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THEORY OF RETRACTS

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PREFACE

The aim of this book is to give an outline of the theory of retracts. This theory is a chapter of topology related to both principal branches: set-theoretic topology and algebraic topology. The main notions of the theory of retracts belong to general topology, but their full elaboration requires the use of algebraic tools. Accordingly, some use will be made of the notions and theorems of algebraic topology, but we shall be quite explicit about just what will be used.

We may distinguish two principal directions in the theory of retracts. First we have the theory of r-maps and their invariants in general topological spaces. Many theorems concerning the extension of maps, in particular various generalizations of the classical theorem of H. Tietze [274], belong to this direction. The first four chapters will outline this general theory of retracts. The second direction in the theory of retracts deals with its geometric aspects. We shall devote four chapters to an exposition of this part of the theory, which will be the main subject matter of this book. Chapter V will contain the theory of compact absolute retracts, called AR-sets, and also the theory of compact absolute neighborhood retracts called ANR-sets. The class of ANR-sets is more inclusive than the class of polyhedra, although the two are intimately related. But, in contrast with the constructive character of the notion of polyhedra — belonging to elementary geometry — the notion of ANR-sets is axiomatic and purely topological. In Chapter VI some examples of ANR-sets will be given with rather paradoxical properties. In Chapter VII some special classes of ANR-sets are considered. Chapter VIII is devoted to the study of some notions related to the problem of classification of spaces, especially of ANR-spaces, from the point of view of the theory of retracts. The last Chapter IX is devoted to a short outlook on the actual state of the theory of retracts. It contains also a list of open problems.

The bibliography given at the end of this book contains papers which contribute to some progress in the theory of retracts. References to this bibliography are made by numbers enclosed in brackets. There are included also several books, among them an excellent monography of S. T. Hu [162] devoted to the theory of retracts and an older book of S. Lefschetz [214] containing an important part of this theory. We have listed also several books on topology of general character, containing some informations about retracts; actually the books of the following authors: P. Alexandroff and H. Hopf [3], R. H. Crowell and R. H. Fox [88], J. Dugundji [104[a]], S. Eilenberg and N. Steenrod [112], P. J. Hilton [146], P. J. Hilton and S. Wylie [148], J. G. Hocking and G. S. Young [150], S. T. Hu [161], W. Hurewicz and H. Wallman [166], K. Kuratowski [200] and [203], W. Rinow [251], G. T. Whyburn [294].

We quote also a number of papers containing some explicit results of importance for the theory of retracts, though not belonging to this theory (for instance, a paper of P. Alexandroff [1] concerning the homological dimension theory or a paper of A. H. Stone [270] concerning paracompact spaces). The bibliography contains some papers concerning the applications of the theory of retracts to other branches of mathematics, as functional analysis [139], differential equations [6], [221], [282] and [283] and theory of semigroups [280], though these applications are not dealt with in this book. For convenience of the reader, the bibliography gives the numbers of pages on which the listed paper is quoted in this book.

I wish to express my thanks to Dr. A. Kirkor, Dr. A. Lelek, Dr. H. Patkowska and Mr. S. Godlewski who gave stimulating advice and read the manuscript. Very valuable aid in preparation of the initial version of this book, edited as lecture notes in 1964 by the University of Wisconsin, was received from Dr. Stephen Slack.

I wish also to thank the Editors of Monografie Matematyczne for the honor and privilege of publishing this volume in its series.

Karol Borsuk

Warsaw, October 1966

CHAPTER I

GENERAL PROPERTIES OF r-MAPS

The theory of retracts can be defined as the theory of some kind of topological properties, called *r-invariants*, intimately related to a class of functions called *r-maps* ([47], p. 1084). These r-maps are much more general than homeomorphisms but they are much more special than arbitrary continuous maps. It follows that all properties of spaces invariant under continuous maps, such as connectedness, compactness, separability, belong to the theory of retracts. This theory also deals with many other topological properties, many of them having a clear-cut geometrical sense.

We agree that by a *space* we shall mean a Hausdorff topological space and that by a *map* we shall mean a continuous map of one space into another. If we do not assume that a function is continuous, we shall refer to it as a *function* or as a *transformation*.

1. r-maps. A map

$$f\colon X \to Y$$

of a space X into a space Y is said to be an *r-map* if there is a map

$$g\colon Y \to X$$

which is a *right inverse* of f, that is such that the composition $fg\colon Y \to Y$ is the identity map i of Y. Thus g assigns in a continuous fashion to every point $y \in Y$ a point $g(y)$ belonging to the inverse image $f^{-1}(y)$ of y by the map f. It is clear that

(1.1) *Every homeomorphism is an r-map.*

Note that the relation $fg = i$ implies that the map $h\colon Y \to g(Y)$ given by the formula $h(y) = g(y)$ for every point $y \in Y$ is a homeomorphism of Y onto the subspace $g(Y)$ of X and that every point y of Y is a value of f, thus f is onto.

Let us observe that

(1.2) $gf(x) = x$ *if and only if* $x \in g(Y)$.

The implication $[gf(x) = x] \Rightarrow [x \in g(Y)]$ is evident. On the other hand, if $x \in g(Y)$, then there is a point $y \in Y$ such that $x = g(y)$ and we infer that $gf(x) = gfg(y) = g(y) = x$.

Now let us prove that

(1.3) *$g(Y)$ is a closed subset of X.*

It is sufficient to prove that for every point $x_0 \in X - g(Y)$ there exists a neighborhood W_0 (in X) contained in $X - g(Y)$. Since (1.2) implies $gf(x_0) \neq x_0$, there is a neighborhood U_0 of x_0 (in X) and a neighborhood V_0 of $gf(x_0)$ (in X) such that $U_0 \cap V_0 = 0$. Then the inverse image $f^{-1}g^{-1}(V_0)$ of V_0 (by the map $gf\colon X \to X$) is a neighborhood of x_0 in X and therefore the set $W_0 = U_0 \cap f^{-1}g^{-1}(V_0)$ is a neighborhood of x_0 in X. Since $W_0 \subset U_0$ and $gf(W_0) \subset gf[f^{-1}g^{-1}(V_0)] \subset V_0$, we infer that $W_0 \cap gf(W_0) = 0$ and (1.2) implies that $W_0 \subset X - g(Y)$.

If there exists an r-map $f\colon X \to Y$, then the space Y is called an *r-image* of the space X. In this case we say that the space X *r-dominates* the space Y and write $X \underset{r}{\geqslant} Y$ or $Y \underset{r}{\leqslant} X$. Evidently, if X and Y are homeomorphic, then $X \underset{r}{\geqslant} Y$ and $X \underset{r}{\leqslant} Y$.

Since g maps Y homeomorphically onto a subset of X, we infer by proposition (1.3) that

(1.4) *Every r-image of X is homeomorphic to a closed subset of X.*

EXAMPLE. A space Y is said to be a *topological divisor* of a non-empty space X if there exists a space Z such that $Y \times Z$ is homeomorphic to X. In such a case there is a homeomorphism h of X onto $Y \times Z$ which assigns to a point $x \in X$ a point $(y(x), z(x)) \in Y \times Z$. Let us set

$$f(x) = y(x) \quad \text{for all } x \in X.$$

Fix a point $z_0 \in Z$ and let

$$g(y) = h^{-1}(y, z_0) \quad \text{for all } y \in Y.$$

We see at once that the map g is a right inverse to f, so that

(1.5) *Each topological divisor of a space is an r-image of the space.*

EXAMPLE. Consider an arbitrary system of spaces $\{X_\mu\}$ where the index μ runs over a set $M \neq 0$. The *Cartesian product* of the spaces X_μ, which we denote by $X = \underset{\mu \in M}{P} X_\mu$, consists of all functions x defined on M with values in $\bigcup_{\mu \in M} X_\mu$ satisfying the condition

$$x(\mu) \in X_\mu \quad \text{for every } \mu \in M.$$

The value of x at μ will be denoted by x_μ and will be called the μth-*coordinate* of x. The function x will be denoted by $\{x_\mu\}$.

The topology in the space $X = \underset{\mu\in M}{P} X_\mu$ is defined as follows: A set $U \subset X$ is said to be a *neighborhood* of a point $x \in M$ provided that there is a finite system $\mu_1, \mu_2, \ldots, \mu_n$ of indices and a system of neighborhoods U_{μ_i} of x_{μ_i} in X_{μ_i} $(1 \leqslant i \leqslant n)$ such that U consists precisely of all those $x \in X$ which satisfy conditions $x_{\mu_i} \in U_{\mu_i}$ for $1 \leqslant i \leqslant n$.

Now let $\{Y_\mu\}$ be another system of spaces labeled by the same set of indices M and let f_μ map X_μ into Y_μ for each $\mu \in M$. Evidently, if we set

(1.6) $$f(\{x_\mu\}) = \{f_\mu(x_\mu)\} \quad \text{for every point } \{x_\mu\} \in X,$$

we obtain a map f of the space $X = \underset{\mu\in M}{P} X_\mu$ into the space $Y = \underset{\mu\in M}{P} Y_\mu$. Moreover, if each map f_μ is an r-map with right inverse g_μ, then setting

$$g(\{y_\mu\}) = \{g_\mu(y_\mu)\} \quad \text{for all points } \{y_\mu\} \in Y,$$

we obtain a right inverse g of f. Thus we see that if all the maps f_μ are r-maps, then f is an r-map. In other words:

(1.7) *If* $Y_\mu \underset{r}{\leqslant} X_\mu$ *for every index* $\mu \in M$, *then* $\underset{\mu\in M}{P} Y_\mu \underset{r}{\leqslant} \underset{\mu\in M}{P} X_\mu$.

If a space Y is homeomorphic to the Cartesian product $\underset{\mu\in M}{P} X_\mu$, then the system of spaces $\{X_\mu\}$ will be said to be a *Cartesian decomposition* of Y.

(1.8) EXAMPLES. As an example of a map which is not an r-map, we may take any map of a segment onto a circle, since the circle is not homeomorphic to any subset of a segment. As an example of a map of a space onto itself which is not an r-map, consider the map f of the closed interval $\langle -3, 3\rangle$ onto itself given by the formula

$$f(x) = \tfrac{1}{6}(x^3 - 3x).$$

Indeed, if g were a right inverse of f, then it would easily follow that $g(-3) = -3$ and $g(3) = 3$. We infer that the homeomorphism g maps $\langle -3, 3\rangle$ onto itself and consequently f also is a homeomorphism, which is not true.

Now let us observe that

(1.9) *The composition of two r-maps is an r-map.*

In fact, let $f_1\colon X \to Y$ and $f_2\colon Y \to Z$ be two r-maps and let $g_1\colon Y \to X$ and $g_2\colon Z \to Y$ be their right inverses. Then

$$f = f_2 f_1\colon Y \to Z \quad \text{and} \quad g = g_1 g_2\colon Z \to X,$$

and we see at once that

$$fg(z) = f_2 f_1 g_1 g_2(z) = f_2 g_2(z) = z \quad \text{for every point } z \in Z.$$

Hence the map g is a right inverse of f.

Moreover, let us observe that

(1.10) *If* $f\colon X \to Y$ *is an r-map,* $Y_0 \subset Y$ *and* $X_0 = f^{-1}(Y_0)$, *then the restriction* $f_0 = f|X_0$ *is an r-map of* X_0 *onto* Y_0.

In fact, if $g\colon Y \to X$ is a right inverse of f, then the partial map $g_0 = g/Y_0$ is a right inverse of f_0, because for every point $y_0 \in Y_0$ we have $g_0(y_0) \in X_0$ and therefore $f_0 g_0(y_0) = fg(y_0) = y_0$.

2. Retractions. The maps called retractions are special kind of r-maps. Suppose that Y is a subset of X. Then a map $f\colon X \to Y$ is said to be a *retraction* ([17], p. 153) if the inclusion $i\colon Y \to X$ is a right inverse of f, i.e. if $f(x) = x$ for all points $x \in Y$. Since f is onto, as an r-map, we say that f *retracts* X *onto* Y. Evidently we may (equivalently) define the retractions on X to be the maps f of X into itself which satisfy the functional equation $ff = f$.

It is clear that every map of the form hr, where r is a retraction and h is a homeomorphism, is an r-map. On the other hand, if $f\colon X \to Y$ is an r-map with a right inverse g, then setting $X_0 = g(Y)$, $r = gf$ and $h = g^{-1}$, we see that r is a retraction of X onto X_0 and that h is a homeomorphism of X_0 onto Y. Since $hr = f$, we conclude that f is of the form hr. Thus

(2.1) *The r-maps are identical with maps of the form hr where r is a retraction and h is a homeomorphism.*

3. Retracts. A subset X_0 of a space X is said to be a *retract* of X ([17], p. 153) if there is a retraction of X onto X_0. From (1.3) we conclude that

(3.1) *Every retract of a space is closed in that space.*

The notion of retract belongs to the general topology. But we shall focus our attention on the geometric side of the theory of retracts and, in consequence, we shall mainly consider the subsets of Euclidean spaces or of Hilbert space. It is therefore convenient to fix our notation concerning these spaces.

Hilbert space is denoted by E^ω and consists of all (real) sequences $x = \{x_k\}$ for which the series $\sum_{k=1}^{\infty} x_k^2$ converges. The number x_k is called the *k-th coordinate* of x. The space E^ω becomes a metric space if we define the distance $\varrho(x, y)$ between points $x = \{x_k\}$ and $y = \{y_k\}$ of E^ω by the formula

$$\varrho(x, y) = \sqrt{\sum_{k=1}^{\infty} (x_k - y_k)^2}.$$

The subset of E^ω consisting of all points $x = \{x_k\}$ with $0 \leqslant x_k \leqslant 1/k$ for $k = 1, 2, \ldots$ is denoted by Q^ω and is called the *Hilbert cube.* Let us recall that Q^ω is a *compactum* (i.e. a metric compact space).

Let $x = \{x_k\}$, $y = \{y_k\}$ be the points of E^ω; we have the following operations:

(3.2) $$\lambda x + \mu y = \{\lambda x_k + \mu y_k\} \quad \text{where } \lambda, \mu \text{ are real numbers;}$$

(3.3) $$x \cdot y = \sum_{k=1}^{\infty} x_k y_k.$$

In particular, $x^2 = x \cdot x = \sum_{k=1}^{\infty} x_k^2$. The number $\sqrt{x^2}$ will be denoted by $\|x\|$. Thus

$$\varrho(x, y) = \|x - y\|.$$

By 0 we mean the point of E^ω every coordinate of which is the number zero.

Euclidean n-dimensional space will be denoted by E^n for $n = 1, 2, 3, \ldots$ This space consists of all points $\{x_k\}$ of E^ω such that $x_k = 0$ for each $k > n$. It follows, in particular, that $E^n \subset E^m$ for every $n \leqslant m$.

A system $x^i = \{x_k^i\}$, $i = 0, 1, \ldots, m$, of points of E^ω is said to be *linearly independent* provided that the linear combination

$$\mu_0 x^0 + \mu_1 x^1 + \ldots + \mu_m x^m, \quad \text{where } \mu_0 + \mu_1 + \ldots + \mu_m = 0,$$

is equal to 0 only if all coefficients μ_i vanish. Then the set σ consisting of all points x of E^ω of the form

$$x = t_0 x^0 + t_1 x^1 + \ldots + t_m x^m, \quad \text{where } t_i \geqslant 0 \text{ and } t_0 + t_1 + \ldots + t_m = 1,$$

is called the *geometric simplex* of dimension m with *vertices* $x^0, x^1, \ldots, x^m$. It will be denoted by $|x^0, x^1, \ldots, x^m|$. In particular, the 1-dimensional simplex $|x^0, x^1|$ is the same as the segment with end-points x^0 and x^1. By a *face* of a geometric simplex σ we understand a geometric simplex σ' with all vertices belonging to the set of vertices of σ. We shall keep the same terminology also for sets isometric with geometric simplexes.

A set which is the union of a finite number of geometric simplexes is said to be a *geometric polyhedron*. The homeomorphic image of a geometric polyhedron will be called a *curvilinear polyhedron*, or shortly, *polyhedron*. For instance, the *disk* (i.e. a homeomorphic image of the triangle) is a curvilinear polyhedron.

(3.4) EXAMPLE. Let K^n denote the ball in E^n given by the inequality $\|x\| \leqslant 1$, where $x \in E^n$. If we set

$$r(x) = \begin{cases} x & \text{for all points } x \in K^n, \\ \dfrac{x}{\|x\|} & \text{for all points } x \in E^n - K^n, \end{cases}$$

then the map $r: E^n \to K^n$ is a retraction. By a similar reasoning we show that the ball

$$K^\omega = \{x \in E^\omega;\ \|x\| \leqslant 1\}$$

is a retract of the space E^ω.

(3.5) *If $G \neq 0$ is an open bounded subset of the Euclidean space E^n, then the set $X = E^n - G$ is not a retract of E^n.*

In order to prove this it is sufficient to show that

(3.6) *If X is a closed subset of E^n and G is one of the bounded components of $E^n - X$, then there is no map $f: \bar{G} \to X$ such that $f(x) = x$ for every point $x \in \bar{G} - G$.*

It is evident that we may restrict our considerations to the case where the point 0 belongs to G and where the diameter $\delta(G)$ of G is $\leqslant 1$. Let K be the ball in E^n with center 0 and radius 1. If there would exist a map $f: \bar{G} \to X$ such that $f(x) = x$ for every point $x \in \bar{G} - G$ then, setting

$$\varphi(x) = \begin{cases} \dfrac{-x}{\|x\|} & \text{for all } x \in K - G, \\ \dfrac{-f(x)}{\|f(x)\|} & \text{for all } x \in \bar{G}, \end{cases}$$

we should obtain a map φ of the ball K into its boundary S. By Brouwer's fixed point theorem, there exists a point $x_0 \in K$ such that $\varphi(x_0) = x_0$. Since $G \cap S = 0$ and $\varphi(x_0) \in S$, the point x_0 belongs to $K - G$ and we infer that $-x_0/\|x_0\| = x_0$. Since $\|x_0\| = 1$, we see that $x_0 = -x_0$, which is impossible, because the point x_0 is in $K - G$, and is, therefore, distinct from 0.

Propositions (3.1) and (3.6) imply

(3.7) *If X is a bounded retract of E^n, where $n > 1$, then the set $E^n - X$ is connected.*

In order to see this it is sufficient to observe that only one component of $E^n - X$ is not bounded.

Moreover, let us observe that (3.1) implies that every bounded retract of E^1 is either a closed segment or it contains only one point. It follows that

(3.8) *If X is a bounded retract of E^1, then $E^1 - X$ contains exactly two components.*

It follows by (3.6) that

(3.9) *The boundary S^{n-1} of the Euclidean n-ball K^n is not a retract of K^n.*

An especially simple proof of this proposition has been recently given by M. W. Hirsch [149].

Let us observe that an analogous proposition for the Hilbert ball

$$K^\omega = \Big\{x = (x_1, x_2, \ldots);\ \sum_{i=1}^{\infty} x_i^2 \leqslant 1\Big\}$$

is not true ([139], p. 50), i.e.

(3.10) *The boundary S^ω of the Hilbert ball K^ω is a retract of K^ω.*

Proof. Let a_n denote, for $n = 0, 1, \ldots$, the point $\{x_i\} \in E^\omega$ with $x_i = 0$ for $i \neq n$ and $x_n = 1$. In particular, $a_0 = 0$. Setting

$$h(t) = (1-t+n)a_n + (t-n)a_{n+1} \quad \text{for} \quad n \leqslant t \leqslant n+1 \text{ and } n = 0, 1, \ldots,$$

we get a homeomorphism h mapping the set T of all non-negative real numbers t onto a closed subset A of K^ω: the union of all segments $|a_n, a_{n+1}|$. The inverse homeomorphism h^{-1} maps A onto T. By the classical theorem of Tietze ([274], p. 11), there exists a continuous extension $f\colon K^\omega \to T$ of the map $h^{-1}\colon A \to T$, and we see at once that setting

$$g(t) = h(t) \quad \text{for every } t \in T,$$

we get a map $g\colon T \to K^\omega$ which is a right inverse of f. Thus f is an r-map of K^ω onto T. Defining

$$\varphi(x) = g\big(f(x)+1\big) \quad \text{for every point } x \in K^\omega,$$

we get a map $\varphi\colon K^\omega \to K^\omega$. If there exists a point $x_0 \in K^\omega$ such that $\varphi(x_0) = x_0$, then $x_0 = \varphi(x_0) \in g(T) \subset A$ and consequently $f(x_0) = h^{-1}(x_0)$ and $\varphi(x_0) = h\big(h^{-1}(x_0)+1\big) \neq h\big(h^{-1}(x_0)\big) = x_0$, contrary to our supposition. Thus we have shown that

$$\varphi(x) \neq x \quad \text{for every point } x \in K^\omega.$$

Now let us denote by $L(x)$ the open half-straight line (in the space E^ω) issuing from $\varphi(x)$ and passing through x, i.e. $L(x)$ consists of all points of E^ω of the form $tx + (1-t)\varphi(x)$ where $0 < t$. It is clear that $L(x)$ intersects the boundary S^ω of the ball K^ω exactly in one point $\psi(x)$ depending continuously on x. Thus we get a map $\psi\colon K^\omega \to S^\omega$. Moreover, for every point $x \in S^\omega$ the intersection of $L(x)$ and S^ω coincides with x. It follows that ψ is a retraction of K^ω to S^ω and the proof of (3.10) is finished.

(3.11) EXAMPLE. Let σ be a triangle in E^3 and let L be a straight line in E^3 which intersects σ in exactly one point a which does not belong to the boundary $\sigma^{\cdot}$ of σ.

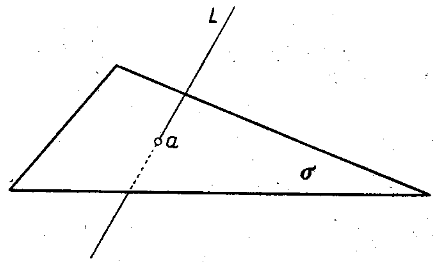


Fig. 1

Let us show that $\sigma^{\cdot}$ is a retract of E^3-L. Let $P(x)$ denote the plane passing through L and through a given point $x \in E^3-L$. It is clear that $P(x)$ intersects $\sigma^{\cdot}$ in two points, of which one, that we shall call $r(x)$, will lie on the same side of L as the point x. The function r so defined is evidently a retraction of E^3-L onto $\sigma^{\cdot}$.

4. Neighborhood retracts. A closed subset X_0 of a space X is said to be a *neighborhood retract* ([21], p. 222) in the space X provided that X_0 is the retract of some open subset of X that contains X_0. Evidently every retract of X is also a neighborhood retract of X.

(4.1) EXAMPLE. Let S^{n-1} be the sphere in E^n defined by the equation $\|x\| = 1$. Setting

$$r(x) = \frac{x}{\|x\|} \quad \text{for every point } x \in E^n-(0),$$

we get a retraction $r\colon E^n-(0) \to S^{n-1}$.

Thus, we see that *S^{n-1} is a neighborhood retract of E^n*. However, as we have already seen, S^{n-1} is not a retract of E^n.

(4.2) *If A is a compact subset of E^n which is a neighborhood retract of E^n, then E^n-A has only a finite number of components.*

Indeed, let U be an open subset of E^n and let $r\colon U \to A$ be a retractino of U to A. We easily see that all but a finite number of components of E^n-A lie in U.

In consequence, if the class of components of E^n-A were infinite, then U would contain a bounded component G of E^n-A. Let us set

$$r'(x) = \begin{cases} r(x) & \text{for every point } x \in G, \\ x & \text{for every point } x \in E^n-G. \end{cases}$$

Then we obtain a map $r'\colon E^n \to E^n-G$ which is a retraction. But by (3.5) this is impossible.

5. Retraction and extension of maps. A map $f: X \to Y$ is said to be an *extension* of a map f_0: $X_0 \to Y$ if X_0 is a subset of the space X and f_0 coincides with the restriction $f|X_0$. The notion of retract is intimately related to the problem of existence of extensions of maps. For the moment we will content ourselves with two almost self-evident theorems:

(5.1) THEOREM. *A subset X_0 of a space X is a retract of X if and only if every map f_0 of X_0 into an arbitrary space Y has a continuous extension f: $X \to Y$.*

Proof. If there exists a retraction r: $X \to X_0$, then for each map f_0: $X_0 \to Y$ we set $f = f_0 r$ and thus obtain the required continuous extension of f_0. The other way round, if every map f_0: $X_0 \to Y$ has a continuous extension f: $X \to Y$, then, in particular, the identity i: $X_0 \to X_0$ has a continuous extension f: $X \to X_0$. Manifestly, f is a retraction of X onto X_0.

(5.2) THEOREM. *A subset Y_0 of a space Y is a retract of Y if and only if for every space X, for every one of its subsets X_0, and for every map f_0: $X_0 \to Y$ with $f_0(X_0) \subset Y_0$, the existence of a continuous extension f': $X \to Y$ implies the existence of a continuous extension f: $X \to Y$ satisfying the condition $f(X) \subset Y_0$.*

Proof. If there exists a retraction r: $Y \to Y_0$ and a continuous extension f': $X \to Y$ of f_0, then setting $f = irf'$, where i: $Y_0 \to Y$ is the inclusion map, we obtain an extension f of f_0 such that $f(X) \subset Y_0$.

On the other hand, let us suppose that for every space X and for every subset X_0 of X, the existence of a continuous extension f' of a map f_0: $X_0 \to Y$ with $f_0(X_0) \subset Y_0$ implies the existence of a map f: $X \to Y$ with $f(X) \subset Y_0$, that extends f_0. If we choose as X the space Y and as X_0 the set Y_0, then we infer that the inclusion i: $Y_0 \to Y$ has a continuous extension f: $Y \to Y$ satisfying the condition $f(Y) \subset Y_0$. We thus see that f is then a retraction of Y to Y_0.

6. *r*-types and *r*-invariants. By an *r-invariant* we understand every property of spaces which is preserved under all *r*-maps. In particular, every hereditary topological property, that is every topological property which passes from a space to all its subsets, is evidently an *r*-invariant, but not conversely. For instance, the connectedness is not hereditary, but is an *r*-invariant. The property of a space X given by the inequality $\dim X \leqslant n$ is evidently hereditary, and therefore it is an *r*-invariant. This is true not only for usual inductive dimension defined for metric separable spaces, but also for the dimension in the more general sense, based onto the notion of the covering. (See, for instance, K. Morita [232], p. 6.) However, the dimension is not an *r*-invariant, because an *r*-map can lower dimension.

In the next chapter we shall see that many combinatorial properties of spaces, especially various properties with a clear geometric sense, belong to r-invariants. As an elementary example of a such property, let us mention here the unicoherence for locally connected, metric, complete spaces. A connected space Y is said to be *unicoherent* provided that for each decomposition of it into the union of two closed connected sets Y_1 and Y_2 their common part $Y_1 \cap Y_2$ is connected. By an elementary argument ([195], p. 309) we see that a locally connected continuum Y is not unicoherent if and only if it contains two points y_1, y_2 and two closed disjoint subsets B_1 and B_2 such that $B_1 \cup B_2$ disconnects Y between y_1 and y_2, but none of the sets B_1 and B_2 does. Now let us consider an r-map

$$f\colon X \to Y$$

of a locally connected continuum X into Y, and let g denote its right inverse. It is evident that Y is a locally connected continuum. Let us assume that it is not unicoherent, and let B_1, B_2, y_1, y_2 be defined as above. Setting

$$A_\nu = f^{-1}(B_\nu) \quad \text{for} \quad \nu = 1, 2,$$

we get two closed disjoint subsets A_1, A_2 of X. Since y_1 and y_2 belong to the same component C_ν of the set $Y - B_\nu$, the set $g(C_\nu)$ is a connected subset of $X - A_\nu$ containing both points $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Hence A_ν does not disconnect X between x_1 and x_2. On the other hand, $A_1 \cup A_2$ disconnects X between x_1 and x_2, because otherwise there would exist a component G of $X - (A_1 \cup A_2)$ containing both points x_1 and x_2, and consequently $f(G)$ would be a connected subset of $Y - (B_1 \cup B_2)$ containing y_1 and y_2, contrary to our hypothesis that $B_1 \cup B_2$ disconnects Y between y_1 and y_2. Thus we see that X is not unicoherent, whence *for locally connected continua the unicoherence is an r-invariant.* In particular, a simple closed curve cannot be a retract of any locally connected unicoherent continuum. One can show that the converse is also true: Every locally connected continuum which is not unicoherent has a retract which is a simple closed curve ([18], p. 184). Without the hypothesis of the local connectedness this last proposition fails. In fact, one easily sees that the closure of the spiral defined in the plane E^2 by the parametric equation

$$x(t) = \big((1+e^{-t})\cos t, (1+e^{-t})\sin t\big), \qquad 0 < t,$$

is a unicoherent, but not locally connected continuum. Moreover, the map f of X given by the formula

$$f(x) = \frac{x}{\|x\|}$$

is a retraction of X to the unit circle $\|x\| = 1$. Consequently, *the unicoherence is not an r-invariant for spaces which are not locally connected.*

Let us observe that the negations of the r-invariants are the same as the properties which pass from a space X to every space Y which r-dominates X. The properties of this kind will be called $\bar{r}$-*invariants*. For instance, the property of a space X given by the inequality $\dim X \geqslant n$ is evidently an $\bar{r}$-invariant.

Now let us introduce the notion of r-type. Two spaces X and Y are said to be *r-equal* if each of them is an r-image of the other, i.e. if both $X \underset{r}{\leqslant} Y$ and $X \underset{r}{\geqslant} Y$ (in symbols, $X \underset{r}{=} Y$). It is clear that this relation is reflexive ($X \underset{r}{=} X$), symmetric (if $X \underset{r}{=} Y$, then $Y \underset{r}{=} X$) and transitive ($X \underset{r}{=} Y$ and $Y \underset{r}{=} Z$ imply $X \underset{r}{=} Z$). Therefore, the class of all spaces decomposes into disjoint classes of r-equal spaces; these classes we call *r-types* ([47], p. 1086). Clearly, two homeomorphic spaces belong to the same r-type, which we also express by saying that they have the same r-type. So the classification of spaces according to their r-type is topological.

Manifestly, the converse is not true. For instance, if we take X to be a triangle and Y to be the union of two triangles, disjoint except for a single common vertex, then X and Y are r-equal but not homeomorphic.

If $X \underset{r}{\leqslant} Y$ but the relation $X \underset{r}{\geqslant} Y$ is false, then we write $X \underset{r}{<} Y$ or $Y \underset{r}{>} X$ and say that X is *r-smaller* than Y or that Y is *r-greater* than X ([61], p. 322). For instance, a segment I is r-smaller than a circle. A segment is also r-smaller than a triod X_1 (the union of three segments disjoint except for a single common endpoint). Also X_1 is r-smaller than the graph X_2 consisting of two parallel segments and of a third segment joining the centers of the first two.

By (1.7), the relations $X \underset{r}{\leqslant} Y$ and $X' \underset{r}{\leqslant} Y'$ imply $X \times X' \underset{r}{\leqslant} Y \times Y'$. But the relation $X \underset{r}{<} Y$ does not imply the relation $X \times X' \underset{r}{<} Y \times X'$ even in the case where $X' = I$ is a segment. In fact, we easily see that for graphs X_1 and X_2 just considered we have $X_1 \underset{r}{<} X_2$ and $X_1 \times I \underset{r}{=} X_2 \times I$.

(6.1) PROBLEM. *Is it true that* $X \underset{r}{<} Y$ *and* $X' \underset{r}{<} Y'$ *imply* $X \times X' \underset{r}{<} Y \times Y'$?

Now we can introduce the notion of an r-invariant and we can say exactly what is the subject of the theory of retracts. A topological property α is said to be an *r-invariant*, or an *r-property* provided that for all spaces X and Y, if X has property α and $X \underset{r}{=} Y$, then Y has property α. The theory of retracts can be defined as the theory of r-invariants

([73], p. 321). Since topology coincides with the theory of topological invariants and since every r-invariant is clearly a topological invariant, it follows that the theory of retracts is a chapter of tolopogy.

It is clear that many basic topological properties such as compactness, connectedness, arcwise connectedness, separability, dimension, etc. belong to the class of r-invariant properties.

On the other hand, there are topological properties which are not r-invariants. Thus the existence of a separating point is not an r-invariant, because a disk (i.e. a set homeomorphic to a triangle) has no such point and yet has the same r-type as the space X consisting of the union of two disks, disjoint, save for a single common boundary point.

Two spaces having the same r-type may be considered as identical from the point of view of the theory of retracts, just as two homeomorphic spaces are considered as identical from a topological point of view. However, there are some cases where r-equality implies topological equality. For instance, it is easy to see that if two closed manifolds are r-equal, they are also homeomorphic. A space for which the r-type determines the topological type is called an *r-determinable space* ([47], p. 1087).

7. Fixed point property. We say that a space X has the *fixed point property* provided that for each map $\varphi\colon X \to X$ there exists a point $x_0 \in X$ such that $\varphi(x_0) = x_0$.

Let us prove the following simple

(7.1) THEOREM. *The fixed point property is an r-invariant.*

We will actually prove more, namely that

(7.2) *The fixed point property is an $\underline{r}$-invariant.*

Indeed, suppose that $f\colon X \to Y$ is an r-map and ψ is a map of Y into itself. Let g be the map of Y into X which is right inverse to f. The composition $\varphi = g\psi f$ maps X into X and, hence, has a fixed point x_0 by hypothesis: $\varphi(x_0) = x_0$. We infer that $f(\varphi(x_0)) = f(x_0)$ or that $\psi(f(x_0)) = f(x_0)$ since fg is the identity. Thus we see that $f(x_0)$ is a fixed point of ψ.

8. Some local r-invariants. We have already observed that compactness, connectedness, etc. are r-invariants. The situation is similar if we localize these properties. Let us recall that a space X is said to be *locally compact* at $x_0 \in X$ provided that each neighborhood of x_0 contains a compact neighborhood of x_0. (By a neighborhood of x_0 we mean always a set containing x_0 in its interior. With this usage a neighborhood does not need to be open.) In a similar way we may define *local connectedness* and *local arcwise connectedness*. It is clear that if X is locally arcwise connected at x_0, then X is locally connected at x_0, although the converse is false. If one of these local properties holds for all $x_0 \in X$, then we say simply, for the sake of brevity, that X is locally compact, connected, or arcwise connected as the case may be.

One easily sees that

(8.1) *Every component of a locally arcwise connected space is arcwise connected.*

It is easy to see that none of these three local properties is invariant under all maps. For instance, the set Z of all integers is locally compact, locally connected, and locally arcwise connected, but it may be continuously mapped onto the set Q of rational numbers which has none of these properties.

Now let us prove that

(8.2) *The local compactness, local connectedness, and local arcwise connectedness of a space are $\underline{r}$-invariants.*

By (2.1), it is sufficient to show that these properties are preserved by retractions. We give the proof for local compactness, the proofs for the other properties being quite similar.

Suppose then that $r: X \to X_0$ is a retraction and that X is locally compact at a point $x_0 \in X_0 \subset X$. Let V be a neighborhood of x_0 in X_0. Then the inverse image $U = r^{-1}(V)$ of V is a neighborhood of x_0 in X. By hypothesis, there is a compact neighborhood U_0 of x_0 in X which is contained in U. Then $r(U_0)$ is surely a compact subset of V. Actually it is a neighborhood of x_0 in X_0 since $r(U_0) \supset r(U_0 \frown X_0) = U_0 \frown X_0$. Thus we see that X_0 is locally compact at x_0.

Now let us observe that if a space X has a local property α at all of its points, then each open subset of X will also have the property α at all of its points. It follows from this and from our remarks above that:

(8.3) *Every neighborhood retract of a space which is locally compact (resp.: locally connected, locally arcwise connected) is also locally compact (resp.: locally connected, locally arcwise connected).*

Another local property of a space X at its point x_0, which is an $\underline{r}$-invariant, is the property that the *order* of X at x_0 (in the sense of Menger and Urysohn ([203], p. 200)) is less than a given cardinal number $\mathfrak{m}_0$. Let us recall that the order of X at x_0 is defined as the smallest cardinal number $\mathfrak{m}$ with the property that each neighborhood U of x_0 in X contains a neighborhood V of x_0 in X such that the power of its boundary $\overline{V} \frown \overline{X - V}$ is less than or equal to $\mathfrak{m}$. It is evident that if X_0 is a subset of X, then the order of a point $x_0 \in X_0$ in the space X_0 is less than or equal to its order in the space X. It follows at once that

(8.4) *If $X \underset{r}{\geqslant} Y$, then for every cardinal number $\mathfrak{m}$ the power of the subset of Y consisting of all points with order $\geqslant \mathfrak{m}$ in Y is less than or equal to the power of the subset of X consisting of all points with order $\geqslant \mathfrak{m}$ in X.*

9. Function spaces. Given two spaces X and Y, we denote by Y^X the set of all maps of X into Y. We make a space out of this set by defining a subbasis of it. Let us recall that by an *open basis* of a space Z one understands a system of open subsets of Z such that for each point $z_0 \in Z$ and for each neighborhood U of z_0 in Z there exists an element W of the basis such that $z_0 \in W \subset U$. By a *subbasis* of Z one understands a system $\mathscr{B}$ of open sets such that the collection of all finite intersections of its elements is an open basis for Z.

Let us observe the following fact:

(9.1) *Let $\mathscr{B}$ be a subbasis for a space Z and $\hat{\mathscr{B}}$ a subbasis for another space $\hat{Z}$. A function $f\colon Z \to \hat{Z}$ is continuous if for every point $z_0 \in Z$ and for every element $\hat{G}$ of $\hat{\mathscr{B}}$ containing $f(x_0)$ there exists an element G of $\mathscr{B}$ such that $z_0 \in G$ and $f(G) \subset \hat{G}$.*

Following R. H. Fox [129], we introduce a subbasis $\mathscr{B}$ of Y^X in the following way:

For each compact set $X_0 \subset X$ and for each open subset V of Y, let $G(X_0, V)$ denote the collection of all maps $f \in Y^X$ such that $f(X_0) \subset V$. The family $\mathscr{B}$ of all such sets $G(X_0, V)$ is taken as a subbasis for the space Y^X (the "compact-open topology" in Y^X).

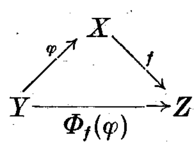
If the space X is compact and Y is metric, then we may introduce in Y^X a metric ϱ, compatible with the compact-open topology defined above, setting

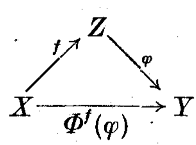
$$\varrho(\varphi, \psi) = \sup_{x \in X} \varrho\big(\varphi(x), \psi(x)\big) \quad \text{for} \quad \varphi, \psi \in Y^X.$$

Suppose now that X, Y and Z are three arbitrary spaces and that $f\colon X \to Z$ is a map. Let us set

$$\Phi_f(\varphi) = f\varphi \quad \text{for} \quad \varphi \in X^Y,$$
$$\Phi^f(\varphi) = \varphi f \quad \text{for} \quad \varphi \in Y^Z.$$

We see, from the diagrams below, that $\Phi_f\colon X^Y \to Z^Y$ and that Φ^f:





$Y^Z \to Y^X$.

We now show that

(9.2) *Φ_f and Φ^f are continuous.*

Indeed, let $\varphi_0 \in X^Y$ and let H be an element of the subbasis $\mathscr{B}$ of Z^Y (as defined above) containing the map $\Phi_f(\varphi_0) = f\varphi_0 \in Z^Y$. Thus there is a compact set $Y_0 \subset Y$ and an open subset V of Z containing the set $f_0(Y_0)$ and such that all maps $\psi \in Z^Y$ with $\psi(Y_0) \subset V$ belong to H. Since f is continuous and $f\varphi_0(Y_0) \subset V$, there is an open neighborhood U of the set $\varphi_0(Y_0)$ in X such that $f(U) \subset V$. Now let us consider the set $G \subset X^Y$ consisting of all maps $\varphi \in X^Y$ with $\varphi(Y_0) \subset U$. In order to prove that Φ_f is continuous it suffices, by (9.1), to show that Φ_f takes G into H. For that purpose, let us observe that if $\varphi \in G$, then the map $\Phi_f(\varphi) = f\varphi$ satisfies the inclusion $(f\varphi)(Y_0) = f(\varphi(Y_0)) \subset f(U) \subset V$, so that $\Phi_f(\varphi) \in H$ as it was required. By a similar argument we may prove the continuity of Φ^f.

We shall refer to Φ_f and Φ^f as the maps *induced* by f.

Now let X_0 be a subset of the space X. We assign to each map $\varphi\colon Y \to X_0$ the map $\psi\colon Y \to X$ given by the formula $\psi(y) = \varphi(y)$ for $y \in Y$, i.e. $\psi = i\varphi$, where $i\colon X_0 \to X$ is the inclusion map. Then the correspondence $\varphi \to \psi$, which is in fact identical with Φ_i, is a map of X_0^Y into X^Y.

Similarly, we assign to each map $\varphi \in Y^X$ the restriction $\psi = \varphi | X_0 \in Y^{X_0}$. Then the correspondence $\varphi \to \psi$, which is in fact identical with Φ_i where $i\colon X_0 \to X$ is the inclusion map, is a map of Y^X into Y^{X_0}.

As a corollary of (9.2), let us prove that:

(9.3) *If $f\colon X \to Z$ is an r-map and if $g\colon Z \to X$ is the right inverse of f, then the induced maps $\Phi_f\colon X^Y \to Z^Y$ and $\Phi^g\colon Y^X \to Y^Z$ are also r-maps.*

In order to prove this, let us consider the maps Φ_g and Φ^f which map Z^Y into X^Y and Y^Z into Y^X, respectively. Then for each map $\varphi \in Z^Y$ we have $(\Phi_f \Phi_g)(\varphi) = fg\varphi = \varphi$ and similarly for each map $\psi \in Y^Z$ we have $(\Phi^g \Phi^f)(\psi) = \psi fg = \psi$. Thus Φ_g is a right inverse of Φ_f and Φ^f is a right inverse of Φ^g.

Finally, let us observe that (9.1) implies that

(9.4) *The topological type of Y^X depends only on the topological types of X and Y.*

Indeed, suppose that $g\colon X \to \hat{X}$ and $h\colon Y \to \hat{Y}$ are homeomorphisms. Then the transformation χ of Y^X into $\hat{Y}^{\hat{X}}$, given by the formula

$$\chi(\varphi) = h\varphi g^{-1} \quad \text{for} \quad \varphi \in Y^X,$$

and the transformation $\hat{\chi}$ of $\hat{Y}^{\hat{X}}$ into Y^X, given by the formula

$$\hat{\chi}(\hat{\varphi}) = h^{-1}\hat{\varphi} g \quad \text{for} \quad \hat{\varphi} \in \hat{Y}^{\hat{X}},$$

are inverse to each other. Moreover,

$$\chi = \Phi_h \Phi^{g^{-1}} \quad \text{and} \quad \hat{\chi} = \Phi_{h^{-1}} \Phi^g$$

and we infer by (9.2) that χ and $\hat{\chi}$ are continuous.

10. Maps of pairs. In order to apply some notions of algebraic topology to the theory of retracts, it is convenient to consider not only maps of spaces, but also *maps of pairs of spaces*. By a *pair* of spaces, (X, X_0), we understand a pair consisting of a space X and of one of its subsets X_0. A pair of the form $(X, 0)$ will be identified with the space X. Also, if X_0 consists of a single point x_0, then we will write rather (X, x_0) than (X, X_0).

By a map $\varphi\colon (X, X_0) \to (Y, Y_0)$ we understand a map $\varphi\colon X \to Y$ satisfying the condition $\varphi(X_0) \subset Y_0$. Such a map will be called an *r-map* if there is a map $\psi\colon (Y, Y_0) \to (X, X_0)$ such that $\varphi\psi$ is the identity map on the pair (Y, Y_0). In particular, if $(Y, Y_0) \subset (X, X_0)$, i.e. if $Y \subset X$ and $Y_0 \subset X_0$, then a map $\varphi\colon (X, X_0) \to (Y, Y_0)$ is called a *retraction* if the inclusion $i\colon (Y, Y_0) \to (X, X_0)$ is a right inverse of φ. In this case, (Y, Y_0) is said to be a *retract* of (X, X_0) ([112], p. 30). In an obvious manner we may define the notion of *r-image*, *r-domination*, and *r-equality* for pairs. Thus, for example, $(X, X_0) \underset{r}{\leqslant} (Y, Y_0)$ means that there is an r-map of (Y, Y_0) onto (X, X_0).

The subset of the space Y^X, which consists of all those maps $\varphi \in Y^X$ for which $\varphi(X_0) \subset Y_0$, will be denoted by $(Y, Y_0)^{(X, X_0)}$. If $X_0 = 0$, then $(Y, Y_0)^{(X, X_0)}$ coincides with Y^X, even if $Y_0 \neq 0$. In a manner similar to that given above, a map $f\colon X \to Z$, lying now in $(Z, Z_0)^{(X, X_0)}$, will induce maps denoted, in analogy to the earlier situation, by Φ_f and Φ^f. Thus

$$\Phi_f\colon (X, X_0)^{(Y, Y_0)} \to (Z, Z_0)^{(Y, Y_0)}$$

and

$$\Phi^f\colon (Y, Y_0)^{(Z, Z_0)} \to (Y, Y_0)^{(X, X_0)}.$$

One easily sees from (9.3) that

(10.1) *If $f\colon (X, X_0) \to (Z, Z_0)$ is an r-map with a right inverse map $g\colon (Z, Z_0) \to (X, X_0)$, then both maps*

$$\Phi_f\colon (X, X_0)^{(Y, Y_0)} \to (Z, Z_0)^{(Y, Y_0)}$$

and

$$\Phi^g\colon (Y, Y_0)^{(X, X_0)} \to (Y, Y_0)^{(Z, Z_0)}$$

are r-maps.

11. Homotopies. Two maps f_0 and f_1 in $(Y, Y_0)^{(X, X_0)}$ are said to be *homotopic* provided that for every $t \in \langle 0, 1\rangle$ there is a map $f_t \in (Y, Y_0)^{(X, X_0)}$, with f_t depending continuously on t. We say that the family of maps $\{f_t\}$ is a *homotopy joining f_0 with f_1*. We observe that the subset of the space $(Y, Y_0)^{(X, X_0)}$ consisting of all the maps f_t is a continuous image of the interval $\langle 0, 1\rangle$. We easily see that the relation "f_0 is homotopic to f_1" between pairs of elements of $(Y, Y_0)^{(X, X_0)}$ is a reflexive, symmetric, and transitive relation on $(Y, Y_0)^{(X,X_0)}$ and hence induces a decomposition of this space into disjoint classes of homotopic maps. These classes are called *homotopy classes*.

The homotopy class containing a given map $f \in (Y, Y_0)^{(X,X_0)}$ will be denoted by $[f]$ and the map f will be called a *representative* of this homotopy class. Evidently $[f] = [f']$ if and only if the maps f and f' are homotopic. The set of all homotopy classes of maps $f\colon (X, X_0) \to \to (Y, Y_0)$ will be denoted by $[Y, Y_0]^{[X,X_0]}$.

In the special case where $X_0 = Y_0 = 0$, the pairs (X, X_0) and (Y, Y_0) coincide with the spaces X and Y, respectively, and thus we can speak about a homotopy joining two maps $f_0, f_1 \in Y^X$ in the space Y^X.

It is sometimes useful to consider two other notions related to the notion of homotopy: the *weak homotopy* and the *strong homotopy*. We say that two maps $f_0, f_1 \in (Y, Y_0)^{(X,X_0)}$ are *weakly homotopic* provided they belong to one component of the space $(Y, Y_0)^{(X,X_0)}$. Evidently the homotopy implies the weak homotopy. We say that two maps $f_0, f_1 \in (Y, Y_0)^{(X,X_0)}$ are *strongly homotopic* provided there exists a map

$$\varphi\colon (X \times \langle 0, 1\rangle, X_0 \times \langle 0, 1\rangle) \to (Y, Y_0),$$

where $\langle 0, 1\rangle$ denotes the interval $0 \leqslant t \leqslant 1$, i.e. a map assigning to every pair (x, t) with $x \in X$ and $0 \leqslant t \leqslant 1$ a point $\varphi(x, t) \in Y$ and satisfying two conditions:

(11.1) $\qquad \varphi(x, t) \in Y_0 \quad$ for every point $x \in X_0$ and $0 \leqslant t \leqslant 1$,

(11.2) $\varphi(x, 0) = f_0(x) \quad$ and $\quad \varphi(x, 1) = f_1(x) \quad$ for every point $x \in X$.

Let us prove the following

(11.3) THEOREM. *Two strongly homotopic maps are homotopic. In the case where X satisfies the first axiom of countability, two homotopic maps are strongly homotopic.*

Proof. First let us suppose that $f_0, f_1 \in (Y, Y_0)^{(X,X_0)}$ are strongly homotopic. Then there is a map $\varphi \in (Y, Y_0)^{(X \times \langle 0,1\rangle, X_0 \times \langle 0,1\rangle)}$ satisfying both conditions (11.1) and (11.2). Let us assign to every $0 \leqslant t \leqslant 1$ the map $f_t \in (Y, Y_0)^{(X,X_0)}$ given by the formula

$$f_t(x) = \varphi(x, t) \quad \text{for every point } x \in X.$$

By (11.2), this notation is unambiguous. It remains to show that f_t depends continuously on the parameter $0 \leqslant t \leqslant 1$. In order to show that, let us consider a number $t_0 \in \langle 0, 1\rangle$ and let H be an element of the subbasis $\mathscr{B}$ of Y^X (as defined in Section 9), containing the map f_{t_0}. Then there exists a compact set $A \subset X$ and an open subset V of Y containing the set $f_{t_0}(A) = \varphi(A, t_0)$ such that all maps $f \in (Y, Y_0)^{(X,X_0)}$ which satisfy the condition $f(A) \subset V$ belong to H. Since φ is continuous and A is compact, we infer that there exists a neighborhood G of t_0 in the interval $\langle 0, 1\rangle$

such that $\varphi(A\times G)\subset V$, i.e. $f_t(A)\subset V$ for every $t\in G$. It follows that $f_t\in V_0\subset V$ for every $t\in G$, and we infer by (9.1) that f_t depends continuously on t. Hence f_0 and f_1 are homotopic.

Now let us assume that X satisfies the first axiom of countability and that the maps f_0 and f_1 are homotopic. Then there exists a map assigning to every $0\leqslant t\leqslant 1$ a map $f_t\in(Y, Y_0)^{(X,X_0)}$. Setting

$$\varphi(x, t) = f_t(x) \quad \text{for every } x\in X \text{ and } 0\leqslant t\leqslant 1,$$

we get a function $\varphi\colon (X\times\langle 0,1\rangle, X_0\times\langle 0,1\rangle)\to(Y, Y_0)$. It remains to show that φ is continuous.

Since X, and consequently also $X\times\langle 0,1\rangle$, satisfy the first axiom of countability, it is sufficient to show ([176], p. 225) that for every sequence of points $(x_n, t_n)\in X\times\langle 0,1\rangle$ convergent to (x_0, t_0) the values $\varphi(x_n, t_n)$ converge to $\varphi(x_0, t_0)$. Let V be a neighborhood of the point $\varphi(x_0, t_0) = f_{t_0}(x_0)$ in the space Y. Since f_{t_0} is a map, the set V is a neighborhood of the point $f_{t_0}(x_n)$ for almost all indices n. Let A denote the set consisting of all such points x_n and of the point x_0. It is clear that A is a compact subset of X and V is a neighborhood of the set $f_{t_0}(A)$ in the space Y. Since f_t depends continuously on t, we infer that there exists a neighborhood G (in the interval $\langle 0,1\rangle$) of t_0 such that $f_t(A)\subset V$ for every $t\in G$. If we recall that $x_n\in A$ and $t_n\in G$ for almost all indices n, we infer that $f_{t_n}(x_n)\in V$ for almost all indices n. Thus $\lim_{n\to\infty}\varphi(x_n, t_n) = \varphi(x_0, t_0)$ and the proof is finished.

12. Deformation retracts. A pair (A, A_0) contained in the pair (X, X_0) is said to be a *deformation retract* of (X, X_0) provided that there is a homotopy $\{f_t\}$ joining in the space $(X, X_0)^{(X,X_0)}$ the identity map $f_0 = i$ with a map f_1 such that the map $r\colon (X, X_0)\to(A, A_0)$ defined by the formula

$$r(x) = f_1(x) \quad \text{for every point } x\in X$$

is a retraction.

More generally, a pair (Y, Y_0) will be said to be a *deformation r-image of a pair* (X, X_0) if there exists a map $f\colon (X, X_0)\to(Y, Y_0)$ and a map $g\colon (Y, Y_0)\to(X, X_0)$ which is a right inverse of f and such that gf is homotopic to the identity map on (X, X_0). We say that f is a *deformation r-map* of (X, X_0) into (Y, Y_0). In particular, if $X_0 = Y_0 = 0$, we get the notion of *deformation retract* ([23], p. 91) and of *deformation r-image* of a space. We easily see that the deformation r-images of a space X coincide with the spaces homeomorphic to deformation retracts of X.

(12.1) Example. Let S^{n-1} be the sphere in E^n defined by the equation $\|x\| = 1$. It is plain that S^{n-1} is a deformation retract of the set $E^n-(0)$. However, if we remove from E^n a point a such that $0\neq\|a\|\neq 1$, then S^{n-1} is a retract of $E^n-(0)-(a)$ but is not a deformation retract of this set.

This apparently obvious assertion is a consequence of some general theorems, which will be proved later on (II, (4.10)).

(12.2) EXAMPLE. *Let S^n be an n-dimensional Euclidean sphere and let a, b be two distinct points of this sphere. Then the pair* $\big(Z, (a, a)\big)$, *where* $Z \doteq \big(S^n \times (a)\big) \cup \big((a) \times S^n\big) \subset S^n \times S^n$, *is a deformation retract of* $\big(S^n \times S^n - (b, b), (a, a)\big)$.

In order to prove this we shall consider S^n as the space obtained from the n-dimensional cube I^n by identification of all the points belonging to the boundary $S^{n-1} = (I^n)^\bullet$ of I^n. We may assume that a is the point of S^n obtained from S^{n-1} by this identification. In other words, there is a map $g\colon I^n \to S^n$ such that $g(S^{n-1}) = (a)$ and g maps $I^n - S^{n-1}$ onto $S^n - (a)$ homeomorphically. It follows that $g^{-1}(b)$ consists of one point $c \in I^n - S^{n-1}$.

Now let us consider the $2n$-dimensional cube $I^{2n} = I^n \times I^n$. If we set $\gamma(x, y) = \big(g(x), g(y)\big)$ for each $(x, y) \in I^{2n} = I^n \times I^n$, we obtain a map $\gamma\colon I^{2n} \to S^n \times S^n$. Evidently, the boundary of $I^n \times I^n$ is a $(2n-1)$-dimensional sphere S^{2n-1} which is in fact the union $(I^n \times S^{n-1}) \cup (S^{n-1} \times I^n)$. Under the map γ the set $I^n \times S^{n-1}$ goes onto $S^n \times (a)$ and $S^{n-1} \times I^n$ goes onto $(a) \times S^n$. Thus, $\gamma(S^{2n-1}) = Z$.

Since (c, c) belongs to the interior of I^{2n}, we easily see that there exists a map $\varphi\colon \big(I^{2n} - (c, c)\big) \times \langle 0, 1\rangle \to I^{2n} - (c, c)$ such that

$$\varphi\big((x, x'), t\big) = (x, x') \quad \text{for every } (x, x') \in S^{2n-1} \text{ and every } t \in \langle 0, 1\rangle,$$

$$\varphi\big((x, x'), 0\big) = (x, x') \quad \text{for every } (x, x') \in I^{2n} - (c, c),$$

$$\varphi\big((x, x'), 1\big) \in S^{2n-1} \quad \text{for every point } (x, x') \in I^{2n} - (c, c).$$

In fact, it is sufficient to move each point $(x, x') \in I^{2n} - (c, c)$ along the ray from (c, c) over (x, x') to the point in which this ray pierces the sphere S^{2n-1}.

Finally, let us set, for all points $p \in \big(S^n \times S^n - (b, b)\big)$ and for all $t \in \langle 0, 1\rangle$,

$$f_t(p) = \gamma\varphi\big(\gamma^{-1}(p), t\big).$$

We obtain a homotopy joining the identity f_0 with the retraction f_1 of the set $S^n \times S^n - \big((b, b)\big)$ to the set Z.

Manifestly, $f_t(a, a) = (a, a)$ for all t and so the proof is complete.

By a similar argument, one can prove ([59], p. 100) that

(12.3) *The pair consisting of the set*

$$W = \big(S^n \times (a) \times (a)\big) \cup \big((a) \times S^n \times (a)\big) \cup \big((a) \times (a) \times S^n\big)$$

and of the point (a, a, a) is a deformation retract of the pair consisting of the set $S^n \times S^n \times S^n - W'$, *where*

$$W' = \big(S^n \times (b) \times (b)\big) \cup \big((b) \times S^n \times (b)\big) \cup \big((b) \times (b) \times S^n\big),$$

and of the point (a, a, a).

13. Contractibility. A set $A \subset X$ is said to be *contractible in the space* X *to a set* $B \subset X$ provided that the inclusion $i\colon A \to X$ is homotopic (in the space X^A) to a map with values belonging to B. If A is contractible to a set B which consists of a single point, then A is said to be *contractible in* X. In particular, a space X may be contractible in itself. Then we write $X \in \mathbf{C}$.

(13.1) EXAMPLE. *Each convex subset A of a linear space X is contractible in itself.*

Indeed, if we fix $a \in A$, then setting

$$f_t(x) = ta + (1-t)x \quad \text{for} \quad x \in A \text{ and } t \in \langle 0, 1 \rangle,$$

we obtain a homotopy $\{f_t\}$ joining, in the space $(A, a)^{(A,a)}$, the identity map f_0 with the map f_1 of A onto (a).

(13.2) THEOREM. *If the space X is contractible in itself, then each r-image of X is also contractible in itself.*

Proof. Let $f\colon X \to Y$ be an r-map with a right inverse $g\colon Y \to X$. Since X is contractible in itself, there is a homotopy $\{f_t\} \subset X^X$ joining the identity map f_0 with a map f_1 of X onto a point. If we put

$$g_t(y) = (ff_tg)(y) \quad \text{for every} \quad y \in Y,$$

then we obtain a family $\{g_t\} \subset Y^Y$ such that g_0 is the identity map and g_1 maps Y onto a point. Moreover, the map g_t depends continuously on the parameter t, because, as we have already shown, the operations Φ_f and Φ^g are continuous.

Thus we see that the contractibility of a space in itself is an $\underline{r}$-invariant.

14. h-maps and homotopy domination. It is convenient to introduce the notion of an h-map, being a generalization of r-map (cf. [9] and [252]). We say that a map $f\colon (X, X_0) \to (Y, Y_0)$ is an *h-map* provided that there exists a map $g\colon (Y, Y_0) \to (X, X_0)$ such that the composition fg mapping (Y, Y_0) into itself is homotopic to the identity ([67], p. 321). We say that g is a *homotopic right inverse of* f.

If there is an h-map of (X, X_0) into (Y, Y_0), then we shall say ([288], p. 1133) that (X, X_0) *homotopically dominates* (Y, Y_0) or, shortly, (X, X_0) *h-dominates* (Y, Y_0). In symbols we express this as follows:

$$(X, X_0) \underset{h}{\geqslant} (Y, Y_0) \quad \text{or} \quad (Y, Y_0) \underset{h}{\leqslant} (X, X_0).$$

It is clear that every r-map is an h-map and hence, if the relation $(X, X_0) \underset{r}{\geqslant} (Y, Y_0)$ holds, then so does the relation $(X, X_0) \underset{h}{\geqslant} (Y, Y_0)$.

Note that if $X_0 = Y_0 = 0$, then the former relations specialize to relations $X \underset{h}{\geqslant} Y$ or $Y \underset{h}{\leqslant} X$.

Let us observe that the relation $(X, X_0) \underset{h}{\geqslant} (Y, Y_0)$ does not generally imply the relation $(X, X_0) \underset{r}{\geqslant} (Y, Y_0)$. Indeed, if $X = (0)$ and $Y = \langle 0, 1\rangle$, then $X \underset{r}{\geqslant} Y$ is false, but $X \underset{h}{\geqslant} Y$ is true since the map f of $\langle 0, 1\rangle$ into itself taking every point onto 0 is homotopic with the identity.

The relation $\underset{h}{\geqslant}$ is clearly transitive. Because, if we have h-maps $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (Y, Y_0) \to (Z, Z_0)$ with homotopy right inverses f' and g', respectively, then we see that $f'g'$ is a homotopy right inverse of gf.

If both relations $(X, X_0) \underset{h}{\geqslant} (Y, Y_0)$ and $(X, X_0) \underset{h}{\leqslant} (Y, Y_0)$ hold, then we say that (X, X_0) and (Y, Y_0) are *homotopically equal* and write $(X, X_0) \underset{h}{=} (Y, Y_0)$. It is clear that the relation $(X, X_0) \underset{r}{=} (Y, Y_0)$ implies $(X, X_0) \underset{h}{=} (Y, Y_0)$, but not conversely.

Following Hurewicz ([164], p. 125), we shall say that two pairs (X, X_0) and (Y, Y_0) are *homotopically equivalent* if we can find the maps $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (Y, Y_0) \to (X, X_0)$ such that each of the compositions fg and gf is homotopy to the appropriate identity. We see that the relation of homotopy equivalence is more restrictive than the notion of homotopic equality. Now, if (X, X_0) and (Y, Y_0) are homotopically equivalent, then we shall write $(X, X_0) \underset{h}{\cong} (Y, Y_0)$. It is easy to see that $\underset{h}{=}$ and $\underset{h}{\cong}$ are both equivalence relations, so that the class of all spaces can be decomposed, on the one hand, into disjoint classes of homotopically equal spaces and, on the other hand, into classes of homotopically equivalent spaces (called *homotopy types*). The latter classes are not larger than the former ones since homotopy equivalence implies homotopy equality.

We see that

(14.1) *If Y is a deformation r-image of a space X, then $X \underset{h}{\cong} Y$ and therefore $X \underset{h}{=} Y$.*

Indeed, there is by hypothesis a map $f\colon X \to Y$ having a right inverse $g\colon Y \to X$ such that $gf\colon X \to X$ is homotopic to the identity. Moreover, since the map $fg\colon Y \to Y$ is equal to the identity, we infer that $X \underset{h}{\cong} Y$. In particular,

(14.2) *If Y is a deformation retract of X, then $X \underset{h}{\cong} Y$.*

One can also prove ([126], p. 45), applying a remark due to H. Samelson [256], that two homotopically equivalent spaces can be embedded in a third space in such a manner that they are both deformation retracts of it. Some other conditions characterizing deformation retracts (of polyhedra) are given in [154] and [160]. Compare also [286].

15. Local contractibility. A space X is said to be *locally contractible at a point* $x_0 \in X$ ([19], p. 952) provided that each neighborhood U of x_0 contains a neighborhood U_0 of x_0 which is contractible in U to a point ([21], p. 235). It is evident that the local contractibility at a point x_0 implies the local arcwise connectivity at this point. A space X is said to be *locally contractible* if it is locally contractible at each of its points. For the sake of brevity we shall write $X \in \mathbf{LC}$ if X is a locally contractible space. We see that every open subset of a locally contractible space is itself locally contractible.

Since the local contractibility implies the local arcwise connectedness, we infer by (8.1) that

(15.1) *Every component of a locally contractible space is arcwise connected.*

(15.2) EXAMPLE. *Each convex subset X of a linear space M is locally contractible.*

Indeed, if $x_0 \in X$, then setting

$$\varphi_t(x) = tx_0 + (1-t)x \quad \text{for} \quad x \in X \text{ and } t \in \langle 0, 1 \rangle,$$

we obtain a homotopy $\{\varphi_t\}$ which contracts the set X in itself to the point x_0. Then, for each neighborhood U of x_0 in X, there is a neighborhood U_0 of x_0 such that the values $\varphi_t(x)$ belong to U for all $x \in U_0$ and $t \in \langle 0, 1 \rangle$.

Now we prove the following

(15.3) THEOREM. *If X_0 is a retract of X and if X is locally contractible at $x_0 \in X_0$, then X_0 is locally contractible at x_0.*

Proof. Let $r: X \to X_0$ be a retraction and let U be a neighborhood of x_0 in X_0. Then the set $V = r^{-1}(U)$ is a neighborhood of x_0 in X. Since the space X is locally contractible at x_0, there is a neighborhood V_0 of x_0 which is contained in V and which is contractible in V to a point. This means that there is a homotopy $\{\varphi_t\} \subset V^{V_0}$ which joins the inclusion map $\varphi_0: V_0 \to V$ with a constant map φ_1 (taking V_0 into a point of V). Since $V = r^{-1}(U)$, we infer that the maps $\varphi'_t: V_0 \to U$, defined by the formula

$$\varphi'_t(x) = (r\varphi_t)(x) \quad \text{for} \quad x \in V_0,\ t \in \langle 0, 1 \rangle,$$

constitute a homotopy $\{\varphi'_t\} \subset U^{V_0}$. We observe that $U_0 = V_0 \cap X_0$ is a neighborhood of the point x_0 in X_0. Moreover, the restrictions $\psi_t = \varphi'_t | U_0$

constitute a homotopy $\{\psi_t\} \subset U^{U_0}$ joining the map ψ_0, which is the inclusion of U_0 into U, with the map ψ_1, which takes U_0 into one point of U. This shows that U_0 can be contracted in U to a point. Thus the proof is concluded.

(15.4) COROLLARY. *Every r-image of a locally contractible space is locally contractible.*

In other words, *local contractibility is an r-invariant.*

16. Homotopically trivial spaces. A space X will be said to be *homotopically trivial over a space A* provided that each map $\varphi: A \to X$ is homotopic to a constant map. We say that *X is locally homotopically trivial over a space A at a point $x_0 \in X$* provided that for each neighborhood U of x_0 in X, there is a neighborhood U_0 of x_0 contained in U such that every map $\varphi: A \to U$ with values in U_0 is homotopic in U^A to a constant map. If this condition is satisfied at every point of X, then we shall say that X *is locally homotopically trivial over A.*

(16.1) THEOREM. *If a space X is homotopically trivial over a space A, then every r-image of X is also homotopically trivial over A.*

Proof. Let $f: X \to Y$ be an r-map of X onto a space Y and let $g: Y \to X$ be a right inverse of f. Suppose that ψ is a map of A into Y. Then $\varphi = g\psi: A \to X$ and since X is homotopically trivial over A, the map φ is homotopic to a constant. Since, as we have seen, the operation $\Phi_f: X^A \to Y^A$, which assigns to each map $\varphi': A \to X$ the map $f\varphi': A \to Y$, is continuous (I, (9.1)), it follows that $f\varphi = fg\psi = \psi$ is homotopic to a constant. This completes the proof.

(16.2) THEOREM. *If X_0 is a retract of a space X and if the space X is locally homotopically trivial over a space A at a point $x_0 \in X_0$, then the space X_0 is locally homotopically trivial over A at the point x_0.*

Proof. Let $r: X \to X_0$ be a retraction and let U be a neighborhood of the point x_0 in the space X_0; then the set $V = r^{-1}(U)$ is a neighborhood of x_0 in X. Let us denote by r_0 the map of V into U given by the formula

$$r_0(x) = r(x) \quad \text{for every point } x \in V.$$

Since the space X is locally homotopically trivial over A at x_0, we infer that there exists a neighborhood V_0 of x_0 contained in V such that every map $\psi \in V^A$ having values in V_0 is homotopic (in V^A) to a constant map. The set $U_0 = V_0 \cap X_0 \subset U \subset V$ is a neighborhood of x_0 in X_0.

Suppose now that $\varphi: A \to U$ is a map having values in U_0. Then if we put $\psi = i\varphi$, where $i: U \to V$ is the inclusion map, we obtain a map ψ of A into V with values in V_0. It follows that ψ is homotopic to a constant map. Since the operation Φ_{r_0} is continuous, we infer that the map $r_0\psi = r_0 i\varphi = \varphi$ is homotopic to a constant. Thus the proof of the theorem is completed.

From (16.2) and (2.1) we obtain the following

(16.3) COROLLARY. *Each r-image of a space which is locally homotopically trivial over a given space A, is locally homotopically trivial over A.*

Thus we see that the local homotopic triviality of a space over a given space is an $\underline{r}$-invariant.

17. n-connectedness and local n-connectedness. A space X is said to be *n-connected* provided that it is homotopically trivial over the n-dimensional sphere S^n. A space is said to be *locally n-connected* ([210], p. 119) if it is locally homotopically trivial over S^n. Examining these definitions we see that 0-connectedness is nothing else but the arcwise connectedness while local 0-connectedness is the same as local arcwise connectedness. Similarly it is clear that 1-connectedness is a reformulation of the notion of simple connectedness.

If a space X is k-connected for each $k = 0, 1, 2, \ldots, n$, then we shall write $X \in \mathbf{C}^n$ while if X is locally k-connected for $k = 0, 1, 2, \ldots, n$, then we shall write $X \in \mathbf{LC}^n$. If $X \in \mathbf{C}^n$ for every n, then we write $X \in \mathbf{C}^\infty$. Similarly, $X \in \mathbf{LC}^\infty$ will mean that $X \in \mathbf{LC}^n$ for every n. We recall that $X \in \mathbf{C}$ (resp. $X \in \mathbf{LC}$) indicates that X is contractible (resp. locally contractible). We then observe that

(17.1) $\mathbf{C}$ *implies* $\mathbf{C}^\infty$ *which implies* $\mathbf{C}^n$ *for every* $n = 0, 1, 2, \ldots,$

(17.2) $\mathbf{LC}$ *implies* $\mathbf{LC}^\infty$ *which implies* $\mathbf{LC}^n$ *for every* $n = 0, 1, 2, \ldots,$

(17.3) $\mathbf{C}^n$ *implies* $\mathbf{C}^m$ *for* $n \geqslant m$,

(17.4) $\mathbf{LC}^n$ *implies* $\mathbf{LC}^m$ *for* $n \geqslant m$,

(17.5) *The properties* $\mathbf{C}, \mathbf{LC}, \mathbf{C}^\infty, \mathbf{LC}^\infty, \mathbf{C}^n, \mathbf{LC}^n$ *are $\underline{r}$-invariants.*

(17.6) EXAMPLE. *The n-dimensional Euclidean sphere S^n has the property $\mathbf{C}^{n-1}$ but has not the property $\mathbf{C}^n$. Each polyhedron has the property* $\mathbf{LC}$ *and consequently,* $\mathbf{LC}^\infty$ *and* $\mathbf{LC}^n$.

EXAMPLE. We give an example of a space X_n having property $\mathbf{LC}^{n-1}$ but not $\mathbf{LC}^n$ and of a space X_∞ having property $\mathbf{LC}^\infty$ but not $\mathbf{LC}$. In order to do this, let us denote by a_k, where $k = 1, 2, 3, \ldots$, the point of Hilbert space E^ω given by the formula

$$a_k = \left(\frac{2k+1}{2k(k+1)}, 0, 0, 0, \ldots\right).$$

The points a_k clearly converge to the origin of E^ω, which we shall denote by a_0. Let S_k^n denote the n-dimensional sphere in E^ω consisting of all points $x = \{x_i\}$ such that $\varrho(x, a_k) = 1/2k(k+1)$ and such that $x_i = 0$ for $i > n+1$. It is easy to see that the set

$$X_n = (a_0) \cup \bigcup_{k=1}^{\infty} S_k^n \tag{17.7}$$

satisfies $\mathbf{LC}^{n-1}$ but not $\mathbf{LC}^n$.

Moreover, the set

$$X_\infty = (a_0) \cup \bigcup_{k=1}^{\infty} S_k^k \tag{17.8}$$

satisfies the condition $\mathbf{LC}^\infty$ but does not satisfy $\mathbf{LC}$.

CHAPTER II

ALGEBRAIC RELATIONS INDUCED BY MAPS

In this chapter we shall consider the homomorphisms induced in homology and homotopy groups by continuous maps, in particular by h-maps and r-maps. Incidentally, we shall recall the principal notions and theorems concerning these groups, which will be necessary in the sequel.

1. r-homomorphisms. A homomorphism f of a group $\mathfrak{A}$ into a group $\mathfrak{B}$ is called an *r-homomorphism* ([76], p. 331) if there is a homomorphism $g: \mathfrak{B} \to \mathfrak{A}$ which is a *right inverse* of f, i.e. such that fg is the identity on $\mathfrak{B}$.

We observe the following simple facts.

(1.1) *Every r-homomorphism $f: \mathfrak{A} \to \mathfrak{B}$ is an epimorphism (is onto) and the right inverse $g: \mathfrak{B} \to \mathfrak{A}$ is a monomorphism (is one-to-one).*

(1.2) *If $f: \mathfrak{A} \to \mathfrak{B}$ is an r-homomorphism, then $\mathfrak{B}$ is isomorphic to the subgroup $\mathfrak{A}_0 = g(\mathfrak{B})$ of $\mathfrak{A}$.*

(1.3) *The composition of two r-homomorphisms is an r-homomorphism.*

If there is an r-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then we say that $\mathfrak{B}$ is an *r-image* of $\mathfrak{A}$. From (1.2) we see that each r-image of $\mathfrak{A}$ may be identified with a subgroup of $\mathfrak{A}$.

