n as a simplicial polytope with the weak topology. Obviously, $\pi_n(K_n) = G$. Next, let B be a set of elements of $\pi_{n+1}(K_n)$ which generate $\pi_{n+1}(K_n)$. Take an $(n+2)$ -simplex E_b for $b \in B$ and let f_b be a mapping of the boundary S_b of E_b into K_n which represents the element b . By subdividing E_b we may assume that f_b is simplicial. We attach E_b to K_n under the mapping f_b for each b . The CW-complex $K_{n+1} = K_n \cup (\cup_{b \in B} E_b)$ has a simplicial subdivision under which K_n is a subcomplex. It is evident that $\pi_i(K_{n+1}) = 0$, $0 \leq i \leq n+1$ and $i \neq n$, and $\pi_n(K_{n+1}) = G$. By repeating this process we obtain a simplicial polytope K_r with the weak topology for each r , $n < r$, such that K_{r-1} is a subcomplex of K_r and $\pi_j(K_r) = 0$, $0 \leq j \leq r$ and $j \neq n$, and $\pi_n(K_r) = G$. Let $K' = \cup_{i=0}^{\infty} K_i$. Then K' is a simplicial polytope with weak topology and $\pi_j(K') = 0$, $j \neq n$, and $\pi_n(K') = G$. Let $K(G, n)$ be a simplicial polytope K' with metric topology. It is well known that K' and $K(G, n)$ have the same homotopy type. This completes the proof. $\square$

36-3. COROLLARY. Let $n > 0$. If G is finitely generated, then, for each $q > n$, there is a finite simplicial polytope K such that $\pi_j(K) = 0$, $0 \leq j \leq q$ and $j \neq n$, and $\pi_n(K) = G$.

PROOF. It is well known that if N is the product of spaces X and Y , then $\pi_i(N) \approx \pi_i(X) \oplus \pi_i(Y)$. Thus we may assume that $G = Z$ or $G = Z_m$, where Z is the additive group of integers, $Z = Z/mZ$ and $m > 1$. We shall follow the proof of Lemma 36-2. In this case the polytope M_n consists of one n -sphere S^n . If $G = Z$, then $K_n = S^n$. If $G = Z_m$, then K_n is obtained by attaching an $(n+1)$ -cell E to S^n under a mapping of the boundary of E into S^n with degree m . In each case $\pi_{n+1}(K_n)$ is finitely generated. Thus K_{n+1} is a finite polytope. Suppose that a finite polytope K_r which satisfies the conclusion of the corollary for $q = r$ is constructed. If $n > 1$, then $\pi_{r+1}(K_r)$ is finitely generated since K_r is finite and simply connected. If $n = 1$ and $G = Z_m$, then the universal covering complex $\tilde{K}$ of K_r is finite and hence $\pi_{r+1}(\tilde{K}) = \pi_{r+1}(K_r)$ is finitely generated. (Note that if $n = 1$ and $G = Z$ then a 1-sphere is taken as K_r for each r .) $\square$

36-4. TERMINOLOGY. Throughout this appendix we denote by $K(G, n)$ a fixed Eilenberg-MacLane simplicial polytope with metric topology. For each $q > n$, we denote by $K(G, n; q)$ the q -section of $K(G, n)$. Note that $\pi_i(K(G, n; q)) = 0$, $0 \leq i < q$ and $i \neq n$, and $\pi_n(K(G, n; q)) = G$.

36-5. THEOREM (Kodama [6]). Let X be a paracompact space with covering dimension $< q$. Then $d(X; G) \leq n$ if and only if for each $m \geq n$ and each closed set A of X every mapping of A into $K(G, m; q)$ is extendable over X .

The proof depends on a sequence of lemmas.

Throughout the remainder of this section we denote by q and m fixed integers such that $q > m$, by K , $K(G, m; q)$ and by k_0 a fixed vertex of K .

36-6. LEMMA. A finite dimensional simplicial polytope with metric topology is an ANR (paracompact). (See Section 4-3 for the definition.)

The proof is found in Hanner [1].

Let M be a $(q-1)$ -dimensional simplicial polytope with the weak topology and let f be a mapping of M into K . Denote by M^i the i -section of M for each integer i . Since $\pi_i(K) = 0$ for $i < m$, the restriction $f|_{M^{m-1}}$ is homotopic to a constant mapping. Since K is an ANR (paracompact) by Lemma 36-6, the homotopy extension theorem shows that there is a mapping $f' : M \rightarrow K$ such that $f'(M^{m-1}) = k_0$ and $f \sim f'$. For each oriented m -simplex σ of M denote by $z(\sigma)$ the element of $\pi_m(K) = G$ which is represented by the mapping $f'|_{\sigma}$. Define a cochain $z(f')$ of M by $z(f')(\sigma) = z(\sigma)$ for each oriented m -simplex σ of M .

36-7. LEMMA. $z(f')$ is an m -cocycle.

PROOF. Let τ be an oriented $(m+1)$ -simplex of M and $\partial\tau$ the boundary of τ . For notational convenience we use $\partial\tau$ for both an oriented m -sphere and an m -chain. By definition $\delta(z(f'))(\tau) = z(f')(\partial\tau)$, where δ is the coboundary operator. Since $f'|_{\partial\tau}$ is extendable over τ , $z(f')(\partial\tau) = 0$ by the homotopy addition theorem. $\square$

36-8. LEMMA. If $f' \sim f'' : M \rightarrow K$ and $f'(M^{m-1}) = f''(M^{m-1}) = k_0$, then $z(f')$ and $z(f'')$ are cohomologous.

PROOF. There is a homotopy $F : M \times I \rightarrow K$ connecting f' and f'' such that $F(M^{m-2} \times I) = k_0$, because $\pi_i(K) = 0$, $i < m$. For each oriented $(m-1)$ -simplex σ of M let $c(\sigma)$ be a unique element of G determined by the mapping $F|_{\sigma \times I}$.[†] Let c be the $(m-1)$ -cochain of M whose value on σ is $c(\sigma)$. A straight forward computation shows that $z(f'') - z(f') = \delta c$. Hence, $z(f')$ and $z(f'')$ are cohomologous. $\square$

By the preceding lemma a mapping $f : M \rightarrow K$ determines a unique element $\xi(f)$ of $H^m(M; G)$ represented by a cocycle $z(f')$.

36-9. LEMMA. Let f and g be mappings of M into K , then $f \sim g$ if and only if $\xi(f) = \xi(g)$.

PROOF. By Lemma 36-8 we may assume that $f(M^{m-1}) = g(M^{m-1}) = k_0$. If $f \sim g$, then there is a homotopy $F : M \times I \rightarrow K$ connecting f and g such that $F(M^{m-2} \times I) = k_0$. The proof of 36-8 shows that $\xi(f) = \xi(g)$.

[†] Here I and $\sigma \times I$ are considered as cells with orientations defined by $\partial I = \{1\} - \{0\}$ and $\partial(\sigma \times I) = \partial\sigma \times I + (-1)^{m-1}\sigma \times (\partial I)$.

To prove the converse let $\xi(f) = \xi(g)$. Then there is an $(m-1)$ -cochain c such that $z(g) - z(f) = \delta c$. Define a mapping

$$F: (M \times \{0\}) \cup (M \times \{1\}) \cup (M^{m-2} \times I) \rightarrow K.$$

by $F|_{M \times \{0\}} = f$, $F|_{M \times \{1\}} = g$ and $F(M^{m-2} \times I) = k_0$. Take an oriented $(m-1)$ -simplex σ of M . Let h_σ be a mapping of $\sigma \times I$, again an oriented m -cell, into K representing the element $c(\sigma)$ of $\pi_m(K)$. Define

$$F': (M \times \{0\}) \cup (M \times \{1\}) \cup (M^{m-1} \times I) \rightarrow K$$

by $F' = F$ on $(M \times \{0\}) \cup (M \times \{1\})$ and $F'|_{\sigma \times I} = h_\sigma$ for each $(m-1)$ -simplex σ of M . For each oriented m -simplex τ of M the element of $\pi_m(K)$ represented by $F'|_{\partial(\tau \times I)}$ is zero, because the element is equal to $(z(g) - z(f) - \delta c)(\tau)$. Thus, F' is extendable over $(M \times \{0\}) \cup (M \times \{1\}) \cup (M^m \times I)$. Since $\pi_i(K) = 0$ for each i , $m < i \leq \dim M$, F' is extendable over $M \times I$. Thus, $f \sim g$. $\square$

36-10. LEMMA. If $e \in H^m(M; G)$, then there is a mapping $f: M \rightarrow K$ such that $\xi(f) = e$.

PROOF. Let z be an m -cocycle of M which represents the element e . Then there is a mapping $f: M \rightarrow K$ such that $f(M^{m-1}) = k_0$ and for each oriented m -simplex σ of M $f|_\sigma$ represents the element $z(\sigma)$ of $\pi_m(K)$. It is evident that f is extendable over M . Denote its extension by f again. Then, $\xi(f) = e$. $\square$

36-11. LEMMA. Let L be a $(q-1)$ -dimensional simplicial polytope with weak topology and let $g: L \rightarrow M$ be a mapping. For any mapping $f: M \rightarrow K$ $\xi(fg) = g^*\xi(f)$, where g^* is the homomorphism: $H^m(M; G) \rightarrow H^m(L; G)$ induced by g .

PROOF. We may assume that g is simplicial and $f(M^{m-1}) = k_0$ by Lemma 36-8 and Lemma 36-9. Hence, $fg(L^{m-1}) = k_0$. The lemma is a consequence of the definitions of $\xi(fg)$, $\xi(f)$ and the induced homomorphism g^* . $\square$

36-12. LEMMA. Let N be a subcomplex of M and let f be a mapping of N into K . Then f is extendable over M if and only if the element $\xi(f)$ belongs to the subgroup $i^*H^m(M; G)$ of $H^m(N; G)$, where i is the inclusion mapping: $N \subset M$.

PROOF. Suppose that f has an extension $g: M \rightarrow K$. We may assume that $g(M^{m-1}) = k_0$. In view of the construction of cocycles $z(g)$ and $z(f)$, $z(g)(\sigma) = z(f)(\sigma)$ for any oriented m -simplex σ of N . Thus we have $i^*\xi(g) = \xi(f)$. To prove the converse, suppose that $\xi(f) = i^*e$ for some element e of $H^m(M: G)$, there is an m -cocycle z of M which represents the element e such that $z(\sigma) = z(f)(\sigma)$ for each oriented m -simplex σ of N . (Note that we may assume $f(N^{m-1}) = k_0$ by the homotopy extension theorem.) Define a mapping $g: M^{m-1} \cup N \rightarrow K$ by $g(M^{m-1}) = k_0$ and $g|N = f$. The mapping g has an extension over M^m whose restriction to each oriented m -simplex σ of $M - N$ represents the element $z(\sigma)$. Denote this extension by g again. It is evident that g is extendable over M . $\square$

36-13. PROOF OF THEOREM 35-6. To prove sufficiency: Let A be a closed set of X and let $m \geq n$. We have to prove that $i^*: H^m(X: G) \rightarrow H^m(A: G)$ is onto, where i is the inclusion: $A \subset X$. Take an element e of $H^m(A: G)$. Let $\mathcal{V}$ be an lfc covering (see Definition 34-8) of A with order $\leq q$ such that there is an element $e_{\mathcal{V}}$ of $H^m(N_{\mathcal{V}}: G)$ whose image under the projection: $H^m(N_{\mathcal{V}}: G) \rightarrow H^m(A: G)$ is the element e , where $N_{\mathcal{V}}$ is the nerve of $\mathcal{V}$. In view of Lemma 36-10 there is a mapping $f_{\mathcal{V}}: N_{\mathcal{V}} \rightarrow K$ such that $\xi(f_{\mathcal{V}}) = e_{\mathcal{V}}$. We may assume that $f_{\mathcal{V}}$ is simplicial. Let $\varphi_{\mathcal{V}}$ be a canonical mapping of A into $N_{\mathcal{V}}$ (see Section 1-5). By the hypothesis the mapping $f_{\mathcal{V}}\varphi_{\mathcal{V}}: A \rightarrow K$ has an extension $g: X \rightarrow K$. Since X is paracompact, there is an lfc covering $\mathcal{U}$ with order $\leq q$ satisfying the following conditions: (1) For each member U of $\mathcal{U}$ there is a vertex u of K such that $U \subset g^{-1}(\text{St}(u))$, where $\text{St}(u)$ is the open star of u (see Definition 1-2), and (2) $\mathcal{U}|A$ is a refinement of $\mathcal{V}$. Let $\varphi_{\mathcal{U}}$ be a canonical mapping of (X, A) into $(M_{\mathcal{U}}, N_{\mathcal{U}})$, where $(M_{\mathcal{U}}, N_{\mathcal{U}})$ is the pair of the nerves of $\mathcal{U}$ and $\mathcal{U}|A$. Denote by $\pi_{\mathcal{V}}^{\mathcal{U}}$ a projection: $N_{\mathcal{U}} \rightarrow N_{\mathcal{V}}$. Note that for each point x of A there is a simplex of $N_{\mathcal{V}}$ in which both the points $\varphi_{\mathcal{V}}(x)$ and $\pi_{\mathcal{V}}^{\mathcal{U}}\varphi_{\mathcal{U}}(x)$ are contained. Thus $\pi_{\mathcal{V}}^{\mathcal{U}}\varphi_{\mathcal{U}}|A \sim \varphi_{\mathcal{V}}$ and hence $f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}\varphi_{\mathcal{U}}|A \sim f_{\mathcal{V}}\varphi_{\mathcal{V}}$. Next, for each vertex α of $M_{\mathcal{U}}$, choose a vertex u_{α} of K such that, if U_{α} is the member of $\mathcal{U}$ corresponding to the vertex α , then $U_{\alpha} \subset g^{-1}(\text{St}(u_{\alpha}))$. This choice is possible by condition (1). Define a mapping $g_{\mathcal{U}}: M_{\mathcal{U}}^0 \rightarrow K$ by $g_{\mathcal{U}}(\alpha) = u_{\alpha}$ for each vertex α of $M_{\mathcal{U}}$. If vertexes $\alpha_0, \dots, \alpha_k$ span a simplex of $M_{\mathcal{U}}$, then $\cap_{i=0}^k U_{\alpha_i} \neq \emptyset$ and hence the vertices u_{α_i} , $i = 0, \dots, k$, span a simplex of K . Therefore $g_{\mathcal{U}}$ is extendable over $M_{\mathcal{U}}$ simplicially. Denote its extension by $g_{\mathcal{U}}$ again. In view of the construction of $g_{\mathcal{U}}$, for each vertex α of $N_{\mathcal{U}}$ there is a simplex of K in which both the points $g_{\mathcal{U}}(\alpha)$ and $f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}(\alpha)$ are contained. Hence $g_{\mathcal{U}}|N_{\mathcal{U}} \sim f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}$. Thus we have $\pi_{\mathcal{V}}^{\mathcal{U}}e_{\mathcal{V}} = \pi_{\mathcal{V}}^{\mathcal{U}}*\xi(f_{\mathcal{V}})$

$= \xi(f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}) = \xi(g_{\mathcal{U}} | N_{\mathcal{U}})$ by Lemma 36-11. Also we have $i_{\mathcal{U}}^*\xi(g_{\mathcal{U}}) = \xi(g_{\mathcal{U}} | N_{\mathcal{U}})$, where $i_{\mathcal{U}}$ is the inclusion: $N_{\mathcal{U}} \subset M_{\mathcal{U}}$. Therefore the element of $H^m(X; G)$ represented by the element $\xi(g_{\mathcal{U}})$ of $H^m(M_{\mathcal{U}}; G)$ is mapped to the element e of $H^m(A; G)$ by i^* . Thus i^* is onto.

To prove necessity: Let f be a mapping of a closed set A of X into K . By an argument similar to that in the proof of sufficiency we can find an lfc covering $\mathcal{V}$ of A subject to the following: there is a simplicial mapping $f_{\mathcal{V}}: N_{\mathcal{V}} \rightarrow K$ such that $f_{\mathcal{V}}\phi_{\mathcal{V}} \sim f$, where $N_{\mathcal{V}}$ is the nerve of $\mathcal{V}$ and $\phi_{\mathcal{V}}$ is a canonical mapping: $A \rightarrow N_{\mathcal{V}}$. Since $i^*: H^m(X; G) \rightarrow H^m(A; G)$ is onto, we can take the following lfc covering $\mathcal{U}$ of X ; order of $\mathcal{U}$ less than or equal to q , $\mathcal{U} | A$ refines $\mathcal{V}$ and $\pi_{\mathcal{V}}^{\mathcal{U}}*\xi(f_{\mathcal{V}}) = \xi(f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}) \in i_{\mathcal{U}}^*H^m(M_{\mathcal{U}}; G)$, where $(M_{\mathcal{U}}, N_{\mathcal{U}})$ is the pair of nerves of $\mathcal{U}$ for (X, A) , $i_{\mathcal{U}}: N_{\mathcal{U}} \subset M_{\mathcal{U}}$ and $\pi_{\mathcal{V}}^{\mathcal{U}}$ is a projection: $N_{\mathcal{U}} \rightarrow N_{\mathcal{V}}$. By Lemma 36-12 there is a mapping $g_{\mathcal{U}}: M_{\mathcal{U}} \rightarrow K$ such that $i_{\mathcal{U}}^*\xi(g_{\mathcal{U}}) = \xi(f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}})$. Let $\phi_{\mathcal{U}}: (X, A) \rightarrow (M_{\mathcal{U}}, N_{\mathcal{U}})$ be a canonical mapping. Since $g_{\mathcal{U}} \sim f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}: N_{\mathcal{U}} \rightarrow K$ and $\pi_{\mathcal{V}}^{\mathcal{U}}\phi_{\mathcal{U}} | A \sim \phi_{\mathcal{V}}: A \rightarrow N_{\mathcal{V}}$, we have $g_{\mathcal{U}}\phi_{\mathcal{U}} | A \sim f_{\mathcal{V}}\pi_{\mathcal{V}}^{\mathcal{U}}\phi_{\mathcal{U}} | A \sim f_{\mathcal{V}}\phi_{\mathcal{V}} \sim f$. Since $g_{\mathcal{U}}\phi_{\mathcal{U}} | A$ has an extension $g_{\mathcal{U}}\phi_{\mathcal{U}}$ over X , f is extendable over X . $\square$

36-14. THEOREM. Let X be a paracompact space. If G is a finitely generated abelian group, then $d(X; G) = d_{\mathcal{F}}(X; G)$.