A group $\mathfrak{B}$ is said to be a *divisor* of a group $\mathfrak{A}$ provided that there exists a group $\mathfrak{C}$ such that $\mathfrak{A}$ and $\mathfrak{B}\times\mathfrak{C}$ are isomorphic. We recall that $\mathfrak{B}\times\mathfrak{C}$ (where $\mathfrak{B}$ and $\mathfrak{C}$ are any groups) is the group consisting of all ordered pairs (y, z) with $y \in \mathfrak{B}$ and $z \in \mathfrak{C}$ and with multiplication defined by the formula

$$(y, z)\cdot(y', z') = (yy', zz').$$

(1.4) THEOREM. *Each divisor $\mathfrak{B}$ of a group $\mathfrak{A}$ is an r-image of $\mathfrak{A}$.*

Proof. Let $h: \mathfrak{A} \to \mathfrak{B}\times\mathfrak{C}$ be an isomorphism. Then h can be described by means of the formula

$$h(x) = \big(f(x), f'(x)\big)$$

where f: $\mathfrak{A} \to \mathfrak{B}$ and f': $\mathfrak{A} \to \mathfrak{C}$ are homomorphisms. Let e denote the neutral element of $\mathfrak{C}$ and let

$$g(y) = h^{-1}(y, e) \quad \text{for every } y \in \mathfrak{B}.$$

Then it is clear that g: $\mathfrak{B} \to \mathfrak{A}$ is a right inverse of f: $\mathfrak{A} \to \mathfrak{B}$ and that f is therefore an r-homomorphism.

A *direct divisor* of a group $\mathfrak{A}$ will mean a subgroup $\mathfrak{A}'$ of $\mathfrak{A}$ such that there is another subgroup $\mathfrak{A}''$ of $\mathfrak{A}$ satisfying the following condition:

Each $x \in \mathfrak{A}$ can be uniquely represented in the form $x = x' \cdot x''$ with $x' \in \mathfrak{A}'$ and $x'' \in \mathfrak{A}''$ and moreover $x' \cdot x'' = x'' \cdot x'$ for every $x' \in \mathfrak{A}'$ and $x'' \in \mathfrak{A}''$. Clearly, the map h: $\mathfrak{A} \to \mathfrak{A}' \times \mathfrak{A}''$ given by $h(x) = (x', x'')$ is an isomorphism; consequently a direct divisor of $\mathfrak{A}$ is, as it is suggested by the name, a divisor of $\mathfrak{A}$. But we can prove a little more, namely

(1.5) THEOREM. *A group $\mathfrak{B}$ is a divisor of a group $\mathfrak{A}$ if and only if $\mathfrak{B}$ is isomorphic to a direct divisor of $\mathfrak{A}$.*

We have already seen that if $\mathfrak{B}$ is isomorphic to a direct divisor of $\mathfrak{A}$, then it is a divisor of $\mathfrak{A}$. It remains to show that if there is a group $\mathfrak{C}$ and an isomorphism h: $\mathfrak{A} \to \mathfrak{B} \times \mathfrak{C}$, then $\mathfrak{B}$ is isomorphic to a direct divisor of $\mathfrak{A}$. Let b denote the neutral element of $\mathfrak{B}$ and c the neutral element of $\mathfrak{C}$, and let $\mathfrak{B}' = \mathfrak{B} \times (c)$ and $\mathfrak{C}' = (b) \times \mathfrak{C}$. It is evident that $\mathfrak{B}'$ and $\mathfrak{C}'$ are direct divisors of the group $\mathfrak{B} \times \mathfrak{C}$. Their images $\mathfrak{A}' = h^{-1}(\mathfrak{B}')$ and $\mathfrak{A}'' = h^{-1}(\mathfrak{C}')$ by the isomorphism h^{-1}: $\mathfrak{B} \times \mathfrak{C} \to \mathfrak{B}$ are direct divisors of the group $\mathfrak{A}$. Since the group $\mathfrak{B}$ is isomorphic to $\mathfrak{B}'$ and the group $\mathfrak{B}'$ is isomorphic to $\mathfrak{A}'$, we infer that $\mathfrak{B}$ is isomorphic to a direct divisor of $\mathfrak{A}$.

In the sequel we shall mostly consider Abelian groups. Thus, unless the contrary is stated, a group will always mean an Abelian group. In conformity with general practice, we shall use additive notations for (Abelian) groups. Thus "$+$" denotes the operation of composition, 0 the neutral element, and $-x$ the inverse of x.

Now let us prove the following

(1.6) THEOREM. *Let $\mathfrak{A}$ be an Abelian group and f: $\mathfrak{A} \to \mathfrak{B}$ be an epimorphism. Then f is an r-homomorphism if and only if the kernel $\mathfrak{A}'$ of f is a direct divisor of $\mathfrak{A}$. In that case $\mathfrak{A}$ is isomorphic to $\mathfrak{A}' \times \mathfrak{B}$.*

Proof. Suppose that $\mathfrak{A}' = \mathrm{Ker}(f)$ is a direct divisor of $\mathfrak{A}$. Then there is a subgroup $\mathfrak{A}''$ of $\mathfrak{A}$ such that each $x \in \mathfrak{A}$ is uniquely expressible as $x = x' + x''$ where $x' \in \mathfrak{A}'$ and $x'' \in \mathfrak{A}''$. We notice that $\varphi = f|\mathfrak{A}''$: $\mathfrak{A}'' \to \mathfrak{B}$ is in fact an isomorphism. Indeed, if $y \in \mathfrak{B}$ then, since f is an epimorphism, there is an $x \in \mathfrak{A}$ with $f(x) = y$. If $x = x' + x''$ (as above), then $f(x) = f(x'')$ since $x' \in \mathfrak{A}' = \mathrm{Ker}(f)$. This shows that φ is an epimorphism. In order to see that φ is a monomorphism, suppose that $\varphi(x'') = \varphi(\bar{x}'') = y$ where x'', $\bar{x}'' \in \mathfrak{A}''$. Then $f(\bar{x}'') - f(x'') = 0$ and therefore $\bar{x}'' - x'' \in \mathfrak{A}' = \mathrm{Ker}(f)$. Hence

$x'' = \bar{x}''$ by the uniqueness of decomposition of elements of $\mathfrak{A}$. Setting $g = \varphi^{-1}$, we see that g is the required right inverse of f: $\mathfrak{A} \to \mathfrak{B}$. Since $\mathfrak{B}$ is isomorphic to $\mathfrak{A}''$, we see that $\mathfrak{A}$ is isomorphic to $\mathfrak{A}' \times \mathfrak{B}$.

To show the converse, suppose that f is an r-homomorphism. Let g: $\mathfrak{B} \to \mathfrak{A}$ be a right inverse of f. Let us set $\mathfrak{A}' = \mathrm{Ker}(f)$ and $\mathfrak{A}'' = g(\mathfrak{B})$. We show that each $x \in \mathfrak{A}$ has a unique representation in the form $x = x' + x''$ where $x' \in \mathfrak{A}'$ and $x'' \in \mathfrak{A}''$. Indeed, if we put $x' = x - gf(x)$ and $x'' = gf(x)$, then we surely have $x = x' + x''$. Moreover, since $f(x') = f(x) - fgf(x) = f(x) - f(x) = 0$, it follows that $x' \in \mathfrak{A}'$. It is clear that $x'' \in \mathfrak{A}''$. This representation of x is unique in fact, because if $x = \bar{x}' + \bar{x}''$ were another such representation, then we should have $f(x) = f(\bar{x}'')$ since $\bar{x}' \in \mathrm{Ker}(f)$ and hence, putting $\bar{x}'' = g(b)$ with $b \in \mathfrak{B}$, we get $gf(x) = gf(\bar{x}'') = gf(g(b)) = g(b) = \bar{x}''$. Thus $\bar{x}'' = x''$ and therefore $x' = \bar{x}'$. This completes the proof.

(1.7) Corollary. *A group $\mathfrak{B}$ is an r-image of an Abelian group $\mathfrak{A}$ if and only if $\mathfrak{B}$ is a divisor of $\mathfrak{A}$.*

If $\mathfrak{B}$ is an r-image of a group $\mathfrak{A}$, then we write $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$ or $\mathfrak{A} \underset{r}{\geqslant} \mathfrak{B}$. It follows by (1.3) that the relation "$\underset{r}{\leqslant}$" is transitive, i.e. $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$ and $\mathfrak{C} \underset{r}{\leqslant} \mathfrak{B}$ imply $\mathfrak{C} \underset{r}{\leqslant} \mathfrak{A}$. It is clear by (1.2) that if $\mathfrak{A}$ is an Abelian group, then every group $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$ is also Abelian. By Corollary (1.7), we see that for Abelian groups the relation $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$ is equivalent to the condition that $\mathfrak{B}$ is a divisor of $\mathfrak{A}$. If both relations $\mathfrak{A} \underset{r}{\leqslant} \mathfrak{B}$ and $\mathfrak{B} \underset{r}{\leqslant} \mathfrak{A}$ hold, then the groups $\mathfrak{A}$ and $\mathfrak{B}$ will be said to be *r-equal* (in symbols, $\mathfrak{A} \underset{r}{=} \mathfrak{B}$). It is easy to see that if $\mathfrak{A}$ is an Abelian group with a finite system of generators, then the relation $\mathfrak{A} \underset{r}{=} \mathfrak{B}$ means just the same as the isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$. However, for arbitrary Abelian groups the analogous statement fails, as it was shown by E. Sąsiada ([257], p. 331).

We shall say that a homomorphism f: $\mathfrak{A} \to \mathfrak{B}$ is *left invertible relatively to a homomorphism* φ: $\mathfrak{C} \to \mathfrak{A}$ provided that there is a homomorphism ψ: $\mathfrak{B} \to \mathfrak{A}$ with $\psi f \varphi = \varphi$. If $\mathfrak{A} = \mathfrak{C}$ and φ is the identity on $\mathfrak{A}$, this notion coincides with the ordinary notion of left inverse of f. Let us prove that

(1.8) *If a homomorphism f: $\mathfrak{A} \to \mathfrak{B}$ is left invertible relatively to a homomorphism φ: $\mathfrak{C} \to \mathfrak{A}$, then* $\mathrm{Ker}(\varphi) = \mathrm{Ker}(f\varphi)$.

Proof. Since $x \in \mathrm{Ker}(\varphi)$ implies $\varphi(x) = 0$, and this implies $f(\varphi(x)) = 0$, it follows that $\mathrm{Ker}(\varphi) \subset \mathrm{Ker}(f\varphi)$. If the homomorphism ψ: $\mathfrak{B} \to \mathfrak{A}$ satisfies the condition $\psi f \varphi = \varphi$, then the relation $z \in \mathrm{Ker}(f\varphi)$ implies $(f\varphi)(z) = 0$ which implies $(\psi f \varphi)(z) = 0$; thus $z \in \mathrm{Ker}(\varphi)$. Hence $\mathrm{Ker}(f\varphi) \subset \mathrm{Ker}(\varphi)$ and the proof is complete.

2. Homology and cohomology groups. We do not intend to develop here systematically homology theory. In this section we shall present the main relevant ideas axiomatically, and in the next one — we shall discuss some notions which allow to introduce homology groups of metric spaces on a constructive way.

In the homology theory one assigns to every pair of spaces (X, X_0) (where $X_0 \subset X$), to each integer n and to each Abelian group $\mathfrak{A}$ an Abelian group $H_n(X, X_0; \mathfrak{A})$, called the n-th *homology group of the pair* (X, X_0) *over the coefficient group* $\mathfrak{A}$, and also an Abelian group $H^n(X, X_0; \mathfrak{A})$, called the n-th *cohomology group of* (X, X_0) *over* $\mathfrak{A}$. For $X_0 = 0$ we have $H_n(X, \mathfrak{A})$ and $H^n(X, \mathfrak{A})$, i.e. the homology and cohomology groups of X over $\mathfrak{A}$.

If, for a space X, the group $H_n(X, \mathfrak{A})$ is trivial for every group of coefficients $\mathfrak{A}$, then we shall say that X is *acyclic in dimension* n. If X is acyclic in all dimensions, then we say that X is *acyclic*. In particular, any space X is acyclic in all dimensions $n < -1$ and if $X \neq 0$, then it is acyclic in dimension $n = -1$ as well. Moreover, if X consists of a single point, then it is acyclic (in every dimension).

There are several ways to construct the homology and cohomology groups of a space. The most important of these are:

1. The construction based on the nerves of coverings, first given by P. S. Alexandroff and extended by E. Čech.

2. The construction given by L. Vietoris, based on the notion of true cycles, and applicable to metric spaces.

3. The construction based on the notion of singular chains, that is, of continuous images of simplicial chains.

The forth way to develop the theory of homology is the axiomatic way used first by S. Eilenberg and N. Steenrod [112]. Unfortunately, only for rather special classes of spaces the groups introduced by these ways are isomorphic to each other. In particular, it is so in the case of polyhedra.

If $\mathfrak{R}$ denotes the additive group of rational numbers, then the rank of the group $H_n(X, \mathfrak{R})$ is called the n-th *Betti number* of the space X and will be denoted by $p_n(X)$. If X is a polyhedron, then $p_n(X)$ is finite and actually vanishes for almost all n. In particular, $p_n(X) = 0$ for all $n > \dim X$. It is also true that $p_n(X)$ is the rank of $H_n(X, \mathfrak{N})$, $\mathfrak{N}$ being the additive group of integers. This group $H_n(X, \mathfrak{N})$ is called the n-th *Betti group* of X and is denoted by $H_n(X)$. The subgroup of $H_n(X)$ consisting precisely of the elements of finite order is called the n-th *torsion group* of X. For a polyhedron X, the Betti groups of X are, with a finite number of exceptions, trivial. The exceptional groups, if any, are finitely generated.

Concerning homology and cohomology groups we shall take as known the following properties:

(2.1) *Each continuous map*

$$f\colon (X, X_0) \to (Y, Y_0)$$

induces homomorphisms

$$f_*\colon H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A}),$$
$$f^*\colon H^n(Y, Y_0; \mathfrak{A}) \to H^n(X, X_0; \mathfrak{A}).$$

(2.2) *The identity map $i\colon (X, X_0) \to (X, X_0)$ induces the identity homomorphisms i_* and i^* on $H_n(X, X_0; \mathfrak{A})$ and $H^n(X, X_0; \mathfrak{A})$.*

(2.3) *If $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (Y, Y_0) \to (Z, Z_0)$ are maps, then the composition gf induces homomorphisms satisfying the conditions*

$$(gf)_* = g_* f_* \quad \text{and} \quad (gf)^* = f^* g^*.$$

(2.4) *If two maps $f, g\colon (X, X_0) \to (Y, Y_0)$ are homotopic, then the induced homomorphisms are equal, that is, $f_* = g_*$ and $f^* = g^*$.*

The converse of (2.4) is false, generally speaking, but in an important case, called the *case of Hopf*, the converse can be proved ([166], p. 149). Consequently we assume as known

(2.5) Theorem of Hopf. *If X is a compact metric space of dimension $\leqslant n$, if S^n is the n-dimensional sphere, and if two maps $f, g\colon X \to S^n$ induce the same homomorphisms ($f_* = g_*$) of $H_n(X, \mathfrak{A}) \to H_n(S^n, \mathfrak{A})$ for every group of coefficients $\mathfrak{A}$, then f and g are homotopic.*

Finally we assume as known the following elementary proposition:

(2.6) *Suppose that $X = X_1 \cup X_2$, where X_1 and X_2 are closed subsets of X, and suppose that $X_1 \cap X_2$ contains only one point x_0. Setting*

$$\psi_1(x) = x, \quad \psi_2(x) = x_0 \quad \text{for all } x \in X_1$$

and

$$\psi_1(x) = x_0, \quad \psi_2(x) = x \quad \text{for all } x \in X_2,$$

we obtain the maps $\psi_1, \psi_2 \in X^X$. Then the induced homomorphisms ψ_{1*}, ψ_{2*} of $H_n(X, \mathfrak{A})$ satisfy the condition $\psi_{1*} + \psi_{2*} = i_*$, where i_* denotes the identity homomorphism of $H^n(X, \mathfrak{A})$.

3. Homology theory in metric spaces. Properties (2.1) - (2.6) of the homology and cohomology groups will be necessary in later considerations. Moreover we shall need the notion of infinite cycle and of true cycle. This last concept makes it possible to introduce homology groups of metric spaces in a constructive way, as it was done by L. Vietoris [278].

Suppose that X is a metric space with a metric ϱ and ε is a positive number. By an *n-dimensional oriented ε-simplex* of X we understand a system $\sigma = (a_0, a_1, \ldots, a_n)$ of $n+1$ points $a_i \in X$ such that $\varrho(a_i, a_j) \leqslant \varepsilon$ for $i, j = 0, 1, 2, \ldots, n$. We refer to the a_i's as *vertices* of σ, although we admit the possibility that vertices with distinct subscripts may be, in fact, the same point of X. If $i_0, i_1, \ldots, i_n$ is an even permutation of the subscripts $0, 1, \ldots, n$, then we shall regard $\sigma = (a_0, \ldots, a_n)$ and $(a_{i_0}, \ldots, a_{i_n})$ as the same simplex, while, if that permutation is odd, then we shall say that $(a_{i_0}, \ldots, a_{i_n})$ is *opposite* to σ and we shall denote it by $-\sigma$. Sometimes it will be convenient to write $1\cdot\sigma$ and $-1\cdot\sigma$ instead of σ and $-\sigma$, respectively. If $n = 0$ and $\sigma = (a_0)$, then we cannot speak of odd permutations of the subscripts. Nevertheless we will define $-\sigma$ also in this case; $-\sigma$ is the pair $(-1, a_0)$. Finally, by definition, there are two (-1)-dimensional oriented ε-simplexes, namely the number 1 and the number -1.

If some of the vertices (at least two) of a simplex σ are identical, then $\sigma = -\sigma$. The simplexes σ satisfying the equation $\sigma = -\sigma$ will be called *degenerate simplexes*.

We denote by $\Sigma^n(X, \varepsilon)$ the collection of all n-dimensional ε-simplexes of the space X. From each pair σ, $-\sigma$ of n-dimensional ε-simplexes we choose one and say that it is *positively oriented*; the other we say is *negatively oriented*. This choice can be done in an arbitrary way, but once we have done it we keep it throughout our discussion. The simplexes which we have called positively oriented form a subcollection of $\Sigma^n(X, \varepsilon)$ which we denote by $\Sigma^n_+(X, \varepsilon)$. Those which we have called negatively oriented form a subcollection of $\Sigma^n(X, \varepsilon)$ which we denote by $\Sigma^n_-(X, \varepsilon)$. The set

$$\Sigma^n_0(X, \varepsilon) = \Sigma^n_+(X, \varepsilon) \cap \Sigma^n_-(X, \varepsilon)$$

consists of all simplexes σ for which $\sigma = -\sigma$, that is, of all degenerate simplexes.

Now, let $\mathfrak{A}$ be an Abelian group. We understand by an *n-dimensional ε-chain* in X over the group $\mathfrak{A}$ a function

$$\varkappa\colon [\Sigma^n(X, \varepsilon) - \Sigma^n_0(X, \varepsilon)] \to \mathfrak{A}$$

satisfying the following two conditions:

1. $\varkappa(-\sigma) = -\varkappa(\sigma)$,
2. $\varkappa(\sigma) = 0$ for all but a finite number of $\sigma \in \Sigma^n(X, \varepsilon) - \Sigma^n_0(X, \varepsilon)$.

Since the chains, thus defined, have a fixed domain and the Abelian group $\mathfrak{A}$ as range, we may define addition of chains by means of the formula

$$(\varkappa_1 + \varkappa_2)(\sigma) = \varkappa_1(\sigma) + \varkappa_2(\sigma) \quad \text{for every simplex} \quad \sigma \in \Sigma^n(X, \varepsilon) - \Sigma^n_0(X, \varepsilon),$$

where the addition on the right-hand side is the addition of elements of the group $\mathfrak{A}$. It is then clear that the collection $C_n(X, \mathfrak{A}, \varepsilon)$ of all n-dimensional ε-chains forms an Abelian group. For $n < -1$ we define $C_n(X, \mathfrak{A}, \varepsilon)$ to be the group consisting of the zero element alone.

The ε-simplexes σ for which $\varkappa(\sigma) \neq 0$ are said to be *simplexes of the ε-chain $\varkappa$*, and their vertices — the *vertices of $\varkappa$*.

If $a \in \mathfrak{A}$ and $\sigma \in \Sigma^n(X, \varepsilon) - \Sigma^n_0(X, \varepsilon)$, then by $a\sigma$ we mean the chain which assigns the value a to σ, the value $-a$ to $-\sigma$, and zero to all other simplexes. We note that $a\sigma = -a(-\sigma)$. If σ is degenerate, then $a\sigma$ is defined to be the zero chain. By this convention every chain $\varkappa$ may be expressed in the form

(3.1) $\varkappa = a_1\sigma_1 + a_2\sigma_2 + \ldots + a_k\sigma_k$ *where the coefficients* a_i *belong to* $\mathfrak{A}$ *and* $\sigma_i \in \Sigma^n(X, \varepsilon)$.

On the other hand, any expression of the above form is a chain of $C_n(X, \mathfrak{A}, \varepsilon)$. We note that this expression for $\varkappa$ is not unique, generally speaking.

The basic operation on a chain is the taking of its *boundary*. For a simplex $\sigma = (a_0, \ldots, a_n)$ we define the boundary $\partial\sigma$ of σ by the formula

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n),$$

so that $\partial\sigma$ is an $(n-1)$-dimensional chain over the group of integers $\mathfrak{N}$. The formula is only meaningful if $n > 0$. It assigns, in particular, to a degenerate n-dimensional simplex the zero element of the group $C_{n-1}(X, \mathfrak{N}, \varepsilon)$. In case $n = 0$, if $\sigma = \pm(a_0)$, then $\partial\sigma$ is the (-1)-dimensional chain ± 1. For $n \leqslant -1$, we define $\partial\sigma$ to be the zero element of the group $C_{n-1}(X, \mathfrak{N}, \varepsilon)$.

We extend ∂ to operate on chains in the following manner: if $\varkappa$ is an element of $C_n(X, \mathfrak{A}, \varepsilon)$ given by (3.1) above, then $\partial\varkappa$ is the $(n-1)$-dimensional chain over the group $\mathfrak{A}$ given by the formula

$$\partial\varkappa = \sum_{i=0}^{k} a_i(\partial\sigma_i) \tag{3.2}$$

where $a_i(\partial\sigma_i)$ is the chain obtained from $\partial\sigma_i$ by multiplying each of its coefficients (in an expansion of the form (3.1)) by a_i.

It is clear that $\partial\colon C_n(X, \mathfrak{A}, \varepsilon) \to C_{n-1}(X, \mathfrak{A}, \varepsilon)$ is a homomorphism by virtue of our definitions. The elements of the kernel of ∂, which we denote by $Z_n(X, \mathfrak{A}, \varepsilon)$, are called *$n$-dimensional ε-cycles in X over $\mathfrak{A}$*.

The image of $C_n(X, \mathfrak{A}, \varepsilon)$ in $C_{n-1}(X, \mathfrak{A}, \varepsilon)$ under ∂ is denoted by $B_{n-1}(X, \mathfrak{A}, \varepsilon)$. The elements of $B_{n-1}(X, \mathfrak{A}, \varepsilon)$ are called the *$(n-1)$-dimensional ε-boundaries in X over $\mathfrak{A}$*.

It is easy to prove that $B_{n-1}(X, \mathfrak{A}, \varepsilon) \subset Z_{n-1}(X, \mathfrak{A}, \varepsilon)$ or, equivalently, that

(3.3) $\partial\partial\varkappa = 0$ *for every chain* $\varkappa \in C_n(X, \mathfrak{A}, \varepsilon)$.

In order to use these notions for construction of homology groups of the space X, it is necessary to apply some kind of a limit process.

A sequence $\boldsymbol{\varkappa} = \{\varkappa_i\}$ of n-dimensional chains in X is said to be an *infinite n-dimensional chain in X* provided that the following two conditions hold:

1. There is a sequence $\{\varepsilon_i\}$ of positive numbers converging to zero and a sequence $\{\mathfrak{A}_i\}$ of Abelian groups $\mathfrak{A}_i$ such that $\varkappa_i \in C_n(X, \mathfrak{A}_i, \varepsilon_i)$.

2. There is a compact subset X_0 of X such that every vertex of every $\varkappa_i$ lies in X_0.

The sequence $\{\varepsilon_i\}$ is called the *majorant* of $\boldsymbol{\varkappa}$ and X_0 is called the *carrier* of $\boldsymbol{\varkappa}$. We observe that if $\{\varepsilon_i'\}$ is a sequence of positive numbers converging to zero and $\varepsilon_i' \geqslant \varepsilon_i$ for all i, then $\{\varepsilon_i'\}$ is also a majorant of $\boldsymbol{\varkappa}$. Moreover, any compact subset X_0' of X which contains X_0 is also a carrier of $\boldsymbol{\varkappa}$. We also observe that the infinite chain $\boldsymbol{\varkappa} = \{\varkappa_i\}$, with $\varkappa_i = 0$ for $i = 1, 2, \ldots$, is actually an infinite chain; we will denote it by 0.

If for two infinite n-dimensional chains $\boldsymbol{\varkappa} = \{\varkappa_i\}$ and $\boldsymbol{\varkappa}' = \{\varkappa_i'\}$ in X the coefficients of $\varkappa_i$ and of $\varkappa_i'$ belong to the same group $\mathfrak{A}_i$, for $i = 1, 2, \ldots$, then the sequence $\{\varkappa_i + \varkappa_i'\}$ is an infinite n-dimensional chain in X; it is denoted by $\boldsymbol{\varkappa} + \boldsymbol{\varkappa}'$. Similarly, by $\boldsymbol{\varkappa} - \boldsymbol{\varkappa}'$ we denote the infinite chain $\{\varkappa_i - \varkappa_i'\}$.

If $\gamma = \{\gamma_i\}$ is an infinite chain in X such that γ_i is a cycle for $i = 1, 2, \ldots$, then γ is said to be an *infinite cycle* in X. In particular, for every infinite n-dimensional chain $\boldsymbol{\varkappa} = \{\varkappa_i\}$, the sequence $\{\partial\varkappa_i\}$ is an infinite $(n-1)$-dimensional cycle which we denote by $\partial\boldsymbol{\varkappa}$. Two infinite cycles $\gamma = \{\gamma_i\}$ and $\gamma' = \{\gamma_i'\}$ in X with the same group $\mathfrak{A}_i$ of coefficients for γ_i and γ_i', $i = 1, 2, \ldots$, are said to be *homologous in X* (in symbols: $\gamma \sim \gamma'$ in X) provided that there is an infinite chain $\boldsymbol{\varkappa}$ in X such that $\partial\boldsymbol{\varkappa} = \gamma - \gamma'$.

An infinite cycle γ is said to be *essential* provided that it has a carrier in which it is not homologous to the cycle 0.

(3.4) *If X is a compactum lying in a metric space Y and if γ is an infinite cycle in X which is not homologous to zero in X, then there exists a neighborhood U of X in Y such that γ is not homologous to zero in U.*

(3.5) *If γ is an infinite cycle homologous to zero in a metric space X and if B is a carrier of γ, then there exists in X a compactum $A_0 \supset B$ such that $\gamma \sim 0$ in A_0 and that for every set $A \subset A_0$ the homology $\gamma \sim 0$ in A implies $A \cup B = A_0$.*

The following proposition ([41], p. 546) is known under the name of the *Phragmèn-Brouwer theorem*:

(3.6) *Let X be the union of two closed subsets X_1 and X_2 and let γ be an infinite n-dimensional cycle in $X_1 \cap X_2$ not homologous to 0 in $X_1 \cap X_2$ and such that for $\nu = 1, 2$, there is an infinite chain $\varkappa_\nu$ in X with $\partial\varkappa_\nu = \gamma$. Then $\varkappa_1 - \varkappa_2$ is an infinite $(n+1)$-dimensional cycle in X which is not homologous to 0 in X.*

An infinite cycle $\gamma = \{\gamma_i\}$ in X is said to be a *true cycle* provided the coefficients of all cycles γ_i belong to the same group and the infinite cycle $\{\gamma_i'\}$, where $\gamma_i' = \gamma_{i+1} - \gamma_i$, is homologous to 0 in X.

The collection $Z_n(X, \mathfrak{A})$ of all n-dimensional true cycles in X over $\mathfrak{A}$ constitutes an Abelian group under an addition given by the formula

$$\{\gamma_i\} + \{\gamma_i'\} = \{\gamma_i + \gamma_i'\}. \tag{3.7}$$

Those elements of $Z_n(X, \mathfrak{A})$ which are homologous to zero form a subgroup of $Z_n(X, \mathfrak{A})$ which we denote by $B_n(X, \mathfrak{A})$. The factor group $Z_n(X, \mathfrak{A})/B_n(X, \mathfrak{A})$ is called the *n-dimensional homology group of X over* $\mathfrak{A}$ and is denoted by $H_n(X, \mathfrak{A})$.

This definition of the homology group is due to Vietoris [278]. Slightly modifying the definitions given above we can define the groups $H_n(X, X_0; \mathfrak{A})$ for pairs (X, X_0) but we shall not need these more general notion and will, therefore, omit the details of the definition.

Let us see how a map $f\colon X \to Y$ induces a homomorphism of $H_n(X, \mathfrak{A})$ into $H_n(Y, \mathfrak{A})$. Suppose that $\sigma = (a_0, a_1, \ldots, a_n)$ is an n-dimensional simplex. Then f associates with σ an n-dimensional simplex in Y, denoted by $f(\sigma)$, given by

$$f(\sigma) = \big(f(a_0), f(a_1), \ldots, f(a_n)\big).$$

It is possible that $f(\sigma)$ is degenerate even if σ is not degenerate. Moreover, f assigns to each n-dimensional chain

$$\varkappa = a_1\sigma_1 + a_2\sigma_2 + \ldots + a_k\sigma_k$$

an n-dimensional chain, denoted by $f(\varkappa)$, given by the formula

$$f(\varkappa) = a_1 f(\sigma_1) + a_2 f(\sigma_2) + \ldots + a_k f(\sigma_k). \tag{3.8}$$

We see that $\varkappa \in C_n(X, \mathfrak{A}, \varepsilon)$ does not imply $f(\varkappa) \in C_n(Y, \mathfrak{A}, \varepsilon)$. However, $f(\varkappa)$ belongs to $C_n(Y, \mathfrak{A}, \eta)$ where η depends on f and ε. Moreover, let us observe, since f is uniformly continuous on any compact set, the sequence $\{f(\varkappa_i)\}$, which is associated by f with an infinite chain $\boldsymbol{\varkappa} = \{\varkappa_i\}$, also is an infinite chain. This infinite chain, denoted by $f(\boldsymbol{\varkappa})$, lies in Y. Moreover, we see that one of the carriers for $f(\boldsymbol{\varkappa})$ is the image under f of a carrier for $\boldsymbol{\varkappa}$. It is not difficult to prove that f commutes with ∂, that is, $\partial f = f\partial$.

From this relation $f\partial = \partial f$ we readily infer that f assigns to infinite cycles (resp.: true cycles, homologous cycles) in X infinite cycles (resp.: true cycles, homologous cycles) in Y. Consequently, f *induces* a homomorphism

$$f_*\colon H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A}).$$

One shows readily the following proposition:

(3.9) *Let $\varkappa$ be an infinite chain in a compactum A and let X be a metric space. Then for every homotopy $\{f_t\}$ in X^A:*

(a) *There exists in X an infinite chain λ such that $\partial\lambda = \partial f_0(\varkappa) - \partial f_1(\varkappa)$ and if $\partial\varkappa = 0$, then $\lambda = 0$.*

(b) *The infinite cycle $f_0(\varkappa) - f_1(\varkappa) - \lambda$ is homologous to 0 in X.*

In particular, it follows, if two maps $f, f'\colon X \to Y$ are homotopic, then for each true cycle γ of X, the true cycles $f(\gamma)$ and $f'(\gamma)$ in Y are homologous. In the case of Hopf (where X is compact, $\dim X \leqslant n$ and $Y = S^n$) we have the following proposition:

(3.10) *If X is a compactum of dimension $\leqslant n$ and if the maps $f, f'\colon X \to S^n$ are such that $f(\gamma) \sim f'(\gamma)$ in S^n for every n-dimensional infinite cycle γ in X, then the maps f and f' are homotopic.*

Let us quote an important theorem of P. S. Alexandroff ([1], p. 195) which illustrates the relation between these homological concepts and the notion of the dimension:

(3.11) THEOREM. *The dimension of the compact metric space X is $\geqslant n$ if and only if there is an infinite n-dimensional chain $\varkappa = \{\varkappa_i\}$ in X such that the infinite $(n-1)$-dimensional cycle $\{\partial\varkappa_i\}$ is essential.*

The just given characterization of the dimension of compacta leads us, in a natural way, to the notion of the *modular dimension* in the sense of P. S. Alexandroff ([1], p. 194). Let m be an integer greater than 1. By the *dimension modulo m* of a compactum X we understand the greatest integer $n = \Delta^m(X)$ such that there exists in X an infinite $(n-1)$-dimensional essential cycle $\{\gamma_i\}$ homologous to zero in X with coefficients belonging to the group $\mathfrak{N}_m$ of integers reduced modulo m. According to (3.11), the dimension $\Delta^m(X)$ is less than or equal to the usual dimension $\dim X$ of X.

4. Homomorphisms induced by h-maps and r-maps. By use of (2.1), (2.2), (2.3) and (2.4) we can prove the following

(4.1) THEOREM. *If $f\colon (X, X_0) \to (Y, Y_0)$ is an h-map and if $g\colon (Y, Y_0) \to (X, X_0)$ is a homotopic right inverse of f, then the induced homomorphisms*

$$f_*\colon H_n(X, X_0; \mathfrak{A}) \to H_n(Y, Y_0; \mathfrak{A})$$

and

$$g^*: H^n(X, X_0; \mathfrak{A}) \to H^n(Y, Y_0; \mathfrak{A})$$

are r-homomorphisms.

Indeed, the map fg: $Y \to Y$ is homotopic to the identity and consequently both $(fg)_* = f_*g_*$ and $(fg)^* = g^*f^*$ are identity homomorphisms. This is just what we needed to prove.

From Theorem (4.1) and from Corollary (1.7) we infer:

(4.2) *If Y is an h-image of a space X, then the homology and cohomology groups of Y are isomorphic to certain direct divisors of the corresponding groups of the space X.*

In particular we infer that

(4.3) $X \underset{h}{\geqslant} Y$ *implies that* $p_n(X) \geqslant p_n(Y)$ *for* $n = 0, 1, 2, \ldots$

Moreover, it follows by (4.2) that

(4.4) *The corresponding homology and cohomology groups of two h-equal spaces are r-equal,*

and

(4.5) *If* $X \underset{h}{=} Y$ *and if the group* $H_n(X, \mathfrak{A})$ *has a finite number of generators, then it is isomorphic to the group* $H_n(Y, \mathfrak{A})$.

We observe that, since r-maps are special kinds of h-maps, all the theorems of this paragraph hold for r-maps. Specifically we have the following propositions:

(4.6) $X \underset{r}{\geqslant} Y$ *implies* $p_n(X) \geqslant p_n(Y)$ *for* $n = 0, 1, 2, \ldots$

(4.7) *The r-equal spaces have equal Betti numbers and their homology groups and cohomology groups are r-equal.*

The question whether for r-equal compacta the corresponding homology and cohomology groups are necessarily isomorphic remains open. Some partial results have been obtained by T. Ganea [133]. As a simple corollary of (4.7) we get the following proposition:

(4.8) *If* $X \underset{r}{=} Y$ *and if* $H_n(X, \mathfrak{A})$ *has a finite number of generators, then* $H_n(X, \mathfrak{A})$ *and* $H_n(Y, \mathfrak{A})$ *are isomorphic.*

With a little stronger hypothesis we get a sharper theorem:

(4.9) *If X and Y have the same homotopy type, then their homology and cohomology groups are isomorphic.*

Indeed, by hypothesis there are maps f: $X \to Y$ and g: $Y \to X$ such that both composites fg and gf are homotopic to the appropriate identities. But in this case, $(fg)_* = f_*g_*$, $(gf)_* = g_*f_*$, $(fg)^* = g^*f^*$ and

$(gf)^* = f^* g^*$ are equal to the appropriate identity homomorphisms. It follows that f_*, g_*, f^* and g^* are isomorphisms which completes the proof.

As we have already seen, each deformation r-image of a space X, and hence each deformation retract of X, has the same homotopy type as X. In view of the last result we obtain the proposition

(4.10) *The homology and cohomology groups of each deformation r-image (in particular of each deformation retract) of a space X are isomorphic to the corresponding groups of X.*

From this we see that the homology and cohomology groups of a space which is contractible in itself are the same as those of a one-point space, i.e. they are trivial. In other words,

(4.11) *Each space which is contractible in itself is acyclic.*

Now let us prove the following statement:

(4.12) *Let r be a retraction of Z to X and let $\varphi\colon Y \to Z$ be a map such that $\varphi(Y) \subset X$. Then the kernel* $\operatorname{Ker}(\varphi_*)$ *of the homomorphism $\varphi_*\colon H_n(Y, \mathfrak{A}) \to H_n(Z, \mathfrak{A})$ induced by φ is the same as the kernel* $\operatorname{Ker}\big((r\varphi)_*\big)$ *of the homomorphism $(r\varphi)_*\colon H_n(Y, \mathfrak{A}) \to H_n(X, \mathfrak{A})$ induced by $r\varphi$.*

Proof. Let i denote the inclusion $i\colon X \to Z$. Then $ir\varphi = \varphi\colon Y \to Z$ and consequently $(ir\varphi)_* = i_* r_* \varphi_* = \varphi_*\colon H_n(Y, \mathfrak{A}) \to H_n(Z, \mathfrak{A})$. It follows that the homomorphism $r_*\colon H_n(Z, \mathfrak{A}) \to H_n(X, \mathfrak{A})$ is left invertible relative to the homomorphism φ_* (in the sense of Section 1). It follows by (1.8) that $\operatorname{Ker}(\varphi_*) = \operatorname{Ker}(r_*\varphi_*) = \operatorname{Ker}\big((r\varphi)_*\big)$.

(4.13) *If $X \subset Y \subset Z$, X is a retract of Z and if the inclusion $i\colon Y \to Z$ is homotopic to a map $\varphi\colon Y \to Z$ such that $\varphi(Y) \subset X$, then the group $H_n(Y, \mathfrak{A})$ is isomorphic to the group $H_n(X, \mathfrak{A}) \times \operatorname{Ker}(i_*)$, where* $\operatorname{Ker}(i_*)$ *denotes the kernel of the homomorphism $i_*\colon H_n(Y, \mathfrak{A}) \to H_n(Z, \mathfrak{A})$ induced by i.*

Proof. Let r be a retraction of Z to X. Setting $\hat{r} = ri$, we get a retraction of Y to X which induces an r-homomorphism:

$$\hat{r}_* = r_* i_*\colon H_n(Y, \mathfrak{A}) \to H_n(X, \mathfrak{A}).$$

If follows, by Theorem (1.6), that the group $H_n(Y, \mathfrak{A})$ is isomorphic to $H_n(X, \mathfrak{A}) \times \operatorname{Ker}(\hat{r}_*)$. Since i and φ are homotopic, we infer by (2.4) that $i_* = \varphi_*$, and consequently $\operatorname{Ker}(\hat{r}_*) = \operatorname{Ker}\big((ri)_*\big) = \operatorname{Ker}(r_* i_*) = \operatorname{Ker}(r_*\varphi_*) = \operatorname{Ker}\big((r\varphi)_*\big)$. But $\varphi(Y) \subset X$ and we infer by (4.12) that $\operatorname{Ker}\big((r\varphi)_*\big) = \operatorname{Ker}(\varphi_*)$. Thus it is shown that

$$\operatorname{Ker}(\hat{r}_*) = \operatorname{Ker}(\varphi_*) = \operatorname{Ker}(i_*),$$

and the proof is finished.

Some more detailed relations between homology groups of a space and of its retract are given in [157].

5. Join of maps. Let x_0 be a point of a space X and y_0 a point of another space Y. Two maps φ_1, φ_2: $(X, x_0) \to (Y, y_0)$ are said to be *separated* provided that the sets $G_1 = \varphi_1^{-1}(Y-(y_0))$ and $G_2 = \varphi_2^{-1}(Y-(y_0))$ are disjoint. Setting

$$\varphi(x) = \begin{cases} \varphi_i(x) & \text{for every } x \in G_i,\ i = 1, 2, \\ y_0 & \text{for every } x \in X - G_1 - G_2, \end{cases}$$

we obtain a map φ: $(X, x_0) \to (Y, y_0)$ which is called the *join* of the separated maps φ_1 and φ_2 ([35], p. 1402). It will be denoted by $\varphi_1 \cdot \varphi_2$. Clearly $\varphi_1 \cdot \varphi_2 = \varphi_2 \cdot \varphi_1$.

Now suppose that z_0 is a point of a space Z and let f be a map of (Z, z_0) into (X, x_0). Then $\varphi_1 f$ and $\varphi_2 f$ are separated maps of (Z, z_0) into (Y, y_0) whose join is exactly φf, where φ denotes the join $\varphi_1 \cdot \varphi_2$.

The operation Φ^f which assigns to each map φ': $(X, x_0) \to (Y, y_0)$ the map $\varphi' f$: $(Z, z_0) \to (Y, y_0)$ is continuous (I, (9.2)), and we infer that

(5.1) $$\Phi^f(\varphi_1 \cdot \varphi_2) = \Phi^f(\varphi_1) \cdot \Phi^f(\varphi_2).$$

Similarly if f: $(Y, y_0) \to (Z, z_0)$, then $f\varphi, f\varphi_1, f\varphi_2$ map (X, x_0) into (Z, z_0) and the former map is the join of the latter two maps. Rephrasing this in terms of the operation Φ_f we have

(5.2) $$\Phi_f(\varphi_1 \cdot \varphi_2) = \Phi_f(\varphi_1) \cdot \Phi_f(\varphi_2).$$

The following theorem illuminates the homological meaning of the operation of the join of separated maps of metric spaces:

(5.3) THEOREM. *If the maps φ_1, φ_2: $(X, x_0) \to (Y, y_0)$ are separated and if $\varphi = \varphi_1 \cdot \varphi_2$, then the induced homomorphisms*

$$\varphi_*: H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A}), \qquad \varphi_{i*}: H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A}),$$
$$\varphi^*: H^n(Y, \mathfrak{A}) \to H^n(X, \mathfrak{A}), \qquad \varphi_i^*: H^n(Y, \mathfrak{A}) \to H^n(X, \mathfrak{A})$$

(where $i = 1, 2$) satisfy the relations

$$\varphi_* = \varphi_{1*} + \varphi_{2*}, \qquad \varphi^* = \varphi_1^* + \varphi_2^*.$$

Proof. Since φ_1 and φ_2 are separated, the sets $G_i = \varphi_i^{-1}(Y-(y_0))$ are disjoint. We observe that the decomposition of the space X into the set $X_0 = X-(G_1 \cup G_2)$ and the individual points of $G_1 \cup G_2$ is an upper semi-continuous decomposition of X (see, for instance, [203], p. 42). Let X' denote the decomposition space of X relative to this decomposition and let α be the natural map $X \to X'$ which associates with each $x \in X$ the element $\alpha(x)$ of the decomposition space, which considered as a subset of X contains x. We note that α maps $X-X_0$ into $X'-(\alpha(x_0))$. For simplicity, let us write x' instead of $\alpha(x)$.

Let us set $X'_i = X' - a(G_i)$ for $i = 1, 2$. Then $X' = X'_1 \cup X'_2$, where X'_1, X'_2 are closed and $X'_1 \cap X'_2 = (x'_0)$. Setting

$$\psi_i(x') = \begin{cases} x' & \text{for all } x' \in a(G_i), \\ x'_0 & \text{for all } x' \in X' - a(G_i), \end{cases}$$

we obtain two maps ψ_1 and ψ_2 of X' into X' which are separated and their join is the identity i: $X' \to X'$. It follows from (2.6) that the induced homomorphisms

$$\psi_{i*}\colon H_n(X', \mathfrak{A}) \to H_n(X', \mathfrak{A}) \quad (i = 1, 2),$$
$$\psi_i^*\colon H^n(X', \mathfrak{A}) \to H^n(X', \mathfrak{A}) \quad (i = 1, 2)$$

are such that $\psi_{1*} + \psi_{2*}$ and $\psi_1^* + \psi_2^*$ are the identity homomorphisms on their respective groups. If we now set

$$\varphi'(x') = (\varphi a^{-1})(x') \quad \text{for all } x' \in X' - (x'_0),$$
$$\varphi'(x'_0) = y_0,$$

then we obtain a map φ': $X' \to Y$ and we may easily check that

$$\varphi_i = \varphi' \psi_i a \quad (i = 1, 2)$$

and

$$\varphi = \varphi' a.$$

From this we infer that

$$\varphi_* = (\varphi' a)_* = \varphi'_* (\psi_{1*} + \psi_{2*}) a_* = \varphi'_* \psi_{1*} a_* + \varphi'_* \psi_{2*} a_* = \varphi_{1*} + \varphi_{2*}.$$

By similar reasoning we may conclude that $\varphi^* = \varphi_1^* + \varphi_2^*$ and this completes the proof of the theorem.

We shall now consider the use of the join of separated maps in defining the homotopy join of homotopy classes of maps.

Let G be an open subset of $X - (x_0)$. A map φ: $(X, x_0) \to (Y, y_0)$ is said to be *concentrated* on G provided that $\varphi(x) = y_0$ for every $x \in X - G$. Suppose that G_1 and G_2 are two disjoint open subsets of $X - (x_0)$ and that φ'_1 and φ'_2 are two maps of (X, x_0) into (Y, y_0) concentrated on G_1 and G_2, respectively. Then it is clear that φ'_1 and φ'_2 are separated and the join $\varphi = \varphi'_1 \cdot \varphi'_2$ of φ'_1 and φ'_2 is well defined.

Moreover, let us observe that

(5.4) *If G_1 and G_2 are two disjoint open subsets of $X - (x_0)$ and if $\{\varphi_{it}\} \subset (Y, y_0)^{(X, x_0)}$ is a family of maps concentrated on G_i and continuously dependent on $t \in \langle 0, 1 \rangle$, then the join $\varphi_{1t} \cdot \varphi_{2t}$ continuously depends on $t \in \langle 0, 1 \rangle$.*

Two maps φ_1 and φ_2 of (X, x_0) into (Y, y_0) are said to be *separable by the open disjoint subsets* G_1 and G_2 of $X - (x_0)$ provided that there are maps φ'_1 and φ'_2 of (X, x_0) into (Y, y_0) homotopic to φ_1 and φ_2 respectively such that φ_i' is concentrated on G_i $(i = 1, 2)$. When this is the case, the join $\varphi = \varphi_1' \cdot \varphi_2'$ of φ_1' and φ_2' is defined. The homotopy class $[\varphi]$ of φ is called the *homotopy join* ([69], p. 203) of the homotopy classes $[\varphi_1]$ and $[\varphi_2]$ of φ_1 and φ_2 *relative to* G_1 *and* G_2. We cannot assert that for arbitrarily given φ_1 and φ_2 we may always find such φ_1' and φ_2'. Neither we do assert that the homotopy class $[\varphi]$ is uniquely defined. Indeed, it depends, among other things, on the sets G_1 and G_2. However, if there is no danger of confusion, we shall say for the sake of simplicity that $[\varphi]$ is the *homotopy join* of $[\varphi_1]$ and $[\varphi_2]$.

From the last theorem, which concerned the relation between the homomorphisms of the homology and cohomology groups induced by two separated maps and their join, we infer the following

(5.5) THEOREM. *If the homotopy class* $[\varphi]$ *is a homotopy join of the homotopy classes* $[\varphi_1]$ *and* $[\varphi_2]$, *then the induced homomorphisms* φ_*, $\varphi_{1*}, \varphi_{2*}$ *of the homology groups and the induced homomorphisms* $\varphi^*, \varphi_1^*, \varphi_2^*$ *of the cohomology groups satisfy the following relations*:

$$\varphi_* = \varphi_{1*} + \varphi_{2*}, \qquad \varphi^* = \varphi_1^* + \varphi_2^*.$$

6. Homotopy groups. The operation of homotopy join may be used as a basis for introducing the homotopy groups. To this end we assume that X is the n-dimensional Euclidean sphere S^n which consists, as usual, of all those points $x \in E^{n+1}$ that $\|x\| = 1$. Let x_0 be an arbitrary point of X and let G be an open ball in X that does not contain the point x_0. The reader can easily verify that if y_0 is a point in the space Y, then each map $f\colon (X, x_0) \to (Y, y_0)$ is homotopic to a map concentrated on G. Consequently, if G_1 and G_2 are two disjoint open balls in X neither of which contains x_0, then for each pair of maps f_1, f_2 of (X, x_0) into (Y, y_0) there are two maps f_1' and f_2' homotopic to f_1, f_2, respectively, and concentrated on G_1 and G_2, respectively. Further, in this case (where $X = S^n$), we are going to show that the homotopy class $[f']$ of the join $f' = f_1' \cdot f_2'$ depends only on the homotopy classes $[f_1]$ and $[f_2]$ for given G_1 and G_2. Indeed, in order to prove this it is sufficient to observe that $X - G_i$ is contractible in itself to x_0 and that this contraction can be obtained by a homotopy $\{\varphi_{it}\}$ of X into itself which begins with the identity φ_{i0} and meets conditions:

$$\varphi_{it}(X - G_i) \subset X - G_i, \qquad \varphi_{it}(x_0) = x_0, \qquad \text{and} \qquad \varphi_{i1}(X - G_i) = (x_0).$$

The existence of such a homotopy $\{\varphi_{it}\}$ $(i = 1, 2)$ is clear. Suppose now that f_i' and f_i'' are both homotopic to f_i and are both concentrated on G_i. Then there is a homotopy $\{f_{it}\} \subset (Y, y_0)^{(X, x_0)}$ such that $f_{i0} = f_i'$ and $f_{i1} = f_i''$. Setting

$$\begin{aligned}
g_{it}(x) &= f_i' \varphi_{i,3t}(x) && \text{for} \quad x \in X \text{ and } 0 \leqslant t \leqslant \tfrac{1}{3}, \\
g_{it}(x) &= f_{i,3t-1}' \varphi_{i,1}(x) && \text{for} \quad x \in X \text{ and } \tfrac{1}{3} \leqslant t \leqslant \tfrac{2}{3}, \\
g_{it}(x) &= f_i'' \varphi_{i,3-3t}(x) && \text{for} \quad x \in X \text{ and } \tfrac{2}{3} \leqslant t \leqslant 1,
\end{aligned}$$

we get a homotopy $\{g_{it}\} \subset (Y, y_0)^{(X,x_0)}$ joining $g_{i0} = f_i'$ with $g_{i1} = f_i''$. If we observe that $\varphi_{i,3t}(X - G_i) \subset X - G_i$ for $0 \leqslant t \leqslant \frac{1}{3}$, $\varphi_{i,1}(X - G_i) = (x_0)$ and $\varphi_{i,3-3t}(X - G_i) \subset X - G_i$ for $\frac{2}{3} \leqslant t \leqslant 1$, then we infer that $g_{it}(x) = y_0$ for every point $x \in X - G_i$ and every $0 \leqslant t \leqslant 1$. Since $g_{i0} = f_i'$ and $g_{i1} = f_i''$, we conclude that $\{g_{it}\}$ is a homotopy joining f_i' with f_i'' and that g_{it} is concentrated on G_i for every $0 \leqslant t \leqslant 1$. It follows by (5.4) that the family $(g_{1t} \cdot g_{2t})$ is a homotopy joining the map $f_1' \cdot f_2'$ with $f_1'' \cdot f_2''$.

Thus we have shown that for given disjoint open balls G_1, G_2 in $X = S^n$ the homotopy class $[f']$ of the join $f' = f_1' \cdot f_2'$ of two maps f_1', f_2' homotopic to f_1, f_2 and concentrated on G_1, G_2 respectively depends only on the homotopy classes $[f_1]$ and $[f_2]$. We shall denote this class by $[f_1]_{G_1} \cdot_{G_2} [f_2]$ and call it the *homotopy join of* $[f_1]$ *and* $[f_2]$ *with respect to* G_1 *and* G_2.

Now let us show that for $n > 1$ the homotopy class $[f_1]_{G_1} \cdot_{G_2} [f_2]$ does not depend on the choice of the balls G_1, G_2, i.e. that if G_1', G_2' is another pair of disjoint open balls in S^n (with $n > 1$) that does not contain the point x_0, then

$$[f_1]_{G_1} \cdot_{G_2} [f_2] = [f_1]_{G_1'} \cdot_{G_2'} [f_2].$$

This is so because, first of all, the geometry of the sphere S^n shows that for each $t \in \langle 0,1 \rangle$ we can find a pair of disjoint open balls G_{1t}, G_{2t} in $X = S^n$, neither containing the point x_0, such that $G_{10} = G_1, G_{20} = G_2$, $G_{11} = G_1', G_{21} = G_2'$ and such that the centers and radii of the G_{it} depend continuously on the parameter t. It follows readily that there exists a homotopy $\{\varphi_t\} \subset (X, x_0)^{(X,x_0)}$ such that φ_t is a homeomorphism of S^n onto itself for every $0 \leqslant t \leqslant 1$ and joins the identity map $\varphi_0 = i$ with a map φ_1 which, restricted to G_i', is a homeomorphism of G_i' onto G_i $(i = 1, 2)$. Then if we set $f_{it} = f_i'\varphi_t$, so $\{f_{it}\}$ is a homotopy in $(Y, y_0)^{(X,x_0)}$ that joins f_i' to the map $f_i'\varphi_1$ which is concentrated on G_i'. Moreover, for every t, the maps f_{1t} and f_{2t} are separated. We infer that the family of joins $\{f_{1t} \cdot f_{2t}\}$ is a homotopy joining $f_1' \cdot f_2'$ with $(f_1'\varphi_1) \cdot (f_2'\varphi_2)$. Consequently, $[f_1]_{G_1} \cdot_{G_2} [f_2]$ coincides with

$$[f_1'\varphi_1]_{G_1'} \cdot_{G_2'} [f_2'\varphi_2] = [f_1]_{G_1'} \cdot_{G_2'} [f_2].$$

In the case $n = 1$, the situation is somewhat different. Let x_1, x_2, x_1', x_2' be the centers of the balls G_1, G_2, G_1', G_2', respectively. The construction of the continuous families $\{G_{it}\}$ of open balls used in the proof in the case $n > 1$ can be carried out in the present case, $n = 1$, if and only if the triple x_0, x_1, x_2 has the same orientation on S^1 as the triple x_0, x_1', x_2'. When this condition is met, we can conclude by the above argument that

$$[f_1]_{G_1} \cdot_{G_2} [f_2] = [f_1]_{G_1'} \cdot_{G_2'} [f_2],$$

while if the triples are oppositely oriented, then we have

$$[f_1]_{G_1}\cdot_{G_2}[f_2] = [f_1]_{G_2'}\cdot_{G_1'}[f_2].$$

Thus, in the case $n = 1$ the operation of homotopy join depends only on the orientation of the triple x_0, x_1, x_2, while in the case $n > 1$ it does not depend on the balls G_1 or G_2 anyhow.

We now show that the operation of homotopy join is associative. To this end, suppose that G_1 and G_2 are two open disjoint balls in $X = S^n$ neither of which contains x_0. Let x_i be the center of G_i $(i = 1, 2)$. We can easily find balls $G_1', G_2', G_1'', G_2'', G_3''$ with centers $x_1', x_2', x_1'', x_2'', x_3''$ such that

$$G_1'' \subset G_1' - G_2', \quad G_2'' \subset G_1' \cap G_2', \quad G_3'' \subset G_2' - G_1'.$$

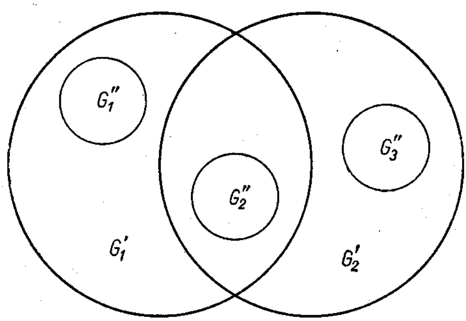


Fig. 2

Moreover, in the case $n = 1$ we may choose these balls so that the triples (x_0, x_1'', x_2'), (x_0, x_1'', x_2''), (x_0, x_2'', x_3''), (x_0, x_1', x_3'') are all consistently oriented with the triple (x_0, x_1, x_2). Now, let us consider three maps f_1, f_2, and f_3 of (X, x_0) into (Y, y_0). Then there are maps f_1'', f_2'', f_3'' homotopic to f_1, f_2, f_3, respectively, and concentrated on G_1'', G_2'', G_3'', respectively. By the choice of our balls, the joins

$$(f_1''\cdot f_2'')\cdot f_3'' \quad \text{and} \quad f_1''\cdot(f_2''\cdot f_3'')$$

are well defined and equal. But $(f_1''\cdot f_2'')\cdot f_3''$ is a representative of the homotopy class $([f_1]_{G_1''}\cdot_{G_2''}[f_2])_{G_1'}\cdot_{G_3''}[f_3]$, and $f_1''\cdot(f_2''\cdot f_3'')$ is a representative of the homotopy class $[f_1]_{G_1''}\cdot_{G_2'}([f_2]_{G_2''}\cdot_{G_3''}[f_3])$. If $n > 1$, then the choice of balls on which the maps are concentrated is immaterial, and we infer that

$$([f_1]_{G_1}\cdot_{G_2}[f_2])_{G_1}\cdot_{G_2}[f_3] = [f_1]_{G_1}\cdot_{G_2}([f_2]_{G_1}\cdot_{G_2}[f_3]).$$

If, however, $n = 1$, then the last relation is a consequence of the hypothesis that the triples (x_0, x_1'', x_2''), (x_0, x_1'', x_2'), (x_0, x_2'', x_3''), (x_0, x_1', x_3'') are all consistently oriented with the triple (x_0, x_1, x_2). Thus the associativity of the homotopy join relative to G_1 and G_2 is proved.

It is clear that the map f_0 of (X, x_0) into (Y, y_0), which assigns to every point $x \in X$ the point $y_0 \in Y$, is a representative of the neutral element for the operation of homotopy join. That is,

$$[f_0]_{G_1 \cdot G_2}[f] = [f]_{G_1 \cdot G_2}[f_0] = [f] \quad \text{for all maps } f\colon (X, x_0) \to (Y, y_0).$$

Now let us prove that for every map $f\colon (X, x_0) \to (Y, y_0)$ there is a map f^- of (X, x_0) into (Y, y_0) such that $[f]_{G_1 \cdot G_2}[f^-] = [f_0]$.

The manner the balls G_1, G_2 are chosen, is irrelevent except that in the case $n = 1$ we must keep the orientation of the triple (x_0, x_1, x_2). Therefore we may take G_1 and G_2 to be open half spheres in $X = S^n$ the common boundary B of which contains the point x_0. If $x \in X$, then we denote by x^- the point of X which is symmetric to x relative to the "equatorial" plane in E^{n+1} containing B. We may suppose that f is concentrated on G_1. Then we define f^- by the formula

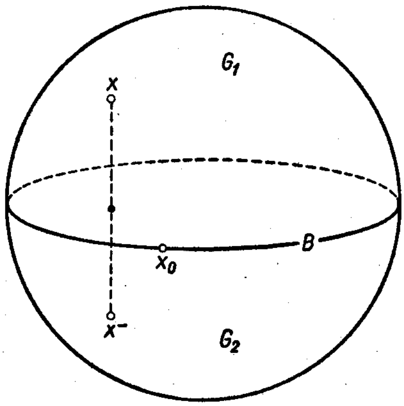


Fig. 3

$$f^-(x) = f(x^-) \quad \text{for every } x \in X.$$

It is clear that f^- is concentrated on G_2 and is separated from f. It remains to show that the map $f' = f \cdot f^-$ is homotopic to the constant map f_0.

In order to do this let us denote by Q^{n+1} the ball in E^{n+1} having $X = S^n$ for its boundary. Thus Q^{n+1} consists of all those points $x \in E^{n+1}$ for which $\|x\| \leqslant 1$. We prove the homotopy of f_0 with f' by showing that we may extend f' to a map f'' of Q^{n+1} into Y. In order to do this we have only to set

$$f''(z) = f'(x)$$

for each point z of the segment joining x with x^-. This definition is not ambiguous since $f'(x) = f'(x^-)$.

We have shown that the homotopy classes of maps of (S^n, x_0) into (Y, y_0) form a group relative to the operation of homotopy join with respect to a pair of open balls G_1, G_2.

Manifestly, for each point $x_0' \in S^n$ there exists a homeomorphism $h\colon (S^n, x_0) \to (S^n, x_0')$ which maps the given disjoint open balls G_1, G_2 onto two other disjoint open balls G_1', G_2'. If we assign to every map $f\colon (S^n, x_0) \to (Y, y_0)$ the map $fh^{-1}\colon (S^n, x_0') \to (Y, y_0)$, then to the join of two maps f_1, f_2 concentrated on G_1 and G_2, respectively, corresponds the join of maps $f_1 h^{-1}$ and $f_2 h^{-1}$ concentrated on G_1' and G_2', respectively. Thus we infer that this correspondence induces an isomorphism between the just considered group of homotopy classes of maps of (S^n, x_0) into (Y, y_0) and the analogous group of homotopy classes of maps of (S^n, x_0') into (Y, y_0). Consequently the algebraic structure of this group does not depend on the choice of the point x_0.

As we have already seen, the choice of the open disjoint balls G_1, G_2 is immaterial in the case $n > 1$. In the case $n = 1$, the homotopy join with respect to G_1 and G_2 depends in general on the orientation (on S_1) of the triple of points x_0, x_1, x_2, where x_i is the center of G_i. However, also in this case the algebraic structure of the considered group does not depend on the choice of balls G_1, G_2, because a homeomorphism $h\colon (S^1, x_0) \to (S^1, x_0)$ changing the orientation of S^1 induces an isomorphism between the group of homotopy classes of maps $f\colon (S^1, x_0) \to (Y, y_0)$ with the operation given by the join with respect to G_1, G_2 and the analogous group with the operation given by the join with respect to $h(G_1)$, $h(G_2)$ which is the same as the join with respect to G_2, G_1. Thus in all cases this group does not depend on the choice of the point x_0 and of the disjoint open balls G_1, G_2. This group is denoted by $\pi_n(Y, y_0)$ and is called the n-th *homotopy group of* Y *with base point* y_0. If $n > 1$, then $\pi_n(Y, y_0)$ is Abelian. The group $\pi_1(Y, y_0)$ is called also the *fundamental group of* Y *with base point* y_0. Simple examples show that $\pi_1(Y, y_0)$ may be not Abelian, for example, if Y is the lemniscate. If the space Y is arcwise connected, then $\pi_n(Y, y_0)$ does not depend on the choice of the base point y_0.

If we recall that $Y \in \mathbf{C}^n$ denotes that each map $\varphi\colon S^k \to Y$ with $k \leqslant n$ is homotopic to a constant, we see that

(6.1) *$Y \in \mathbf{C}^n$ if and only if the group $\pi_k(Y, y_0)$ is trivial for $k = 0, 1, 2, \ldots, n$ and for every point $y_0 \in Y$.*

Since $Y \in \mathbf{C}^\infty$ denotes that $Y \in \mathbf{C}^n$ for every n, we have

(6.2) *$Y \in \mathbf{C}^\infty$ if and only if the group $\pi_n(Y, y_0)$ is trivial for every $n = 0, 1, \ldots$ and for every point $y_0 \in Y$.*

In particular, if Y consists of a single point y_0, then all homotopy groups of Y are trivial.

7. Homomorphisms induced by maps. Suppose we are given a map

$$f\colon (Y, y_0) \to (Z, z_0).$$

We recall that the operation Φ_f, which assigns to a map $\varphi \in (Y, y_0)^{(X,x_0)}$, the map $f\varphi \in (Z, z_0)^{(X,x_0)}$, is a continuous operation. Moreover, we have seen that if $\varphi_1, \varphi_2 \in (Y, y_0)^{(X,x_0)}$ are separated, then

$$\Phi_f(\varphi_1 \cdot \varphi_2) = \Phi_f(\varphi_1) \cdot \Phi_f(\varphi_2)$$

and therefore we have the following

(7.1) THEOREM. *Each map $f\colon (Y, y_0) \to (Z, z_0)$ induces, by means of the operation Φ_f, a homomorphism*

$$\Phi_{f*}\colon \pi_n(Y, y_0) \to \pi_n(Z, z_0).$$

Some relations between the fundamental group of a space and of its retract have been studied by T. Ganea [132].

We note the following properties of the homomorphism Φ_{f*}:

(7.2) *The identity map on (Y, y_0) induces the identity homomorphism on $\pi_n(Y, y_0)$.*

(7.3) *Homotopic maps induce the same homomorphism.*

(7.4) *All homotopy groups of a contractible space are trivial* (a consequence of (7.3)).

(7.5) *The homomorphism induced by the composition of two maps is the composition of the induced homomorphisms.*

(7.6) *If $f\colon (Y, y_0) \to (Z, z_0)$ has the homotopic right inverse*

$$g\colon (Z, z_0) \to (Y, y_0),$$

then Φ_{g*} is a right inverse of Φ_{f*} (a consequence of (7.2), (7.3), (7.5)).

(7.7) *If (Z, z_0) is an h-image of (Y, y_0), then $\pi_n(Z, z_0)$ is an r-homomorphic image of $\pi_n(Y, y_0)$ (for all n).*

(7.8) *The h-equal spaces have r-equal homotopy groups.*

(7.9) *Homotopically equivalent spaces have isomorphic homotopy groups.*

(7.10) *The r-maps induce r-homomorphisms* (a particular case of (7.7)).

(7.11) *The r-equal spaces have r-equal homotopy groups* (a particular case of (7.8)).

8. Homomorphism of Hurewicz. Let us recall an important theorem of W. Hurewicz concerning a relation between the groups of homotopy $\pi_n(Y, y_0)$ of a space Y with the base point y_0 and the Betti groups $H_n(Y)$ of Y.

Since the nth Betti group $H_n(X)$ of the n-dimensional sphere $X = S^n$ is the infinite cyclic group, there exists an element $\gamma \in H_n(X)$ (defined uniquely up to the sign) which is a generator of the group $H_n(X)$. Let us fix such a generator γ and let us consider a map $f\colon (X, x_0) \to (Y, y_0)$ and the homomorphism $f_*\colon H_n(X) \to H_n(Y)$ induced by f. As we already know, the homomorphism f_* depends only on the homotopy class $[f]$ of f, and this homotopy class is an element of the group $\pi_n(Y, y_0)$. Let us set

$$\theta([f]) = f_*(\gamma) \quad \text{for every } [f] \in \pi_n(Y, y_0).$$

If we recall that, by (5.3), the homomorphism induced by a homotopic join of two maps is the sum of the homomorphisms induced by these maps, we can easily infer that the function θ is a homomorphism of the group $\pi_n(Y, y_0)$ into the group $H_n(Y)$. This homomorphism is called the *homomorphism of Hurewicz.*

The theorem of Hurewicz ([163], p. 522) asserts that

(8.1) *If $Y \in \mathbf{C}^{n-1}$ and $Y \in \mathbf{LC}^{n-1}$ (where $n > 1$), then the homomorphism of Hurewicz $\theta\colon \pi_n(Y, y_0) \to H_n(Y)$ is an isomorphism.*

By an easy induction, we get the following statement from this theorem:

(8.2) *If $Y \in \mathbf{LC}^{n-1}$ (where $n > 1$), the fundamental group of Y with the base point y_0 is trivial, and the Betti groups $H_k(Y)$ are trivial for $k = 1, 2, \ldots, n-1$, then the homotopy group $\pi_n(Y, y_0)$ is isomorphic to the Betti group $H_n(Y)$.*

If we recall that for an arcwise connected space the homotopy groups do not depend on the choice of the base point, we can get in particular

(8.3) *For an arcwise connected space $X \in \mathbf{LC}^\infty$ with trivial fundamental group and trivial Betti groups, all homotopy groups are trivial.*

9. Maps into manifolds. The following proposition ([166], p. 75) belongs to elementary statements concerning the properties of maps into Euclidean spaces:

(9.1) *Let Z be a metric space of dimension $< m$ and let y_1 be a point of the Euclidean m-dimensional space E^m. Then for every map $\varphi\colon Z \to E^m$ and for every $\varepsilon > 0$ there exists a map $\psi\colon Z \to E^m$ such that*

$$\varrho\big(\varphi(z), \psi(z)\big) < \varepsilon \text{ and } \psi(z) \neq y_1 \quad \text{for every point } z \in Z.$$

Let us apply proposition (9.1) in order to prove the following proposition:

(9.2) *Let Z be a metric space of dimension $< m$ and let y_1 be a point of the Euclidean m-dimensional space E^m. Then for every map $\varphi\colon Z \to E^m$ and for every open ball V in E^m with center y_1 there exists a homotopy $\{\varphi_t\} \subset (E^m)^Z$ satisfying the following conditions:*

(i) $\varphi_0 = \varphi$,

(ii) $\varphi_t(z) = \varphi(z)$ *for every point* $z \in \varphi^{-1}(E^m - V)$ *and every* $0 \leqslant t \leqslant 1$,

(iii) $\varphi_t(z) \in V$ *for every point* $z \in \varphi^{-1}(V)$ *and every* $0 \leqslant t \leqslant 1$,

(iv) *The distance of the point y_1 from the set $\varphi_1(Z)$ is positive.*

Proof. Let V_a denote the open ball in E^m with center y_1 and radius $a > 0$. Let ε be a positive number less than 1/3 of the radius of the ball V. Setting

$$Z_1 = \varphi^{-1}(E^m - V_\varepsilon), \qquad Z_2 = \varphi^{-1}(E^m - V_{2\varepsilon}),$$

we get two closed subsets Z_1, Z_2 of the space Z such that Z_2 is contained in the interior of Z_1. Moreover,

$$\varphi^{-1}(E^m - V) \subset Z_2 \quad \text{and} \quad \varrho\big(\varphi(z), y_1\big) \geqslant \varepsilon \quad \text{for every point } z \in Z_1.$$

Since Z_2 is a subset of the interior of the set Z_1, there exists a continuous real-valued function ϑ defined in Z and satisfying the following conditions:

$$0 \leqslant \vartheta(z) \leqslant 1 \quad \text{for every point } z \in Z,$$

$$\vartheta(z) = \begin{cases} 0 & \text{for every point } z \in Z_2, \\ 1 & \text{for every point } z \in Z - Z_1. \end{cases}$$

Let ψ be a map of Z into E^m satisfying (9.1). Setting

(9.3) $$\varphi_t(z) = 2t\cdot\vartheta(z)\psi(z) + \big(1 - 2t\cdot\vartheta(z)\big)\varphi(z) \quad \text{for} \quad 0 \leqslant t \leqslant \tfrac{1}{2} \text{ and } z \in Z,$$

we get a continuous family of maps $\{\varphi_t\} \subset (E^m)^Z$ joining $\varphi_0 = \varphi$ with $\varphi_{\frac{1}{2}}$. Let us observe that the map $\varphi_{\frac{1}{2}}$ satisfies the condition

(9.4) $$\varphi_{\frac{1}{2}}(Z) \subset E^m - (y_1).$$

In fact, if $z \in Z_1$, then $\varrho\big(\varphi(z), y_1\big) \geqslant \varepsilon$ and $\varrho\big(\varphi(z), \psi(z)\big) < \varepsilon$. Consequently, the point $\varphi_{\frac{1}{2}}(z)$, belonging by (9.3) to the segment $|\varphi(z), \psi(z)|$, is distinct from y_1. If, however, $z \in Z - Z_1$, then $\vartheta(z) = 1$, whence $\varphi_{\frac{1}{2}}(z) = \psi(z) \neq y_1$. Thus (9.4) is proved.

Moreover, let us observe that

$$\varphi_t(z) = \varphi(z) \quad \text{for every point } z \in \varphi^{-1}(E^m - V) \text{ and } 0 \leqslant t \leqslant \tfrac{1}{2},$$

because $\varphi^{-1}(E^m - V) \subset Z_2$ implies that $\vartheta(z) = 0$ for $z \in \varphi^{-1}(E^m - V)$.

Now let us set

$$r_t(y) = t\left(\frac{\varepsilon(y - y_1)}{|y - y_1|} + y_1\right) + (1 - t)y \quad \text{for every } y \in V_\varepsilon - (y_1) \text{ and } 0 \leqslant t \leqslant 1,$$

$$r_t(y) = y \quad \text{for every point } y \in E^m - V_\varepsilon \text{ and } 0 \leqslant t \leqslant 1.$$

We easily see that if we complete formula (9.3) by the formula

$$\varphi_t(z) = r_{2t-1}\varphi_{\frac{1}{2}}(z) \quad \text{for every point } z \in Z \text{ and for } \tfrac{1}{2} \leqslant t \leqslant 1,$$

then we get a homotopy $\{\varphi_t\} \subset (E^m)^Z$ satisfying conditions (i), (ii) and (iii) of (9.2). Also condition (iv) is satisfied, because all values of the map r_1 belong to the set $E^m - V_\varepsilon$.

Thus the proof of (9.2) is finished.

(9.5) THEOREM. *Let X_0 be a subset of a metric space X of dimension $< m$ and let y_0, y_1 be two distinct points of an m-dimensional manifold Y. Then for every map $f\colon (X, X_0) \to (Y, y_0)$ and for every positive number ε there exists a map $g\colon (X, X_0) \to (Y, y_0)$ homotopic to f and satisfying both conditions:*

(a) $\varrho(f(x), g(x)) < \varepsilon$ *for every point* $x \in X$.

(b) *The distance of the point y_1 from the set $g(X)$ is positive.*

Moreover, if X_1 is a subset of X such that the distance of the point y_1 from the set $f(X_1)$ is positive, then the map g can be chosen so that it satisfies the following condition

(c) $f(x) = g(x)$ *for every point* $x \in X_1$.

Proof. Since Y is a manifold, there exists a homeomorphism h mapping E^m onto an open neighborhood $U \subset Y - (y_0)$ of the point y_1 in the space Y. Let V be an open ball in E^m with center $h^{-1}(y_1)$ and with the radius so small that the diameter of the set $h(V) \subset Y$ is less than ε. Consider the map φ defined on the set $Z = f^{-1}(U) \subset X - X_0$ by the formula

$$\varphi(z) = h^{-1}f(z) \quad \text{for every point } z \in Z.$$

By (9.2) there exists a homotopy $\{\varphi_t\} \subset (E^m)^Z$ satisfying the conditions (i)-(iv) of (9.2), where we substitute $h^{-}(y_1)$ instead of y_1. Now let us set

$$(9.6) \qquad f_t(x) = \begin{cases} h\varphi_t(x) & \text{for every point } x \in Z, \\ f(x) & \text{for every point } x \in X - Z. \end{cases}$$

We infer by (ii) that one obtains on this way a homotopy $\{f_t\} \subset (Y, y_0)^{(X, X_0)}$ joining the map $f_0 = f$ with the map $f_1 = g$ given by the formulas:

$$g(x) = \begin{cases} h\varphi_1(x) & \text{for every point } x \in Z, \\ f(x) & \text{for every point } x \in X - Z. \end{cases}$$

It follows by (ii) and (iii) that $f(x) \neq g(x)$ implies that both points $f(x)$ and $g(x)$ belong to $h(V)$ and consequently condition (a) is satisfied. Moreover, it follows by (iv) that there exists an open ball $\hat{V}$ in E^m with center $h^{-1}(y_1)$

and the radius η which does not contain any point of the set $\varphi_1(Z)$. We infer by (9.6) that none of the values of the map $f_1 = g$ belongs to the set $h(\hat{V})$ being a neighborhood of the point y_1 in Y. Hence condition (b) is satisfied.

In order to finish the proof, let us observe that if X_1 is a subset of X such that the distance of the point y_1 from the set $f(X_1)$ is positive, then the neighborhood U of the point y_1 in Y can be chosen so that $U \cap f(X_1) = 0$. Then $X_1 \subset X - Z$ and (9.6) implies that $f_t(x) = f(x)$ for every point $x \in X_1$ and $0 \leqslant t \leqslant 1$. In particular, $g(x) = f_1(x) = f(x)$ for every point $x \in X_1$. Thus the proof of Theorem (9.5) comes to an end.

10. Join of maps into spheres. Suppose now that we are given a pair (X, X_0), where X is a metric space, and X_0 is a closed subset of X. We study the properties of the homotopy join of homotopy classes of maps $f\colon (X, X_0) \to (Y, y_0)$, where $Y = S^n$ and $y_0 \in Y$.

First we observe, if f_0 is the map of X which takes all points of X into y_0, then for each map $f\colon (X, X_0) \to (Y, y_0)$ the homotopy class $[f]$ coincides with the homotopy join of the classes $[f]$ and $[f_0]$. Consequently $[f_0]$ acts as a neutral element with respect to the operation of homotopy join. Evidently, if $f_0'\colon (X, X_0) \to (Y, y_0)$ and $f_0'(X) \neq Y$, then f_0' is homotopic to f_0 and hence $f_0' = [f_0]$.

We now prove the following

(10.1) THEOREM. *For each map $f\colon (X, X_0) \to (Y, y_0)$ there is a map $f'\colon (X, X_0) \to (Y, y_0)$ such that $[f_0]$ is a homotopy join of $[f]$ and $[f']$.*

Proof. We begin with expressing Y as the union of two closed hemispheres Q_1 and Q_2 whose common boundary B contains the point y_0. It is easy to see that there exists a homotopy $\{\varphi_t\} \subset (Y, y_0)^{(Y, y_0)}$ such that $\varphi_0 = i$ is the identity map and φ_1 satisfies the condition $\varphi_1(Q_2) = y_0$. Let U_i denote the interior of the hemisphere Q_i and $G_i = f^{-1}(U_i)$ for $i = 1, 2$. Moreover, we denote by H the set $X - (G_1 \cup G_2)$ and by ψ the map of Y into itself which assigns to $y \in Y$ the point $\psi(y)$ of Y symmetric to y with respect to the n-plane of E^{n+1} determined by B.

Let us set

$$f' = \psi f, \quad f_t = \varphi_t f, \quad f_t' = \varphi_t f' \quad \text{for} \quad t \in \langle 0, 1 \rangle.$$

Then we have $f(x) = f'(x)$ for every point $x \in H = X - G_1 - G_2$ and consequently $f_t(x) = f_t'(x)$ for $x \in H$ and $t \in \langle 0, 1 \rangle$. If we set

$$f_t''(x) = \begin{cases} f_t(x) & \text{for} \quad x \in X - G_2, \\ f_t'(x) & \text{for} \quad x \in X - G_1, \end{cases}$$

then we obtain a continuous family $\{f_t''\}$ of maps of (X, X_0) into (Y, y_0) since f_t'' is well defined in $(X-G_1)\cap(X-G_2) = X-G_1-G_2$. Let us observe that all the values of f_0'' belong to Q_1 and so this map is homotopic to the constant map f_0. Moreover, we have

$$f_1''(x) = \begin{cases} \varphi_1(f(x)) & \text{for} \quad x\in G_1, \\ \varphi_1(f'(x)) & \text{for} \quad x\in G_2, \\ y_0 & \text{for} \quad x\in X-G_1-G_2. \end{cases}$$

It follows that f_1'' is the join of the maps $\varphi_1 f$ and $\varphi_1 f'$ which are concentrated on G_1 and G_2, respectively. Since $\varphi_1 f$ is homotopic to f and $\varphi_1 f'$ is homotopic to f', it follows that the homotopy class $[f_0] = [f_0'']$ is a homotopy join of $[f]$ and $[f']$.

In order to insure the possibility of making the homotopy join, let us assume that $\dim X < 2n$. Since the Cartesian product $Y\times Y$ of two n-spheres $Y = S^n$ is a $2n$-dimensional manifold, it follows from Theorem (9.5) that for each pair of maps φ_1, φ_2: $(X, X_0)\to(Y, y_0)$ and for each point $y_1\in Y$ distinct from y_0 there are maps φ_1', φ_2': $(X, X_0)\to(Y, y_0)$ with φ_i' homotopic to φ_i, $i = 1, 2$, and such that (y_1, y_1) is not in the image of the map f': $(X, X_0)\to\big(Y\times Y, (y_0, y_0)\big)$ defined by the formula

$$f'(x) = \big(\varphi_1'(x), \varphi_2'(x)\big).$$

As we have already seen (I, (12.2)), the pair $\big(Z, (y_0, y_0)\big)$, where

$$Z = \big(Y\times(y_0)\big)\cup\big((y_0)\times Y\big),$$

is a deformation retract of the pair consisting of the set

$$M = Y\times Y-(y_1, y_1)$$

and of the point (y_0, y_0). This means that there is a homotopy $\{\psi_t\}\subset\big(M, (y_0, y_0)\big)^{(M,(y_0,y_0))}$ such that $\psi_0 = i$, the identity, and $\psi_1(M) = Z$, with the restriction $\psi_1|Z$ being the identity map on Z. Setting $f_t = \psi_t f'$, we obtain a homotopy $\{f_t\}$ joining f' with $f_1 = f''$ which has the form

$$f'' = (\varphi_1'', \varphi_2''): (X, X_0)\to\big(Y\times Y, (y_0, y_0)\big),$$

and such that all values of f'' are in the set Z. We infer that the maps φ_1'', φ_2'': $(X, X_0)\to(Y, y_0)$ are homotopic to the maps φ_1, φ_2 respectively and that φ_1'' is concentrated on the open set

$$G_1 = f''^{-1}\big[\big(Y\times(y_0)\big)-(y_0, y_0)\big]$$

and φ_2'' is concentrated on the open set

$$G_2 = f''^{-1}\big[\big((y_0)\times Y\big)-(y_0, y_0)\big].$$

Since G_1 and G_2 are disjoint, it follows that the homotopy class of the map $\varphi'' = \varphi_1''\cdot\varphi_2''$ is a homotopy join of the homotopy classes $[\varphi_1]$ and $[\varphi_2]$. Thus we have proved the following

(10.2) THEOREM. *If* $\dim X < 2n$, *then for each pair* φ_1, φ_2 *of maps of* (X, X_0) *into* (S^n, y_0) *there is a map* φ *of* (X, X_0) *into* (S^n, y_0) *such that the homotopy class* $[\varphi]$ *is a homotopy join of the homotopy classes* $[\varphi_1]$ *and* $[\varphi_2]$.

Remark. If $\dim X \geqslant 2n$, then the homotopy join of the homotopy classes of maps of X into S^n need not exist ([43], p. 736). For instance, if X is a 4-dimensional manifold being the Cartesian product of the 2-dimensional sphere S^2 with itself, then the homotopy join of the maps φ_1, φ_2: $X \to S^2$ given by the formulas

$$\varphi_1(x_1, x_2) = x_1, \quad \varphi_2(x_1, x_2) = x_2 \quad \text{for every point } (x_1, x_2)\in X$$

does not exist. In order to see this let us consider a true cycle γ which is a representative of a generator of the Betti group $H_2(S^2)$. Let us pick out a point $a\in S^2$ and let ψ_1 and ψ_2 denote two maps of S^2 into $S^2\times S^2$ given by the formulas

$$\psi_1(x) = (x, a), \quad \psi_2(x) = (a, x) \quad \text{for every point } x\in S^2.$$

Then ψ_1 and ψ_2 map γ onto true cycles $\gamma_1 = \psi_1(\gamma)$ and $\gamma_2 = \psi_2(\gamma)$. If there would exist a homotopy join φ: $X \to S^2$ of the maps φ_1 and φ_2 then we should infer by Theorem (5.3) that φ assigns to each of the true cycles γ_1 and γ_2 a true cycle homologous to γ in S^2. But this is impossible, by a theorem of Hopf ([153], p. 436).

11. Cohomotopy groups. In order to define the homotopy groups of a space we considered maps of (S^n, x_0) into the space and the operation of homotopy join. In the view to obtain the cohomotopy groups of a pair of spaces (X, X_0) ([35] and [266]) we consider maps f: $(X, X_0) \to (Y, y_0)$ where $Y = S^n$ and $y_0\in Y$. By Theorem (10.2), if X is a metric space of dimension $< 2n$, then for every two homotopy classes belonging to $[Y, y_0]^{[X, X_0]}$ there exists a homotopy class being their homotopy join.

Before we shall prove (by a little more restrictive hypotheses) the uniqueness and the associativity of the operation of the homotopy join, let us prove a proposition which may be considered as a substitute of the associativity.

First we establish an auxiliary proposition concerning the maps of a space of dimension $< 2n$ into the Cartesian product of three n-dimensional spheres.

(11.1) *If X is a metric space of dimension $< 2n$ and $Y = S^n$, then for every closed subset X_0 of X, for every two distinct points $y_0, y_1 \in Y$, and for every map $f: (X, X_0) \to \big(Y \times Y \times Y, (y_0, y_0, y_0)\big)$ there exists a map $\hat{f}: (X, X_0) \to \big(Y \times Y \times Y, (y_0, y_0, y_0)\big)$ homotopic to f and such that none of its values belongs to the set*

$$W_1 = \big(Y \times (y_1) \times (y_1)\big) \cup \big((y_1) \times Y \times (y_1)\big) \cup \big((y_1) \times (y_1) \times Y\big).$$

Proof. Let

$$f(x) = \big(\varphi_1(x), \varphi_2(x), \varphi_3(x)\big) \quad \text{for every point } x \in X.$$

Setting

$$f_1(x) = \big(\varphi_2(x), \varphi_3(x)\big) \quad \text{for every point } x \in X,$$

we get a map $f_1: (X, X_0) \to \big(Y \times Y, (y_0, y_0)\big)$. It follows by (9.5) that there exists a map $f_1': (X, X_0) \to \big(Y \times Y, (y_0, y_0)\big)$ homotopic to f_1 and such that the distance d_1 of the point $(y_1, y_1) \in Y \times Y$ from the set $f_1'(X)$ is positive. Let

$$f_1'(x) = \big(\varphi_2'(x), \varphi_3'(x)\big) \quad \text{for every point } x \in X.$$

It is plain that the map f' given by the formula

$$f'(x) = \big(\varphi_1(x), \varphi_2'(x), \varphi_3'(x)\big) \quad \text{for every point } x \in X$$

is homotopic to f and it satisfies the condition

(11.2) $\varrho\big(f'(x), (y, y_1, y_1)\big) > d_1$ *for every point $x \in X$ and every point $y \in Y$.*

Now let us consider the map $f_2: (X, X_0) \to \big(Y \times Y, (y_0, y_0)\big)$ given by the formula

$$f_2(x) = \big(\varphi_1(x), \varphi_3'(x)\big) \quad \text{for every point } x \in X.$$

Applying (9.5) for the second time, we infer that there exists a map $f_2': (X, X_0) \to \big(Y \times Y, (y_0, y_0)\big)$ homotopic to f_2 and satisfying the following two conditions:

(11.3) $\varrho\big(f_2'(x), f_2(x)\big) < \frac{1}{2} d_1$ *for every point $x \in X$,*

(11.4) *There exists a positive number $d_2 < \frac{1}{2} d_1$ such that $\varrho\big(f_2'(x), (y_1, y_1)\big) > d_2$ for every point $x \in X$.*

Let $f_2'(x) = \big(\varphi_1''(x), \varphi_3''(x)\big)$ for every point $x \in X$. Then the map f'' given by the formula

$$f''(x) = \big(\varphi_1''(x), \varphi_2'(x), \varphi_3''(x)\big) \quad \text{for every point } x \in X$$

is homotopic to f' (hence also to f) and it satisfies the condition

$\varrho(f''(x),(y_1,y,y_1)) > d_2$ for every point $x \in X$ and every point $y \in Y$. Moreover, since

$$\varrho(f''(x), f'(x)) = \varrho(f_2'(x), f_2(x)) \quad \text{for} \quad x \in X,$$

we infer by (11.2) and (11.3) that

(11.5) $\varrho(f''(x),(y,y_1,y_1)) > d_2$ *and* $\varrho(f''(x),(y_1,y,y_1)) > d_2$ *for every point* $x \in X$ *and for every point* $y \in Y$.

Now we consider the map $f_3\colon (X,X_0) \to (Y \times Y, (y_0, y_0))$ given by the formula

$$f_3(x) = (\varphi_1''(x), \varphi_2'(x)) \quad \text{for every point } x \in X.$$

Applying (9.5) for the third time, we infer that there exists a map $f_3'\colon (X,X_0) \to (Y \times Y, (y_0,y_0))$ homotopic to f_3 and satisfying two conditions

(11.6) $$\varrho(f_3'(x), f_3(x)) < \tfrac{1}{2} d_2,$$

(11.7) $$f_3'(x) \neq (y_1, y_1) \quad \text{for every point } x \in X.$$

Let $f_3'(x) = (\varphi_1'''(x), \varphi_2'''(x))$ for every point $x \in X$. Setting

$$\hat{f}(x) = (\varphi_1'''(x), \varphi_2'''(x), \varphi_3''(x)) \quad \text{for every point } x \in X,$$

we get a map $\hat{f}\colon (X,X_0) \to (Y \times Y \times Y, (y_0,y_0,y_0))$ homotopic to f'' (hence homotopic also to f) and, since $\varrho(\hat{f}(x), f''(x)) = \varrho(f_3'(x), f_3(x))$, we infer by (11.5), (11.6), and (11.7) that for every point $x \in X$ and for every point $y \in Y$ the value $\hat{f}(x)$ is distinct from (y,y_1,y_1), (y_1,y,y_1), and (y_1,y_1,y), i.e. $\hat{f}(X) \subset Y \times Y \times Y - W_1$. Thus the proof of (11.1) is finished.

Now let us prove the following proposition:

(11.8) *If X is a metric space of dimension $< 2n$, then for every three maps $\varphi_1, \varphi_2, \varphi_3$ of (X,X_0) into (Y,y_0) where $Y = S^n$, there exist maps $\varphi, \varphi', \varphi''$ of (X,X_0) into (Y,y_0) such that:*

1. $[\varphi']$ *is a homotopy join of* $[\varphi_1]$ *and* $[\varphi_2]$,
2. $[\varphi'']$ *is a homotopy join of* $[\varphi_2]$ *and* $[\varphi_3]$,
3. $[\varphi]$ *is a homotopy join of* $[\varphi']$ *and* $[\varphi_3]$ *and it is also a homotopy join of* $[\varphi_1]$ *and of* $[\varphi'']$.

Proof. Setting $f(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x))$ for every point $x \in X$, we obtain a map $f\colon (X,X_0) \to (Y \times Y \times Y, (y_0,y_0,y_0))$. By (11.1), there is a map f of (X,X_0) into $(Y \times Y \times Y, (y_0,y_0,y_0))$ homotopic to f and such that

$$f'(X) \subset Y \times Y \times Y - W_1.$$

We have already noticed (I, (12.3)) that the pair $(W_0, (y_0, y_0, y_0))$, where

$$W_0 = (Y\times(y_0)\times(y_0))\cup((y_0)\times Y\times(y_0))\cup((y_0)\times(y_0)\times Y),$$

is a deformation retract of the pair

$$(Y\times Y\times Y - W_1, (y_0, y_0, y_0)).$$

It follows that f' is homotopic to a map $f'' = (\varphi_1'', \varphi_2'', \varphi_3'')$ whose values lie in W_0. Since f' is homotopic to $f = (\varphi_1, \varphi_2, \varphi_3)$, we infer that φ_i is homotopic to φ_i'' for $i = 1, 2, 3$. Let us now observe that the set $W_0 - (y_0, y_0, y_0)$ consists of three components. We infer that there exist in the space X three open disjoint sets G_1, G_2, G_3 not intersecting X_0 and such that φ_i'' is concentrated on G_i, $i = 1, 2, 3$. If we set

$$\varphi'(x) = \begin{cases} \varphi_i''(x) & \text{for} \quad x\in G_i,\ i = 1, 2, \\ y_0 & \text{for} \quad x\in X - G_1 - G_2, \end{cases}$$

$$\varphi''(x) = \begin{cases} \varphi_i''(x) & \text{for} \quad x\in G_i,\ i = 2, 3, \\ y_0 & \text{for} \quad x\in X - G_2 - G_3, \end{cases}$$

$$\varphi(x) = \begin{cases} \varphi_i''(x) & \text{for} \quad x\in G_i,\ i = 1, 2, 3, \\ y_0 & \text{for} \quad x\in X - G_1 - G_2 - G_3, \end{cases}$$

then we obtain maps $\varphi, \varphi', \varphi''$ of (X, X_0) into (Y, y_0). From this definition it readily follows that the map φ' is the join of φ_1'' and φ_2'', the map φ'' is the join of φ_2'' and φ_3'', and the map φ is the join of φ' and φ_3'' and it is also the join of φ_1'' and φ''. Thus the proof of (11.8) is accomplished.

Now let us prove that the operation of homotopy join is unique when $\dim X < 2n-1$.

Suppose then that φ_1, φ_2 are two maps of (X, X_0) into (Y, y_0) concentrated on two disjoint open sets G_1 and G_2, respectively. Further suppose that φ_1', φ_2' are maps homotopic to φ_1, φ_2, respectively, with φ_1', φ_2' concentrated on the disjoint open sets G_1' and G_2', respectively. Let us show that the join $\varphi_1\cdot\varphi_2$ is homotopic to the join $\varphi_1'\cdot\varphi_2'$.

To this aim let $\{\varphi_{it}\}$ be a homotopy joining $\varphi_{i0} = \varphi_i$ to the map $\varphi_{i1} = \varphi_i'$ in the functional space $(Y, y_0)^{(X,X_0)}$ $(i = 1, 2)$. Setting

$$f_t(x) = (\varphi_{1t}(x), \varphi_{2t}(x)) \quad \text{for} \quad x\in X,$$

we get a continuous family of maps joining, in the space $(Y\times Y, (y_0, y_0))^{(X,X_0)}$, the map $f_0 = (\varphi_1, \varphi_2)$ to the map $f_1 = (\varphi_1', \varphi_2')$. Let us observe that the values of both f_0 and f_1 lie in the set

$$Z = (Y\times(y_0))\cup((y_0)\times Y).$$

The family $\{f_t\}$ may be considered as one map (I, (11.3)):

$$\psi\colon \big(X\times\langle 0,1\rangle, X_0\times\langle 0,1\rangle\big)\to\big(Y\times Y, (y_0,y_0)\big)$$

defined by the formula $\psi(x,t) = f_t(x)$. Since $Y\times Y$ is a manifold of dimension $2n$ and the dimension of the set $X\times\langle 0,1\rangle$ is less than $2n$, and since $\psi\big(X\times(0)\cup X\times(1)\big)$ is a subset of the set Z lying at a positive distance from the point (y_1,y_1), we infer by (9.5) that there is a map

$$\psi'\colon (X\times\langle 0,1\rangle, X_0\times\langle 0,1\rangle)\to\big(Y\times Y, (y_0,y_0)\big)$$

homotopic to ψ and such that ψ and ψ' are equal on the set $X\times(0)\cup X\times(1)$, and that $\psi'(X\times\langle 0,1\rangle)$ misses the point (y_1,y_1) distinct from (y_0,y_0).

As we have shown (I, (12,2)), there is a retraction r of $Y\times Y-(y_1,y_1)$ to Z. It follows that $\psi''=r\psi'$ is a map of $(X\times\langle 0,1\rangle, X_0\times\langle 0,1\rangle)$ into $\big(Z,(y_0,y_0)\big)$ such that

$$\psi''(x,0) = r\psi'(x,0) = r\psi(x,0) = rf_0(x) = f_0(x) = \big(\varphi_1(x),\varphi_2(x)\big),$$

$$\psi''(x,1) = r\psi'(x,1) = r\psi(x,1) = rf_1(x) = f_1(x) = \big(\varphi_1'(x),\varphi_2'(x)\big)$$

for every point $x\in X$.

Define a map α of $\big(Z,(y_0,y_0)\big)$ into (Y,y_0) by means of the formula

$$\alpha(y,y_0) = \alpha(y_0,y) = y \quad \text{for every point } y\in Y.$$

Setting

$$\varphi_t''(x) = \alpha\psi''(x,t) \quad \text{for every } x\in X \text{ and } t\in\langle 0,1\rangle,$$

we see that the map $\varphi_0''=\alpha f_0$ is the join of φ_1 and φ_2 while $\varphi_1''=\alpha f_1$ is the join of φ_1' and φ_2'. Since $\{\varphi_t''\}$ is a homotopy in $(Y,y_0)^{(X,X_0)}$ which joins $\varphi_1\cdot\varphi_2$ to $\varphi_1'\cdot\varphi_2'$, it follows that the operation of homotopy join is single valued. Summarizing, we have the following

(11.9) THEOREM. *If the dimension of the metric space X is $<2n-1$, then for each pair of homotopy classes $[\varphi_1],[\varphi_2]\subset(S_n,y_0)^{(X,X_0)}$ there is precisely one homotopy class $[\varphi]$ which is the homotopy join of $[\varphi_1]$ and $[\varphi_2]$.*

We have seen that the operation of homotopy join is commutative and (now genuinely) associative, that there is a neutral element for this operation and an inverse class for each homotopy class. Consequently we have the following

(11.10) THEOREM. *Let X_0 be a closed subset of a metric space X of dimension $<2n-1$ and $y_0\in S^n$. Then the set of all homotopy classes of maps $\varphi\colon(X,X_0)\to(S^n,y_0)$ is an Abelian group relative to the operation of homotopy join.*

Let us denote this group by $\pi^n(X, X_0, y_0)$. We easily see that this group *does not depend on* y_0. Because, if y_0' is another point of S^n and h is a rotation of S^n taking y_0 onto y_0', then the correspondence Φ_h assigning to the map φ the map $\varphi' = h\varphi$, induces a one-to-one correspondence between homotopy classes $[\varphi]$ of maps φ: $(X, X_0) \to (Y, y_0)$ and classes $[\varphi']$ of maps φ': $(X, X_0) \to (Y, y_0')$. This correspondence preserves homotopy joins and so is an isomorphism of $\pi^n(X, X_0, y_0)$ onto $\pi^n(X, X_0, y_0')$. Thus we shall write $\pi^n(X, X_0)$ rather that $\pi^n(X, X_0, y_0)$. This group $\pi^n(X, X_0)$ is called the *n*th-*cohomotopy group of the pair* (X, X_0). If $X_0 = 0$, then the group $\pi^n(X, X_0)$ is said to be *n*th-*cohomotopy group of* X.

12. Homomorphisms induced by maps. Let X and X' be two metric spaces of dimensions $< 2n-1$ and let X_0 be a closed subset of X and X_0' a closed subset of X'.

As formerly, $Y = S^n$ and $y_0 \in Y$. As we know (I, Section 9), every map

$$f\colon (X', X_0') \to (X, X_0)$$

induces a continuous operation

$$\Phi^f\colon (Y, y_0)^{(X,X_0)} \to (Y, y_0)^{(X',X_0')};$$

namely Φ^f assigns to every map φ: $(X, X_0) \to (Y, y_0)$ the map φf:$(X', X_0') \to (Y, y_0)$. Since Φ^f is continuous (I, Section 9), it assigns to a pair of homotopic maps a pair of homotopic maps. Thus Φ^f induces a correspondence between homotopy classes and since Φ^f commutes with the operation of join (by (5.1)), it follows that Φ^f induces a homomorphism

$$\Phi^{f*}\colon \pi^n(X, X_0) \to \pi^n(X', X_0').$$

That means

(12.1) *Every map* f: $(X', X_0') \to (X, X_0)$ *induces a homomorphism*

$$\Phi^{f*}\colon \pi^n(X, X_0) \to \pi^n(X', X_0').$$

Moreover,

(12.2) *The identity map* i: $(X, X_0) \to (X, X_0)$ *induces the identity homomorphism* Φ^{i*} *on* $\pi^n(X, X_0)$.

Further,

(12.3) *Homotopic maps induce the same homomorphism.*

In order to see this, suppose that f, g: $(X', X_0') \to (X, X_0)$ are homotopic. Then the homotopy classes of φf and φg assigned by Φ^{f*} and Φ^{g*} to a map φ: $(X, X_0) \to (Y, y_0)$ are the same.

Suppose that f: $(X', X_0') \to (X, X_0)$ and g: $(X'', X_0'') \to (X', X_0')$ are two maps where X'' is metric and has dimension $< 2n-1$ and $x_0'' \in X''$. Then we have

(12.4) $\Phi^{(fg)*} = \Phi^{g*}\Phi^{f*}$, *that means, the composition of two maps induces a homomorphism which is the composition* (in reverse order!) *of the homomorphism induced by each of the two maps.*

We recall that $(X, X_0) \underset{h}{\leqslant} (X', X'_0)$ means that there is a map $f\colon (X', X'_0) \to (X, X_0)$ and a map $g\colon (X, X_0) \to (X', X'_0)$ such that the composition $fg\colon (X, X_0) \to (X, X_0)$ is homotopic to the identity map. Then it follows from (12.4) that Φ^{f*} is a right inverse of Φ^{g*}. In other words,

(12.5) *If* $(X, X_0) \underset{h}{\leqslant} (X', X'_0)$, *then* $\pi^n(X, X_0) \underset{r}{\leqslant} \pi^n(X', X'_0)$.

From (12.5) if follows that

(12.6) $(X, X_0) \underset{r}{\leqslant} (X', X'_0)$ *implies* $\pi^n(X, X_0) \underset{r}{\leqslant} \pi^n(X', X'_0)$;

(12.7) $(X, X_0) \underset{h}{=} (X', X'_0)$ *implies* $\pi^n(X, X_0) \underset{r}{=} \pi^n(X', X'_0)$;

(12.8) $(X, X_0) \underset{r}{=} (X', X'_0)$ *implies* $\pi^n(X, X_0) \underset{r}{=} \pi^n(X', X'_0)$.

Moreover, if $(X, X_0) \underset{h}{\simeq} (X', X'_0)$, then Φ^{f*} and Φ^{g*} are, by (12.4), right inverses of each other, so that

(12.9) $(X, X_0) \underset{h}{\simeq} (X', X'_0)$ *implies that* $\pi^n(X, X_0)$ *is isomorphic to* $\pi^n(X', X'_0)$.

From (12.9) it follows in particular that

(12.10) *All the cohomotopy groups of a contractible space are trivial.*

13. Homotopy dependence of maps. The definition of homotopy and cohomotopy groups was based on the operation of join for the homotopy classes of maps of spheres into a space, or of a space into a sphere. Using a theorem of H. Hopf ([153], p. 436) one can show ([43], p. 737) that it is impossible to introduce reasonably a group operation for homotopy classes of maps of an arbitrary space into another arbitrary space (compare Section 10, Remark). However, in several rather general cases it is possible to introduce a relation with some definite algebraic content, for homotopy classes of maps. These algebraic content is contained in the notion of the homotopy dependence of maps [58].

Consider a pair of spaces (Y, Y_0) with $Y \neq 0$ and let U be an arbitrary set of elements u called *indices*. Setting $Y_u = Y$ and $Y_{0u} = Y_0$ for every $u \in U$, let us denote by Y^U the Cartesian product $\underset{u \in U}{P} Y_u$ and by Y_0^U the Cartesian product $\underset{u \in U}{P} Y_{0u}$. The set Y^U consists of all systems $\{y_u\}$ with $u \in U$ and $y_u \in Y$. The elements y_u are said to be *coordinates* of the point $\{y_u\}$. The set Y_0^U consists of all points $\{y_u\}$ with all coordinates y_u in Y_0.

If $U = 0$ and $Y_0 \neq 0$, then $Y^U = Y_0^U$ consists of exactly one point (the empty system of coordinates). If, however, $U = 0$ and $Y_0 = 0$, then Y^U consists of exactly one point and the set Y_0^U is empty.

Every map

$$(13.1) \qquad \vartheta\colon (Y^U, Y_0^U) \to (Y, Y_0)$$

(for arbitrary set of indices U) will be said to be a *multimap* on the pair (Y, Y_0).

Let us observe that if we assign to every index $u \in U$ and arbitrary set V_u of indices v_u and a multimap

$$\vartheta_u\colon (Y^{V_u}, Y_0^{V_u}) \to (Y, Y_0),$$

then, setting $\hat{U} = \bigcup_{u \in U} V_u$ and replacing in $\vartheta(\{y_u\})$ each variable y_u by the map ϑ_u, we get a multimap $\hat{\vartheta}$ on (Y, Y_0) of the form

$$\hat{\vartheta}\colon (Y^{\hat{U}}, Y^{\hat{U}}) \to (Y, Y_0).$$

This fact may be briefly formulated as follows:

(13.2) *Each composite of multimaps on (Y, Y_0) is also a multimap on (Y, Y_0).*

Now let us consider another pair of spaces (X, X_0) and let M be a subset of $(Y, Y_0)^{(X, X_0)}$. Let ϑ be a multimap on (Y, Y_0) given by (13.1). Hence ϑ assigns to each point $\{y_u\} \in Y^U$ a value $\vartheta(\{y_u\}) \in Y$ which belongs to Y_0 if $\{y_u\} \in Y_0^U$. Let α be a function assigning to every $u \in U$ a map $\varphi_u \in M$. Setting

$$(13.3) \qquad \varphi(x) = \vartheta\big(\{\varphi_u(x)\}\big) \quad \text{for every point } x \in X,$$

we get a map $\varphi\colon (X, X_0) \to (Y, Y_0)$. This map φ will be denoted by $\vartheta(\{\varphi_u\})$. Let us observe that

(13.4) *If $\varphi_u, \varphi'_u \in (Y, Y_0)^{(X, X_0)}$ are homotopic for every $u \in U$, then the maps $\vartheta(\{\varphi_u\})$ and $\vartheta(\{\varphi'_u\})$ are homotopic.*

All maps $\psi\colon (X, X_0) \to (Y, Y_0)$ homotopic to a map φ of the form (13.3) (where ϑ is an arbitrary multimap on (Y, Y_0) and $\varphi_u \in M$ for every $u \in U$) are said to be *homotopically dependent on the set of maps M*. The set of all such maps will be denoted by $\omega(M)$. It is evident that

$$(13.5) \qquad M \subset \omega(M),$$

$$(13.6) \qquad M \subset M' \quad \textit{implies} \quad \omega(M) \subset \omega(M').$$

If $X_0 \neq 0$ and $Y_0 = 0$, then the set $(Y, Y_0)^{(X, X_0)}$ is empty, whence also $\omega(M) = 0$. Thus we can assume in the sequel that either $X_0 = 0$ or $Y_0 \neq 0$.

If $M = 0$, then the set U (as a set of arguments of a function $\alpha\colon U \to M$) is empty. If $X_0 \neq 0$, then $Y_0 \neq 0$ and both sets Y^U, Y_0^U consist of exactly one point, and every multimap ϑ assigns to this point a point of Y_0. Hence in this case the set $\omega(0)$ consists of all maps $\psi\colon (X, X_0) \to (Y, Y_0)$ homotopic to a constant map with the value in Y_0. If $X_0 = 0$, then $(Y, Y_0)^{(X,X_0)} = Y^X$ and instead of the pair (Y^U, Y_0^U) we have a set consisting only of one point. Then every multimap ϑ assigns to this point an arbitrary point of Y. In this case, $\omega(0)$ consists of all maps $\psi\colon X \to Y$ homotopic to a constant.

If M consists only of one map φ_0, then every map given by (13.3) can be represented in the form $\vartheta_0\varphi_0$, whence the maps ψ homotopically dependent on M (i.e. on the map φ_0) are the same as the maps homotopic to maps of the form $\vartheta_0\varphi_0$, where $\vartheta_0 \in (Y, Y_0)^{(Y,Y_0)}$. These maps ψ are said to be *homotopy multiples* of the map φ_0 ([57], p. 81, and [147], p. 358).

The basic property of the homotopy dependence relation is given by the following proposition:

(13.7) $$\omega(\omega(M)) = \omega(M) \quad \text{for every set } M \subset (Y, Y_0)^{(X,X_0)}.$$

Since (13.5) implies that $\omega(M) \subset \omega(\omega(M))$, proposition (13.7) will be proved if we show that $\omega(\omega(M)) \subset \omega(M)$, i.e. that each map homotopically dependent on $\omega(M)$ is homotopically dependent on M. Since every map belonging to $\omega(M)$ is homotopic to a map of the form (13.3), we infer by (13.4) that $\omega(\omega(M)) = \omega(A)$, where A denotes the set of all maps φ given by formula (13.3), i.e. $\omega(\omega(M))$ is the set of maps which are obtained if one replaces in each multimap ϑ on (Y, Y_0) every argument y_u by a map ψ_u of the form $\vartheta_u(\{\varphi_u\})$, where ϑ_u is a multimap of (Y, Y_0). But (13.2) implies that we get this way a map of the form $\hat{\vartheta}(\{\varphi_{\hat{u}}\})$, where $\hat{\vartheta}$ is a multimap on (Y, Y_0). Hence every map ψ homotopically dependent on A is also homotopically dependent on M and the proof of (13.7) is finished.

It follows by (13.4) that the homotopy dependence of a map φ on M is in fact a relation between homotopy class $[\varphi]$ of φ and the homotopy classes of maps belonging to M. Now let us denote by $\boldsymbol{M}$ the collection of all homotopy classes of maps belonging to M and by $\lambda(\boldsymbol{M})$ the collection of all homotopy classes of maps dependent on M. Thus we get an operation λ assigning to every subset $\boldsymbol{M}$ of the set $\boldsymbol{N} = [Y, Y_0]^{[X,X_0]}$ of all homotopy classes of maps of (X, X_0) into (Y, Y_0) a subset $\lambda(\boldsymbol{M})$ of the set $\boldsymbol{N}$. It follows by (13.5), (13.6), and (13.7) that the operation λ satisfies the following conditions:

(13.8) $$\boldsymbol{M} \subset \lambda(\boldsymbol{M}) \subset \boldsymbol{N} \quad \text{for every set } \boldsymbol{M} \subset \boldsymbol{N}.$$

(13.9) $$\text{If } \boldsymbol{M} \subset \boldsymbol{M}' \subset \boldsymbol{N}, \text{ then } \lambda(\boldsymbol{M}) \subset \lambda(\boldsymbol{M}').$$

(13.10) $$\lambda(\lambda(\boldsymbol{M})) = \lambda(\boldsymbol{M}) \quad \text{for every set } \boldsymbol{M} \subset \boldsymbol{N}.$$

An operation λ defined on all subsets of a set N and satisfying conditions (13.8), (13.9), and (13.10) is said to be the *dependence operation* ([67], p. 324) and the set N in which a such operation is defined is said to be a *dependence domain.* If $\lambda(0) = N$, then λ assigns to every set $M \subset N$ the whole set N. In this case, the dependence domain N will be said to be *trivial.*

As we have shown, the set $[Y, Y_0]^{[X,X_0]}$ of all homotopy classes of maps $\varphi \in (Y, Y_0)^{(X,X_0)}$ is a dependence domain with the operation of dependence λ defined as the operation assigning to each set $M \subset [Y, Y_0]^{[X,X_0]}$ the set $\lambda(M)$ consisting of all homotopy classes of maps homotopically dependent on the set of representatives of the homotopy classes belonging to M. First let us consider some special cases.

As we have shown, if $M = 0$, then $\omega(M)$ consists of all maps ψ homotopic to constant maps with values in Y_0 (if $X_0 \neq 0$) or with values in Y (if $X_0 = 0$). It follows that for homotopic dependence operation the set $\lambda(0)$ coincides with the subset of $[Y, Y_0]^{[X,X_0]}$ consisting of all homotopy classes containing constant maps with values in Y_0 (if $X_0 \neq 0$) or with values in Y (if $X_0 = 0$).

If $X_0 = 0$, then the dependence domain $[Y, Y_0]^{[X,X_0]}$ coincides with $[Y]^{[X]}$. In the case where the space X, or the space Y, is contractible in itself, every map $\varphi \in Y^X$ is homotopic to a constant and we infer that

(13.11) *If the space X or the space Y is contractible in itself, then the dependence domain $[Y]^{[X]}$ is trivial.*

14. λ-morphisms. Let N_1 and N_2 be two dependence domains, with dependence operations λ_1 and λ_2 respectively. A function

$$f\colon N_1 \to N_2$$

will be called a *λ-morphism* (or *λ-homomorphism*, as in [67], p. 325) provided that

$$f\lambda_1(M) \subset \lambda_2 f(M) \qquad \text{for every set } M \subset N_1.$$

It is clear that if $f\colon N_1 \to N_2$ and $g\colon N_2 \to N_3$ are λ-morphisms, then their composition $gf\colon N_1 \to N_3$ is also a λ-morphism.

A λ-morphism $f\colon N_1 \to N_2$ is said to be an *$r\lambda$-morphism* if there exists a λ-morphism $g\colon N_2 \to N_1$ being the right inverse to f, i.e. such that

$$fg\colon N_2 \to N_2$$

is the identity map. A special case of the $r\lambda$-morphisms are *λ-isomorphisms*, i.e. such λ-morphisms $f\colon N_1 \to N_2$ for which there exists an inverse λ-morphism, i.e. a λ-morphism $g\colon N_2 \to N_1$ such that both λ-morphisms $gf\colon N_1 \to N_1$ and $fg\colon N_2 \to N_2$ are identity maps.

Let us prove the following

(14.1) THEOREM. *Let $f \in (X, X_0)^{(\hat{X}, \hat{X}_0)}$. If we assign to every homotopy class $[\varphi] \in [Y, Y_0]^{[X, X_0]}$ the homotopy class*

$$\Phi^{f*}([\varphi]) = [\varphi f] \in [Y, Y_0]^{[\hat{X}, \hat{X}_0]},$$

then we get a function

$$\Phi^{f*}\colon [Y, Y_0]^{[X, X_0} \to [Y, Y_0]^{[X, X_0]}$$

which is a λ-morphism. Moreover, if $g \in (\hat{X}, \hat{X}_0)^{(X', X'_0)}$, then $\Phi^{fg*} = \Phi^{g*}\Phi^{f*}$.

Proof. In order to prove that Φ^{f*} is a λ-morphism it is sufficient to show that for every set $M \subset (Y, Y_0)^{(X, X_0)}$

$$\Phi^f[\omega(M)] \subset \omega[\Phi^f(M)]. \tag{14.2}$$

Let $\psi \in \omega(M)$; then there exists a set U of indices, a function α assigning to every $u \in U$ a map $\varphi_u \in M$, and a multimap ϑ on the pair (Y, Y_0) with arguments $\{y_u\} \in (Y^U, Y_0^U)$ such that ψ is homotopic to the map φ given by the formula

$$\varphi(x) = \vartheta(\{\varphi_u(x)\}) \quad \text{for every point } x \in X.$$

Then $\Phi^f(\psi) = \psi f$ is homotopic to the map φf given by the formula

$$\varphi f(\hat{x}) = \vartheta(\{\varphi_u f(\hat{x})\}) \quad \text{for every point } \hat{x} \in \hat{X}.$$

But this map belongs to the set $\omega[\Phi^f(M)]$ and consequently the inclusion (14.2) is satisfied. Hence Φ^{f*} is a λ-morphism. The relation $\Phi^{fg*} = \Phi^{g*}\Phi^{f*}$ is an immediate consequence of the definition of the operation Φ^{f*}. Thus the proof of Theorem (14.1) comes to the end.

The λ-morphism

$$\Phi^{f*}\colon [Y, Y_0]^{[X, X_0]} \to [Y, Y_0]^{[\hat{X}, \hat{X}_0]}$$

is said to be *induced* by the map $f\colon (\hat{X}, \hat{X}_0) \to (X, X_0)$. It is clear that:

(14.3) *The identity map $i\colon (X, X_0) \to (X, X_0)$ induces the identity λ-morphism $\Phi^{i*}\colon [Y, Y_0]^{[X, X_0]} \to [Y, Y_0]^{[X, X_0]}$.*

(14.4) *Two homotopic maps $f, g\colon (\hat{X}, \hat{X}_0) \to (X, X_0)$ induce the same λ-morphism $\Phi^{f*} = \Phi^{g*}\colon [Y, Y_0]^{[X, X_0]} \to [Y, Y_0]^{[\hat{X}, \hat{X}_0]}$.*

It holds, by Theorem (14.1), the following

(14.5) COROLLARY. *If $f\colon (\hat{X}, \hat{X}_0) \to (X, X_0)$ is an h-map, then the homotopically right inverse map $g\colon (X, X_0) \to (\hat{X}, \hat{X}_0)$ induces a λ-morphism*

$$\Phi^{g^*}\colon [Y, Y_0]^{[\hat{X}, \hat{X}_0]} \to [Y, Y_0]^{[X, X_0]}$$

such that the λ-morphism

$$\Phi^{f^*}\colon [Y, Y_0]^{[X, X_0]} \to [Y, Y_0]^{[\hat{X}, \hat{X}_0]}$$

is the right inverse of Φ^{g^*}.

(14.6) COROLLARY. *If two pairs of spaces (X, X_0) and $(\hat{X}, \hat{X}_0)$ are homotopically equivalent, then, for every pair of spaces (Y, Y_0), the dependence domains $[Y, Y_0]^{[X, X_0]}$ and $[Y, Y_0]^{[\hat{X}, \hat{X}_0]}$ are λ-isomorphic.*

CHAPTER III

EXTENSION OF MAPS

As was already mentioned, a large part of the theory of retracts is devoted to the endeavour to generalize the classical theorem of Tietze on the extension of continuous real functions. An important step in this direction is due to J. Dugundji [102] who proved a generalization of Tietze's theorem to maps whose values belong to linear locally convex spaces. In order to prove Dugundji's theorem we give some preliminary definitions and theorems.

1. Coverings. By a *covering* of a space X we understand a class $\boldsymbol{U} = \{U_\mu\}$ of subsets U_μ of X, where μ runs through a set of indices M, such that

$$X = \bigcup_{\mu \in M} U_\mu.$$

If all the sets U_μ are open in X, then $\boldsymbol{U}$ is said to be an *open covering* of X. If every U_μ is closed, then $\boldsymbol{U}$ is said to be a *closed covering* of X. An other covering $\boldsymbol{V} = \{V_\nu\}$, $\nu \in N$, is said to be a *refinement* of U provided that for each $\nu \in N$ there is a $\mu \in M$ for which $V_\nu \subset U_\mu$. A covering U of X is said to be *locally finite* if, for each $x \in X$, there is a neighborhood G of x in X such that $G \cap U_\mu \neq 0$ for a finite number of $\mu \in M$ at most. A space X for which every open covering has a locally finite open refinement is said to be *paracompact.* By a theorem of A. H. Stone ([270], p. 979), which we assume without proof, *every metric space is paracompact.*

Let G be an open subset of a space X and $\boldsymbol{U} = \{U_\mu\}$, $\mu \in M$, be an open covering of G. We say that $\boldsymbol{U}$ is *canonical* with respect to X provided that the following two conditions are satisfied:

(1.1) *$\boldsymbol{U}$ is locally finite.*

(1.2) *For each point $a \in X - G$ and each neighborhood V_a of a in X there is a neighborhood W_a of a in X such that $U_\mu \cap W_a \neq 0$ implies $U_\mu \subset V_a$.*

Let us observe that (1.2) implies the following statement:

(1.3) *If $a \in \overline{G} \cap (X - G)$, then every neighborhood V_a of a contains infinitely many sets U_μ.*

Indeed, let W_a be a neighborhood of a promised by condition (1.2). Then $U_\mu \cap W_a \neq 0$ implies $U_\mu \subset V_a$. If V_a would contain only a finite number of sets U_μ, then there would exist an index μ_0 for which $a \in \overline{U}_{\mu_0}$ since $a \in \overline{G}$. Let b be a point of U_{μ_0} distinct from a, and let V'_a be the neighborhood of a obtained by removing b from V_a. By (1.2), there is a neighborhood W'_a of a such that $U_\mu \cap W'_a \neq 0$ implies $U_\mu \subset V'_a$. But this is not true for U_{μ_0}, and consequently V_a must contain infinitely many U_μ.

We apply the theorem of Stone in order to prove the

(1.4) THEOREM. *If the space X is metric, then for each open subset G of X there exists a canonical covering of G with respect to X.*

Proof. If $G = X$, then any locally finite open covering of X is canonical. Suppose then that $G \neq X$. It is evident that if $x \in G$ and K_x is the open ball in X with center x and radius $\frac{1}{2}\varrho(x, X-G)$, then $K_x \subset G$ and $\boldsymbol{K} = \{K_x\}$ is an open covering of G. By the theorem of Stone, there is a locally finite open refinement $\boldsymbol{U} = \{U_\mu\}$ of $\boldsymbol{K}$. We shall show that $\boldsymbol{U}$ is the required canonical covering.

If V_a is a neighborhood of a point $a \in X-G$, then there is an $\varepsilon > 0$ such that the open ball (in X) with center a and radius ε lies in V_a. Let W_a be the open ball with center a and radius $\frac{1}{3}\varepsilon$. Suppose that there is a point x in $U_\mu \cap W_a$. We have to show that $U_\mu \subset V_a$. Since $\boldsymbol{U}$ refines $\boldsymbol{K}$, we can find a point $y \in G$ for which $U_\mu \subset K_y$. Then

$$\varrho(a, y) \leqslant \varrho(a, x) + \varrho(x, y) < \varrho(a, x) + \tfrac{1}{2}\varrho(a, y)$$

since $x \in K_y$ implies $\varrho(x, y) < \frac{1}{2}\varrho(x, X-G) \leqslant \frac{1}{2}\varrho(a, y)$. Moreover, since $\varrho(a, x) < \frac{1}{3}\varepsilon$, we infer that $\varrho(a, y) < \frac{2}{3}\varepsilon$. Hence, if $z \in U_\mu \subset K_y$, we have

$$\varrho(a, z) \leqslant \varrho(a, y) + \varrho(y, z) < \tfrac{2}{3}\varepsilon + \tfrac{1}{2}\varrho(y, X-G)$$
$$< \tfrac{2}{3}\varepsilon + \tfrac{1}{2}\varrho(a, y) < \tfrac{2}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon.$$

This means that $z \in V_a$ and the proof is completed.

By the dimension of a normal space X we understand the *covering dimension* of X ([232], p. 6). Specifically, we say $\dim X \leqslant n$ provided that each open covering of X has a locally finite open refinement such that no point of X lies in more than $n+1$ elements of the refinement. If $\dim X \leqslant n$ is true but $\dim X \leqslant n-1$ is false, then we say $\dim X = n$. If $\dim X \leqslant n$ is satisfied for no integer n whatever, then we say that the dimension of X is *infinite* and write $\dim X = \infty$. For a separable metric space X, this definition is equivalent to the classical definition of Menger and Urysohn (see, for instance, [166], p. 67). As has been shown by K. Morita ([233], p. 222), if X is a paracompact space, then $\dim(X \times \langle 0, 1\rangle) = \dim X + 1$.

Since every locally finite open refinement of a canonical covering is also canonical, we infer that for an open set G of dimension $\leqslant n$ we can find a canonical covering such that each point of G lies in at most $n+1$ of the sets of this covering.

2. Polytopes. A space X (not necessarily metric) is said to be a *polytope* if there is a collection $\mathscr{T}$ of geometric simplexes σ such that

(2.1) $$X = \bigcup_{\sigma\in\mathscr{T}} \sigma.$$

(2.2) *Each face of a simplex σ of $\mathscr{T}$ is also a simplex in $\mathscr{T}$.*

(2.3) *If σ_1 and σ_2 are in $\mathscr{T}$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.*

(2.4) *A subset G of X is open if and only if $G \cap \sigma$ is open in σ for each $\sigma \in \mathscr{T}$.*

The collection $\mathscr{T}$ is called a *triangulation* of X and the vertices of the simplexes in $\mathscr{T}$ are called the *vertices* of $\mathscr{T}$.

Observe that if we are given a set X satisfying (2.1), (2.2), and (2.3), then condition (2.4) imposes a topology upon X. The topology thus defined is called the *weak topology* upon X. The polytope X is then a *simplicial* CW-*complex* in the sense of J. H. C. Whitehead ([289], p. 223).

Let us prove the following

(2.5) THEOREM. *If $\mathscr{T}$ is a triangulation of a polytope X and f is a function defined on X with values in a topological space Y, then f is continuous if and only if the restriction $f|\sigma$ is continuous for each σ in $\mathscr{T}$.*

Proof. The necessity of the condition is obvious. On the other hand, if the condition is satisfied, then for every open subset V of Y the common part of $f^{-1}(V)$ and σ, where σ is a simplex in $\mathscr{T}$, is open in σ. Consequently $f^{-1}(V)$ is open in X by condition (2.4). Hence f is continuous.

A triangulation $\mathscr{T}'$ of a space X is said to be a *subdivision* of a triangulation $\mathscr{T}$ of X provided that each simplex of $\mathscr{T}$ is the union of simplexes of $\mathscr{T}'$. In particular, if we replace each simplex of $\mathscr{T}$ by its barycentric subdivision, we get a triangulation $\mathscr{T}'$ which is said to be a *barycentric subdivision of $\mathscr{T}$*.

If $\mathscr{T}_0$ is a subcollection of $\mathscr{T}$ satisfying condition (2.2) (with $\mathscr{T}$ replaced by $\mathscr{T}_0$), then $X_0 = \bigcup_{\sigma\in\mathscr{T}_0} \sigma$ is a polytope with triangulation $\mathscr{T}_0$. We say that X_0 is a *subpolytope* of X. If $\mathscr{T}_0$ contains almost all (i.e., all with a possible finite number of exceptions) of the simplexes of $\mathscr{T}$, then we say that X_0 is *almost full.*

Let n be an integer. The subset X^n of X which is the union of all simplexes of $\mathscr{T}$ of dimensions $\leqslant n$ is said to be *n-skeleton* of X relative to the triangulation $\mathscr{T}$. It is clear that X^n is a subpolytope of X.

3. Finite and locally finite polytopes. If the triangulation of a polytope X is finite, then X is evidently a compactum homeomorphic to a polyhedron in the elementary geometric sense. (Compare I, Section 3.) The following theorem indicates that the converse is also true:

(3.1) THEOREM. *If a polytope X has an infinite triangulation, then it is not compact.*

Proof. Let us select a point a_σ in the interior of each non-empty simplex σ in $\mathscr{T}$. That is, $a_\sigma \in \sigma - \sigma^\bullet$, where $\sigma^\bullet$ denotes the boundary of σ. Let A be the collection of all these points a_σ. If we set $G_\sigma = (X-A) \cup (a_\sigma)$, then we see at once that the common part of the set G_σ with each simplex σ' of $\mathscr{T}$ is open in σ'; indeed, $G_\sigma \cap \sigma'$ consists of all points of σ' with a finite number of points removed from its faces. Consequently the sets G_σ are open. The family $\{G_\sigma\}$ is then an open covering of X which does not contain a finite subcovering. Indeed, the covering is infinite by hypothesis, and if we omit G_σ, then the remaining sets do not cover a_σ. This completes the proof.

It follows by Theorem (3.1) that if $\mathscr{T}'$ is a triangulation of X, then for every subdivision $\mathscr{T}'$ of $\mathscr{T}$ the collection of simplexes lying on a simplex $\sigma \in \mathscr{T}$ is finite; it constitutes a triangulation of σ.

For each point $x \in X$, we denote by $\mathrm{St}_{\mathscr{T}}(x)$ the union of the interiors of all simplexes σ of $\mathscr{T}$ for which $x \in \sigma$. This set, also denoted by $\mathrm{St}(x)$ if there is no danger of confusion, is called the *star of x relative to $\mathscr{T}$.* It is plain that $\mathrm{St}(x)$ is an open neighborhood of x in X. Moreover, the stars of all vertices of $\mathscr{T}$ consitute an open covering of X. The reader can verify the

(3.2) THEOREM. *Suppose that $x_0, x_1, \ldots, x_n$ are vertices of a triangulation $\mathscr{T}$. Then there is a simplex of $\mathscr{T}$ with precisely these vertices if and only if*

$$\mathrm{St}(x_0) \cap \mathrm{St}(x_1) \cap \ldots \cap \mathrm{St}(x_n) \neq 0.$$

A triangulation $\mathscr{T}$ of a polytope X is said to be *locally finite* provided that each simplex of $\mathscr{T}$ is the face of a finite number of simplexes of $\mathscr{T}$ at most. This condition is equivalent to the condition that for each $x \in X$, $\mathrm{St}(x)$ is a finite union of interiors of simplexes belonging to $\mathscr{T}$. It is also equivalent to the condition that each vertex of $\mathscr{T}$ belongs to a finite number of simplexes of $\mathscr{T}$ at most. From these remarks it follows the following

(3.3) THEOREM. *If a polytope has a locally finite triangulation, then each point has a neighborhood which is a polyhedron. In particular, a polytope having a locally finite triangulation is a locally compact space.*

We shall prove the converse of this theorem.

(3.4) THEOREM. *If a polytope X is locally compact, then each triangulation of X is locally finite.*

Proof. Let $x_0 \in X$ and let Z be the class of all simplexes of the triangulation $\mathscr{T}$ of X which contain x_0. Suppose that V is a compact neighborhood of x_0. We choose a point $a_\sigma \in (\sigma - \sigma^\bullet) \cap V$ for each simplex $\sigma \in Z$. Let A be the set of all such points a_σ and let $G_\sigma = (V - A) \cup (a_\sigma)$. It is easy to see that the sets G_σ are open in V so that the family $\{G_\sigma\}$ is an open covering of V. Since V is compact, this covering has a finite subcovering. Since, however, a_σ is only covered by G_σ, it follows that this subcovering is in fact the entire covering. That means, $\{G_\sigma\}$ is a finite family or, what is the same, Z is finite. Consequently, $\mathscr{T}$ is locally finite by the definition.

4. Metrizability of polytopes. We begin by proving the following

(4.1) THEOREM. *If a polytope X is metrizable, then each triangulation of X is locally finite.*

Proof. Let ϱ be a metric for the polytope X. Suppose, to the contrary, that X has a triangulation $\mathscr{T}$ which is not locally finite. Then there exists a point $x_0 \in X$ which is a vertex of an infinite sequence of distinct simplexes $\sigma_1, \sigma_2, \ldots$ of $\mathscr{T}$. Without loss of generality every σ_n may be assumed to be of positive dimension. In the interior of each σ_n we pick out a point x_n such that $\varrho(x_n, x_0) < 1/n$. Now let us observe that the set $G = X - \bigcup_{n=1}^{\infty} (x_n)$ is open in X, since for each simplex σ of $\mathscr{T}$ the set $\sigma - G$ is finite, so that $\sigma \cap G$ is open in σ. It follows that G is a neighborhood of x_0 which is impossible since no x_n belongs to G and $\lim_{n\to\infty} x_n = x_0$.

On the other hand, we have the following

(4.2) THEOREM. *If a polytope X has a locally finite triangulation $\mathscr{T}$, then X is metrizable.*

Proof. Let M be the set of all vertices μ of $\mathscr{T}$. Let $F(\mathscr{T})$ be the set of all real-valued functions defined on M and vanishing for all but a finite number of $\mu \in M$. If the addition and scalar multiplication are defined pointwise and the norm is defined by the formula

$$\|f\| = \sqrt{\sum_{\mu \in M} f(\mu)^2} \quad \text{for} \quad f \in F(\mathscr{T}),$$

it is clear that $F(\mathscr{T})$ is a normed linear space. Moreover, a basis for $F(\mathscr{T})$ consists of the functions a_μ, $\mu \in M$, defined by the condition

$$a_\mu(\mu') = \begin{cases} 1 & \text{if} \quad \mu' = \mu, \\ 0 & \text{if} \quad \mu' \neq \mu. \end{cases} \tag{4.3}$$

Now let us construct a homeomorphism h of the space X onto a subset of $F(\mathscr{T})$ in the following manner.

Suppose that x is a point of X lying in a simplex σ of $\mathscr{T}$ which has the vertices $\mu_0, \mu_1, \ldots, \mu_n$. Let $t_0, t_1, \ldots, t_n$ denote the barycentric coordinates of x in the simplex σ corresponding to $\mu_0, \mu_1, \ldots, \mu_n$, respectively. We recall that every t_i is not less that 0 and that $\sum_{i=1}^{n} t_i = 1$.

We set

$$h(x) = t_0 a_{\mu_0} + t_1 a_{\mu_1} + \ldots + t_n a_{\mu_n}. \tag{4.4}$$

It should be established that this definition is unambiguous. Indeed, if σ is the unique simplex of $\mathscr{T}$ containing x in its interior, σ is a proper face of each simplex $\sigma' \in \mathscr{T}$ containing x and any coordinate t_i occurring in (4.4) in connection with this larger simplex σ' is zero, unless it is a t_i arising from σ.

From the linear independence of the functions a_μ it follows that h is a one-to-one transformation $h: X \to h(X)$. It remains to check that h and h^{-1} are continuous.

Let W_μ denote the closure of the star $\mathrm{St}(\mu)$ of the vertex μ in $\mathscr{T}$. Since the triangulation $\mathscr{T}$ is locally finite, W_μ is a polyhedron and the restriction $h|W_\mu$ is a simplicial one-to-one map of the polyhedron W_μ onto a polyhedron $h(W_\mu) \subset h(X)$. Consequently h maps $\mathrm{St}(\mu)$ topologically onto the set $h(\mathrm{St}(\mu))$. Since the stars $\mathrm{St}(\mu)$, as μ runs over M, constitute an open covering of X, it remains to prove that the images $h(\mathrm{St}(\mu))$ of the stars $\mathrm{St}(\mu)$ are open in $h(X)$. To this end let us observe that a point $p \in h(X)$, which is a real function defined on the set M, does not belong to the set $h(\mathrm{St}(\mu))$ if and only if the value $p(\mu)$ is equal to zero. The set of all such functions, i.e., the set $h(X) - h(\mathrm{St}(\mu))$ is therefore closed in $h(X)$ and we infer that $h(\mathrm{St}(\mu))$ is open in $h(X)$.

Thus h is a homeomorphism of X onto a subset of a metric space (in fact, of a normed linear space) and hence X is metrizable.

(4.5) THEOREM. *A polytope X with a triangulation $\mathscr{T}$ is separable if and only if $\mathscr{T}$ is finite or countable.*

Proof. If $\mathscr{T}$ is finite or countable, then the separability of X is obvious. If $\mathscr{T}$ is uncountable, then the set of vertices of $\mathscr{T}$ is also uncountable. Let $\mathscr{T}'$ denote the barycentric subdivision of the triangulation $\mathscr{T}$. Evidently each vertex μ of $\mathscr{T}$ is also a vertex of $\mathscr{T}'$ and the stars (in the triangulation $\mathscr{T}'$) of two different vertices of $\mathscr{T}$ are always disjoint. These stars constitute an uncountable family of open, disjoint subsets of X. Hence X is not separable.

5. Null-triangulations. A triangulation $\mathscr{T}$ of a polytope X is said to be a *null-triangulation* if it is countable and the diameters of its simplexes (arranged into a sequence) converge to zero.

(5.1) THEOREM. *Every metric, separable but not compact polytope X has a null-triangulation.*

We begin the proof with the following lemma:

(5.2) LEMMA. *Let ε be a positive number and let $\mathscr{T}_0$ be a triangulation of the boundary $\sigma^{\bullet}$ of a geometric simplex σ such that no simplex in $\mathscr{T}_0$ has diameter $\geqslant \varepsilon$. Then there is a triangulation $\mathscr{T}_1$ of σ which contains $\mathscr{T}_0$ and is such that no simplex of $\mathscr{T}_1$ has diameter $\geqslant \varepsilon$.*

The proof of this lemma may be left to the reader. As to the theorem, let $\mathscr{T}$ be a triangulation of X. It follows by (3.1) and (4.5) that $\mathscr{T}$ is countable and hence the vertices of $\mathscr{T}$ may be arranged into an infinite sequence $\{a_n\}$. Let σ be a simplex of $\mathscr{T}$ and let a be a vertex of $\mathscr{T}$. We say that σ and a are *adjacent* if there is a simplex σ' in $\mathscr{T}$ having σ as a face and a as a vertex. Since $\mathscr{T}$ is locally finite (by (4.1)), we infer that for each simplex of $\mathscr{T}$ there is only a finite number of vertices which are adjacent to it and that each vertex has only a finite number of adjacent simplexes.

The n-dimensional simplexes of the triangulation $\mathscr{T}$ may be arranged into a sequence $\{\sigma_j^n\}$. The required subdivision is obtained stepwise by considering first $\{\sigma_j^1\}$, then $\{\sigma_j^2\}$, etc. First we perform a subdivision such that if σ_j^1 is adjacent to the vertex a_i, then the diameter of each of the resulting simplexes composing σ_j^1 is less than $1/i$. This has the effect of subdividing the boundaries of 2-simplexes σ_k^2 adjacent to a_i. By Lemma (5.2) this subdivision of the boundaries may be extended to a subdivision of these 2-simplexes, so that the resulting simplexes have diameter $< 1/i$.

We proceed by induction; assume that for an $n \geqslant 1$ a subdivision of all simplexes σ_j^k, $k \leqslant n$, has been found such that the diameter of all resulting simplexes lying on those simplexes σ_j^k which are adjacent to a_i is less than $1/i$. Thus the boundaries of all $(n+1)$-simplexes adjacent to a_i have been subdivided, and by Lemma (5.2) we may extend this subdivision, without increasing the diameter of simplexes, to the $(n+1)$-simplexes themselves.

We obtain in this manner a subdivision $\mathscr{T}_0$ of the given triangulation $\mathscr{T}$ such that the diameter of any simplex of $\mathscr{T}_0$ which lies in a simplex of $\mathscr{T}$ adjacent to the vertex a_1 is less than $1/i$. Since each vertex is adjacent to only a finite number of simplexes of $\mathscr{T}$, we infer that the diameters of the simplexes of $\mathscr{T}_0$ converge to zero.

6. Nerve of a covering. Let $\boldsymbol{G} = \{G_\mu\}$, $\mu \in M$, be a covering of a space X. We assume that $\mu \neq \mu'$ implies $G_\mu \neq G_{\mu'}$. We may assume that the elements of the family $\boldsymbol{G}$ make up the basis of a vector space F; that means, we consider F as the free real module with basis $\boldsymbol{G}$. Then the points $p \in F$ can be uniquely expressed in the form

$$p = \sum_{\mu \in M} t_\mu G_\mu,$$

where the coefficients t_μ are real and vanish for all but a finite number of $\mu \in M$. If $\mu_0, \mu_1, \ldots, \mu_n$ are distinct indices, then by the simplex σ in F with vertices $G_{\mu_0}, \ldots, G_{\mu_n}$ we shall mean the set $\sigma(G_{\mu_0}, G_{\mu_1}, \ldots, G_{\mu_n})$ of all points $p \in F$ such that

$$p = \sum_{i=0}^{n} t_{\mu_i} G_{\mu_i},$$

where every $t_{\mu_i} \geqslant 0$ and $\sum_{i=0}^{n} t_{\mu_i} = 1$. The number t_{μ_i} is called the *barycentric coordinate* of p with respect to the vertex G_{μ_i}. Since the vertices G_μ are linearly independent, the intersection of two simplexes is again a simplex, namely the simplex spanned by the vertices lying in the intersection.

In particular, let us identify these abstract simplexes with usual geometric simplexes. Then the collection of all simplexes σ having as vertices points corresponding to elements of the covering with non-empty intersection is a triangulation $\mathscr{T}$ of a polytope W. Thus, σ is in $\mathscr{T}$ if and only if its vertices $G_{\mu_0}, \ldots, G_{\mu_n}$ are such that $\bigcap_{i=0}^{n} G_{\mu_i} \neq 0$. The polytope W with the weak topology is called the *nerve of the covering* $\boldsymbol{G} = \{G_\mu\}$ and is denoted by $N(\boldsymbol{G})$.

Suppose now that $\boldsymbol{G} = \{G_\mu\}$ is a locally finite open covering of a metric space X. Then each point $p \in X$ belongs to one at least, but to finitely many sets G_μ at most and consequently $\sum_{\mu \in M} \varrho(p, X - G_\mu)$ is well defined and strictly positive. Thus setting

$$\tau_\mu(p) = \frac{\varrho(p, X-G_\mu)}{\sum_{\mu' \in M} \varrho(p, X-G_{\mu'})},$$

we get a real function τ_μ on X. It is clear that $\tau_\mu(p) \geqslant 0$ and $\sum_{\mu \in M} \tau_\mu(p) = 1$ for each point $p \in X$; also $\tau_\mu(p) > 0$ if and only if $p \in G_\mu$. Moreover, since the covering $\boldsymbol{G}$ is locally finite, there exists for every point $p_0 \in X$ a neighborhood U_0 (in the space X) such that $U_0 \cap G_\mu \neq 0$ holds only for a finite system $\mu_0, \mu_1, \ldots, \mu_n$ of indices μ. Thus, for $p \in U_0$,

$$\tau_\mu(p) = \frac{\varrho(p, X-G_\mu)}{\varrho(p, X-G_{\mu_0}) + \ldots + \varrho(p, X-G_{\mu_n})}$$

and consequently the function τ_μ in continuous. Moreover, if $p \in U_0$, then $\tau_\mu(p) = 0$ for every $\mu \neq \mu_0, \mu_1, \ldots, \mu_n$. It follows that setting

$$\varkappa(p) = \sum_{\mu \in M} \tau_\mu(p) G_\mu, \tag{6.1}$$

we get a function $\varkappa\colon X \to N(\boldsymbol{G})$. All the values of $\varkappa$ on the neighborhood U_0 of p_0 belong to such simplexes σ of the triangulation $\mathscr{T}$ of $N(\boldsymbol{G})$ that all vertices of σ are contained in the system $G_{\mu_0}, G_{\mu_1}, \ldots, G_{\mu_n}$. It is plain that all such simplexes σ constitute a finite triangulation $\mathcal{T}_{p_0}$ of a subpolytope $N_0 \subset N(G)$. Thus the function $\varkappa$ restricted to U_0 is the same as the function $\varkappa_0$: $U_0 \to N_0$ defined by the formula

$$\varkappa_0(p) = \tau_{\mu_0}(p)G_{\mu_0} + \ldots + \tau_{\mu_n}(p)G_{\mu_n}. \tag{6.2}$$

Since the polytope N_0, having a finite triangulation, is a polyhedron, its weak topology is equivalent with the topology given by the barycentric coordinates. Thus the continuity of $\tau_\mu(p)$ and formula (6.2) imply the continuity of $\varkappa_0$, and since the function $\varkappa$ on the neighborhood U_0 of the point p_0 has the same values as the function $\varkappa_0$, we infer that $\varkappa$ is continuous at each point p_0 of X. The map $\varkappa$: $X \to N(G)$ is said to be the *canonical map* of the space X into the nerve $N(G)$ of the locally finite open covering G of X.

7. Generalization of Tietze's Theorem. In a linear space Y, we shall understand by the *convex hull* $C(A)$ of a subset A of Y the set of all points $y \in Y$ of the form

$$y = \sum_{i=1}^{n} t_i a_i,$$

where the points a_i are in A, and the coefficients t_i are $\geqslant 0$ and their sum is equal to 1. It is easy to see that $C(A)$ is equal to the intersection of all the convex subsets of Y which contain A.

If Y is a topological linear space and for each point $y \in Y$ and for each neighborhood V of y there is a convex neighborhood U of y contained in V, then we say that Y is a *locally convex space*. For example, every normed linear space is a locally convex space.

The theorem of Tietze asserts that each real continuous function defined on a closed subset of a metric space X can be extended to a real continuous function defined on all points of X. The generalization of this theorem proved by J. Dugundji ([102], p. 357) shows that for the range space we can take any locally convex space as well. More precisely, we have the following

(7.1) Generalized Theorem of Tietze. *Let A be a closed subset of a metric space X and let Y be a locally convex linear space. For every map f: $A \to Y$ there exists a continuous extension $\bar{f}$: $X \to Y$ of f. Moreover, all the values of $\bar{f}$ can be taken from the convex hull $C(f(A))$ of the set $f(A)$.*

Proof. By (1.4) there is a canonical covering $\{G_\mu\}$, where $\mu \in M$, of the set $X - A$. Let us select in each set G_μ a point x_μ and let us assign to it a point $a_\mu \in A$ such that

$$\varrho(x_\mu, a_\mu) < 2\varrho(x_\mu, A).$$

Then $f(a_\mu)$ is a point of the linear space Y. Taking account of the function τ_μ defined in Section 6, we see that if $x \in X-A$, the numbers $\tau_\mu(x)$ vanish for all save a finite number of indices $\mu \in M$. Consequently, if we define

$$\bar{f}(x) = \begin{cases} f(x) & \text{for} \quad x \in A, \\ \sum_{\mu \in M} \tau_\mu(x) \cdot f(a_\mu) & \text{for} \quad x \in X-A, \end{cases}$$

then we obtain a function $\bar{f}\colon X \to Y$ which extends f and has its values in $C(f(A))$. It remains to prove that $\bar{f}$ is continuous.

Since the covering $\{G_\mu\}$ of the set $X-A$ is locally finite and since A is closed, there exists, for each point $p \in X-A$, a neighborhood $U \subset X-A$ of the point p which intersects only a finite number of the sets G_μ, say the sets $G_{\mu_1}, G_{\mu_2}, \ldots, G_{\mu_n}$. Then, for every $x \in U$, $\tau_\mu(x)$ can fail to vanish only if μ is equal to μ_i for some i, $1 \leqslant i \leqslant n$. Since each of $\tau_{\mu_1}, \ldots, \tau_{\mu_n}$ depends continuously on x, it follows that $\bar{f}$ is continuous at points $p \in X-A$. Moreover, it is evident that $\bar{f}$ is continuous at all interior points of A. Consequently, we have only to check that $\bar{f}$ is continuous for every point $p \in A \cap \overline{X-A}$. Thus, if V is a neighborhood of $\bar{f}(p) = f(p)$ in Y, then we must find a neighborhood U_0 of p in X such that $\bar{f}(U_0) \subset V$. Without loss of generality we may assume that V is convex since the space Y is locally convex.

Let $K(a)$ be the open ball in X with center p and radius α. Since f is continuous in A, there is a positive number ε such that

$$f(A \cap K(\varepsilon)) \subset V.$$

Since $\{G_\mu\}$ is a canonical covering, we infer that there is a neighborhood U_0 of p in X such that $U_0 \subset K(\varepsilon)$ and $G_\mu \cap U_0 \neq 0$ implies that $G_\mu \subset K(\frac{1}{3}\varepsilon)$. Since $x_\mu \in G_\mu$, we see that

$$\varrho(p, a_\mu) \leqslant \varrho(p, x_\mu) + \varrho(x_\mu, a_\mu) < \tfrac{1}{3}\varepsilon + 2\varrho(x_\mu, A) < \varepsilon.$$

We conclude that $\bar{f}(x) = f(x) \in V$ for $x \in A \cap U_0$, and for points $x \in (X-A) \cap U_0$ we can find indices $\mu_1, \ldots, \mu_n$ such that $x \in \bigcap_{i=1}^{n} G_{\mu_i}$ and that x does not belong to any G_μ with $\mu \neq \mu_i$, $1 \leqslant i \leqslant n$. Then $\tau_{\mu_i}(x) > 0$ for $i = 1, 2, \ldots, n$ and $\tau_\mu(x) = 0$ for all other indices μ. Consequently, $\bar{f}(x) = \sum_{i=1}^{n} \tau_{\mu_i}(x) f(a_{\mu_i})$. Since $x \in G_{\mu_i} \cap U_0$, it follows that $f(a_{\mu_i}) \in V$, and since V is convex, it also follows that $\bar{f}(x) \in V$. Thus $\bar{f}(U_0) \subset V$ and so the proof of the continuity of $\bar{f}$ is complete.

8. Embedding of a metrizable space in a normed linear space. In order to apply Dugundji's generalization of Tietze's theorem to the theory of extension of maps with values in an arbitrary metrizable space it is convenient to prove first the following theorem of Kuratowski and Wojdysławski ([197], p. 543 and [296], p. 186):

(8.1) THEOREM. *For each metrizable space X there is a normed linear space Z and a homeomorphism h of X onto a subset $h(X)$ of Z which is closed in its convex hull $C(h(X))$.*

Proof. If ϱ is a metric for X, let us set

$$\varrho'(x, y) = \frac{\varrho(x, y)}{1+\varrho(x, y)} \quad \text{for} \quad x, y \in X.$$

Then ϱ' is also a metric for X and relative to ϱ' the space X has diameter $\leqslant 1$. Consequently we may assume without loss of generality that X has diameter $\leqslant 1$.