PROOF. By Lemma 35-3 we may assume that $\dim X > 0$. We denote by K_m a finite simplicial polytope such that $\pi_j(K_m) = 0$, $0 \leq j \leq q$, $j \neq m$, and $\pi_m(K_m) = G$, where $\dim X \leq q$ and $m < q$. Such a polytope exists by Corollary 36-3. An argument similar to that in the proof of Theorem 36-5 shows that the following are equivalent: (i) $d(X; G) \leq n$, (ii) $d_{\mathcal{F}}(X; G) \leq n$, and (iii) for each $m \geq n$ and each closed set A of X every mapping of A into K_m is extendable over X . Lemma 36-12 in which the complex K is replaced by the complex K_m is true and hence (i) and (iii) are equivalent. Let f be a mapping of a closed set A into K_m . Since K_m is a finite polytope, there is a finite open covering $\mathcal{V}$ of A and a mapping $f_{\mathcal{V}}$ from the nerve $N_{\mathcal{V}}$ of $\mathcal{V}$ into K_m such that $f_{\mathcal{V}}\phi_{\mathcal{V}} \sim f$. Thus the proof of Theorem 36-5 can be applied with a trivial change to prove the equivalence of (ii) and (iii). $\square$

36-15. COROLLARY. If X is paracompact, then $\dim X = d(X; Z) = d_{\mathcal{F}}(X; Z)$, where Z is the additive group of integers.

PROOF. It is evident that $\dim X \geq d(X; Z)$. Suppose that $q = \dim X > d(X; Z)$. Since $q - 1 \geq d(X; Z)$, we can take a $(q - 1)$ -sphere S

as the finite polytope K_{q-1} in the proof of Theorem 36-14. Since $\dim X = q$, there is a closed set A and a mapping $f: A \rightarrow S$ such that f is not extendable over X . This contradiction shows that $\dim X = d(X: Z)$.

36-16. PROBLEM. Is it true that $d(X: G) = d_{\mathcal{F}}(X: G)$ for any normal or paracompact space X and any abelian group G ?

36-17. REMARK. In a way similar to the cohomological case, homological dimension is defined. For a compact space there is not a large difference between the homological and the cohomological dimension. (See Kodama [3].) For instance, denote by $d_*(X: G)$ the least integer n such that for each $m \geq n$, and each closed set A of X , the homomorphism: $H_m(A: G) \rightarrow H_m(X: G)$ induced by the inclusion: $A \subset X$ is a monomorphism, where H_m means the unrestricted Čech homology group. If X is compact, then $\dim X = d_*(X: R_1)$ is true, where R_1 is the additive group of rationals modulo one. This is guaranteed by the following theorem.

HOPF'S EXTENSION THEOREM (HOMOLOGY). Let X be a compact space of covering dimension $\leq n+1$, A a closed set of X , and f a mapping of A into an n -sphere S^n , then f is extendable over X if and only if Kernel $\{i_*: H_n(A: R_1) \rightarrow H_n(X: R_1)\}$ is mapped to zero in $H_n(S^n: R_1)$ under f_* , where $i: A \subset X$.

The proof is found in Hurewicz and Wallman [1]. Unfortunately this theorem fails in case X is noncompact. (See Dowker [5] or Kodama [5].) On the other hand, Hopf's theorem in terms of cohomology is true for noncompact spaces as follows:

HOPF'S EXTENSION THEOREM (COHOMOLOGY). Let X be a paracompact space of covering dimension $\leq n+1$, A a closed set of X , and f a mapping of A into S^n . Then f is extendable over X if and only if $f^*H^n(S^n: Z) \subset i^*H^n(X: G)$.

This theorem is proved in Dowker [5] and it is also evident by the proof of Corollary 36-15.

36-18. REMARK. If X is paracompact and perfectly normal, then the dimensional restriction in Theorem 36-5 is removed, because a simplicial polytope of any dimension is an ANR (paracompact and perfectly normal).

Finally, Theorem 36-14 is generalized as follows: Let $\mathcal{L}$ be the collection of all locally finite coverings of a space X and denote by $H_{\mathcal{L}}^*$ the Čech cohomology group based on $\mathcal{L}$. Then the cohomological dimension $d_{\mathcal{L}}(X; G)$ based on $\mathcal{L}$ is defined by using the groups $H_{\mathcal{L}}^*$. Then, if G is finitely generated and X is a (normal) space, then $d_{\mathcal{L}}(X; G) = d_{\mathcal{F}}(X; G)$ (see Kodama [7]). Also we have $\dim X = d_{\mathcal{L}}(X; Z) = d_{\mathcal{F}}(X; Z)$.

37. $D(X; G)$ and $d(X; G)$

37-1. DEFINITION. Let (A_i, B_i) , $i = 1, 2$, be pairs of closed sets of a space X such that $A_2 \subset A_1$ and $B_2 \subset B_1$, and let $i: (A_2, B_2) \subset (A_1, B_1)$. If an element e_1 of $H^n(A_1, B_1; G)$ is mapped into an element e_2 of $H^n(A_2, B_2; G)$ under i^* , then e_2 is called the *restriction* of e_1 to (A_2, B_2) and we write $e_1 \mid (A_2, B_2) = e_2$.

37-2. REDUCTION LEMMA. Let X be paracompact, let A be closed in X and let e be an element of $H^n(X; G)$. If $e \mid A = 0$, then there is a closed neighborhood B of A such that $e \mid B = 0$.

PROOF. Let $\mathcal{U}$ be an lfc covering of X such that there is an element $e_{\mathcal{U}}$ of $H^n(M_{\mathcal{U}}; G)$ which represents the element e and the restriction of $e_{\mathcal{U}}$ to $N_{\mathcal{U}}$ is zero, where $M_{\mathcal{U}}$ and $N_{\mathcal{U}}$ are the nerves of $\mathcal{U}$ and $\mathcal{U} \mid A$. Let $\bar{M}_{\mathcal{U}}$ be the second barycentric subdivision of $M_{\mathcal{U}}$ and let $\bar{L}_{\mathcal{U}}$ be the subcomplex of $\bar{M}_{\mathcal{U}}$ consisting of all simplexes of $\bar{M}_{\mathcal{U}}$ which intersect $N_{\mathcal{U}}$. Since $N_{\mathcal{U}}$ is a deformation retract of $\bar{L}_{\mathcal{U}}$, the homomorphism $i^*: H^n(\bar{L}_{\mathcal{U}}; G) \rightarrow H^n(N_{\mathcal{U}}; G)$ induced by the inclusion $i_{\mathcal{U}}: N_{\mathcal{U}} \subset \bar{L}_{\mathcal{U}}$ is an isomorphism. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 A & \xrightarrow{i} & \varphi_{\mathcal{U}}^{-1} \bar{L}_{\mathcal{U}} & \xrightarrow{j} & X \\
 \downarrow \varphi_{\mathcal{U}}' & & \downarrow \varphi_{\mathcal{U}}'' & & \downarrow \varphi_{\mathcal{U}} \\
 N_{\mathcal{U}} & \xrightarrow{i_{\mathcal{U}}} & \bar{L}_{\mathcal{U}} & \xrightarrow{j_{\mathcal{U}}} & M_{\mathcal{U}}
 \end{array}$$

where i, j and $j_{\mathcal{U}}$ are the inclusions and $\varphi_{\mathcal{U}}'$ and $\varphi_{\mathcal{U}}''$ are the restrictions of a canonical mapping $\varphi_{\mathcal{U}}$ of X into $M_{\mathcal{U}}$. Since $\varphi_{\mathcal{U}}^* e_{\mathcal{U}} = e$ by Lemma 34-11, we have

$$j^* e = j^* \varphi_{\mathcal{U}}^* e_{\mathcal{U}} = \varphi_{\mathcal{U}}''^* j_{\mathcal{U}}^* e_{\mathcal{U}} = \varphi_{\mathcal{U}}''^* i_{\mathcal{U}}^{*-1} (j_{\mathcal{U}} i_{\mathcal{U}})^* e_{\mathcal{U}} = 0$$

Thus the closed neighborhood $\varphi_{\mathcal{U}}^{-1} \bar{L}_{\mathcal{U}}$ of A satisfies the conclusion of the lemma. $\square$

37-3. EXTENSION LEMMA. Let A be a closed set of a paracompact space X and let $e \in H^n(A:G)$. There is a closed neighborhood B and an element e' of $H^n(B:G)$ such that $e' \mid A = e$.

The proof is given by a similar argument to that in the preceding lemma.

37-4. COROLLARY. Let A be closed in a paracompact space X and let B be a closed neighborhood of A . Let e be an element of $H^n(X, B:G)$ whose restriction to (X, A) is zero, then there is a closed neighborhood C of A contained in B such that $e \mid (X, C) = 0$.

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccc} \cdots \rightarrow H^{n-1}(X:G) \rightarrow H^{n-1}(B:G) & \xrightarrow{\delta'} & H^n(X, B:G) & \xrightarrow{j^*} & H^n(X:G) \rightarrow \cdots \\ & \parallel & \downarrow & & \downarrow & \parallel \\ \cdots \rightarrow H^{n-1}(X:G) \rightarrow H^{n-1}(A:G) & \xrightarrow{\delta} & H^n(X, A:G) \rightarrow H^n(X:G) \rightarrow \cdots \end{array}$$

Each line is exact because they are the cohomology sequences of (X, B) and (X, A) respectively (see Lemma 34-4). Since $e \mid (X, A) = 0$, $j^*e = 0$ and hence there is an element e_1 of $H^{n-1}(B:G)$ such that $\delta'e_1 = e$. Set $e_2 = e_1 \mid A$. Since $\delta e_2 = \delta'e_1 \mid (X, A) = e \mid (X, A) = 0$, there is an element e_3 of $H^{n-1}(X:G)$ such that $e_3 \mid A = e_2$. Set $e_4 = e_1 - (e_3 \mid B)$. Then $e_4 \mid A = e_1 \mid A - e_3 \mid A = e_2 - e_2 = 0$. By Lemma 37-2 there is a closed neighborhood C of A contained in B such that $e_4 \mid C = 0$. The set C is the required one, because, if $\delta'': H^{n-1}(C:G) \rightarrow H^n(X, C:G)$ is the co-boundary homomorphism, then $e \mid (X, C) = \delta'e_1 \mid (X, C) = \delta'e_4 \mid (X, C) = \delta''(e_4 \mid C) = 0$. $\square$

The following theorem is a useful generalization of Lemma 34-6.

37-5. THEOREM (Wallace [1]). Let (X, A) and (Y, B) be pairs of paracompact spaces and let $f: (X, A) \rightarrow (Y, B)$ be a closed mapping such that the restriction $f \mid X - A$ is a homeomorphism of $X - A$ onto $Y - b$, then $f^*: H^n(Y, B:G) \rightarrow H^n(X, A:G)$ is an isomorphism.

PROOF. At first, let us prove that f^* is a monomorphism. Let $e \in H^n(Y, B:G)$ and $f^*e = 0$. By the extension lemma, 37-3, there is a closed neighborhood B_1 of B and an element e_1 of $H^n(Y, B_1:G)$ such that $e_1 \mid (Y, B) = e$. Put $A_1 = f^{-1}(B_1)$. Let e_2 be the image of e_1 under

the homomorphism: $H^n(Y, B_1; G) \rightarrow H^n(X, A_1; G)$ induced by f . Since $e_2 \mid (X, A) = f^*e = 0$, by the reduction lemma, 37-2, there is a closed neighborhood A_2 of A contained in A_1 such that $e_2 \mid (X, A_2) = 0$. Put $B_2 = f(A_2)$. Since f is a closed mapping, B_2 is a closed neighborhood of B contained in B_1 . Let $h: H^n(Y, B_2; G) \rightarrow H^n(X, A_2; G)$ be the homomorphism induced by f . Then h is an isomorphism, because; there is an open neighborhood V of B such that $\bar{V} \subset B$, then

$$f \mid X - f^{-1}V: (X - f^{-1}V, A_2 - f^{-1}V) \rightarrow (Y - V, B_2 - V)$$

is a homeomorphism and by 34-5,

$$H^n(X, A_2; G) \approx H^n(X - f^{-1}V, A_2 - f^{-1}V; G)$$

and

$$H^n(Y, B_2; G) \approx H^n(Y - V, B_2 - V; G)$$

hence h is an isomorphism. Put $e_3 = e_1 \mid (Y, B_2)$. Then we have $h(e_3) = e_2 \mid (X, A_2) = 0$. Since h is 1:1, $e_3 = 0$. Thus we have $e = e_1 \mid (Y, B) = e_3 \mid (Y, B) = 0$. This proves that f^* is a monomorphism. That f^* is onto is proved similarly. $\square$

Let X be a paracompact space and let A be closed in X . We denote by X_0 the space obtained from X by contracting the closed set A to a point p_0 ; that is, X_0 is the decomposition space determined by the decomposition $\{A, x: x \in X - A\}$. The space X_0 is also denoted by X/A . Let f be an identification mapping: $(X, A) \rightarrow (X_0, p_0)$. It is evident that f is a closed mapping and X_0 is paracompact.

37-6. COROLLARY. Let A be a closed set of a paracompact space X and let $X_0 = X/A$ and let p_0 be the point corresponding to A . If $f: (X, A) \rightarrow (X_0, p_0)$ is the identification mapping, then $f^*: H^n(X_0, p_0; G) \rightarrow H^n(X, A; G)$ is an isomorphism.

37-7. THEOREM (Skljarenko [1]). If X is paracompact, then $D(X; G) = d(X; G)$.

We need the following lemmas.

37-8. LEMMA. Let τ be a discrete family of closed sets of a paracompact space X and let $F = \cup \{A: A \in \tau\}$. If e is an element of $H^n(X; G)$ such that $e \mid A = 0$ for each $A \in \tau$, then $e \mid F = 0$.

This is evident, because $H^n(F; G)$ is the direct product of the groups $H^n(A; G)$, $A \in \tau$.

37-9. LEMMA. Let A and B be closed sets of a paracompact space X and let $m > d(X:G)$. If e is an element of $H^m(X:G)$ such that $e|A = e|B = 0$, then $e|A \cup B = 0$. Consequently, if $A \cup B = X$ and $e|A = e|B = 0$, then $e = 0$.

PROOF. Consider the following Mayer-Vietoris sequence (see Lemma 34-11):

$$\begin{aligned} \cdots \rightarrow H^{m-1}(A:G) \oplus H^{m-1}(B:G) \xrightarrow{i} H^{m-1}(A \cup B:G) \rightarrow H^m(A \cup B:G) \\ \xrightarrow{j} H^m(A:G) \oplus H^m(B:G) \rightarrow \cdots \end{aligned}$$

Note that the homomorphisms i and j are induced by the inclusions. (See Eilenberg and Steenrod [1, Chap. I].) Since $d(X:G) \leq m-1$, $d(A:G) \leq m-1$ and $d(B:G) \leq m-1$. Hence i is onto and j is therefore 1:1. Let $e' = e|A \cup B$. Then $e'|A = e'|B = 0$ and hence $j|e' = 0$. Since j is 1:1, we can conclude $e' = 0$. $\square$

Now we are going to prove Theorem 37-7. Since $d(X:G) \leq D(X:G)$ by Lemma 35-2, it is sufficient to prove $d(X:G) \geq D(X:G)$. Let $d(X:G) \leq n$. Let A be closed in X and consider the following exact sequence:

$$\cdots \rightarrow H^m(X:G) \xrightarrow{i^*} H^m(A:G) \xrightarrow{\delta} H^{m+1}(X, A:G) \xrightarrow{j^*} H^{m+1}(X:G) \rightarrow \cdots$$

If $m \geq n$, then i^* is onto and hence j^* is 1:1. Thus, if we prove $H^{m+1}(X:G) = 0$, then we can conclude that $H^{m+1}(X, A:G) = 0$. Let $e \in H^{m+1}(X:G)$. We have to prove $e = 0$. For each point x of X the restriction of e to the point x is zero, because $H^{m+1}(x:G) = 0$. Thus, in view of Lemma 37-3, there is an open neighborhood V_x of x such that the restriction of e to the closure $\bar{V}_x$ of V_x is zero. Since X is paracompact, by Theorem 2-22, there is a σ -discrete closed covering $\mathcal{H} = \{\mathcal{H}_i: i = 1, 2, \dots\}$ which refines the covering $\{V_x: x \in X\}$. Put $F_i = \cup\{F: F \in \mathcal{H}_i\}$, $i = 1, 2, \dots$. Since $e|F = 0$ for each F of $\mathcal{H}_i$ and $\mathcal{H}_i$ is a discrete family, we have $e|F_i = 0$ by Lemma 37-8. Let us construct a sequence of open sets $\{U_i: i = 1, 2, \dots\}$ satisfying the following conditions: $F_i \cup \bar{U}_{i-1} \subset U_i$ and $e|\bar{U}_i = 0$ for $i = 1, 2, \dots$, where $U_0 = \phi$ and $\bar{U}_i$ is the closure of U_i . In view of Lemma 37-2, there is an open neighborhood U_1 of F_1 such that $e|\bar{U}_1 = 0$. Then $e|F_2 \cup \bar{U}_1 = 0$ by 37-9. Suppose that an open neighborhood U_i of $F_i \cup \bar{U}_{i-1}$ is constructed such that $e|\bar{U}_i = 0$. Lemma 37-9, shows $e|F_{i+1} \cup \bar{U}_i = 0$. By Lemma 37-2, there is an open neighborhood U_{i+1} of $F_{i+1} \cup \bar{U}_i$ such that $e|\bar{U}_{i+1} = 0$. Next, put $W_i = \bar{U}_i - U_{i-1}$ for $i = 2$,

3, ... Then $\{W_{2i}: i = 1, 2, \dots\}$ and $\{W_{2i+1}: i = 1, 2, \dots\}$ are discrete families of closed sets and $e \mid W_i = 0$ for each i . Thus, if $S = \cup \{W_{2i}\}$ and $T = \cup \{W_{2i+1}\}$, then $e \mid S = e \mid T = 0$ by Lemma 37-9. Since $S \cup T = X$, Lemma 37-9, also shows $e = 0$. $\square$

37-10. THEOREM.[†] Let X be a paracompact space and let G be a finitely generated abelian group, then $D_{\mathcal{F}}(X: G) = D(X: G) = d_{\mathcal{F}}(X: G) = d(X: G)$.

PROOF. In view of Theorem 36-14 and the preceding theorem, observe $D(X: G) = d(X: G) = d_{\mathcal{F}}(X: G)$. By Lemma 35-4, $D_{\mathcal{F}}(X: G) = D(\beta X: G)$ and $d_{\mathcal{F}}(X: G) = d(\beta X: G)$. Since $D(\beta X: G) = d(\beta X: G)$ by 37-7, we have $D_{\mathcal{F}}(X: G) = D(X: G) = d_{\mathcal{F}}(X: G) = d(X: G)$. $\square$

37-11. DEFINITION. A *CW cell complex* M is a topological space which is a union of n -cells σ_{α}^n , $\alpha \in \Gamma_n$ and $n = 0, 1, \dots$, and satisfies the following conditions: (i) if $M^n = \cup \{\sigma_{\gamma}^m: \gamma \in \Gamma_m \text{ and } m \leq n\}$ (called the n -section of M), then $\sigma_{\alpha}^n \cap M^{n-1}$ is the boundary $(n-1)$ -sphere σ_{α}^{n-1} of σ_{α}^n and it is an exact union of a finite number of $(n-1)$ -cells σ_{β}^{n-1} , and (ii) M has the weak topology, that is, a subset A of M is closed in M if and only if for each cell σ_{α}^n the intersection $A \cap \sigma_{\alpha}^n$ is closed in σ_{α}^n . If there are no cells of dimension $> m$, then M is m -dimensional. A *subcomplex* of M is a closed subset of M which is a union of a subcollection of the cells of M . Let σ be a cell of M . We denote by $\overline{\text{St}}(\sigma)$ the union of all cells intersecting σ . $\overline{\text{St}}(\sigma)$ is a subcomplex.

37-12. THEOREM. If X is paracompact, then $d(X: G) \leq n$ if and only if for every open covering $\mathcal{U}$ of X there is a finite dimensional CW cell complex M and a mapping $f: X \rightarrow M$ satisfying the following conditions: (i) for each cell σ of M the set $f^{-1}(\overline{\text{St}}(\sigma))$ is contained in some member of $\mathcal{U}$, and (ii) for each $m > n$ and each cell σ of M $H^m(f^{-1}\sigma, f^{-1}\sigma: G) = 0$, where $\dot{\sigma}$ is the boundary of σ .

For the proof we need the following lemma due to Wojdyslawski [1].

[†] More generally, it is true that if X is a (normal) space and G is finitely generated, then $D_{\mathcal{F}}(X: G) = D_{\mathcal{L}}(X: G) = d_{\mathcal{F}}(X: G) = d_{\mathcal{L}}(X: G)$. See Remark 36-17.

37-13. LEMMA. Let X be a bounded metric space, then there is a Banach space B and an isometric imbedding $\lambda: X \rightarrow B$ such that $\lambda(X)$ is a closed subset of the convex hull C of $\lambda(X)$ in B . (By the *convex hull* we mean the intersection of all convex subsets of B containing $\lambda(X)$.)

PROOF. Let B be the set of all bounded (continuous) real valued functions defined on X . With the usual addition and scalar multiplication B forms a Banach space. For instance, we define a norm of f of B by taking $\|f\| = \sup\{|f(x)| : x \in X\}$. Let ρ be a metric function of X . For each point a of X , define a function f_a by $f_a(x) = \rho(a, x)$ for $x \in X$. It is evident that f_a is a bounded real valued function. Define $\lambda: X \rightarrow B$ by $\lambda(a) = f_a$ for $a \in X$. For $a \in X$ and $b \in X$ we have

$$\begin{aligned}\rho(a, b) &= |f_a(b) - f_b(b)| \\ &\leq \|f_a - f_b\| = \sup\{|\rho(a, x) - \rho(b, x)| : x \in X\} \\ &\leq \rho(a, b)\end{aligned}$$

Hence λ is an isometric imbedding. Let us prove that $\lambda(X)$ is a closed set of the convex hull C of $\lambda(X)$ in B . Let f be a point of C which does not belong to $\lambda(X)$. We shall prove that f has an open neighborhood V which does not intersect $\lambda(X)$. This shows $\lambda(X)$ is closed in C . Since C is the convex hull of $\lambda(X)$, there are points $a_1, a_2, \dots, a_k$ of X and real numbers $t_1, t_2, \dots, t_k$ such that $\sum_{i=1}^k t_i = 1$ and $f = \sum_{i=1}^k t_i f_i$, $f_i = \lambda(a_i)$. Since $f \neq f_i$ for $i = 1, 2, \dots, k$, we can find a positive number ε such that $2\varepsilon < \|f - f_i\|$ for $i = 1, 2, \dots, k$. Put $V = \{g : g \in C \text{ and } \|f - g\| < \varepsilon\}$. Then V is an open neighborhood of f in C . Assume that there is a point x of X such that $\lambda(x) = h \in V$. Since $\|f_i - h\| \geq \|f_i - f\| - \|f - h\| > \varepsilon$ for each i and λ is isometric, we have $f_i(x) = \rho(a_i, x) = \|f_i - h\| > \varepsilon$ for each i . Thus $\|f - h\| \geq |f(x) - h(x)| = |f(x)| = \sum_{i=1}^k t_i f_i(x) > (\sum_{i=1}^k t_i)\varepsilon = \varepsilon$. This contradicts the assumption that h belongs to V . Thus $V \cap \lambda(X) = \emptyset$ and $\lambda(X)$ is closed in C . $\square$

37-14. PROOF OF THEOREM 37-12. The necessity is evident: If $\mathcal{V}$ is an lfc covering of finite order which is a star refinement of $\mathcal{U}$, then the nerve M of $\mathcal{V}$ and a canonical mapping $f: X \rightarrow M$ satisfying conditions (i) and (ii).

To prove sufficiency, let g be a mapping of a closed set A of X into $K(G, m; q)$ (see Section 36-4), where $m \geq n$ and $q > \dim X$. We have to prove that g is extendable over X . Put $K = K(G, m; q)$. By Lemma 37-13 we can consider K as a closed set of a convex set C in a Banach space.

Since K is an ANR (paracompact), there is an open neighborhood H of K in C and a retraction $\mu: H \rightarrow K$. For each point k of K , take a spherical neighborhood $V(k)$ of k in C contained in H . Let $\mathcal{U}$ be an open covering of X such that for each member U of $\mathcal{U}$ $g(U \cap A)$ is contained in some $V(k)$. Let M and f be a CW cell complex and a mapping satisfying the conditions (i) and (ii). Denote by N the subcomplex consisting of all cells intersecting $f(A)$. At first, we construct a mapping $h: N \rightarrow H$ such that there is some $V(k)$ containing both $hf(x)$ and $g(x)$ for each point x of A . For this purpose, let v be a vertex of N . Take a cell σ_v of N containing v and a point x_v of $A \cap f^{-1}(\sigma_v)$. Define a mapping $h: N \rightarrow H$ by $h(v) = g(x_v)$ for each vertex v of N . If $v_i, i = 1, \dots, j$, are the vertexes of a cell σ of N , then the points $x_{v_i}, i = 1, \dots, j$, are in the set $f^{-1}(\overline{\text{St}}(\sigma))$. (See Definition 37-11.) By condition (i) there is some member U of $\mathcal{U}$ containing $f^{-1}(\overline{\text{St}}(\sigma))$ and hence the set $\cup \{h(x_{v_i}): i = 1, \dots, j\}$ is contained in some $V(k)$. Since $V(k)$ is convex, h is extended over σ linearly. We denote the extension of h over X again by h . It is evident that for each $x \in A$ $hf(x)$ and $g(x)$ are both in some $V(k)$. Set $F = f^{-1}(N)$ and define a mapping $h_1: F \rightarrow K$ by $h_1(x) = \mu hf(x)$ for $x \in F$, where μ is a retraction: $H \rightarrow K$. Then we have $g \sim h_1 f|A: A \rightarrow K$, because $g \sim hf|A: A \rightarrow H$. Thus, for our purpose, we may assume that g is a mapping of F into K and $g = h_1$. Next, let $k = \max\{\dim M, q\}$ and put $K' = K(G, m; k)$. We shall prove that the mapping g has an extension: $X \rightarrow K'$. Because $\dim X < q$, every mapping of X into K' is deformed into a mapping of X into K . (K is the q -section of K' .) This gives a complete proof of the theorem. Since $\pi_i(K') = 0$ for $i < m$, g is extendable over the set $f^{-1}(M^m \cup N)$, where M^m is the m -section of M . Let σ be an $(m+1)$ -cell of $M - N$. By condition (ii) in the theorem $H^{m+1}(f^{-1}\sigma, f^{-1}\dot{\sigma}; G) = 0$ and hence the homomorphism: $H^m(f^{-1}\sigma; G) \rightarrow H^m(f^{-1}\dot{\sigma}; G)$ is onto. Thus the mapping $g|f^{-1}\dot{\sigma}$ is extended over $f^{-1}\sigma$. (See Section 36-13, proof of necessity.) Hence g is extendable over $M^{m+1} \cup N$. By repeating this process we know that g is extendable over X . $\square$

38. Sum Theorem

38-1. DEFINITION. Let X be a topological space and let $\{F_\alpha\}$ be a closed covering of X . It is said that X has the *weak topology with respect to* $\{F_\alpha\}$. (This notion is due to Morita [12].) If the following are satisfied: (i) For any subfamily $\{F_\beta\}$ of $\{F_\alpha\}$, the union $\cup \{F_\beta\}$ is closed in X , and

a subset F of $\cup\{F_\beta\}$ is closed in $\cup\{F_\beta\}$ if and only if the intersection of F with each member F_β of $\{F_\beta\}$ is closed relative to the subspace topology of F_β .

It is evident that any CW cell complex has the weak topology with respect to the closed covering consisting of all cells. Also, if $\{F_\alpha\}$ is a locally finite closed covering of a topological space X , then X has the weak topology with respect to $\{F_\alpha\}$.

38-2. LEMMA. Let X be a space having the weak topology with respect to a closed covering $\{F_\alpha\}$ of X and let Y be a space, then $f: X \rightarrow Y$ is continuous if and only if the restriction $f|F_\alpha$ is continuous for each F_α .

The proof is trivial.

38-3. SUM THEOREM (Okuyama [1] and Kodama [6]). Let X be a paracompact space.

i. If $\{A_i: i = 1, 2, \dots\}$ is a countable closed covering of X , then $d(X: G) = \text{lub}\{d(A_i: G): i = 1, 2, \dots\}$.

ii. If X has the weak topology with respect to a closed covering $\{A_\lambda: \lambda \in \Gamma\}$, then $d(X: G) = \text{lub}\{d(A_\lambda: G): \lambda \in \Gamma\}$.

iii. If A is a closed set of X and $X - A$ is paracompact, then $d(X: G) \leq \max\{d(A: G), d(X - A: G)\}$. Moreover, if A is G_δ , then equality holds.

PROOF (Morita [12]). (i) Let $m \geq \text{lub}\{D(A_i: G): i = 1, 2, \dots\}$ and let B be a closed set of X . It is sufficient to prove that, if f is a mapping of B into K ($= K(G, m; q)$, $q > \dim X$), then f is extendable over X . Let us construct a sequence of open sets $\{U_i: i = 1, 2, \dots\}$ satisfying $\overline{U_i} \cup A_{i+1} \subset U_{i+1}$ and such that f has an extension $f_i: \overline{U_i} \rightarrow K$ with $f_i| \overline{U_{i-1}} = f_{i-1}$ for each i , where $U_0 = B$ and $f_0 = f$. For $i \leq k$, suppose that we find mappings f_i and open sets U_i subject to these conditions. Since $d(A_{k+1}: G) \leq m$, $f_k| \overline{U_k} \cap A_{k+1}$ has an extension $h_k: A_{k+1} \rightarrow K$. Define $f_{k+1}: \overline{U_k} \cup A_{k+1} \rightarrow K$ by

$$f_{k+1}| \overline{U_k} = f_k \quad \text{and} \quad f_{k+1}| A_{k+1} = h_k$$

Since K is an ANR (paracompact), there is an open neighborhood U_{k+1} of $\overline{U_k} \cup A_{k+1}$ such that f_{k+1} is extendable over $\overline{U_{k+1}}$. Denote its extension again by f_{k+1} . Finally, define $g: X \rightarrow K$ by $g| \overline{U_i} = f_i$ for each i . It is evident that g is a continuous extension of f .

(ii) This is an immediate consequence of Theorem 36-5 and Lemma 38-2.

(iii) The first part is proved by an argument similar to that in (i). The second part is a consequence of (i) and the first part. $\square$

38-4. THEOREM.[†] Let X and Y be paracompact and let f be a closed mapping of X onto Y . Suppose that $d(f^{-1}(y): G) \leq k$ for each point y of Y , then $d(X: G) \leq \text{Ind } Y + k$, where $\text{Ind } Y$ is the large inductive dimension of Y .

PROOF (by induction with respect to $\text{Ind } Y$). Suppose that the theorem is true in case $\text{Ind } Y = n - 1$. Let $\text{Ind } Y = n$. Let A be a closed set of X . It is sufficient to prove that for each $m \geq n + k$ and each mapping $g: A \rightarrow K (= K(G, m; q), q > \dim X)$ g is extendable over X . Let $g: A \rightarrow K$. We may assume that A is the closure of an open set, because g is extendable over some closed neighborhood of A . Since $d(f^{-1}(y): G) \leq k \leq m$ for each $y \in Y$, then $g|_{f^{-1}(y) \cap A}$ has an extension $g_y: f^{-1}(y) \rightarrow K$. We define a mapping $h_y: A \cup f^{-1}(y) \rightarrow K$ by $h_y|_A = g$ and $h_y|_{f^{-1}(y)} = g_y$. Then h_y is extendable over some open neighborhood V_y of $A \cup f^{-1}(y)$ in X . Denote its extension by h_y again. Since the mapping f is closed, the set $U_y = Y - f(X - V_y)$ is an open neighborhood of y for each $y \in Y$. In view of the paracompactness of Y and the fact $\text{Ind } Y = n$, there exists a locally finite open covering $\mathcal{W} = \{W_\alpha: \alpha \in \Gamma\}$ of Y such that $\{\bar{W}_\alpha: \alpha \in \Gamma\}$ refines $\{U_y: y \in Y\}$ and $\text{Ind}(\bar{W}_\alpha - W_\alpha) \leq n - 1$ for each $\alpha \in \Gamma$, where $\bar{W}_\alpha$ is the closure of W_α . Put $F_\alpha = f^{-1}(\bar{W}_\alpha)$ and $G_\alpha = f^{-1}(W_\alpha)$. For each $\alpha \in \Gamma$, choose a point y of Y such that $F_\alpha \subset V_y$ and set $h_\alpha = h_y|_{F_\alpha}$. Assume that the index set Γ is well ordered by $<$. Set

$$T_\alpha = \cup \{F_\beta: \beta < \alpha, \beta \in \Gamma\} \quad \text{and} \quad S_\alpha = T_\alpha \cup F_\alpha$$

Let us assume that for each $\alpha < \gamma \in \Gamma$ there is an extension $g_\alpha: S_\alpha \cup A \rightarrow K$ of g such that if $\beta < \alpha$ then we have $g_\alpha|_{S_\beta \cup A} = g_\beta$. Define $\bar{g}_\gamma: T_\gamma \cup A \rightarrow K$ by $\bar{g}_\gamma|_{S_\alpha \cup A} = g_\alpha$ for each $\alpha < \gamma$. Since the family $\{A, F_\alpha: \alpha < \gamma\}$ is locally finite, $\bar{g}_\gamma$ is continuous. We shall prove $\bar{g}_\gamma$ has an extension $g_\gamma: S_\gamma \cup A \rightarrow K$. For this purpose, set $B = T_\gamma - \cup \{G_\alpha: \alpha < \gamma\}$, $A_0 = \text{interior of } A$, $A_b = \text{boundary of } A$ (recall that A is the closure of an open set), $F = F_\gamma - A_0 - \cup \{G_\alpha: \alpha < \gamma\}$ and $H = F \cap (A_b \cup B)$. Note that B is a closed subset of $\cup \{f^{-1}(\bar{W}_\alpha - W_\alpha): \alpha < \gamma\}$. Since $\text{Ind}(\bar{W}_\alpha - W_\alpha) \leq n - 1$, we have $d(B: G) \leq n - 1 + k \leq m - 1$ by the induction hypotheses. Hence,

[†] In Theorems 38-3 and 38-4, $d(*: G)$ can be replaced by $D(*: G)$ in view of Theorem 37-5.