Let us consider the set Z of all bounded real continuous functions defined on X. Let us set

$$\varrho(f_1, f_2) = \sup_{x \in X} |f_1(x) - f_2(x)| \qquad (\text{for } f_1, f_2 \in Z)$$

and

$$|f| = \sup_{x \in X} |f(x)| \qquad (\text{for } f \in Z).$$

Then we see that Z, with the obvious algebraic operations, is a normed linear space.

In order to define a homeomorphism $h\colon X \to h(X) \subset Z$ we define a function $f_x \in Z$, associated with the point $x \in X$, by means of the formula

$$f_x(y) = \varrho(x, y),$$

and let us set

$$h(x) = f_x \quad \text{for every point } x \in X.$$

Then h is an isometry. Indeed,

$$\varrho(f_{x_1}, f_{x_2}) \geqslant |\varrho(x_1, x_2) - \varrho(x_2, x_2)| = \varrho(x_1, x_2)$$

and, for any $y \in X$, we have

$$|f_{x_1}(y) - f_{x_2}(y)| = |\varrho(x_1, y) - \varrho(x_2, y)| \leqslant \varrho(x_1, x_2),$$

so that $\varrho(f_{x_1}, f_{x_2}) \leqslant \varrho(x_1, x_2)$. These inequalities show that $\varrho(f_{x_1}, f_{x_2}) = \varrho(x_1, x_2)$ proving that h is an isometry as required.

It remains to prove that $h(X)$ is closed in its convex hull $C(h(X))$. To this end let $f \in C(h(X))$ and suppose $f = \lim_{n\to\infty} f_{x_n}$, where $f_{x_n} \in h(X)$. Then, since f is in the convex hull of $h(X)$, f is a linear combination of elements of $h(X)$, i.e. there are points $a_0, \ldots, a_k \in X$ and real positive numbers $\lambda_0, \ldots, \lambda_k$ such that

$$f = \sum_{i=0}^{k} \lambda_i f_{a_i}, \quad \text{where} \quad \sum_{i=0}^{k} \lambda_i = 1.$$

Without loss of generality we may assume that the a_i are distinct and that some λ_i, say λ_0, satisfies the condition $\lambda_0 \geqslant 1/(k+1)$. Then

$$\varrho(f, f_{x_n}) \geqslant |f(x_n) - f_{x_n}(x_n)| = |f(x_n)| \geqslant \lambda_0 f_{a_0}(x_n) \geqslant \frac{1}{k+1}\varrho(a_0, x_n).$$

Since $\lim_{n\to\infty} f_{x_n} = f$, it follows that $\lim_{n\to\infty} x_n = a_0$ and hence that $f = f_{a_0} \in h(X)$. This completes the proof.

(8.2) COROLLARY. *Let A be a closed subset of a metric space X and let f be a map of A into a metric space Y. Then there is a homeomorphism h of Y onto a closed subset $h(Y)$ of a metric space Z_0 such that the map $hf\colon A \to Z_0$ has a continuous extension $g\colon X \to Z_0$.*

Simply, we take Z_0 to be the convex hull $C(h(Y))$ in the normed linear space Z exhibited in Theorem (8.1) and apply the generalized Tietze's theorem (7.1).

9. Extension of maps with values belonging to an $\mathbf{LC}^n$-space. Let us recall that a space Y is said to be an $\mathbf{LC}^n$-space (I, Section 17) if it is locally k-connected for $k = 0, 1, \ldots, n$, i.e. if for every point $y_0 \in Y$ and for every neighborhood V of y_0 there exists a neighborhood V_0 of y_0 contained in V and such that each map of a sphere of dimension $k \leqslant n$ into V_0 is homotopic to a constant in V^{S^k}. Let us prove the following theorem of Kuratowski ([196], p. 273), in a generalized form due to Dugundji ([103], p. 232):

(9.1) THEOREM. *For every metric space Y, the following four conditions are equivalent:*

(1) *$Y \in \mathbf{LC}^n$, where n is an integer $\geqslant 0$.*

(2) *If A is a closed subset of a metric space X and* $\dim(X-A) \leqslant n+1$, *then for every map $f\colon A \to Y$ there exists a neighborhood U of A in X such that f has a continuous extension $\hat{f}\colon U \to Y$.*

(3) *If V is a neighborhood of a point $y \in Y$, then there exists a neighborhood $V_0 \subset V$ of y such that; for every metric space X and for every closed subset A of X satisfying the condition* $\dim(X-A) \leqslant n+1$, *every map $f\colon A \to V_0$ has a continuous extension $\hat{f}\colon X \to V$.*

(4) *If V is a neighborhood of a point $y \in Y$, then there exists a neighborhood $V_0 \subset V$ of y such that every map of a metric space X of dimension $\leqslant n$ into V_0 is homotopic to a constant in V^X.*

Proof. First let us prove that (1) implies (2). Consider a canonical covering $G = \{G_\mu\}$, $\mu \in M$, of the set $X-A$ such that each point $x \in X-A$ belongs to $n+2$ sets G_μ at most. Let $N = N(G)$ be the nerve of this covering. As we have shown, this nerve is a polytope with a triangulation $\mathcal{T}$

consisting of all simplexes of the form $\sigma(G_{\mu_0}, G_{\mu_1}, \ldots, G_{\mu_k})$ with $G_{\mu_0} \cap G_{\mu_1} \cap \ldots \cap G_{\mu_k} \neq 0$. Consequently, $\dim \sigma \leqslant n+1$ for all $\sigma \in \mathscr{T}$. This polytope N was defined in an abstract way and thus we can assume that it is disjoint with the set A. Let N^k denote the k-skeleton of $\mathscr{T}$, i.e. the polytope defined as the union of all simplexes of $\mathscr{T}$ of dimensions $\leqslant k$.

Consider now the space Z consisting of all points of the set $A \cup N$ and where the neighborhoods of a point $z \in Z$ are defined as subsets U_z of Z such that:

(a) If $z \in A - \overline{X - A}$, then $U_z \cap A$ is a neighborhood of z in X.

(b) If $z \in N$, then $U_z \cap N$ is a neighborhood of z in N.

(c) If $z \in A \cap \overline{X - A}$, then $U_z \cap A$ is a neighborhood of z in A and there exists a neighborhood V of z in the space X such that every simplex $\sigma = \sigma(G_{\mu_0}, G_{\mu_1}, \ldots, G_{\mu_k}) \in \mathscr{T}$, where $G_{\mu_0} \cup G_{\mu_1} \cup \ldots \cup G_{\mu_k} \subset V$, is contained in U_z.

Applying property $\mathbf{LC}^n$ of the space Y, let us show that

(9.2) *For every map f: $A \to Y$ there is a neighborhood W of A (in the space Z) such that f can be extended to a map $\bar{f}$: $W \to Y$.*

In order to prove it, let us assign to each vertex G_μ of the triangulation $\mathscr{T}$ a point $a_\mu \in A$ such that

$$\varrho(a_\mu, G_\mu) < 2 \operatorname*{Sup}_{x \in G_\mu} \varrho(x, A).$$

Let us observe that setting

$$f_0(x) = f(x) \quad \text{for every point } x \in A,$$
$$f_0(G_\mu) = f(a_\mu) \quad \text{for every vertex } G_\mu \text{ of } \mathscr{T},$$

we get a map f_0: $A \cup N^0 \to Y$ (cf. Section 7).

Setting $W_0 = Z$, let us assume that for an index k with $0 \leqslant k \leqslant n$ a neighborhood W_k of the set A in Z and a map

$$f_k\colon A \cup (N^k \cap W_k) \to Y$$

are already defined. Since $Y \in \mathbf{LC}^n$, there exists for every point $a \in A$ a neighborhood $H_a \subset W_k$ of a in the space Z such that, for each $(k+1)$- dimensional simplex $\sigma = \sigma(G_{\mu_0}, G_{\mu_1}, \ldots, G_{\mu_{k+1}}) \in \mathscr{T}$ lying in H_n the map $f_k|\sigma^\bullet$, has a continuous extension onto the whole simplex σ with values in Y.

Let ε_σ denote the greatest lower bound for diameters of the sets of values (on σ) for all such extensions. Then there exists a continuous extension $f_{k\sigma}$ of the restriction $f_k|\sigma^\bullet$ to σ such that

(9.3) $$f_{k\sigma}(\sigma) \subset Y \quad \text{and} \quad \delta[f_{k\sigma}(\sigma)] \leqslant 2\varepsilon_\sigma.$$

Now let us denote by W_{k+1} the union of the set A and of all simplexes of the triangulation $\mathscr{T}$ contained in at least one of the sets H_a. Manifestly,

$W_{k+1} \subset W_k$ and W_{k+1} is a neighborhood of A in the space Z. Let us extend the restriction $f_k|(A \cup (N^k \cap W_{k+1}))$ to a function

$$f_{k+1}: A \cup (N^{k+1} \cap W_{k+1}) \to Y$$

setting

$$f_{k+1}(x) = f_{k\sigma}(x) \tag{9.4}$$

for every point x which belongs to a $(k+1)$-dimensional simplex $\sigma \in \mathscr{T}$ lying in W_{k+1}. The continuity of f_{k+1} at every point a of the set $(A - \overline{W_{k+1} - A}) \cup (N^{k+1} \cap W_{k+1} - A)$ is evident. Since f_{k+1} is an extension of the map $f_k|(A \cup (N^k \cap W_{k+1}))$, the continuity of f_{k+1} at a point a of the set $A \cap \overline{W_{k+1} - A}$ will be proved if we show that for every positive ε there exists a neighborhood U of a in Z such that for each $(k+1)$-dimensional simplex $\sigma = \sigma(G_{\mu_0}, G_{\mu_1}, \ldots, G_{\mu_{k+1}}) \in \mathscr{T}$ lying in U the distance of every point $y \in f_{k+1}(\sigma) = f_{k\sigma}(\sigma)$ from $f(a)$ is less than ε. Since $Y \in \mathbf{LC}^n$, there exists a neighborhood V_ε of the point $f(a)$ in the space Y such that every map $\varphi: \sigma^\bullet \to V_\varepsilon$ has a continuous extension $\bar{\varphi}$ to σ such that $\varrho(\bar{\varphi}(x), f(a)) < \frac{1}{6}\varepsilon$ for every point $x \in \sigma$. Since f_k is continuous, there exists a neighborhood U_ε of a in Z such that $f_k(\sigma^\bullet) \subset V_\varepsilon$ for every $(k+1)$-dimensional simplex $\sigma \in U_\varepsilon$. It follows by (9.3) and (9.4) that for each such simplex

$$\delta[f_{k+1}(\sigma)] < \tfrac{2}{3}\varepsilon.$$

Let us select a point $y_\sigma \in f_k(\sigma^\bullet)$; we infer that for every point $y \in f_{k+1}(\sigma)$

$$\varrho(y, f(a)) \leqslant \delta[f_{k+1}(\sigma)] \cup \varrho(y_\sigma, f(a)) < \tfrac{2}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon,$$

and consequently the continuity of f_{k+1} is proved.

Thus, after $n+1$ steps, we arrive to a map $\bar{f} = f_{n+1}$ of a neighborhood $W = W_{n+1}$ of the set A in the space Z with values in Y. Hence proposition (9.2) is proved.

Now let us prove that

(9.5) *There exists a map $\varphi: X \to Z$ satisfying the condition $\varphi(x) = x$ for every point $x \in A$.*

In order to obtain such a map φ, let us apply the canonical map $\varkappa$ (as defined in Section 6) of the set $X - A$ into the nerve of the covering $\{G_\mu\}$. Setting

$$\varphi(x) = \begin{cases} x & \text{for every point } x \in A, \\ \varkappa(x) & \text{for every point } x \in X - A, \end{cases}$$

we get a transformation $\varphi: X \to Z$. The continuity of φ at points of the set $A - \overline{X - A}$ is obvious, and at points of the set $X - A$, it is a consequence of the continuity of the map $\varkappa$. In order to show that φ is continuous also at every point x_0 of the set $A \cap \overline{X - A}$, it is sufficient to recall that the covering $\{G_\mu\}$ is canonical and consequently the values of φ, for arguments belonging to a sufficiently small neighborhood of the point x_0, belong either to the same neighborhood, or to simplexes of the triangulation $\mathcal{T}$ lying in an arbitrarily small neighborhood of x_0 in the space Z.

The implication (1) $\Rightarrow$ (2) is an easy consequence of propositions (9.2) and (9.5). In fact, since $W = W_{n+1}$ is a neighborhood of the set A in the space Z and since the map $\varphi\colon X \to Z$ satisfies the condition $\varphi(x) = x$ for every point $x \in A$, we infer that the set $U = \varphi^{-1}(W)$ is a neighborhood of A in the space X. The map φ carries U onto a subset of the set W in which the map $\bar{f}$ is defined. Setting

$$\hat{f}(x) = \bar{f}\varphi(x) \qquad \text{for every point } x \in U,$$

we get a map $\hat{f}\colon U \to Y$. This map is an extension of the map f, because for every point $x \in A$ we have $\varphi(x) = x$ and $\bar{f}(x) = f(x)$. Thus the proof of the implication (1) $\Rightarrow$ (2) is finished.

In order to prove the implication (2) $\Rightarrow$ (3) suppose to the contrary that a space Y satisfies condition (2), but does not meet condition (3). Then, for a point $b \in Y$ and for a neighborhood V of this point, there exist, for every $i = 1, 2, \ldots$, a metric space X_i and a closed subset A_i of it such that $\dim(X_i - A_i) \leqslant n+1$ and that there exists a map f_i of the set A_i into the ball $K(1/i)$ with center b and radius $1/i$ such that f_i cannot be extended to a map of X_i into V. It is clear that we can assume that the spaces X_i are disjoint with one another, and that the metric ϱ_i in the space X_i is such that the diameter of X_i is $\leqslant 1$. Let X_0 denote the space consisting of only one point a which does not belong to $\bigcup_{i=1}^{\infty} X_i$. Setting $X = \bigcup_{i=0}^{\infty} X_i$ and

$$\varrho(x, x') = \frac{1}{i}\,\varrho_i(x, x') \qquad \text{for every} \quad x, x' \in X_i,\ i = 1, 2, \ldots,$$

$$\varrho(x, x') = \frac{1}{\operatorname{Min}(i, i')} \qquad \text{for} \quad x \in X_i,\ x' \in X_{i'},\ i \neq i' \neq 0 \neq i,$$

$$\varrho(x, a) = \frac{1}{i} \qquad \text{for} \quad x \in X_i,\ i = 1, 2, \ldots,$$

we readily see that ϱ is a metric in the space X by which the set $A = X_0 \cup \bigcup_{i=1}^{\infty} A_i$ is closed in X, and the dimension of the set $X - A$ is $\leqslant n+1$. If we set

$$f(x) = f_i(x) \qquad \text{for every point } x \in A_i,$$

$$f(a) = b,$$

then we obtain a map $f\colon A \to Y$. By condition (2), there exists a continuous extension $\hat{f}$ of f onto a neighborhood U of A with values belonging to Y. By the definition of the space X, we infer that

$$X_i \subset U \quad \text{for almost all indices } i,$$

and, by the continuity of $\hat{f}$, we infer that

$$\hat{f}(X_i) \subset V \quad \text{for almost all indices } i.$$

It follows that for almost all indices i the restriction $\hat{f}|X_i$ is a continuous extension of the map f_i on the whole space X_i, with values belonging to V, which contradicts our hypothesis. Thus the proof of the implication (2) $\Rightarrow$ (3) is finished.

Now let us assume that the space Y satisfies condition (3) and let X be a metric space of dimension $\leqslant n$. Then the Cartesian product $X \times \langle 0, 1\rangle$ is a metric space of dimension $\leqslant n+1$ and the set $A = X \times (0) \cup X \times (1)$ is closed in $X \times \langle 0, 1\rangle$. By condition (3), each neighborhood V of a point $y \in Y$ contains a neighborhood V_0 of this point such that every map $f\colon A \to V_0$ has a continuous extension $\hat{f}\colon X \times \langle 0, 1\rangle \to V$. Now, if φ is an arbitrary map of X into V_0, then setting

$$f(x, 0) = \varphi(x) \quad \text{and} \quad f(x, 1) = y \quad \text{for every point } x \in X,$$

we get a map $f\colon A \to V_0$. If $\hat{f}\colon X \times \langle 0, 1\rangle \to V$ is a continuous extension of f, then setting

$$\varphi_t(x) = \hat{f}(x, t) \quad \text{for every point } x \in X \text{ and } 0 \leqslant t \leqslant 1,$$

we get a continuous family $\{\varphi_t\}$ of maps joining in the space V^X the map $\varphi_0 = \varphi$ with the constant map φ_1. Thus the implication (3) $\Rightarrow$ (4) is proved.

Finally, let us observe that condition (4) restricted to the case $X = S^k$, for $k = 0, 1, \ldots, n$, is the same as condition (1). Hence (4) $\Rightarrow$ (1) and so the proof of the theorem is complete.

(9.6) COROLLARY. *For metric spaces of dimension less than or equal to n the condition* $\mathbf{LC}^n$ *is equivalent to the local contractibility.*

CHAPTER IV

ABSOLUTE RETRACTS AND ABSOLUTE NEIGHBORHOOD RETRACTS IN METRIC SPACES

The notions of absolute retract and absolute neighborhood retract were first introduced for compact metric spaces. In the next chapter we shall consider these notions in this initial sense, but in this chapter we consider them for arbitrary metrizable spaces following C. H. Dowker [98] and J. Dugundji [103].

1. Spaces $\mathrm{AR}(\mathfrak{M})$ **and** $\mathrm{ANR}(\mathfrak{M})$**.** We shall write $X \in \mathfrak{M}$ to indicate that the space X is metrizable: that is, we shall consider the class $\mathfrak{M}$ of all metrizable spaces. A space X will be called an *absolute retract for metrizable spaces* provided that $X \in \mathfrak{M}$ and for each homeomorphism h mapping X onto a closed subset $h(X)$ of a metrizable space Y, the set $h(X)$ is a retract of Y. In symbols, $X \in \mathrm{AR}(\mathfrak{M})$. Similarly, a space X is said to be an *absolute neighborhood retract for metrizable spaces* (in symbols, $X \in \mathrm{ANR}(\mathfrak{M})$) provided that $X \in \mathfrak{M}$ and for each homeomorphism h, mapping X onto a closed subset $h(X)$ of a space $Y \in \mathfrak{M}$, $h(X)$ is a neighborhood retract in Y.

Clearly, $X \in \mathrm{AR}(\mathfrak{M})$ implies $X \in \mathrm{ANR}(\mathfrak{M})$.

2. Elementary properties of spaces $\mathrm{AR}(\mathfrak{M})$**.** The spaces $\mathrm{AR}(\mathfrak{M})$ are characterized by the following

(2.1) THEOREM. *In order that a metrizable space X be an* $\mathrm{AR}(\mathfrak{M})$*-space it is necessary that X be the r-image of a convex subset of a normed linear space and it is sufficient that X be an r-image of a convex subset of a locally convex linear space.*

Proof. Suppose that $X \in \mathrm{AR}(\mathfrak{M})$. By the theorem of Kuratowski and Wojdysławski (III, (8.1)), there is a homeomorphism h mapping X onto a closed subset Y of a convex subset Q of a normed linear space. By the definition of an $\mathrm{AR}(\mathfrak{M})$, there is a retraction $r\colon Q \to Y$. Then $h^{-1}r\colon Q \to X$ is the required r-map of Q onto X.

Now suppose that X is an r-image of a convex subset Q of a locally convex linear space. Let $f\colon Q \to X$ be an r-map and let $g\colon X \to Q$ be a right inverse of f. Let h be a homeomorphism of X onto a closed subset of a metric space X'. Then $\varphi = gh^{-1}$ is a map of $h(X)$ into Q and by the generalized theorem of Tietze (III, (7.1)) there is a continuous extension of φ to $\varphi'\colon X' \to Q$. Setting $r(x') = h\big(f(\varphi'(x'))\big)$ for each $x' \in X'$, we obtain a map $r\colon X' \to h(X)$ such that for $y = h(x) \in h(X)$,

$$r(y) = (hf\varphi'h)(x) = (hfgh^{-1}h)(x) = h(x) = y.$$

Thus r is a retraction of the space X' to $h(X)$ and so the proof of the theorem is complete.

(2.2) COROLLARY. *Every r-image of an* AR($\mathfrak{M}$)*-space is an* AR($\mathfrak{M}$)*-space.*

(2.3) COROLLARY. *Each* AR($\mathfrak{M}$)*-space is contractible in itself and locally contractible.*

Corollary (2.3) follows from the following facts: (i) That each convex subset of a linear space is contractible in itself (I, (13.1)) and locally contractible (II, (15.2)) and (ii) both of these properties are $\underline{r}$-invariants (I, (13.2) and (15.4)).

(2.4) COROLLARY. *The groups of homology, cohomology, homotopy and cohomotopy of an* AR($\mathfrak{M}$)*-space are trivial.*

3. Elementary properties of spaces ANR($\mathfrak{M}$). The ANR($\mathfrak{M}$)-spaces are characterized in the following way:

(3.1) THEOREM. *In order that a metrizable space X be an* ANR($\mathfrak{M}$) *it is necessary that X be an r-image of an open subset of a convex set lying in a normed linear space; it is sufficient that X be an r-image of an open subset of a convex set lying in a locally convex linear space.*

Proof. Suppose $X \in$ ANR($\mathfrak{M}$). By the theorem of Kuratowski and Wojdysławski (III, (8.1)), there is a homeomorphism h mapping X onto a closed subset of a convex subset Q of a normed linear space. Then there is a retraction r of an open neighborhood U of $h(X)$ in Q onto $h(X)$. Then $h^{-1}r\colon U \to X$ is an r-map of U onto X.

Now suppose that X is an r-image of a set U which is open in a convex subset Q of a locally convex linear space. Let $f\colon U \to X$ be an r-map and let $g\colon X \to U$ be a right inverse for f. Consider a homeomorphism h mapping X onto a closed subset $h(X)$ of a metric space X'. Then $\varphi = gh^{-1}$ maps $h(X)$ into $U \subset Q$ and so, by the generalized theorem of Tietze, there is a continuous extension φ' of φ mapping X' into Q. Let U' be the inverse image of U under φ'. Then U' is a neighborhood of $h(X)$ in X'. Setting

$$r(x') = hf\varphi'(x') \quad \text{for} \quad x' \in U',$$

we obtain a map $r\colon U' \to h(X)$ such that for $y = h(x) \in h(X)$ we have

$$r(y) = hf\varphi'h(x) = hfgh^{-1}h(x) = h(x) = y.$$

Thus r is a retraction of U' to $h(X)$ and the proof of the theorem is finished.

(3.2) COROLLARY. *Every r-image of an* ANR($\mathfrak{M}$)*-space is an* ANR($\mathfrak{M}$)*-space.*

(3.3) COROLLARY. *Each* ANR($\mathfrak{M}$)*-space is locally contractible.*

We have only to recall that open sets in convex subsets of normed linear spaces are locally contractible and that this property is preserved by r-maps; Corollary (3.3) then follows from Theorem (3.1).

Some other local properties of ANR($\mathfrak{M}$)-spaces have been studied by J. Dugundji [103].

(3.4) COROLLARY. *Every neighborhood retract of an* ANR($\mathfrak{M}$)*-space is an* ANR($\mathfrak{M}$)*-space.*

Proof. Let $X \epsilon$ ANR($\mathfrak{M}$). By Theorem (3.1), there is an r-map $f\colon G \to X$ where G is an open subset of a convex set Q lying in a normed linear space Z. Let $g\colon X \to G$ be a right inverse of f. If X_0 is a neighborhood retract of X, then X_0 is closed in X and there is an open neighborhood U of X_0 in X and a retraction $r\colon U \to X_0$. The set $H = f^{-1}(U)$ is open in Q. Setting $f_0(z) = r(f(z))$ for $z \epsilon H$, we obtain a map $f_0\colon H \to X_0$. This map is an r-map since, setting $g_0(x) = g(x)$ for every point $x \epsilon X_0$, we get a map $g_0\colon X_0 \to H$ which is a right inverse of f_0. Indeed, for $x \epsilon X_0$ we have

$$f_0 g_0(x) = rfg(x) = r(x) = x.$$

This completes the proof.

4. Absolute retracts and extension of maps. First we have the following, almost obvious,

(4.1) THEOREM. *Let X be a closed subset of a metrizable space X'. If $X \epsilon$* AR($\mathfrak{M}$)*, then every map $f\colon X \to Y$ has a continuous extension $f'\colon X' \to Y$. If $X \epsilon$* ANR($\mathfrak{M}$)*, then there is a neighborhood U of X in X' such that every map $f\colon X \to Y$ has a continuous extension $f'\colon U \to Y$.*

Proof. The inclusion map $i\colon X \to X'$ is a homeomorphism embedding X as a closed subset of X'. Hence $X \epsilon$ AR($\mathfrak{M}$) implies that there is a retraction $r\colon X' \to i(X) = X$. Setting $f' = fr$, we get the required extension. If, however, $X \epsilon$ ANR($\mathfrak{M}$), then there exists a retraction r of a neighborhood U of X (in the space X') onto X. Thus the formula $f' = fr$ gives a continuous extension $f'\colon U \to Y$.

Now we prove the following

(4.2) THEOREM. *Let Y be a metrizable space. Then*

(i) *Y is an* AR($\mathfrak{M}$)*-space if and only if, for each closed subset X of a metrizable space X', every map $f\colon X \to Y$ admits a continuous extension $f'\colon X' \to Y$.*

(ii) *Y is an* ANR($\mathfrak{M}$)*-space if and only if, for each closed subset X of a metrizable space X', every map f: $X \to Y$ admits a continuous extension mapping a neighborhood U of X in X' into Y.*

Proof. By the theorem of Kuratowski and Wojdysławski (III, (8.1)) we may assume, without loss of generality, that Y is a closed subset of a convex set Q lying in a normed linear space.

By the generalized theorem of Tietze, for each map f: $X \to Y$ there is a map f'': $X' \to Q$ such that $f''(x) = f(x)$ for every point $x \in X$. If we suppose $Y \in \mathrm{AR}(\mathfrak{M})$, then there is a retraction r: $Q \to Y$ and we infer that $f' = rf''$ is the required extension of f. If $Y \in \mathrm{ANR}(\mathfrak{M})$, then there is a neighborhood V of Y in Q and a retraction r: $V \to Y$. We let U be the inverse image of V under f'' and define f' by

$$f'(x) = rf''(x) \quad \text{for} \quad x \in U.$$

We obtain an extension f' of f mapping U into Y as required. This proves the necessity of the conditions in (i) and (ii).

To prove the sufficiency in (i) let us suppose that every continuous map f of a closed subset X of a metrizable space X' into Y has a continuous extension f': $X' \to Y$. Then, in particular, the identity map of Y admits an extension f': $Q \to Y$. But under these conditions, f' is a retraction. It follows by (2.1) that $Y \in \mathrm{AR}(\mathfrak{M})$.

To prove the sufficiency in (ii), suppose that every map f: $X \to Y$ admits an extension f': $U \to Y$ where U is a neighborhood of X in X'. Then, in particular, taking $X = Y$, $X' = Q$, we infer that the identity i: $Y \to Y$ admits an extension i': $U \to Y$. Under these conditions, i' is a retraction and we infer by (3.1) that $Y \in \mathrm{ANR}(\mathfrak{M})$. Thus the proof of the theorem is complete.

5. Spaces of maps with values in an ANR($\mathfrak{M}$)-space. We prove the following

(5.1) THEOREM. *If X_0 is a subset of a compactum X and y_0 is a point of an* ANR($\mathfrak{M}$)*-space Y, then the functional space $(Y, y_0)^{(X,X_0)}$ is an* ANR($\mathfrak{M}$)*-space.*

Proof. By (3.1) we can find an r-map f of an open subset U of a convex set Q, lying in a normed linear space Z, onto the space Y. Let g: $Y \to U$ be a right inverse of f and let z_0 denote the point $g(y_0)$. Since the space Z is linear, we can define, for every two maps φ, ψ: $(X, X_0) \to (Z, z_0)$ and real λ, μ, the linear combination $\lambda\varphi+\mu\psi$ setting

$$(\lambda\varphi+\mu\psi)(x) = \lambda\varphi(x)+\mu\psi(x)+(1-\lambda-\mu)z_0.$$

Hence the space $(Z, z_0)^{(X,X_0)}$ is linear. Moreover, it is normed, because the hypothesis that X is compact makes it possible to assign to every map $\varphi \in (Z, z_0)^{(X, X_0)}$ its norm $|\varphi|$ given by the formula

$$|\varphi| = \operatorname*{Sup}_{x \in X} \varrho\big(\varphi(x), z_0\big).$$

Moreover, $(Q, z_0)^{(X, X_0)}$ is a convex subset of this space, while $(U, z_0)^{(X, X_0)}$ is open in $(Q, z_0)^{(X, X_0)}$. Since f is an r-map, the operation

$$\Phi_f\colon (U, z_0)^{(X, X_0)} \to (Y, y_0)^{(X, X_0)},$$

which assigns to $\varphi \in (U, z_0)^{(X, X_0)}$ the map $\Phi_f(\varphi) = f\varphi \in (Y, y_0)^{(X, X_0)}$ is an r-map by I, (10.1). Consequently, the space $(Y, y_0)^{(X, X_0)}$ is an r-image of the set $(U, z_0)^{(X, X_0)}$ which is an open subset of the convex set $(Q, z_0)^{(X, X_0)}$ lying in the normed linear space $(Z, z_0)^{(X, X_0)}$. If we observe that the hypothesis that X is compact implies the metrizability of $(Y, y_0)^{(X, X_0)}$, we infer by (3.1) that the space $(Y, y_0)^{(X, X_0)}$ is an ANR($\mathfrak{M}$).

It follows by (3.3) that, by our hypotheses, the space $(Y, y_0)^{(X, X_0)}$ is locally contractible. By means of I, (8.1), we infer that every component of the space $(Y, y_0)^{(X, X_0)}$ is arcwise connected (compare [170], p. 187). It follows that in this case the homotopy classes of maps of (X, X_0) into (Y, y_0) coincide with the components of the space $(Y, y_0)^{(X, X_0)}$, i.e. the weak homotopy in the sense given in I, Section 11, is the same as homotopy. Moreover, since the space X is metric, it satisfies the first axiom of countability and by I, (11.3), we infer that in our case the homotopy is the same as strong homotopy. Thus we get the following

(5.2) COROLLARY. *If X_0 is a subset of a compactum X and y is a point of an* ANR($\mathfrak{M}$)*-space Y, then for maps of (X, X_0) into (Y, y_0) the notions of homotopy, weak homotopy, and of strong homotopy are equivalent to each other.*

Another consequence of Theorem (5.1) is a theorem due to M. Pavel [245] concerning families of equicontinuous maps of a compactum X into a compact ANR($\mathfrak{M}$)-space Y. Let us recall that a family $F \subset Y^X$ is said to be *equicontinuous* if for every $\varepsilon > 0$ there exists an $\eta > 0$ such that for every two points $x_1, x_2 \in X$ with $\varrho(x_1, x_2) < \eta$ and for every $f \in F$, $\varrho\big(f(x_1), f(x_2)\big) < \varepsilon$. It is known that the closure of such family F is a compactum. Now, if we observe that a compactum lying in a locally connected space is contained in a finite number of components of this space, we get from Theorem (5.1) the following

(5.3) COROLLARY. *Each family of equicontinuous maps of a compactum X into a compact* ANR($\mathfrak{M}$)*-space Y is contained in a finite number of components of Y^X.*

Some other properties of the class of components of the space Y^X, where X is a locally compact, separable metric space and Y is a separable ANR($\mathfrak{M}$), have been given by K. Kuratowski [202] and by J. R. Jackson [170].

6. Addition of $AR(\mathfrak{M})$-spaces and of $ANR(\mathfrak{M})$-spaces. The following theorem ([7], p. 194, and [21], p. 226) gives a relation between the $AR(\mathfrak{M})$ and $ANR(\mathfrak{M})$-properties of two sets, of their union, and of their common part:

(6.1) THEOREM. *Suppose that the metrizable space X is the union of two closed subsets X_1 and X_2 and let $X_0 = X_1 \cap X_2$. Then*

(i) *If $X_0, X_1, X_2 \in AR(\mathfrak{M})$, then $X \in AR(\mathfrak{M})$.*

(ii) *If $X_0, X_1, X_2 \in ANR(\mathfrak{M})$, then $X \in ANR(\mathfrak{M})$.*

(iii) *If $X, X_0 \in AR(\mathfrak{M})$, then $X_1, X_2 \in AR(\mathfrak{M})$.*

(iv) *If $X, X_0 \in ANR(\mathfrak{M})$, then $X_1, X_2 \in ANR(\mathfrak{M})$.*

Proof. In order to prove (i) it is sufficient to show that if X is a closed subset of a metric space Z and if $X_0, X_1, X_2 \in AR(\mathfrak{M})$, then X is a retract of Z. Let us set

$$Z_0 = \{z \in Z;\ \varrho(z, X_1) = \varrho(z, X_2)\},$$

$$Z_1 = \{z \in Z;\ \varrho(z, X_1) < \varrho(z, X_2)\},$$

$$Z_2 = \{z \in Z;\ \varrho(z, X_1) > \varrho(z, X_2)\}.$$

It is evident that $Z = Z_0 \cup Z_1 \cup Z_2$, the set $X_0 \subset Z_0$ is closed in Z_0 and $X_i \cap Z_0 = X_0$ for $i = 1, 2$. Hence there exists a retraction r_0: $Z_0 \to X_0$. Moreover, the set $X_i \cup Z_0$ is closed in $Z_i \cup Z_0$ for $i = 1, 2$ and we infer by (4.2) that the map r_i: $X_i \cup Z_0 \to X_i$, given by the formulas

$$r_i(z) = \begin{cases} z & \text{for every} \quad z \in X_i, \\ r_0(z) & \text{for every} \quad z \in Z_0, \end{cases}$$

has a continuous extension f_i: $Z_i \cup Z_0 \to X_i$. It is sufficient to set $r(z) = f_i(z)$ for $z \in Z_i \cup Z_0$, $i = 1, 2$, to obtain a retraction r: $Z \to X$.

Passing to (ii), we need to show that if X is a closed subset of a metric space Z and if $X_0, X_1, X_2 \in ANR(\mathfrak{M})$, then there exists in Z a neighborhood U of the set X such that X is a retract of U. Consider the sets Z_0, Z_1, Z_2 as defined in the proof of (i). Then X_0 is a closed subset of the set Z_0 and hence there is a neighborhood W_0 of the set X_0 in the space Z_0, closed in Z_0, and a retraction r_0: $W_0 \to X_0$. Setting

$$r_i(z) = \begin{cases} r_0(z) & \text{for every point } z \in W_0, \\ z & \text{for every point } z \in X_i, \end{cases}$$

we obtain a retraction r_i of the set $X_i \cup W_0$ (which is closed in $Z_0 \cup Z_i$) into the set X_i, $i = 1, 2$. Since $X_i \in ANR(\mathfrak{M})$, we infer by (4.2) that there exists a continuous extension r_i' of r_i to a neighborhood V_i of $X_i \cup W_0$

in $Z_0 \cup Z_i$ with r_i' having values in X_i. It is clear that V_i contains a closed neighborhood U_i of X_i in the space $Z_0 \cup Z_i$ such that $U_i \cap Z_0 \subset W_0$. Since

$$U_1 \cap U_2 \subset U_1 \cap (Z_0 \cup Z_1) \cap (Z_0 \cup Z_2) = U_1 \cap Z_0 \subset W_0,$$

the formula

$$r(z) = r_i'(z) \quad \text{for} \quad z \in U_i,\ i = 1, 2,$$

defines a retraction r of the set $U = U_1 \cup U_2$, which is a neighborhood of X in the space Z, onto the set X. Thus the proof of (ii) is complete.

In order to prove (iii) let us observe that the condition $X_0 \in \mathrm{AR}(\mathfrak{M})$ implies that there exists a retraction r_i: $X_i \to X_0$ for $i = 1, 2$. If we set

$$r(x) = \begin{cases} x & \text{for} \quad x \in X_1, \\ r_2(x) & \text{for} \quad x \in X_2, \end{cases}$$

then we obtain a retraction r: $X \to X_1$. Since $X \in \mathrm{AR}(\mathfrak{M})$, we infer by (2.2) that $X_1 \in \mathrm{AR}(\mathfrak{M})$. Similar reasoning shows that $X_2 \in \mathrm{AR}(\mathfrak{M})$.

In order to prove (iv) let us observe that $X_0 \in \mathrm{ANR}(\mathfrak{M})$ implies that there exists a neighborhood U_0 of the set X_0 in the space X such that, for $i = 1, 2$, there exists a retraction r_i: $X_i \cap U_0 \to X_0$. Setting

$$r(x) = \begin{cases} x & \text{for} \quad x \in X_1, \\ r_2(x) & \text{for} \quad x \in X_2 \cap U_0, \end{cases}$$

we obtain a retraction r: $U_0 \cup X_1 \to X_1$. Since $U_0 \cup X_1$ is a neighborhood of X_1 in X, and $X \in \mathrm{ANR}(\mathfrak{M})$, it follows by (3.4) that $X_1 \in \mathrm{ANR}(\mathfrak{M})$. A similar argument shows that $X_2 \in \mathrm{ANR}(\mathfrak{M})$.

(6.2) COROLLARY. *Every polyhedron is an* ANR($\mathfrak{M}$)*-space.*

Proof. Let X be an n-dimensional polyhedron and $\mathscr{T}$ be a triangulation of X. By III, (3.1), the triangulation $\mathscr{T}$ is finite. If the triangulation $\mathscr{T}$ contains only $k = 1$ simplex, then the theorem is obvious.

Suppose that $k = m > 1$ and that for polyhedra having triangulations with $k < m$ simplexes the theorem is true. Let σ be an n-simplex of $\mathscr{T}$ and let $\mathscr{T}_1$ be $\mathscr{T}$ with σ deleted. By inductive hypothesis the polyhedron X_1 of $\mathscr{T}_1$ is an ANR($\mathfrak{M}$). Moreover, $X_0 = \sigma^{\bullet}$ is also an ANR($\mathfrak{M}$) since, as a subset of X_1 relative to $\mathscr{T}_1$, it is a polyhedron that has a triangulation with fewer than m simplexes. Moreover, $X_2 = \sigma$ is an AR($\mathfrak{M}$) because it is homeomorphic to a convex subset of a Euclidean space. Hence, since $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2$, Theorem (6.1) implies that $X \in \mathrm{ANR}(\mathfrak{M})$.

Remark. As has been shown by S. D. Liao [218], every locally compact polytope is an ANR($\mathfrak{M}$) as well. See also [236], p. 95.

7. Cartesian products of AR($\mathfrak{M}$)-sets and of ANR($\mathfrak{M}$)-sets. Suppose that $\{X_n\}$ is a sequence of metrizable spaces. Then their Cartesian product

$$X = \mathop{P}_{n=1}^{\infty} X_n$$

is also metrizable. In fact, if ϱ_n is a metric for X_n and $x = \{x_n\}$, $x' = \{x_n'\}$ are points of the product space X, then the formula

$$\varrho(x, x') = \sum_{n=1}^{\infty} n^{-2} \frac{\varrho_n(x_n, x_n')}{1+\varrho_n(x_n, x_n')}$$

defines a metric ϱ for X. It is plain that the points $z_k = \{x_{kn}\}$ of X converge to the point $z = \{x_n\}$ of X if and only if $\lim_{k\to\infty} \varrho_n(x_{kn}, x_n) = 0$ for $n = 1, 2, \ldots$

We now prove the following

(7.1) THEOREM. *The Cartesian product* $X = \mathop{P}_{n=1}^{\infty} X_n$ *is an* AR($\mathfrak{M}$)*-space if and only if every factor* X_n *is an* AR($\mathfrak{M}$)*-space.*

Proof. We have already seen that the projection $\varphi_j\colon X \to X_j$ given by the formula $\varphi(\{x_n\}) = x_j$ is an r-map. Consequently $X \in \mathrm{AR}(\mathfrak{M})$ implies that $X_j \in \mathrm{AR}(\mathfrak{M})$, $j = 1, 2, \ldots$

To show the converse, suppose that every $X_n \in \mathrm{AR}(\mathfrak{M})$. Without loss of generality we may assume that X_n is a closed subset of a convex set Q_n lying in a normed linear space Z_n and that there is a retraction $r_n\colon Q_n \to X_n$, $n = 1, 2, \ldots$ The product $Z = \mathop{P}_{n=1}^{\infty} Z_n$ is linear and locally convex, while the set $Q = \mathop{P}_{n=1}^{\infty} Q_n$ is a convex subset of Z that contains X as a closed subset. We obtain a retraction $r\colon Q \to X$ by means of the formula

$$r(z) = \{r_n(x_n)\},$$

where $z = \{x_n\} \in Q$. We have shown thereby that $X \in \mathrm{AR}(\mathfrak{M})$.

The product of a countable number of ANR($\mathfrak{M}$)-spaces need not be an ANR($\mathfrak{M}$)-space. For example, if we let X_n be the two-point discrete space $\{0, 1\}$ for $n = 1, 2, \ldots$, then $X = \mathop{P}_{n=1}^{\infty} X_n$ is a Cantor discontinuum which, failing to be locally connected, fails also to be an ANR($\mathfrak{M}$). However we may prove the following

(7.2) THEOREM. *The Cartesian product* $X = \mathop{P}_{n=1}^{\infty} X_n$ *is an* ANR($\mathfrak{M}$)*-space if and only if every* $X_n \in \mathrm{ANR}(\mathfrak{M})$ *and almost every* $X_n \in \mathrm{AR}(\mathfrak{M})$.

Proof. First let us assume that $X \in \mathrm{ANR}(\mathfrak{M})$. Let φ_m be the mth projection $X \to X_m$ given by the formula $\varphi_m(\{x_n\}) = x_m$. Since φ_m is an r-map, it follows from (3.2) that $X_m \in \mathrm{ANR}(\mathfrak{M})$ for $m = 1, 2, \ldots$ By the theorem of Kuratowski and Wojdysławski (III, (8.1)) we can assume that X_n is a closed subset of convex set Q_n lying in a normed linear space Z_n. Let $Q = \mathop{P}\limits_{n=1}^{\infty} Q_n$. Then Q is metrizable and contains X as a closed subset. Since $X \in \mathrm{ANR}(\mathfrak{M})$, there is a neighborhood U of X in Q and a retraction $r\colon U \to X$. Let us select a point $a = \{a_n\}$ in X. Since U is a neighborhood of a in Q, there exists an index n_0 such that all points $x = \{x_n\} \in Q$ with $x_i = a_i$ for $1 \leqslant i \leqslant n_0$ belong to U. Now let us consider an index $m > n_0$ and let us set

$$\psi_m(x) = \{a_1, a_2, \ldots, a_{m-1}, x, a_{m+1}, \ldots\} \quad \text{for every } x \in Q_m.$$

Then ψ_m is a map of Q_m into U. Setting

$$r_m = \varphi_m r \psi_m,$$

we obtain a map $r_m\colon Q_m \to X_m$ such that for $x \in X_m$ we have

$$r_m(x) = \varphi_m r \psi_m(x) = \varphi_m \psi_m(x) = x.$$

It follows that r_m is a retraction of Q_m to X_m. Since Q_m is a convex subset of a normed linear space, we infer by (2.1) that $X_m \in \mathrm{AR}(\mathfrak{M})$ for every $m > n_0$.

Now let us assume that $X_n \in \mathrm{ANR}(\mathfrak{M})$ for every $n = 1, 2, \ldots$ and that there is an index n_0 such that $X_m \in \mathrm{AR}(\mathfrak{M})$ for every $m > n_0$. We have to prove that $X = \mathop{P}\limits_{n=1}^{\infty} X_n$ is an ANR($\mathfrak{M}$)-space.

First let us show that the Cartesian product $X_1 \times X_2$ of two ANR($\mathfrak{M}$)-spaces is an ANR($\mathfrak{M}$)-space. By the theorem of Kuratowski and Wojdysławski we can assume that X_i is a closed subset of a convex subset Q_i of a normed linear space Z_i $(i = 1, 2)$. Then $X_1 \times X_2$ is a closed subset of the set $Q_1 \times Q_2$ which is a convex set in the normed linear space $Z_1 \times Z_2$.

Let U_i be a neighborhood of X_i in Q_i and let $r_i\colon U_i \to X_i$ be a retraction. Then $U_1 \times U_2$ is a neighborhood of the set $X_1 \times X_2$ in $Q_1 \times Q_2$. Moreover, setting

$$r(z_1, z_2) = \big(r_1(z_1), r_2(z_2)\big) \quad \text{for} \quad (z_1, z_2) \in U_1 \times U_2,$$

we obtain a retraction of $U_1 \times U_2$ to $X_1 \times X_2$. Therefore $X_1 \times X_2 \in \mathrm{ANR}(\mathfrak{M})$.

It follows at once that the Cartesian product of any finite number of ANR($\mathfrak{M}$)-spaces is an ANR($\mathfrak{M}$)-space.

Let us observe that $X = \mathop{P}\limits_{n=1}^{\infty} X_n$ is homeomorphic to the Cartesian product of $\mathop{P}\limits_{n=1}^{n_0} X_n$ and of $\mathop{P}\limits_{n=n_0+1}^{\infty} X_n$. By the preceding remarks we see that

$\mathop{P}\limits_{n=1}^{n_0} X_n$ is an ANR($\mathfrak{M}$), while the factor $\mathop{P}\limits_{n=n_0+1}^{\infty} X_n$ is an AR($\mathfrak{M}$), which is the product of AR($\mathfrak{M}$)-spaces. Consequently X is the product of an ANR($\mathfrak{M}$)-set and an AR($\mathfrak{M}$)-set and hence it is an ANR($\mathfrak{M}$)-set.

8. Extension of a homotopy. One of the most important properties of ANR($\mathfrak{M}$)-spaces is given by the following ([39], p. 103)

(8.1) THEOREM. *Let X be a closed subset of a metrizable space X' and let Y be an* ANR($\mathfrak{M}$)*-space. Let $\{f_t\}$, $t\in\langle 0,1\rangle$, be a continuous family of maps $f_t\colon X\to Y$ such that f_0 has a continuous extension $f_0'\colon X'\to Y$. Then there is a continuous family $\{f_t'\}$ of maps $f_t'\colon X'\to Y$ such that f_t' extends f_t for each $t\in\langle 0,1\rangle$.*

First we prove the following lemma:

(8.2) LEMMA. *If X is a closed subset of a metric space X', then for every neighborhood V of the set*

$$Z = \big(X'\times(0)\big)\cup\big(X\times\langle 0,1\rangle\big)$$

in the space $Z' = X'\times\langle 0,1\rangle$ there exists a map $\varphi\colon Z'\to V$ which is the identity on Z.

Proof. Let us assign to every $x\in X$ the segment $L_x = (x)\times\langle 0,1\rangle$ in Z. Since L_x is compact and V is a neighborhood of Z, there is an open neighborhood U_x of x in X' such that $U_x\times\langle 0,1\rangle\subset V$. Then the set $U=\bigcup_{x\in X} U_x$ is an open neighborhood of X in X' and hence $U\times\langle 0,1\rangle$ is open in Z' and it is contained in V.

Now let us consider a map $\alpha\colon X'\to\langle 0,1\rangle$ which has the value 0 on $X'-U$, and 1 on X. Such a map exists by the classical Urysohn's lemma (see, for instance, [200], p. 126). If we set

$$\varphi(x,t) = \big(x,\alpha(x)t\big) \quad \text{for} \quad (x,t)\in Z',$$

we obtain the map promised in the lemma.

Proof of Theorem (8.1). Using the notation of Lemma (8.2), consider the function $f\colon Z\to Y$ defined by the conditions

$$f(x,0) = f_0'(x) \quad \text{for} \quad x\in X',$$

$$f(x,t) = f_t(x) \quad \text{for} \quad x\in X \text{ and } t\in\langle 0,1\rangle.$$

It follows by I, (11.3), that f is a map. Since $Y\in$ ANR($\mathfrak{M}$) and since Z is closed in the metrizable space Z', there is by (4.2) a continuous extension f' of f to a neighborhood V of Z in Z' which has the values in Y. Let $\varphi\colon Z'\to V$ be the map of the lemma. Setting

$$f_t'(x) = f'\big(\varphi(x,t)\big) \quad \text{for every } x\in X' \text{ and } t\in\langle 0,1\rangle,$$

we obtain the required family $\{f'_t\}$. Thus the proof of the theorem is complete.

Some generalizations of Theorem (8.1) have been given by C. H. Dowker ([97], p. 205, and [100], pp. 101, 106, and 115), by S. T. Hu([156], p. 232), and also by S. A. Chow ([87], p. 234).

Suppose that X_0 is a closed subset of a compact metric space X and let $y_0 \epsilon Y$, where Y is an ANR$(\mathfrak{M})$. We have proved in (5.1) that the functional space $(Y, y_0)^{(X, X_0)}$ is an ANR$(\mathfrak{M})$-space. Consequently, two maps which are in the same component of $(Y, y_0)^{(X, X_0)}$ can be joined by a continuous family of maps in this space. By Theorem (8.1) it results the following

(8.3) COROLLARY. *If X_0 is a closed subset of a compact metric space X and $Y \epsilon$ ANR$(\mathfrak{M})$, then either every map belonging to a component of Y^{X_0} can be extended to a map belonging to Y^X or none of them can be so extended.*

The following corollary (H. Samelson [256], p. 448) is important in the theory of deformation retracts:

(8.4) COROLLARY. *Let X_0 be a deformation retract of an ANR$(\mathfrak{M})$-space X. Then there exists a homotopy $\{\varphi_t\} \subset X^X$ such that φ_0 is the identity map, $\varphi_t(x) = x$ for every point $x \epsilon X_0$ and $0 \leqslant t \leqslant 1$, and the map $r\colon X \to X_0$, given by the formula $r(x) = \varphi_1(x)$ for every point $x \epsilon X$, is a retraction of X to X_0.*

Proof. Since X_0 is a deformation retract of X, there exists a homotopy $\{\psi_t\} \subset X^X$ such that ψ_0 is the identity and the map $r\colon X \to X_0$, defined by the formula $r(x) = \psi_1(x)$ for every point $x \epsilon X$, is a retraction of X to X_0. Consider the closed subset

$$Y_0 = \big(X \times (0)\big) \cup (X_0 \times \langle 0, 1\rangle) \cup \big(X \times (1)\big)$$

of the Cartesian product $Y = X \times \langle 0, 1\rangle$. Setting

$$\psi(x, t) = \psi_t(x) \quad \text{for every } x \epsilon X,\ t \epsilon \langle 0, 1\rangle,$$

we get (by I, (11.3)) a map $\psi\colon Y \to X$. Let us observe that the partial map $\psi | Y_0\colon Y_0 \to X$ is homotopic to the map $\hat{\varphi}\colon Y_0 \to X$ given by the formulas

$$\hat{\varphi}(x, 0) = x, \quad \hat{\varphi}(x, 1) = \psi(x, 1) \quad \text{for every } x \epsilon X,$$

$$\hat{\varphi}(x, t) = x \quad \text{for every } x \epsilon X_0 \text{ and } 0 \leqslant t \leqslant 1.$$

In fact, setting

$$\hat{\varphi}_s(x, t) = \psi[\hat{\varphi}(x, t), s] \quad \text{for every} \quad (x, t) \epsilon Y_0 \text{ and } 0 \leqslant s \leqslant 1,$$

$$\hat{\psi}_s(x, t) = \psi[x, (1-t)s + t] \quad \text{for every} \quad (x, t) \epsilon Y_0 \text{ and } 0 \leqslant s \leqslant 1,$$

we get two homotopies $\{\hat{\varphi}_s\}, \{\hat{\psi}_s\} \subset X^{Y_0}$ such that $\hat{\varphi}_0 = \hat{\varphi}$, $\hat{\psi}_0 = \psi|Y_0$. Moreover, if $(x, t) \in Y_0$ and $t \neq 1$, then

$$\hat{\varphi}_1(x, t) = \psi[\hat{\varphi}(x, t), 1] = \psi(x, 1) = \hat{\psi}_1(x, t),$$

and if $(x, t) \in Y$ and $t = 1$, then

$$\hat{\varphi}_1(x, t) = \psi[\hat{\varphi}(x, 1), 1] = \psi[\psi(x, 1), 1] = rr(x) = r(x)$$
$$= \psi(x, 1) = \hat{\psi}_1(x, t).$$

Hence $\hat{\varphi}_1 = \hat{\psi}_1$ and we infer that $\hat{\varphi}_0 = \hat{\varphi}$ and $\hat{\psi}_0 = \psi|Y_0$ are homotopic in X^{Y_0}, and (8.1) implies that $\hat{\varphi}$ has a continuous extension $\varphi\colon Y \to X$. So it remains to set

$$\varphi_t(x) = \varphi(x, t) \quad \text{for every point} \quad x \in X \text{ and } 0 \leqslant t \leqslant 1,$$

in order to obtain a homotopy $\{\varphi_t\} \subset X^X$ satisfying the required conditions.

9. Contractible ANR($\mathfrak{M}$)**-spaces.** Let us prove the following

(9.1) THEOREM. *A space is an* AR($\mathfrak{M}$)*-space if and only if it is an* ANR($\mathfrak{M}$)*-space which is contractible in itself.*

Proof. We have seen that every AR($\mathfrak{M}$)-space is contractible and is an ANR($\mathfrak{M}$)-space.

On the other hand, suppose that $X \in$ ANR($\mathfrak{M}$) and that X is contractible in itself. We may assume that X is a closed subset of a convex set Q lying in a normed linear space Z. Since X is contractible, there is a continuous family $\{f_t\}$ of maps $f_t \in X^X$ such that f_0 carries X onto a single point $a \in X$ and f_1 is the identity. Setting $f_0'(x) = a$ for $x \in Q$, we get a continuous extension $f_0'\colon Q \to X$ of the map f_0. Applying Theorem (8.1), we obtain a continuous family $\{f_t'\}$ of extensions f_t' of f_t mapping Q into X.

In particular, $f_1'\colon Q \to X$ extends the identity f_1 and hence f_1' is a retraction. Hence $X \in$ AR($\mathfrak{M}$) and so the proof is complete.

10. Open subsets of ANR($\mathfrak{M}$)**-spaces.** Let us prove two theorems of O. Hanner.

(10.1) FIRST THEOREM OF HANNER ([143], p. 391). *Every open subset of an* ANR($\mathfrak{M}$)*-space is an* ANR($\mathfrak{M}$)*-space.*

Proof. Let G be an open subset of a space $X \in$ ANR($\mathfrak{M}$) and let h be a homeomorphism mapping G onto a closed subset $h(G)$ of a metric space Y. Then h^{-1} maps $h(G)$ onto $G \subset X$. Since $X \in$ ANR($\mathfrak{M}$), there is by (4.2) a continuous extension f of h^{-1} to a neighborhood V of the set $h(G)$ in the space Y with values belonging to X. Since G is open in the space X, the set $U = f^{-1}(G)$ is a neighborhood of $h(G)$ in V and hence also a neighborhood of $h(G)$ in Y. Setting

$$r(y) = h\big(f(y)\big) \quad \text{for} \quad y \in U,$$

we obtain a map $r\colon U \to h(G)$ which is a retraction. Indeed, if $y \in h(G)$, then $r(y) = h\big(h^{-1}(y)\big) = y$. Thus $h(G)$ is a neighborhood retract in Y and hence $G \in \text{ANR}(\mathfrak{M})$.

(10.2) SECOND THEOREM OF HANNER ([143], p. 392). *If a metrizable space X is the countable union of open sets G_i, $i = 1, 2, \dots$, which are* ANR($\mathfrak{M}$)*-spaces, then X is an* ANR($\mathfrak{M}$)*-space.*

Proof. In order to prove that $X \in \text{ANR}(\mathfrak{M})$ it is sufficient to snow that for every metrizable space Y containing X as a closed subset, X is a neighborhood retract in Y. We shall first consider some special cases:

(i) There exists a natural number n such that $X = G_1 \cup G_2 \cup \dots \cup G_n$. Evidently it is sufficient to show our proposition assuming that $n = 2$ and that none of the sets G_1, G_2 is the whole space X.

By First Theorem of Hanner $G_0 = G_1 \cap G_2$ is an ANR($\mathfrak{M}$)-space. Consider the sets

$$H_0 = \{y \in Y;\ \varrho(y, X - G_1) = \varrho(y, X - G_2)\},$$

$$H_1 = \{y \in Y;\ \varrho(y, X - G_1) \geqslant \varrho(y, X - G_2)\},$$

$$H_2 = \{y \in Y;\ \varrho(y, X - G_1) \leqslant \varrho(y, X - G_2)\}.$$

These sets are closed in Y and

$$Y = H_1 \cup H_2, \qquad H_0 = H_1 \cap H_2.$$

Since

(10.3) $$H_0 \cap X \subset G_0,$$

it follows that G_0 is closed in $G_0 \cup H_0$.

Because X is closed in Y, the closure $\bar{G}_0$ of G_0 in Y lies in X. Hence (10.3) implies

$$\bar{G}_0 \cap (G_0 \cup H_0) \subset X \cap (G_0 \cup H_0) = G_0.$$

Since $G_0 \in \text{ANR}(\mathfrak{M})$, there is a neighborhood V of the set G_0 in the set $G_0 \cup H_0$ and a retraction $r_0\colon V \to G_0$. Since V is a neighborhood of the set G_0 in $G_0 \cup H_0$ and the set H_0 is closed in Y, we infer, using (10.3), that there is a neighborhood U of X in Y such that

$$U \cap (G_0 \cup H_0) \subset Y.$$

Consequently $(U \cap H_1) \cap (U \cap H_2) = U \cap H_0 \subset V$. Setting

$$\varphi_i(y) = \begin{cases} r_0(y) & \text{for} \quad y \in (G_0 \cup H_0) \cap H_i \cap U, \\ y & \text{for} \quad y \in G_i - G_0, \end{cases}$$

we obtain a map φ_i of the set $(G_i - G_0) \cup [(G_0 \cup H_0) \cap H_i \cap U] = (X \cap H_i) \cup (H_0 \cap U)$ which is closed in $U \cap H_i$, into the set $G_i \in \mathrm{ANR}(\mathfrak{M})$. Hence, by (4.2), there is a neighborhood W_i of the set $(X \cap H_i) \cup (H_0 \cap U)$ in $U \cap H_i$ and a continuous extension

$$\varphi_i' \colon W_i \to G_i$$

of the map φ_i.

Since $[(U \cap H_0) \cup (X \cap H_1)] \cup [(U \cap H_0) \cup (X \cap H_2]) = (U \cap H_0) \cup X$ and $(U \cap H_1) \cup (U \cap H_2) = U$, we infer that the set $W = W_1 \cup W_2$ is a neighborhood of X in Y and the sets $W_1 = W \cap H_1$ and $W_2 = W \cap H_2$ are closed in W. Since

$$\varphi_1'(y) = \varphi_2'(y) = r_0(y) \qquad \text{for every point } y \in U \cap H_0,$$

and since $W_1 \cap W_2 \subset (U \cap H_1) \cap (U \cap H_2) = U \cap H_0$, we infer that setting

$$r(y) = \varphi_i'(y) \qquad \text{for every point } y \in W_i,\ i = 1, 2,$$

we get a map $r\colon W \to X$. Moreover, for every point $y \in X \cap W_i$ either $y \in G_i - G_0$ or $y \in G_0 \cap H_i \cap U$ and, in both cases, $r(y) = \varphi_i'(y) = y$. It follows that r is a retraction of W to X. Thus X is a retract of the set W which is a neighborhood of X in Y. This completes the proof.

(ii) The sets $G_1, G_2, \ldots$ are pairwise disjoint. If only a finite number of them is not empty, then we have a special case of (i). Thus we may assume (removing the empty terms) that $G_i \neq 0$ for every index $i = 1, 2, \ldots$ Since $G_i = X - \bigcup_{j \neq i} G_j$, we infer that G_i is closed in X, and so it is also closed in Y. Setting

$$U_i = \{y \in Y;\ \varrho(y, G_i) < \varrho(y, X - G_i)\},$$

we get a sequence $\{U_i\}$ of open and disjoint subsets of Y such that $G_i \subset U^i$ for every $i = 1, 2, \ldots$ Then G_i is a closed subset of U_i, and consequently there is a retraction

$$r_i\colon V_i \to G_i,$$

where V_i is an open subset of U_i containing G_i. Setting

$$V = \bigcup_{i=1}^{\infty} V_i,$$

we get a neighborhood V of X in the space Y and the function r defined by the formula

$$r(y) = r_i(y) \qquad \text{for every point } y \in V_i$$

is a retraction of V to X.

Now let us pass to the general case. Setting

$$P_i = G_1 \cup G_2 \cup \ldots \cup G_i \qquad \text{for } i = 1, 2, \ldots,$$

we get an increasing sequence $\{P_i\}$ of open subsets of the space X such that

$$X = \bigcup_{i=1}^{\infty} P_i.$$

Moreover, it follows according to case (i) that $P_i \in \mathrm{ANR}(\mathfrak{M})$. Thus we may assume that $X \neq P_i$ for $i = 1, 2, \ldots$ Setting

$$Q_i = \{x \in X;\ \varrho(x, X - P_i) > 1/i\},$$

we get an open subset of P_i. Moreover, since $P_i \subset P_{i+1}$ and since $X = \bigcup_{i=1}^{\infty} P_i$, we see at once that

$$\bar{Q}_i \subset Q_{i+1} \quad \text{and} \quad X = \bigcup_{i=1}^{\infty} Q_i.$$

Now let us set

$$R_i = \begin{cases} Q_i & \text{for} \quad i = 1, 2, \\ Q_i - \bar{Q}_{i-2} & \text{for} \quad i = 3, 4, \ldots \end{cases}$$

It is evident that

$$X = \bigcup_{i=1}^{\infty} R_i \quad \text{and} \quad R_j \cap R_{i+2} = 0 \quad \text{for} \quad i = 1, 2, \ldots \text{ and } j \leqslant i.$$

Moreover, R_i is an open subset of the set $P_i \in \mathrm{ANR}(\mathfrak{M})$ and we infer by (10.1) that $R_i \in \mathrm{ANR}(\mathfrak{M})$. According to the case (ii), the sets $R = \bigcup_{i=1}^{\infty} R_{2i-1}$ and $R' = \bigcup_{i=1}^{\infty} R_{2i}$ are ANR($\mathfrak{M}$)-spaces and they are open in the space X, which is their union. It follows by (i) that the space X is an ANR($\mathfrak{M}$). Thus the proof of Theorem (10.2) is finished.

(10.4) COROLLARY. *If every point of a separable metric space X has a neighborhood which is an* ANR($\mathfrak{M}$), *then* $X \in \mathrm{ANR}(\mathfrak{M})$.

Thus for separable metric spaces the property of being an ANR($\mathfrak{M}$) is a local property ([143], p. 392). For compact metric spaces an analogous proposition was proved earlier by T. Yajima [298].

If we recall that the Cartesian product of two ANR($\mathfrak{M}$)-spaces is an ANR($\mathfrak{M}$)-space and that every point of a bundle space X ([161], p. 65) over a base space B and a director space D has a neighborhood homeomorphic to the Cartesian product of an open subset of B by D, we infer that

(10.5) *If X is a metric separable bundle space with a base $B \in \mathrm{ANR}(\mathfrak{M})$ and a director space $D \in \mathrm{ANR}(\mathfrak{M})$, then $X \in \mathrm{ANR}(\mathfrak{M})$.*

(10.6) PROBLEM. *Is it true that a metrizable space in which every point has a neighborhood being an* ANR($\mathfrak{M}$) *is necessarily an* ANR($\mathfrak{M}$)?

CHAPTER V

ABSOLUTE RETRACTS AND ABSOLUTE NEIGHBORHOOD RETRACTS IN COMPACTA

In the previous chapters we have considered the notion of retracts in a rather general setting, such a point of view being necessary for many applications. However, the principal aim of this book is a theory of spaces of the possibly most pleasant nature defined in a purely topological way and sufficiently general in order to include all polyhedra. Hence in this chapter we shall confine our attention to compacta, that is, compact metric spaces.

1. AR-spaces and ANR-spaces. A space X will be called an *absolute retract*, or an AR-*space* ([17], p. 159), symbolically, $X \in \mathrm{AR}$ if X is compact and $X \in \mathrm{AR}(\mathfrak{M})$. Similarly, we say that X is an *absolute neighborhood retract* or an ANR-*space* ([21], p. 222), written $X \in \mathrm{ANR}$, provided that X is compact and $X \in \mathrm{ANR}(\mathfrak{M})$.

(1.1) THEOREM. AR-*spaces are precisely the r-images of the Hilbert cube.* ANR-*spaces coincide with compact r-images of open subsets of the Hilbert cube.*

Proof. The Hilbert cube Q^ω is a compact convex subset of Hilbert space E^ω and hence every r-image of Q^ω is compact and it is an $\mathrm{AR}(\mathfrak{M})$-space. Also, $\mathrm{ANR}(\mathfrak{M})$-spaces are the same as r-images of open subsets of convex sets lying in a normed linear space. Consequently, every compact r-image of an open subset of Q^ω is a compact $\mathrm{ANR}(\mathfrak{M})$-space.

Now let us assume that $X \in \mathrm{AR}$. By classical Urysohn's theorem (see, for instance, [200], p. 136) there is a homeomorphism h mapping X onto a subset of Q^ω. Since X is compact, the set $h(X)$ is closed in Q^ω, and since $X \in \mathrm{AR}(\mathfrak{M})$, it follows that there is a retraction $r\colon Q^\omega \to h(X)$. Then the map $h^{-1}r\colon Q^\omega \to X$ is an r-map so that X is an r-image of Q^ω.

Finally let us assume that $X \in \mathrm{ANR}$. As above, there is a homeomorphism h of X onto a closed subset $h(X)$ of Q^ω. Since $X \in \mathrm{ANR}(\mathfrak{M})$, there is an open neighborhood U of $h(X)$ in Q^ω and a retraction $r\colon U \to h(X)$. The map $h^{-1}r\colon U \to X$ is an r-map, so that X is an r-image of an open subset U of Q^ω. This completes the proof.

(1.2) *A compactum X is an* AR-*space if and only if for each homeomorphism h of X onto a subset of a compactum Y the set $h(X)$ is a retract of Y.*

Indeed, if $X \in$ AR, then the set $h(X)$ is an AR($\mathfrak{M}$)-set closed in Y and consequently $h(X)$ is a retract of Y. Conversely, let X be a compactum such that for each homeomorphism $h\colon X \to h(X) \subset Y$, where Y is a compactum, $h(X)$ is a retract of Y. By Urysohn's theorem there is a homeomorphism $h\colon X \to h(X) \subset Q^{\omega}$. Consider a retraction $r\colon Q^{\omega} \to h(X)$. Then $h^{-1}r$ is an r-map of Q^{ω} onto X and (1.1) implies that $X \in$ AR.

By a similar argument we can show that

(1.3) *A compactum X is an* ANR *if and only if for each homeomorphism h of X onto a subset of a compactum Y the set $h(X)$ is a neighborhood retract of Y.*

2. Elementary properties of AR-spaces and ANR-spaces. It will be useful to collect the various properties of AR-spaces and ANR-spaces which follow from theorems about AR($\mathfrak{M}$)-spaces and ANR($\mathfrak{M}$)-spaces and the theorems above.

(2.1) *Every r-image of an* AR *is an* AR *and every r-image of an* ANR *is an* ANR.

This follows by (1.1) and I,(1.9).

(2.2) *Every polyhedron is an* ANR.

This follows by IV,(6.2).

(2.3) *$X \in$* AR *if and only if $X \in$* ANR *and X is contractible in itself.*

This follows by IV,(9.1).

(2.4) *All the homology, cohomology, homotopy and cohomotopy groups of an* AR *are trivial.*

This is a special case of IV,(2.4).

(2.5) *Every neighborhood retract of an* ANR *is an* ANR.

This follows by IV,(3.4), taking into account that every neighborhood retract of a space is closed in this space.

(2.6) *Every* ANR *is locally contractible. In particular, $X \in$* ANR *implies $X \in \mathbf{LC}^{\infty}$.*

This follows by IV,(3.3), and I,(17.2).

It follows at once that

(2.7) *Every* ANR-*space has only a finite number of components.*

(2.8) *Every map of an* AR *into itself has a fixed point.*

This follows by (1.1) and I,(7.2), and from the fact that Q^ω has the fixed-point property.

(2.9) *The union of two* AR (*resp.* ANR)*-spaces the common part of which is an* AR (*resp.* ANR) *space is itself an* AR (*resp.* ANR) *space.*

This follows by IV,(6.1), (i) and (ii).

(2.10) *Each open subset of an* ANR *is an* ANR($\mathfrak{M}$).

This is a special case of First Theorem of Hanner (IV,(10.1)).

(2.11) *Each component of an* ANR *is an* ANR.

This follows from (2.10) since each component of a locally connected compactum X is open and closed in X.

(2.12) *If the union of two compacta and their common part are both* AR (*resp.* ANR) *spaces, then each of these compacta is an* AR (*resp.* ANR) *space.*

This follows from IV,(6.1), (iii) and (iv).

(2.13) *The Cartesian product of a sequence of spaces is an* AR*-space if and only if every its factor is an* AR*-space.*

This is a special case of IV,(7.1).

(2.14) *The Cartesian product of a sequence of spaces is an* ANR*-space if and only if all the factors are* ANR*-spaces and almost all factors are* AR*-spaces.*

This is a special case of IV,(7.2).

(2.15) *A compactum X is an* ANR *if and only if every point of X has neighborhoods (even arbitrarily small neighborhoods) which are* ANR($\mathfrak{M}$)*-spaces.*

Proof. It follows by First Theorem of Hanner (IV,(10.1)) that every point of an ANR($\mathfrak{M}$) has arbitrarily small neighborhoods being ANR($\mathfrak{M}$) -sets. To show the converse, suppose that X is a compactum such that each $x \in X$ has a neighborhood U_x which is an ANR($\mathfrak{M}$)-set. The interiors of these neighborhoods U_x are also ANR($\mathfrak{M}$)-sets which are open. These interiors cover X and since X is compact, a finite number of ANR($\mathfrak{M}$)-sets cover X. From Second Theorem of Hanner (IV,(10.2)) it follows that $X \in$ ANR($\mathfrak{M}$). Since X is compact, we infer that $X \in$ ANR.

From (2.15) we see that the property of being an ANR-space is a consequence of the local structure of a space. However let us mention that there exist ANR-spaces containing points for which do not exist neighborhoods being ANR-sets with arbitrarily small diameter. Such an example will be given later on (VII, Section 4).

(2.16) *If* $X \in \mathrm{AR}$ *and* X *is a subset of* $X' \in \mathfrak{M}$, *then every map* $f\colon X \to Y$ *has a continuous extension* $f'\colon X' \to Y$.

(2.17) *If* $X \in \mathrm{ANR}$ *and* X *is a subset of* $X' \in \mathfrak{M}$, *then there is a neighborhood* U *of* X *in* X' *such that every map* $f\colon X \to Y$ *has a continuous extension* $f'\colon U \to Y$.

(2.18) *A compactum* Y *is an* AR-*space if and only if for each closed subset* X *of a metrizable space* X' *and each map* $f\colon X \to Y$ *there is a continuous extension* f' *of* f *mapping* X' *into* Y.

(2.19) *A compactum* Y *is an* ANR-*space if and only if for each closed subset* X *of a metrizable space* X' *and each map* $f\colon X \to Y$ *there is a neighborhood* U *of* X *in* X' *such that* f *has a continuous extension* $f'\colon U \to Y$.

(2.20) *If an* ANR-*set* X *lies in the Euclidean n-space* E^n, *then* $E^n - X$ *has only a finite number of components.*

This is an immediate consequence of I,(4.2).

(2.21) *If an* AR-*set* X *lies in the Euclidean n-space* E^n, *then the set* $E^n - X$ *is connected for* $n > 1$ *and it has two components, for* $n = 1$.

This is an immediate consequence of I,(3.7), and I,(3.8).

3. A theorem on extension of maps. We prove the following

(3.1) THEOREM. *If* $Y \in \mathrm{ANR}$ *and* $\varepsilon > 0$, *then there is an* $\eta > 0$ *such that for every closed subset* X_0 *of a metric space* X *and for all maps* $f_1, f_2 \in Y^{X_0}$ *with* $\varrho(f_1, f_2) \leqslant \eta$ *if* f_1 *has an extension* $f_1' \in Y^X$, *then* f_2 *has an extension* $f_2' \in Y^X$ *such that* $\varrho(f_1', f_2') \leqslant \varepsilon$.

Proof. Without loss of generality we may assume $Y \subset Q^\omega \subset E^\omega$. Since $Y \in \mathrm{ANR}$, there is a neighborhood U of Y in E^ω and a retraction $r\colon U \to Y$. Since Y is compact, there is an $\eta > 0$ such that $\eta \leqslant \frac{1}{2}\varepsilon$, that the generalized ball

$$K = \{y \in E^\omega;\ \varrho(y, Y) \leqslant \eta\}$$

lies in the set U and that $\varrho\big(y, r(y)\big) \leqslant \frac{1}{2}\varepsilon$ for all $y \in K$ (since $r(y) = y$ for $y \in Y$). If we set

$$\varphi(x) = f_1(x) - f_2(x) \quad \text{for every} \quad x \in X_0,$$

we infer that the range of φ is a subset of the ball $K_0 = \{y \in E^\omega;\ \varrho(y, 0) \leqslant \eta\}$, because $\varrho(f_1, f_2) \leqslant \eta$. Since K_0 is convex, it follows from the generalized theorem of Tietze (III,(7.1)) that there is a continuous extension $\varphi'\colon X \to K_0$ of the map φ. Consequently, if f_1' is a given extension of f_1, then

$$\varrho\big(f_1'(x), f_1'(x) - \varphi'(x)\big) = \varrho\big(0, \varphi'(x)\big) \leqslant \eta \quad \text{for} \quad x \in X, \tag{3.2}$$

and therefore $f_1'(x) - \varphi'(x) \in K$. Hence the formula

$$f_2'(x) = r\big(f_1'(x) - \varphi'(x)\big)$$

defines a map $f_2'\colon X \to Y$ which extends to f_2 because, for $x \in X_0$, we have $f_2'(x) = r\big(f_2(x)\big) = f_2(x)$. Moreover, from equation (3.2) and the inequality $\varrho\big(y, r(y)\big) \leqslant \frac{1}{2}\varepsilon$ we infer that $\varrho\big(f_1'(x), f_2'(x)\big) \leqslant \varepsilon$ for all $x \in X$. Thus the proof is complete.