$d(H \cap B) \leq m - 1$. Now we have two mappings defined on the set H , $\bar{g}_\gamma | H$ and $h_\gamma | H$. We shall prove that these mappings are homotopic. Then the homotopy extension theorem shows that $\bar{g}_\gamma | B$ has an extension $h: F \rightarrow K$, because h_γ is defined on the set $F \subset F_\gamma$. Define $g_\gamma: S_\gamma \cup A \rightarrow K$ by

$$g_\gamma | T_\gamma \cup A = \bar{g}_\gamma \quad \text{and} \quad g_\gamma | F = h$$

Obviously g_γ is continuous. Finally, define a mapping $\bar{g}: X \rightarrow K$ by $\bar{g} | S_\alpha \cup A = g_\alpha$ for each $\alpha \in \Gamma$. Then $\bar{g}$ is a continuous extension of g . It remains to prove that $\bar{g}_\gamma | H$ and $h_\gamma | H$ are homotopic. Let us assume the following fact: If C is paracompact and $d(C: G) \leq m - 1$, then $d(C \times I: G) \leq m$, where $I = [0, 1]$. This fact will be proved in Section 42. Since $d(H \cap B: G) \leq m - 1$, we have $d((H \cap B) \times I: G) \leq m$. Let us define

$$h: \{(H \cap B) \times (\{0\} \cup \{1\})\} \cup \{(H \cap B \cap A_b) \times I\} \rightarrow K$$

by

$$h(x, 0) = \bar{g}_\gamma(x)$$

$$h(x, 1) = h_\gamma(x) \quad \text{for } x \in H \cap B$$

$$h(x, t) = \bar{g}_\gamma(x) \quad \text{for } x \in H \cap B \cap A_b \text{ and } t \in I$$

Since $\bar{g}_\gamma | H \cap A_b = h_\gamma | H \cap A_b$, h is continuous. By Theorem 36-5, h has an extension $h': (H \cap B) \times I \rightarrow K$. Define $h'': H \times I \rightarrow K$ by $h'' | (H \cap B) \times I = h'$ and $h''(x, t) = \bar{g}_\gamma(x)$ for $x \in H \cap A_b$. Obviously h'' is a homotopy connecting $\bar{g}_\gamma | H$ and $h_\gamma | H$. $\square$

38-5. DEFINITION. The *strong cohomological dimension* $D_s(X: G)$ of a space X is the least integer n such that, for each pair consisting of a closed set A and an open set U containing A , there is an open set V satisfying $A \subset V \subset \bar{V} \subset U$ and $D(\bar{V} - V: G) \leq n - 1$.

38-6. THEOREM. If X is a paracompact space, then $D_s(X: G) - 1 \leq D(X: G) \leq D_s(X: G)$.

PROOF. If $D(X: G) \leq n$, then $D(F: G) \leq n$ for each closed set F of X and hence $D_s(X: G) \leq n + 1$. It remains to prove that $D(X: G) \leq D_s(X: G)$. Let $D(X: G) = n + 1$. Then there is a closed set A of X and a mapping $f: A \rightarrow K (= K(G, n; q), q > \dim X)$ such that f has no extension over X . Extend f over some closed neighborhood F of A and denote its extension by f again. Assume that $D_s(X: G) \leq n$. Then there is an open set U such that $A \subset U \subset F$ and $D(\bar{U} - U: G) \leq n - 1$. Since $D((\bar{U} - U) \times I: G) \leq n$,

if we define $h: (\bar{U} - U) \times (\{0\} \cup \{1\}) \rightarrow K$ by $h(x, 0) = f(x)$ and $h(x, 1) = k_0$ (= a fixed point of K), then h has an extension $h_1: (\bar{U} - U) \times I \rightarrow K$. Define $h_2: (\bar{U} - U) \times I \cup \bar{U} \times \{0\} \rightarrow K$ by $h_2|_{(\bar{U} - U) \times I} = h_1$ and $h_2(x, 0) = f(x)$ for $x \in \bar{U}$. The homotopy extension theorem shows h_2 has an extension $h_3: \bar{U} \times I \rightarrow K$. Define $h_4: \bar{U} \times I \cup X \times \{1\} \rightarrow K$ by $h_4|_{\bar{U} \times I} = h_3$ and $h_4(x, 1) = k_0$ for $x \in X - U$. Then h_4 has an extension $h_5: X \times I \rightarrow K$. Since $h_5(x, 0) = f(x)$ for $x \in \bar{U}$, an extension of f over X does exist. This contradiction shows that $D_s(X; G) \geq n + 1 = D(X; G)$. $\square$

38-7. REMARK. Theorems 38-3, 38-4 and 38-6 are true for a (normal) space X if G is finitely generated and $d(*: G)$ or $D(*: G)$ is replaced by $d_{\mathcal{A}}(*: G)$. (If G is finitely generated and X is normal, then $d_{\mathcal{A}}(X; G) = d_{\mathcal{F}}(X; G) = D_{\mathcal{A}}(X; G) = D_{\mathcal{F}}(X; G)$. Compare Section 36-17.) Thus, if we put $G = Z$ in Theorem 38-3, then we obtain the sum theorem, Section 9-10. Also, Theorem 38-4 for $G = Z$ implies Theorem 21-1. Moreover, we know that $D_s(X; Z) \leq \text{Ind } X$ for a (normal) space X . This is proved by a simple induction with respect to $\text{Ind } X$ and by using Theorem 38-6. Since $\dim X = D_{\mathcal{A}}(X; Z) \leq D_s(X; Z)$, this fact implies Theorem 10-1.

38-8. PROBLEM. If we replace $\text{Ind } Y$ by $d(Y; G)$ in Theorem 38-4, then the theorem does not generally hold: Consider the continuum M_{z_p} to be constructed in Section 40. Put $X = M_{z_p} \times M_{z_p}$, $Y = M_{z_p}$ and let f be the projection of X onto Y . Denote by Q_p the additive group of p -adic rationals modulo one, where p is a prime. Then it is known that $d(X; Q_p) = 3$ and $d(Y; Q_p) = 1$ (see Section 40). Thus we have $d(X; Q_p) > d(Y; Q_p) + \sup\{d(f^{-1}(y); Q_p): y \in Y\} = 2$. On the other hand, $D_s(Y; Q_p) = 2$. Hence, the relation $d(X; Q_p) = D_s(Y; Q_p) + \sup\{d(f^{-1}(y); Q_p): y \in Y\}$ holds. Thus the following problem arises: Under the same hypothesis as in Theorem 38-4, is the relation $d(X; G) \leq D_s(Y; G) + k$ true?

38-9. LEMMA (Kodama [1]). If X is a 2-dimensional compact ANR (metric), then $D(X; G) = D_s(X; G) = \dim X$ for any abelian group G .

PROOF. There is a point x_0 of X such that $\dim F = 2$ for any closed neighborhood F of x_0 . Since X is an ANR and any closed neighborhood of x_0 is not an acyclic curve, there is a homeomorph S of a 1-sphere which is contractible in X . Consider the following exact sequence:

$$\cdots \rightarrow H^1(X; G) \xrightarrow{i^*} H^1(S; G) \xrightarrow{\delta} H^2(X, S; G) \rightarrow \cdots$$

The homomorphism i^* is trivial because S is contractible in X . Hence δ is 1:1. Since $H^1(S; G) \approx G$, we can conclude $H^2(X, S; G) \neq 0$. Thus, $2 = D(X; G) \leq D_s(X; G) \leq \dim X$. $\square$

38-10. PROBLEM. As discussed in Problem 38-8, the equality $d(X; G) = D_s(X; G)$ is not generally true even if X is a continuum. Thus the problem arises: Determine a continuum X such that $d(X; G) = D_s(X; G)$ holds for each abelian group G . This equality is true for a 2-dimensional compact ANR (metric) and a simplicial polytope. We conjecture that if X is a compact ANR (metric), then the equality $D(X; G) = D_s(X; G)$ holds for any group G .

38-11. There is an interesting corollary of Theorem 38-6.

THEOREM (Kodama [8]). Let X be a paracompact space with $\text{Ind } X < \infty$, then, for each k , $0 \leq k \leq D(X; G)$, there is a closed subset X_k of X such that $D(X_k; G) = k$.

PROOF. This is proved by induction with respect to the number $\text{Ind } X$. If $\text{Ind } X = 0$, then the theorem is trivially true. Suppose that the theorem holds in case $\text{Ind } X \leq n - 1$. Let $\text{Ind } X = n$ and $D(X; G) = m$. For each pair (F, U) , F closed, U open and $F \subset U$, choose an open set $V_{(F, U)}$ such that

$$F \subset V_{(F, U)} \subset \bar{V}_{(F, U)} \subset U \quad \text{and} \quad \text{Ind}(\bar{V}_{(F, U)} - V_{(F, U)}) \leq n - 1$$

If $D(\bar{V}_{(F, U)} - V_{(F, U)}; G) \leq m - 2$ for each pair (F, U) , then $D(X; G) \leq m - 1$ in view of Theorem 38-6. Thus there is some pair (F, U) such that $D(\bar{V} - V; G) = m$ or $m - 1$, where $V = V_{(F, U)}$. By induction hypothesis there is a closed set X_k of $\bar{V} - V$ such that $D(X_k; G) = k$, $0 \leq k \leq m - 1$. This completes the proof. $\square$

39. Bockstein's Theorem I

39-1. This section is devoted to discussing Bockstein's first theorem. The second theorem is described in Section 41. These theorems are unquestionably most important in algebraic dimension theory for compact spaces. The original proof of the theorems by Bockstein is very long and complicated (Bockstein [1], [2]). Dyer [1] gave an excellent alternate proof. The proof given here is a restatement of Dyer's proof.

The elementary properties of the tensor product and the torsion product of groups (all groups being abelian) are assumed. For standard references see Hilton and Wylie [1] and Cartan and Eilenberg [1].

Let $\{G_\lambda\}$ and $\{H_\omega\}$ be direct systems of groups over the directed sets Λ and Ω respectively. Then $\{G_\lambda \otimes H_\omega\}$ and $\{G_\lambda * H_\omega\}$ are the direct systems over $\Lambda \times \Omega$, where $\otimes$ and $*$ are the tensor product and torsion product respectively. Then we have the Lemmas which follow.

39-2. LEMMA. $\varinjlim \{G_\lambda \otimes H_\omega\} \approx \varinjlim G_\lambda \otimes \varinjlim H_\omega$ and $\varinjlim \{G_\lambda * H_\omega\} \approx \varinjlim G_\lambda * \varinjlim H_\omega$.

The proof is found in Cartan and Eilenberg [1, Chapter V, 9.2 and 9.4].

39-3. LEMMA. If $0 \rightarrow H' \rightarrow H \rightarrow H'' \rightarrow 0$ is an exact sequence of groups and G is a group, then the sequence

$$0 \rightarrow H' * G \rightarrow H * G \rightarrow H'' * G \rightarrow H' \otimes G \rightarrow H \otimes G \rightarrow H'' \otimes G \rightarrow 0$$

is exact.

See Hilton and Wylie [1, 5.5.4] for the proof.

39-4. LEMMA. Let (X, A) be a pair of compact spaces and G a group, then the sequence

$$0 \rightarrow H^n(X, A) \otimes G \rightarrow H^n(X, A; G) \rightarrow H^{n+1}(X, A) * G \rightarrow 0 \quad (1)$$

is exact. Here $H^*(X, A)$ is the Čech group with coefficients in $\mathbb{Z}$.

PROOF. If (X, A) is a pair (M, N) of finite polytopes, then the theorem is well known (see Hilton and Wylie [1, 5.4.24]). Moreover, the sequence (1) is natural with respect to a mapping: $(M, N) \rightarrow (M', N')$, that is, if (M, N) and (M', N') are finite triangulable pairs and $f: (M, N) \rightarrow (M', N')$ is a mapping, then the following diagram is commutative:

$$\begin{array}{ccccccc} 0 \rightarrow H^n(M', N') \otimes G & \rightarrow & H^n(M', N'; G) & \rightarrow & H^{n+1}(M', N') * G & \rightarrow & 0 \\ & \downarrow f_1^* & & \downarrow f^* & & \downarrow f_2^* & \\ 0 \rightarrow H^n(M, N) \otimes G & \rightarrow & H^n(M, N; G) & \rightarrow & H^{n+1}(M, N) * G & \rightarrow & 0 \end{array}$$

where f^* , f_1^* and f_2^* are induced by the mapping f . For a compact pair (X, A) there is an inverse limiting system $\{(M_\alpha, N_\alpha)\}$ of finite triangulable

pairs whose inverse limit is (X, A) (see Eilenberg and Steenrod [1, Chapter X, 10.1]). Thus by Lemma 39-2, the sequence

$$\begin{aligned} 0 \rightarrow \varinjlim \{H^n(M_\alpha, N_\alpha)\} \otimes G &\rightarrow \varinjlim \{H^n(M_\alpha, N_\alpha; G)\} \\ &\rightarrow \varinjlim \{H^{n+1}(M_\alpha, N_\alpha)\} * G \rightarrow 0 \end{aligned}$$

is exact. By Theorem 34-12, $H^n(X, A; G) \approx \varinjlim \{H^n(M_\alpha, N_\alpha; G)\}$. This completes the proof. $\square$

39-5. LEMMA. If $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ is an exact sequence of groups and (X, A) is a paracompact pair, then the sequence

$$\cdots \rightarrow H^n(X, A; G') \rightarrow H^n(X, A; G) \rightarrow H^n(X, A; G'') \rightarrow H^{n+1}(X, A; G') \rightarrow$$

is exact.

The lemma is well known in case (X, A) is a triangulable pair (see Hilton and Wylie [1, 5.6.5]). The lemma follows from the same argument as in Section 39-4.

Throughout this appendix the following notation will be used:

R is the additive group of rationals.

R_p is the additive group of rationals whose denominators are coprime with p .

Z_p is the cyclic group of order p .

Q_p is the additive group of p -adic rationals reduced modulo one.

Here p is a prime. For a group G we denote by $(G)_p$ the p -primary part of G .

39-6. DEFINITION. A group G is said to have *property* $P(p)$, p a prime, if there is some element of $G/(G)_p$ which is not divisible by p ; equivalently, if there is an element $g \in G$ such that $p \cdot g' - g \notin (G)_p$ for any element $g' \in G$.

The following useful lemmas are due to Dyer [1]. The proofs are omitted. Let $\bar{G}$ and H be groups.

39-7. LEMMA.

1. If $(H)_p \neq 0$ and $\bar{G}$ has property $P(p)$, then $H \otimes \bar{G} \neq 0$
2. If $H \otimes Q_p \neq 0$, then H has property $P(p)$

39-8. LEMMA. $H \otimes \bar{G} \neq 0$ if and only if one of the following conditions is satisfied:

1. Both H and $\bar{G}$ contain elements of infinite order

For some prime p :

2_p. H has property $P(p)$ and $(\bar{G})_p \approx \oplus Q_p$, that is, $(G)_p$ is the direct sum of copies of Q_p

3_p. $(H)_p \approx \oplus Q_p$ and $\bar{G}$ has property $P(p)$

4_p. Both H and $(\bar{G})_p$ contain elements not divisible by p

5_p. Both $(H)_p$ and $\bar{G}$ contain elements not divisible by p

Moreover, $(H \otimes \bar{G})_p \neq 0$ if and only if one of the conditions (2_p), (3_p), (4_p) and (5_p) holds.

Further, $H * \bar{G} \neq 0$ if and only if for some prime p :

6_p. $(H)_p \neq 0$ and $(\bar{G})_p \neq 0$

The following corollary is immediate.

39-9. COROLLARY.

1. Any torsion group does not have property $P(p)$

2. If G has property $P(p)$, then G contains an element not divisible by p

3. If G contains an element not divisible by p and every element of $(G)_p$ is divisible by p , then G has property $P(p)$

4. If there is a homomorphism of G onto Z , then G has property $P(p)$ for every prime p

PROOF. (1) and (2) follow from Lemma 39-7. To prove (3) let g be an element of G which is not divisible by p . If for some element $g' \in G$, $p \cdot g' - g \in (G)_p$, then there is an element g'' of (G) such that $p \cdot g' - g = p \cdot g''$ and $g = p(g' - g'')$. Thus g is divisible by p . Finally, let $h: G \rightarrow Z$ be an epimorphism. Let g be an element in $h^{-1}(1)$. If $p \cdot g' - g \in (G)_p$ for some $g' \in G$, then $1 = p \cdot h(g')$ which is impossible. $\square$

39-10. DEFINITION. Let $\mathcal{G}$ be the collection of all groups R , R_p , Z_p and Q_p for every prime p . For a group G we define the *subcollection* $\gamma(G)$ of G as follows:

$R \in \gamma(G)$ if and only if G has an element of infinite order

$R_p \in \gamma(G)$ if and only if G has property $P(p)$

$Z_p \in \gamma(G)$ if and only if $(G)_p$ contains an element not divisible by p

$Q_p \in \gamma(G)$ if and only if $(G)_p \approx \oplus Q_p$

39-11. BOCKSTEIN'S THEOREM I. If X is compact, then $D(X: G) = \text{lub}\{D(X: H): H \in \gamma(G)\}$.

PROOF. First we show that $D(X: G) \leq \text{lub}\{D(X: H): H \in \gamma(G)\}$. Let $D(X: G) = n$. Since $H^n(X, A: G) \neq 0$ for some closed set A , by Lemma 39-4, either $H^n(X, A) \otimes G \neq 0$ or $H^{n+1}(X, A) * G \neq 0$. Thus, in view of Lemma 39-8, one of the conditions (1), (2_p) , (3_p) , (4_p) , or (5_p) for $H = H^n(X, A)$ and (6_p) for $H = H^{n+1}(X, A)$ in Lemma 39-8 holds. If (1) is true, then $R \in \gamma(G)$ and $H^n(X, A)$ contains an element of infinite order. Thus, $H^n(X, A: R) \neq 0$ by Lemma 39-8 and $D(X: R) \geq n$. If (2_p) holds, then $Q_p \in \gamma(G)$ and $H^n(X, A: Q_p) \neq 0$. Thus, $D(X: Q_p) \geq n$. If (3_p) holds, then $R_p \in \gamma(G)$ and hence $D(X: R_p) \geq n$. If (4_p) holds, then $Z_p \in \gamma(G)$ and $D(X: Z_p) \geq n$. If (5_p) is true and (4_p) fails, then G has property $P(p)$ by Corollary 39-9(3) and hence $R_p \in \gamma(G)$. Thus, $H^n(X, A: R_p) \neq 0$ by Lemma 39-8 and Corollary 39-9(2), and $D(X: R_p) \geq n$. If (6_p) is true, then either $Q_p \in \gamma(G)$ or $Z_p \in \gamma(G)$. Thus either $D(X: Q_p) \geq n$ or $D(X: Z_p) \geq n$. This completes the proof of the first assertion.

To complete the proof, we now show that $D(X: G) \geq \text{lub}\{D(X: H): H \in \gamma(G)\}$. If $R \in \gamma(G)$ and $D(X: R) = n$, then $H^n(X, A: R) \neq 0$ for some closed set A of X and hence $H^n(X, A) \otimes R \neq 0$ by Lemma 39-4. Thus, $H^n(X, A)$ contains an element of infinite order and therefore $H^n(X, A: G) \neq 0$ by Lemma 39-8, that is, $D(X: G) \geq n$. If $R_p \in \gamma(G)$ and $D(X: R_p) = n$, then either (1), (3_p) , or (5_p) in Lemma 39-8 holds for $\bar{G} = R_p$ and $H = H^n(X, A)$. Since G has property $P(p)$, either (1), (3_p) , or (5_p) holds for $\bar{G} = G$. Thus, $H^n(X, A: G) \neq 0$. If $Z_p \in \gamma(G)$ and $D(X: Z_p) = n$, then either $H^n(X, A)$ contains an element not divisible by p or $(H^{n+1}(X, A))_p \neq 0$. In any case we have $H^n(X, A: G) \neq 0$ and $D(X: G) \geq n$. Finally, if $Q_p \in \gamma(G)$ and $D(X: Q_p) = n$, then $G \approx \oplus Q_p$ and either (2_p) or (6_p) holds for $\bar{G} = Q_p$. Thus, $H^n(X, A: G) \neq 0$. This completes the proof. $\square$

39-12. THEOREM. If X is compact, then:

1. $D(X: Z) = \text{lub}\{D(X: R): p \text{ a prime}\}$
2. $\max\{D(X: R), D(X: Z_p)\} \leq D(X: R_p)$
3. $D(X: Z_p) - 1 \leq D(X: Q_p) \leq D(X: Z_p)$
4. $D(X: Q_p) \leq \max\{D(X: R), D(X: R_p) - 1\}$
5. $D(X: R_p) \leq \max\{D(X: R), D(X: Q_p) + 1\}$

PROOF. Note that $\gamma(R) = \{R\}$ and $\gamma(Z) = \{R, R_p: p \text{ a prime}\}$. Thus (1) and half of (2) follow from Theorem 39-11. Let $D(X: Z_p) = n$. Since $H^n(X, A: Z_p) \neq 0$ for some closed set A , either (4_p) , (5_p) or (6_p) in Lemma 39-8 holds for $\bar{G} = Z_p$ and $H = H^n(X, A)$ or $H = H^{n+1}(X, A)$. If (4_p) or (5_p) holds, then $H^n(X, A) \otimes R_p \neq 0$, because R_p has property $P(p)$. Thus, $D(X: R_p) \geq n$. If (6_p) holds, then we find $H^n(X, A: Q_p) \neq 0$ and hence, $D(X: Q_p) \geq n$. Since (4) implies $D(X: Q_p) \leq D(X: R_p)$, it is sufficient to prove (3), (4) and (5). Consider the following short exact sequences:

$$0 \rightarrow Z_p \rightarrow Q_p \xrightarrow{i} Q_p \rightarrow 0 \quad (\text{i})$$

$$0 \rightarrow R_p \rightarrow R \rightarrow Q_p \rightarrow 0 \quad (\text{ii})$$

Here the homomorphism i is defined by $i(g) = p \cdot g$ for $g \in Q_p$. The following exact sequences are induced by (i) and (ii) (see Section 39-5):

$$\begin{aligned} \cdots \rightarrow H^m(X, A: Z_p) \rightarrow H^m(X, A: Q_p) \rightarrow H^m(X, A: Q_p) \\ \rightarrow H^{m+1}(X, A: Z_p) \rightarrow \cdots \end{aligned} \quad (\text{iii})$$

$$\begin{aligned} \cdots \rightarrow H^m(X, A: R_p) \rightarrow H^m(X, A: R) \rightarrow H^m(X, A: Q_p) \\ \rightarrow H^{m+1}(X, A: R) \rightarrow \cdots \end{aligned} \quad (\text{iv})$$

Let $D(X: Q_p) = n$. By the exactness of (iii), if $m > n$, then $H^{m+1}(X, A: Z_p) = 0$ for every closed set A of X . Thus $D(X, Z_p) - 1 \leq D(X: Q_p)$. Let $H^n(X, A: Q_p) \neq 0$. Then either (2_p) or (6_p) in Lemma 39-8 holds for, $\bar{G} = Q_p$. If (2_p) holds, then $H^n(X, A)$ has property $P(p)$ and hence $H^n(X, A: Z_p) \neq 0$. If (6_p) holds, then $(H^{n+1}(X, A))_p \neq 0$ and $H^n(X, A: Z_p) \neq 0$. Thus, $D(X: Q_p) \leq D(X: Z_p)$. Finally (4) and (5) follow from the exactness of the sequence (iv). $\square$

39-13. PROBLEM. In the proofs of Theorem 39-11 and Theorem 39-12 the compactness of the space X is essential. Because Lemma 39-4 (universal coefficient theorem) fails for noncompact spaces.[†] Thus the following problem arises: If X is noncompact (for example if X is separable metric), then are Theorems 39-11 or 39-12 still true?

39-14. PROBLEM. Let $\mathcal{H}$ be the collection of the groups Z, R, R_p, Z_p and Q_p for every prime p . Suppose that the numbers $n(G)$ for each G of $\mathcal{H}$ are given such that $n(Z) = k$ and they are subject to the relations in (1)–(5) in Theorem 39-12 (that is, for instance, (1): $n(Z) = \text{lub}\{n(R_p): p \text{ a}$

[†] For instance, see Kodama [6, p. 354].

prime}, (2): $\max\{n(R), n(Z_p)\} \leq n(R_p)$ and so on). Does there exist a k -dimensional compact metric space X such that $n(G) = D(X: G)$ for each $G \in \mathcal{H}$? Kuzuminov [1] gave a partial solution for this problem. In particular, if the number system $\{n(G): G \in H\}$ is given such that $n(Z) = k$, $n(G) \geq [k/2]$ for every G of H and they satisfy relations (1)–(5) in Theorem 39-12, there is a k -dimensional compact metric space X such that $n(G) = D(X: G)$ for every $G \in \mathcal{H}$.

40. Cantor Manifolds

40-1. DEFINITION. A compact space X is called a *Cantor n -manifold with respect to a group G* if $D(X: G) = n$ and, whenever A and B are closed in X such that $A \cup B = X$ and the interiors of A and B are both nonvoid, $D(A \cap B: G) \geq n - 1$. A Cantor manifold with respect to Z is a usual Cantor manifold (see Section 28-5).

40-2. THEOREM. If X is compact and $D(X: G) = n$, then X contains a Cantor n -manifold A with respect to G .

PROOF. Since $D(X: G) = n$, there is a closed set F and a mapping $f: F \rightarrow K(G, n - 1: q)$, $q > \dim X$, such that f is not extendable over X . The proof in Hurewicz and Wallman [1, Theorem VI 8] shows that there is a closed set A such that f is not extendable over $A \cup F$ and if B is a proper closed set of A then f is extendable over $B \cup F$. Suppose that there are proper closed sets A_1 and A_2 of A such that $A = A_1 \cup A_2$ and $D(A_1 \cap A_2: G) < n - 1$. An argument similar to that in the proof of Theorem 38-4 shows that f is extendable over $A \cup F$.

40-3. COROLLARY. If X is compact and $D(X: G) = n$, then there is a point x of X and an open neighborhood U of x such that if V is an open neighborhood of x contained in U then the homomorphism

$$H^n(X, X - V: G) \rightarrow H^n(X, X - U: G)$$

induced by the inclusion $(X, X - U) \subset (X, X - V)$ is nontrivial. Consequently, if X and Y are compact, then $D(X \times Y: G) = \text{lub}\{n: H^n((X, A) \times (Y, B): G) \neq 0 \text{ for some closed sets } A \text{ and } B \text{ of } X \text{ and } Y\}$, where $(X, A) \times (Y, B) = (X \cup Y, X \times B \cup A \times Y)$.

PROOF. In view of the preceding theorem and Theorem 36-5 there are closed sets F and A satisfying the following: (i) there is a nonzero element a of $H^{n-1}(F; G)$ such that, if $j^*: H^{n-1}(A \cup F; G) \rightarrow H^{n-1}(F; G)$ is the homomorphism induced by $j: F \subset A \cup F$, then $a \notin \text{Image } j^*$, and (ii) A is minimal with respect to the property (i), that is, if A' is a proper closed set of A and $j: F \subset A' \cup F$, then $a \in \text{Image } j^*$. Let x be a point of $A - F$ and $U = X - A$. We shall prove that x and U satisfy the conclusion of the corollary. Let V be an open neighborhood of x contained in U and put $F' = X - V$ and $B = A \cup F$. Consider the following commutative diagram.

$$\begin{array}{ccccc}
 H^n(X, F'; G) & \xrightarrow{i_0} & H^n(X, F; G) & & \\
 \swarrow i_5 & & \nearrow \delta_1 & & \downarrow i_3 \\
 & & H^{n-1}(F; G) & & \\
 \nearrow i_2 & & \nwarrow \delta_2 & & \\
 H^{n-1}(F' \cap B; G) & & H^{n-1}(F; G) & & \\
 \nwarrow \delta_3 & & \nearrow i_1 & & \\
 H^n(F' \cup B, F'; G) & \xrightarrow{i_6} & H^n(B, F' \cap B; G) & \xrightarrow{i_1} & H^n(B, F; G)
 \end{array}$$

Here δ indicates the coboundary homomorphism and the other homomorphisms are induced by the inclusions. We have to prove that i_0 is nontrivial. Since $F' \cap B$ is a proper closed set of B , there is $a' \in H^{n-1}(F' \cap B; G)$ such that $i_2 a' = a$. Because $i_1 \delta_3 a' = \delta_2 i_2 a' = \delta_2 a \neq 0$, we know $\delta_3 a' \neq 0$. Since i_6 is an isomorphism by Theorem 37-5 and $H^{n+1}(X, F' \cup B; G) = 0$, $i_4 = i_6 i_5$ is onto by exactness of the cohomology sequence of the triple $(X, F' \cup B, F')$. Thus there is an $a_1 \in H^n(X, F'; G)$ such that $i_4 a_1 = \delta_3 a'$. Then we know $i_0 a_1 \neq 0$, because $i_3 i_0 a_1 = i_1 i_4 a_1 = i_1 \delta_3 a' = \delta_2 a \neq 0$. This completes the proof. $\square$

The remainder of this section is devoted to constructing Cantor manifolds with respect to various groups.

40-4. DEFINITION. Let (a, b) be an ordered pair of integers such that $a > 0$ and $b \geq 0$. Let S be a 1-sphere. By a *Möbius band mod* (a, b) we understand the continuum $M(a, b)$ obtained from the product $S \times I$ by identifying on the circumference $S_0 = S \times \{0\}$ points corresponding to each other under the rotation through angle $2\pi/a$ and by identifying on

the circumference $S_1 = S \times \{1\}$ points corresponding to each other under the rotation through angle $2\pi/b$. Let h be the identification mapping. Put $T_0 = h(S_0)$ and $T_1 = h(S_1)$. We call T_0 the *boundary* of $M(a, b)$. If $b = 0$, then we understand that T_1 is one point. (For example, $M(1, 0)$ is a 2-cell.) In other cases, T_0 and T_1 are both 1-spheres. It is evident that $M(a, b)$ has a simplicial subdivision. We give a fixed triangulation to $M(a, b)$. Let s be a 2-simplex and $\dot{s}$ the boundary of s . There is a mapping $f: (M(a, b), T_0) \rightarrow (s, \dot{s})$ such that $f|_{T_0}$ is an homeomorphism and $f(M(a, b) - T_0) \subset s - \dot{s}$.

40-5. LEMMA.

1. $H^2(M(a, b), T_0) \approx Z_b$, where $Z_0 = Z$. If u is a generator of $H^2(M(a, b), T_0)$, then the homomorphism f^* maps a generator of $H^2(s, \dot{s})$ into $\pm a \cdot u$
2. If $a > 1$ and a is not divisible by b , then the coboundary homomorphism $H^1(T_0) \rightarrow H^2(M(a, b), T_0)$ is nontrivial
3. If σ is a 2-simplex of $M(a, b)$, then T_0 is a retract of $M(a, b) - \text{Interior } \sigma$

A simple proof is omitted.

Let $\alpha = (a_1, a_2, \dots)$ be a sequence of positive integers. Put $M_1 = M(a_1, 0)$. For each $i, i \leq k$, suppose that a simplicial 2-polytope M_i and a mapping $\pi_{i-1}^i: M_i \rightarrow M_{i-1}$ are constructed. Let $\tau_h, h = 1, \dots, n_k$, be all 2-simplexes of M_k . Replace each τ_h by a Möbius band $M_h(a_{k+1}, 0) \bmod (a_{k+1}, 0)$ such that

$$M_h(a_{k+1}, 0) \cap M_{h'}(a_{k+1}, 0) = T_h \cap T_{h'} = \tau_h \cap \tau_{h'} \quad \text{for } h \neq h'$$

where T_h and $T_{h'}$ are the boundaries of $M_h(a_{k+1}, 0)$ and $M_{h'}(a_{k+1}, 0)$. Put $M_{k+1} = \bigcup_{h=1}^{n_k} M_h(a_{k+1}, 0)$. Then M_{k+1} is a simplicial polytope and there is a mapping $\pi_k^{k+1}: M_{k+1} \rightarrow M_k$ such that if Δ_k is the 1-section of M_k then the mapping $\pi_k^{k+1}|(\pi_k^{k+1})^{-1}\Delta_k$ is a homeomorphism. We obtain the inverse limiting system $\{M_k; \pi_{k-1}^k\}$ of 2-polytopes. We denote by $M(\alpha)$ its limit space and by π_k the projection: $M(\alpha) \rightarrow M_k$. It is evident that $\pi_k|(\pi_k)^{-1}\Delta_k$ is a homeomorphism. We call the set $(\pi_1)^{-1}T_0$ the *boundary* of $M(\alpha)$ and denote it by T , where T_0 is the boundary of M_1 .

40-6. LEMMA. $M(\alpha)$ is a Cantor 2-manifold.

PROOF. By Lemma 40-5(1), $H^2(M_1, T_0) \approx \mathbb{Z}$. Let u be a generator of $H^2(M_1, T_0)$. Put $\pi_1^k = \pi_1^2 \cdots \pi_{k-1}^k$ and $T_k = (\pi_1^k)^{-1}T_0 (= \pi_k(T))$. Take an oriented 2-simplex τ of M_k and denote by v_τ the element of $H^2(M_k, T_k)$ which is represented by the cocycle taking 1 on τ and 0 on the other 2-simplex of M_k . Then v_τ is of infinite order. A simple calculation shows $(\pi_1^k)^*u = \pm (\prod_{i=2}^k a_i) \cdot v_\tau$, where $a = (a_1, a_2, \dots)$. Thus we know that $H^2(M(a), T)$ contains an element of infinite order. To prove that $M(a)$ is a Cantor manifold, observe that the mapping $\pi_1 | T: T \rightarrow T_0$ is not extendable over $M(a)$ since $H^1(T) \rightarrow H^2(M(a), T)$ is nontrivial by 40-5(2), the assertion follows from Hopf's extension theorem (cohomology) in Section 36-16. Let A and B be proper closed sets whose union is $M(a)$. Since the open set $M(a) - A$ is not empty, for some k there is a 2-simplex s of M_k such that $s \cap \pi_k(A) = \emptyset$. Since T_k is a retract of the set $M_k - \text{Interior } s$ by Lemma 40-5(3), $\pi_1 | T$ has an extension $f_1: A \cup T \rightarrow T_0$. Similarly, $\pi_1 | T$ has an extension $f_2: B \cup T \rightarrow T_0$. Assume that $\dim(A \cap B) < 1$, then the mappings $f_1 | A \cap B$ and $f_2 | A \cap B$ are homotopic. The homotopy extension theorem shows that $\pi_1 | T$ is extendable over $M(a)$. Thus, $\dim(A \cap B) \geq 1$ and $M(a)$ is a Cantor manifold. $\square$

Let α_0 be a sequence of all integers of the form q^k , where q ranges over all primes and k ranges over all positive integers, and let α_p be the subsequence of α_0 consisting of all integers co-prime with p . We denote $M(\alpha_0)$ and $M(\alpha_p)$ by M_R and M_{R_p} respectively.

40-7. THEOREM. The cohomological dimensions of M_R and M_{R_p} are indicated in the following table:

	R	R_p	Z_p	Q_p	R_q	Z_q	Q_q
M_R	2	2	1	1	2	1	1
M_{R_p}	2	2	2	2	2	1	1

Here q is a prime different from p . Moreover, $D_s(M_R: G) = D_s(M_{R_p}: G) = 2$ for any group G .