(3.3) COROLLARY. *Let Y_0 be a closed subset of a space Y. If $Y, Y_0 \in \mathrm{ANR}$, then there exists a neighborhood U of Y_0 in Y and a homotopy $\{\omega_t\} \subset Y^U$ such that*

(i) $\omega_0(y) = y$ *for every point* $y \in U$,

(ii) $\omega_t(y) = y$ *for every point* $y \in Y_0$ *and* $0 \leqslant t \leqslant 1$,

(iii) ω_1 *is a retraction of* U *to* Y_0.

Proof. Since $Y_0 \in \mathrm{ANR}$, there exists a map r retracting a neighborhood U_0 of Y_0 in Y to Y_0. Now let U be a neighborhood of Y_0 contained in U_0. Setting

$$X = U \times \langle 0,1 \rangle,$$
$$X_0 = Y_0 \times \langle 0,1 \rangle \cup U \times (0) \cup U \times (1),$$

let us consider two maps $f_1, f_2 \in Y^{X_0}$ given by the formulas

$$f_1(y,t) = y \quad \text{for every} \quad (y,t) \in X_0,$$
$$f_2(y,t) = \begin{cases} y & \text{for every} \quad (y,t) \in Y_0 \times \langle 0,1 \rangle \cup U \times (0), \\ r(y) & \text{for every} \quad (y,t) \in U \times (1). \end{cases}$$

Now let η be a positive number. It is evident that the neighborhood U of Y_0 in Y may be chosen so that $\varrho(f_1, f_2) < \eta$. Since X_0 is closed in X and since f_1 has a continuous extension f_1' to X with values in Y (given by the same formula $f_1'(y,t) = y$), we infer by Theorem (3.1) that there exists a continuous extension $f_2'\colon X \to Y$ of f_2. Setting

$$\omega_t(y) = f_2'(y,t) \quad \text{for} \quad (y,t) \in U \times \langle 0,1 \rangle,$$

we get a homotopy $\{\omega_t\} \subset Y^U$ satisfying conditions (i), (ii), and (iii).

4. ANR-spaces and polyhedra. Now we shall discuss some simple relations between ANR-spaces and polyhedra. Consider, for every natural number m, the map φ_m of the Hilbert cube Q^ω given by

$$\varphi_m\big((x_1, x_2, \ldots)\big) = (x_1, x_2, \ldots, x_m, 0, 0, \ldots).$$

Clearly φ_m is a retraction of Q^ω to the m-dimensional cube Q^m consisting of all points $x \in Q^\omega$ of the form $x = (x_1, x_2, \ldots, x_m, 0, 0, \ldots)$. It is clear that there is a sequence $\{\varepsilon_m\}$ of positive numbers converging to zero such that

$$\varrho\big(\varphi_m(x), x\big) \leqslant \varepsilon_m \quad \text{for every} \quad x \in Q^\omega$$

(for example, $\varepsilon_m = \sqrt{\sum_{i=m+1}^{\infty}(1/i^2)}$). A set $A \dot{\subset} Q^\omega$ will be called a *prism* in Q^ω provided that there are a natural number m and a polyhedron $P \subset Q^m$ such that $A = \varphi_m^{-1}(P)$. The polyhedron P is called a *base* of the prism A; observe that P is a deformation retract of A and that A is homeomorphic to the Cartesian product $P \times Q^\omega$. Since P and Q^ω are ANR-spaces, we infer that

(4.1) LEMMA. *Every prism in Q^ω is an* ANR-*space.*

Suppose that $P \subset Q^m$ is a base of the prism A and let k be a natural number. The set $P_k \subset Q^{m+k}$ of all points $x \in Q^{m+k}$ for which $\varphi_m(x) \in P$ is a polyhedron (homeomorphic to $P \times Q^k$) and is also a base of A. We infer the following

(4.2) LEMMA. *A prism A' in Q^ω is contained in the prism A in Q^ω if and only if there are number m and a base $P' \subset Q^m$ of A' contained in a base $P \subset Q^m$ of A.*

Now let us prove the following

(4.3) LEMMA. *Let X be a compactum in Q^ω and let $\varepsilon > 0$ be a given number. Then there is a prism $A \subset Q^\omega$ which is a neighborhood of X in Q^ω and such that $\varrho(x, X) < \varepsilon$ for every point $x \in A$.*

Proof. If $X = 0$, we may assume $A = 0$. If $X \neq 0$, then consider an index m such that $\varepsilon_m < \frac{1}{3}\varepsilon$. The set $X_m = \varphi_m(X)$ is a non-empty compactum lying in Q^m. Consider a polyhedron $P_m \subset Q^m$ which is a neighborhood of X_m in Q^m such that

$$\varrho(x, X_m) < \varepsilon_m \quad \text{for every} \quad x \in P_m.$$

Since φ_m is continuous, the prism $A = \varphi_m^{-1}(P_m)$ is a neighborhood of X in Q^ω. Moreover, $\varphi_m(x) \in P_m$ for $x \in A$ and

$$\varrho(x, X_m) \leqslant \varrho\big(x, \varphi_m(x)\big) + \varrho\big(\varphi_m(x), X_m\big) < 2\varepsilon_m$$

(because $\varrho\big(x, \varphi_m(x)\big) < \varepsilon_m$ by the definition of ε_m). Since the distance from a point $x \in X_m$ to X is $\leqslant \varepsilon_m$, we infer that $\varrho(x, X) < 3\varepsilon_m < \varepsilon$ for $x \in A$, and this completes the proof.

New let us prove the following

(4.4) THEOREM. ANR-*spaces coincide with the r-images of prisms in Q^ω.*

Proof. Since a prism in Q^ω is an ANR-space, every r-image of a prism is also an ANR. To show the converse, suppose $X \in$ ANR; we have to show that X is the r-image of a prism. Without loss of generality we may assume $X \subset Q^\omega$. Since $X \in$ ANR, there is a neighborhood U of X in Q^ω and a retraction $r\colon U \to X$. By Lemma (4.3) there is a prism A in Q^ω which is a neighborhood of X and lies in U, i. e., $X \subset A \subset U$. The restriction $r|A$ is a retraction of A to X and so X is the r-image of a prism in Q^ω.

(4.5) COROLLARY. *For every space* $X \in$ ANR *there is a polyhedron* P *which homotopically dominates* X, *i. e.* $P \underset{h}{\geqslant} X$.

Proof. The theorem asserts that there is a prism A in Q^ω such that $A \underset{r}{\geqslant} X$, and consequently $A \underset{h}{\geqslant} X$. On the other hand, if P is a base of A, we know that P is a deformation retract of A. It follows by I,(14.2), that $A \underset{h}{\simeq} P$ and consequently $A \underset{h}{=} P$, whence $P \underset{h}{\geqslant} X$.

(4.6) COROLLARY. *For each* ANR-*space* X, *there is a polyhedron* P *such that the groups of homology, cohomology, homotopy and cohomotopy of* X *are r-images of the corresponding groups of* P.

Thus the homological and homotopic properties of ANR-spaces show much similarity with the analogous properties of polyhedra. One can prove also that several homological properties of position of compacta in ANR-spaces show a close similarity to analogous properties of compacta which lie in polyhedra. A result in this direction has been recently found by A. Deleanu [95].

5. Approximation of ANR-sets by two prisms. A more precise relation between the algebraic properties of polyhedra and of ANR-spaces is a consequence of the following

(5.1) THEOREM. *For every* ANR-*space* X *lying in the Hilbert cube* Q^ω *there are two prisms* A *and* A' *such that*

(1) $X \subset A' \subset A$,

(2) *there is a retraction* $r: A \to X$,

(3) *the map* $f: A' \to A$ *given by* $f(x) = r(x)$ *for* $x \in A'$ *is homotopic to the inclusion* $i: A' \to A$.

Proof. As in the proof of the previous theorem, we may find a prism A in Q^ω which is a neighborhood of X in Q^ω and satisfies condition (2) of this theorem. Since r is the identity on X, we may find a neighborhood U of X which is contained in A and satisfies the condition

$$(1-t)x+t\cdot r(x) \in A \quad \text{for all} \quad x \in U \text{ and } t \in \langle 0,1\rangle.$$

Then by Lemma (4.3) there is a prism A' for which $X \subset A' \subset U$. If we set

$$f(x) = r(x) \quad \text{for} \quad x \in A'$$

and

$$f_t(x) = t\cdot r(x)+(1-t)\cdot x \quad \text{for} \quad x \in A' \text{ and } t \in \langle 0,1\rangle,$$

then we obtain a map $f: A' \to A$ and a continuous family $\{f_t\}$ of maps joining (in the space $A^{A'}$) the map $f_0 = i$ to the map $f_1 = f$. This completes the proof.

(5.2) THEOREM. *For every space* $X \in$ ANR *there are two polyhedra* P *and* $P' \subset P$ *such that for each index* $n \geqslant 0$ *and every group of coefficients* $\mathfrak{A}$ *the homology group* $H_n(X, \mathfrak{A})$ *is isomorphic to the image* $\mathrm{Im}(j_*)$ *of the homomorphism*

$$j_*\colon H_n(P', \mathfrak{A}) \to H_n(P, \mathfrak{A})$$

induced by the inclusion $j\colon P' \to P$.

Proof. We may assume that X is a subset of the Hilbert cube Q^ω. By Theorem (5.1) we can find prisms A and A' such that $X \subset A' \subset A \subset Q^\omega$ and a retraction $r\colon A \to X$ such that the map $f\colon A' \to A$, given by the formula $f(x) = r(x)$ (for $x \in A'$), is homotopic to the inclusion $i\colon A' \to A$. Applying Lemma (4.2), we may assume that the base P' of A' is contained in the base P of A. Since the values of f are in X, it follows by II,(4.13), that the group $H_n(X, \mathfrak{A})$ is isomorphic to $H_n(A', \mathfrak{A})|\mathrm{Ker}(i_*)$, where $i_*\colon H_n(A', \mathfrak{A}) \to H_n(A, \mathfrak{A})$ is the homomorphism induced by the inclusion map $i\colon A' \to A$.

It remains to show that $H_n(A', \mathfrak{A})|\mathrm{Ker}(i_*)$ is isomorphic to $H_n(P', \mathfrak{A})|\mathrm{Ker}(j_*)$. To this end we observe first that the projection φ of A onto its base P is homotopic (in the space A^A) to the identity map of A. Moreover, we may choose this homotopy so, that it simultaneously deforms the prism A' onto its base P'. It readily follows that the isomorphism

$$\varphi'_*\colon H_n(A', \mathfrak{A}) \to H_n(P', \mathfrak{A})$$

induced by the projection φ' of $A' \to P'$ maps the group $\mathrm{Ker}(i_*)$ onto $\mathrm{Ker}(j_*)$ and from this we infer that $H_n(A', \mathfrak{A})|\mathrm{Ker}(i_*)$ is isomorphic to $H_n(P', \mathfrak{A})|\mathrm{Ker}(j_*)$. The proof of Theorem (5.2) is thus complete.

This theorem states that the homology groups of an ANR are determined by the homology groups of a polyhedron and the kernels of the homomorphisms induced by the inclusion map of this polyhedron into a larger one. Since, for polyhedra, the homology groups as defined by Čech or Vietoris and also the singular groups are all isomorphic to the groups determined by any simplicial triangulation of the polyhedron, we have ([211], p. 18, [180], p. 96, and [222], p. 30) the following

(5.3) COROLLARY. *For* ANR-*spaces, the homology groups of Čech or of Vietoris are isomorphic to the corresponding singular groups.*

6. Embedding of compacta in AR-spaces. According to Urysohn's theorem every compactum may be topologically embedded in the Hilbert cube, which is an AR-space. Another embedding of compacta in AR-spaces is given by the following ([38], p. 240)

(6.1) THEOREM. *For each compactum* X *there is an* AR-*space* X' *and a homeomorphism* h *of* X *onto a subset* $h(X)$ *of* X' *for which* $X' - h(X)$ *is a polytope.*

We may assume that $X \neq 0$. Let us observe first that, since $X'-h(X)$ is separable, any triangulation of $X'-h(X)$ can have at most a countable number of simplexes (III,(4.5)). This triangulation is finite only if the set $X'-h(X)$ is compact. In this case, we can obtain an AR-set X'' for which $X''-h(X)$ is non-compact by attaching to X' a segment having only one point in common with X'. Therefore we can assume that the triangulation is countable, and also — by III,(5.1) — that it is a null-triangulation. Consequently, since the case $X = 0$ is trivial, we may (equivalenty) reformulate Theorem (6.1) as follows:

(6.2) THEOREM. *Each compactum $X \neq 0$ is homeomorphic to a subset X' of an* AR*-space X'' such that $X''-X'$ is a polytope having a null-triangulation.*

Before giving the proof we establish the following

(6.3) LEMMA. *Let $\mathscr{T}$ be a triangulation of a polyhedron X and let X_1, X_2 be two disjoint subpolyhedra of this triangulation and let I be an interval $\langle a, \beta\rangle$. Then there is a triangulation $\mathscr{T}'$ of $X \times I$ and a retraction f of $X \times I$ linear on each simplex of $\mathscr{T}'$ and given by the formula*

$$f(x, t) = \big(x, \theta(x, t)\big) \qquad (\text{for } x \in X \text{ and } t \in I),$$

where $\theta(x, t)$ is a function with values in I such that

(1) $\theta(x, t) = a$ *if* $x \in X_1$ *or* $t = a$,

(2) $\theta(x, t) = t$ *if* $x \in X_2$.

Proof. We easily see that $X \times I$ is a polyhedron. Indeed, one triangulation $\mathscr{T}'$ of $X \times I$ consists of some simplexes σ with the vertices (x_0, t_0), $\ldots, (x_n, t_n)$ where the points $x_0, \ldots, x_n$, which need not be distinct, are vertices of a simplex of the triangulation $\mathscr{T}$ and each t_i is either a or β. We define a map f on the set of vertices of $\mathscr{T}'$ by the formulas

$$f\big((x_i, t_i)\big) = \begin{cases} (x_i, a) & \text{for} \quad x_i \in X_1, \\ (x_i, t_i) & \text{for} \quad x_i \in X - X_1. \end{cases}$$

This map f is then extended to a map linear on the simplexes of $\mathscr{T}'$; we also denote the extension by f. Then f is the required retraction since $ff = f$. This completes the proof of the lemma and now we are ready for the

Proof of Theorem (6.1). We assume, without loss of generality, that $X \subset Q^\omega$. As before let φ_m be the projection of Q^ω onto Q^m given by the formula

$$\varphi_m(\{x_n\}) = (x_1, x_2, \ldots, x_m, 0, 0, \ldots).$$

For each natural number m, let P_m be a polyhedron in Q^m which is a neighborhood in Q^m of the set $\varphi_m(X)$ and let

$$Z_m = \varphi_m^{-1}(P_m).$$

We see that Z_m is a prism which is a neighborhood of X in Q^ω. It is clear that we may inductively choose the polyhedra P_m, so that

$$Z_1 = Q^\omega, \tag{6.4}$$

$$Z_{m+1} \subset \operatorname{Int} Z_m, \tag{6.5}$$

$$\bigcap_{m=1}^{\infty} Z_m = X. \tag{6.6}$$

From (6.5) and the definitions of φ_m and Z_m it follows that the polyhedra $\varphi_m(Z_{m+1})$ and $\varphi_m(\overline{Q^\omega - Z_m}) = \overline{Q^m - P_m}$ are disjoint. From Lemma (6.3) we conclude that there is a map θ_m of $Q^m \times I_m$ into I_m, where $I_m = \langle 0, 1/(m+1)\rangle$, such that $\theta_m(x_1, \ldots, x_{m+1}, 0, \ldots) = 0$ for $(x_1, \ldots, x_m, 0, \ldots) \in \overline{Q^m - P_m}$, $\theta_m(x_1, \ldots, x_{m+1}, 0, \ldots) = x_{m+1}$ for $(x_1, \ldots, x_m, 0, \ldots) \in \varphi_m(Z_{m+1})$. Moreover, the map f_m defined on the polyhedron $Q^{m+1} = Q^m \times I_m$ by the formula

$$f_m\big((x_1, x_2, \ldots, x_m, x_{m+1}, 0, 0, \ldots)\big) = \big(x_1, \ldots, x_m, \theta(x_1, \ldots, x_{m+1}), 0, 0, \ldots\big)$$

is a retraction linear on every simplex of a triangulation $\mathcal{T}'$ of Q^{m+1} and such that

$$f_m(x) = \varphi_m(x) \quad \text{if} \quad \varphi_m(x) \in \overline{Q^m - P_m}, \tag{6.7}$$

$$f_m(x) = x \quad \text{if} \quad \varphi_m(x) \in \varphi_m(Z_{m+1}). \tag{6.8}$$

From (6.4), (6.5), and (6.6) it follows that $Q^\omega - X$ is the union of the compacta $B_m = \overline{Z_m - Z_{m+1}}$ which satisfy the conditions

(6.9) $B_m \cap B_{m+1} = 0$ *for* $m = 1, 2, \ldots$ *and* $k = 2, 3, \ldots$

(6.10) $B_m \cap B_{m+1} = Z_{m+1} \cap (\overline{Q^\omega - Z_{m+1}})$ *for* $m = 1, 2, \ldots$

(6.11) *If U is a neighborhood of X in Q^ω, then for all but a finite number of indices m the set B_m lies in U.*

(6.12) *$\varphi_{m+1}(B_m)$ is a polyhedron contained in B_m for* $m = 1, 2, \ldots$

(6.13) $f_m[\varphi_{m+1}(B_m)] \subset \varphi_{m+1}(B_m)$.

From (6.10) it follows for any $x \in B_m \cap B_{m+1}$ that

$$\varphi_m\big(\varphi_{m+1}(x)\big) = \varphi_m(x) \in \varphi_m(Z_{m+1})$$

and

$$\varphi_{m+1}\big(\varphi_{m+2}(x)\big) = \varphi_{m+1}(x) \in \varphi_{m+1}(\overline{Q^\omega - Z_{m+1}}) = \overline{Q^{m+1} - P_{m+1}}.$$

Thus we infer from (6.7) and (6.8) that

$$f_m(\varphi_{m+1}(x)) = \varphi_{m+1}(x) = \varphi_{m+1}(\varphi_{m+2}(x)) = f_{m+1}(\varphi_{m+2}(x))$$
$$\text{for every point } x \in B_m \cap B_{m+1}.$$

Consequently, by means of (6.9) the formulas

$$f(x) = \begin{cases} f_m(\varphi_{m+1}(x)) & \text{for} \quad x \in B_m, \\ x & \text{for} \quad x \in X \end{cases}$$

define uniquely a function f on the whole cube Q^ω.

Relation (6.11) implies that f is continuous in $Q^\omega - X$. Moreover, for every point $x = \{x_n\}$ in B_m we have

$$f(x) = f_m(\varphi_{m+1}(x)) = f_m((x_1, \ldots, x_{m+1}, 0, 0, \ldots))$$
$$= (x_1, x_2, \ldots, x_m, \theta(x_1, x_2, \ldots, x_{m+1}), 0, 0, \ldots)$$

and hence

$$\varrho(x, f(x))^2 \leqslant \sum_{k=1}^{\infty} \frac{1}{(m+k)^2}.$$

This inequality together with the definition of f implies that f is continuous at every point of X.

By the definition of f the set $X'' = f(Q^\omega)$ is the union of X and of the sets $f(B_m) = f_m(\varphi_{m+1}(B_m))$. Moreover, (6.12) and (6.13) imply that $f_m(\varphi_{m+1}(B_m)) \subset \varphi_{m+1}(B_m) \subset B_m$ and, since f_m is linear on every simplex of a triangulation of Q^{m+1}, we infer that all sets $f(B_m) \subset B_m$ are polyhedra. Recalling (6.9) and (6.11), we conclude that the set $X'' - X = \bigcup_{m=1}^{\infty} f(B_m)$ is a polytope.

In order to complete the proof of (6.1) it is sufficient by (1.1) to show that f is a retraction of Q^ω, i.e. that $f(f(x)) = f(x)$ for all $x \in Q^\omega$. If $x \in X$, then $f(x) = x$ so that $f(f(x)) = f(x)$. If $x \in Q^\omega - X$, then there is an index m with $x \in B_m$. Then $f(x) = f_m(\varphi_{m+1}(x)) \in \varphi_{m+1}(B_m) \subset B_m$. Moreover, the map f_m, which is a retraction, satisfies the condition $f_m f_m = f_m$ and hence for every point $x \in B_m$ we have

$$f(f(x)) = f_m(\varphi_{m+1}(f(x))) = f_m(f(x)) = f_m(f_m(\varphi_{m+1}(x)))$$
$$= f_m(\varphi_{m+1}(x)) = f(x).$$

Thus we have shown that f is a retraction of Q^ω to $X'' = f(Q^\omega)$. It follows by (1.1) that $X'' \in \mathrm{AR}$. This completes the proof of (6.1), and consequently also of (6.2).

7. A characterization of ANR-**spaces.** Let us prove the following theorem:

(7.1) THEOREM. *A compactum Y is an* ANR-*space if and only if it satisfies the following condition:*

(I) *For each metrizable polytope W having a null-triangulation $\mathscr{T}$ and for each uniformly continuous map f_0 of the 0-skeleton W_0 of W (relative to $\mathscr{T}$) into Y there is a uniformly continuous extension f of f_0 to an almost full subpolytope W_1 of W (relative to $\mathscr{T}$).*

Proof. We can assume that $Y \subset Q^\omega$. In order to show that condition (I) is necessary let us prove the following stronger proposition:

(7.2) *If $Y \in$ ANR, then for every metrizable polytope W and for each triangulation $\mathscr{T}$ of W, with diameters of simplexes converging to zero, and for each subpolytope W' of W (relative to $\mathscr{T}$) which contains W^0, every uniformly continuous map $f'\colon W' \to Y$ has a uniformly continuous extension which maps an almost full subpolytope W_1 of W (relative to $\mathscr{T}$) into Y.*

Since $Y \in$ ANR, there is a retraction r of a closed neighborhood U of Y in Q^ω to Y. Let σ be an arbitrary simplex of $\mathscr{T}$. For the purposes of this part of the proof, let $f_0 = f'$. We denote by A_σ the smallest convex subset of Q^ω which contains the set $f_0(\sigma \cap W')$. It is clear that

$$(7.3) \qquad \sigma' \subset \sigma \quad \text{implies} \quad A_{\sigma'} \subset A_\sigma \quad \text{for every } \sigma, \sigma' \in \mathscr{T}.$$

Let $\mathscr{T}_1$ be the collection of all simplexes $\sigma \in \mathscr{T}$ such that $A_\sigma \subset U$. Since the map f_0 is uniformly continuous, the diameters of the sets A_σ converge to zero and hence almost all simplexes of $\mathscr{T}$ belong to $\mathscr{T}_1$. Let W_1 denote the subpolytope of W having $\mathscr{T}_1$ as its triangulation and let $\mathscr{T}(k)$ denote the collection of all simplexes of dimension $\leqslant k$ which belong to $\mathscr{T}_1$. Setting

$$W'_k = W' \cup \bigcup_\sigma \{\sigma;\ \sigma \in \mathscr{T}(k)\},$$

we have $W'_0 = W'$ and hence the map f_0 is defined on the whole set W'_0 with its values in $Y \subset U$. Now let us assume that for an index k there is a uniformly continuous extension $f_k\colon W'_k \to U$ of the map f_0 satisfying the condition $f_k(\sigma \cap W'_k) \subset A_\sigma$ for every $\sigma \in \mathscr{T}$. This is obvious if $k = 0$.

Now consider a simplex σ of $\mathscr{T}(k+1)$ which is not contained in $\mathscr{T}(k)$. Then f_k is defined on the boundary $\sigma^\bullet$ of σ and we have $f_k(\sigma^\bullet) \subset A_\sigma$. Since A_σ is a convex subset of Q^ω, we infer that the map $f_k|\sigma^\bullet$ can be extended to a continuous map of the whole simplex σ into the set A_σ. Thus we may extend f_k onto each simplex σ of $\mathscr{T}(k+1)$ which is not already contained in $\mathscr{T}(k)$ and so we obtain a map f_{k+1} defined on the whole set W'_{k+1}.

Recalling that the diameters of the sets A_σ converge to zero, we infer that the map f_{k+1} is uniformly continuous. Moreover, since f_{k+1} extends f_k which is an extension of f_0, it follows that f_{k+1} extends f_0 and (7.3) implies that f_{k+1} satisfies the condition

$$f_{k+1}(\sigma \cap W'_{k+1}) \subset A_\sigma \quad \text{for each} \quad \sigma \in \mathscr{T}.$$

Setting $f(x) = r(f_k(x))$ for each $x \in W'_k$, $k = 0, 1, \ldots$, we see at once that the function f is a uniformly continuous map of W_1 into Y which extends f_0. Thus the proof of (7.2) is complete.

In order to finish the proof of Theorem (7.1) it remains to show that a compactum $Y \neq 0$ satisfying (I) is an ANR. By Theorem (6.2) there is a set $Y'' \in$ AR such that $Y \subset Y'' \subset Q^\omega$ and such that $W = Y'' - Y$ is a polytope with a countable triangulation $\mathscr{T}$ whose simplexes have diameters converging to zero. Let us assign to each vertex x of $\mathscr{T}$ a point $f_0(x) \in Y$ such that $\varrho(f_0(x), x) = \varrho(x, Y)$. It is plain that f_0 is a uniformly continuous map of the 0-skeleton W^0 of $\mathscr{T}$ into Y. By our hypothesis there exists an almost full subpolytope W' of W and a uniformly continuous map f': $W' \to Y$ such that $f'(x) = f_0(x)$ for every $x \in W^0 \cap W'$. It is clear that the set $W' \cup Y$ is a neighborhood in Y'' of Y. Setting

$$r(x) = \begin{cases} x & \text{for} \quad x \in Y, \\ f(x) & \text{for} \quad x \in W', \end{cases}$$

we readily see that the map r of the set $W' \cup Y$ into Y defined by these formulas is a retraction. Thus Y is a neighborhood retract in the AR-space Y'' and we infer by (2.5) that $Y \in$ ANR. The proof of Theorem (7.1) is thus complete.

8. Condition of Lefschetz. A positive number η is said to satisfy the *condition of Lefschetz* for a pair of spaces (Y, Y_0) and for $\varepsilon > 0$ provided that for every polyhedron W, every triangulation $\mathscr{T}$ of W, and every subpolyhedron W' of this triangulation containing all vertices of $\mathscr{T}$, every map f': $W' \to Y_0$, such that $\delta[f'(\sigma \cap W')] \leqslant \eta$ for each simplex $\sigma \in \mathscr{T}$, has a continuous extension f: $W \to Y$ such that $\delta[f(\sigma)] \leqslant \varepsilon$ for each simplex $\sigma \in \mathscr{T}$.

Let us prove the following theorem (due to S. Lefschetz [210], [211], [213], and [214], p. 83):

(8.1) THEOREM. *A compactum Y is an* ANR*-space if and only if there exists for every $\varepsilon > 0$ a number $\eta > 0$ satisfying the condition of Lefschetz for (Y, Y) and ε.*

Proof. We can assume that the compactum $Y \in$ ANR is a subset of the Hilbert cube $Q^\omega \subset E^\omega$. Since $Y \in$ ANR, we see at once that for every

$\varepsilon > 0$ there exists a positive $\eta \leqslant \frac{1}{4}\varepsilon$ such that Y is a retract of the generalized ball

(8.2) $$K = \{y \in E^{\omega};\ \varrho(y, Y) \leqslant \eta\}$$

under a retraction $r\colon K \to Y$ satisfying the condition

(8.3) $$\varrho\big(y, r(y)\big) \leqslant \tfrac{1}{4}\varepsilon \quad \text{for every point } y \in K.$$

Now let us consider a map $f'\colon W' \to Y$ of a subpolyhedron W' of W (in the triangulation $\mathscr{T}$), satisfying the condition

$$\delta[f'(\sigma \cap W')] \leqslant \eta \quad \text{for each simplex} \quad \sigma \in \mathscr{T}.$$

Let A_σ denote the convex hull (in the space E^{ω}) of the set $f'(\sigma \cap W')$. Then

(8.4) $$\delta(A_\sigma) \leqslant \eta \quad \text{for each simplex } \sigma \in \mathscr{T}.$$

Consider now the polyhedron W_k defined as the union of the polyhedron W' and of all k-dimensional simplexes $\sigma_1^k, \sigma_2^k, \ldots, \sigma_{n_k}^k$ of the triangulation $\mathscr{T}$. Since W' contains all vertices of the triangulation $\mathscr{T}$, the polyhedron W_0 coincides with W' and consequently $f'\colon W_0 \to Y \subset E^{\omega}$. Setting $f_0 = f'$, let us assume that for an index $k \geqslant 0$ a map $f_k\colon W_k \to E^{\omega}$ is already defined, satisfying the condition

(8.5) $$f_k(\sigma \cap W_k) \subset A_\sigma \quad \text{for each simplex } \sigma \in \mathscr{T}.$$

It is clear that this condition is satisfied for $k = 0$. Now we define a map $f_{k+1}\colon W_{k+1} \to E^{\omega}$ in the following manner: If $\sigma = \sigma_i^{k+1}$ is a simplex of the triangulation $\mathscr{T}$ lying in W_{k+1}, but not in W', then f_k is already defined on the boundary $\sigma^{\bullet}$ of the simplex σ and the condition $f_k(\sigma^{\bullet}) \subset A_\sigma$ is satisfied. Since A_σ is a convex subset of the space E^{ω}, we infer that the restriction $f_k|\sigma^{\bullet}$ has a continuous extension which is a map of σ into the set A_σ. If we define f_{k+1} on σ as such an extension of the map $f_k|\sigma^{\bullet}$, we get a continuous map $f_{k+1}\colon W_{k+1} \to E^{\omega}$ and it is cleat that $f_{k+1}(\sigma \cap W_{k+1}) \subset A_\sigma$ for every simplex $\sigma \in \mathscr{T}$.

Let m denote the dimension of the polyhedron W. Then $W_m = W$ and the map $f_m\colon W \to E^{\omega}$ satisfies condition (8.5) with $k = m$. By this construction, the values of the map f_m at the vertices of the triangulation $\mathscr{T}$ belong to Y. We conclude, by (8.2), (8.4), and (8.5), that $f_m(W) \subset K$. Moreover, for $x \in W'$ the map f_m is the same as the map f_0 and consequently $f_m(x) \in Y$. Setting $f = rf_m$, we get a continuous extension $f\colon W \to Y$ of the map f_0. Moreover, we infer by (8.3), (8.4), and (8.5) that for every simplex $\sigma \in \mathscr{T}$ the diameter of the set $f(\sigma)$ is less than or equal to ε. Thus the necessity of the condition of Lefschetz is proved.

Let us pass to the proof that the condition of Lefschetz is also sufficient. By (6.2) we can assume that the compactum Y is contained in a space $\hat{Y} \in AR$ such that the set $\hat{Y} - Y$ is a polytope having a null-triangulation $\hat{\mathscr{T}}$. It follows that each neighborhood of Y in the space $\hat{Y}$ contains almost all simplexes of the triangulation $\hat{\mathscr{T}}$. Applying the condition of Lefschetz, we readily infer that there exist two decreasing sequences of positive numbers $\{\eta_n\}$ and $\{\xi_n\}$ converging to zero and such that

(8.6) $$\xi_n \leqslant \tfrac{1}{3}\eta_n \quad \text{for every} \quad n = 1, 2, \ldots,$$

and that for every polyhedron W with a triangulation $\mathscr{T}$, and for every subpolyhedron W' of this triangulation, containing all vertices of $\mathscr{T}$ we have:

(8.7) *If a map $f_0: W' \to Y$ satisfies the condition $\delta[f_0(\sigma \cap W')] \leqslant \eta_n$ for each simplex $\sigma \in \mathscr{T}$, then there exists a continuous extension $f: W \to Y$ of f_0 such that $\delta[f(\sigma)] \leqslant 1/(n+1)$ for every simplex σ of the triangulation $\mathscr{T}$.*

(8.8) *If a map f_0: $W' \to Y$ satisfies the condition $\delta[f_0(\sigma \cap W')] \leqslant \xi_n$ for each simplex $\sigma \in \mathscr{T}$, then there exists a continuous extension f: $W \to Y$ of f_0 such that $\delta[f(\sigma)] \leqslant \frac{1}{3}\eta_n$ for every simplex σ of the triangulation $\mathscr{T}$.*

Now let us assign to every vertex x of the triangulation $\hat{\mathscr{T}}$ a point $\varphi(x) \in Y$ such that

(8.9) $$\varrho\big(x, \varphi(x)\big) = \varrho(x, Y).$$

Let us denote by K_n the generalized ball defined by the formula

(8.10) $$K_n = \{y \in \hat{Y};\ \varrho(y, Y) \leqslant \tfrac{1}{3}\xi_n\}.$$

In particular, the ball K_1 is a neighborhood of the set Y in the space $\hat{Y}$, and consequently almost all simplexes of the triangulation $\hat{\mathscr{T}}$ are subsets of K_1 with diameters $\leqslant \frac{1}{3}\xi_1$. Let $\mathscr{T}_0$ denote the collection of all simplexes $\sigma \in \hat{\mathscr{T}}$ satisfying these conditions and let Y_0 denote the union of the set Y and of all simplexes of the triangulation $\mathscr{T}_0$. Then

(8.11) *The set $Y_0 \subset K_1$ is a neighborhood of Y in the space $\hat{Y}$.*

(8.12) *$Y_0 - Y$ is a polytope with the triangulation $\mathscr{T}_0$.*

(8.13) *For each simplex $\sigma \in \mathscr{T}_0$ the diameter of the set of values of φ on vertices of σ is $\leqslant \xi_1$.*

Manifestly, the polytope $Y_0 - Y$ can be represented as the union of a sequence $W_0, W_1, \ldots$ of polyhedra satisfying the following conditions:

(8.14) $W_0 = 0,\ W_n \subset W_{n+1} - \overline{Y_0 - W_{n+1}}$.

(8.15) *The set $Z_n = \overline{W_n - W_{n-1}}$ is contained in K_n for $n = 1, 2, \ldots$*

(8.16) *The simplexes of $\mathscr{T}_0$ lying in Z_n constitute a triangulation $\mathscr{T}_n$ of Z_u.*

(8.17) *If* $\sigma \epsilon \mathscr{T}_0$ *and* $\sigma \frown W_n \neq 0$, *then* $\sigma \frown \overline{Y_0 - W_{n+1}} = 0$.

(8.18) *If* $\sigma \epsilon \mathscr{T}_n$, *then* $\delta(\sigma) \leqslant \frac{1}{3}\xi_n$.

It follows by (8.9), (8.10), (8.15), (8.16), and (8.18) that

(8.19) *If* $\sigma \epsilon \mathscr{T}_n$, *then* φ *maps the set of vertices of* σ *onto a subset of* Y *with diameter* $\leqslant \xi_n$.

Let $W_n^\bullet$ denote the boundary of W_n in the space Y_0. It is evident that the set W_n is a polyhedron being the union of some simplexes of the triangulation $\mathscr{T}_n$. By (8.14) the polyhedra $W_0^\bullet, W_1^\bullet, W^\bullet, \ldots$ are disjoint one to another. Let M_n denote the set of all vertices of the triangulation $\mathscr{T}_n$. It follows by (8.19) that $\delta[\varphi(\sigma \frown M_n)] \leqslant \xi_n$ for each simplex $\sigma \epsilon \mathscr{T}_n$. We infer by (8.8) that there exists a continuous extension $\varphi_n \colon W_n^\bullet \to Y$ of the restriction $\varphi|(M_n \frown W_n^\bullet)$ satisfying the condition

(8.20) $\delta[\varphi_n(\sigma)] \leqslant \frac{1}{3}\eta_n$ for every simplex $\sigma \epsilon \mathscr{T}_n$ such that $\sigma \subsetneq W_n^\bullet$.

It follows by (8.14) that the boundary of the set $Z_n = \overline{W_n - W_{n-1}}$ (in the space Y_0) coincides with the union of the polyhedra $W_n^\bullet$ and $W_{n-1}^\bullet$. Setting

$$(8.21) \qquad f_n'(x) = \begin{cases} \varphi_n(x) & \text{for every point } x \epsilon W_n^\bullet, \\ \varphi_{n-1}(x) & \text{for every point } x \epsilon W_{n-1}^\bullet, \\ \varphi(x) & \text{for every point } x \epsilon M_n, \end{cases}$$

we obtain a map f_n' of the set $N_n = W_n^\bullet \smile W_{n-1}^\bullet \smile M_n$ into Y. If $\sigma \epsilon \mathscr{T}_n$, then we infer by (8.17) that either $\sigma \frown W_n^\bullet = 0$ or $\sigma \frown W_{n-1}^\bullet = 0$. In both these cases, (8.19), (8.20), and (8.6) imply that the diameter of the set $f_n'(\sigma \frown N_n)$ is $\leqslant \eta_{n-1}$. It follows by (8.7) that f_n' has a continuous extension

$$f_n \colon Z_n \to Y$$

satisfying the condition

$$(8.22) \qquad \delta[f_n(\sigma)] \leqslant \frac{1}{n} \quad \text{for every simplex } \sigma \epsilon \mathscr{T}_n.$$

It follows by (8.21) that

$$f_n(x) = f_{n+1}(x) = \varphi_n(x) \quad \text{for every point } x \epsilon W_n^\bullet = Z_n \frown Z_{n+1}.$$

Hence setting

$$r(x) = f_n(x) \quad \text{for every point } x \epsilon Z_n,\ n = 1, 2, \ldots,$$

$$r(x) = x \quad \text{for every point } x \epsilon Y,$$

we obtain a function $r\colon Y_0 \to Y$. It is clear that this function is continuous at every point of the set $Y_0 - Y$, and also at every point of the set

$Y-\overline{Y_0-Y}$. If, however, $x \in Y \cap \overline{Y_0-Y}$, then for every sufficiently small neighborhood U of x in the space Y_0 every point $x' \in U$ belongs either to Y or to one of the sets Z_n with arbitrarily great index n. In the first case, the distance $\varrho(r(x), r(x')) = \varrho(x, x')$ is arbitrarily small. In the second case, the point x' belongs to a simplex $\sigma \in \mathscr{T}_n$. Let p be one of the vertices of σ. Then we infer by (8.9), (8.10), (8.15), (8.18), and (8.22) that

$$\varrho(r(x), r(x')) = \varrho(x, f_n(x')) \leqslant \varrho(x, x') + \varrho(x', p) + \varrho(p, \varphi(p)) + \delta[f_n(\sigma)]$$
$$\leqslant \varrho(x, x') + \frac{1}{3}\xi_n + \frac{1}{3}\xi_n + \frac{1}{n};$$

hence, for x' lying in a sufficiently small neighborhood U of x, the distance $\varrho(r(x), r(x'))$ is arbitrarily small.

Thus we have proved that the function $r\colon Y_0 \to Y$ is continuous. Moreover, since Y_0 is a neighborhood of Y in the space $\hat{Y}$ and since $r(x) = x$ for every point $x \in Y$, we infer that r is a retraction and Y is a retract of its neighborhood Y_0 in the space Y. If we recall that $\hat{Y} \in \mathrm{AR}$, we infer by (2.5) that $Y \in \mathrm{ANR}$. Thus the proof of Theorem (8.1) is finished.

9. Matching of sets. Let X_1 and X_2 be two disjoint compacta and let φ be a map of a closed subset X_0 of X_1 into X_2. Then by the *matching* of X_1 and X_2 by φ ([30], p. 250) we understand the decomposition space ([203], p. 46) of the upper semicontinuous decomposition $\mathscr{D}$ of $X_1 \cup X_2$ into the following sets:

a) the individual points of the set $(X_1-X_0) \cup (X_2-\varphi(X_0))$,

b) the sets $(x) \cup \varphi^{-1}(x)$ with $x \in \varphi(X_0)$.

This decomposition space will be denoted by $X_1 \overset{\varphi}{\cup} X_2$; by elementary theorems on decompositions, it is a compactum and the function

$$\psi\colon X_1 \cup X_2 \to X_1 \overset{\varphi}{\cup} X_2$$

assigning to every point $x \in X_1 \cup X_2$ the element $\psi(x)$ of the decomposition $\mathscr{D}$, containing x, is continuous. This map ψ is said to be the *natural map* of $X_1 \cup X_2$ onto $X_1 \overset{\varphi}{\cup} X_2$. It is clear that the restriction $\psi|(X_1-X_0)$ is a homeomorphism of X_1-X_0 onto the subset $\psi(X_1-X_0)$ of $X_1 \overset{\varphi}{\cup} X_2$. Moreover, let us notice that the restriction $\psi|X_2$ is a homeomorphism of X_2 onto the set $X_0 \overset{\varphi}{\cup} X_2 \subset X_1 \overset{\varphi}{\cup} X_2$.

Let us prove the following theorem ([287], p. 1125; compare also [30], p. 250):

(9.1) THEOREM. *Let X_0, X_1, X_2 be* ANR-*sets such that $X_1 \cap X_2 = 0$ and $X_0 \subset X_1$, and let φ be a map of X_0 into X_2. Then $X' = X_1 \overset{\varphi}{\cup} X_2$ is an* ANR-*set.*

The proof, due to J. H. C. Whitehead [287], is based on three lemmas:

(9.2) LEMMA. *If* X_1, X_2 *and* $X_0 \subset X_1$ *are* ANR's *and* $\varphi \in X_2^{X_0}$, *then there exists a neighborhood* U *of the set* $X_0 \overset{\varphi}{\cup} X_2$ *in the space* $X' = X_1 \overset{\varphi}{\cup} X_2$ *and a homotopy* $\{g_t\} \subset X'^U$ *such that*

(i) $g_0(x') = x'$ *for every point* $x' \in U$,

(ii) $g_t(x') = x'$ *for every point* $x' \in X_0 \overset{\varphi}{\cup} X_2$ *and* $0 \leqslant t \leqslant 1$,

(iii) g_1 *is a retraction of* U *to* $X_0 \overset{\varphi}{\cup} X_2$.

Proof. By (3.3) there exists a neighborhood V of X_0 in X_1 and a homotopy $\{\omega_t\} \subset X_1^V$ such that ω_0 is the inclusion map of V in X_1, ω_1 is a retraction of V to X_0 and that $\omega_t(x) = x$ for every $x \in X_0$ and $0 \leqslant t \leqslant 1$. Consider now the natural map ψ: $X_1 \cup X_2 \to X'$ and let us set

$$U = \psi(V) \cup (X_0 \overset{\varphi}{\cup} X_2),$$

$$g_t(x') = \psi\omega_t\psi^{-1}(x') \quad \text{for} \quad x' \in U - (X_0 \overset{\varphi}{\cup} X_2) \text{ and } 0 \leqslant t \leqslant 1,$$

$$g_t(x') = x' \quad \text{for} \quad x' \in X_0 \overset{\varphi}{\cup} X_2 \text{ and } 0 \leqslant t \leqslant 1.$$

Then we get a homotopy $\{g_t\} \subset X'^U$ satisfying conditions (i), (ii), and (iii).

(9.3) LEMMA. *Let* X *and* $A \subset X$ *be* ANR-*sets and let* Y *be a metric space. Let* ε *be a positive number and* $\{f_t\} \subset Y^A$ *be a homotopy such that for each point* $a \in A$ *the diameter of the set* F_a *of all points* $f_t(a)$ *with* $0 \leqslant t \leqslant 1$ *is less than* ε. *If the map* f_0 *has an extension* $\hat{f}_0 \in Y^X$, *then there exists a homotopy* $\{\hat{f}_t\} \subset Y^X$ *such that* $\hat{f}_t$ *is an extension of* f_t *and that for each point* $x \in X$ *the diameter of the set* $\hat{F}_x$ *of all points* $\hat{f}_t(x)$ *with* $0 \leqslant t \leqslant 1$ *is less than* ε.

Proof. Since A is a compactum, there exists a positive number $\varepsilon' < \varepsilon$ such that

(9.4) $$\delta(F_a) < \varepsilon' \quad \textit{for every point } a \in A.$$

Since the set

$$Z = \big(X \times (0)\big) \cup (A \times \langle 0, 1\rangle)$$

in an ANR, there exists a neighborhood U of Z in the space $X \times \langle 0, 1\rangle$ and a retraction r: $U \to Z$. Setting

(9.5) $$g(z) = \begin{cases} \hat{f}_0(x) & \text{if} \quad z = (x, 0) \text{ with } x \in X, \\ f_t(x) & \text{if} \quad z = (x, t) \text{ with } x \in A \text{ and } 0 \leqslant t \leqslant 1, \end{cases}$$

we get a map g: $Z \to Y$. The compactness of X implies that for positive and sufficiently small values of η the following condition is satisfied:

(9.6) *If* $x \in X$ *and* $\varrho(x, A) < \eta$, *then* $(x) \times \langle 0, 1\rangle \subset U$.

Moreover, since r is a retraction of U to Z and g is continuous, we infer by (9.4) that for the number η sufficiently small the following statement holds:

(9.7) *For every point $x \in X$ such that $\varrho(x, A) < \eta$ the diameter of the set $gr\big((x) \times \langle 0, 1\rangle\big)$ is less than ε.*

Now let us consider a continuous, real-valued function α defined in the space X and satisfying the condition:

$$0 \leqslant \alpha(x) \leqslant 1 \textit{ for every point } x \in X,$$

(9.8)
$$\alpha(x) = \begin{cases} 0 & \textit{if} \quad \varrho(x, A) \geqslant \eta, \\ 1 & \textit{if} \quad x \in A. \end{cases}$$

It follows by (9.6) and (9.8) that

$$\big(x, t\alpha(x)\big) \in U \quad \text{for every point } x \in X \text{ and } 0 \leqslant t \leqslant 1.$$

Hence, setting

$$\hat{f}_t(x) = gr\big(x, t\alpha(x)\big) \quad \text{for every point } x \in X \text{ and for } 0 \leqslant t \leqslant 1,$$

we get a homotopy $\{\hat{f}_t\} \subset Y^X$ satisfying the condition $\hat{f}_0 = f_0$. Moreover, $\hat{f}_t$ is an extension of f_t and if $x \in X$ and $\varrho(x, A) \geqslant \eta$, then $\alpha(x) = 0$ and consequently the set $\hat{F}_x$ consists of only one point $gr(x, 0)$. If, however, $x \in X$ and $\varrho(x, A) < \eta$, then (9.7) implies that the diameter of the set $\hat{F}_x = gr\big((x) \times \langle 0, 1\rangle\big)$ is less than ε. Hence $\delta(\hat{F}_x) < \varepsilon$ for every point $x \in X$ and the proof of Lemma (9.3) is finished.

(9.9) LEMMA. *Let X be a compactum and $Y \subset X$ be an* ANR-*set such that:*

(a) *There is a neighborhood U of Y in X and a homotopy $\{g_t\} \subset X^U$ such that $g_0(x) = x$ for every point $x \in U$, $g_t(y) = y$ for every point $y \in Y$ and $0 \leqslant t \leqslant 1$, and that g_1 is a retraction of U to Y.*

(b) *For every $\varepsilon > 0$ and for every open neighborhood V of Y there exists an $\eta > 0$ satisfying the condition of Lefschetz for the pair $(X, X - V)$ and for ε.*

Then $X \in$ ANR.

Proof. By Theorem (8.1) it is sufficient to show that for every $\varepsilon > 0$ there is a number $\eta > 0$ satisfying the condition of Lefschetz for the pair (X, X) and for ε.

Given a positive number t, let us denote by U_t the set of all points $x \in X$ with $\varrho(x, Y) < t$. First let us prove that

(9.10) *For every $\varepsilon > 0$ there exists a positive number η satisfying the condition of Lefschetz for the pair $(X, U_{3\eta})$ and for ε.*

Since $Y \in$ ANR, we infer by (8.1) that there exists a positive number $\varepsilon_1 < \varepsilon$ satisfying the condition of Lefschetz for (Y, Y) and for $\frac{1}{3}\varepsilon$.

Since $g_t(y) = y$ for every $y \in Y$ and for $0 \leqslant t \leqslant 1$, there exists a positive number $\eta < \frac{1}{3}\varepsilon_1$ such that $U_{3\eta} \subset U$ and that for every point $x \in U_{3\eta}$ the diameter of the set of all points $g_t(x)$, with $0 \leqslant t \leqslant 1$, is less than $\frac{1}{3}\varepsilon_1$. It follows that

$$(9.11) \qquad \delta[g_1(A)] \leqslant \delta(A) + \tfrac{2}{3}\varepsilon_1 \quad \text{for every set } A \subset U_{3\eta}.$$

Consider now a polyhedron W with a triangulation $\mathscr{T}$ and a subpolyhedron W' of the triangulation $\mathscr{T}$ containing all vertices of $\mathscr{T}$. Let $f'\colon W' \to U_{3\eta}$ be a map such that

$$(9.12) \qquad \delta[f'(\sigma \cap W')] < \eta \quad \text{for each simplex } \sigma \in \mathscr{T}.$$

Setting

$$f_t(p) = g_{1-t}f'(p) \quad \text{for} \quad p \in W' \text{ and } 0 \leqslant t \leqslant 1,$$

we get a homotopy $\{f_t\} \subset X^{W'}$ such that for every point $p \in W'$ the set F_p of all points $f_t(p)$ with $0 \leqslant t \leqslant 1$ has the diameter less than $\frac{1}{3}\varepsilon_1$. Moreover, we infer by (9.12) and (9.13) that

$$\delta[f_0(\sigma \cap W')] = \delta[g_1 f'(\sigma \cap W')] \leqslant \delta[f'(\sigma \cap W')] + \tfrac{2}{3}\varepsilon_1$$
$$< \eta + \tfrac{2}{3}\varepsilon_1 < \varepsilon_1$$

for each simplex $\sigma \in T$. Since the values of f_0 belong to Y and since ε_1 satisfies the condition of Lefschetz for the pair (Y, Y) and for $\frac{1}{3}\varepsilon$, we infer that f_0 has a continuous extension $\hat{f}_0 \in X^W$ such that

$$\delta[\hat{f}_0(\sigma)] < \tfrac{1}{3}\varepsilon \quad \text{for each simplex } \sigma \in T.$$

If we recall that $f_t = g_{1-t}f'$ and that the diameter of the set of all points $g_t(x)$, with $0 \leqslant t \leqslant 1$, is less than $\frac{1}{3}\varepsilon_1$, we infer, by Lemma (9.3), that there exists a homotopy $\{\hat{f}_t\} \subset X^W$ such that $\hat{f}_t$ is an extension of f_t and that for every point $p \in W$, the diameter of the set of all points $\hat{f}_t(p)$, with $0 \leqslant t \leqslant 1$, is less than $\frac{1}{3}\varepsilon_1$. It follows that

$$\delta[\hat{f}_1(\sigma)] < \delta[\hat{f}_0(\sigma)] + \tfrac{2}{3}\varepsilon_1 < \tfrac{1}{3}\varepsilon + \tfrac{2}{3}\varepsilon = \varepsilon.$$

If we observe that

$$\hat{f}_1(p) = g_0 f'(p) = f'(p) \quad \text{for every point } p \in W',$$

we infer that the number η satisfies the condition of Lefschetz for the pair $(X, U_{3\eta})$ and for ε, and thus the proof of (9.10) is complete.

In order to finish the proof of Lemma (9.9), let us assign to a given number $\varepsilon > 0$ a positive number $\eta \leqslant \varepsilon$ satisfying (9.11). Condition (b) implies that there exists a positive number $\eta' \leqslant \eta$ satisfying the condition of Lefschetz for the pair $(X, X - U_\eta)$ and for η.

Now let us consider a polyhedron W with a triangulation $\mathscr{T}$ and a subpolyhedron W' of the triangulation $\mathscr{T}$ containing all vertices of $\mathscr{T}$, and let f': $W' \to X$ be a map such that

(9.13) $$\delta[f'(\sigma \cap W')] < \eta' \quad \text{for each simplex } \sigma \in \mathscr{T}.$$

Let $\mathscr{T}_1$ denote the triangulation consisting of all faces of the simplexes $\sigma \in \mathscr{T}$ such that $f'(\sigma \cap W') \cap (X - U_{2\eta}) \neq 0$. Setting $\mathscr{T}_2$ equal to the set of all faces of simplexes belonging to $\mathscr{T} - \mathscr{T}_1$ and $\mathscr{T}_0 = \mathscr{T}_1 \cap \mathscr{T}_2$, we obtain three subtriangulations $\mathscr{T}_0, \mathscr{T}_1, \mathscr{T}_2$ of the triangulation $\mathscr{T}$ such that $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$. Let W_ν $(\nu = 0, 1, 2)$ denote the polyhedron with the triangulation $\mathscr{T}_\nu$ and let $W'_\nu = W_\nu \cap W'$. Since $\eta' \leqslant \eta$, it follows by (9.13) that

$$f'(W'_1) \subset X - U_\eta \quad \text{and} \quad f'(W'_2) \subset U_{3\eta}.$$

Since η' satisfies the condition of Lefschetz for the pair $(X, X - U_\eta)$ and for η, we infer that the restriction $f'_1 = f'|W'_1$ has a continuous extension f_1: $W_1 \to X$ satisfying the condition

(9.14) $$\delta[f_1(\sigma)] < \eta \quad \text{for each simplex } \sigma \in \mathscr{T}_1.$$

In particular, if $\sigma \in \mathscr{T}_0$, then $f'(\sigma \cap W') \subset U_{2\eta}$ and since on the set $\sigma \cap W'$ the map f_1 is identical with f', we infer that

(9.15) $$f_1(W_0) \subset U_{3\eta}.$$

Now let us consider the map f''_2 defined as equal to f_1 on W_0 and as equal to f' on W'_2.

Since $f'(W'_2) \subset U_{3\eta}$, we infer by (9.15) that f''_2 maps the set $W''_2 = W_0 \cup W'_2$ into $U_{3\eta}$. Moreover, it follows by (9.13) and (9.14) that

$$\delta[f''_2(\sigma \cap W''_2)] < \eta \quad \text{for every simplex } \sigma \in \mathscr{T}_2.$$

Since η satisfies (9.10), we infer that f''_2 has a continuous extension f_2: $W_2 \to X$ such that

$$\delta[f_2(\sigma)] < \varepsilon \quad \text{for each simplex } \sigma \in \mathscr{T}_2.$$

Setting

$$f(x) = \begin{cases} f_1(x) & \text{for} \quad x \in W_1, \\ f_2(x) & \text{for} \quad x \in W_2, \end{cases}$$

we get a continuous extension f: $W \to X$ of the map f' such that

$$\delta[f(\sigma)] < \varepsilon \quad \text{for each simplex } \sigma \in \mathscr{T}.$$

It follows by (8.1) that $X \in \text{ANR}$, and so the proof of Lemma (9.9) is finished.

Proof of Theorem (9.1). By Lemma (9.9), we have to show that the space $X = X_1 \overset{\varphi}{\cup} X_2$ and its subset $Y = X_0 \overset{\varphi}{\cup} X_2$ (homeomorphic to X_2, consequently being an ANR-set) satisfy conditions (a) and (b), as given in (9.9). By Lemma (9.2), condition (a) is satisfied. It remains to prove that condition (b) is satisfied.

Consider the natural map

$$\psi\colon X_1 \cup X_2 \to X_1 \overset{\varphi}{\cup} X_2.$$

Since $X_1 \cup X_2$ is a compactum, there exists a function λ assigning to every positive number ε a positive number $\lambda(\varepsilon)$ such that

(9.16) $$\varrho(x, y) < \lambda(\varepsilon) \quad \text{implies} \quad \varrho\big(\psi(x), \psi(y)\big) < \varepsilon.$$

Since $X_1 \epsilon$ ANR, we infer by (8.1) that for a given $\varepsilon > 0$ there exists a positive number ε' satisfying the condition of Lefschetz for (X_1, X_1) and for $\lambda(\varepsilon)$. Let V be an open neighborhood of the set $X_0 \overset{\varphi}{\cup} X_2$ in the space X. Then $X_1 - \psi^{-1}(V)$ is a compactum contained in $X_1 - X_0$, and since the restriction $\psi|(X_1 - X_0)$ is a homeomorphism, we infer that there exists a positive number η such that if $x, x' \epsilon X_1 - \psi^{-1}(V)$ and $\varrho\big(\psi(x), \psi(x')\big) < \eta$, then $\varrho(x, x') < \varepsilon'$. Now let W be a polyhedron with a triangulation $\mathscr{T}$ and W' a subpolyhedron in $\mathscr{T}$ containing all vertices of $\mathscr{T}$. Let $f'\colon W' \to X - V$ be a map satisfying the condition $\delta[f'(\sigma \cap W')] < \eta$ for every simplex $\sigma \epsilon \mathscr{T}$. Then $\bar{f} = \psi^{-1} f'$ maps W' into $X_1 - \psi^{-1}(V)$ and $\delta[\bar{f}(\sigma \cap W')] < \varepsilon'$ for each simplex $\sigma \epsilon \mathscr{T}$. It follows that there exists a continuous extension $\hat{f}\colon W \to X_1$ of the map $\bar{f}$ such that $\delta[\hat{f}(\sigma)] < \lambda(\varepsilon)$ for each simplex $\sigma \epsilon \mathscr{T}$. Setting $f(p) = \psi\hat{f}(p)$ for every point $p \epsilon W$, we get a map $f\colon W \to X$ such that $\delta[f(\sigma)] < \varepsilon$ for each simplex $\sigma \epsilon \mathscr{T}$. Moreover, if $p \epsilon W'$, then $f(p) = \psi\bar{f}(p) = \psi\psi^{-1} f'(p) = f'(p)$, that means that f is a continuous extension of the map f'. Thus condition (b) is satisfied and the proof of Theorem (9.1) is finished.

(9.17) Remark. *If the sets X_0, X_1 and X_2, which appear in the formulation of Theorem* (9.1) *are* AR-*sets, then similar argument shows that the set $X_1 \overset{\varphi}{\cup} X_2$ is an* AR-*set.*

The operation of matching makes it possible to obtain various examples of ANR's with rather paradoxical properties. Some of them will be considered in the next chapter. Here let us mention only the example which we get by matching the 3-dimensional Euclidean sphere S^3 with a space consisting of only one point along a simple (wild) arc $L \subset S^3$ such that the fundamental group of $S^3 - L$ is not trivial. As has been shown by T. Ganea [137], the ANR-space X obtained in this way is not a manifold, but for every $\varepsilon > 0$ there exists a map of X onto S^3 such that the diameter of the set $f^{-1}(p)$ is less than ε for every point $p \epsilon S^3$.

10. Finite-dimensional ANR-**spaces.** Now let us consider the case of finite dimensional ANR's.

(10.1) THEOREM. *Finite dimensional* ANR-*spaces are the same as the r-images of polyhedra.*

Proof. We have already seen that the r-image of a polyhedron is an ANR-space. Conversely, if $X \in$ ANR, then, by the Menger-Nöbeling embedding theorem (see, for instance, [166], p. 56), there is a homeomorphism h mapping X onto a compactum $h(X)$ lying in a Euclidean space E^n, where $n = 2\dim X + 1$. Since $X \in$ ANR, there is a neighborhood U of $h(X)$ in E^n and a retraction $r: U \to h(X)$. Clearly, there is a polyhedron W such that $h(X) \subset W \subset U$. Then the restriction $f = h^{-1}r|W$ is an r-map of W onto X.

(10.2) THEOREM. *Finite dimensional* AR-*spaces are the same as the r-images of geometric simplexes.*

Proof. Since every geometric simplex σ is an AR, it follows by (2.1) that every r-image of σ is an AR. On the other hand, if $X \in$ AR, then by the Menger-Nöbeling embedding theorem there is a homeomorphism h mapping X onto a subset $h(X)$ of a geometric simplex σ. Since $h(X) \in$ AR, there exists a retraction r: $\sigma \to h(X)$. Then the map $h^{-1}r$: $\sigma \to X$ is an r-map of σ onto X.

(10.3) THEOREM. *An n-dimensional compactum X is an* ANR-*space if and only if* $X \in \mathbf{LC}^n$.

Proof. We know already, by (2.6), that $X \in$ ANR implies $X \in \mathbf{LC}$, and consequently also $X \in \mathbf{LC}^n$ (by I, (17.2)). On the other hand, if $\dim X \leqslant n$, then $X \in \mathbf{LC}^n$ implies $X \in \mathbf{LC}$ (by III, (9.6)). By the Menger-Nöbeling embedding theorem, there is a homeomorphism h mapping X onto a subset $h(X)$ of a geometric simplex σ. Since $X \in \mathbf{LC}$ implies $X \in \mathbf{LC}^\infty$, we infer by III, (9.1), that $h(X)$ is a neighborhood retract in σ and we infer by (2.5) that $h(X)$, and consequently also X, is an ANR-space.

Since every ANR-space is locally contractible (2.6), and since local contractibility implies the condition $\mathbf{LC}^\infty$ (by I, (17.2)), we get the following

(10.4) COROLLARY. *A finite-dimensional compactum X is an* ANR-*space if and only if it is locally contractible.*

First a proof of this proposition is given in [21], p. 240. A noteworthy generalization of it (for arbitrary metric spaces) is given recently by J. Dugundji ([104], p. 189). By (2.3) it follows the

(10.5) COROLLARY. *A finite-dimensional compactum* $X \neq 0$ *is an* AR-*set if and only if it is contractible in itself and locally contractible.*

(10.6) THEOREM. *An n-dimensional* ($n \geqslant 0$) *compactum X is an* AR-*set if and only if* $X \in \mathbf{LC}^n$ *and* $X \in \mathbf{C}^n$.

Proof. Since the contractibility $\mathbf{C}$ implies the condition $\mathbf{C}^n$ and the local contractibility $\mathbf{LC}$ implies $\mathbf{LC}^n$ (by I, (17.1), and (17.2)), we infer by (2.3) and (2.6) that both the conditions are necessary. Now let us assume that X is an n-dimensional compactum and that $X \in \mathbf{LC}^n$ and $X \in \mathbf{C}^n$.

It follows by (10.3) that $X \in \mathrm{ANR}$. In order to prove that $X \in \mathrm{AR}$ it is sufficient — by (2.3) — to show that X is contractible in itself.

By the Menger–Nöbeling embedding theorem ([166], p. 56) we can assume that X is a subset of a geometric m-dimensional simplex σ, where $m = 2n+1$. Let $\mathscr{T}$ be a triangulation of the set $\sigma - X$ with diameters of simplexes converging to zero and let N^k be the k-dimensional skeleton of this triangulation. In particular, $N^m \cup X = \sigma$ and consequently, for $k = m$, there is a homotopy

$$\varphi_k\colon X \times \langle 0, 1\rangle \to N^k \cup X$$

contracting X to a point. Now let us assume that for a $k > n+1$ there exists such a homotopy φ_k. Let σ^k be a k-dimensional simplex of $\mathscr{T}$ and let b denote its barycenter. Since $\dim(X \times \langle 0, 1\rangle) \leqslant n+1 < k$, we easily see (II, (9.2)) that the homotopy φ_k can be modified in the set $\varphi_k^{-1}(\sigma^k)$, without changing its values in the set $\varphi_k^{-1}(\sigma^{k\bullet})$, so that the point b will not belong to the set of its values. If we compose this homotopy (in the set $\varphi_k^{-1}(\sigma^k)$ only) with the projection of the set $\sigma^k - b$ from b onto the boundary $\sigma^{k\bullet}$, and if we apply this procedure to each k-dimensional simplex of the triangulation $\mathscr{T}$, we obtain from the homotopy φ_k a homotopy $\varphi_{k-1}\colon X \times \langle 0, 1\rangle \to N^{k-1} \cup X$ contracting X to a point. Repeating this procedure $m-n-1$ times, we obtain a homotopy

$$\varphi_{n+1}\colon X \times \langle 0, 1\rangle \to N^{n+1} \cup X$$

contracting X to a point.

New let us show that X is a retract of $N^{n+1} \cup X$. Since $X \in \mathrm{ANR}$, there exists a retraction r_0 to X of a set A which is the union of X and of almost all simplexes of the triangulation $\mathscr{T}$; we can assume that all vertices of $\mathscr{T}$ belong to A. Let P denote the polyhedron consisting of all simplexes of $\mathscr{T}$ which do not lay in A, and let P^k denote the k-dimensional skeleton of P (by the triangulation $\mathscr{T}$). Since $P^0 \subset A$, the retraction r_0 is defined on the set $A \cup P^0$. Let us assume that for some integer k, with $0 \leqslant k \leqslant n$, the retraction

$$r_k\colon A \cup P^k \to X$$

is already defined. If σ^{k+1} is a $(k+1)$-dimensional simplex of $\mathscr{T}$ lying in P^{k+1} (but not in A), then r_k is defined on $(\sigma^{k+1})^\bullet$ and since $X \in \mathbf{C}^n$, there exists a continuous extension of the restriction $r_k|(\sigma^{k+1})^\bullet$ onto σ^{k+1} with values in X. Applying this procedure to all simplexes $\sigma^{k+1} \in \mathscr{T}$ lying in P^{k+1}, but not in A (their number is finite), we get a retraction r_{k+1} of the set

$A \cup P^{k+1}$ to X. Thus in $n+1$ steps we obtain a retraction r_{n+1} of the set $A \cup P^{n+1} \supset X \cup N^{n+1}$ to X.

It remains to set $\psi = r_{n+1}\varphi_{n+1}$ in order to obtain a homotopy ψ: $X \times \langle 0, 1\rangle \to X$ contracting the set X to a point.

Thus the proof of Theorem (10.6) is finished.

(10.7) Remark. As it has been shown by S. D. Liao ([219], p. 142), proposition (10.3) and (10.6) hold also without the hypothesis of compactness. More precisely, among n-dimensional metric separable spaces, the condition $\mathbf{LC}^n$ characterizes the ANR($\mathfrak{M}$)-spaces, and the conditions $\mathbf{LC}^n$ and $\mathbf{C}^n$ — the AR($\mathfrak{M}$)-spaces. Compare also [103], p. 244, [183], and [196].

As we have seen (II, (6.1)), the condition $X \in \mathbf{C}^\infty$ is equivalent to the triviality of all homotopy groups of X. Applying (2.4) and II, (7.4), we get the following

(10.8) THEOREM. *A finite-dimensional* ANR*-space* X *is an* AR*-space if and only if the fundamental group* $\pi_1(X)$ *and all Betti groups* $H_k(X)$ *for* $k = -1, 0, 1, \ldots$ *are trivial.*

It is clear, if $\dim X = n$, the triviality of groups $H_k(X)$ for $k > n$ is automatically satisfied.

Applying Theorem (10.8), let us give an example (due to E. Begle [10], p. 386) of a polyhedron $X \in$ AR which admitts a decomposition into the union of two polyhedra $X_1, X_2 \in$ AR with the common part $X_0 = X_1 \cap X_2$ being a polyhedron which is not an AR-space. Let $\mathscr{T}$ be a triangulation of a Poincaré sphere, i.e. of a 3-dimensional manifold P with the homology groups of the 3-dimensional Euclidean sphere, but with non-trivial fundamental group. Let G denote the interior of a 3-dimensional simplex of $\mathscr{T}$. Setting $X_0 = P - G$, let us consider two cones X_1, X_2 with the common base X_0 such that $X_1 \cap X_2 = X_0$. The polyhedra X_1, X_2, as contractible in itself, are AR-sets. Moreover, we easily see that the fundamental group of X_0 is not trivial, and consequently X_0 is not an AR-space. However, by (10.8), $X \in$ AR, because the fundamental group of X and also all Betti groups of X are trivial.

11. Locally contractible compactum which is not an ANR**-space.** As we have seen, the condition of the local contractibility characterizes ANR-spaces among finite-dimensional compacta. The problem arises ([111], p. 248) whether a similar proposition holds also for compacta of the infinite dimension. Since, by (4.6), almost all Betti numbers of an ANR-space vanish, the negative answer to this problem is included in the following

(11.1) THEOREM ([48], p. 176). *There exists a locally contractible compactum* X *such that the* n*-th Betti number* $p_n(X)$ *is positive for every* $n = 0, 1, \ldots$

Proof. Let us consider the following subsets of the Hilbert cube Q^ω:

$$X_0 = \{x = \{x_i\};\ x_1 = 0\},$$

$$X_k = \left\{x = \{x_i\}; \frac{1}{k+1} \leqslant x_1 \leqslant \frac{1}{k} \text{ and } x_i = 0 \text{ for } i > k\right\} \quad \text{for} \quad k = 1, 2, \ldots$$

It is plain that X_0 is homeomorphic to Q^ω and X_k — to the Euclidean k-cube Q^k. Let $X_k^\bullet$ denote the boundary of X_k (i. e. the image under such a homeomorphism of the boundary $Q^{k\bullet}$ of the k-cube Q^k) for $k = 1, 2, \ldots$ Setting

$$X = X_0 \cup \bigcup_{k=1}^{\infty} X_k^\bullet,$$

we easily see that X is a compactum. Moreover, the map r_m defined for every $m = 1, 2, \ldots$ by the formulas

$$r_m(\{x_i\}) = \begin{cases} \left(\dfrac{1}{m+1}, x_2, \ldots, x_m, 0, 0, \ldots\right) & \text{for} \quad \{x_i\} \in X_0 \cup \bigcup\limits_{k=m+1}^{\infty} X_k^\bullet, \\ (x_1, x_2, \ldots) & \text{for} \quad \{x_i\} \in X_m^\bullet, \\ \left(\dfrac{1}{m}, x_2, x_3, \ldots\right) & \text{for} \quad \{x_i\} \in \bigcup\limits_{k=1}^{m-1} X_k^\bullet \end{cases}$$

is a retraction of the set X to $X_m^\bullet$. Since the Betti number $p_{m-1}(X_m^\bullet)$ is equal to 1, we infer by II, (4.3), that

$$p_{m-1}(X) \geqslant 1 \quad \text{for every} \quad m = 1, 2, \ldots$$

In order to finish the proof, it remains to show that X is locally contractible. The local contractibility of X at every point $a \in \bigcup_{k=1}^{\infty} X_k^\bullet$ is evident, because such a point has a polyhedral neighborhood in X. Now let us consider a point $a \in X_0$ and a positive number ε. Let $m > 1$ be an integer such that $3/m < \varepsilon$ and let U and V denote the neighborhoods of a in the space X given by the formulas:

$$U = \{x \in X;\ \varrho(a, x) < \varepsilon\} \quad \text{and} \quad V = \{x \in X;\ \varrho(a, x) < 1/2m\}.$$

In order to finish our proof it is sufficient to show that V is contractible to a point in U.

The set V is evidently disjoint with at least one of the sets

$$A_m = \{x = \{x_i\};\ x_m = 0\}; \quad B_m = \{x = \{x_i\};\ x_m = 1/m\}.$$

Now let us consider the following two cases:

Case 1. $V \cap B_m = 0$. In this case, let us set

$$\varphi(\{x_i\}, t) = \begin{cases} (x_1, \ldots, x_{m-1}, (1-3t)x_m, x_{m+1}, \ldots) \\ \qquad \text{for} \quad 0 \leqslant t \leqslant \frac{1}{3} \text{ and } \{x_i\} \epsilon V, \\ ((2-3t)x_1, x_2, \ldots, x_{m-1}, 0, x_{m+1}, \ldots) \\ \qquad \text{for} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \text{ and } \{x_i\} \epsilon V, \\ (0, a_2 - 3(1-t)(a_2 - x_2), a_3 - 3(1-t)(a_3 - x_3), \ldots, \\ a_{m-1} - 3(1-t)(a_{m-1} - x_{m-1}), (3t-2)a_m, a_{m+1} - \\ \qquad -3(1-t)(a_{m+1} - x_{m+1}), \ldots) \text{ for } \frac{2}{3} \leqslant t \leqslant 1 \text{ and } \{x_i\} \epsilon V. \end{cases}$$

Thus we get a homotopy in X deforming the set V to the point a. By this homotopy every point $x = \{x_i\} \epsilon V$ runs, for $0 \leqslant t \leqslant \frac{1}{3}$, along the segment L_1 joining the point $\varphi(x, 0) = x$ with the point $\varphi(x, \frac{1}{3}) = (x_1, \ldots, x_{m-1}, 0, x_{m+1}, \ldots)$, for $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$ – through the segment L_2 joining the point $\varphi(x, \frac{1}{3})$ with the point $\varphi(x, \frac{2}{3}) = (0, x_2, \ldots, x_{m-1}, 0, x_{m+1}, \ldots)$ and for $\frac{2}{3} \leqslant t \leqslant 1$ – through the segment L_3 joining the point $\varphi(x, \frac{2}{3})$ with the point $\varphi(x, 1) = a$. The lengths of L_1 and L_2 are $\leqslant 1/m$, and the length of L_3 is $\leqslant 1/m + a_m < 2/m$. Since $1/m < \frac{1}{3}\varepsilon$, we infer that $L_1 \cup L_2 \cup L_3$ is a subset of U.