PROOF. In view of Theorem 39-12 it is sufficient to prove that (1) $D(M_R: R) = 2$, (2) $D(M_R: Z_r) = 1$ for every prime r , (3) $D(M_{R_p}: Q_p) = 2$ and (4) $D(M_{R_p}: Z_q) = 1$. Assertion (1) follows from the proof of Lemma 40-6, that is, if T is the boundary of M_R , then $H^2(M_R, T)$ contains elements

of infinite order. Assertion (3) is proved similarly, because if T is the boundary of M_{R_p} , then $H^2(M_{R_p}, T)$ contains an element which is of infinite order and not divisible by p and hence $H^2(M_{R_p}, T)$ has property $P(p)$. To prove assertion (2), let $\{M_i: \pi_i^i: \pi_{i-1}^i\}$ be the inverse limiting system used in the construction of M_R . For each 2-simplex σ of M_R , we shall show that the homomorphism $(\pi_k | \pi_k^{-1}\sigma)^*: H^2(\sigma, \dot{\sigma}: Z_r) \rightarrow H^2(\pi_k^{-1}\sigma, \pi_k^{-1}\dot{\sigma}: Z)$ is trivial. Then (2) follows from Theorem 37-12. The sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ which is used in the construction of M_R contains an integer α_l such that $l > k$ and α_l is a power of the prime r . Put $\pi = \pi_k^{k+1} \dots \pi_l^l$, $N = \pi^{-1}\sigma$ and $\dot{N} = \pi^{-1}\dot{\sigma}$. Choose an oriented 2-simplex τ of $N - \dot{N}$ and let v_τ be the element of $H^2(N, \dot{N}: Z_r)$ which is represented by the 2-cocycle taking 1 on τ and 0 on the other 2-simplex of $N - \dot{N}$. The elements v_τ for $\tau \in N - \dot{N}$ generate $H^2(N, \dot{N}: Z_r)$. It is easy to show that the homomorphism $(\pi | N)^*: H^2(\sigma, \dot{\sigma}: Z_r) \rightarrow H^2(N, \dot{N}: Z_r)$ maps a generator of $H^2(\sigma, \dot{\sigma}: Z_r)$ into $\pm (\prod_{i=k+1}^l \alpha_i) \cdot v_\tau$ and hence $(\pi | N)^*$ is trivial. Assertion (4) is proved similarly. Finally, to prove $D_s(M_R: G) = 2$, let A be a proper closed set of M_R and U an open set containing A . There is an integer k such that $\pi_k(M_R - U)$ and $\pi_k(A)$ are disjoint and each simplex of M_k which intersects $\pi_k(A)$ is contained in $M_k - \pi_k(M_R - U)$. Let N be a subcomplex of M_k consisting of all simplexes intersecting $\pi_k(A)$ and put $V = \text{Interior } N$ and $W = \pi_k^{-1}V$. Then we have $A \subset W \subset \bar{W} \subset U$. Since $\bar{W} - V$ is a 1-dimensional subcomplex of M_k , $\pi_k | \bar{W} - W: \bar{W} - W \rightarrow \bar{V} - V$ is a homeomorphism. Thus, $D(\bar{W} - W: G) = 1$. This completes the proof. $\square$

Next, let $b = (b_1, b_2, \dots)$ be a sequence of positive integers such that $b_i > 1$ and b_{i+1} is divisible by b_i for $i = 1, 2, \dots$. Put $c_i = b_i/b_{i-1}$, $i = 2, 3, \dots$. Let M_1 be a Möbius band $M(1, b_1) \bmod (1, b_1)$. For $i \leq k$, suppose that a 2-polytope M_i and a mapping $\pi_{i-1}^i: M_i \rightarrow M_{i-1}$ are constructed. Let τ_h , $h = 1, \dots, n_k$, be all 2-simplexes of M_k . We replace every τ_h by a Möbius band $M_h(c_{k+1}, b_{k+1}) \bmod (c_{k+1}, b_{k+1})$ such that

$$M_h(c_{k+1}, b_{k+1}) \cap M_{h'}(c_{k+1}, b_{k+1}) = T_h \cap T_{h'} = \tau_h \cap \tau_{h'}$$

where T_h and $T_{h'}$ are the boundaries of $M_h(c_{k+1}, b_{k+1})$ and $M_{h'}(c_{k+1}, b_{k+1})$. Put $M_{k+1} = \bigcup_{h=1}^{n_k} M_h(c_{k+1}, b_{k+1})$. Then M_{k+1} is a simplicial polytope and there is a mapping $\pi_k^{k+1}: M_{k+1} \rightarrow M_k$ such that if Δ_k is the 1-section of M_k then the mapping $\pi_k^{k+1} | (\pi_k^{k+1})^{-1}\Delta_k$ is a homeomorphism. We obtain the inverse limiting system $\{M_k: \pi_k^k\}$ of 2-polytopes. We denote by $M(b)$ its limit space. We call the set $(\pi_1)^{-1}T_0$ the *boundary*

of $M(b)$, where $\pi_k: M(b) \rightarrow M_k$, $k = 1, 2, \dots$, is the projection and T_0 is the boundary of M_1 .

Let p be a prime > 1 and let $b_1 = (p, p, \dots, p, \dots)$ and $b_p = (p, p^2, \dots, p^{2^l}, \dots)$. We denote $M(b_1)$ and $M(b_p)$ by M_{Z_p} and M_{Q_p} respectively.†

40-8. THEOREM. The continua M_{Z_p} and M_{Q_p} are Cantor 2-manifolds. The cohomological dimensions of M_{Z_p} and M_{Q_p} are indicated in the following table:

	R	R_p	Z_p	Q_p	R_q	Z_q	Q_q
M_{Z_p}	1	2	2	1	1	1	1
M_{Q_p}	1	2	1	1	1	1	1

Here q is a prime different from p . Moreover, $D_s(M_{Z_p}; G) = D_s(M_{Q_p}; G) = 2$ for any group G .

PROOF. We shall prove the assertion of the theorem for M_{Q_p} . The assertion for M_{Z_p} is proved similarly. Let $\{M_k: \pi_k^{-1}\}$ be the inverse limiting system used in the construction of M_{Q_p} . Let σ be a 2-simplex of M and put $(N, \dot{N}) = (\pi^{-1}\sigma, \pi^{-1}\dot{\sigma})$, where $\pi = \pi_{k+1}^{k+1} \dots \pi_{l-1}^l$, $l > k$. Take an oriented 2-simplex τ of $N - \dot{N}$ and denote by v_τ the element of $H^2(N, \dot{N})$ which is represented by the cocycle taking 1 on τ and 0 on the other 2-simplexes of $N - \dot{N}$. Then the element v_τ , $\tau \in N - \dot{N}$, generate the group $H^2(N, \dot{N})$. Also v_τ is of order $p^{2^{l-1}}$ (observe that the l -th integer in the sequence b_p used in the construction of M_{Q_p} is $p^{2^{l-1}}$). Put $n_l = 2^{k-1} + 2^k + \dots + 2^{l-2}$. A simple calculation shows that the homomorphism $(\pi | N)^*: H^2(\sigma, \dot{\sigma}) \rightarrow H^2(N, \dot{N})$ maps a generator of $H^2(\sigma, \dot{\sigma})$ into $\pm p^{n_l} \cdot v_\tau$. Also we know that the element $p^{n_l} \cdot v_\tau$ is of order $p^{2^{k-1}}$. Put $(L, \dot{L}) = (\pi_k^{-1}\sigma, \pi_k^{-1}\dot{\sigma})$, where π_k is the projection: $M_{Q_p} \rightarrow M_k$. These considerations show that (i) the order of every element of $H^2(L, \dot{L})$ is a power of the prime p and is divisible by p and (ii) $H^2(L, \dot{L})$ contains an element of order $p^{2^{k-1}}$. Thus, if $H^2(L, \dot{L}; G) \neq 0$, then the group G must have property $P(p)$. That M_{Q_p} is a Cantor manifold and $D_s(M_{Q_p}; G) = 2$ for any group G is proved by an argument similar to the proof of Theorem 40-7. □

† M_{Z_p} was given by Pontryagin [1]. M_R , M_{R_p} and M_{Q_p} are due to Boltyanskii [1] and Kodama [2], [3]. Kuzuminov [1] constructed other interesting examples with similar properties.

41. Bockstein's Theorem II

41-1. LEMMA. Let (X, A) and (Y, B) be pairs of compact spaces, then the following sequence is exact:

$$\begin{aligned}
 0 \rightarrow \sum_{i+j=n} H^i(X, A) \otimes H^j(Y, B) &\rightarrow H^n((X, A) \times (Y, B)) \\
 &\rightarrow \sum_{i+j=n+1} H^i(X, A) * H^j(Y, B) \rightarrow 0
 \end{aligned}
 \tag{1}$$

Furthermore, if F is a field, then

$$\sum_{i+j=n} H^i(X, A; F) \otimes H^j(Y, B; F) \approx H^n((X, A) \times (Y, B); F)$$

Here $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$.

Sequence (1) is called the *Künneth sequence* of $(X, A) \times (Y, B)$ with dimension n .

PROOF. If (X, A) and (Y, B) are pairs of finite polytopes, then the lemma is well known (see Hilton and Wylie [1, 5.7.12c and 5.7.25c]). Also, in this case, sequence (1) is natural with respect to a mapping $\pi \times \pi': (X, A) \times (Y, B) \rightarrow (X', A') \times (Y', B')$, where $\pi: (X, A) \rightarrow (X', A')$ and $\pi': (Y, B) \rightarrow (Y', B')$. Thus the lemma is proved as in the proof of Lemma 39-4. $\square$

41-2. LEMMA. If $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$ is exact, then:

1. If G has property $P(p)$ and G'' is a torsion group, then G' has property $P(p)$
2. If G'' is a torsion group, then the homomorphism: $G' \otimes Q_p \rightarrow G \otimes Q_p$ induced by the homomorphism: $G' \rightarrow G$ is onto
3. If $(G)_p = 0$ and $(G'')_p \neq 0$, then $G' \otimes Z_p \neq 0$

PROOF. We prove only part (3). By Lemma 39-3, the sequence

$$0 \rightarrow G' * Z_p \rightarrow G * Z_p \rightarrow G'' * Z_p \rightarrow G' \otimes Z_p \rightarrow G \otimes Z_p \rightarrow G'' \otimes Z_p \rightarrow 0$$

is exact. By Lemma 39-8 we have $G * Z_p = 0$ and $G'' * Z_p \neq 0$. Thus, $G' \otimes Z_p \neq 0$. $\square$

41-3. BOCKSTEIN'S THEOREM II. If X and Y are compact, then:

1. If F is a field, then $D(X \times Y: F) = D(X: F) + D(Y: F)$
2. $D(X \times Y: Q_p) = \max\{D(X: Q_p) + D(Y: Q_p), D(X \times Y: Z_p) - 1\}$
3. $D(X \times Y: R_p) = \min\{\max\{D(X \times Y: R), D(X \times Y: Z_p), D(X: Q_p) + D(Y: Q_p) + 1\}, \max\{D(X \times Y: R), D(X \times Y: Z_p), D(X: Q_p) + D(Y: R_p), D(X: R_p) + D(Y: Q_p)\}\}$

PROOF. (1) This is a consequence of Corollary 40-3 and Lemma 41-1.

(2) To prove that $D(X \times Y: Q_p) \leq \max\{\}$: Let $D(X \times Y: Q_p) = n$. There are closed sets A and B of X and Y such that $H^n((X, A) \times (Y, B): Q_p) \neq 0$. In view of Lemma 39-4 and Lemma 39-8 either (a) $H^n((X, A) \times (Y, B))$ has property $P(p)$ or (b) $(H^{n+1}((X, A) \times (Y, B)))_p \neq 0$. If (a) is true, then Lemma 41-2(2) is applied to the relative Künneth sequence of $(X, A) \times (Y, B)$ with dimension n . Since $H^n((X, A) \times (Y, B)) \otimes Q_p \neq 0$, $(H^i(X, A) \otimes H^j(Y, B)) \otimes Q_p \neq 0$ for some i and j , $i + j = n$. Thus, both $H^i(X, A)$ and $H^j(Y, B)$ have property $P(p)$ and hence $H^i(X, A: Q_p) \neq 0 \neq H^j(Y, B: Q_p)$. Therefore $D(X: Q_p) + D(Y: Q_p) \geq i + j = n$. If (b) holds, then either

$$\sum_{i+j=n+1} (H^2(X, A) \otimes H^j(Y, B))_p \neq 0$$

or

$$\sum_{i+j=n+2} (H^i(X, A) * H^j(Y, B))_p \neq 0$$

by Lemma 39-4. In view of Lemma 39-8, for some integers i and j , $i + j = n$, one of the following holds:

- i. $(H^{i+1}(X, A))_p \approx \oplus Q_p$ and $H^j(Y, B)$ has property $P(p)$
- ii. $H^{i+1}(X, A)$ has property $P(p)$ and $(H^j(Y, B))_p \approx \oplus Q_p$
- iii. $(H^{i+1}(X, A))_p$ and $H^j(Y, B)$ have elements not divisible by p
- iv. $H^{i+1}(X, A)$ and $(H^j(Y, B))_p$ have elements not divisible by p
- v. $(H^{i+1}(X, A))_p \neq 0 \neq (H^{j+1}(Y, B))_p$

If (i) holds, then $H^i(X, A: Q_p) \neq 0 \neq H^j(Y, B: Q_p)$ and hence $D(X: Q_p) + D(Y: Q_p) \geq i + j = n$. Similarly, if either (ii) or (iii) holds, then $D(X: Q_p) + D(Y: Q_p) \geq n$. If (iii) holds, then $H^{i+1}(X, A: Z_p) \neq 0 \neq H^j(Y, B: Z_p)$ and hence $D(X: Z_p) + D(Y: Z_p) - 1 \geq n$ by Lemma 39-8. Similarly, if (iv) holds, then $D(X: Z_p) + D(Y: Z_p) - 1 \geq n$. Thus, we know that $D(X \times Y: Q_p) \leq \max\{\}$.

(2a) To prove that $D(X \times Y: Q_p) \geq \max\{i, j\}$: Let $D(X: Q_p) = i$ and $D(Y: Q_p) = j$, $i + j = n$. Then for some closed sets A and B of X and Y one of the following holds: (i) $H^i(X, A)$ and $H^j(Y, B)$ both have property $P(p)$. (ii) $H^i(X, A)$ has property $P(p)$ and $(H^{j+1}(Y, B))_p \neq 0$. (iii) $(H^{i+1}(X, A))_p \neq 0$ and $H^j(Y, B)$ has property $P(p)$. (iv) $(H^{i+1}(X, A))_p \neq 0 \neq (H^{j+1}(Y, B))_p$. If (ii) holds, then $(H^i(X, A) \otimes H^{j+1}(Y, B))_p \neq 0$ and hence $D(X \times Y: Q_p) \geq n$. Similarly, if (iii) holds, then $D(X \times Y: Q_p) \geq n$. If (i) holds, and (ii) and (iii) are false, then $(H^s(X, A) * H^t(Y, B))_p = 0$ for each s and t , $s + t = n + 1$. By tensoring the relative Künneth sequence of $(X, A) \times (Y, B)$ with dimension n by Q_p (compare Lemma 41-2) we know

$$\sum_{s+t=n} H^s(X, A) \otimes H^t(Y, B) \otimes Q_p \approx H^n((X, A) \times (Y, B)) \otimes Q_p$$

Since $H^i(X, A)$ and $H^j(Y, B)$ both have property $P(p)$, $H^i(X, A) \otimes H^j(Y, B) \otimes Q_p \neq 0$ by Lemma 39-7 and hence $H^n((X, A) \times (Y, B))$ has property $P(p)$. Therefore, $D(X \times Y: Q_p) \geq n$. If (iv) holds, then $(H^{i+1}(X, A) * H^{j+1}(Y, B))_p \neq 0$. If $(H^{n+1}((X, A) \times (Y, B)))_p \neq 0$, then $D(X \times Y: Q_p) \geq n$. If $(H^{n+1}((X, A) \times (Y, B)))_p = 0$, then Theorem 40-2(3) is applied to the relative Künneth sequence of $(X, A) \times (Y, B)$ with dimension $n + 1$. Thus we know that

$$\sum_{s+t=n+1} H^s(X, A) \otimes H^t(Y, B) \otimes Z_p \neq 0$$

and hence for some s and t , $s + t = n + 1$, $H^s(X, A: Z_p) \neq 0 \neq H^t(Y, B: Z_p)$. This implies that $D(X, Z_p) + D(Y, Z_p) \geq s + t = n + 1$. This completes the proof of assertion (2).

(3) To prove that $D(X \times Y: R_p) \leq \min\{\max\{i, j\}, \max\{i, j\} + 1\}$: Since $D(X \times Y: R_p) \leq \max\{D(X \times Y: R), D(X \times Y: Q_p) + 1\}$ by Theorem 39-12(5), in view of (2), we know

$$D(X \times Y: R_p) \leq \max\{D(X \times Y: R), D(X \times Y: Z_p), D(X: Q_p) + D(Y: Q_p) + 1\}$$

Thus it is enough to prove

$$D(X \times Y: R_p) \leq \max\{D(X \times Y: R), D(X \times Y: Z_p), D(X: Q_p) + D(Y: R_p), D(X: R_p) + D(Y: Q_p)\}$$

Let $D(X \times Y: R_p) = n$. For some closed sets A and B of X and Y either (i) $H^n((X, A) \times (Y, B))$ contains an element of infinite order, or (ii)

$(H^n((X, A) \times (Y, B)))_p \neq 0$. If (i) holds, then $D(X \times Y: R) \geq n$. If (ii) holds, then either

$$\sum_{i+j=n} (H^i(X, A) \otimes H^j(Y, B))_p \neq 0$$

or

$$\sum_{i+j=n+1} (H^i(X, A) * H^j(Y, B))_p \neq 0$$

Hence, one of the followings holds for some i and j , $i+j=n$: (a) $(H^i(X, A))_p \approx \oplus Q_p$ and $H^j(Y, B)$ has property $P(p)$. (b) $H^i(X, A)$ has property $P(p)$ and $(H^j(Y, B))_p \approx \oplus Q_p$. (c) $(H^i(X, A))_p$ and $H^j(Y, B)$ both contain elements not divisible by p . (d) $H^i(X, A)$ and $(H^j(Y, B))_p$ both contain elements not divisible by p . (e) $(H^{i+1}(X, A))_p \neq 0 \neq (H^{j+1}(Y, B))_p$. If (a) or (b) holds, then $D(X: R_p) + D(Y: Q_p) \geq n$ or $D(X: Q_p) + D(Y: R_p) \geq n$. If (c) or (d) holds, then $D(X \times Y: Z_p) \geq n$. If (e) holds, then $D(X: Q_p) + D(Y: R_p) \geq n$ and $D(X: R_p) + D(Y: Q_p) \geq n$.

(3a) To prove that $D(X \times Y: R_p) \geq \min\{\max\{ \}, \max\{ \}\}$: Assume that $D(X \times Y: R_p) < \min\{\max\{ \}, \max\{ \}\}$. We shall prove that this assumption leads to a contradiction. Since $D(X \times Y: R_p) \geq \max\{D(X \times Y: R), D(X \times Y: Z_p)\}$ by Theorem 39-12(2), the assumption implies: (i) $D(X \times Y: R_p) < D(X: Q_p) + D(Y: Q_p) + 1$ and (ii) $D(X \times Y: R_p) < \max\{D(X: Q_p) + D(Y: R_p), D(X: R_p) + D(Y: Q_p)\}$. Let $i = D(X: Q_p)$, $i' = D(X: R_p)$, $j = D(Y: Q_p)$, $j' = D(Y: R_p)$ and $n = D(X \times Y: R_p)$. Then (i) states that $n < i + j + 1$. For some closed sets A and B of X and Y one of the following holds: (a) $(H^{i+1}(X, A))_p \neq 0 \neq (H^{j+1}(Y, B))_p$. (b) $(H^{i+1}(X, A))_p \neq 0$ and $H^j(Y, B)$ has property $P(p)$. (c) $H^i(X, A)$ has property $P(p)$ and $(H^{j+1}(Y, B))_p \neq 0$. (d) Both $H^i(X, A)$ and $H^j(Y, B)$ have property $P(p)$. If (a) holds, then $(H^{i+1}(X, A) * H^{j+1}(Y, B))_p \neq 0$ and hence $H^{i+j+1}((X, A) \times (Y, B))$ contains an element of infinite order or an element of order p by the exactness of the relative Künneth sequence of $(X, A) \times (Y, B)$ with dimension $i + j + 1$. In each case we have $n = D(X \times Y: R_p) \geq i + j + 1 > n$. Similarly, (b) and (c) lead a contradiction. If (b) and (c) fail and (d) is true, then $H^{i+j}((X, A) \times (Y, B))$ has property $P(p)$ (compare the second part of the proof of (2)) and hence $n = D(X \times Y: R_p) \geq i + j > n - 1$. Thus the statement (i) implies that $n = i + j$ and (d) is true. Note that (ii) implies $i + j' > i + j$ or $i' + j > i + j$. If the first statement is true, then for some closed set B' of Y either $(H^{j'}(Y, B'))_p \neq 0$ or $H^{j'}(Y, B')$ contains an element of infinite order. Since $H^i(X, A)$ has property $P(p)$, $H^{i+j'}((X, A) \times (Y, B'))$ either contains an element of infinite order or an element of order p . In each case we have

$i + j = n = D(X \times Y: R_p) \geq i + j' > i + j$. The second statement similarly leads to a contradiction. $\square$

The following theorems follow from Bockstein's theorem II.

41-4. THEOREM. If X and Y are compact, then $\dim(X \times Y) = \dim X + \dim Y$ if and only if

1. $\dim X = D(X: R)$ and $\dim Y = D(Y: R)$
- or, for some prime p :
2. $\dim X = D(X: Z_p)$ and $\dim Y = D(Y: Z_p)$
 3. $\dim X = D(X: R_p)$ and $\dim Y = D(Y: Q_p)$
- or
4. $\dim X = D(X: Q_p)$ and $\dim Y = D(Y: R_p)$

41-5. THEOREM. If X is compact, then $\dim(X \times X) \geq 2 \cdot \dim X - 1$.

41-6. PROBLEM. Does the relation in Bockstein's theorem II hold for noncompact spaces? Let X_0 be the set of all rational points in Hilbert space. It is shown in Hurewicz and Wallman [1, Example II, 11] that $\dim(X_0 \times X_0) = \dim X_0 = 1$. Hence, $D(X_0 \times X_0: G) = D(X_0: G) = 1$ for any group G . Therefore, in a generalized Bockstein's theorem, at least one of X and Y has to be compact. Finally, put $X = X_0 \times M_{Q_p}$. Since $\dim(M_{Q_p} \times M_{Q_p}) = 3$, we have $\dim(X \times X) = 4$. On the other hand, $\dim X = 3$. Thus, Theorem 41-5 does not generally hold for separable metric spaces. Note that the space X is not complete. We do not know whether or not Theorem 41-5 holds for complete separable metric spaces.

In the remainder of this section we discuss some properties of the Cantor manifolds M_R , M_{R_p} , M_{Z_p} and M_{Q_p} constructed in Section 40. The following is a consequence of Theorem 40-7, Theorem 40-8, and Bockstein's Theorem II (41-3).

41-7. THEOREM. Let X be a compact space with covering dimension n , then:

1. $D(X: R) = n$ if and only if $\dim(X \times M_R) = n + 2$
2. $D(X: R_p) = n$ if and only if $\dim(X \times M_{R_p}) = n + 2$
3. $D(X: Z_p) = n$ if and only if $\dim(X \times M_{Z_p}) = n + 2$
4. $D(X: Q_p) = n$ if and only if $\dim(X \times M_{Q_p}) = n + 2$

41-8. THEOREM.[†] Let f be a mapping of a compact metric space X onto a space Y . Let G be one of the groups R, Z_p and R_p for every prime p , then:

1. If $\text{ord } f \leq n$, then $D(X: G) \leq D(Y: G) + n$
2. If f is open and has finite order, then $D(X: G) = D(Y: G)$

We need the following lemma which is a consequence of Bockstein's Theorem II.

41-9. LEMMA.[‡] If X is compact and G is one of the groups R, R_p and Z_p, p a prime, then $D(X: G) + 2 = D(X \times M_G: G)$.

PROOF OF THEOREM 41-8. We shall prove only part (2) that is, in case $G = R_p$. The other parts are treated similarly. Take an integer k such that

$$k \geq \max\{\dim X - D(X: R_p), \dim Y - D(Y: R_p)\}$$

and denote by $(M_{R_p})^k$ the k -fold product $M_{R_p} \times \cdots \times M_{R_p}$. Since the mapping $f \times 1: X \times (M_{R_p})^k \rightarrow Y \times (M_{R_p})^k$ is open and has finite order, we know that $\dim(X \times (M_{R_p})^k) = \dim(Y \times (M_{R_p})^k)$. We shall prove that $\dim(X \times (M_{R_p})^k) = D(X: R_p) + 2k$ and $\dim(Y \times (M_{R_p})^k) = D(Y: R_p) + 2k$. Then the theorem is immediate. Let $D(X: R_p) = m$. Then Lemma 41-9 implies $D(X \times (M_{R_p})^k) = m + 2k$ and hence $\dim(X \times (M_{R_p})^k) \geq m + 2k$. Assume that $\dim(X \times (M_{R_p})^k) = r > m + 2k$. Then either $\dim(X \times (M_{R_p})^{k-1}) = r - 2$ or $r - 1$. If $\dim(X \times (M_{R_p})^{k-1}) = r - 2$, then $D(X \times (M_{R_p})^{k-1}: R_p) = r - 2$ by Theorem 41-7(2) and hence, $D(X: R_p) = r - 2k > m$ by Lemma 41-9. Thus, $\dim(X \times (M_{R_p})^{k-1}) = r - 1$ has to be true. By the same argument we have $\dim(X \times (M_{R_p})^{k-j}) = r - j$, $j = 1, \dots, k$. But this means that $\dim X = r - k > m + k \geq \dim X$. Thus, $\dim(X \times (M_{R_p})^k) = m + 2k$. Similarly, $\dim(Y \times (M_{R_p})^k) = D(Y: R_p) + 2k$. This completes the proof. $\square$

42. Products of Compact Metric Spaces and Paracompact Spaces

42-1. THEOREM. If X is a compact metric space and Y is a paracompact space, then $D(X \times Y: G) = \text{lub}\{n: H^n((X, A) \times (Y, B): G) \neq 0 \text{ for some closed sets } A \text{ and } B \text{ of } X \text{ and } Y\}$.

[†] This is due to Kodama [4]. Kuzuminov [1] proved this theorem for any group G .

[‡] This theorem is true in case X is paracompact. See Kodama [9]. Hence, Theorem 41-8 holds if X and Y are metrizable and f is closed.

This section is devoted to proving this theorem. If Y is compact, then the theorem is a consequence of Corollary 40-3. But, in that proof, the compactness of both X and Y is essential.

PROOF OF 42-1. First, we prove assertion (A):

$D(X \times Y; G) = \text{lub}\{n: H^n((A_1, A_2) \times (B_1, B_2); G) \neq 0 \text{ for some closed sets } A_2 \subset A_1 \subset X \text{ and } B_2 \subset B_1 \subset Y\}$.

To prove (A), it is sufficient to show that, whenever $m > n$ and $H^m((A_1, A_2) \times (B_1, B_2); G) = 0$ for any closed sets $A_2 \subset A_1 \subset X$ and $B_2 \subset B_1 \subset Y$, $D(X \times Y; G) \leq n$. For this purpose, if $\mathcal{U}$ is an open covering of $X \times Y$, then a CW cell complex P and a mapping $f: X \times Y \rightarrow P$ satisfying the following conditions are constructed: For each cell σ of P , (i) $f^{-1}\sigma$ is contained in some member of $\mathcal{U}$ and (ii) the pair $(f^{-1}\sigma, f^{-1}\bar{\sigma})$ has the form $(A_1, A_2) \times (B_1, B_2)$ for closed sets $A_2 \subset A_1 \subset X$ and $B_2 \subset B_1 \subset Y$. Then assertion (A) follows from Theorem 37-12. The construction is divided into four steps.

STEP 1. Since X is a compact metric space, there is an inverse limiting system $\{M_i; \pi_{i-1}^i\}$ of finite simplicial polytopes such that (i) $\dim M_i \leq \dim X$, $i = 1, 2, \dots$, (ii) the projection $\pi_{i+1}^i: M_{i+1} \rightarrow M_i$ is linear on each simplex of M_{i+1} and (iii) $\text{inv lim}\{M_i\}$ is homeomorphic to X . This fact follows from the theorems of Section 27. Denote by $\pi_i^j: M_j \rightarrow M_i$, $j > i$, the composition of π_k^{k+1} , $k = i, \dots, j-1$, and by μ_i the projection: $X \rightarrow M_i$. We have $\mu_i = \pi_i^j \mu_j$ for $j > i$. Let $\mathcal{U}_i$, $i = 1, 2, \dots$, be the open covering of X consisting of the inverse images of all open stars of M_i under μ_i . We may assume without loss of generality that $\{\mathcal{U}_i; i = 1, 2, \dots\}$ forms a cofinal system of open coverings of X .

STEP 2. Let $\mathcal{U}$ be a given open covering of $X \times Y$. There exists an lfc covering $\mathcal{V} = \{W_\alpha \mid \alpha \in \Omega\}$ of Y (see Section 34-9) with order $\leq \dim Y + 1$ subject to the following conditions:

i. For each $\alpha \in \Omega$, there is an open covering $\mathcal{U}_{i(\alpha)}$ of X such that the collection $\mathcal{V} = \{U \times W_\alpha: U \in \mathcal{U}_{i(\alpha)} \text{ and } \alpha \in \Omega\}$ is a locally finite open covering of $X \times Y$ which is a star refinement of $\mathcal{U}$. (See Step 1 for $U_{i(\alpha)}$.)

ii. If $\Omega_\alpha = \{\beta: W_\alpha \cap W_\beta \neq \emptyset \text{ and } \beta \in \Omega\}$, then $\text{lub}\{i(\beta): \beta \in \Omega_\alpha\} < \infty$ for each $\alpha \in \Omega$.

To show this, take a locally finite open covering $\mathcal{W}'$ of Y satisfying condition (i) and let $\mathcal{W}$ be an lfc covering of Y such that $\mathcal{W}$ is a refinement of $\mathcal{W}'$ and each member of $\mathcal{W}$ intersects only a finite number of members of $\mathcal{W}'$; then it is evident that $\mathcal{W}$ satisfies conditions (i) and (ii).

STEP 3. Let N be the nerve of $\mathcal{W}$ with the weak topology. Denote by w_α the vertex of N corresponding to a member W_α of $\mathcal{W}$. Let T^0 be the topological sum of the sets $M_{i(\alpha)} \times w_\alpha$, $\alpha \in \Omega$. Suppose that T^l is constructed for $0 \leq l < j$. For a j -simplex σ of N , put $i(\sigma) = \max\{i(\mu) : \mu \text{ is a } (j-1)\text{-face of } \sigma\}$. Let T^j be the topological sum of the sets $M_{i(\sigma)} \times \sigma$, where σ ranges over all j -simplexes of N . For a 1-simplex $s = (w_\alpha, w_\beta)$ of N , letting $i(s) = \max\{i(\alpha), i(\beta)\}$, the projections $\pi_{i(\alpha)}^{i(s)}$ and $\pi_{i(\beta)}^{i(s)}$ induce the mapping g_s of the subset $M_{i(s)} \times (w_\alpha \cup w_\beta)$ of T^1 into T^0 . If we identify the corresponding points of T^1 and T^0 under these mappings g_s , s a 1-simplex of N , then we obtain a set P_1 . Denote by g_1 the identification mapping: $T^0 \cup T^1 \rightarrow P_1$. Since the projection π_i^j , $i < j$, is linear on each simplex of M_j , it is evident that P_1 is a CW cell complex (see Definition 37-11). The closure finiteness of P_1 is guaranteed by condition (ii), which is satisfied by the covering $\mathcal{W}$ (see Step 2). (Note that $g_1 \mid T^0$ is the inclusion mapping and a subdivision of T^0 is considered as a subcomplex of P_1 .) For each k , $0 \leq k < j$, suppose that a CW cell complex P_k and a mapping $g_k: \cup_{i=0}^k T^i \rightarrow P_k$ are constructed such that (i) P_{k-1} is a subset of P_k , (ii) $g_{k-1} = g_k \mid \cup_{i=0}^{k-1} T^i$ and (iii) if σ is a k -simplex of N and μ is an l -face of σ , then $g_k(x, y) = g_l(\pi_{i(\mu)}^{i(\sigma)}(x), y)$ for each point $(x, y) \in M_{i(\sigma)} \times \mu \subset T^k$. Put $S_j = \cup \{M_{i(\sigma)} \times \dot{\sigma} : \sigma \text{ is a } j\text{-simplex of } N\}$, where $\dot{\sigma}$ is the boundary of σ . Then S_j is a subcomplex of T^j . Define a mapping $h: S_j \rightarrow P_{j-1}$ by $h(x, y) = g_{j-1}(\pi_{i(\mu)}^{i(\sigma)}(x), y)$ for $x \in M_{i(\sigma)}$ and $y \in \mu$, where μ is a $(j-1)$ -face of σ . By identifying the corresponding points of $S_j \subset T^j$ and P_{j-1} under the mapping h , we obtain a CW cell complex P_j . Let $g': T^j \rightarrow P_j$ be the identification mapping. Define a mapping $g_j: \cup_{i=0}^j T^i \rightarrow P_j$ by $g_j \mid T^j = g'$ and $g_j \mid \cup_{i=0}^{j-1} T^i = g_{j-1}$. It is evident that P_j and g_j satisfy conditions (i) and (ii) for $k = j$. To prove that (iii) is satisfied, let σ be a j -simplex of N and τ an l -face of σ . We have to prove $g_j(x, y) = g_l(\pi_{i(\tau)}^{i(\sigma)}(x), y)$ for $x \in M_{i(\sigma)}$ and $y \in \tau$. If μ is a $(j-1)$ -face of σ containing τ , then $g_j(x, y) = g_{j-1}(\pi_{i(\mu)}^{i(\sigma)}(x), y) = g_l(\pi_{i(\tau)}^{i(\mu)} \pi_{i(\mu)}^{i(\sigma)}(x), y) = g_l(\pi_{i(\tau)}^{i(\sigma)}(x), y)$. Thus we obtain a CW cell complex P_j for each j satisfying (i), (ii), and (iii). We denote by P the CW cell complex P_j for $j = \dim Y$ and by g the identification mapping: $\cup_i T^i \rightarrow P$.

STEP 4. Let φ be a canonical mapping of Y into N . Put $B_\sigma = \varphi^{-1}(\sigma)$ for each simplex σ of N . Define a mapping $\bar{g}_\sigma: X \times B_\sigma \rightarrow T^j$ by $\bar{g}_\sigma(x, y)$

$= (\mu_{i(\sigma)}(x), \varphi(y))$ for $x \in X$ and $y \in B$, where $\dim \sigma = j$. (See Step 3 for T^j and see Step 1 for $\mu_{i(\sigma)}$.) The mapping $\bar{g}_\sigma$ is compatible with the identification mapping $g: \cup_i T^i \rightarrow P$ in the following sense: If σ is a j -simplex of N and μ is a k -face of σ , then $g(\bar{g}_\sigma(x, y)) = g(\bar{g}_\mu(x, y))$ for $(x, y) \in X \times B_\mu$. This holds because $g(\bar{g}_\sigma(x, y)) = g(\mu_{i(\sigma)}(x), \varphi(y)) = g_j(\mu_{i(\sigma)}(x), \varphi(y)) = g_k(\pi_{i(\mu)}^{i(\sigma)} \mu_{i(\sigma)}(x), \varphi(y)) = g_k(\mu_{i(\mu)}(x), \varphi(y)) = g(\bar{g}_\mu(x, y))$. Define $f: X \times Y \rightarrow P$ by $f(x, y) = g(\bar{g}_\sigma(x, y))$ for $(x, y) \in X \times B_\sigma$, $\sigma \in N$. It is easy to show that f is continuous. Moreover, the mapping f has the following property: For each cell τ of P there are simplexes σ and $v_{i(\sigma)}$ of N and $M_{i(\sigma)}$ such that

$$f^{-1}\tau = \mu_{i(\sigma)}^{-1}(v_{i(\sigma)}) \times \varphi^{-1}(\sigma)$$

$$f^{-1}\dot{\tau} = (\mu_{i(\sigma)}^{-1}(v_{i(\sigma)}), \mu_{i(\sigma)}^{-1}(\dot{v}_{i(\sigma)})) \times (\varphi^{-1}(\sigma), \varphi^{-1}(\dot{\sigma}))$$

Finally, it is evident that for each cell τ of P , $f^{-1}(\tau)$ is contained in some member of $\mathcal{U}$. Thus, assertion (A) has been proved.