Case 2. $V \cap A_m = 0$. In this case we set

$$\varphi(\{x_i\}, t) = \begin{cases} \left(x_1, \ldots, x_{m-1}, \dfrac{1}{m} + (3t-1)\left(\dfrac{1}{m} - x_m\right), x_{m+1}, \ldots\right) \\ \qquad \text{for} \quad 0 \leqslant t \leqslant \frac{1}{3} \text{ and } \{x_i\} \epsilon V, \\ \left((2-3t)x_1, x_2, \ldots, x_{m-1}, \dfrac{1}{m}, x_{m+1}, \ldots\right) \\ \qquad \text{for} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \text{ and } \{x_i\} \epsilon V, \\ \Big(0, a_2 - 3(1-t)(a_2 - x_2), a_3 - 3(1-t)(a_3 - x_3), \ldots, \\ a_{m-1} - 3(1-t)(a_{m-1} - x_{m-1}), \dfrac{3}{m}(1-t) + (3t-2)a_m, \\ a_{m+1} - 3(1-t)(a_{m+1} - x_{m+1}), \ldots\Big) \text{ for } \frac{2}{3} \leqslant t \leqslant 1 \text{ and } \{x_i\} \epsilon V. \end{cases}$$

One sees, as in Case 1, that φ is a homotopy contracting the set V in the set U to the point a.

Thus X is locally contractible at every of its points and the proof of theorem is finished.

(11.2) COROLLARY. *There exists a locally contractible compactum Y which is contractible in itself but is not an* ANR-*space.*

In fact, the cone Y with a base X satisfying (11.1) is a locally contractible compactum which can be contracted in itself to its vertex. Manifestly, the base X of Y is a neighborhood retract for Y. Since X is not an ANR-space, we infer by (2.5) that Y is not an ANR-space.

12. Upper semicontinuous decompositions of ANR's. Now let us prove the following theorem due to S. Smale ([265], p. 604):

(12.1) THEOREM. *Let f be a map of an* ANR-*space X onto a compactum Y such that $f^{-1}(y)\in$* AR *for each point $y\in Y$. Then $y\in \mathbf{LC}^{\infty}$. Moreover, if $X\in$* AR, *then $Y\in \mathbf{C}^{\infty}$.*

First let us introduce some notation. For every point $y\in Y$ and for every positive number ε, we let

(12.2) $$V(y,\varepsilon) = \{y'\in Y;\ \varrho(y,y') < \tfrac{1}{2}\varepsilon\}.$$

It is evident that $V(y,\varepsilon)$ is an open subset of Y with diameter $\leqslant \varepsilon$. Setting

(12.3) $$U(y,\varepsilon) = f^{-1}\big(V(y,\varepsilon)\big),$$

we get an open subset of X containing $f^{-1}(y)$. Let us observe that for each point $y\in Y$ there is an open subset $G(y,\varepsilon)$ of X such that

(12.4) $$f^{-1}(y) \subset G(y,\varepsilon) \subset U(y,\varepsilon),$$

(12.5) $$G(y,\varepsilon) \text{ is contractible in } U(y,\varepsilon).$$

In order to prove this, let us assume that X is a subset of the Hilbert cube Q^{ω}. Since $X\in$ ANR, there exists a retraction r of a neighborhood G of X in Q^{ω} to X. Moreover, since $f^{-1}(y)\in$ AR, there exists a retraction r_0 of Q^{ω} to $f^{-1}(y)$. Now let us show that if a is a positive, sufficiently small number, then it is sufficient to set

$$G(y,\varepsilon) = \big\{x\in X;\ \varrho\big(x, f^{-1}(y)\big) < a\big\}$$

in order to obtain an open subset of X satisfying (12.4) and (12.5). It is evident that $G(y,\varepsilon)$ is open and that for values of a sufficiently small condition (12.4) is satisfied. Moreover, setting

$$\vartheta_t(x) = (1-t)x + tr_0(x) \quad \text{for every } x\in G(y,\varepsilon) \text{ and } 0\leqslant t\leqslant 1,$$

we get a homotopy $\{\vartheta_t\}$ which carries the set $G(y,\varepsilon)$ in Q^{ω} to the set $f^{-1}(y)$. It is plain that for sufficiently small values of a the values of $\vartheta_t(x)$ belong to G and the values of the maps

$$\chi_t(x) = r\vartheta_t(x) \quad \text{for every} \quad x\in G(y,\varepsilon)$$

belong to $U(y,\varepsilon)$ for every $0\leqslant t\leqslant 1$. Thus $\{\chi_t\}$ is a homotopy which carries the set $G(y,\varepsilon)$ to $f^{-1}(y)$ in the set $U(y,\varepsilon)$. Since the set $f^{-1}(y)$ is contractible in itself, we infer that condition (12.5) is satisfied.

Since $f^{-1}(y) \dot{\subset} G(y, \varepsilon)$, there exists an open neighborhood $W(y, \varepsilon)$ of y in the space Y such that

(12.6) $$f^{-1}\big(W(y, \varepsilon)\big) \subset G(y, \varepsilon).$$

Since the space Y is compact, we infer that there exists a positive number $\eta = \eta(\varepsilon) < \frac{1}{2}\varepsilon$ such that

(12.7) *Every subset of Y with diameter $< \eta$ is contained in one at least of the sets $W(y, \varepsilon)$.*

It follows by (12.6), (12.3), and (12.4) that

(12.8) $$W(y, \varepsilon) \subset V(y, \varepsilon).$$

We shall assume in the sequel of this section that X, Y, f, ε, $\eta = \eta(\varepsilon)$, $G(y, \varepsilon)$, $U(y, \varepsilon)$, $V(y, \varepsilon)$, $W(y, \varepsilon)$ keep the meaning just fixed. Moreover, by P we shall denote always a polyhedron. Now, using this notation, let us prove two lemmas:

(12.9) LEMMA. *The set of all maps of the form fg, where $g \in X^P$, is dense in the space Y^P.*

Proof. If $\dim P = 0$, then P consists of a finite number of points $p_1, p_2, \ldots, p_k$. Let $\psi \in Y^P$. Since $f(X) = Y$, there exists for every index $i = 1, 2, \ldots, k$ a point $x_i \in X$ such that $f(x_i) = \psi(p_i)$. Setting

$$g(p_i) = x_i \quad \text{for every} \quad i = 1, 2, \ldots, k,$$

we get a map $g: P \to X$ such that $fg(p_i) = \psi(p_i)$ for $i = 1, 2, \ldots, k$. Thus, in this case, each map $\psi: P \to Y$ is of the form fg.

Now let us assume that the maps of the form fg constitute a dense part of the space Y^P if $\dim P \leqslant n$ and let us pass to the case where $\dim P = n+1$.

Let $\psi \in Y^P$. For every $\varepsilon > 0$, there exists a triangulation $\mathscr{T}$ of P such that ψ maps each simplex $\sigma_0 \in \mathscr{T}$ onto a subset of Y with diameter less than $\eta = \eta(\varepsilon)$. It follows by (12.7) that there is a point $y_0 \in Y$ such that $\psi(\sigma_0) \subset W(y_0, \varepsilon)$. Let P' denote the n-dimensional skeleton of the triangulation $\mathscr{T}$. Since P' is an n-dimensional polyhedron, the restriction $\psi' = \psi|P'$ can be approximated by the maps of the form fg', where $g' \in X^{P'}$. If fg' is sufficiently close to the map ψ', then $\varrho(fg', \psi') < \varepsilon$ and $\psi(\sigma_0) \cup fg'(\sigma_0 \cap P') \subset W(y_0, \varepsilon)$, for every $(n+1)$-dimensional simplex $\sigma_0 \in \mathscr{T}$.

It follows by (12.6) that $f^{-1}[fg'(\sigma_0 \cap P')] \subset G(y_0, \varepsilon)$ and we infer by (12.5) that the set $g'(\sigma_0 \cap P') \subset f^{-1}[fg'(\sigma_0 \cap P')]$ is contractible in $U(y_0, \varepsilon)$. Consequently g' has a continuous extension onto the whole simplex σ_0 with values belonging to $U(y_0, \varepsilon)$. Applying this procedure to each $(n+1)$-dimensional simplex $\sigma_0 \in \mathscr{T}$, we get a map $g: P \to X$, which is an extension of the map g'. Moreover, for each point $p \in P$, if $p \in P'$, then $fg(p) = fg'(p)$ and $\varrho(fg(p), \psi(p)) < \varepsilon$, because $\varrho(fg', \psi') < \varepsilon$. If, however, $p \in P - P'$, then p belongs to the interior of an $(n+1)$-dimensional simplex $\sigma_0 \in \mathscr{T}$, and then $g(p) \in U(y_0, \varepsilon)$. It follows by (12.3) that $fg(p) \in V(y_0, \varepsilon)$. Hence

$$\varrho(fg(p), \psi(p)) \leqslant \varrho(fg(p), y_0) + \varrho(y_0, \psi(p)) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

because $\varrho(fg(p), y_0) < \frac{1}{2}\varepsilon$, and (12.8) and (12.2) imply $\varrho(y_0, \psi(p)) < \frac{1}{2}\varepsilon$. Thus it is shown that $\varrho(fg, \psi) < \varepsilon$ and the proof of Lemma (12.9) is finished.

(12.10) LEMMA. *For every map $\psi \in Y^P$ and for every positive ε there is a positive λ such that for every two maps $g, g' \in X^P$ satisfying the conditions*

$$\varrho(fg, \psi) < \lambda, \qquad \varrho(fg', \psi) < \lambda,$$

there exists a homotopy $\{g_t\} \subset X^P$ such that $g_0 = g$, $g_1 = g'$ and that for each point $p \in P$ the diameter of the set

$$L_p = \{y = fg_t(p);\ 0 \leqslant t \leqslant 1\}$$

is less than ε.

Proof. If $\dim P = 0$, then P is a finite set. One sees at once, by (12.7), that for sufficiently small values of λ the points $fg(p)$ and $fg'(p)$ belong both to one of the sets $W(y, \varepsilon)$. Then (12.6) implies that $g(p)$, $g'(p) \in f^{-1}(W(y, \varepsilon)) \subset G(y, \varepsilon)$ and we infer by (12.5) that there exists a homotopy $\{g_t\}$ carrying the point $g(p)$ to the point $g'(p)$ in the set $U(y, \varepsilon)$. It follows by (12.3) that $L_p \subset V(y, \varepsilon)$ and we infer by (12.2) that the diameter of the set L_p is less than ε.

Now let us assume that the lemma holds if the dimension of P is $\leqslant n$ and let us pass to the case where $\dim P = n+1$. Consider a triangulation $\mathscr{T}$ of P with diameters of simplexes so small that for every simplex $\sigma \in \mathscr{T}$ the diameter of the set $\psi(\sigma)$ is less than $\frac{1}{3}\eta$. If λ is a positive number sufficiently small, then the diameter of the set $fg(\sigma) \cup fg'(\sigma)$ is also less than $\frac{1}{3}\eta$ for each simplex $\sigma \in \mathscr{T}$.

Let P' denote the n-dimensional skeleton of the triangulation $\mathscr{T}$. By our hypothesis, if λ is a sufficiently small number, then there exists a homotopy $\{g'_t\} \subset X^{P'}$ such that $g'_0 = g|P'$, $g'_1 = g'|P'$ and that for each $p \in P'$ the diameter of the set $\{fg'_t(p);\ 0 \leqslant t \leqslant 1\}$ is less than $\frac{1}{3}\eta$. Setting

$$g''(p, 0) = g(p), \qquad g''(p, 1) = g'(p) \qquad \text{for every point } p \in P,$$
$$g''(p, t) = g'_t(p) \qquad \text{for every point } p \in P' \text{ and } 0 \leqslant t \leqslant 1,$$

we get a map of the subset $R = (P \times (0)) \cup (P' \times \langle 0, 1\rangle) \cup (P \times (1))$ of the Cartesian product $P \times \langle 0, 1\rangle$ into X. Let σ_0 be an $(n+1)$-dimensional simplex of the triangulation $\mathscr{T}$. Then the diameter of the set $fg''(\sigma_0 \times \langle 0, 1\rangle \cap R)$ is less than η. It follows by (12.7) that there exists a point $y \in Y$ such that $fg''(\sigma_0 \times \langle 0, 1\rangle \cap R) \subset W(y, \varepsilon)$. By virtue of (12.6) we have

$$g''(\sigma_0 \times \langle 0,1\rangle \cap R) \subset f^{-1}fg''(\sigma_0 \times \langle 0, 1\rangle \cap R) \subset f^{-1}\big(W(y, \varepsilon)\big) \subset G(y, \varepsilon),$$

and we infer by (12.5) that the set $g''(\sigma_0 \times \langle 0, 1\rangle \cap R)$ is contractible in the set $U(y, \varepsilon)$. Consequently, the map $g''|(\sigma_0 \times \langle 0, 1\rangle \cap R)$ has a continuous extension onto the whole set $\sigma_0 \times \langle 0, 1\rangle$ with values belonging to $U(y, \varepsilon)$. Applying this procedure to each $(n+1)$-dimensional simplex $\sigma_0 \in \mathcal{T}$, we get a map $\hat{g}\colon P \times \langle 0, 1\rangle \to X$. Setting $g_t(p) = \hat{g}(p, t)$ for every $(p, t) \in P \times \langle 0, 1\rangle$, we get a homotopy $\{g_t\} \subset X^P$ joining the map $g_0 = g$ with the map $g_1 = g'$. Moreover, since for every point $p \in P$ all values $fg_t(p)$, where $0 \leqslant t \leqslant 1$, belong to one of the sets $f\big(U(y, \varepsilon)\big) = V(y, \varepsilon)$, the diameter of the set L_p is $< \varepsilon$ and the proof of Lemma (12.10) is complete.

P r o o f o f T h e o r e m (12.1). Let ε be a positive number and let ψ be a map of the n-dimensional sphere S^n onto a subset of Y with diameter less than $\eta = \eta(\varepsilon)$. In order to prove that $Y \in \mathbf{LC}^\infty$, it is sufficient to show that there exists a homotopy $\{\psi_t\} \subset Y^{S^n}$ such that $\psi_0 = \psi$, $\psi_1 = \text{const}$ and that the diameter of the set of all values of all maps ψ_t is $\leqslant \varepsilon$.

Let us denote S^n by P. Since the diameter of the set $\psi(P)$ is $< \eta$, we infer by (12.7) that there exists a point $y \in Y$ such that $\psi(P) \subset W(y, \varepsilon)$. By Lemma (12.9), there exists in X^P a sequence $g_1, g_2, \ldots$ of maps such that

$$\lim_{i\to\infty} \varrho(fg_i, \psi) = 0. \tag{12.11}$$

Manifestly we can assume that all sets $fg_i(P)$ are contained in the set $W(y, \varepsilon)$ since otherwise we could replace the given sequence of maps $\{g_i\}$ by one of its subsequences. It follows by (12.6) that

$$g_i(P) \subset f^{-1}\big(W(y, \varepsilon)\big) \subset G(y, \varepsilon) \quad \text{for} \quad i = 1, 2, \ldots \tag{12.12}$$

We infer by (12.5) that the set $g_i(P)$ is contractible in $U(y, \varepsilon)$. Consequently the map g_i is homotopic to a constant in the set $U(y, \varepsilon)$. Moreover, applying Lemma (12.10), we may assume that for every index $i = 1, 2, \ldots$ there exists a homotopy $\{g_{i,t}\} \subset Y^P$ such that

$$g_{i,0} = fg_i \quad \text{and} \quad g_{i,1} = fg_{i+1},$$

and that for every point $p \in P$ the diameter of the set $\{g_{i,t}(p);\ 0 \leqslant t \leqslant 1\}$ is less than some number $\varepsilon_i > 0$ with $\lim_{i\to\infty} \varepsilon_i = 0$.

It follows at once that setting

$$\psi_t(p) = g_{i,(i+1)(1-it)}(p) \quad \text{for every point } p \in P \text{ and } \frac{1}{i+1} < t \leqslant \frac{1}{i},$$

$$\psi_0(p) = \psi(p) \quad \text{for every point } p \in P,$$

we get a homotopy $\{\psi_t\} \subset Y^P$. Since (by (12.8)) $\psi(P) \subset W(y, \varepsilon) \subset V(y, \varepsilon)$ and since $\lim_{i\to\infty} \varepsilon_i = 0$, we infer by (12.11) that there is an index i_0 such that

$$g_{i,t}(P) \subset V(y, \varepsilon) \quad \text{for every} \quad i \geqslant i_0.$$

It follows that the maps ψ_t, for $0 \leqslant t \leqslant 1/i_0$ constitute a homotopy with values in the set $V(y, \varepsilon)$ which joins the map $\psi_0 = \psi$ with the map $\psi_{1/i_0} = g_{i_0,0} = fg_{i_0}$. Since g_{i_0} is homotopic to a constant in the set $U(y, \varepsilon)$, we infer by (12.2) that the map fg_{i_0} is homotopic to a constant in the set $V(y, \varepsilon)$. Consequently the map ψ is also homotopic to a constant in the set $V(y, \varepsilon)$. Since the diameter of the set $V(y, \varepsilon)$ is $\leqslant \varepsilon$, the proof that $Y \in \mathbf{LC}^\infty$ is terminated.

Moreover let us observe that in the case where $X \in$ AR any condition concerning the diameter of the set $\psi(S^n)$ is superfluous in the proof of the homotopy of ψ to a constant. It follows that in this case $Y \in \mathbf{C}^\infty$. Thus the proof of Theorem (12.1) comes to an end.

Using the notion of the upper semicontinuous decomposition and its hyperspace (see, for instance, [194], p. 171, or [203], p. 46), one can express Theorem (12.1) as follows:

(12.13) *For each upper semicontinuous decomposition of an* ANR*-space X into* AR*-s the hyperspace Y of this decomposition satisfies the condition* $\mathbf{LC}^\infty$. *Moreover, if $X =$* AR, *then $Y \in \mathbf{C}^\infty$.*

Some relations between the homology and homotopy properties of a space $X \in \mathrm{ANR}(\mathfrak{M})$ and of the decomposition space of an upper-semi-continuous decomposition of X into elements being $\mathrm{AR}(\mathfrak{M})$-spaces have been studied also by Y. Kodama [184].

From Theorem (12.1) and Theorems (10.3) and (10.6) we deduce the following two corollaries:

(12.14) COROLLARY. *If $X \in$* ANR *and Y is a metric, finite-dimensional space and if a map $f\colon X \to Y$ satisfies the condition $f^{-1}(y) \in$* AR *for every point $y \in Y$, then $Y \in$* ANR.

(12.15) COROLLARY. *If $X \in$* AR *and Y is a metric finite-dimensional space and if a map $f\colon X \to Y$ satisfies the condition $f^{-1}(y) \in$* AR *for every point $y \in Y$, then $Y \in$* AR.

(12.16) PROBLEM. *Do Corollaries* (12.14) *and* (12.15) *remain true if we omit the hypothesis that the dimension of the space Y is finite?*

13. Plane AR**-sets.** By Corollary (10.4) the study of finite-dimensional ANR-s is reduced to the study of locally contractible, compact subsets of Euclidean spaces, that is of bounded neighborhood retracts of Euclidean spaces. As we have already proved (I, (4.2)), if $X \subset E^n$ and

X is a bounded neighborhood retract in E^n, then E^n-X has only a finite number of components. If $X \in \mathrm{AR}$, then the set E^n-X is connected for $n>1$, and it consists of exactly two components if $n=1$.

From these remarks we see that every AR-set in E^1 is either a point or a closed segment, while every ANR-set in E^1 is a finite union of closed segments and of isolated points. Already in the plane E^2 the situation is much more complicated. The following theorem ([20], p. 211) characterizes plane AR-sets:

(13.1) THEOREM. *A subset of the Euclidean plane E^2 is an* AR-*set if and only if it is a locally connected, non-empty continuum which does not decompose E^2.*

The proof of this theorem is based on three lemmas.

(13.2) LEMMA. *If X is a continuum in E^k and E^k-X is connected, then for every $\varepsilon>0$ there exists an $\eta>0$ such that if the distance of each point of the boundary $Z^\bullet$ of a compactum $Z \subset E^k$ from X is less than η, then the distance of each point of Z from X is less than ε.*

Proof. Otherwise there would exist an $\varepsilon>0$ and a bounded sequence of points $\{a_n\} \subset E^k$ such that for $n=1,2,\ldots$:

(1) $\varrho(a_n, X)>\varepsilon$.

(2) There exists a compactum $Z_n \subset E^k$ such that $a_n \in Z_n$ and that $\varrho(x, X)<1/n$ for every point $x \in Z_n^\bullet$.

Let a be a limit point of the sequence $\{a_n\}$. Then $\varrho(a, X) \geqslant \varepsilon$ and there exists a positive number α and a connected unbounded subset A of E^k-X such that $a \in A$ and that $\varrho(x, X)>\alpha$ for every point $x \in A$. Moreover, there exists an index n such that $1/n<\mathrm{Min}(\alpha, \frac{1}{2}\varepsilon)$ and that $\varrho(a, a_n)<\frac{1}{2}\varepsilon$. We infer by (1) that the distance of each point of the segment $|a, a_n|$ from X is greater that $\frac{1}{2}\varepsilon$ and consequently $|a, a_n| \cap Z_n^\bullet=0$. Hence $a \in Z_n-Z_n^\bullet$. Since the connected set A contains the point a and being unbounded it contains also a point of E^k-Z_n, we infer that there exists a point $c \in A \cap Z_n^\bullet$. But this is impossible, because $c \in A$ implies $\varrho(c, X)>\alpha$ and $c \in Z_n^\bullet$ implies $\varrho(c, X)<1/n<\alpha$.

(13.3) LEMMA. *Let X be a non-degenerate continuum in E^2 such that the set E^2-X is connected and let F be a finite subset of X. Then for every positive ε there exists a polyhedron $P \subset E^2$ which is a disk satisfying the following three conditions:*

(i) $X \subset P$.

(ii) *If $x \in P$, then $\varrho(x, X) \leqslant \varepsilon$.*

(iii) *The boundary $P^\bullet$ of the disk P is the union of a finite number of arcs $L_1, L_2, \ldots, L_k$ with ends belonging to $X-F$, with interiors lying in the region $G=E^2-X$, and with diameters less than ε.*

Proof. Consider a positive number η and a triangulation $\mathscr{T}$ of the plane E^2 with diameters of simplexes less than η. One readily sees that the triangulation $\mathscr{T}$ can be chosen so that each triangle of it either contains in its interior a point $x \in X$ or it is disjoint to X. Let A denote the union of all triangles $\sigma \in \mathscr{T}$ such that $\sigma \cap X \neq 0$. Manifestly A is a connected polyhedron containing X in its interior. This polyhedron may decompose the plane E^2, but if we add to it all bounded components of the set $E^2 - A$, then we get a polyhedron B being a disk. It is easy to see (by Lemma (13.2)) that for η sufficiently small, every point $x \in B$ satisfies the condition $\varrho(x, X) \leqslant \varepsilon$. Moreover, we can assume that $\eta < \frac{1}{2}\varepsilon$.

Now let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be these triangles of the triangulation $\mathscr{T}$ which lie in B but have at least one side included in the simple closed broken line $B^{\bullet}$ being the boundary of the disk B. Let $M_i = |a_i, b_i|$ be a side of σ_i, $i = 1, 2, \ldots, k$, such that $M_i \subset B^{\bullet}$. Manifestly, $\sigma_i \cap X \neq 0$ and $M_i \cap X = 0$. Then there exists a point $x_i \in X$ lying in the interior of the triangle σ_i. Let us show that the point x_i can be selected so that it does not belong to the set F and that there exists in the triangle σ_i a polygonal arc D_i with endpoints a_i and b_i such that $D_i - (a_i) - (b_i) \subset \sigma_i - \sigma_i^{\bullet}$ and that the disk (in E^2) bounded by $D_i \cup M_i$ has with X only the point x_i in common. In order to show this, consider a point $z_i \in X - F$ lying in the interior of σ_i. On the segment M_i there exist evidently two points a_i', b_i'

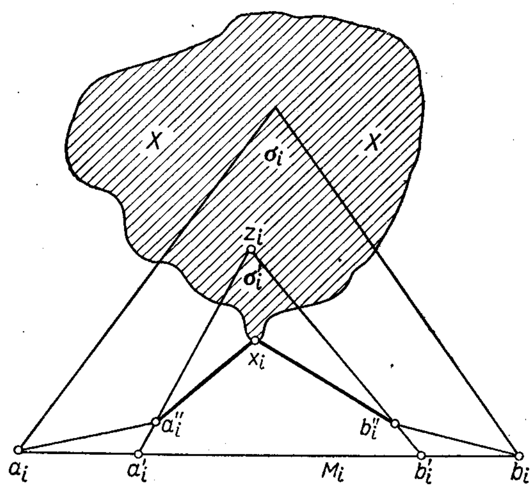


Fig. 4

such that a_i' lies between a_i and b_i' and that the triangle $\sigma_i' = |a_i', b_i', z_i|$ is disjoint with F. Let x_i be a point of the set $X \cap T$ with the minimal distance from the segment M_i. If we select two points $a_i'' \in |a_i', z_i| - (a_i')$ and $b_i'' \in |b_i', z_i| - (b_i')$ lying sufficiently close to the segment M_i, then we see at once that the point x_i and the polygonal arc

$$D_i = |a_i, a_i''| \cup |a_i'', x_i| \cup |x_i, b_i''| \cup |b_i'', b_i|$$

satisfy all required conditions.

If we replace in the broken line $B^{\bullet}$ each of the segments M_i by the broken line D_i, then, as it is easy to see, we obtain a simple closed broken line C which bounds a polygonal disk P satisfying condition (i). Since $\varrho(x, X) < \varepsilon$ for each point $x \epsilon B$, condition (ii) is also satisfied. Moreover, it follows by our construction, that the points $x_1, x_2, \ldots, x_k \epsilon X - F$ decompose the broken line C into arcs $L_1, L_2, \ldots, L_k$, each of them is contained in one or in the union of two triangles of the triangulation $\mathscr{T}$ with a common vertex. It follows by $\eta < \frac{1}{2}\varepsilon$ that the diameter of each arc L_i is less than ε. Thus condition (iii) is also satisfied and the proof of Lemma (13.3) is finished.

The polygonal disk satisfying conditions (i), (ii), and (iii) (with $F = 0$) will be said to be an *ε-approximation* of the continuum X.

(13.4) LEMMA. *Let X be a plane non-degenerate continuum which does not decompose E^2. Then there exists a sequence $\{P_m\}$ of polygonal disks such that:*

(i) *P_m is a $\dfrac{1}{m}$-approximation of the continuum X.*

(ii) *$P_{m+1} \subset P_m$ for every $m = 1, 2, \ldots$*

Proof. It follows by Lemma (13.3) that there exists a polygonal disk P_1 being a 1-approximation of X. Consequently, it remains to show that if we have a polygonal disk P_m being a $\dfrac{1}{m}$-approximation of X, then we can construct a polygonal disk $P_{m+1} \subset P_m$ being a $\dfrac{1}{m+1}$-approximation of X. In order to show it let us fix an orientation on the plane E^2. It induces an orientation on the boundary $P_m^{\bullet}$ of the disk P_m. By our hypothesis,

$$P_m^{\bullet} = L_{m,1} \cup L_{m,2} \cup \ldots \cup L_{m,n_m},$$

where $L_{m,i}$ is a polygonal arc (oriented as P_m) with the initial point a_i and the terminal point b_i, both belonging to X, and with the interior lying in the set $E^2 - X$. The diameter of the arc $L_{m,i}$ is less than $1/m$. Moreover, we can assume that the arcs $L_{m,1}, L_{m,2}, \ldots, L_{m,n_m}$ are distinct and that they are ordered according to the fixed orientation of the curve P_m. It follows that $a_{i+1} = b_i$ for $i = 1, 2, \ldots, n_m$, where a_{n_m+1} denotes the point a_1. Now let us select an integer $q \geqslant 1$ and let us pick up on the arc $L_{m,i}$ two points a_i' and b_i' such that the segments $|a_i, a_i'|$ and $|b_i, b_i'|$

are included in $L_{m,i}$ and each of them has the diameter less than the minimum of two numbers: $\frac{1}{2}\varrho(a_i, b_i)$ and $1/2(m+q)$. Thus the broken line $L_{m,i}$ consists of two segments $|a_i, a_i'|$, $|b_i, b_i'|$ and of an arc $L'_{m,i}$ lying in the set $E^2 - X$ with the initial point a_i' and the terminal point b_i'.

Now let us select a positive number ε smaller than $1/2(m+q)$ and also less than the distance of every point of $L'_{m,i}$ from the set X. Let F denote the set consisting of points $a_1, b_1, a_2, b_2, \ldots, a_{n_m}, b_{n_m}$. By Lemma (13.3), there exists a polygonal disk P satisfying conditions (i), (ii), and (iii) of this lemma. Let us orient the boundary $P^\bullet$ of the disk P according to the fixed orientation of the plane E^2 (consequently, also according to the orientation of the curve $P^\bullet_m$); it consists of the arcs $L_1, L_2, \ldots, L_k$ satisfying condition (iii), and we can assume that their order is consistent with the orientation of P. Moreover, let us observe that each of the arcs L_j $(j = 1, 2, \ldots, k)$ is disjoint with each of the arcs $L'_{m,i}$ $(i = 1, 2, \ldots, n_m)$. Manifestly the set $X \cup L_{m,i}$ disconnects E^2 into two regions, from which one is bounded; we denote it by G_i. If ε is sufficienlty small, then there exist

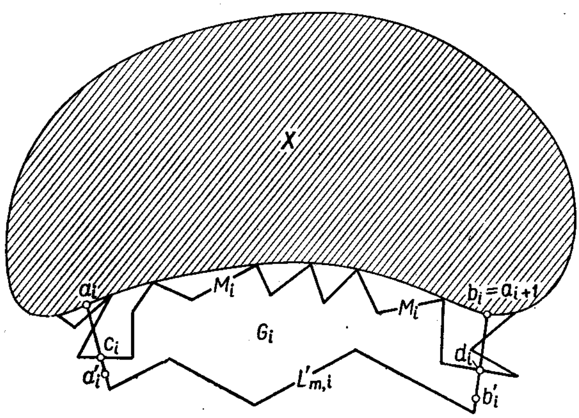


Fig. 5

among the arcs L_j several ones which lie in G_i. Then there exists on the boundary $P^\bullet$ a polygonal arc M_i (oriented consistently to $P^\bullet$) lying in the closure of the region G_i and such that its initial point c_i belongs to the segment $|a_i, a_i'|$ and its terminal point d_i belongs to the segment $|b_i', b_i|$. Since the endpoints of arcs L_j do not belong to F, one readily sees that the set

$$\bigcup_{i=1}^{n_m} (M_i \cup |a_i, c_i| \cup |d_i, b_i|)$$

is a simple closed curve bounding a polygonal disk $P_{m+1} \subset P_m$ such that $X \subset P_{m+1}$. Moreover, our construction implies that the boundary of the disk P_{m+1} is the union of these arcs L_j, which are contained in the arcs

M_i, and of arcs each from which is the union of a segment contained in $|a_i, a'_i|$ or in $|b'_i, b_i|$ and of some arc contained in one of the arcs L_j. Since the diameter of each of these sets $|a_i, a'_i|$, $|b'_i, b_i|$ and L_j is less than $1/2(m+q)$, we infer that the set P_{m+1} is the union of arcs with endpoints belonging to the set X and with interiors lying in the set E^2-X, and the diameters of these arcs are $\leqslant 1/(m+q)$. If the number q is sufficiently great, then we infer by Lemma (13.2) that the polygonal disk $P_{m+1} \subset P_m$ is a $\dfrac{1}{m+1}$-approximation of the continuum X and the proof of Lemma (13.3) is finished.

Now let us pass to the

Proof of Theorem (13.1). As we have already seen (2.21), every plane AR-set is a locally connected continuum which does not disconnect the plane. Now let us assume that $X \subset E^2$ is a locally connected continuum such that the set E^2-X is connected. We can assume that X is not degenerate. By Lemma (13.4), there exists a decreasing sequence $\{P_n\}$ of disks satisfying the conditions of this lemma. Let us define a sequence $\{r_m\}$ of retractions

$$r_m\colon P_1 \to P_m$$

in the following manner: the retraction r_1 is the identity map. Assume that for an index m the retraction r_m carrying the set P_1-P_m into $P^{\bullet}_m$ is already defined. The boundary $P^{\bullet}_m$ of the disk P_m is the union of n_m arcs $L_{m,i}$ with endpoints belonging to X and with interiors lying in the set

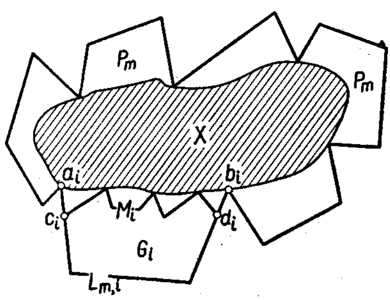


Fig. 6

E^2-X. Let us keep the notation used in the proof of Lemma (13.4), and let G_i denote the bounded region of the set $E^2-(X \cup L_{m,i})$. The boundary $P^{\bullet}_{m+1}$ of the disk P_{m+1} is the union of n_m arcs of the form $N_i = M_i \cup |a_i, c_i| \cup |d_i, b_i| \subset \bar{G}_i$. Let us denote by φ_i a retraction defined on the set $\bar{G}_i$ as a map which is the identity on the closure of each bounded component of the set $E^2-(X \cup N_i)$ and which carries all other points of the set $\bar{G}_i$ into N_i. The existence of such a map φ_i is obvious, because N_i is an AR-set. Setting

$$\varphi(x) = \begin{cases} \varphi_i(x) & \text{for every point } x \in \bar{G}_i, \\ x & \text{for every point } x \in X, \end{cases}$$

we get a retraction $\varphi\colon P_m \to P_{m+1}$. Let us observe that φ maps the points of the set $P_m - P_{m+1}$ into the set $P^{\bullet}_{m+1}$. Setting $r_{m+1} = \varphi r_m$, we obtain a retraction

$$r_{m+1}\colon P_1 \to P_{m+1}.$$

It is clear that r_{m+1} maps the set $P_1 - P_{m+1}$ into the set $P^{\bullet}_{m+1}$.

Formerly the hypothesis that the continuum X is locally connected was not utilized. Let us observe that local connectedness of the set X implies that the diameters of the regions G_i converge to zero. In fact, since the diameter of the arc $L_{m,i}$ is less than or equal to $1/m$, the local connectedness of X implies that there exists a sequence $\{\varepsilon_m\}$ of positive numbers converging to zero such that the ends a_i and b_i of the arc $L_{m,i}$ can be joined in the continuum X by an arc $\hat{L}_{m,i}$ with diameter less than ε_m. It follows that the diameter of the region G_i is less than $1/m + \varepsilon_m$; hence it converges to zero when m increases without bound.

Now let us observe that the inductive definition of the maps r_m implies that

$$r_{m+k}(x) \in \bar{G}_i \quad \text{for every point } x \in \bar{G}_i \text{ and every } k = 1, 2, \ldots$$

It follows that

$$\varrho\big(r_{m+k}(x), r_m(x)\big) < \frac{1}{m} + \varepsilon_m \quad \text{for every } k = 1, 2, \ldots,$$

and consequently the sequence $\{r_m\}$ is uniformly convergent to a map $r = \lim_{m\to\infty} r_m$ of the disk P_1 into X, and this map is the identity on X. Thus r is a retraction of the disk P_1 to the set X and consequently $X \in \mathrm{AR}$.

Remark. Let us observe that the retraction $r\colon P_1 \to X$ maps the set $P_1 - X$ into the boundary of X. Moreover, there exists a retraction ψ of the whole plane E^2 to the disk P_1 such that

$$\psi(E^2 - P_1) \subset P^{\bullet}_1.$$

Setting

$$r_0(x) = r\psi(x) \quad \text{for every point } x \in E^2,$$

we get a retraction $r_0\colon E^2 \to X$ mapping the set $E^2 - X$ into the boundary of the set X. We get the following

(13.5) Corollary. *If X is a non-empty and locally connected continuum in E^2, which does not disconnect E^2, then there exists a retraction $r_0\colon E^2 \to X$ satisfying the condition $r_0(E^2 - X) \subset X \cap \overline{E^2 - X}$.*

Let us recall that among locally connected curves the dendrites are characterized by the unicoherence ([203], p. 225) and also by the acyclicity ([22], p. 230) and that ([281], pp. 125 and 137) every dendrite is homeomorphic to a subset of the plane E^2. These facts and Theorem (13.1) imply the following proposition ([20], p. 211):

(13.5) COROLLARY. *One-dimensional* AR*-spaces coincide with dendrites.*

Another consequence of Theorem (13.1) is the following

(13.6) COROLLARY. *One-dimensional* ANR*-spaces coincide with the locally connected compacta of dimension* 1 *with the finite first Betti number.*

Indeed, since every ANR-space is a locally connected compactum with finite Betti numbers, the proof reduces to showing that these conditions are sufficient. In order to do it, let us observe that a 1-dimensional compactum X having the finite first Betti number contains only a finite number of simple closed curves. By virtue of the local connectedness of X, we infer that every point $x \in X$ has an arbitrarily small neighborhood being a locally connected continuum which does not contain any simple closed curve. This means that such a neighborhood is a dendrite, whence it is an AR-set. As we have already shown (2.15) this implies that X is an ANR-space.

14. Plane ANR**-sets.** Now let us prove the following ([21], p. 242)

(14.1) THEOREM. *A subset X of the plane E^2 is an* ANR*-set if and only if X is a locally connected compactum such that the set E^2-X has a finite number of components.*

Proof. The necessity of these conditions has been already proved (I, (4.2) and (8.2)). In order to show that they are also sufficient consider a locally connected compactum $X \subset E^2$ such that the set E^2-X consists of a finite number of components. The local connectedness of the compactum X implies that X has only a finite number of components. It is evident that each of them decomposes E^2 into a finite number of regions. Thus in the sequel we can restrict our considerations to the case where X is a continuum. If we add to X all bounded components of the set E^2-X, we get a locally connected continuum X_0 which does not decompose the plane E^2. It is clear that the set $G_0 = E^2-X_0$ coincides with the unbounded component of the set E^2-X. By Corollary (13.4), there exists a retraction r_0 of the closure $\overline{G}_0$ of the set G_0 to its boundary $G_0^{\bullet}$.

Consider now all bounded components $G_1, G_2, \ldots, G_m$ of the set E^2-X. Let us pick up a point a_i in each of the sets G_i, $i = 1, 2, \ldots, m$, and let α_i denote the inversion of the plane E^2 with the pole a_i. Manifestly, α_i maps the set $E^2-(a_i)$ topologically onto itself and the image $Y_i = \alpha_i(X)$ of the set X is a locally connected continuum and the set $H_i = \alpha_i(G_i-(a_i))$

is the unbounded component of the set E^2-Y_i. As we have seen (13.4), there exists a retraction s_i of the closure of the region H_i onto its boundary. It is clear that setting

$$r_i(x) = \alpha_i s_i \alpha_i(x) \quad \text{for every point } x \in \bar{G}_i-(a_i),$$

we get a retraction r_i of the set $\bar{G}_i-(a_i)$ to the boundary of the region G_i. Now it is sufficient to set

$$r(x) = \begin{cases} r_0(x) & \text{for every point } x \in G_0, \\ r_i(x) & \text{for every point } x \in G_i-(a_i),\ i = 1, 2, \ldots, m, \\ x & \text{for every point } x \in X, \end{cases}$$

in order to obtain a retraction

$$r\colon E^2-\bigcup_{i=1}^{m}(a_i) \to X.$$

Since the set $E^2-\bigcup_{i=1}^{m}(a_i)$ is a neighborhood of the set X in the plane E^2, we conclude that X is a neighborhood retract in the space E^2, and consequently $X \in$ ANR. Thus the proof of the theorem is terminated.

Remark. Let us observe that the just defined retraction r maps the set G_0 onto a subset of the boundary $X^\bullet = X \cap \overline{E^2-X}$ of the continuum X, and also it maps each of the regions $G_i-(a_i)$, $i = 1, 2, \ldots, m$, onto a subset of $X^\bullet$. If the compactum X is not connected (but locally connected) and such that the set E^2-X has a finite number of components, then there exists a neighborhood U of X such that the distinct components of X lie in distinct components of U and that none of the components of the set E^2-X is contained in U. Applying the just used construction to each component of X, we infer that there exists a retraction of the set U to X which maps the set $U-X$ onto a subset of the boundary $X^\bullet$ of X. Thus we get the following

(14.2) COROLLARY. *If $X \subset E^2$ is a locally connected compactum and if the set E^2-X consists of a finite number of components, then there exists a retraction of a neighborhood U of X in E^2 to X such that the image of the set $U-X$ is contained in the boundary $X^\bullet$ of X.*

Let us mention that one proves by an elementary argument ([29], p. 138) that every connected, plane, non-empty ANR-set can be represented as the union of two AR-sets. An analogous proposition holds also for all polyhedra lying in E^3 and acyclic in dimensions 0 and 1 ([31], p. 56).

15. n-dimensional subsets of an n-dimensional ANR-**space.** It is evident that the separability of an n-dimensional polyhedron implies that it does not contain an uncountable family of disjoint n-dimensional cubes. On the other hand, there are n-dimensional compacta which do contain such families. For example, the Cartesian product of an n-dimensional cube Q^n and a Cantor discontinuum C is an n-dimensional compactum $Q^n \times C$ which contains the uncountable family of disjoint n-dimensional cubes of the form $Q^n \times (x)$ with $x \in C$. Now we shall prove that for ANR-spaces the situation is the same as for polyhedra. More precisely, we have the following *First Theorem on Families of* ANR-*sets* ([70], p. 685, and [71], p. 115):

(15.1) THEOREM. *Let $\{X_\mu\}$, with μ running over an uncountable set M, be a family of n-dimensional* ANR-*sets lying in an n-dimensional* ANR-*space X. Then there are distinct indices μ and μ' in M such that* $\dim(X_\mu \cap X_{\mu'}) = n$.

Proof. Without loss of generality we may assume that X is a subset of Hilbert space E^ω. Then, since $X \in$ ANR, there is a neighborhood U of X in E^ω and a retraction $r\colon U \to X$. We assume also that for distinct indices $\mu \neq \mu'$ we have $X_\mu \neq X_{\mu'}$, since the theorem is trivial otherwise. Since $\dim X_\mu = n$, we infer by II, (3.11), and (3.4) that

(15.2) *There exists in X_μ an infinite n-dimensional chain $\varkappa_\mu = \{\varkappa_{\mu i}\}$, a positive number ε_μ and a compact carrier $A_\mu \subset X$ of the infinite cycle $\{\partial\varkappa_{\mu i}\}$ such that this infinite cycle is not homologous to zero in the generalized ball $Q_\mu = \{x \in E^\omega;\ \varrho(x, A_\mu) \leqslant \varepsilon_\mu\}$.*

It readily follows that

(15.3) *If C is a compactum and if the infinite cycle $\{\partial\varkappa_{\mu i}\}$ is homologous to zero in $Q_\mu \cup C$, then* $\dim C \geqslant n$.

In order to show it, let us observe that the boundary B_μ of the set $C - Q_\mu$ in the space $C \cup Q_\mu$ is non-empty. Let $\{\lambda_i\}$ be an infinite chain in $Q_\mu \cup C$ such that $\partial\lambda_i = \partial\varkappa_{\mu i}$ for every $i = 1, 2, \ldots$ Consider a vertex $x \in C - Q_\mu$ belonging to a simplex σ of λ_i with at least one vertex lying in Q_μ. Let us assign to each such vertex x a point $\varphi_i(x) \in B_\mu$ such that $\varrho\big(x, \varphi_i(x)\big)$ is equal to the distance of x from B_μ. Setting $\varphi_i(x) = x$ for all other vertices of the chain λ_i, we get from the chain λ_i a chain $\varphi_i(\lambda_i)$ lying in the set $C \cup Q_\mu$ with the same boundary $\partial\varphi_i(\lambda_i) = \partial\varkappa_{\mu i}$ (because the function φ_i assigns to each vertex x of the chain λ_i lying in Q_μ the same point x). We immediately see that $\{\varphi_i(\lambda_i)\}$ is an infinite chain and that each simplex of the chain $\varphi_i(\lambda_i)$ lies in one at least of the sets Q_μ and C. Thus we can assume that the chains λ_i themselves have this property already.

Now we can decompose the chain λ_i into the sum of two chains $\lambda_i = \lambda_i' + \lambda_i''$, where λ_i' lies in C and λ_i'' in Q_μ. Then $\partial\varkappa_{\mu i} = \partial\lambda_i' + \partial\lambda_i''$, and consequently $\partial\lambda_i' = \partial\varkappa_{\mu i} - \partial\lambda_i''$, where the chain $\partial\lambda_i'$ lies in the set C, and the chain $\partial\varkappa_{\mu i} - \partial\lambda_i''$ lies in the set Q_μ. If follows that $\{\partial\lambda_i'\}$ is an infinite cycle lying in the set $C \cap Q_\mu$ and it is homologous to the infinite cycle $\{\partial\varkappa_{\mu i}\}$

in the set Q_μ. Consequently $\{\partial\lambda_i'\}$ is an $(n-1)$-dimensional infinite cycle in the set C homologous to zero in C but not homologous to zero in the set $C \cap Q_\mu \subset Q_\mu$, which is its carrier. By II,(3.11), this implies the inequality $\dim C \geqslant n$. Thus proposition (15.3) is proved.

Now let us observe that we can replace in (15.2) the number ε_μ by an arbitrarily given positive number ε less than ε_μ. Since the set of indices M is uncountable, we infer that there exists a positive number ε such that the inequality $\varepsilon \leqslant \varepsilon_\mu$ holds for all indices belonging to an uncountable subset M_0 of the set M. Replacing ε_μ by ε for $\mu \in M_0$, we get

(15.4) $$\varepsilon_\mu = \varepsilon \quad \text{for every index } \mu \in M_0.$$

The compacta X_μ can be considered as points of the space 2^X of all (non-empty) subcompacta of X (with the Hausdorff metric; see for instance [203], p. 20). Since 2^X is a compactum and since the set M_0 is uncountable, there is a sequence $\{\mu_m\}$ of distinct indices in M_0 and an index $\nu \in M_0$ such that

(15.5) $$\lim_{m\to\infty} X_{\mu_m} = X_\nu.$$

Moreover, we can assume that $\mu_m \neq \nu$ for $m = 1, 2, \ldots$ and consequently

(15.6) $$X_{\mu_m} \neq X_\nu \quad \text{for} \quad m = 1, 2, 3, \ldots$$

Since X_ν is an ANR-set, there is a neighborhood V of X_ν in E^ω and a retraction $s\colon V \to X_\nu$. From (15.5) it follows that there is a natural number m_0 such that for the index $\mu_0 = \mu_{m_0}$ the following statements hold:

(15.7) *The segment from x to $s(x)$ is contained in $U \cap V$ for all $x \in X_{\mu_0}$.*

(15.8) *The diameter of the image under the map r of the segment from x to $s(x)$ is $< \varepsilon$ for all $x \in X_\mu$.*

Let Z be the compactum obtained by taking the union of all segments from x to $s(x)$ as x runs over A_{μ_0} (recall (15.7)). Define the functions f_t, $t \in \langle 0, 1\rangle$, on X_{μ_0} by means of the formula

(15.9) $$f_t(x) = r\big((1-t)x + ts(x)\big) \quad \text{where} \quad t \in \langle 0, 1\rangle \text{ and } x \in X_{\mu_0}.$$

Then the family $\{f_t\}$ is a homotopic deformation of X_{μ_0} to X_ν in the space X. Note that f_0 is the inclusion $X_{\mu_0} \to X$ and that $f_1 = s|X_{\mu_0}$. From this, together with (15.8), (15.9) and (15.4), we see that $f_t(x) \in r(Z) \subset Q_{\mu_0}$ for all $x \in A_{\mu_0}$. It follows by II,(3.9), that there exists in the space X an infinite $(n+1)$-dimensional chain $\lambda = \{\lambda_i\}$ such that

(15.10) $$\partial\lambda = \varkappa_{\mu_0} - s(\varkappa_{\mu_0}) - \alpha,$$

where $\alpha = \{a_i\}$ is an infinite n-dimensional chain lying in the set $r(Z) \subset Q_{\mu_0}$. Thus we see that

(15.11) $$\gamma = \varkappa_{\mu_0} - s(\varkappa_{\mu_0}) - \alpha$$

is an infinite n-dimensional cycle lying in the compactum

$$Y = X_{\mu_0} \cup X_\nu \cup r(Z)$$

and homologous to zero in X. Since $\dim X = n$, we infer by II,(3.11), that the infinite cycle γ is not essential, and consequently there exists in Y an infinite $(n+1)$-dimensional chain $\beta = \{\beta_i\}$ such that

(15.12) $$\partial\beta_i = \varkappa_{\mu_0 i} - s(\varkappa_{\mu_0 i}) - a_i \quad \text{for} \quad i = 1, 2, \ldots$$

Let us set

$$Y_1 = X_{\mu_0}, \qquad Y_2 = X_\nu \cup r(Z).$$

Since $\partial\gamma = \partial\partial\lambda = 0$, we infer by (15.11) that

$$\partial\varkappa_{\mu_0} = \partial s(\varkappa_{\mu_0}) + \partial\alpha,$$

and since $\partial\varkappa_{\mu_0}$ lies in Y_1 and $\partial s(\varkappa_{\mu_0}) + \partial\alpha$ lies in Y_2, we infer that

(15.13) *The infinite cycle $\partial\varkappa_{\mu_0}$ lies in $Y_1 \cap Y_2$.*

Let $\{\varepsilon_i\}$ be a majorante of the infinite chain β. Since Y_1 and Y_2 are compacta, there exists a sequence $\{\eta_i\}$ of positive numbers convergent to zero and such that if $x_1 \in Y_1$, $x_2 \in Y_2$ and $\varrho(x_1, x_2) < \varepsilon_i$, then $\varrho(x_\nu, Y_1 \cap Y_2) < \eta_i$ for $\nu = 1, 2$. Consider for every $i = 1, 2, \ldots$ a function ψ_i defined in the set Y and satisfying the following two conditions:

(15.14) $\psi_i(x) = x$ *if* $\varrho(x, Y_1 \cap Y_2) \geqslant \eta_i$,

(15.15) $\psi_i(x)$ *is a point of* $Y_1 \cap Y_2$ *such that* $\varrho(x, \psi_i(x)) = \varrho(x, Y_1 \cap Y_2)$ *if* $\varrho(x, Y_1 \cap Y_2) < \eta_i$.

It is clear that the sequence $\hat{\beta} = \{\psi_i(\beta_i)\}$ is an infinite chain in Y satisfying the following condition:

(15.16) *For each simplex σ of the chain $\psi_i(\beta_i)$ all vertices of σ belong to one of the sets Y_1 and Y_2 at least.*

Equality (15.12) gives

(15.17) $\partial\hat{\beta} = \{\psi_i(\partial\beta_i)\} = \{\psi_i(\varkappa_{\mu_0 i}) - \psi_i(s(\varkappa_{\mu_0 i}) + a_i)\}$, *where* $\psi_i(\varkappa_{\mu_0 i})$ *lies in* Y_1 *and* $\psi_i(s(\varkappa_{\mu_0 i}) + a_i)$ *lies in* Y_2.

Moreover, it follows by (15.13) and (15.15) that

(15.18) $$\partial\varkappa_{\mu_0} \sim \partial\{\psi_i(\varkappa_{\mu_0 i})\} \quad \text{in} \quad Y_1 \cap Y_2.$$

Condition (15.16) implies that there is a decomposition

(15.19) $$\psi_i(\beta_i) = \beta_i' - \beta_i'' \quad \text{for} \quad i = 1, 2, \ldots$$

such that the chain β_i' lies in Y_1 and the chain β_i'' lies in Y_2. From (15.17) and (15.19) we infer that

$$(15.20) \qquad \psi_i(\varkappa_{\mu_0 i})-\partial\beta_i' = \psi_i\big(s(\varkappa_{\mu_0 i})+a_i\big)-\partial\beta_i''.$$

Since the chain of the left-hand side lies in Y_1, and the chain of the right-hand side lies in Y_2, it follows that the chain $\psi_i(\varkappa_{\mu_0 i})-\partial\beta_i'$ lies in the set

$$(15.21) \quad Y_1 \cap Y_2 = X_{\mu_0} \cap \big(X_\nu \cup r(Z)\big) = (X_{\mu_0} \cap X_\nu) \cup \big(X_{\mu_0} \cap r(Z)\big).$$

Since $\partial\big(\psi_i(\varkappa_{\mu_0 i})-\partial\beta_i'\big) = \partial\psi_i(\varkappa_{\mu_0 i})$, we infer that the infinite cycle $\partial\{\psi_i(\varkappa_{\mu_0 i})\}$ is homologous to zero in the set $Y_1 \cap Y_2$. It follows by (15.18) and (15.21) that the infinite cycle $\partial\varkappa_{\mu_0}$ is homologous to zero in the set $(X_{\mu_0} \cap X_\nu) \cap \big(X_{\mu_0} \cap r(Z)\big) \subset Q_{\mu_0} \cup (X_{\mu_0} \cap X_\nu)$. It follows, by (15.3), that $\dim(X_{\mu_0} \cap X_\nu) \geqslant n$ and since $\dim X_{\mu_0} = n$, we infer that $\dim(X_{\mu_0} \cap X_\nu) = n$. This completes the proof.

An important generalization of First Theorem on Families of ANR-sets has been given by K. Sieklucki ([262], p. 433):

(15.22) THEOREM. *Let $\{X_\mu\}$, with μ running over an uncountable set M, be a family of n-dimensional compacta lying in an n-dimensional* ANR-*space X. Then there are two distinct indices μ, $\mu' \in M$ such that* $\dim(X_\mu \cap X_{\mu'}) = n$.

The proof of this theorem is similar to the proof of (15.1) but a little more complicated.

We remark that if the dimension of a compactum X at a point x_0 is greater than n, then there is an $\varepsilon > 0$ such that for any positive number $\mu < \varepsilon$ the set $X_\mu = \{x \in X;\ \varrho(x, x_0) = \mu\}$ is a compactum of dimension $\geqslant n$. Consequently, *a compactum of dimension greater than n contains an uncountable family of disjoint compacta each of dimension $\geqslant n$.*

16. Umbrellas theorem. By an *n-dimensional umbrella* we understand the union of an n-dimensional ball $Q = Q^n$ and of an arc L such that the set $L \cap Q$ consists of only one point a, which is an endpoint of the arc L and an inner point of Q. The point a is called the *center* of this umbrella. Let us notice that there exist n-dimensional locally connected continua X such that each point $x \in X$ is the center of an n-dimensional umbrella lying in X. For instance, if C is the well-known universal curve of W. Sierpiński ([263], p. 630), then each point $x \in C$ is the center of a 1-dimensional umbrella lying in C. The Cartesian product $X = C \times S^{n-1}$, where S^{n-1} is an $(n-1)$-dimensional sphere, is a locally connected n-dimensional continuum with the property that every point of it is the center of an n-dimensional umbrella lying in X. A different situation is in the case of ANR-spaces. We have the following

(16.1) UMBRELLAS THEOREM ([16], p. 106). *If* $X \in \mathrm{ANR}$ *and* $\dim X = n$, *then the set* T *of centers of all* n*-dimensional umbrellas lying in* X *is of the first category of Baire in* X.

Proof. Let us suppose that T is of the second category of Baire in the space X. Consider an n-dimensional Euclidean ball Q and a point a of its interior. Then, for each point $x \in T$, there is a homeomorphism h_x of Q onto a subset $h_x(Q)$ of X such that $h_x(a) = x$ and that x is an endpoint of an arc L_x issuing from x and such that $L_x - (x) \subset X - h_x(Q)$. Since the space X^Q of all maps of Q into X is separable, there exists in T a sequence of points $x_1, x_2, \ldots$ such that the homeomorphisms $h_{x_1}, h_{x_2}, \ldots$ constitute a dense subset in the set of all homeomorphisms h_x, with $x \in T$. Then the point x_i belongs to the set

$$F_i = \overline{X - h_{x_i}(Q)} \cap h_{x_i}(Q). \tag{16.2}$$

Evidently the set F_i is closed and it does not contain any open non-empty subset of X. Since T is of the second category of Baire, there exists a point

$$x_0 \in T - \bigcup_{i=1}^{\infty} F_i. \tag{16.3}$$

Now let us observe that

$$x_0 \in T - \bigcup_{i=1}^{\infty} h_{x_i}(Q). \tag{16.4}$$

For if it were not so, then there would exist an index $i > 0$ such that $x_0 \in h_{x_i}(Q)$. Then — by (16.2) and (16.3) — the set $h_{x_i}(Q)$ would be a neighborhood of x_0 in the space X, which is impossible because x_0 is the center of an arbitrarily small n-dimensional umbrella lying in X and the set $h_{x_i}(Q)$ — as homeomorphic to Q — does not contain any n-dimensional umbrella.

Now let us consider the Cartesian product $Q^{n+1} = 0 \times \langle 0, 1 \rangle$. It is homeomorphic to the $(n+1)$-dimensional ball and its boundary Q^{n+1} is given by the formula

$$(Q^{n+1})^{\bullet} = \big(Q \times (0)\big) \cup (Q^{\bullet} \times \langle 0, 1 \rangle) \cup \big(Q \times (1)\big).$$

Let us set, for every index $i = 0, 1, \ldots,$

$$\varphi_i(p, 0) = h_{x_0}(p), \quad \varphi_i(p, 1) = h_{x_i}(p) \quad \text{for every } p \in Q. \tag{16.5}$$

Thus we get a map φ_i of the compact subset $\big(Q \times (0)\big) \cup \big(Q \times (1)\big)$ of the space Q^{n+1} into X. Setting

$$\bar{\varphi}_0(p, t) = h_{x_0}(p) \quad \text{for every point } (p, t) \in Q^{n+1},$$

we get a continuous extension of the map φ_0 onto Q^{n+1}, with values in X. Now let us recall that for every $\eta > 0$ there exists an index $i_\eta > 0$ such that $\varrho(h_{x_0}, h_{x_{i_\eta}}) < \eta$. Since $X \in \mathrm{ANR}$, we infer by Theorem (3.1) that for every $\varepsilon > 0$ the number η may be chosen so that the map φ_{i_η} has an extension $\bar{\varphi}_{i_\eta} \in X^{Q^{n+1}}$ satisfying the condition

$$\varrho(\bar{\varphi}_{i_\eta}, \bar{\varphi}_0) < \varepsilon.$$

Since the point x_0 does not belong to the set $\bar{\varphi}_0(Q^\bullet \times \langle 0, 1\rangle) = h_{x_0}(Q^\bullet)$, we infer that for sufficiently small value of ε

(16.6) *The point x_0 does not belong to $\bar{\varphi}_{i_\eta}(Q^\bullet \times \langle 0, 1\rangle)$.*

It follows by (16.4), (16.5), and (16.6) that

(16.7) $$x_0 \in h_{x_0}(Q) - \bar{\varphi}_{i_\eta}\big(Q^\bullet \times \langle 0, 1\rangle \cup Q \times (1)\big).$$

Now let us consider an infinite $(n-1)$-dimensional cycle γ^{n-1} in $Q^\bullet$ (with integer coefficients) which is not homologous to 0 in $Q^\bullet$. Then there exists in Q an infinite n-dimensional chain $\varkappa^n$ such that $\partial\varkappa^n = \gamma^{n-1}$. Thus $\gamma^{n-1} \sim 0$ in Q. On the other hand, it is clear that

(16.8) *γ^{n-1} is not homologous to zero in any proper subset of Q.*

Let a_ν $(\nu = 0, 1)$ denote the map of Q into $Q \times (\nu)$ assigning to each point $p \in Q$ the point $(p, \nu) \in Q \times (\nu)$. Then in $Q^\bullet \times \langle 0, 1\rangle$ there is an infinite n-dimensional chain λ^n such that

$$\partial\lambda^n = a_0(\gamma^{n-1}) - a_1(\gamma^{n-1}).$$

Setting

$$\gamma^n = a_0(\varkappa^n) - \lambda^n - a_1(\varkappa^n),$$

we get an infinite n-dimensional cycle γ^n in $(Q^{n+1})^\bullet$ which is not homologous to 0 in $(Q^{n+1})^\bullet$, but which is homologous to zero in Q^{n+1}. It follows that

$$\bar{\varphi}_{i_\eta}(\gamma^n) = h_{x_0}(\varkappa^n) - \bar{\varphi}_{i_\eta}(\lambda^n) - h_{x_{i_\eta}}(\varkappa^n)$$

is an infinite n-dimensional cycle in the set $\bar{\varphi}_{i_\eta}(Q^{n+1})^\bullet$ homologous to 0 in X. Moreover,

$$\partial h_{x_0}(\varkappa^n) = h_{x_0}(\partial\varkappa^n) = h_{x_0}(\gamma^{n-1}) = \partial\bar{\varphi}_{i_\eta}(\lambda^n) + \partial h_{x_{i_\eta}}(\varkappa^n)$$

is an infinite $(n-1)$-dimensional cycle in the set $h_{x_0}(Q^\bullet)$ which is homologous to 0 in the both sets $\bar{\varphi}_{i_\eta}\big(Q \times (0)\big) = h_{x_0}(Q)$ and $\bar{\varphi}_{i_\eta}\Big((Q^\bullet \times \langle 0, 1\rangle) \cup \big(Q \times (1)\big)\Big)$. Moreover, since h_{x_0} is a homeomorphism, we infer by (16.7) and (16.8) that the infinite cycle $h_{x_0}(\gamma^{n-1})$ is not homologous to 0 in the common part of the sets $h_{x_0}(Q)$ and $\bar{\varphi}_{i_\eta}\Big((Q^\bullet \times \langle 0, 1\rangle) \cup \big(Q \times (1)\big)\Big)$. It follows by Phragmén-Brouwer Theorem (II,(3.6)) that the infinite n-dimensional cycle $\bar{\varphi}_{i_\eta}(\gamma^n)$ is not homologous to 0 in its carrier $\bar{\varphi}_{i_\eta}\big((Q^{n+1})^\bullet\big)$. As we have already seen, it is homologous to zero in the space X, and consequently, by the theorem of Alexandroff (II,(3.11)), the dimension of X exceeds n, contrary to our hypothesis. Thus the proof of Umbrellas Theorem is finished.

CHAPTER VI

PATHOLOGIES AMONG ANR-SPACES

We have seen in the last chapter that ANR-spaces and polyhedra share many topological properties. In particular, we have shown that all r-properties of polyhedra are properties of ANR-spaces having finite dimension, because every such space is an r-image of a polyhedron.

In this chapter we shall consider some examples of ANR-spaces whose topological properties are profoundly different from the topological properties of polyhedra.

1. The singularity of Peano ([30], p. 255, and [51], p. 28). A space X is said to show the *singularity of Peano* provided that there is a closed subset X_0 of X such that it is contractible to a point in X and if $\{f_t\}$ is any homotopy contracting X_0 to a point, then the dimension of the set $\bigcup_{0\leqslant t\leqslant 1} f_t(X_0)$ swept out by this homotopy is necessarily greater than $1+\dim X_0$. We shall prove in Chapter VII that polyhedra are free of the singularity of Peano. However, there exist ANR-spaces having this singularity. We shall now construct an example of such a space.

Let Q^2 be a 2-dimensional disk and L a segment lying in the interior of Q^2. Since a solid 3-dimensional ball Q^3 is a Peano space (i.e. a locally connected continuum), it follows by the Hahn-Mazurkiewicz theorem that there is a map φ mapping L onto Q^3. Let us show that the space

$$X = Q^2 \overset{\varphi}{\cup} Q^3.$$

has the singularity of Peano. Indeed, this space is an AR-space by V, (9.17). Moreover, its dimension is clearly three. The boundary S of Q^2 is a simple closed curve which we can consider as lying in X. Let $\{f_t\}$, $t\in\langle 0,1\rangle$, be a homotopy contracting S to a point. We will show that the set F of values of $\{f_t\}$ is the whole space X, showing thereby that the space X has the singularity of Peano.

Suppose, by way of contradiction, that F is not all of X, so that there exists a point $a_0\in X-F$. By the definition of X, the set $Q^2-(L\cup S)$ is dense in X. Since F is a compactum, the set $X-F$ is open in X and we conclude that there is a point $a_1 \in Q^2 - (L \cup S \cup F)$. One can easily construct a retraction r of $Q^2 - (a_1)$ to S under which L is mapped onto a single point a_2 of S. Then, setting

$$r_1(p) = \begin{cases} r(p) & \text{for} \quad p \in Q^2 - L - (a_1), \\ a_2 & \text{for} \quad p \in L \overset{\varphi}{\cup} Q^3, \end{cases}$$

we obtain a retraction of the set $X-(a_1)$ to S. It follows that if we set

$$g_t(x) = r_1 f_t(x) \quad \text{for every } x \in S \text{ and } 0 \leqslant t \leqslant 1,$$

then we obtain a homotopy $\{g_t\}$ which contracts the simple closed curve S in itself to a point, which is impossible. This contradiction shows that X does indeed have the singularity of Peano.

Manifestly one can generalize this example by replacing the disk Q^2 by a $(k+1)$-dimensional ball Q^{k+1} (where $k \geqslant 1$) and Q^3 by an m-dimensional ball Q (where $m > k+1$). In this way one can construct an AR-space X in which every homotopy in X that contracts the boundary of Q^{k+1} to a point must sweep out the entire m-dimensional space X.

2. The singularity of Alexandroff. In 1925, P. Urysohn has introduced the important notion of the Cantor manifold (see, for instance, [166], p. 93) and he assigned to each compactum some numbers called coefficients of Urysohn. Let us recall the definitions of these notions.

An n-dimensional compactum X is said to be a *Cantor manifold* provided that the dimension of each closed subset X_0 of X decomposing X is not less than $n-1$. For every $k = 0, 1, \ldots$ and every compactum X, the *k-th coefficient of Urysohn* $d_k(X)$ ([277], p. 353) is defined as the greatest lower bound of the set of all positive numbers ε such that there exists a finite covering of X by closed sets with diameters less than ε and with the nerve of the dimension less than k. It is known ([166], p. 67) that

(2.1) *The compacta of dimension $\geqslant n$ are characterized by the inequality $d_n(X) > 0$.*

The notion of the Cantor manifold is a far reaching generalization of the classical notion of manifold. We easily see that each n-dimensional Cantor manifold X, which is a polyhedron, satisfies the following condition (formulated in 1957 by P. Alexandroff [2], p. 70):

(2.2) CONDITION (A). *For every couple of disjoint subsets A, B of X, with non-empty interiors, there exists a positive number ε such that every compactum $X_0 \subset X-(A \cup B)$ which decomposes X between each point $a \in A$ and each point $b \in B$ has the $(n-1)$-th coefficient of Urysohn $d_{n-1}(X_0)$ greater than ε.*

However, Condition (A) is not satisfied by all Cantor manifolds;

P. Alexandroff has given a simple example of a 2-dimensional Cantor manifold which does not meet Condition (A). Such an example can be defined as the union X of all rectangles R_n, $n = 0, 1, 2, \ldots$, given in E^2 by the formulas:

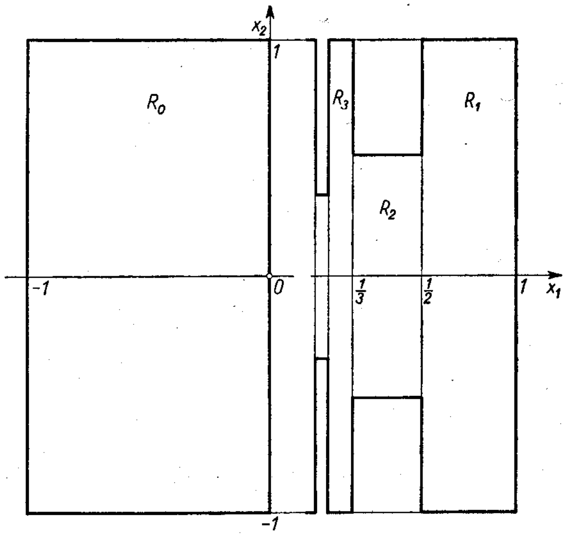


Fig. 7

$$R_0 = \{(x_1, x_2) \in E^2;\ -1 \leqslant x_1 \leqslant 0,\ -1 \leqslant x_2 \leqslant 1\},$$

$$R_{2k-1} = \left\{(x_1, x_2) \in E^2;\ \frac{1}{2k} \leqslant x_1 \leqslant \frac{1}{2k-1},\ -1 \leqslant x_2 \leqslant 1\right\},$$

$$R_{2k} = \left\{(x_1, x_2) \in E^2;\ \frac{1}{2k+1} \leqslant x_1 \leqslant \frac{1}{2k},\ \frac{-1}{k+1} \leqslant x_2 \leqslant \frac{1}{k+1}\right\}$$

for $k = 1, 2, \ldots$

It is evident that X is a 2-dimensional Cantor manifold without Condition (A). However it is not an ANR-set and, as has been recently proved by A. Lelek ([216], p. 241), every 2-dimensional ANR-space being a Cantor manifold satisfies Condition (A). Moreover, A. Lelek has constructed an example of a 3-dimensional AR-space being a Cantor manifold, which does not satisfy Condition (A).

The construction given by A. Lelek is the following: Let Q_1^3 and Q_2^3 be two 3-dimensional cubes having only an edge L in common. It is plain that the set $P = Q_1^3 \cup Q_2^3$ is an AR-set. Let φ be a map of L onto a square Q^2, disjoint to P. Then the space $X = P \overset{\varphi}{\cup} Q^2$ is an AR-space (by V,(9.17)) and it is clear that $\dim X = 3$. In order to show that X is a Cantor manifold let us observe that any closed subset X_0 of X with the dimension less than 2 cannot contain the whole set $L \overset{\varphi}{\cup} Q^2$ homeomorphic to Q^2. Since L is the common edge of Q_1^3 and of Q_2^3, we infer that the locally connected space X contains a simple arc joining in the set $X-X_0$ a point $a_1 \in Q_1^3-L$ with a point $a_2 \in Q_2^3-L$. Since Q_1^3 and Q_2^3 are Cantor manifolds, we conclude that every point $x \in X-X_0$ can be joined in the set $X-X_0$ by an arc with at least one of the points a_1 and a_2. Consequently X_0 does not disconnects the space X, and thus the proof that X is a 3-dimensional Cantor manifold is terminated.

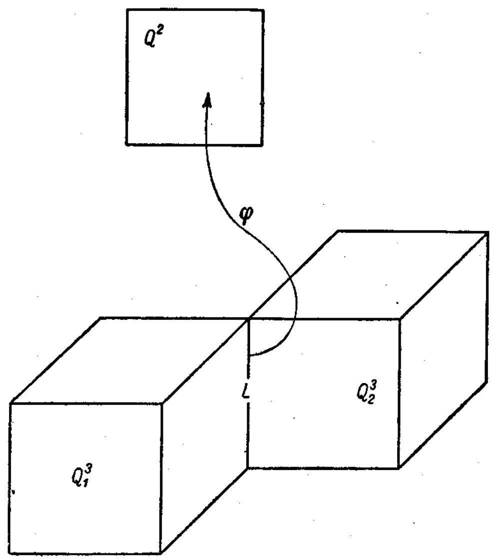


Fig. 8

Now let us show that the space X does not satisfy Condition (A). Consider a 3-dimensional ball K_i, lying in the interior of the cube Q_i^3 for $i = 1, 2$. In Q_1^3 there exists evidently a rectangle R arbitrarily narrow, and consequently having the Urysohn coefficient $d_2(R)$ arbitrarily small, which decomposes Q_1^3 between the points of the ball K_1 and the points of the edge L. One easily sees that by the operation of the matching $\overset{\varphi}{\cup}$, the rectangle R will be topologically mapped onto a compact subset X_0 of X with the second coefficient of Urysohn arbitrarily small. Since the set X_0 decomposes the space X between each point of the ball K_1 and each point of the ball K_2, we infer that the space X does not satisfy the condition (A).

Thus we have shown that X is a 3-dimensional Cantor manifold which is an AR-space, but which does not satisfy condition (A). We say that the space X has the *singularity of Alexandroff*.

3. The singularity of Brouwer. In 1910, L. E. J. Brouwer ([83], p. 426) constructed a plane continuum which was the common boundary of three plane regions. Later, K. Kuratowski ([194], p. 138) showed that any plane continuum with this property must be rather nasty. Namely, it is either an indecomposable continuum or the union of two indecomposable continua. In E^3 the situation is quite different. Indeed, in 1953 M. Lubański ([220], p. 29) constructed an ANR-set in E^3 which is the common boundary of three regions. We shall now carry out this construction.

The construction is similar to that involved in constructing the *Lakes of Wada* ([299], p. 60). Indeed, we shall begin with a solid 3-simplex in which two holes have been made and then bore out tubular feelers from the exterior and from each of the two holes, so that the feelers get "dense" in the simplex. In more detail, let A_1 be the polyhedron obtained from a 3-dimensional simplex $\sigma_0 \subset E^3$ by removal from its interior of the interiors ${}_1G_1$ and ${}_1G_2$ of two disjoint 3-simplexes. We assume that the diameter of σ_0 is $<\frac{1}{2}$ and we let ${}_1G_0 = E^3 - \sigma_0$. Evidently, for $n = 1$, the following two conditions are satisfied:

1_n. The set A_n is a polyhedron and is *strongly connected in dimension* 3, that means, for each triangulation $\mathcal{T}$ of A_n and for each pair of 3-simplexes σ and σ' of $\mathcal{T}$ there exists in $\mathcal{T}$ a finite sequence of 3-simplexes the first of which is σ, the last being σ', and such that simplexes with successive indices intersect in a common 2-dimensional face.

2_n. The set $E^3 - A_n$ has three components ${}_nG_0, {}_nG_1, {}_nG_2$ such that the distance from any $x \in A_n$ to each of these components is $< 2^{-n}$.