The proof of the theorem is now straightforward. Let $D(X \times Y: G) = n$. By assertion (A) there are closed sets $A \subset A_1 \subset X$ and $B \subset B_1 \subset Y$ such that $H^n(A_1, A) \times (B_1, B): G \neq 0$. Put

$$F = A_1 \times B \cup A \times B_1$$

$$F_1 = X \times B \cup A \times Y$$

$$F_2 = A_1 \times B_1 \cup X \times B \cup A \times Y$$

Since $(A_1 \times B_1) \cap F_1 = F$ and the inclusion mapping: $(A_1 \times B_1, F) \subset (F_2, F_1)$ is closed, we know $H^n(F_2, F_1: G) \approx H^n(A_1 \times B_1, F: G)$ by Theorem 37-5. Consider the cohomology sequence of the triple $(X \times Y, F_2, F_1)$:

$$\cdots \rightarrow H^n(X \times Y, F_1: G) \rightarrow H^n(F_2, F_1: G) \rightarrow H^{n+1}(X \times Y, F_2: G) \rightarrow \cdots$$

Since $D(X \times Y: G) = n$, $H^{n+1}(X \times Y, F_2: G) = 0$ and hence, $H^n(X \times Y, F_1: G) \neq 0$. This completes the proof. $\square$

There are immediate corollaries of Theorem 42-1, one of which is used in the proof of Theorem 38-4. Let X be a finite simplicial polytope and let Y be a paracompact space. In this case, we may suppose that the covering $\mathcal{U}_i$ of X in Step 1 of the proof of Theorem 42-1 consists of all open stars of the i -th barycentric subdivision of X and each projection $\pi_{i-1}^i: M_i \rightarrow M_{i-1}$ is the identity mapping. The cell complex P constructed in the proof of the theorem is a subdivision of the complex $X \times N_{\mathcal{W}}$, where

$N_{\mathcal{W}}$ is the nerve of the covering $\mathcal{W}$ of Y . Thus P is triangulable. Let $\mathcal{W}'$ be an lfc refinement of $\mathcal{W}$ and let $\pi_{\mathcal{W}'}^{\mathcal{W}'}: N_{\mathcal{W}'} \rightarrow N_{\mathcal{W}}$ be a projection, where $N_{\mathcal{W}'}$ is the nerve of $\mathcal{W}'$. Define a mapping $\bar{\pi}_{\mathcal{W}'}^{\mathcal{W}'}: X \times N_{\mathcal{W}'} \rightarrow X \times N$ by $\bar{\pi}_{\mathcal{W}'}^{\mathcal{W}'}(x, y) = (x, \pi_{\mathcal{W}'}^{\mathcal{W}'}(y))$ for $(x, y) \in X \times N_{\mathcal{W}'}$. The following is evident.

42-2. COROLLARY. Let (X, A) be a pair of finite simplicial polytopes and let (Y, B) be a pair of paracompact spaces. Then $H^n((X, A) \times (Y, B); G)$ is the direct limit of the directed system $\{H^n((X, A) \times (M_{\mathcal{W}}, N_{\mathcal{W}}); G): \pi_{\mathcal{W}'}^{\mathcal{W}'}\}$, where $(M_{\mathcal{W}}, N_{\mathcal{W}})$ is the pair of the nerves of $\mathcal{W}$ for (Y, B) and $\mathcal{W}$ ranges over all lfc open coverings of Y .

Next, consider the case where $X = I (= [0, 1])$. Let $D(I \times Y; G) = n$. There are closed sets A and B of I and Y such that $H^n((I, A) \times (Y, B); G) \neq 0$. We may assume that $A = \dot{I} (= \{0\} \cup \{1\})$. By the preceding corollary

$$H^n((I, \dot{I}) \times (Y, B); G) = \lim_{\rightarrow} \{H^n((I, \dot{I}) \times (M_{\mathcal{W}}, N_{\mathcal{W}}); G): \pi_{\mathcal{W}'}^{\mathcal{W}'}\}$$

It is well known that there is a natural isomorphism: $H^n((I, \dot{I}) \times (M_{\mathcal{W}}, N_{\mathcal{W}}); G) \approx H^{n-1}(M_{\mathcal{W}}, N_{\mathcal{W}}; G)$. Thus we find $H^{n-1}(Y, B; G) \neq 0$ and $D(I \times Y; G) \leq D(Y; G) + 1$. The converse relation, $D(I \times Y; G) \geq D(Y; G) + 1$ is proved similarly. Thus the following theorem has been proved.

42-3. THEOREM (Morita [5], Kodama [6]). If X is a locally finite polytope and Y is a paracompact space, then $D(X \times Y; G) = \dim X + D(Y; G)$. Consequently, the equality $\dim(X \times Y) = \dim X + \dim Y$ holds.

42-4. THEOREM. If X is a 2-dimensional compact ANR (metric) and Y is a paracompact space, then $D(X \times Y; G) \geq 2 + D(Y; G)$. Consequently the equality $\dim(X \times Y) = 2 + \dim Y$ holds.

PROOF. By Lemma 38-9, there is a homeomorph A of a 1-sphere which is contractible in X . Denote by (E, S) a pair consisting of a 2-cell and its boundary. Let $f: S \rightarrow A$ be the homeomorphism. Since A is contractible in X , f has an extension $g: E \rightarrow X$. Denote by $h: X \rightarrow E$ an extension of f . Then $hg: (E, S) \rightarrow (E, S)$ is homotopic to the identity. Take a closed set B of Y such that $H^n(Y, B; G) \neq 0$, where $D(Y; G) = n$. Consider the mappings $\bar{g} = g \times 1: (E, S) \times (Y, B) \rightarrow (X, A) \times (Y, B)$ and $\bar{h} = h \times 1: (X, A) \times (Y, B) \rightarrow (E, S) \times (Y, B)$. Since $H^{n+2}((E, S) \times (Y, B); G) \neq 0$ by Corollary 42-2 and $\bar{h}\bar{g}$ is homotopic to the identity, the homomorphism $\bar{g}^* \bar{h}^*$ is nontrivial by the homotopy axiom, Lemma 34-5. Thus, $H^{n+2}((X, A) \times (Y, B); G) \neq 0$. This completes the proof. $\square$

42-5. PROBLEM. Let X be a 1-dimensional compact metric space. Then it is easy to prove that $D(X: G) = 1$ for every group G . Thus, by Bockstein's Theorem II we know that $\dim(X \times Y) = \dim Y + 1$ for every compact space Y . Is this equality true for every paracompact space Y ? More generally, what is a compact metric space X such that the equality $\dim(X \times Y) = \dim X + \dim Y$ for every paracompact space Y ?

43. Subset Theorem and Product Theorem

43-1. THEOREM. Let X be a hereditarily paracompact space. Then, for any subspace Y of X and any group G , the relation $D(Y: G) \leq D(X: G)$ holds.

PROOF. Let $D(X: G) = n$, $m \geq n$ and B be a closed set of Y . It is sufficient to prove that for each mapping $f: B \rightarrow K (= K(G, m; q))$, $q > \dim X$ f is extendable over Y . Let $f: B \rightarrow K$ be a mapping. There is an open neighborhood M of B in Y and an extension $g: M \rightarrow K$ of f . Let $\mathcal{U} = \{U_\alpha: \alpha \in \Omega\}$ be the open covering of K consisting of open stars. Take open sets V_0 and V_α of X such that $V_0 \cap Y = Y - B$ and $V_\alpha \cap Y = g^{-1}(U_\alpha)$, $\alpha \in \Omega$, and let $\mathcal{V} = \{V_0, V_\alpha: \alpha \in \Omega\}$. Since the subspace $Z = \cup \{V: V \in \mathcal{V}\}$ is paracompact, there is a locally finite open covering $\mathcal{W} = \{W\}$ of Z such that the closure $\bar{W}$ of each W in X is contained in some V of $\mathcal{V}$. Since $D(\bar{W}: G) \leq n$, we know $D(Z: G) \leq n$ by the sum theorem, Section 38-3. Let T be the nerve of $\mathcal{W}$ and $\varphi: Z \rightarrow T$ a canonical mapping. Consider the subcollection $\mathcal{W}' = \{W: W \in \mathcal{W}, W \cap A \neq \emptyset\}$. Then the nerve T' of $\mathcal{W}'$ is a subcomplex of T and $\varphi(A) \subset T'$. By the construction of $\mathcal{W}$ there is a mapping $g': T' \rightarrow K$ such that $g' \varphi|_A$ and f are homotopic. We may assume that $f = g' \varphi|_A$, because K is an ANR (paracompact). Put $Z' = \varphi^{-1}(T')$. Then Z' is closed in Z . Since $D(Z: G) \leq n$, the mapping $g' \varphi|_{Z'}: Z' \rightarrow K$ is extendable over Z . This completes the proof. $\square$

43-2. THEOREM. If X is a paracompact space and Y is a subspace with the star-finite property, then $D(Y: G) \leq D(X: G)$.

PROOF. The proof is carried out in the same way as in the preceding theorem until we obtain a collection $\mathcal{V} = \{V_0, V_\alpha: \alpha \in \Omega\}$ of open sets in X , and therefore we use the same notation. Since Y has the star-finite property, there is an open covering $\mathcal{W} = \{W\}$ of Y such that $\mathcal{W}$ is star-

finite and $\{\bar{W}\}$ refines $\mathcal{V}$, where $\bar{W}$ is the closure of W in X . Since $\mathcal{W}$ is star-finite, $\mathcal{W}$ is decomposed into a union of countable subfamilies $\mathcal{W}_\gamma$, $\gamma \in \Gamma$, whose fields $Y_\gamma = \cup\{W_i: W_i \in \mathcal{W}_\gamma\}$, $\gamma \in \Gamma$, are pairwise disjoint and, therefore, are open-closed sets in Y . Put $X_\gamma = \cup\{\bar{W}_i: W_i \in \mathcal{W}_\gamma\}$. Since X_γ is an F_σ set in X , it is paracompact and $D(X_\gamma: G) \leq n$. An argument similar to that in Theorem 43-1 shows that the mapping $f|_{Y_\gamma \cap B}: Y_\gamma \cap B \rightarrow K$ is extendable to a mapping $g_\gamma: X_\gamma \rightarrow K$. Define $h: Y \rightarrow K$ by $h|_{Y_\gamma} = g_\gamma|_{Y_\gamma}$, $\gamma \in \Gamma$. Then h is the required extension of f .

43-3. PROBLEM. Let βX be the Stone-Čech compactification of X . In view of Lemma 35-4 and Theorem 37-10 we know that if X is paracompact and G is finitely generated then $D(X: G) = D(\beta X: G)$. Thus the following question arises: For every group G , does the equality $D(X: G) = D(\beta X: G)$ hold? This question is open even in case G is countably generated. As we show below, if G is countably generated, then the relation $D(X: G) \leq D(\beta X: G)$ holds. However, we do not know whether or not this relation holds for an uncountable group G .

43-4. THEOREM. Let X be perfectly normal and paracompact and let Y be metrizable. If G is countably generated, then $D(X \times Y: G) \leq D(\beta X \times Y: G)$.

If G is countably generated, then there is an Eilenberg-MacLane space $K(G, m)$ which is a locally-finite and countable simplicial polytope (see Whitehead [2]; also, compare Lemma 36-2). An argument similar to that in the proof of Theorem 43-1 shows that the theorem follows from the next lemma.

43-5. LEMMA. Let X and Y be spaces as in Theorem 43-4. Let $\mathcal{U}$ be a star-finite and countable open covering of $X \times Y$. Then there is an open F_σ subset W of $\beta X \times Y$ and a star-finite countable open covering $\mathcal{V}$ of W such that W contains $X \times Y$ and $\mathcal{V} \cap (X \times Y)$ refines $\mathcal{U}$.

PROOF. Let $\mathcal{B} = \cup \mathcal{B}_i$ be an open basis of Y such that $\mathcal{B}_i = \{B_{i\alpha}: \alpha \in \Omega_i\}$ is a locally finite open covering of Y for $i = 1, 2, \dots$ and $\text{St}(y, \mathcal{B}_i)$, $i = 1, 2, \dots$, is a neighborhood basis at each point y of Y . Put $A(\alpha_1, \dots, \alpha_i) = \cap_{k=1}^i B_{k\alpha_k}$, $\alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i$. Note that for each j , $j = 1, 2, \dots$, the collection $\{A(\alpha_1, \dots, \alpha_i): \alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i \text{ and } i = 1, \dots, j\}$ is locally

finite in Y . Let $\mathcal{U} = \{U_k: k = 1, 2, \dots\}$ be a star-finite and countable open covering of $X \times Y$. If $A(\alpha_1, \dots, \alpha_i) \neq \phi$, then we put

$$G(\alpha_1, \dots, \alpha_i: k) = \cup \{P: P \times A(\alpha_1, \dots, \alpha_i) \subset U_k, \quad P \text{ is open in } X\},$$

$$k = 1, 2, \dots$$

If $A(\alpha_1, \dots, \alpha_i) = \phi$, let $G(\alpha_1, \dots, \alpha_i: k) = \phi$. It is easy to show that the collection

$$\{G(\alpha_1, \dots, \alpha_i: k) \times A(\alpha_1, \dots, \alpha_i): \alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i,$$

$$i = 1, 2, \dots, \quad k = 1, 2, \dots\}$$

forms an open covering of $X \times Y$ which refines $\mathcal{U}$. Note that each $G(\alpha_1, \dots, \alpha_i: k)$ is an F_σ set of X by the perfect normality of X . By the property of the Stone-Čech compactification βX , there is an open F_σ set $K(\alpha_1, \dots, \alpha_i: k)$ of βX such that $K(\alpha_1, \dots, \alpha_i: k) \cap X = G(\alpha_1, \dots, \alpha_i: k)$. Put

$$L(\alpha_1, \dots, \alpha_i: k) = K(\alpha_1, \dots, \alpha_i: k) \times A(\alpha_1, \dots, \alpha_i), \quad \alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i,$$

$$i = 1, 2, \dots$$

Then $L(\alpha_1, \dots, \alpha_i: k)$ is an open F_σ set in $\beta X \times Y$ whose intersection with X is contained in U_k for each k . Put

$$V_k = \cup \{L(\alpha_1, \dots, \alpha_i: k): \alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i, \quad i = 1, 2, \dots\},$$

$$k = 1, 2, \dots$$

and $W = \cup_k V_k$. We shall prove that the subspace W and its open covering $\mathcal{V} = \{V_k\}$ satisfy the conclusion of the lemma. For each $k = 1, 2, \dots$, let H_k be the largest open set of $\beta X \times Y$ such that $H_k \cap (X \times Y) = U_k$. Obviously $V_k \subset H_k$ for $k = 1, 2, \dots$. Since $X \times Y$ is dense in $\beta X \times Y$, the collections $\{H_k\}$ and $\{U_k\}$ are similar and therefore $\mathcal{V}$ is countable and star-finite. To complete the proof, it is enough to show that V_k , $k = 1, 2, \dots$, is an F_σ set of $\beta X \times Y$. Consider the collection

$$\mathcal{V}_{kj} = \{L(\alpha_1, \dots, \alpha_i: k): \alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i, \quad i = 1, \dots, j\}$$

$$\text{for } j = 1, 2, \dots$$

Since the collection $\{A(\alpha_1, \dots, \alpha_i): \alpha_1 \in \Omega_1, \dots, \alpha_i \in \Omega_i, i = 1, \dots, j\}$ is locally finite in Y for each j , $\mathcal{V}_{kj}$ is locally finite in $\beta X \times Y$. Since $L(\alpha_1, \dots, \alpha_i: k)$ is an F_σ set in $\beta X \times Y$, $\cup \{L: L \in \mathcal{V}_{kj}\}$ is an F_σ set and hence $V_k = \cup \{L: L \in \mathcal{V}_{kj} \text{ and } j = 1, 2, \dots\}$ is an F_σ set in $\beta X \times Y$. $\square$

43-6. THEOREM.[†] If X is paracompact and G is countably generated, then $D(X: G) \leq D(\beta X: G)$.

[†] More generally, we can prove that, if X is countably paracompact and G is countably generated, then $D_{\mathcal{G}}(X: G) \leq D(\beta X: G)$. See Remark 36-17.

PROOF. As in Theorem 43-4, it is enough to prove the following fact: If $\mathcal{U}$ is a countable star-finite open covering of X , then there is a subspace H of βX and its open covering $\mathcal{V}$ such that (i) H is locally compact, paracompact, and contains X and (ii) $\mathcal{V} \cap X$ refines $\mathcal{U}$. Let $\mathcal{U} = \{U_k: k = 1, 2, \dots\}$ be a countable star-finite open covering of X . Take a closed covering $\{F_k\}$ of X such that $F_k \subset U_k$ for each k . Let W_k be the largest open set of βX such that $W_k \cap X = U_k$. Then $\{W_k\}$ and $\mathcal{U}$ are similar. Put $H = \cup_k \bar{F}_k$, where $\bar{F}_k$ is the closure of F_k in βX . For each k , $\bar{F}_k \subset W_k$. Hence, $\mathcal{V} = \{H \cap W_k\}$ is a countable, star-finite, open covering of H and $\mathcal{V} \cap X$ refines $\mathcal{U}$. Also, $\{\bar{F}_k\}$ is a star-finite, closed covering of H . Thus, H is locally compact and paracompact. This completes the proof. $\square$

The following theorem is a consequence of Theorem 43-4 and Lemma 43-5.

43-7. THEOREM. Let X and Y be spaces as in Theorem 43-4. Then $\dim(X \times Y) \leq \dim X + \dim Y$. Moreover, if Y has the star-finite property and G is finitely generated, then $D(X \times Y: G) \leq D(X: G) + D(Y: G)$.

PROOF. By Theorem 26-4, $\dim(\beta X \times Y) \leq \dim \beta X + \dim Y = \dim X + \dim Y$. Thus, the first half of the theorem is a consequence of Lemma 43-5. Let Y be a metrizable space with the star-finite property, then, $\beta X \times Y$ is a subspace of $\beta X \times \beta Y$ with the star-finite property. Thus, $D(\beta X \times Y: G) \leq D(\beta X \times \beta Y: G)$ by Theorem 43-2. If G is finitely generated, then we know by Bockstein's theorem, Section 41-3, that $D(\beta X \times \beta Y: G) \leq D(\beta X: G) + D(\beta Y: G)$. In view of Lemma 35-4 and Theorem 37-10, $D(X: G) = D(\beta X: G)$ and $D(Y: G) = D(\beta Y: G)$. This completes the proof. $\square$

43-8. REMARK. In Theorem 43-4 and Theorem 43-6 we cannot replace βX by any compactification. To see that this is so, let X_0 be the metric space constructed in Roy [1]. Then, $\text{ind } X_0 = 0$ and $\dim X_0 = 1$. Let γX_0 be the Freudenthal compactification. Since $\text{ind } \gamma X_0 = 0$, $\dim \gamma X_0 = D(\gamma X_0: G) = 0$ for any group G . However, $D(X_0: G) = 1$ holds for any group G . Finally, we note that the equality in Theorem 43-7 does not generally hold even in case $G = Z_p$, p a prime, and X and Y are both separable metric spaces.

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