Now suppose that we have inductively constructed a decreasing system of polyhedra $A_1, A_2, \ldots, A_n$ $(n \geqslant 1)$ so that A_k satisfies 1_k and 2_k above. We indicate the construction of A_{n+1}. We begin by selecting in the interior of A_n three disjoint systems of distinct points $a_1, \ldots, a_m$; $b_1, \ldots, b_m$; $c_1, \ldots, c_m$ such that for every $x \in A_n$ there are indices i, j, k for which

$$\varrho(x, a_i) < 2^{-n-1}, \quad \varrho(x, b_j) < 2^{-n-1}, \quad \varrho(x, c_k) < 2^{-n-1}.$$

By use of 2_n we construct three polygonal arcs L_i, M_i and N_i each of diameter $< 2^{-n}$ such that a_i (resp. b_i, c_i) is one end of L_i (resp. M_i, N_i) while the other end is a boundary point of ${}_nG_0$ (resp. ${}_nG_1, {}_nG_2$) and such that the interiors of these arcs lie in the interior of A_n. Further, these arcs can be and are so constructed that they are all disjoint.

It is clear that we can find a small polyhedral tubular neighborhood for each of these arcs the diameter of which is $< 2^{-n}$ and which is homeomorphic to a 3-simplex. These neighborhoods can be and are so constructed that the boundary of each of them intersects the boundary of A_n in a 2-dimensional disk, and such that these neighborhoods are pairwise disjoint. We denote by $U(L_i)$, $U(M_i)$, $U(N_i)$ the tubular neighborhoods of L_i, M_i, N_i, respectively.

We now let A_{n+1} be the closure of the set

$$A_n - \bigcup_{i=1}^{m} \big(U(L_i) \cup U(M_i) \cup U(N_i)\big).$$

It is clear that A_{n+1} is a subset of A_n which satisfies conditions 1_{n+1} and 2_{n+1}. Then the set $A = \bigcap_{n=1}^{\infty} A_n$ is the example of Lubański.

In order to see that A has the required properties, let us observe that we may find a retraction $r_n\colon A_n \to A_{n+1}$ such that $\varrho\big(x, r_n(x)\big) < 2^{-n}$ for every $x \in A_n$. Indeed, we let r_n be the identity map on A_{n+1} and on each $U(L_i)$ (resp. $U(M_i)$, $U(N_i)$), let r_n retract the set $U(L_i)$ to the disk $\overline{U(L_i)^{\bullet} - \big(U(L_i) \cap A_n^{\bullet}\big)}$, the set $U(M_i)$ to $\overline{U(M_i)^{\bullet} - \big(U(M_i) \cap A_n^{\bullet}\big)}$, and the set $U(N_i)$ to the disk $\overline{U(N_i)^{\bullet} - \big(U(N_i) \cap A_n^{\bullet}\big)}$, respectively. We let r_n' be the composition $r_n, r_{n-1}, \ldots, r_1\colon A_1 \to A_{n+1}$, which is a retraction. Since r_n moves no point by as much as 2^{-n}, it follows that the sequence $\{r_n'\}$ converges to a map r, which is a retraction

$$r\colon A_1 \to A.$$

Thus A is a retract of the polyhedron A_1 and consequently it is an ANR-space. Moreover, $E^3 - A$ has three components, namely,

$$G_0 = \bigcup_{n=1}^{\infty} {}_nG_0, \quad G_1 = \bigcup_{n=1}^{\infty} {}_nG_1, \quad G_2 = \bigcup_{n=1}^{\infty} {}_nG_2,$$

and A is their common boundary by virtue of conditions 2_n.

A compactum which is the common boundary of at least three regions in a Euclidean space is said to have the *singularity of Brouwer*. One may, however, define this singularity by internal properties of the space. For, by the Alexander-Pontrjagin duality theorem, a compact subset X of Euclidean space E^n $(n > 1)$ decomposes E^n into k regions if and only if the $(n-1)$-st Betti number of X is $k-1$. If X is the common boundary of these regions, then no proper closed subset of X will decompose E^n and hence its $(n-1)$-st Betti number vanishes. These facts enable us to make the following definition:

A space X shows the *singularity of Brouwer of type* (n, k) provided that for each proper closed subset X' of X the difference of the n-th Betti number of X and the n-th Betti number of X' is at least $k+1$.

The construction of Lubański clearly generalizes to show that there exists for every $n > 1$ and every $k \geqslant 1$ an ANR-set in E^{n+1} having the singularity of Brouwer of the type (n, k).

4. The singularity of Mazurkiewicz. Polyhedra have finite triangulations and consequently each polyhedron can be represented as a finite union of simplexes. Given any particular triangulation of a polyhedron, if one applies a finite number of times the process of barycentric subdivision one obtains a decomposition into simplexes of arbitrarily small diameter. In view of this simple fact one is naturally led to the following question: Can every ANR-space be expressed as a finite union of AR-sets having arbitrarily small diameters? It is easy to show that for ANR-spaces of dimension 1 the answer is positive. However, for ANR-spaces of dimension >1 the answer is negative. Indeed, in 1934, a 2-dimensional ANR-space was described ([77], p. 111) which was not a finite (or countable) union of AR-sets. At the same time an AR-space of dimension 2 was described which cannot be decomposed into a finite (or countable) number of AR-sets of arbitrarily small diameters. A space which cannot be expressed as a finite or countable union of AR-sets of arbitrarily small diameter will be said to have the *singularity of Mazurkiewicz*. Below we give an example of a space having this singularity.

Let Q be a 2-dimensional disk lying in E^2 which we consider as a plane decomposing E^3 into two open half-spaces one called the *positive half-space* and the other the *negative half-space*. Let Q' and Q'' be disjoint polygonal disks lying in the interior of Q and let L be a polygonal arc in the interior of $Q-(Q'\cup Q'')$ except for its two end-points one of which, called a', lies in $Q'^{\bullet}$ and the other $a''\in Q''^{\bullet}$. Finally let $b\in Q''^{\bullet}$ with $b\neq a''$.

Consider the set $M=Q'\cup L\cup Q''^{\bullet}$ and a positive number ε. For every point $x\in M$, let K_x^+ denote the ball in E^3 tangent to the plane E^2 at the point x with center in the positive half-space and radius equal to $\varepsilon\varrho(x,b)$. Similarly, let K_x^- be the ball in E^3 tangent to E^2 at x with center in the negative half-space and radius equal to $\varepsilon\varrho(x,b)$. It is easy to see that for ε sufficiently small the two isometric sets

$$A^+=A^+(Q',Q'',L,b,\varepsilon)=\bigcup_{x\in M}K_x^+,\qquad A^-=A^-(Q',Q'',L,b,\varepsilon)=\bigcup_{x\in M}K_x^-$$

are each homeomorphic to the set which we obtain from a solid 3-ball by identifying two points on its boundary. Moreover, we see that

$$A^+\cap E^2=A^-\cap E^2=M.$$

Now let us denote by

$$B^+=B^+(Q',Q'',L,b,\varepsilon)\quad\text{and}\quad B^-=B^-(Q',Q'',L,b,\varepsilon)$$

the sets which we obtain from the boundaries of A^+ and A^-, respectively, by removing the interior of the disk Q'. We easily see that for $\varepsilon>0$ sufficiently small the following conditions are satisfied:

(4.1) *The circle $Q''^{\bullet}$ is contractible to the circle $Q'''^{\bullet}$ in the set B^+ (and also in B^-) but is not contractible to $Q'''^{\bullet}$ in any proper subset of B^+ (resp. of B^-).*

(4.2) *Each of the sets $(Q-Q')\cup B^+$ and $(Q-Q')\cup B^-$ is an* AR*-set.*

(4.3) *The set B^+ is a retract of A^+ and the set B^- is a retract of A^-.*

Let us extend the definition of the sets $A^+ = A^+(Q', Q'', L, b, \varepsilon)$ and $A^- = A^-(Q', Q'', L, b, \varepsilon)$ to the case where the disk Q'' degenerates to the set consisting of only one point a''. In this case we set $b = a''$, $M = Q'\cup L$ and the sets $A^+ = \bigcup_{x\in M} K_x^+$, $A^- = \bigcup_{x\in M} K_x^-$ are homeomorphic to a 3-ball, and the sets $B^+ = B^+(Q', Q'', L, b, \varepsilon)$, $B^- = B^-(Q', Q'', L, b, \varepsilon)$ are disks. Properties (4.1) – (4.3) remain valid, but instead of the circle $Q'''^{\bullet}$ we have the set consisting of one point b.

By an easy induction on $k = 1, 2, \ldots$ one can construct a system

$$Q_{k,0}, Q_{k,1}, \ldots, Q_{k,2m_k}, L_{k,1}, L_{k,2}, \ldots, L_{k,2m_k}$$

of subsets of the interior of the disk Q such that

(4.4) *The sets $Q_{k,i}$, $k\geqslant 1$, $0\leqslant i\leqslant 2m_k$, are disjoint with one another. The sets $Q_{k,i}$, $k\geqslant 1$, $0\leqslant i\leqslant 2m_k$, are disks, and each of the sets $Q_{k,2m_k}$ consists of one point only.*

(4.5) *The sets $L_{k,j}$, $k\geqslant 1$, $0\leqslant j\leqslant 2m_k$, are disjoint polygonal arcs.*

(4.6) *The set $L_{l,j}\cap [\bigcup_{k=1}^{\infty}\bigcup_{i=0}^{2m_k} Q_{k_i}]$ consists of the two endpoints of the arc $L_{l,j}$: one endpoint $a_{l,j}\in Q^{\bullet}_{l,j}$, and the other $b_{l,j-1}\in Q^{\bullet}_{l,j-1}$.*

(4.7) *The distance of each point $x\in Q-\bigcup_{l=1}^{k-1}\bigcup_{i=0}^{2m_l} Q_{l,i}$ from the set $\bigcup_{i=0}^{2m_k} Q_{k,i}$ is less than $1/k$.*

(4.8) *For each point $x\in Q$ which does not lie on any $Q_{k,i}$ and for each natural number n, there exists an index $l > n$ such that $\varrho(x, Q_{l,0}) < 1/n$.*

(4.9) *The diameters of the sets $Q_{k,i}$ and $L_{k,i}$ are less than $1/k$ for all k and all i.*

From (4.9) we infer the existence of a positive number $\varepsilon_k\leqslant 1/k$ such that for every positive number $\eta\leqslant 2\varepsilon_k$ the sets

$$A^+_{k,j}(\eta) = A^+(Q_{k,j-1}, Q_{k,j}, L_{k,j}, b_{k,j}, \eta),$$
$$A^-_{k,j}(\eta) = A^-(Q_{k,j-1}, Q_{k,j}, L_{k,j}, b_{k,j}, \eta),$$
$$B^+_{k,j}(\eta) = B^+(Q_{k,j-1}, Q_{k,j}, L_{k,j}, b_{k,j}, \eta),$$
$$B^-_{k,j}(\eta) = B^-(Q_{k,j-1}, Q_{k,j}, L_{k,j}, b_{k,j}, \eta)$$

satisfy the following conditions:

(4.10) *The diameters of $A^+_{k,j}(\eta)$ and of $A^-_{k,j}(\eta)$ are less than $27/k$ and each of these sets is homeomorphic to the set obtained by identifying two boundary points of a solid 3-ball,*

because for every point $x \in Q_{k,j-1} \cup Q_{k,j} \cup L_{k,j}$ the distance $\varrho(x, b_{k,j})$ is $\leqslant 3/k$ and the diameter of the ball K^+_x is equal to $2\eta \cdot \varrho(x, b_{k,j}) \leqslant 12/k^2 \leqslant 12/k$.

(4.11) $A^+_{k,j}(\eta) \cap A^+_{k,m}(\eta) = A^-_{k,j}(\eta) \cap A^-_{k,m}(\eta) = 0$ *if* $|m-j| > 1$.

(4.12) *If* $k \neq l$ *and* $\eta \leqslant 2\varepsilon_k$, $\eta' \leqslant 2\varepsilon_l$, *then* $A^+_{k,i}(\eta) \cap A^+_{l,j}(\eta') = A^-_{k,i}(\eta) \cap A^-_{l,j}(\eta') = 0$ *for* $1 \leqslant i \leqslant 2m_k$ *and* $1 \leqslant j \leqslant 2m_l$.

Now let us show that

(4.13) *The set X given by the formula*

$$X = \Big[\bigcap_{k=1}^{\infty} \bigcap_{i=0}^{2m_k} (Q - Q_{k,i})\Big] \cup \Big[\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} \big(B^+_{k,2j-1}(\varepsilon_k) \cup B^-_{k,2j}(\varepsilon_k)\big)\Big]$$

is a 2-dimensional AR*-space.*

First let us observe that (4.10) and the inequality $\varepsilon_k \leqslant 1/k$ imply that the diameter of the set $A^+_{k,2j-1}(2\varepsilon_k) \cup A^-_{k,2j}(2\varepsilon_k)$ is less than $54/k$. Consider a 3-dimensional ball Q^3 in E^3 containing $Q \cup X$ and a retraction r of Q^3 to Q. Since r is uniformly continuous on Q^3, we infer that there exists a sequence $\{\alpha_k\}$ of positive numbers converging to zero and such that

$$\delta\big[r\big(A^+_{k,2j-1}(2\varepsilon_k) \cup A^-_{k,2j}(2\varepsilon_k)\big)\big] < \alpha_k \quad \text{for} \quad k = 1, 2, \ldots;\ j = 1, 2, \ldots, m_k.$$

It follows that for every $k = 1, 2, \ldots$ and $j = 1, 2, \ldots, m_k$ there exists a disk $D_{k,j} \subset Q$ such that

$$r\big(A^+_{k,2j-1}(2\varepsilon_k) \cup A^-_{k,2j}(2\varepsilon_k)\big) \subset D_{k,j}, \tag{4.14}$$

$$\delta(D_{k,j}) < 2\alpha_k. \tag{4.15}$$

Moreover, we see at once that

$$Q_{k,2j-1} \cup L_{k,2j-1} \cup Q_{k,2j-1} \cup L_{k,2j} \cup Q_{k,2j} \subset D_{k,j}$$

and that

(4.16) $D_{k,j} \cup A^+_{k,2j-1}(\varepsilon_k) \cup A^-_{k,2j}(\varepsilon_k)$ *is an* AR*-set.*

Since the common part of the set $A^+_{k,2j-1}(\varepsilon_k) \cup A^-_{k,2j}(\varepsilon_k)$ with the set $B^+_{k,2j-1}(2\varepsilon_k) \cup B^-_{k,2j}(2\varepsilon_k)$ is contained in $D_{k,j}$, we infer that the restriction

$$r|[B^+_{k,2j-1}(2\varepsilon_k) \cup B^-_{k,2j}(2\varepsilon_k)]$$

has a continuous extension

$$r_{k,j}\colon A^+_{k,2j-1}(2\varepsilon_k) \cup A^-_{k,2j}(2\varepsilon_k) \to D_{k,j} \cup A^+_{k,2j-1}(\varepsilon_k) \cup A^-_{k,2j}(\varepsilon_k)$$

satisfying the condition

$$r_{k,j}(x) = x \quad \text{for every point } x \in A^+_{k,2j-1}(\varepsilon_k) \cup A^-_{k,2j}(\varepsilon_k).$$

If we recall that $\lim\limits_{k\to\infty} a_k = 0$, we see that setting

$$r'(x) = \begin{cases} r(x) & \text{for every point } x \in Q^3 - \bigcup\limits_{k=1}^{\infty} \bigcup\limits_{j=1}^{m_k} [A^+_{k,2j-1}(2\varepsilon_k) \cup A^-_{k,2j}(2\varepsilon_k)], \\ r_{k,j}(x) & \text{for every point } x \in A^+_{k,2j-1}(2\varepsilon_k) \cup A^-_{k,2j}(2\varepsilon_k), \end{cases}$$

we get a retraction r' of the ball Q^3 to the set

$$Y = Q \cup \Big[\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} \big(A^+_{k,2j-1}(\varepsilon_k) \cup A^-_{k,2j}(\varepsilon_k)\big)\Big].$$

Thus Y is an AR-set. In order to finish the proof of (4.13), it is sufficient to show that there exists a retraction of Y to X. By (4.3), there exists a retraction $s_{k,2j-1}$ of the set $A^+_{k,2j-1}(\varepsilon_k)$ to $B^+_{k,2j-1}(\varepsilon_k)$ and a retraction $s_{k,2j}$ of the set $A^-_{k,2j}(\varepsilon_k)$ to $B^-_{k,2j}(\varepsilon_k)$. Since the diameters of the sets $A^+_{k,2j-1}(\varepsilon_k)$ and $A^-_{k,2j}(\varepsilon_k)$ are less than $27/k$, we infer at once that the map $s\colon Y \to X$ given by the formulas

$$s(x) = \begin{cases} x & \text{for every point } x \in X, \\ s_{k,2j-1}(x) & \text{for every point } x \in A^+_{k,2j-1}(\varepsilon_k), \\ s_{k,2j}(x) & \text{for every point } x \in A^-_{k,2j}(\varepsilon_k) \end{cases}$$

is a retraction of Y to X. Thus (4.13) is proved.

Now let us observe that (4.8) and (4.9) imply that for every neighborhood U (in the set $Q \cap X$) of a point $x \in Q \cap X$ and for every natural number n there exists an index $l > n$ such that $Q^{\bullet}_{l,0} \subset U$. Hence, from (4.1), we infer that every AR-set $Z \subset X$ which contains U must also contain the set

$$\bigcup_{j=1}^{m_l} (B^+_{l,2j-1} \cup B^-_{l,2j}).$$

It follows by (4.7) and (4.9) that the distance from a point $z \in Q \cap X$ to Z is $< 1/n$ for every n and hence $z \in Z$. Thus $Q \cap X \subset Z$ and we infer by (4.1) that $Z = X$.

Thus we have shown that if an AR-set Y is a proper subset of X, then $Q \cap Y$ does not contain any non-empty open subset of $Q \cap X$. Since $Q \cap X$ is a compactum, the classical theorem of Baire implies that

(4.17) *The space X is an* AR*-set which cannot be decomposed into a finite or countable number of* AR*-sets distinct from X.*

Hence X has the singularity of Mazurkiewicz.

In particular we conclude that X contains points for which no sufficiently small neighborhood is an AR-set.

By a slight modification of the space X we may obtain a space X' with the following property:

(4.18) *X' is a 2-dimensional* ANR-*set which is not decomposable into a finite or countable family of* AR-*sets.*

Proof. Let x_1 and x_2 be two distinct points of X and let X' be the space obtained from X by identifying x_1 and x_2. Let x_0' be the point of X' resulting from identification of x_1 and x_2 and let ψ denote the natural map of X onto X', i.e. the map assigning to every point $x \in X-(x_1)-(x_2)$ the same point (in X') and to both points x_1 and x_2 the point x_0'. Now let us consider an AR-set A' lying in X' and containing the point x_0' and let $A = \psi^{-1}(A')$. Then A is a compactum and A' results from A by the identification of points x_1 and x_2. If both those points belong to one component of A, then the first Betti number of A' would be positive, which is impossible because $A' \in \mathrm{AR}$. It follows that A decomposes into two compacta having only the point x_0' in common. By V, (2.12), those compacta are AR-sets, each of them distinct from X' (because of $A' \neq X'$). Thus, if it were possible to decompose X' into a finite or countable number of AR-sets, then for one of them, called A', the set $\psi^{-1}(A')$ would decompose into two AR-sets each distinct from X. The other AR-sets that compose X' may be considered as lying in X. But, under these circumstances, we have obtained a finite or countable decomposition of X into AR-sets which is impossible by (4.17). Thus the proof of (4.18) is finished.

We remark that by a suitable "thickening" of the construction given above one may construct a 2-dimensional AR-space X which lies in E^3 such that the 1-dimensional Betti number of every proper two dimensional subset is infinite [52]. This says that, in some sense, the space X is a 2-dimensional *indecomposable* ANR-space.

Recently R. Molski ([230], p. 132) has generalized the construction given in [52] onto higher dimensions, in order to obtain an n-dimensional indecomposable ANR-space for every $n = 2, 3, \ldots$

Let us observe that all plane ANR-sets are free from the singularity of Mazurkiewicz, because every plane ANR-set is the union of a finite number of connected ANR's, and — as we have already noticed (V, Section 14) — every connected, non-empty, plane ANR-set is the union of two AR-sets.

As we have already observed, the singularity of Mazurkiewicz is impossible with polyhedra. It is easy to prove ([40], p. 431) the following, more exact theorem: *Every connected n-dimensional polyhedron can be represented as the union of $n+1$ polyhedra which are* AR-*sets.*

On the other hand, there exist ([32], p. 142) among polyhedra such 2-dimensional AR-sets which cannot be represented as the union of two proper subpolyhedra being AR-sets.

5. Some open problems. The following questions remain still open: Let $X, Y \in$ ANR and $Z = X \times Y$,

(5.1) *If X has the singularity of Peano, Brouwer or Mazurkiewicz, then has Z the same singularity?*

(5.2) *If Z has one of the singularities of Peano or Brouwer, have X or Y the same singularity?*

(5.3) *If a polyhedron is represented as a Cartesian product, is every factor free from the singularities of Peano, Brouwer, and Mazurkiewicz?*

The examples given above in discussing the singularities of Peano, Brouwer, and Mazurkiewicz have been known for some time. Another type of peculiarity of ANR-spaces was recently discovered by R. H. Bing and myself [15]. We shall describe this phenomenon without going into much detail.

From the theorem of Hahn and Mazurkiewicz we know that every locally connected compactum contains an arc. In particular, ANR-spaces contain arcs. One is led to ask: *Does every* ANR-*space of sufficiently high dimension contain a disk?* It is easy (compare Section 6) to construct 2-dimensional AR-spaces which contain no disks, but the construction of a 3-dimensional ANR-space which fail to contain a disk is much more difficult.

The main idea of the construction [15] is the following:

The required space X is defined as a continuous image of a solid 3-ball Q^3 or, what is the same, as a decomposition space of Q^3 under a suitable upper semi-continuous decomposition. The non-degenerate elements (i.e. elements containing more than one point) of the decomposition form a countable collection of arcs whose diameters converge to zero. These arcs have the property that each disk (i.e. each topological image of the 2-dimensional simplex) lying in Q^3 intersects at least one of the arcs and their common part is a disconnected set. In order to obtain such a sequence of arcs one considers certain sets in E^3 which will be called *pseudo-chains*.

Let A and B be two disjoint compacta lying in E^3. Let us say that A *links* B if A is not contractible (to a point) in the set E^3-B. By a *usual chain* in E^3 we will understand a finite system $T_0, \ldots, T_k$ of solid tori such that T_i links T_j if and only if $i-j$ is congruent to 1 modulo $k-1$. In the interior of T_j we construct a copy A_i of Antoine's necklace (see, for instance, [150], p. 177), that is, a set homeomorphic to the Cantor discontinuum and having the property that a simple closed curve in E^3-T_i that links T_i links also the set A_i. By a theorem of G. T. Whyburn ([290], p. 57) we may find an arc L_i for which $A_i \subset L_i \subset T_i$. The set $L_0 \cup L_1 \cup \ldots \cup L_k$ will be called a *pseudo-chain replacing the usual chain* $T_0 \cup T_1 \cup \ldots \cup T_k$. The arcs L_i will be called the *links* of this pseudo-chain.

One can show that each disk in E^3 the boundary of which links the chain $T_0 \cup T_1 \cup \ldots \cup T_k$ must intersect one of the arcs L_i in a disconnected set.

Now one proves that there exists in the interior of Q^3 a sequence Φ of disjoint pseudo-chains the links of which have diameters converging to zero and such that the boundary of exery disk lying in the interior of Q^3 links at least one of the pseudo-chains belonging to Φ.

Finally, in order to obtain the arcs for our decomposition of Q^3 one completes the sequence of all links of all pseudo-chains belonging to Φ by a suitably constructed countable family of arcs whose endpoints lie on the boundary of Q^3 and whose diameters converge to zero. Then it follows from Theorem V, (12.1), that the resulting decomposition space of Q^3 is an AR-space. One easily shows that this space is 3-dimensional. The proof that it does not contain any disk is a little more complicated.

6. Families of locally r-incomparable AR-**sets.** The examples of ANR-spaces having the various singularities of Peano, Brouwer, and Mazurkiewicz show that the variety of topological phenomena to be found in ANR-spaces is much greater than in polyhedra, which have none of these singularities (compare [51] and [56]). We shall now prove a theorem which may help to explain this fact.

First we observe that the class of topological types of polyhedra is a countable class. This follows from the fact that every polyhedron may be considered as a set in a Euclidean space having a finite triangulation such that the vertices of the triangulation have only rational coordinates. It is evident that the class of all such triangulations is countable. It follows that the class of topological types of polyhedra is only countable. On the other hand, as we will now show, the class of r-types of ANR-spaces is of the power $2^{\aleph_0}$.

We say that two spaces are *locally r-incomparable* ([73], p. 333) provided that no open subset of one is an r-image of a subset of the other. Since every r-image of a space is homeomorphic to a closed subset of this space, we infer that two spaces are locally r-incomparable if and only if no open subset of one can be topologically embedded into the other.

Now let us prove ([72], p. 294, and [231], p. 142) the following

(6.1) THEOREM. *For every natural number n there is in E^{n+1} a family of $2^{\aleph_0}$ n-dimensional locally r-incomparable* AR-*sets.*

Proof. We assume first that $n > 1$ and consider the case $n = 1$ at the end of the proof.

By a *k-star* with *radius ε* and *center a* we mean the union of k line segments ($k > 1$), disjoint, save for a common endpoint a, each segment having length $\varepsilon > 0$.

Let Q^n be the solid n-ball in E^n consisting of all $x \in E^n$ for which $\|x\| \leqslant 1$, and let $\{k_i\}$ be an arbitrary increasing sequence of natural numbers > 1. We easily see that there is a sequence $\{A_\nu\}$ of sets A_ν lying in the interior of Q^n such that

1. There exists an increasing sequence of indices $1 = \nu_1 < \nu_2 < \ldots$ and a sequence of positive numbers ε_ν converging to zero such that for $\nu_i \leqslant \nu \leqslant \nu_{i+1}$ the set A_ν is a k_i-star with radius ε_ν lying in the interior of Q^n.

2. $A_\nu \cap A_{\nu'} = 0$ for $\nu \neq \nu'$.

3. For $i = 1, 2, \ldots$ the distance from each point $x \in Q^n$ to the set $\bigcup\limits_{\nu_i \leqslant \nu < \nu_{i+1}} A_\nu$ is less than $1/i$.

Let a_ν denote the center of the star A_ν. For each $t \in \langle 0, \varepsilon_\nu \rangle$, let A_ν^t be the set consisting of all $x \in A_\nu$ for which $\varrho(a_\nu, x) = t$. Evidently A_ν^0 consists only of the point a_ν, while for $0 < t \leqslant \varepsilon_\nu$ the set A_ν^t consists of k_i points. By condition 2, the sets A_ν^t are pairwise disjoint and by condition 1 the decomposition $\mathscr{A}$ of Q^n into the sets A_ν^t and the individual points of the set $Q^n - \bigcup\limits_{\nu=1}^{\infty} A_\nu$ is an upper semi-continuous decomposition.

Let us show that the resulting decomposition space, which we denote by $X = X(\{A_\nu\})$, is an AR-space. In order to see it let us consider the space E^n as hyperplane of the space E^{n+1} consisting of all points of the form $(\xi_1, \xi_2, \ldots, \xi_n, 0)$, where $\xi_1, \xi_2, \ldots, \xi_{n+1}$ denote the Cartesian coordinates in E^{n+1}. In particular, every point a_ν is of the form $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0)$. Let B_ν denote the cone with the base A and the vertex $a_\nu' = (\alpha_1, \alpha_2, \ldots, \alpha_n, \varepsilon_\nu)$, i.e. the union of all segments in E^{n+1} of the form $|a_i', x|$ with $x \in A_\nu$. One readily sees that the set

$$R = Q^n \cup \bigcup_{\nu=1}^{\infty} B_\nu$$

is an AR-space. Setting

$$a_{\nu,t}' = (\alpha_1, \alpha_2, \ldots, \alpha_n, t) \quad \text{for} \quad 0 \leqslant t \leqslant \varepsilon_\nu,\ \nu = 1, 2, \ldots,$$

let us denote by B_ν^t the union of all segments $|a_{\nu,t}', x|$ with $x \in A_\nu^t$. Manifestly the sets B_ν^t are pairwise disjoint AR-sets and the decomposition $\mathscr{B}$ of R into the sets B_ν^t and the individual points of the set $R - \bigcup\limits_{\nu=1}^{\infty} B_\nu$ is upper semi-continuous. Moreover, the map assigning to every element of $\mathscr{B}$ the element of $\mathscr{A}$ contained in it is a homeomorphism of the decomposition space Y of $\mathscr{B}$ onto the decomposition space X of $\mathscr{A}$. Applying Theorem V, (12.1), we infer that Y is an AR-space. Hence X is also an AR-space. Moreover, one easily sees that X is homeomorphic to a subset of E^{n+1}. One may prove this assertion by observing that, for each index ν, the identification of the points of the set A_ν^t (where $\nu_i \leqslant \nu < \nu_{i+1}$) leads to a set homeomorphic with the figure obtained from Q^n if one replaces the interiors of n-dimensional simplexes $\sigma_1, \ldots, \sigma_{k_i} \subset Q^n - Q^{n\bullet}$, by all segments in $E^{n+1} \supset E^n$, which join the points of $\sigma_\mu^\bullet$, $\mu = 1, 2, \ldots, k_i$, to a point $b_\nu \neq a_\nu$ lying on the straight line in E^{n+1} perpendicular to E^n and passing through a_ν.

We easily see that the dimension of $X = X(\{A_\nu\})$, obtained in this way, is equal to n and that X contains the $(n-1)$-dimensional sphere $Q^{n\bullet}$. This sphere is contractible in X since X is an AR-space, but one can show that it is not contractible in any proper subset of X. One says that X is a *membrane suspended upon the sphere* $Q^{n\bullet}$.

In passing from Q^n to the decomposition space X each of the stars A_ν corresponds to a segment L_ν. From condition 3 it follows that for each increasing sequence of indices $\{i_j\}$ the segments L_ν obtained from the k_{i_j}-stars $\{A_\nu\}$ are dense in the space X, more specifically, each (non-empty) open subset of X contains such segments. Moreover, let us observe that for each point x in the interior of a segment L_ν (with $\nu_i \leqslant \nu < \nu_{i+1}$) there exists an arbitrarily small open neighborhood U_x (in the space X) of the point x such that the set $U_x \cap L_\nu = L'_\nu$ is an open segment and that the set $U_x - L'_\nu$ contains exactly $2k_i$ components in the case $n = 2$, and k_i components in the case $n > 2$, for which L'_ν is their common boundary (in U_x). It is easy to show that the points of X which do not belong to the interior of any L_ν (with $\nu_i \leqslant \nu < \nu_{i+1}$) have not this property.

Now let $\{k'_\nu\}$ be another increasing sequence of indices containing an infinity of numbers which do not belong to the sequence $\{k_\nu\}$ and let $\{A'_\nu\}$ denote the sequence of stars corresponding to the sequence $\{k'_\nu\}$. Since X is a membrane suspended upon $Q^{n\bullet}$ and since each of the segments L_ν (with $\nu_i \leqslant \nu < \nu_{i+1}$) locally decomposes X at the points belonging to $L_\nu - L_\nu^\bullet$ into $2k_i$ components (into k_i components if $n > 2$), one easily proves that no open (non-empty) subset of either of the spaces $X = (\{A_\nu\})$ or $X' = (\{A'_\nu\})$ is homeomorphic to any subset of the other. We infer that the spaces X and X' are locally r-incomparable.

Remark. It easily follows from our construction that every n-dimensional closed subset of $X = X(\{A_\nu\})$ contains a non-empty open subset of X.

Let $\{w_n\}$ be a sequence of all rational numbers such that $n \neq n'$ implies $w_n \neq w_{n'}$. Further, let $\{k_i(t)\}$ denote the increasing sequence of all indices k for which w_k is less than the real number t. Evidently for $t' < t$ the sequence $\{k_i\} = \{k_i(t)\}$ contains an infinity of numbers non appearing in the sequence $\{k'_i\} = \{k_i(t')\}$. As we have already seen the spaces $X_t = X(\{A_\nu\})$ and $X_{t'} = X(\{A'_\nu\})$ are locally r-incomparable. Thus the theorem is completely proved in the case $n > 1$. The proof given above is not valid in the case $n = 1$. For this latter case, an example of a family of $2^{\aleph_0}$ locally r-incomparable dendrites has been given by Sieklucki ([258], p. 334).

As we have already noticed, our construction of the family $\{X_t\}$ implies (for $n > 1$) that each n-dimensional closed subset of X_t contains a non-empty open subset of X_t. Since each subset of an r-image of a space is an r-image of a subset of this space (I,(1.10)), we infer that for $t \neq t'$ no n-dimensional closed subset of X_t is an r-image of a subset of $X_{t'}$. Thus we get the following *Second Theorem on Families of ANR-sets*:

(6.2) THEOREM. *For every natural number $n > 1$ there is in the Euclidean $(n+1)$-space E^{n+1} a family consisting of $2^{\aleph_0}$ n-dimensional, locally r-incomparable* AR-*sets such that no n-dimensional closed subset of one of them is an r-image of a subset of another.*

This proposition fails for $n = 1$, because every 1-dimensional AR-set contains an arc.

7. Universal retracts. A space X is said to be (topologically) *universal for a family of spaces $\{X_\mu\}$*, where μ runs over a class of indices M, provided that

1. X is homeomorphic to some element of the family $\{X_\mu\}$.
2. Every X_μ is homeomorphic to some subset of X.

(7.1) EXAMPLE. The Hilbert cube is universal for the class of all compact metric spaces and also for the class of separable metric spaces. Hence it is universal for the class of all AR-sets or for the class of all ANR-spaces.

(7.2) EXAMPLE. For each natural number n, there is a universal space for the family of all n-dimensional compacta.

The construction of this universal space has been given first by K. Menger ([227], p. 1126), without an exact proof. The proof is due to S. Lefschetz ([209], p. 528).

There is a universal space for the family of all dendrites or, equivalenty (V,(13.5)), for all 1-dimensional AR-spaces. The construction of this universal space is due to Ważewski ([281], p. 137).

It is natural to inquire after the existence of a universal space for the family of all n-dimensional AR-spaces or of all n-dimensional ANR-spaces. It is clear, however, that for the latter family there can be no universal space since, if X is an n-dimensional compactum, then for every closed subset X_0 of X the n-th Betti number of X_0 is less than or equal to the n-th Betti number of X. Since there are obviously n-dimensional ANR-sets with arbitrarily large n-th Betti numbers, any universal space for all n-dimensional ANR's would have an infinite n-th Betti number and so could not be an ANR-space.

Using both theorem on families of ANR's, V,(15.1), and (6.2), we may prove ([72], p. 296) the following

(7.3) THEOREM. *For $n > 1$ a universal n-dimensional* AR-*space does not exist.*

Proof. By (6.2), there is an uncountable family $\{X_\mu\}$ of locally r-incomparable n-dimensional AR-spaces such that no n-dimensional closed subset of one of them is an r-image of a subset of another. We will show that any n-dimensional ANR-space X does not topologically contain every X_μ, and proving this will prove the theorem.

Indeed, if X is an n-dimensional ANR-space that topologically contains every X_μ, then there exists a family $\{X'_\mu\}$ of subsets of X with X'_μ homeomorphic to X_μ. Then, by V,(15.1), there would exist two distinct indices μ and ν such that $\dim(X'_\mu \cap X'_\nu) = n$. Then the spaces X_μ and X_ν would contain subsets Y_μ and Y_ν, respectively, each homeomorphic to $X'_\mu \cap X'_\nu$, contrary to our hypothesis. This completes the proof.

Remark 1. Actually we have shown more, that for each n-dimensional ANR-space X there exists an n-dimensional AR-space which is not homeomorphic to any subset of X.

Remark 2. We notice that there is an n-dimensional AR-space X which topologically contains every n-dimensional polyhedron $P \in$ AR. Indeed, consider in E^{2n+1} the sequence of $(2n+1)$-dimensional balls $\{Q^k\}$ with centers $\big((2k+1)/2k(k+1), 0, 0, \ldots, 0\big)$ and radii $1/2k(k+1)$. It is clear that there is a sequence $\{P_k\}$ of n-dimensional polyhedra which are AR's such that every n-dimensional polyhedron which is an AR is homeomorphic to some P_k. Moreover, we may assume without loss of generality that $P_k \subset Q_k$ and that P_k contains both points $(1/k, 0, \ldots, 0)$ and $\big(1/(k+1), 0, \ldots, 0\big)$. Then the set consisting of the origin and of points belonging to the set $\bigcup_{k=1}^{\infty} P_k$ is a contractible and locally contractible n-dimensional compactum and hence it is an AR-space. This is the demanded space X.

As an application of the theorem of Sieklucki (V,(15.22)) and of (6.2), let us prove the following

(7.4) THEOREM. *For every $n > 1$ and for every n-dimensional space $Y \in$* ANR *there exists an n-dimensional space $X \in$* AR *which does not topologically contain any closed n-dimensional subset of Y.*

Proof. Let $\{X_\mu\}$ be an uncountable family of n-dimensional AR-sets such that for $\mu \neq \nu$ no n-dimensional closed subset of X_μ is an r-image of a subset of X_ν. If Y is an n-dimensional ANR-set for which the theorem were false, then each X_μ would topologically contain an n-dimensional closed subset Y_μ of Y. Since the sets X_μ are locally r-incomparable, $Y_\mu \cap Y_\nu$ has dimension $\leqslant n-1$ whenever $\mu \neq \nu$, contradicting the theorem of Sieklucki (V,(15.22)).

CHAPTER VII

ANR-SPACES SATISFYING SOME SPECIAL CONDITIONS

We have seen that ANR-spaces may have so pathological properties that polyhedra cannot have them. It is therefore natural to seek topological conditions, satisfied by all polyhedra, which will eliminate as much pathology as possible from among ANR's.

1. Condition (Δ). This condition ([33], p. 1086, [34], p. 187, and [37], p. 78), designed to eliminate the singularity of Peano, is defined as follows:

(1.1) CONDITION (Δ). *A space Y satisfies condition (Δ) at a point $y \in Y$ provided that every neighborhood U of y contains a neighborhood V of y such that each compactum $A \subset V$ is contractible to a point in a subset of U having dimension* $\leqslant \dim A+1$. *If a space Y satisfies condition (Δ) at all of its points, then we say, for brevity, that Y satisfies condition (Δ) and we write $Y \in (\Delta)$.*

If Y is locally compact, then $Y \in (\Delta)$ implies evidently that Y is locally contractible. Hence if Y is a finite dimensional compactum, then $Y \in (\Delta)$ implies $Y \in \mathrm{ANR}$. We shall prove in the sequel that a compactum with the singularity of Peano does not satisfy condition (Δ). It follows by VI, Section 1, that there exist 3-dimensional AR-spaces without property (Δ). On the other hand, it is easy to show that for compacta of dimension $\leqslant 2$ condition (Δ) is equivalent to the local contractibility and consequently it characterizes the ANR-spaces among compacta of dimension less than 3.

(1.2) THEOREM. *A compactum Y satisfies condition (Δ) if and only if for every positive number ε there exists a positive number η such that every closed set $A \subset Y$, with diameter less than η, is contractible in a subset of the space Y of dimension* $\leqslant \dim A+1$ *and of diameter less than ε.*

Proof. The sufficiency of the condition is obvious. In order to show that it is also necessary, let us assume that Y is a compactum satisfying condition (Δ). We can assume that Y contains at least two points. Then we can assign to every point $y \in Y$ a neighborhood $U_y \subset Y$ such that every compactum $A \subset U_y$ can be contracted to a point in a set of dimension $\leqslant \dim A+1$ lying in the ball with center y and diameter $\frac{1}{2}\varepsilon$. Setting

$$\lambda(z) = \operatorname{Sup}_{y \in Y} \varrho(z, Y - U_y) \quad \text{for every point } z \in Y,$$

we see at once that λ is a continuous function on Y with positive values. It is evident that the number η equal to the lower bound of the function λ (in the set Y) satisfies the required condition.

2. Maps into a space $Y \in (\Delta)$. Let us prove the following

(2.1) THEOREM. *Let X and Y be compacta and let $Y \in (\Delta)$. The subset of the functional space Y^X consisting of all maps $f \in Y^X$ which satisfy the condition* $\dim f(X) \leqslant \dim X$ *is dense in the space* Y^X.

Proof. We can assume that the dimension n of X is finite and thus we can assume that X is a subset of the Euclidean space E^m, where m is a natural number sufficiently large. As we have previously shown (III,(9.1)), there is for every map $f \in Y^X$, a compact neighborhood U of X in the space E^m such that f has an extension $\bar{f} \in Y^U$. As we know ([166], p. 73), there exists for every positive number α a map φ of X onto a polyhedron $\varphi(X) \subset U$ of dimension n satisfying the condition $\varrho(x, \varphi(x)) < \alpha$ for every point $x \in X$. Manifestly, if α is a sufficiently small number, then the distance (in the space Y^X) from the map $f \in Y^X$ to the map $\bar{f}\varphi \in Y^X$ is arbitrarily small. Consequently the general case is reduced to the special one, where X is a polyhedron.

If $\dim X = 0$, then X is a finite set and the proposition is obvious. Thus we can assume that $\dim X = n > 0$ and that our proposition is true for all polyhedra of dimension less than n. We have to show that for every positive ε there exists a map $f' \in Y^X$ such that $\dim f'(X) \leqslant n$ and that $\varrho(f, f') < 2\varepsilon$. In order to do it, consider a triangulation $\mathscr{T}$ of the polyhedron X with the mesh so small that f maps each simplex of $\mathscr{T}$ onto a subset of Y of diameter less than $\frac{1}{2}\eta$, where the positive number $\eta \leqslant \varepsilon$ is assigned to the number ε, so that each compact set $A \subset Y$ with diameter less than η is contractible in a subset of Y of dimension $\leqslant \dim A+1$ and of diameter less than ε.

By the hypothesis of induction, the $(n-1)$-dimensional skeleton X^{n-1} of X (by the triangulation $\mathscr{T}$) can be mapped into Y by a map $\hat{f}$ satisfying the inequality

$$\varrho(\hat{f}(x), f(x)) < \tfrac{1}{4}\eta \quad \text{for every point } x \in X^{n-1}$$

and the condition

$$\dim \hat{f}(X^{n-1}) \leqslant n-1.$$

The boundary $\sigma^{\bullet}$ of each n-dimensional simplex $\sigma \in \mathscr{T}$ will be mapped by $\hat{f}$ onto a set $\hat{f}(\sigma)$ of dimension less than n, and one easily sees that the diameter of the set $\hat{f}(\sigma^{\bullet})$ is less than η. It follows that the set $\hat{f}(\sigma^{\bullet})$ is contractible to a closed point in a set $A_\sigma \subset Y$ of dimension $\leqslant n$ and of diameter less than ε. This implies that the restriction $\hat{f}|\sigma^{\bullet}$ can be extended to a map $f_\sigma\colon |\sigma| \to A_\sigma$. Setting

$$f'(x) = \begin{cases} \hat{f}(x) & \text{for every point } x \in X^{n-1}, \\ f_\sigma(x) & \text{for every point } x \in \sigma, \end{cases}$$

we get a map $f'\colon X \to Y$ which satisfies the condition $\varrho\big(f(x), f'(x)\big) < 2\varepsilon$ for every point $x \in X$. Moreover, $f'(X) \subset \hat{f}(X^{n-1}) \cup \bigcup_\sigma A_\sigma$ and consequently $\dim f'(X) \leqslant n$. Thus the proof of theorem (2.1) is terminated.

3. Extension of maps with values in a space $Y \in (\Delta)$. Now let us apply the last theorem in order to prove the following

(3.1) THEOREM. *Let X_0, X_1 be two closed and disjoint subsets of a compactum X and let f_0 be a map of X_0 into a space $Y \in$ ANR satisfying condition (Δ). Let $Y^X(f_0)$ denote the subset of the space Y^X consisting of all extensions of the map f_0. The set Ω of all maps $f \in Y^X(f_0)$ satisfying the condition $\dim f(X_1) > \dim X_1$ is a F_σ-set of the first category (of Baire) in the space $Y^X(f_0)$.*

Proof. We can assume that $n = \dim X_1$ is an integer. Let Ω_k $(k = 1, 2, \ldots)$ denote the set consisting of all maps $\varphi \in Y^X(f_0)$ for which the $(n+1)$-th Urysohn's coefficient (as defined in VI, Section 2) of the set $\varphi(X_1)$ is less than $1/k$. One easily shows, using VI, (2.1), that the sets Ω_k are open in the set $Y^X(f_0)$ and that

$$\Omega = Y^X(f_0) - \bigcap_{k=1}^{\infty} \Omega_k.$$

It remains to prove that the set $\bigcap_{k=1}^{\infty} \Omega_k$ is dense in the space $Y^X(f_0)$.

We can assume that Y is a subset of the Hilbert space E^ω and r is a retraction of a neighborhood U of Y in the space E^ω to Y. By Theorem (2.1), for every positive number ε and for every map $\varphi \in Y^X(f_0)$ there exists a map $\psi \in Y^{X_1}$ such that

$$\dim \psi(X_1) \leqslant n; \quad \varrho\big(\varphi(x), \psi(x)\big) < \varepsilon \quad \text{for every point } x \in X_1.$$

The last inequality implies that the map $\chi \in (E^\omega)^{X_1}$ defined by the formula

(3.2) $$\chi(x) = \psi(x) - \varphi(x) \quad \text{for every point } x \in X_1$$

satisfies the condition

(3.3) $$|\chi(x)| \leqslant \varepsilon \quad \text{for every point } x \in X_1.$$

Now let us extend the map χ continuously, first onto the set X_0, setting

(3.4) $$\chi(x) = 0 \quad \text{for every point } x \epsilon X_0,$$

and then onto the whole space X, so that condition (3.3) holds for every point $x \epsilon X$. This is always possible by III, (7.1), because the ball in the space E^ω with center 0 and radius ε is convex. Now, if we set

$$\psi'(x) = \varphi(x) + \chi(x) \quad \text{for every point } x \epsilon X,$$

then we get a map ψ'. By (3.4), the map ψ' coincides on the set X_0 with the map φ and it satisfies the inequality

$$\varrho\big(\varphi(x), \psi'(x)\big) \leqslant \varepsilon \quad \text{for every point } x \epsilon X.$$

If ε is a sufficiently small number, then all values of ψ' belong to U. Setting $\hat{\psi} = r\psi'$, we get a map which coincides with the map φ in the set X_0, and with the map ψ in the set X_1. Moreover, the map $\hat{\psi}$ belongs to $Y^X(f_0)$ and it satisfies the condition

$$\dim \hat{\psi}(X_1) \leqslant n,$$

and therefore $\hat{\psi} \epsilon \bigcap_{n=1}^{\infty} \Omega_k$. Moreover, for sufficiently small values of ε, the distance (in the space Y^X) between the maps $\hat{\psi}$ and φ is arbitrarily small. Thus we see that the set $\bigcap_{k=1}^{\infty} \Omega_k$ is dense in the space $Y^X(f_0)$, and our proof is finished.

It follows by this theorem and by the classical theorem of Baire on the sets of first category in a complete space the

(3.5) Corollary. *Let f be a map of a compactum X into an* ANR*-space Y satisfying condition* (Δ). *Let $X_0, X_1, \ldots$ be a sequence of closed subsets of X and let ε be a positive number. Then there exists a map $f' \epsilon Y^X$ satisfying the following conditions:*

(i) $\varrho(f, f') < \varepsilon$,

(ii) $f'(x) = f(x)$ *for every point* $x \epsilon X_0$,

(iii) $\dim f'(X_n - X_0) \leqslant \dim(X_n - X_0)$ *for every* $n = 1, 2, \ldots$

(3.6) Corollary. *Let Y be an* ANR*-space satisfying condition* (Δ). *For every positive number ε there exists a positive number η such that for every map f_0 of a closed subset X_0 of a compactum X onto a set $f_0(X_0) \subset Y$, which has the diameter less than η, there is a map $f \epsilon Y^X(f_0)$ such that the diameter of the set $f(X)$ is less than ε and the dimension of the set $f(X - X_0)$ is less than or equal to the dimension of the set $X - X_0$.*

Proof. We can assume that Y is a subset of the Hilbert space E^ω. Then there exists a retraction r of a neighborhood U of Y in the space E^ω to Y. Consider a positive number η so small that for every point $y \in Y$ the ball K (in the space E^ω) with center y and radius η is contained in U and that the diameter of the image of this ball under r is less than ε. It is evident that, if we select the center y of the ball K in the set $f_0(X_0)$, then $f_0(X_0) \subset K$. Since K is a convex subset of the Hilbert space E^ω, there exists an extension $f' \in K^X$ of the map f_0. Then the map rf' maps the set X onto a subset of Y with diameter less than ε. Applying (3.5) to the map rf' (after substituting $X_n = X$ for $n = 1, 2, \ldots$), we obtain Corollary (3.6).

(3.7) COROLLARY. *ANR-spaces satisfying condition* (Δ) *are free of the singularity of Peano.*

4. Addition of sets satisfying condition (Δ). Let us prove ([37], p. 83) the following

(4.1) THEOREM. *Let* $Y \in \mathrm{ANR}$ *be the union of two closed sets* Y_1, Y_2 *whose intersection* Y_0 *is an* ANR*-set with property* (Δ). *Then* $Y \in (\Delta)$ *if and only if* $Y_1, Y_2 \in (\Delta)$.

Proof. It follows by V,(2.12), that both sets Y_1 and Y_2 are ANR's. Now let us assume that $Y_1, Y_2 \in (\Delta)$. Then Y satisfies condition (Δ) at every point $y \in Y - Y_0$. It remains to show that Y satisfies condition (Δ) also at every point $y \in Y_0$. Since $Y_0 \in (\Delta)$, there exists by (1.2), for every natural number n, a positive number $\varepsilon_n < 1/n$ such that every closed subset B of Y_0 with diameter less than ε_n is contractible to a point in a subset of Y_0 with diameter less than $1/n$ and with dimension $\leqslant \dim B + 1$. Consider now a closed subset A of Y lying in a ball with center $y \in Y_0$ and radius $\frac{1}{2}\varepsilon_n$, and let us set $B = A \cap Y_0$. Then there exists a map ψ of the Cartesian product $B \times \langle 0, 1 \rangle$ into a closed subset of the set Y_0, with dimension $\leqslant \dim A + 1$ and with diameter less than $1/n$, which satisfies the condition

$$\psi(x, 0) = x \quad \text{and} \quad \psi(x, 1) = y \quad \text{for every point } x \in B.$$

Using these formulas, we can assume that the map ψ is defined not only in the set $B \times \langle 0, 1 \rangle$, but also in the whole set

$$\big(A \times (0)\big) \cup \big(A \times (1)\big) \cup \big(B \times \langle 0, 1 \rangle\big),$$

which is a closed subset of the Cartesian product $A \times \langle 0, 1 \rangle$. The set of values of this map is of dimension $\leqslant \dim A + 1$ and its diameter is less than $1/n + \varepsilon_n < 2/n$. We infer by (3.6) that for values of n sufficiently large, ψ has a continuous extension to each of the sets $(A \cap Y_\nu) \times \langle 0, 1 \rangle$, $\nu = 1, 2$, with values belonging to a set $Z \subset Y$ having the dimension $\leqslant \dim A+1$ and the diameter arbitrarily small. The map ψ, extended in this manner, is a homotopy of the set A in the set Z and thus condition (Δ) at the point y is satisfied.

Now let us assume that the space Y satisfies condition (Δ). Then the set Y_1 satisfies this condition at each point $y \in Y_1 - Y_2$. If, however, $y \in Y_1 \cap Y_2 = Y_0$ and if U_1 is a neighborhood of y in Y_1, then there exists a positive number ε such that

$$(4.2) \qquad \varrho(x, y) < \varepsilon \textit{ and } x \in Y_1 \quad \textit{imply} \quad x \in U_1.$$

By Corollary (3.6) there exists a positive number $\eta \leqslant \frac{1}{2}\varepsilon$ such that for every compact set $B \subset Y$ with the diameter less than η and such that $y \in B$, there is a map $\varphi_B\colon B \to Y_0$ satisfying the conditions:

$$(4.3) \qquad \varphi_B(x) = x \quad \textit{for every point } x \in B \cap Y_0,$$

$$(4.4) \qquad \delta[\varphi_B(B)] < \tfrac{1}{2}\varepsilon \quad \textit{and} \quad \dim \varphi_B(B) \leqslant \dim B.$$

Setting

$$(4.5) \qquad r_B(x) = \begin{cases} \varphi_B(x) & \textit{for every point } x \in B \cap Y_2, \\ x & \textit{for every point } x \in Y_1, \end{cases}$$

we get a retraction $r_B\colon B \cup Y_1 \to Y_1$. It follows by (4.3) and (4.4) that

$$(4.6) \qquad \delta[r_B(B)] < \varepsilon, \quad \dim r_B(B) \leqslant \dim B.$$

Since $Y \in (\Delta)$, there exists a neighborhood V of y in Y such that each compactum $A \subset V$ is contractible to a point in a subset of Y of dimension $\leqslant \dim A+1$ and of diameter $<\eta$. Let $V_1 = V \cap Y_1$. In order to prove that Y_1 satisfies condition (Δ) at the point y, it is sufficient to show that every compact subset A_1 of V_1 containing y is contractible in a subset U_1 having dimension $\leqslant \dim A_1+1$.

Since $A_1 \subset V$, there exists a map

$$\psi\colon A_1 \times \langle 0, 1\rangle \to Y$$

such that

(4.7) $\varrho(x, 0) = x$ *and* $\psi(x, 1) = y$ *for every point* $x \in A_1$.

(4.8) *The diameter of the compact set* $B = \psi(A_1 \times \langle 0, 1\rangle)$ *is less than* η, *and* $\dim B \leqslant \dim A_1+1$.

Now let us set

$$\varphi(x, t) = r_B \psi(x, t) \quad \text{for every } x \in A_1 \text{ and } 0 \leqslant t \leqslant 1.$$

It follows by (4.5) and (4.7) that φ is a homotopy in the set $r_B(B)$, contracting the set A_1 to the point y. Since $y \in A_1 \subset B$ and $r_B(B) \subset Y_1$, we infer by (4.2) and (4.6) that $\varphi(A_1 \times \langle 0, 1\rangle) \subset U_1$. Moreover, (4.6) and (4.8) imply that

$$\dim r_B(B) \leqslant \dim A_1+1.$$

Thus we have shown that condition (Δ) is satisfied by the set Y_1 at each of its points. Consequently, $Y_1 \in (\Delta)$. By an analogous argument we show that $Y_2 \in (\Delta)$.

5. Cartesian product of sets satisfying condition (Δ). Now we shall prove that for compacta of a finite dimension, condition (Δ) is multiplicative, i.e. we shall prove ([37], p. 85) the following

(5.1) THEOREM. *If Y_1 and Y_2 are finite-dimensional compacta satisfying condition* (Δ), *then their Cartesian product $Y_1 \times Y_2$ satisfies also condition* (Δ).

Proof. We can assume that Y_ν ($\nu = 1, 2$) is a subset of an $(n-1)$-dimensional hyperplane H_ν lying in an n_ν-dimensional Euclidean space E_ν. Then the set $Y_1 \times Y_2$ is a subset of the (n_1+n_2)-dimensional Euclidean space $E_0 = E_1 \times E_2$. The points of the space E_0 are of the form $(x_1, x_2, \dots, x_{n_0})$, where $n_0 = n_1 + n_2$. Let us denote by $M_{\nu,i}$ (where $\nu = 0, 1, 2$, and $i = 1, 2, \dots$) the collection of all sets $Q_{\nu,i,j_1,\dots,j_{n_\nu}}$ defined as subsets of the space E_ν consisting of all points $(x_1, \dots, x_{n_\nu}) \in E_\nu$ which satisfy the condition

$$2^{-i} j_\mu \leqslant x_\mu \leqslant 2^{-i}(j_\mu + 1), \qquad \mu = 1, 2, \dots, n_\nu,$$

where each of the indexes j_μ runs through all integers. Manifestly, the set $Q_{\nu,i,j_1,\dots,j_{n_\nu}}$ is isometric to the Cartesian product of n_ν segments with length 2^{-i}. Thus we see that $Q_{\nu,i,j_1,\dots,j_{n_\nu}}$ is a cube with the edge 2^{-i}.

Let us observe that the common part of two cubes belonging to the collection $M_{\nu,i}$ is either empty or a point, or else it is a cube with the edge 2^{-i}, called a *face* of these cubes. It is evident that each cube has only a finite number of faces and the interiors of distinct faces are disjoint. Every face of a face is also a face and the common part of two faces is a face (we consider here the empty set as a (-1)-dimensional face). Moreover, let us observe that every face of a cube, belonging to the collection $M_{0,i}$, is the Cartesian product of a face of a cube belonging to the collection $M_{1,i}$ and of a face of a cube belonging to the collection $M_{2,i}$.

It is clear that each cube belonging to $M_{\nu,i+1}$ is contained in one of the cubes belonging to $M_{\nu,i}$ and each cube belonging to $M_{\nu,i}$ is the union of some cubes belonging to the collection $M_{\nu,i+1}$.

Now let us denote by N_1 the collection of all cubes $Q \in M_{0,1}$ lying in the set $E_0 - (Y_1 \times Y_2)$ and, assuming that for a natural index i a collection N_i of cubes $Q \in M_{0,i}$ is already defined, let us define N_{i+1} as the collection of all cubes $Q \in M_{0,i+1}$ lying in the set $E_0 - (Y_1 \times Y_2)$ and such that their interiors are disjoint to the union of all cubes belonging to the set $N_1 \cup N_2 \cup \dots \cup N_i$.

Since the set Y_ν satisfies condition (Δ), we infer from (3.5) that there exists a neighborhood U_ν of Y_ν in the space E_ν and a retraction $r_\nu: U_\nu \to Y_\nu$ such that

$$\dim r_\nu(Q) \leqslant \dim Q \quad \text{for every face } Q \text{ of a cube } Q' \in M_{\nu,i} \text{ lying in } U_\nu.$$

Consider now a point $z^0 = (y_1^0, y_2^0) \in Y_1 \times Y_2$. For every positive number ε there exists a positive number η such that all cubes $Q \in \bigcup_{i=1}^{\infty} M_{\nu,i}$, lying in the set $E_\nu - Y_\nu$ at the distance less than η from the point y_ν^0, are contained in the ball (in the space E_ν) with center y_ν^0 and radius $\frac{1}{2}\varepsilon$. Let A be a closed subset of the space $Y_1 \times Y_2$ lying in the ball (in the space E_0) with center z^0 and radius $\frac{1}{2}\eta$, and let W_k denote the union of all at most k-dimensional faces of cubes belonging to $\bigcup_{i=1}^{\infty} N_i$, which are contained in the ball with center z^0 and radius ε. We shall asssume that the number ε is so small that each such ball lies in the set $U_1 \times U_2$. Let us show that for $k = n_0$ there exists a map

$$\psi: A \times \langle 0, 1\rangle \to A \cup W_k \tag{5.2}$$

such that

$$\psi(A \times \langle 0, 1\rangle) \cap (Y_1 \times Y_2) = A \tag{5.3}$$

and that

$$\psi(x, 0) = x \quad \textit{and} \quad \psi(x, 1) = \text{const} \quad \textit{for every point } x \in A. \tag{5.4}$$

In order to obtain a map ψ satisfying these conditions (hence being a homotopy in the set $A \cup W_k$ contracting the set A to a point) let us select a point $z^1 \in E_0 - (H_1 \times H_2)$ lying at a distance less than $\frac{1}{2}\eta$ from the point z^0, and let us set, for every point $x \in A$ and $0 \leqslant t \leqslant 1$:

$$\psi(x, t) = \text{point dividing the segment } |x, z^1| \text{ in the ratio } t : 1-t.$$

Since all values of ψ belong to the ball in the space E_0 with center z^0 and radius $\frac{1}{2}\eta$, condition (5.2) is satisfied for $k = n_0 = n_1 + n_2$. Condition (5.3) follows by the hypothesis that the point z^1 does not belong to the hyperplane $H_1 \times H_2$, and condition (5.4) is obviously met.

Now let us show that in the case $k > \dim A + 1$ the existence of a homotopy ψ satisfying conditions (5.2), (5.3), and (5.4) implies that there is a homotopy ψ' satisfying the conditions:

$$\psi': A \times \langle 0, 1\rangle \to A \cup W_{k-1}, \tag{5.2'}$$

$$\psi'(A \times \langle 0, 1\rangle) \cap (Y_1 \times Y_2) = A, \tag{5.3'}$$

$$\psi'(x, 0) = x \quad \textit{and} \quad \psi'(x, 1) = \text{const} \quad \textit{for every point } x \in A. \tag{5.4'}$$

In order to show it, let us order all k-dimensional faces of cubes belonging to the collection $\bigcup_{i=1}^{\infty} N_i$ and lying in the set W_k into a sequence $\{Q_l\}$. The diameters of the sets Q_l converge evidently to zero and each Q_l is of the form $Q_l = Q_{1,l} \times Q_{2,l}$, where $Q_{\nu,l}$ is a face of a cube belonging to the collection $\bigcup_{i=1}^{\infty} M_{\nu,i}$. The map ψ transforms the set

$$F_l = \psi^{-1}(Q_{1,l} \times Q_{2,l}) \subset A \times \langle 0, 1 \rangle$$

onto a subset of the cube Q_l. Let B_l denote the boundary of the cube Q_l and C_l denote the set $\psi^{-1}(B_l)$. Since $\dim A + 1 < k$, one sees readily (applying II,(9.2)) that there exists a map

$$\psi_l\colon F_l \to B_l,$$

which coincides with the map ψ in the set C_l. If we observe that the boundary of the set F_l (in the space $A \times \langle 0, 1 \rangle$) is contained in the set C_l and that the diameters of the sets Q_l converge to zero, we infer that setting

$$\psi'(z) = \begin{cases} \psi_l(z) & \text{for every point } z \in F_l,\ l = 1, 2, \ldots, \\ \psi(z) & \text{for every point } z \in A \times \langle 0, 1 \rangle - \bigcup_{l=1}^{\infty} F_l, \end{cases}$$

we get a map ψ' satisfying conditions (5.2′), (5.3′), and (5.4′).

Iterating this procedure a finite number of times, we shall get a map

$$\hat{\psi}\colon A \times \langle 0, 1 \rangle \to A \cup W_{\hat{k}}$$

which is a homotopy contracting the set A to a point in the set $A \cup W_k$, with $\hat{k} = \dim A + 1$.

Now let us set

$$(5.5) \qquad r(y_1, y_2) = \big(r_1(y_1), r_2(y_2)\big) \quad \textit{for every point } (y_1, y_2) \in U_1 \times U_2.$$

It is clear that r is a retraction of the neighborhood $U_1 \times U_2$ of the set $Y_1 \times Y_2$ in the space E_0 to the set $Y_1 \times Y_2$. It follows that the map $r\hat{\psi}$ is a homotopy contracting A to a point in the set $A \cup r(W_{\hat{k}}) \subset Y_1 \times Y_2$. If we recall that the set $W_{\hat{k}}$ is a subset of the ball in the space E_0 with center z^0 and radius ε, we see that the diameter of the set $A \cup r(W_{\hat{k}})$ is arbitrarily small, provided that the number ε is sufficiently small. Consequently, the proof of the theorem will be finished when we show that the dimension of the set $A \cup r(W_{\hat{k}})$ is less than or equal to $\dim A + 1$.

In order to prove this, let us observe that (5.5) implies that

$$r(Q_l) = r(Q_{1,l} \times Q_{2,l}) = r_1(Q_{1,l}) \times r_2(Q_{2,l})$$

and consequently

$$\dim r(Q_l) = \dim[r_1(Q_{1,l}) \times r_2(Q_{2,l})] \leqslant \dim r_1(Q_{1,l}) + \dim r_2(Q_{2,l})$$
$$\leqslant \dim Q_{1,l} + \dim Q_{2,l} \leqslant \hat{k} = \dim A + 1,$$

for every cube Q_l, which is a face of a cube belonging to the collection $\bigcup_{i=1}^{\infty} N_i$ and lying in the set $W_{\hat{k}}$. It follows that $\dim(A \cup r(W_{\hat{k}})) \leqslant \dim A + 1$ and the proof of the theorem is terminated.

If we notice that the empty set, the set consisting of only one point, and also the segment satisfy condition (Δ), we infer by the last theorem that every Euclidean cube, and consequently every geometric simplex, satisfies this condition. Applying Theorem (4.1) on the addition of sets with property (Δ), we get the following

(5.6) COROLLARY. *Every polyhedron satisfies condition* (Δ).

6. Weak homology. Convergent cycles and their divisors. Now we shall consider cycles with integers as coefficients, as a special case of cycles with rational coefficients, and we shall denote the relation of homology in the rational sense for such cycles by the symbol "$\approx$". By a *convergent cycle* in a space X we understand an infinite cycle $\{\gamma_i\}$ with integral coefficients for which there exists a majorante $\{\varepsilon_i\}$ and a carrier $X_0 \subset X$ such that $\gamma_i \underset{\varepsilon_i}{\approx} \gamma_{i+1}$ in X_0 for every $i = 1, 2, \ldots$ (that is if there exists in X_0 an ε_i-chain $\varkappa_i$ with rational coefficients such that $\partial\varkappa_i = \gamma_i - \gamma_{i+1}$). It is evident that every true cycle with integral coefficients is convergent, but not conversely. For instance, let X denote the projective plane and L a projective straight line in X. From the topological point of view, L is a simple closed curve and consequently there exists in L a 1-dimensional true cycle $\{\gamma_i\}$ such that its homology class is a generator of the Betti group $H_1(L)$. It is easily seen that the sequence

$$\gamma_1, 0, \gamma_2, 0, \gamma_3, 0, \ldots$$

is a convergent cycle in X; however it is not a true cycle (with integral coefficients).

Two convergent cycles γ and γ' are said to be *weakly homologous in* X (in symbols: $\gamma \approx \gamma'$ in X) provided that there exists in the space X an infinite chain $\varkappa = \{\varkappa_i\}$ with rational coefficients such that its boundary $\partial\varkappa$ coincides with $\gamma - \gamma'$. Manifestly $\gamma \sim \gamma'$ in X implies $\gamma \approx \gamma'$ in X, but not conversely.

An integer $l > 1$ is said to be a *divisor* (in the space X) of a convergent cycle γ provided there exists in X a convergent cycle γ' such that $\gamma \approx l\gamma'$ in X.

(6.1) LEMMA. *Every convergent cycle in a compactum X which is not weakly homologous to zero in X has only a finite number of divisors.*

Proof. First let us consider the special case where X is a polyhedron. Let $\mathscr{T}$ be a triangulation of X. It is known that every convergent cycle γ in X is weakly homologous in X to a sequence of successive barycentric subdivisions of a cycle γ of the triangulation $\mathscr{T}$, and the weak homology $\gamma \approx 0$ in X is equivalent to the weak homology $\gamma \approx 0$ in $\mathscr{T}$, i.e. to the existence of a chain in $\mathscr{T}$ with rational coefficients having γ as its boundary. It follows that if l is a divisor of the convergent cycle γ, then there exists in the triangulation $\mathscr{T}$ an integral cycle γ' such that

$$\gamma \approx l\gamma' \quad \text{in} \quad \mathscr{T}. \tag{6.2}$$

Now let us denote by $\hat{B}_n(\mathscr{T})$ the subgroup of the group of integral cycles $Z_n(\mathscr{T})$ consisting of all cycles weakly homologous to zero in $\mathscr{T}$. Evidently the factor-group $Z_n(\mathscr{T})/\hat{B}_n(\mathscr{T})$ has a finite system of linearly independent generators (γ_1), (γ_2), ..., (γ_m). Consequently there exist integers $p_1, p_2, \ldots, p_m$ and $q_1, q_2, \ldots, q_m$ such that

$$(\gamma) = p_1(\gamma_1) + \ldots + p_m(\gamma_m); \qquad (\gamma') = q_1(\gamma_1) + \ldots + q_m(\gamma_m).$$

It follows by (6.2) that

$$p_1(\gamma_1) + \ldots + p_m(\gamma_m) = lq_1(\gamma_1) + \ldots + lq_m(\gamma_m).$$

Evidently, if $p_1(\gamma_1) + \ldots + p_m(\gamma_m)$ is fixed and different from zero, then there exists only a finite set of integers l satisfying this equality.

Now let us pass to the general case where X is an arbitrary compactum. Without loss of generality we may assume that X is a subset of the Hilbert cube Q^ω. Setting

$$\varphi_k(x_1, x_2, \ldots, x_k, x_{k+1}, \ldots) = (x_1, x_2, \ldots, x_k, 0, \ldots)$$

for every point $x = (x_1, x_2, \ldots) \in Q^\omega$, we get a retraction of the cube Q^ω to the k-dimensional Euclidean cube $Q^k = \varphi_k(Q^\omega)$. Now let us assign to every $k = 1, 2, \ldots$ a polyhedron P_k which is a neighborhood of the set $\varphi_k(X)$ in Q^k so small that the distance of each point $x \in P_k$ from the set $\varphi_k(X)$ is less than $1/k$. It is plain that the sets

$$A_k = \varphi_k^{-1}(P_k)$$

(prisms, in the terminology of V, Section 4), constitute a sequence $\{A_k\}$ of neighborhoods of X in the space Q^ω such that

$$X = \bigcap_{k=1}^{\infty} A_k. \tag{6.3}$$

Moreover, it is clear that each convergent cycle lying in the set A_k is homologous in A_k to a convergent cycle lying in P_k. In particular, for a convergent cycle γ lying in the set X, the convergent cycle $\gamma_k = \varphi_k(\gamma)$ lies in the polyhedron P_k and it is homologous to γ in the prism A_k. Manifestly, every divisor of γ (in the set X) is also a divisor of γ_k (in the polyhedron P_k). Consequently, if γ has an infinite number of divisors (in the set X), then γ_k has an infinite number of divisors in the polyhedron P_k. As we have shown, this implies that $\gamma_k \approx 0$ in P_k and consequently also $\gamma \approx 0$ in A_k, for every $k = 1, 2, \ldots$ It follows by (6.3) that $\gamma \approx 0$ in X and thus the proof is complete.

7. True modular cycles. Now let us prove the following two lemmas.

(7.1) LEMMA. *If there exists in a compactum X a convergent integral n-dimensional cycle γ, which is not weakly homologous to zero in X, then, for every integer $m > 1$, there exists in X an infinite n-dimensional cycle modulo m which is not homologous to zero in X.*

Proof. By Lemma (6.1), we can assume that the convergent cycle $\gamma = \{\gamma_i\}$ has no divisor. The convergence of γ implies that there exists, for every $i = 1, 2, \ldots$, an ε_i-chain $\varkappa_i$ in X with rational coefficients such that $\partial\varkappa_i = \gamma_i - \gamma_{i+1}$ and that the sequence $\{\varepsilon_i\}$ converges to 0. Replacing the coefficients of γ_i by their rests modulo m, we get an infinite cycle $\{_m\gamma_i\}$ modulo m. If this infinite cycle would be homologous to zero in X, then there would exist in X, for every $i = 1, 2, \ldots$, an integral η_i-chain λ_i such that $\partial\lambda_i = \gamma_i - m\gamma_i'$ and that the sequence $\{\eta_i\}$ converges to 0. Then $\{\gamma_i'\}$ is an infinite integral cycle such that $\{\gamma_i\} \sim m\{\gamma_i'\}$ in X. Moreover,

$$\partial\varkappa_i = \gamma_i - \gamma_{i+1} = \partial\lambda_i + m\gamma_i' - \partial\lambda_{i+1} - m\gamma_{i+1}',$$

whence $m(\gamma_i' - \gamma_{i+1}') = \partial(\varkappa_i - \lambda_i + \lambda_{i+1})$, and consequently $\gamma_i' - \gamma_{i+1}' \underset{\varepsilon_i+\eta_i}{\approx} 0$ in X. Thus the infinite cycle $\{\gamma_i'\}$ is convergent and the relation $\{\gamma_i\} \sim m\{\gamma_i'\}$ contradicts the hypothesis that γ has no divisor.

(7.2) LEMMA. *If a true n-dimensional integral cycle lying in an n-dimensional compactum X is weakly homologous to zero in X, then it is also homologous to zero in X.*

Proof. We can assume that X lies in an Euclidean space E. Let $\gamma = \{\gamma_i\}$ be a true n-dimensional cycle in X which is not homologous to zero in X. Consider a decreasing sequence of compact neighborhoods $\{A_k\}$ of X in E such that

$$X = \bigcap_{k=1}^{\infty} A_k.$$

Then there exists an index k_0 such that γ is not homologous to zero in A_{k_0}. Let φ be a continuous map of X onto an n-dimensional polyhedron

$P \subset E$ such that for every point $x \epsilon X$ the segment $|x, \varphi(x)|$ lies in the set A_{k_0}. It follows that φ maps the true cycle γ onto a true cycle $\varphi(\gamma)$ homologous to γ in A_{k_0} and, consequently, not homologous to zero in P. Now let us recall that for n-dimensional polyhedra the n-dimensional torsion group is trivial. Consequently the true cycle $\varphi(\gamma)$ is not weakly homologous to zero in P. It follows at once that γ is not weakly homologous to zero in X.

8. True cycles in spaces satisfying condition (Δ). As we have already stated (II, (3.11)), every compactum of dimension $\geqslant n$ contains an n-dimensional infinite chain $\{\varkappa_i\}$ such that its boundary $\{\partial\varkappa_i\}$ has a carrier in which it is not homologous to zero. Now we shall consider the case of compacta satisfying condition (Δ) and prove for them a more precise statement.

Let $\mathscr{T}_i$ denote the triangulation of an n-dimensional simplex σ obtained by the iterating i times the operation of the barycentric subdivision, and let γ_i' denote the chain being the sum of all $(n-1)$-dimensional simplexes of $\mathscr{T}_i$, which lie on the boundary $\sigma^\bullet$ of σ and are oriented consistently with a fixed orientation of $\sigma^\bullet$. The sequence $\{\gamma_i'\}$ is evidently a true $(n-1)$-dimensional integral cycle in $\sigma^\bullet$, which is said to be the *basic cycle* of $\sigma^\bullet$. Now let us consider a map f of $\sigma^\bullet$ into a metric space X. The image $\{\gamma_i\} = \{f(\gamma_i')\}$ of the true cycle $\{\gamma_i'\}$ by the map f is a true $(n-1)$-dimensional cycle in X. Every true cycle $\{\gamma_i\}$ obtained in this way will be called a *spherical cycle* in the space X. If the map $f\colon \sigma^\bullet \to X$ has a continuous extension $f\colon \sigma \to X$, then the spherical cycle $\{\gamma_i\} = \{f(\gamma_i')\}$ is said to be *homotopic to zero* in X. It is evident that a spherical cycle with a carrier contractible in X is homotopic to zero in X, and a spherical cycle homotopic to zero in X is also homologous to zero in X, but the converse is not generally true. For instance, if X is the space, we get from the torus $T = S^1 \times S^1$ removing the interior of a disk $D \subset T$, then the boundary $D^\bullet$ of D contains a spherical cycle which in X is homologous, but not homotopic to zero.

Now let us prove the following

(8.1) THEOREM. *If X is an n-dimensional compactum satisfying condition* (Δ), *then there exists in X an $(n-1)$-dimensional spherical cycle with an $(n-1)$-dimensional carrier A, which is contractible in X and is not weakly homologous to zero in A.*

Proof. We can assume that X lies in an Euclidean space E. Since X, as a locally contractible and finitely dimensional compactum is an ANR-set, there exists a retraction r of a compact neighborhood U of X in E to X. It follows by II, (3.11), that there is in X an n-dimensional infinite chain $\{\varkappa_i\}$ such that the infinite $(n-1)$-dimensional cycle $\{\partial\varkappa_i\}$ has a carrier $X_0 \subset X$ in which it is not homologous to zero. It follows by

II,(3.4), that there exists a compact neighborhood $U_0 \subset U$ of X_0 in the space E such that $\{\partial \varkappa_i\}$ is not homologous to zero in U_0.

By removing from the sequence $\{\varkappa_i\}$ a finite number of terms, we can assume that for every simplex $\sigma = (a_0, a_1, \ldots, a_n)$, which appears in one at least of the chains $\varkappa_i$ with a non-vanishing coefficient, the minimal convex set $C(a_0, a_1, \ldots, a_n)$ containing all points $a_0, a_1, \ldots, a_n$ is included in U_0 and that for every $(n-1)$-dimensional simplex $(a_0', a_1', \ldots, a_{n-1}')$ which appears in one at least of the chains $\partial \varkappa_i$ with a non-vanishing coefficients, $C(a_0', a_1', \ldots, a_{n-1}') \subset U_0$. By virtue of Corollary (3.5), we can assume that the retraction $r\colon U \to X$ maps each set of the form $C(a_{\nu_0}, a_{\nu_1}, \ldots, a_{\nu_k})$ onto a subset of X of dimension less than or equal to k.

Now let us denote by $\varkappa_{i,j}$ the chain (in U), which we get if we replace in the chain $\varkappa_i$ each of its simplexes by the result of j times applied barycentric subdivision. It is easily seen that the sequence $\boldsymbol{\varkappa}_i = \{\varkappa_{i,j}\}$ (where i is fixed and j runs through values 1, 2, ...) is an infinite chain lying in U and that $\partial \boldsymbol{\varkappa}_i$ is a true cycle lying in U_0. Let us observe that the hypothesis that the infinite cycle $\{\partial \varkappa_i\}$ is not homologous to zero in U_0 implies that there exists an infinite set of indices i such that:

(i) $\partial \boldsymbol{\varkappa}_i$ is not homologous to zero in U_0.

(ii) $r(\boldsymbol{\varkappa}_i) = \{r(\varkappa_{i,j})\}$ is an infinite chain in X such that $\partial r(\boldsymbol{\varkappa}_i) = \{\partial r(\varkappa_{i,j})\} = \{r(\partial \varkappa_{i,j})\}$ is a true cycle homologous to $\partial \boldsymbol{\varkappa}_i$ in U_0, and consequently not homologous to zero in U_0.

Now let us consider an arbitrarily given positive number ε. There is an index i satisfying both conditions (i) and (ii) and such that the diameters of all simplexes σ_k of the chain $\varkappa_i = \sum_{k=1}^{p} \alpha_k \sigma_k$ are less than ε. Let $\sigma_{k,j}$ denote the integral chain obtained from σ_k by applying j times the barycentric subdivision. Then

$$\boldsymbol{\varkappa}_i = \sum_{k=1}^{p} \alpha_k \{\sigma_{k,j}\}$$

and consequently

$$\partial r(\boldsymbol{\varkappa}_i) = \sum_{k=1}^{p} \alpha_k \{r(\partial \sigma_{k,j})\}.$$

Now let us observe that each of the sequences $\{r(\partial \sigma_{k,j})\}$ is an $(n-1)$-dimensional spherical cycle lying in the image A_k (by the map r) of the union of all convex sets spanned by the $(n-1)$-dimensional faces of the simplex σ_k. Since r maps such convex sets without raising their dimension, we infer that the dimension of the set A_k is less than n and consequently

$$\dim(A_1 \cup A_2 \cup \ldots \cup A_p) < n.$$

Since the set A_k is a carrier of the $(n-1)$-dimensional spherical cycle $\{r(\partial\sigma_{k,j})\}$ and for a number ε sufficiently small it is contractible in X (because its diameter is arbitrarily small then), our proof will be complete if we show that there exists an index k such that the spherical cycle $\{r(\partial\sigma_{k,j})\}$ is not weakly homologous to zero in the set $A = A_k$.

Suppose, to the contrary, that

$$\{r(\partial\varkappa_{k,j})\} \approx 0 \text{ in } A_k \text{ for every } k = 1, 2, \ldots, p.$$

Then, by Lemma (7.2), it is also $\{r(\partial\sigma_{k,j})\} \sim 0$ in A_k for $k = 1, 2, \ldots, p$, and consequently

$$\partial r(\varkappa_i) \sim 0 \quad \text{in the set } A_1 \cup A_2 \cup \ldots \cup A_p.$$

However, this is impossible, because $\dim(A_1 \cup A_2 \cup \ldots \cup A_p) < n$ and because the $(n-1)$-dimensional cycle $\partial r(\varkappa_i)$ is not homologous to zero in the set U_0. Thus the proof is terminated.

9. Modular dimensions of compacta satisfying condition (Δ). We have already mentioned (II, Section 3) the notion of the dimension modulo $m > 1$, which is defined (for a compactum X) as the greatest integer $n = \Delta^m(X)$ such that there exists in X a closed subset X_0 containing an infinite $(n-1)$-dimensional cycle $\{\gamma_i\}$ modulo m that is homologous to zero in X, but that is not homologous to zero in X_0. It is evident that $\Delta^m(X) \leqslant \dim X$ for every $m = 2, 3, \ldots$

Now let us prove ([37], p. 90) the following

(9.1) THEOREM. *If X is an n-dimensional compactum satisfying condition* (Δ), *then all modular dimensions $\Delta^m(X)$, $m = 2, 3, \ldots$, are equal to n.*

Proof. By Theorem (8.1) there exists in X an $(n-1)$-dimensional spherical cycle $\{\gamma_i\}$ with an $(n-1)$-dimensional carrier X_0 contractible in the space X such that the cycle $\{\gamma_i\}$ is not weakly homologous to zero in X_0. By Lemma (7.1), there exists in the set X_0 an infinite $(n-1)$-dimensional cycle $\{{}_m\gamma_i\}$ modulo m (for each $m = 2, 3, \ldots$) which is not homologous to zero in the set X_0. Since X_0 is contractible in X, we infer that the infinite cycle $\{{}_m\gamma_i\}$ is homologous to zero in the space X and consequently $\Delta^m(X) \geqslant n$. It remains to recall that $\Delta^m(X) \leqslant \dim X = n$ in order to obtain our proposition.

It is known, by an example due to L. Pontrjagin ([248], p. 1105), that there exist compacta for which the modular dimensions differ from the usual dimension. The question whether such examples exist also among ANR-sets remains still open.

The theory of modular dimensions shows some interesting features. For instance, the dimension modulo a prime number m of the Cartesian product is equal to the sum of corresponding dimensions of factors ([248], p. 1105). By Theorem (9.1) we get the following

(9.2) COROLLARY. *If X and Y are compacta satisfying condition* (Δ), *then* $\dim(X\times Y) = \dim X + \dim Y$.

10. Condition (Γ). One of the most important tools of the topology of polyhedra is the fact that every polyhedron has a triangulation, that is a finite covering by geometric simplexes such that the common part of each system of these simplexes is a simplex.

Generalizing this property, we get the notion of a brick decomposition of a space X. If X is a space, then a *brick decomposition* of X is a finite system $X_1, X_2, \ldots, X_k$ of non-empty sets satisfying two following conditions:

(i) $X = X_1 \cup X_2 \cup \ldots \cup X_k$.

(ii) For each system of indices $i_0, i_1, \ldots, i_m$, where $1 \leqslant i_\nu \leqslant k$, the set $X_{i_0} \cap X_{i_1} \cap \ldots \cap X_{i_m}$ is either empty or it is an AR-set.

In particular, it follows by (ii) that all sets X_i are AR-sets.

(10.1) CONDITION (Γ). *A space X is said to satisfy condition* (Γ), *written* $X\in(\Gamma)$, *provided that it admits a brick decomposition.*

It is clear that for every triangulation of a polyhedron the non-empty simplexes of this triangulation constitute a brick decomposition. Consequently all polyhedra satisfy condition (Γ). Moreover, it follows by V,(2.9), that every metric space admitting a brick decomposition is an ANR-space. Thus

(10.2) $X\in(\Gamma)$ *implies* $X\in$ ANR.

The fact that there exist spaces having the singularity of Mazurkiewicz (VI,(4.17)) shows that there are ANR-spaces which do not satisfy condition (Γ). Thus the class of all spaces satisfying condition (Γ) properly contains the class of polyhedra and is properly contained in the class of ANR-spaces.

A brick decomposition of a space allows us to study the topological properties of the space in a combinatorial way, in the same manner that a triangulation makes it possible to study topological properties of a polyhedron.

Let us observe that if $\{X_1, X_2, \ldots, X_k\}$ is a brick decomposition of a space X and $\{Y_1, Y_2, \ldots, Y_l\}$ is a brick decomposition of a space Y, then the sets $X_i\times Y_j$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, l$, constitute a brick decomposition of the space $X\times Y$, because for each system of indices $i_0, j_0, i_1, j_1, \ldots, i_\nu, j_\nu$ the set

$$(X_{i_0}\times Y_{j_0}) \cap (X_{i_1}\times Y_{j_1}) \cap \ldots \cap (X_{i_\nu}\times Y_{j_\nu}) = (X_{i_0}\cap X_{i_1}\cap \ldots \cap X_{i_\nu})\times(Y_{j_0}\cap Y_{j_1}\cap \ldots \cap Y_{j_\nu})$$

is either empty or it is an AR-set. It follows that

(10.3) *The property* (Γ) *is multiplicative.*

It means that $X, Y \in (\Gamma)$ implies $X \times Y \in (\Gamma)$. The question whether the property (Γ) is *additive,* i.e. whether $X, Y, X \cap Y \in (\Gamma)$ implies $X \cup Y \in (\Gamma)$, remains still open.

Our next aim is to show that the homotopy type (in the sense of Hurewicz; I, Section 14) of a finite-dimensional compactum satisfying condition (Γ) is uniquely determined by the combinatorial properties of its brick decomposition. We start with some lemmas concerning relations between an arbitrary finite covering of a compactum by closed sets and a triangulation of a polyhedron.

11. A lemma on embedding in simplexes. Let us prove the following

(11.1) LEMMA. *Let B be a closed subset of a compactum A such that*

$$\dim(A-B) \leqslant p < \infty,$$

and let f_0 be a map of B into the boundary of a $(2p+1)$-dimensional simplex σ. Then f_0 has a continuous extension $f\colon A \to \sigma$ such that the restriction $f|(A-B)$ is a homeomorphism of the set $A-B$ into the interior $\sigma - \sigma^{\bullet}$ of the simplex σ.

Proof. We can consider, instead of a map into the simplex σ, a map into the $(2p+1)$-dimensional Euclidean ball

$$Q = \{x \in E^{2p+1};\ \varrho(x, 0) \leqslant 1\}.$$

By the embedding theorem of Menger and Nöbeling ([166], p. 56), every compactum of dimension less than or equal to p is homeomorphic to a subset of the interior $G = Q - Q^{\bullet}$ of the ball Q. We infer that the lemma is true if $B = 0$. Consequently, we can suppose that $B \neq 0$. Moreover, we can assume that the diameter of A is less than $\frac{1}{2}$. Setting

(11.2) $$A_k = \{x \in A;\ \varrho(x, B) \geqslant 1/k\} \quad \text{for every } k = 1, 2, \ldots,$$

we get a sequence $\{A_k\}$ of compacta such that

(11.3) $$A_1 = A_2 = 0, \qquad A - B = \bigcup_{k=1}^{\infty} A_k.$$

Let us denote by Ω the subset of the space Q^A consisting of all maps f satisfying both conditions:

(11.4) $$f(x) = f_0(x) \quad \text{for every point } x \in B,$$

(11.5) $$\varrho\big(f(x), 0\big) \leqslant 1 - 1/k \quad \text{for} \quad x \in A_k,\ k = 1, 2, \ldots$$

Manifestly, Ω is a closed subset of the complete space Q^A. Hence Ω is a complete space. Moreover, $\Omega \neq 0$, because if we consider any continuous extension $\bar{f} \in Q^A$ of the map f_0, then it is sufficient to set

$$f'(x) = [1-\varrho(x, B)]\cdot\bar{f}(x) \quad \text{for every point } x \in A,$$

in order to obtain a map f' satisfying both conditions (11.4) and (11.5). This is so, since for every point $x \in A_k$ the distance $\varrho(x, B)$ is not less than $1/k$, and consequently $\varrho(f'(x), 0) \leqslant 1-\varrho(x, B) \leqslant 1-1/k$.

Let us denote by $\Omega_{m,n}$, for all natural indices m, n, the subset of the set Ω consisting of all maps f' which satisfy the condition

(11.6) If $x, y \in A_n$ and $\varrho(x, y) \geqslant 1/m$, then $f'(x) \neq f'(y)$.

The set $\Omega_{m,n}$ is evidently an open subset of Ω. Let us prove that the set $\Omega_{m,n}$ is dense in Ω. It is sufficient to show that for every map $f \in \Omega$ and for every positive number $\varepsilon < 1/(n+1)$ there exists a map $f' \in \Omega_{m,n}$ which satisfies the inequality

(11.7) $$\varrho[f'(x), f(x)] \leqslant \varepsilon \quad \text{for every point } x \in A.$$

By the embedding theorem of Menger and Nöbeling, there exists a homeomorphism f'' mapping the set A_n onto a subset of the interior G of the ball Q, so that

(11.8) $$\varrho[f''(x), f(x)] < \tfrac{1}{3}\varepsilon \quad \text{for every point } x \in A_n.$$

Setting

$$g(x) = f''(x) - f(x) \quad \text{for every point } x \in A_n,$$

we get a map g of the set A_n into the ball

$$Q(\tfrac{1}{3}\varepsilon) = \{x \in E^{2p+1};\ \varrho(x, 0) \leqslant \tfrac{1}{3}\varepsilon\}.$$

Since the sets A_n and $\overline{A-A_{n+1}}$ are closed and disjoint, we can extend the definition of g setting

$$g(x) = 0 \quad \text{for every point } x \in \overline{A-A_{n+1}}.$$

Thus we get a map g of the compactum $A_n \cup \overline{A-A_{n+1}}$ into the ball $Q(\frac{1}{3}\varepsilon)$ and there exists a continuous extension g' of the map g to the set A with values belonging to the ball $Q(\frac{1}{3}\varepsilon)$. Setting

(11.9) $$f'''(x) = f(x) + g'(x) \quad \text{for every point } x \in A,$$

we obtain a map f''' of the set A into the ball Q. In fact, f''' coincides in the set $\overline{A-A_{n+1}}$ with the map f, in the set A_n — with the map f'', and for points $x \in \overline{A_{n+1}-A_n}$, we infer by (11.5) that

$$\varrho(f'''(x), 0) \leqslant \varrho(f(x), 0) + \varrho(g'(x), 0) \leqslant 1 - \frac{1}{n+1} + \frac{\varepsilon}{3} < 1 - \varepsilon + \frac{\varepsilon}{3} < 1.$$

Moreover,

(11.10) $\varrho\big(f'''(x), f(x)\big) \leqslant \frac{1}{3}\varepsilon$ for every point $x \epsilon A$,

(11.11) $f'''(x) = f(x)$ for every point $x \epsilon \overline{A - A_{n+1}}$.

Now let us define the required map f' by the formulas:

(11.12) $f'(x) = (1-\frac{2}{3}\varepsilon)f'''(x)$ for every point $x \epsilon A_{n+1}$,

(11.13) $f'(x) = \{1-\frac{2}{3}(n+1)\varepsilon[(n+2)\varrho(x, B)]-1\}f'''(x)$

for $x \epsilon \overline{A_{n+2} - A_{n+1}}$,

(11.14) $f'(x) = f'''(x)$ for $x \epsilon \overline{A - A_{n+2}}$.

In order to show that formulas (11.12) - (11.14) define a map, let us observe that $\varrho(x, B) = 1/(n+1)$ for every point $x \epsilon A_{n+1} \cap \overline{A_{n+2} - A_{n+1}}$. Consequently, the values of the function f' given by formulas (11.12) and (11.13) coincide. Similarly

$$\varrho(x, B) = \frac{1}{n+2} \quad \text{for every point } x \epsilon \overline{A_{n+2} - A_{n+1}} \cap \overline{A - A_{n+2}},$$

and consequently the value of the function f' defined by formula (11.13) is identical with the value given by formula (11.14).

By (11.12), in the set $A_n \subset A_{n+1}$ the map f' is the composition of the homeomorphism f''' and of the homeomorphism h, defined by the formula

$$h(y) = (1-\tfrac{2}{3}\varepsilon)y,$$

mapping the ball Q into itself. Consequently, the map f' restricted to the set A_n is a homeomorphism. By formulas (11.12)-(11.14), the values of the map f' belong to the ball Q and f' is of the form

$$f'(x) = \big(1-\alpha(x)\big)f'''(x),$$

where $0 \leqslant \alpha(x) \leqslant \frac{2}{3}\varepsilon$. Therefore

$$\varrho[f'(x), f'''(x)] = \alpha(x)\cdot\varrho[f'''(x), 0] \leqslant \tfrac{2}{3}\varepsilon.$$

It follows by (11.10) that f' satisfies inequality (11.7).

It remains to prove that $f' \epsilon \Omega_{m,n}$, i.e. that f' satisfies conditions (11.4), (11.5), and (11.6). Condition (11.6) is met, because f' is a homeomorphism on the set A_n. In order to verify condition (11.4) it is sufficient to observe that f' coincides in the set $B \subset \overline{A - A_{n+2}}$ with the map f'''.

In order to verify condition (11.5), let us consider the following cases:

If $x \epsilon A_k \cap \overline{A - A_{n+1}}$, then (11.11), (11.13), and (11.14) give

$$\varrho[f'(x), 0] \leqslant \varrho[f'''(x), 0] = \varrho[f(x), 0] \leqslant 1 - \frac{1}{k}.$$

If $x \in A_k \cap A_{n+1}$, then by (11.12) we infer that

$$\varrho[f'(x), 0] = (1-\tfrac{2}{3}\varepsilon)\,\varrho[f'''(x), 0].$$

If $\varrho[f'''(x), 0] \leqslant \frac{1}{2}$, then (11.3) implies

$$\varrho[f'(x), 0] \leqslant \frac{1}{2} < 1 - \frac{1}{k}.$$

If, however, $\varrho[f'''(x), 0] > \frac{1}{2}$, then by (11.12) and (11.10) we have

$$\varrho[f'(x), 0] = \varrho[f'''(x), 0] - \tfrac{2}{3}\varepsilon\,\varrho[f'''(x), 0] < \varrho[f'''(x), 0] - \tfrac{1}{3}\varepsilon$$
$$\leqslant \varrho[f'''(x), 0] - \varrho[f'''(x), f(x)] \leqslant \varrho[f(x), 0] \leqslant 1 - \frac{1}{k}.$$

Thus inequality (11.5) is proved in all cases and therefore f' belongs to the set $\Omega_{m,n}$. Thus we see that the open set $\Omega_{m,n}$ is dense in the complete space Ω. It follows, by the classical theorem of Baire on the category, that there exists a map $f \in \bigcap_{m,n=1}^{\infty} \Omega_{m,n}$. We infer by (11.6), (11.5), and (11.3) that f maps the set $A - B$ topologically into the interior of the ball Q. Thus Lemma (11.1) is established.

12. Simplicial realizations of finite coverings. Let $\{X_1, \ldots, X_k\}$ be a finite covering of a space X and let $\{Y_1, \ldots, Y_l\}$ be a finite covering of a space Y. These coverings will be said to be *similar* provided that $k = l$ and that for each sequence $i_1, i_2, \ldots, i_m$ of indices, $1 \leqslant i_\nu \leqslant k$, the relation $X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_m} = 0$ is equivalent to the relation $Y_{i_1} \cap Y_{i_2} \cap \ldots \cap Y_{i_m} = 0$. In particular, $X_i = 0$ if and only if $Y_i = 0$. It is clear that the similarity of coverings is a reflexive, symmetric and transitive relation.

Applying the concept of the nerve (as introduced in III, Section 6), we see at once that two coverings $\{X_1, \ldots, X_k\}$ and $\{Y_1, \ldots, Y_k\}$ are similar if and only if the correspondence $X_i \to Y_i$ induces an isomorphism of their nerves.

By a *geometric complex* we understand a finite system $\mathscr{K}$ of geometric simplexes such that the common part of every two of them is their common face. The vertices of simplexes belonging to $\mathscr{K}$ will be said to be *vertices* of $\mathscr{K}$. The union of all simplexes σ belonging to a geometric complex $\mathscr{K}$ is a polyhedron, which will be denoted by $|\mathscr{K}|$. Each triangulation of a polyhedron is evidently a geometric complex, but not conversely, because we do not require that each face of a simplex belonging to a geometric complex belongs to this complex.

A geometric complex $\mathscr{K}$ will be called a *simplicial realization* of the covering $\{X_1, X_2, \ldots, X_k\}$ of a space X provided that the simplexes of $\mathscr{K}$ can be ordered into a finite sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that the covering

$\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ of the polyhedron $|\mathscr{K}|$ is similar to the covering $\{X_1, X_2, \ldots, X_k\}$ of X and there exists a homeomorphism h mapping X onto a subset of $|\mathscr{K}|$ such that

$$h(X_i) = h(X) \cap \sigma_i \quad \text{for each} \quad i = 1, 2, \ldots, k.$$

Now let us prove ([49], p. 224) the following

(12.1) THEOREM. *For every finite closed covering of a finite-dimensional compactum there exists a simplicial realization.*

Proof. Let $\{X_1, X_2, \ldots, X_k\}$ be a closed covering of a compactum X of a finite dimension q. For every number $m = 0, 1, \ldots, k-1$, let

$$\mathfrak{X}^m = \{X_1^m, X_2^m, \ldots, X_{a(m)}^m\}$$

denote the system of all different sets of the form $X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_{k-m}}$, where $1 \leqslant i_1 < i_2 < \ldots < i_{k-m} \leqslant k$. The system $\mathfrak{X}^m$ constitutes a covering of the set

$$X^m = X_1^m \cup X_2^m \cup \ldots \cup X_{a(m)}^m.$$

In particular, $a(k-1) = k$, $X^{k-1} = X$ and the sequence $X_1^{k-1}, \ldots, X_k^{k-1}$ is a permutation of the sequence $X_1, X_2, \ldots, X_k$. We can assume that

$$X_i^{k-1} = X_i \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Now let us consider in the Hilbert space E^ω the sequence of points $e_i = (\delta_i^1, \delta_i^2, \ldots, \delta_i^n, \ldots)$, where δ_i^j denotes 0 for $i \neq j$ and 1 for $i = j$. Let us prove, for $m = 0, 1, \ldots, k-1$, the following proposition:

(12.2_m) *There exists in E^ω a geometric complex $\mathscr{K}^m$ which is a simplicial realization of the covering $\mathfrak{X}^m$ of the set X^m such that all vertices of $\mathscr{K}^m$ belong to the sequence $\{e_i\}$.*

It is clear that proposition (12.2_{k-1}) implies our theorem.

The proof of proposition (12.2_m) is by induction.

If $m = 0$, then $a(m) = 1$ and $X^0 = X_1^0 = X_1 \cap X_2 \cap \ldots \cap X_k$. If this set is empty, then we define the geometric complex $\mathscr{K}^0$ to consist of the (-1)-dimensional simplex only. If however $X^0 \neq 0$, i.e. if $\dim X^0 = q_0 \geqslant 0$, then we obtain ($12.2_0$) if $\mathscr{K}^0$ consists of one $(2q_0+1)$-dimensional simplex σ_0 with the vertices $e_0, e_1, \ldots, e_{2q_0+1}$, because by the embedding theorem of Menger and Nöbeling there exists a homeomorphism h^0 mapping the set X^0 onto a subset of the simplex σ_0.

Now let us assume that proposition (12.2_m) holds for an $m < k-1$. Consequently, there exists a geometric complex $\mathscr{K}^m = \{\sigma_1^m, \ldots, \sigma_{a(m)}^m\}$ with all vertices of simplexes σ_i^m belonging to the sequence $\{e_i\}$, and there is a homeomorphism h^m of the set X^m onto a subset of the polyhedron $|\mathscr{K}^m|$ such that

$$(12.3_m) \qquad h^m(X_i^m) = h^m(X^m) \cap \sigma_i^m \quad \text{for} \quad i = 1, 2, \ldots, \alpha(m).$$

Moreover, for each system of indices $1 \leqslant i_1 < i_2 < \ldots < i_\nu \leqslant \alpha(m)$ the set $X_{i_1}^m \cap X_{i_2}^m \cap \ldots \cap X_{i_\nu}^m$ is empty if and only if the set $\sigma_{i_1}^m \cap \sigma_{i_2}^m \cap \ldots \cap \sigma_{i_\nu}^m$ is empty.

In order to prove proposition (12.2_{m+1}), let us consider the system of sets

$$\mathscr{X}^{m+1} = \{X_1^{m+1}, X_2^{m+1}, \ldots, X_{\alpha(m+1)}^{m+1}\}.$$

It is plain that the set X_λ^{m+1} can be written, for every index $\lambda = 1, 2, \ldots, \alpha(m+1)$, in the form

$$X_\lambda^{m+1} = X_{i_{\lambda,1}} \cap X_{i_{\lambda,2}} \cap \ldots \cap X_{i_{\lambda,k-m-1}} \text{ with } i_{\lambda,\mu} \neq i_{\lambda,\nu} \text{ for } \mu \neq \nu.$$

Let us range all indices $\leqslant k$ which do not belong to the system $i_{\lambda,1}, i_{\lambda,2}, \ldots, i_{\lambda,k-m-1}$ in a sequence $j_{\lambda,1}, j_{\lambda,2}, \ldots, j_{\lambda,m+1}$. Then

$$X_\lambda^{m+1} \cap X^m = \bigcup_{\nu=1}^{m+1} (X_\lambda^{m+1} \cap X_{j_{\lambda,\nu}}).$$

But each of the sets $X_\lambda^{m+1} \cap X_{j_{\lambda,\nu}}$ belongs to the sequence $\mathscr{X}^m$, hence there exists for $\nu = 1, 2, \ldots, m+1$ at least one index $j \leqslant \alpha(m)$ such that

$$X_\lambda^{m+1} \cap X_{j_{\lambda,\nu}} = X_j^m.$$

Let $J(\lambda, \nu)$ denote the collection of all such indices j and let $J(\lambda) = \bigcup_{\nu=1}^{m+1} J(\lambda, \nu)$. It follows by (12.3_m) that

$$h^m(X_j^m) = h^m(X^m) \cap \sigma_j^m \quad \text{for every } j \in J(\lambda, \nu).$$

Hence

$$(12.4) \qquad X_\lambda^{m+1} \cap X^m = \bigcup_{\nu=1}^{m+1} \bigcup_{j \in J(\lambda,\nu)} X_j^m.$$

Let us assign to each set $X_\lambda^{m+1} \in \mathscr{X}^{m+1}$ the simplex σ_λ^{m+1} defined as follows: If $X_\lambda^{m+1} = 0$, and hence also $X_j^m = 0$ for every index $j \in J(\lambda, \nu)$, $\nu = 1, 2, \ldots, m+1$, then we set $\sigma_\lambda^{m+1} = 0$. If, however, $X_\lambda^{m+1} \neq 0$, then σ_λ^{m+1} denotes the simplex with vertices $p_{\lambda,0}^{m+1}, p_{\lambda,1}^{m+1}, \ldots, p_{\lambda,\beta_\lambda}^{m+1}$, which we obtain if we add to all vertices of the simplexes $\sigma_j^m, j \in J(\lambda, \nu)$, $\nu = 1, 2, \ldots, m+1$, some supplementary vertices of the sequence $\{e_i\}$, different from all vertices of $\mathscr{K}^m$ and distinct for different values of the index λ. Moreover, we can assume that the number of supplementary vertices in each simplex σ_λ^{m+1} is positive and so large that $\beta_\lambda \geqslant 2q+1$ for $\lambda = 1, 2, \ldots, \alpha(m+1)$. Let us notice that

$$(12.5) \qquad (\sigma_\lambda^{m+1})^\bullet \cap \bigcup_{\nu=1}^{\alpha(m)} \sigma_\nu^m = \bigcup_{j \in J(\lambda)} \sigma_j^m.$$

Moreover, let us observe that for every $\lambda = 1, 2, \ldots, a(m+1)$, if i is an index $\leqslant a(m)$ which does not belong to the set $J(\lambda)$, that means, for each index i such that $X_i^m \cap X_\lambda^{m+1} = 0$, the simplexes σ_i^m and σ_λ^{m+1} are disjoint. In fact, if σ_i^m intersects σ_λ^{m+1}, then σ_i^m intersects one of the simplexes σ_j^m with $j \in J(\lambda, \nu)$ (because all points e_i are linearly independent) and therefore $X_i^m \cap X_j^m \neq 0$, which implies $X_i^m \cap X_\lambda^{m+1} \neq 0$, contrary to our hypotheses.

Since all vertices of the simplexes σ_λ^{m+1} belong to the sequence $\{e_i\}$, the system $\{\sigma_1^{m+1}, \sigma_2^{m+1}, \ldots, \sigma_{a(m+1)}^{m+1}\}$ is a geometric complex $\mathscr{K}^{m+1}$. It remains to prove that this geometric complex is a simplicial realization of the covering $\{X_1^{m+1}, \ldots, X_{a(m+1)}^{m+1}\}$ of the set X^{m+1}.

First let us show that the covering $\{\sigma_1^{m+1}, \ldots, \sigma_{a(m+1)}^{m+1}\}$ of the polyhedron $|\mathscr{K}^{m+1}|$ is similar to the covering $\{X_1^{m+1}, \ldots, X_{a(m+1)}^{m+1}\}$ of the set X^{m+1}. Let $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ be a sequence of indices $\leqslant a(m+1)$. If there exists an index μ such that $1 \leqslant \mu \leqslant \nu$ and $X_{\lambda_\mu}^{m+1} = 0$, then also $\sigma_{\lambda_\mu}^{m+1} = 0$. Thus both sets $X_{\lambda_1}^{m+1} \cap \ldots \cap V_{\lambda_\nu}^{m+1}$ and $\sigma_{\lambda_1}^{m+1} \cap \ldots \cap \sigma_{\lambda_\nu}^{m+1}$ are empty. Hence we can assume that each of the sets $X_{\lambda_1}^{m+1}, \ldots, X_{\lambda_\nu}^{m+1}$ is not empty.

If $\lambda_1 = \lambda_2 = \ldots = \lambda_\nu$, then $X_{\lambda_1}^{m+1} \cap \ldots \cap X_{\lambda_\nu}^{m+1} = X_{\lambda_1}^{m+1}$ and $\sigma_{\lambda_1}^{m+1} \cap \ldots \cap \sigma_{\lambda_\nu}^{m+1} = \sigma_{\lambda_1}^{m+1}$. In this case, relations $X_{\lambda_1}^{m+1} \cap \ldots \cap X_{\lambda_\nu}^{m+1} = 0$ and $\sigma_{\lambda_1}^{m+1} \cap \ldots \cap \sigma_{\lambda_\nu}^{m+1} = 0$ are equivalent, by the definition of the simplexes σ_λ^{m+1}.

Now let us assume that in the sequence $\lambda_1, \ldots, \lambda_\nu$ there exist at least two different numbers. Let $X_{\lambda_j}^{m+1} = X_{i_{\lambda_j,1}} \cap \ldots \cap X_{i_{\lambda_j,k-m-1}}$, where $i_{\lambda_j,1}, \ldots, i_{\lambda_j,k-m-1}$ is a system of $k-m-1$ different indices $\leqslant k$. By our hypothesis, there exists, for each such system, an index $\varkappa_j$ of the form $i_{\lambda_{j'},l}$ with $j' \neq j$, which does not belong to the system $i_{\lambda_j,1}, \ldots, i_{\lambda_j,k-m-1}$. Then the set $X_{\lambda_j}^{m+1} \cap X_{\varkappa_j}$ belongs to the system $\mathfrak{X}^m$. Let us set

$$(12.6) \qquad X_{\lambda_j}^{m+1} \cap X_{\varkappa_j} = X_{\mu_j}^m.$$

Then

$$X_{\lambda_1}^{m+1} \cap \ldots \cap X_{\lambda_\nu}^{m+1} = X_{\mu_1}^m \cap \ldots \cap X_{\mu_\nu}^m,$$

because $\varkappa_j = i_{\lambda_{j'},l}$ belongs to the system of indices $i_{\lambda_{j'},1}, \ldots, i_{\lambda_{j'},k-m-1}$. It follows by (12.6) that the simplex $\sigma_{\mu_j}^m$ is a face of the simplex $\sigma_{\lambda_j}^{m+1}$.

If $X_{\lambda_1}^{m+1} \cap \ldots \cap X_{\lambda_\nu}^{m+1} \neq 0$, then $X_{\mu_1}^m \cap \ldots \cap X_{\mu_\nu}^m \neq 0$ and the induction hypothesis implies that $\sigma_{\mu_1}^m \cap \ldots \cap \sigma_{\mu_\nu}^m \neq 0$ and, consequently, also $\sigma_{\lambda_1}^{m+1} \cap \ldots \cap \sigma_{\lambda_\nu}^{m+1} \neq 0$.

Conversely, if $\sigma_{\lambda_1}^{m+1} \cap \ldots \cap \sigma_{\lambda_\nu}^{m+1} \neq 0$, then there exists a vertex p common to all simplexes $\sigma_{\lambda_j}^{m+1}, j = 1, 2, \ldots, \nu$. This vertex cannot be a "supplementary vertex", because not all numbers $\lambda_1, \ldots, \lambda_\nu$ are identical and the simplexes with distinct indices have distinct "supplementary vertices". Therefore p is a vertex of the geometric complex $\mathscr{K}^m$. It follows that each simplex $\sigma_{\lambda_j}^{m+1}$, $j = 1, 2, \ldots, \nu$, contains a face $\sigma_{\omega_j}^m$ for which p

is one of the vertices. By the construction of the simplex $\sigma_{\lambda_j}^{m+1}$ we infer that $X_{\omega_j}^m \subset X_{\lambda_j}^{m+1}$. But $p \in \sigma_{\omega_1}^m \cap \ldots \cap \sigma_{\omega_\nu}^m$ implies that $X_{\omega_1}^m \cap \ldots \cap X_{\omega_\nu}^m \neq 0$, and consequently also $X_{\lambda_1}^{m+1} \cap \ldots \cap X_{\lambda_\nu}^{m+1} \neq 0$. Thus we have shown that the covering $\{X_1^{m+1}, \ldots, X_{\alpha(m+1)}^{m+1}\}$ of the set X^{m+1} and the covering $\{\sigma_1^{m+1}, \ldots, \sigma_{\alpha(m+1)}^{m+1}\}$ of the polyhedron $P^{m+1} = |\mathscr{K}^{m+1}|$ are similar.

It remains to show that there exists a homeomorphism h^{m+1} satisfying proposition (12.3_{m+1}). We define such a homeomorphism by extending the homeomorphism h^m over each of the sets X_λ^{m+1}, $\lambda = 1, 2, \ldots, \alpha(m+1)$. We can assume that $X_\lambda^{m+1} \neq 0$. Let us denote by Y_λ^{m+1} the common part of the set X_λ^{m+1} and of the set $\bigcup\limits_{\mu \neq \lambda} X_\mu^{m+1}$. Hence $Y_\lambda^{m+1} = X^m \cap X_\lambda^{m+1}$. The homeomorphism h^m maps the set Y_λ^{m+1} onto a subset of the boundary of the simplex σ_λ^{m+1}. Since the dimension of the simplex σ_λ^{m+1} is greater than $2q$ and the dimension of X_λ^{m+1} does not exceed q, we infer, by Lemma (11.1), that there is an extension of the restriction $h^m | Y_\lambda^{m+1}$ to a homeomorphism h_λ^{m+1} of the set X_λ^{m+1} onto a subset of the simplex σ_λ^{m+1}, assigning to points of the set $X_\lambda^{m+1} - Y_\lambda^{m+1}$ the points lying in the interior of the simplex σ_λ^{m+1}. Since the interiors of different simplexes σ_λ^{m+1} are disjoint with one another, we get a homeomorphism of the set X^{m+1} into the polyhedron P^{m+1} if we set

$$h^{m+1}(x) = h_\lambda^{m+1}(x) \quad \text{for every point} \quad x \in X_\lambda^{m+1}.$$

It follows by (12.4) and (12.5) that

$$\begin{aligned} h^m(Y_\lambda^{m+1}) &= h^m(X^m \cap X_\lambda^{m+1}) = h^m\Big(\bigcup_{j \in J(\lambda)} X_j^m\Big) = \bigcup_{j \in J(\lambda)} h^m(X_j^m) \\ &= h^m(X^m) \cap \bigcup_{j \in J(\lambda)} \sigma_j^m = h^m(X^m) \cap \bigcup_{\nu=1}^{\alpha(m)} \sigma_\nu^m \cap (\sigma_\lambda^{m+1})^\cdot \\ &= h^m(X^m) \cap (\sigma_\lambda^{m+1})^\cdot, \end{aligned}$$

because $h^m(X^m) \subset \bigcup\limits_{\nu=1}^{\alpha(m)} \sigma_\nu^m$.

We infer by the construction of h_{m+1} that

$$\begin{aligned} h^{m+1}(X_\lambda^{m+1}) &= h^{m+1}(X_\lambda^{m+1} - Y_\lambda^{m+1}) \cup h^m(Y_\lambda^{m+1}) \\ &= h^{m+1}(X^{m+1}) \cap [\sigma_\lambda^{m+1} - (\sigma_\lambda^{m+1})^\cdot] \cup h^m(X^m) \cap (\sigma_\lambda^{m+1})^\cdot \\ &= h^{m+1}(X^{m+1}) \cap [\sigma_\lambda^{m+1} - (\sigma_\lambda^{m+1})^\cdot] \cup h^{m+1}(X^{m+1}) \cap (\sigma_\lambda^{m+1})^\cdot \\ &= h^{m+1}(X^{m+1}) \cap \sigma_\lambda^{m+1}. \end{aligned}$$

Thus we see that condition (12.3_{m+1}) is satisfied and the proof of Theorem (12.1) is finished.

(12.7) COROLLARY. *For two similar closed, finite coverings of two compacta of finite dimension there exists a common simplicial realization.*

Proof. Let $\{X_1, X_2, \ldots, X_k\}$ and $\{Y_1, Y_2, \ldots, Y_k\}$ be two similar closed coverings of two finite-dimensional compacta X and Y, respectively. We can assume that X and Y are disjoint subsets of the Hilbert space E^ω. The sets $\{X_1 \cup Y_1, \ldots, X_k \cup Y_k\}$ constitute a covering of the space $X \cup Y$ similar to the covering $\{X_1, \ldots, X_k\}$ of the space X. In fact, the relation $X_{i_1} \cap \ldots \cap X_{i_\nu} = 0$ is equivalent to the relation $Y_{i_1} \cap \ldots \cap Y_{i_\nu} = 0$ and consequently (because $X \cap Y = 0$) it is equivalent also to the relation $(X_{i_1} \cup Y_{i_1}) \cap \ldots \cap (X_{i_\nu} \cup Y_{i_\nu}) = 0$.

By (12.1), there exists a geometric complex $\mathscr{K} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ of a polyhedron P which is a simplicial realization of the covering $\{X_1 \cup Y_1, \ldots, X_k \cup Y_k\}$. It means that the covering $\{\sigma_1, \ldots, \sigma_k\}$ of the polyhedron P is similar to the covering $\{X_1 \cup Y_1, \ldots, X_k \cup Y_k\}$ of the space $X \cup Y$ and that there exists a homeomorphism h mapping the space $X \cup Y$ onto a subset of the polyhedron P in such a manner that

$$(12.8) \qquad h(X_i \cup Y_i) = h(X \cup Y) \cap \sigma_i \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Let us prove that the geometric triangulation $\mathscr{T}$ of the polyhedron P is a common simplicial realization of both coverings $\{X_1, \ldots, X_k\}$ and $\{Y_1, \ldots, Y_k\}$. In fact, by the similarity of the coverings $\{X_1, \ldots, X_k\}$ and $\{X_1 \cup Y_1, \ldots, X_k \cup Y_k\}$ we infer that the covering $\{X_1, \ldots, X_k\}$ of the space X and the covering $\{\sigma_1, \ldots, \sigma_k\}$ of the polyhedron P are similar. The partial homeomorphism $h|X$ maps the space X into the polyhedron P and we infer by (12.8) that

$$h(X_i) = h(X) \cap \sigma_i \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Thus we see that $\mathscr{T}$ is a simplicial realization of the covering $\{X_1, \ldots, X_k\}$ of the space X. By the same argument we see that $\mathscr{T}$ is also a simplicial realization of the covering $\{Y_1, \ldots, Y_k\}$ of the space Y.

13. Spaces satisfying condition (Γ) **as deformation retracts of polyhedra.** As we have seen, the spaces satisfying condition (Γ) constitute a special kind of ANR-spaces including all polyhedra. Now we shall study their homotopy properties. First let us prove the following

(13.1) LEMMA. *If A and $B \subset A$ are AR-spaces, C is a closed subset of A and φ is a map of the set $C \times \langle 0, 1\rangle$ into A such that*

$$\varphi(x, 0) = x, \quad \varphi(x, 1) \in B \quad \textit{for every point } x \in C,$$

$$\varphi(x, 1) = x \quad \textit{for every point } x \in B \cap C,$$

then φ has a continuous extension ψ: $A \times \langle 0, 1\rangle \to A$ satisfying the condition:

$$\psi(x, 0) = x \quad \textit{for every point } x \in A,$$

and such that the map r: $A \to B$, defined by the formula $r(x) = \psi(x, 1)$ for every point $x \in A$, is a retraction of A to B.

Proof. Setting

$$f(x) = \begin{cases} x & \text{for every point } x \in B, \\ \varphi(x, 1) & \text{for every point } x \in C, \end{cases}$$

we get a map $f\colon B \cup C \to B$. Since $B \in \mathrm{AR}$, there exists a continuous extension $r\colon A \to B$ of f. Setting

$$\varphi'(x, 0) = x \qquad \text{for every point } x \in A,$$

$$\varphi'(x, 1) = r(x) \qquad \text{for every point } x \in A,$$

$$\varphi'(x, t) = \varphi(x, t) \qquad \text{for every point } (x, t) \in C \times \langle 0, 1\rangle,$$

we get a map φ' of the closed subset $\big(A \times (0)\big) \cup (C \times \langle 0, 1\rangle) \cup \big(A \times (1)\big)$ of the space $A \times \langle 0, 1\rangle$ into A. Since A is an AR-space, there exists a continuous extension $\psi\colon A \times \langle 0, 1\rangle \to A$ of φ' and we see at once that this extension satisfies our lemma.

(13.2) THEOREM ([49], p. 230). *Let X be a finite-dimensional compactum having a brick decomposition $\{X_1, X_2, \ldots, X_k\}$. If $\mathscr{T} = \{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is a simplicial realization of the covering $\{X_1, X_2, \ldots, X_k\}$, then X is homeomorphic to a deformation retract of the polyhedron $P = \bigcup_{i=1}^{k} \sigma_i$.*

Proof. By our hypothesis, there is a homeomorphism h of the space X onto a subset of the polyhedron P such that

(13.3) $$h(X_i) = h(X) \cap \sigma_i \quad \text{for} \quad i = 1, 2, \ldots, k.$$

It is sufficient to prove that the set $Y = h(X)$ is a deformation retract of the polyhedron P.

For each number $m = 0, 1, \ldots, k-1$ there exists $a(m) = \binom{k}{m}$ different increasing sequences $i_1, i_2, \ldots, i_{k-m}$ with natural terms $\leqslant k$. Let $\vartheta_1, \vartheta_2, \ldots, \vartheta_{a(m)}$ be these sequences. If $\vartheta_\nu = (i_1, \ldots, i_{k-m})$, then we set

(13.4) $$X_\nu^m = X_{i_1} \cap \ldots \cap X_{i_{k-m}}; \quad Y_\nu^m = h(X_\nu^m),$$

(13.5) $$\sigma_\nu^m = \sigma_{i_1} \cap \sigma_{i_2} \cap \ldots \cap \sigma_{i_{k-m}}.$$

The simplexes $\sigma_1^m, \ldots, \sigma_{a(m)}^m$ constitute a geometric complex $\mathscr{K}^m$. We see immediately that the sets $Y_1^m, \ldots, Y_{a(m)}^m$ constitute a brick decomposition of the set $Y^m = \bigcup_{\nu=1}^{a(m)} Y_\nu^m$ similar to the covering $\mathscr{K}^m$ of the polyhedron $P^m = |\mathscr{K}^m|$. Moreover, we infer by (13.3), (13.4), and (13.5) that

(13.6) $$Y_\nu^m = Y^m \cap \sigma_\nu^m \quad \text{for every } \nu = 1, 2, \ldots, a(m).$$

Therefore $\mathcal{K}^m$ is a simplicial realization of the brick decomposition $\{Y_1^m, \ldots, Y_{a(m)}^m\}$ of the set Y^m.

Now we shall proceed by induction, proving, for $m = 0, 1, \ldots, k-1$, that there exists a map $r^m\colon P^m \times \langle 0, 1\rangle \to P^m$, such that

$$(13.7_m) \qquad \begin{array}{l} r^m(x, 0) = x \textit{ for every point } x \in P^m, \\ r^m(x, 1) \textit{ is a retraction of } P^m \textit{ to } Y^m, \\ r^m(x, t) \in \sigma_\nu^m \textit{ for every point } x \in \sigma_\nu^m \textit{ and } 0 \leqslant t \leqslant 1. \end{array}$$

If $m = 0$, then $Y^0 = h(X_1 \cap X_2 \cap \ldots \cap X_k)$ and $P^0 = \sigma_1^0 = \sigma_1 \cap \sigma_2 \cap \ldots \cap \sigma_k$. If $Y^0 = 0$, then $\sigma_1^0 = 0$ and our statement is obvious. If, however, $Y^0 \neq 0$, then Y^0 is an AR-set lying in the simplex σ_1^0. By Lemma (13.1) (where $C = 0$, $B = Y^0$ and $A = \sigma_1^0$) we infer that there exists a map r^0 satisfying conditions (13.7_0).

Now let us assume that for an $m < k-1$ there exists a map

$$r^m\colon P^m \times \langle 0, 1\rangle \to P^m$$

which satisfies (13.7_m). For every index $\mu = 1, 2, \ldots, a(m+1)$, let us consider the set $Y_\mu^{m+1} = Y^{m+1} \cap \sigma_\mu^{m+1}$, which is an AR-set. Then the map r^m is defined on the set $(\sigma_\mu^{m+1} \cap P^m) \times \langle 0, 1\rangle$ and its values belong to the union of all faces of σ_μ^{m+1} which intersect the set Y. Applying Lemma (13.1), we infer that the restriction $r^m|[(\sigma_\mu^{m+1} \cap P^m) \times \langle 0, 1\rangle]$ can be extended to a map $\hat{r}_\mu$ of the set $\sigma_1^{m+1} \times \langle 0, 1\rangle$ into the simplex σ_μ^{m+1}, which is a retraction by deformation of σ_μ^{m+1} to the set Y_μ^{m+1}. Moreover, if $\sigma_{\mu'}^{m+1}$ is another one of the simplexes $\sigma_\mu^{m+1}, \ldots, \sigma_{a(m+1)}^{m+1}$, then $\sigma_\mu^{m+1} \cap \sigma_{\mu'}^{m+1} \neq 0$ if and only if the set $\sigma_\mu^{m+1} \cap \sigma_{\mu'}^{m+1}$ is a face of one of the simplexes $\sigma_1^m, \ldots, \sigma_{a(m)}^m$. Then both maps $\hat{r}_\mu$ and $\hat{r}_{\mu'}$ have the same values on the set $(\sigma_\mu^{m+1} \cap \sigma_{\mu'}^{m+1}) \times \langle 0, 1\rangle$ as the map r^m. It follows that if we extend r^m in this manner to each simplex σ_μ^{m+1}, $\mu = 1, 2, \ldots, a(m+1)$, we get a map r^{m+1} satisfying condition (13.7_{m+1}).

Thus we see that a map r^m satisfying condition (13.7_m) can be constructed for each $m = 0, 1, \ldots, k-1$. In particular the map r^{k-1} is a retraction by deformation of the polyhedron $P^{k-1} = P$ onto the set $Y^{k-1} = Y$ homeomorphic to the space X. Thus theorem (13.2) is established.

14. Homotopy types of spaces satisfying condition (Γ). The Theorem (13.2) just proved implies some corollaries ([49], pp. 232–234).

(14.1) Corollary. *If two finite-dimensional compacta X and X' have similar brick decompositions $\mathcal{X}$ and $\mathcal{X}'$, then there exists a polyhedron P containing two deformation retracts homeomorphic with X and X', respectively.*

Proof. As we have seen in (12.7), there exists a geometric complex $\mathcal{K}$ which is a simplicial realization of both brick decompositions $\mathcal{X}$ and $\mathcal{X}'$.

By Theorem (13.2), the spaces X and X' are homeomorphic to some deformation retracts of the polyhedron P. According to I, (14.2), every deformation retract of a space is homotopically equivalent (in the sense of Hurewicz (I, Section 14)) to this space. It follows

(14.2) COROLLARY. *Every space of a finite dimension, satisfying condition* (Γ) *is homotopically equivalent to a polyhedron.*

If we recall that the relation of the homotopic equivalence is transitive, we get the following

(14.3) COROLLARY. *Two finite-dimensional compacta having similar brick decompositions have the same homotopy type.*

It follows that all finite-dimensional spaces with similar brick decompositions have isomorphic homology and homotopy groups. Thus the topological structure of finitely dimensional spaces satisfying condition (Γ) is in a high degree determined by the combinatorial scheme of their brick decompositions.

Let us observe that if a finite-dimensional compactum X has a brick decomposition $X_1, X_2, \ldots, X_k$, then the nerve P of this decomposition is a polyhedron homotopically equivalent to X. In fact, the nerve P is given in a triangulation $\mathscr{T}$ consisting of all simplexes $(X_{i_0}, X_{i_1}, \ldots, X_{i_\nu})$ with $X_{i_0} \cap X_{i_1} \cap \ldots \cap X_{i_\nu} \neq 0$ and of the (-1)-dimensional simplex. Let $\mathscr{T}'$ denote the barycentric subdivision of $\mathscr{T}$. Let us assign to each vertex X_i of the triangulation $\mathscr{T}$ the union $A(X_i)$ of all simplexes of $\mathscr{T}'$ containing this vertex. One easily sees that for every system of indices $j_0, j_1, \ldots, j_m$ the set $A(X_{j_0}) \cap \ldots \cap A(X_{j_m})$ is either empty or it is a contractible polyhedron; hence it is an AR-set. It follows that the system $\mathscr{A} = \{A(X_1), \ldots, A(X_k)\}$ is a brick decomposition of the polyhedron P. This decomposition is similar to the decomposition $\{X_1, \ldots, X_k\}$, because for each system $(i_0, \ldots, i_\nu)$ of indices $\leqslant k$, the relation $A(X_{i_0}) \cap \ldots \cap A(X_{i_\nu}) \neq 0$ holds if and only if all vertices $X_{i_0}, \ldots, X_{i_\nu}$ belong to one of the simplexes of the triangulation $\mathscr{T}$, that means, if and only if $X_{i_0} \cap \ldots \cap X_{i_\nu} \neq 0$.

We infer by Corollary (14.3) that the space X and the polyhedron P have the same homotopy type. Thus we have the following

(14.4) COROLLARY. *If $\{X_1, X_2, \ldots, X_k\}$ is a brick decomposition of a finite-dimensional compactum X, then X has the same homotopy type as the nerve of this decomposition.*

Let us mention that W. Holsztyński has lately shown ([151], p. 611) that Corollaries (14.2), (14.3), and (14.4) remain true also without the hypothesis of the finite dimension. On the other hand, the question whether every ANR-space has the homotopy type of a polyhedron remains open. Only for ANR-spaces with trivial fundamental group the positive answer to this question has been given by C. B. De Lyra ([96], p. 58).

15. Homogeneous ANR-spaces. A space X is said to be *homogeneous* (*topologically*) provided that for every two points $x_1, x_2 \epsilon X$ there exists a homeomorphism h mapping X onto itself and satisfying the condition $h(x_1) = x_2$. The classical examples of homogeneous spaces are linear spaces and also manifolds (in the classical sense). There exist also various other examples of homogeneous spaces, as the Cantor's discontinuum, the solenoids of van Dantzig [91], the Hilbert cube [175], the universal curve of Sierpiński [4], and the pseudo-arc [12]. The question if each homogeneous ANR-space of a finite dimension is a manifold remains still open. A partial result concerning this question is given ([16], p. 106) by the following

(15.1) THEOREM. *An n-dimensional, connected* ANR-*space is a manifold if and only if it is homogeneous and contains topologically a Euclidean n-ball.*

Proof. The necessity of the condition is obvious. Now let us assume that X is a connected, homogeneous ANR-space and that there exists a homeomorphism φ mapping the n-dimensional Euclidean ball $Q = Q^n$ onto a subset $\varphi(Q)$ of X. Let $Q^{\bullet}$ denote the boundary of Q. If there exists a point $a \epsilon \varphi(Q - Q^{\bullet})$ belonging to the closure of the set $X - \varphi(Q)$; then one sees at once (by the local contractibility of X) that there exists in $X - \varphi(Q^{\bullet})$ a simple arc L joining a point $b \epsilon X - \varphi(Q)$ with the point a. Then there exists a point $c \epsilon L \cap \varphi(Q)$ such that the interior of the subarc L' of L with endpoints b and c lies in $X - \varphi(Q)$. It is evident that the set $L' \cup \varphi(Q)$ is an n-dimensional umbrella in X (as defined in V, Section 16) with center c. It follows, by the homogeneity of X, that every point $x \epsilon X$ is the center of an n-umbrella lying in X. Applying Theorem on Umbrellas (V, (16.1)) we infer that $\dim X > n$, contrary to our hypothesis. Thus the proof of Theorem (15.1) is complete.

(15.2) Remark. One can show ([16], p. 106) that Theorem (15.1) remains true if we replace the hypothesis that X is an ANR-space by the weaker one, that X is a complete, connected, locally contractible space. Also the hypothesis on the homogeneity of X may be replaced by the less restrictive one of the local homogeneity, that means, by condition that for every two points $x_1, x_2 \epsilon X$ there exists a homeomorphism h mapping a neighborhood U_1 of x_1 into X and satisfying the requirement $h(x_1) = x_2$.

16. Disconnection of homogeneous ANR-spaces by compacta. Now let us prove the following

(16.1) THEOREM. *Let X be an n-dimensional homogeneous* ANR-*space and let B be a compact subset of X cyclic in dimension $n-1$ and contractible in a proper subset X' of X which is contractible in X. Then the set $X - B$ is not connected.*

Proof. Let $\gamma = \gamma^{n-1}$ be an $(n-1)$-dimensional infinite cycle in B, which is not homologous to zero in B. Since B is contractible in X', the infinite cycle γ is homologous to zero in X'. By II, (3.5), there exists a compactum $A \subset X'$ such that $B \subset A$ and that γ is homologous to zero in A, but it is not homologous to zero in any other subset of A containing B. In order to prove (16.1) it is sufficient to show that B disconnects X between every point of $A-B$ and every point of $X-A \supset X-X' \neq 0$. Otherwise there exists a point

$$a \in (A-B) \cap \overline{X-A}. \tag{16.2}$$

Let us denote by B_m, $m = 1, 2, \ldots$, the set consisting of all points $x \in X$ with $\varrho(x, B) < 1/m$. It is evident that the sets $A-B_m$ are compact and that $A-B = \bigcup_{m=1}^{\infty}(A-B_m)$. Since the set $(A-B_m) \cap \overline{X-A} \subset A \cap \overline{X-A}$ does not contain any open, non-empty subset of X, we infer that the set

$$(A-B) \cap \overline{X-A} = \bigcup_{m=1}^{\infty}(A-B_m) \cap \overline{X-A}$$

is of the first category (of Baire) in X. Since the space Φ consisting of all homeomorphisms h of X onto itself is separable (as a subset of the separable space X^X), there exists a sequence $\{h_k\}$ of homeomorphisms

$$h_k\colon X \xrightarrow[\text{onto}]{} X$$

dense in Φ. The set $(A-B) \cap \overline{X-A}$ is mapped by the homeomorphism h_k onto the set $h_k[(A-B) \cap \overline{X-A}]$ of the first category in the space X. It follows that there is a point

$$a_0 \in X - \bigcup_{k=1}^{\infty} h_k[(A-B) \cap \overline{X-A}]. \tag{16.3}$$

The homogeneity of X implies that there exists a homeomorphism

$$h_0\colon X \xrightarrow[\text{onto}]{} X$$

such that

$$h_0(a) = a_0 \in X - h_0(B). \tag{16.4}$$

The homology $\gamma \sim 0$ in A implies

$$h_k(\gamma) \sim 0 \ \textit{in}\ h_k(A) \quad \textit{for} \quad k = 0, 1, \ldots \tag{16.5}$$

Since the sequence $h_1, h_2, \ldots$ is dense in the space Φ, we infer that for every positive number η there exists an index $k' = k(\eta)$ such that $\varrho(h_0, h_{k'}) \leqslant \eta$.

Moreover, since the infinite cycle γ is not homologous to zero in any other subset of A containing B, we infer that

(16.6) *If* $h_k(B) \subset C \subset h_k(A)$ *and* $h_k(\gamma) \sim 0$ *in* C, *then*

$$h_k(A-B) \subset C \quad \textit{for} \quad k = 0, 1, 2, \ldots$$

Now let us consider the Cartesian product

$$Z = A \times \langle 0, 1\rangle$$

and its closed subset

$$Z_0 = A \times (0) \cup A \times (1).$$

Setting

$$f_1(x, t) = h_0(x) \quad \text{for} \quad (x, t) \in Z_0,$$

$$f_2(x, 0) = h_0(x) \quad \text{for} \quad x \in A,$$

$$f_2(x, 1) = h_{k'}(x) \quad \text{for} \quad x \in A,$$

we get two maps f_1, f_2: $Z_0 \to X$ satisfying the condition $\varrho(f_1, f_2) \leqslant \eta$. The first of them admits a continuous extension to Z with values in the set $h_0(A)$; we get an extension $\bar{f}_1$ of this kind setting $\bar{f}_1(x, t) = h_0(x)$ for $(x, t) \in Z$. Since A, as a subset of X', is contractible in X and h_0 is a homeomorphism of X onto itself, we infer that the set $h_0(A)$ is contractible in X. It follows by V, (3.1), that for every positive ε the number $\eta > 0$ can be chosen so that f_2 has a continuous extension

$$\bar{f}_2\colon Z \to X$$

such that

$$\varrho\big(\bar{f}_2(x, t), h_0(x)\big) \leqslant \varepsilon \quad \text{for every} \quad (x, t) \in Z.$$

Moreover, since $h_0(A)$ is contractible in X, and since the point a_0 does not belong to $h_0(B)$, we infer that the number ε can be chosen so that

(16.7) *the set* $\bar{f}_2(Z)$ *is contractible in* X,

(16.8) $$a_0 \in X - \bar{f}_2(B \times \langle 0, 1\rangle).$$

Let a_ν ($\nu = 0, 1$) denote the map of A into $A \times (\nu)$ assigning to each point $x \in A$ the point $(x, \nu) \in A \times (\nu)$. Manifestly, there is in the set $B \times \langle 0, 1\rangle$ an infinite n-dimensional chain λ such that

$$\partial \lambda = a_0(\gamma) - a_1(\gamma).$$

It follows that

(16.9) $h_0(\gamma) = f_2\big(a_0(\gamma)\big) \sim \bar{f}^2\big(a_1(\gamma)\big) = h_{k'}(\gamma)$ *in the set* $\bar{f}_2(B \times \langle 0, 1\rangle)$.

Now let us distinguish two cases:

Case 1. $a_0 \in X - h_{k'}(A)$. The infinite cycle $h_0(\gamma)$ is homologous to zero in the set $D = h_{k'}(A) \cup \bar{f}_2(B \times \langle 0, 1\rangle)$. Now let us observe that D does not contain the set $h_0(A-B)$, because a_0 belongs to $h_0(A-B)$ and, by (16.8), a_0 does not belong to the set $h_{k'}(A) \cup \bar{f}_2(B\times\langle 0,1\rangle)$. It follows by (16.6) that the infinite $(n-1)$-dimensional cycle $h_0(\gamma)$ is not homologous to zero in the common part of the sets D and $h_0(A)$, however — by (16.5) — it is homologous to zero in each of them. We infer, by the theorem of Phragmen-Brouwer (II, (3.6)), that there is in the set $D \cup h_0(A) \subset \bar{f}_2(Z)$ an n-dimensional infinite cycle γ' unhomologous to zero in $D \cup h_0(A)$. Moreover, (16.7) implies that $\gamma' \sim 0$ in X and we conclude, by the theorem of Alexandroff (II, (3.11)), that $\dim X > n$, contrary to our hypothesis.

Case 2. $a_0 \in h_{k'}(A)$. In this case the set $h_{k'}(A)$ is a neighborhood of the point a_0, because a_0 belongs neither to the set $h_{k'}\big((A-B) \cap \overline{X-A}\big)$ nor to the set $h_{k'}(B)$. Since the infinite cycle $h_{k'}(\gamma)$ is homologous to zero in the set $D' = \bar{f}_2(B\times\langle 0,1\rangle) \cup h_0(A) \supset h_{k'}(B)$ and h_0 is a homeomorphism of X onto itself, we infer by (16.2), (16.4), and (16.8) that the set D' is not a neighborhood of a_0 in X, and consequently it does not contain the set $h_{k'}(A-B)$. Applying the theorem of Phragmen–Brouwer to the cycle $h_{k'}(\gamma)$ and to its carriers D' and $h_{k'}(A)$, we infer that there exists in the set $D' \cup h_{k'}(A) \subset \bar{f}_2(Z)$ an infinite n-dimensional cycle γ' unhomologous to zero in $D' \cup h_{k'}(A)$. It follows by (16.7) that $\gamma' \sim 0$ in X, which, by the theorem of Alexandroff (II, (3.11)), contradicts the hypothesis $\dim X = n$. Thus the proof of Theorem (16.1) is terminated.

(16.10) THEOREM. *For dimensions $n = 0, 1, 2$, the connected, n-dimensional, homogeneous* ANR-*space coincide with n-dimensional manifolds.*

Proof. It is clear that the conditions are necessary (not only for dimensions $0, 1, 2$). In the case $n = 0$, it is also evident that they are sufficient. In the case $n = 1$ they are sufficient by Theorem (15.1), because every 1-dimensional ANR-space contains arcs. In order to finish the proof in the case $n = 2$, let us apply a characterization of 2-dimensional manifolds due to G. S. Young ([300], p. 986), as locally connected continua X of dimension 2 satisfying the following both conditions:

(i) *No point of X separates X.*

(ii) *There exists an $\varepsilon > 0$ such that every simple closed curve in X, with diameter less than ε, separates X.*

In order to prove that a 2-dimensional homogeneous, connected ANR-space X satisfies (i), let us consider a point $a \in X$ and a sequence $\{a_i\}$ of points of $X-(a)$ dense in $X-(a)$. Since X is a locally connected continuum, there exists, for $i = 1, 2, \ldots$, an arc $L_i \subset X$ joining a with a_i. Since X is 2-dimensional at each of its points, none of the arcs L_i contains an open, non-empty subset of X. Consequently the connected set $X' = \bigcup_{i=1}^{\infty} L_i$ is of the first category in the space X and we infer that there exists a point $b \in X - X'$. Then the connected set X' is dense in $X-(b)$, and thus b does not separate X. Since X is topologically homogeneous, no point separates X and condition (i) is established. Condition (ii) follows immediately by Theorem (16.1). Thus the proof of Theorem (16.10) is finished.

If we recall that no n-dimensional manifold, as a set cyclic in dimension n, is an AR-space, we get from Theorem (16.10) the following

(16.11) COROLLARY. *No* AR-*space of dimension* 1 *and* 2 *is homogeneous.*

(16.12) PROBLEM. *Does there exist a homogeneous* AR-*space* X *with* $2 < \dim X < \infty$?

(16.13) **Remark.** Both Theorems (16.1) and (16.10) remain true if we replace the hypothesis that X is an n-dimensional connected, homogeneous ANR-space by the weaker one, that X is an n-dimensional locally contractible, connected, locally homogeneous, locally compact, separable, metric space ([16], pp. 108 and 110).

Let us mention that, by a result of E. Fadell ([117], p. 531), if we replace the condition of the homogeneity by another, a more restrictive one of the *strong homogeneity*, the answer to the problem analogous to (16.12) will be negative, even for $n = \infty$.

CHAPTER VIII

ON r-CLASSIFICATION OF SPACES

As we have already seen (I, Section 6), all topological spaces can be classified into classes called r-types, where two spaces belong to the same r-type if and only if each of them is an r-image of the other. Now we shall study this classification a little more exactly. In particular, we shall look for some criterion which would make it possible, in some cases, to measure the difference between the topological properties of two given spaces.

1. Partial ordering of r-types. Let us recall (I, Section 6) that two spaces X and Y are said to be *r-equal* (written $X \underset{r}{=} Y$) if each of them is an r-image of the other. If X is an r-image of Y, but the converse is false, then we write $X \underset{r}{<} Y$ or $Y \underset{r}{>} X$. Evidently, the relation "$\underset{r}{<}$" is antisymmetric and transitive. Two spaces X and Y are said to be *r-comparable* if at least one of them is an r-image of the other, i.e. if $X \underset{r}{\leqslant} Y$ or $X \underset{r}{\geqslant} Y$ (I, Section 6). If none of these relations holds, then X and Y are said to be *r-incomparable* ([61], p. 322).

Let us observe that the relations $\underset{r}{<}$, $\underset{r}{\leqslant}$ and the relation of the r-incomparability may be considered also as the relations between r-types. In particular the relation $\underset{r}{<}$ partially orders each collection of r-types and the relation of the r-incomparability allows to speak about r-incomparable r-types of spaces.

Let us formulate some elementary propositions concerning the relations $\underset{r}{<}$ and $\underset{r}{\leqslant}$, which are immediate consequences of the properties of r-maps formulated in Chapter I and Chapter II:

(1.1) *$X \underset{r}{\leqslant} Y$ holds if and only if X is homeomorphic to a retract of Y.*

(1.2) *$X \underset{r}{\leqslant} Y$ implies* $\dim X \leqslant \dim Y$.

(1.3) *$X \underset{r}{\leqslant} Y$ implies that the power of the set of components (or of the set of arcwise connected components) of X is $\leqslant$ to the power of the set of components (or of the set of arcwise connected components respectively) of Y.*

(1.4) *$X \underset{r}{\leqslant} Y$ implies that for every cardinal number* $\mathfrak{m}$ *the power of the set of points of order* $\geqslant \mathfrak{m}$ *of X is less than or equal to the power of the set of points of order* $\geqslant \mathfrak{m}$ *of Y.*

(1.5) *If $X \underset{r}{\leqslant} Y$ and if Y has the fixed-point property, then X has also the fixed-point property.*

(1.6) *$X \underset{r}{\leqslant} Y$ implies that each homology and cohomology group of X is a divisor of the corresponding group of Y.*

It follows in particular that

(1.7) *If $X \underset{r}{\leqslant} Y$, then the Betti numbers of X are less than or equal to the corresponding Betti numbers of Y.*

If we limit ourselves to the arcwise connected spaces, then the homotopy groups are uniquely defined and we have

(1.8) *If $X \underset{r}{\leqslant} Y$, then the homotopy groups of X are r-images of the corresponding homotopy groups of Y.*

Since the homotopy groups of dimension $n > 1$ are Abelian, we infer by II, (1.7), that

(1.9) *If $X \underset{r}{\leqslant} Y$ and $n > 1$, then the group $\pi_n(X)$ is a divisor of the group $\pi_n(Y)$.*

For cohomotopy groups we have the following proposition:

(1.10) *If $X \underset{r}{\leqslant} Y$ and* $\dim Y < 2n-1$, *then the n-th cohomotopy group of X is a divisor of the n-th cohomotopy group of Y.*

2. r-minorants and r-majorants ([61], p. 323). A space X_0 is said to be an *r-minorant* for a family $\mathscr{X}$ of spaces provided that $X_0 \underset{r}{\leqslant} X$ for every space $X \in \mathscr{X}$. A space X_0 satisfying the condition $X_0 \underset{r}{\geqslant} X$ for every space $X \in \mathscr{X}$ is said to be an *r-majorant* for the class $\mathscr{X}$. Manifestly, if $\mathscr{X}$ is an arbitrary family of non-empty spaces, then the space consisting of only one point is an r-minorant for $\mathscr{X}$, and the Cartesian product of all spaces belonging to $\mathscr{X}$ is an r-majorant for $\mathscr{X}$.

If a family of spaces $\mathscr{X}$ contains at least one of its r-minorants, then it will be said to be *r-closed on the left* ([61], p. 324), and if it contains at least one of its r-majorants, then it will be said to be *r-closed on the right.* It is clear that all r-minorants of a family $\mathscr{X}$ belonging to $\mathscr{X}$ are r-equal, and the same holds also for all r-majorants belonging to $\mathscr{X}$.

(2.1) EXAMPLE. *The class of all 0-dimensional compacta is r-closed on the left by a one-point space and it is also r-closed on the right by the Cantor discontinuum.*

(2.2) EXAMPLE. *The class of all n-dimensional compacta (where $n > 0$) is not r-closed on the left.*

In fact, if a compactum X_0 would be an r-minorant belonging to this class, then X_0 would be homeomorphic to a subset of the n-dimensional Euclidean cube Q^n. Since $\dim X_0 = n$, the cube Q^n would be homeomorphic to a subset of X_0. It follows that every n-dimensional compactum would contain topologically the cube Q^n, which is not true. As it has been shown by A. Lelek ([217], p. 285), the class of all 1-dimensional compacta is not r-closed on the right.

(2.3) EXAMPLE. *The class of all n-dimensional* AR-*spaces* $(n > 1)$ *is not r-closed on the left.*

This follows by the same argument as in Example (2.2). The same holds also for the class of all n-dimensional ANR-spaces.

(2.4) EXAMPLE. *The class of all n-dimensional* AR-*spaces is r-closed on the right, if $n = 1$, but it is not r-closed on the right, if $n > 1$.*

In fact, the universal dendrite ([281], p. 57) is a 1-dimensional AR-space which is an r-majorant for the class of all 1-dimensional AR-spaces. If, however, n is greater than 1, then VI, (7.3), implies that no n-dimensional AR-space is an r-majorant for the family of all n-dimensional AR-spaces.

(2.5) EXAMPLE. *Any r-majorant of the class of all n-dimensional* ANR-*spaces is not an* ANR-*space.*

In fact, as we have observed, the power of the set of components of such an r-majorant would be greater than or equal to the power of the set of components of every n-dimensional ANR-space, and consequently it would be infinite. However, as we know (V, (2.7)), the power of the set of all components of an ANR-space is finite.

It follows, in particular, that for every integer $n \geqslant 0$ (finite or infinite) the class of all ANR-spaces of dimension $\leqslant n$ is not r-closed on the right. An analogous statement holds also for the class of all polyhedra of dimension $\leqslant n$.

3. r-increasing and r-decreasing sequences. A sequence $\{X_n\}$ of spaces is said to be *r-increasing* provided that $X_n \underset{r}{<} X_{n+1}$, and it is said to be *$r$-decreasing* provided that $X_n \underset{r}{>} X_{n+1}$ for every $n = 1, 2, \ldots$ The examples of r-increasing sequences of spaces are at hand. For instance, the sequence $\{X_n\}$, where X_n consists of exactly n points, or the sequence $\{E^n\}$ of all Euclidean spaces, or the sequence $\{Q^n\}$ of all Euclidean cubes.

A little more difficult is to give an example of an r-decreasing sequence of spaces. Let $\{X_n\}$ denote the sequence of 2-dimensional polyhedra defined as follows: Consider a regular polygon W_n with $n+2$ sides L_1, $L_2, \ldots, L_{n+2}$ lying in the plane E^2 (Fig. 9). Let σ_i denote the regular triangle such that L_i is one of its sides, and its interior contained in the plane E^2 on the outer side of W_n. Setting

$$X_n = \bigcup_{i=1}^{n+2} \sigma_i,$$

we easily see that X_{n+1} is an r-image of X_n, but the converse is false. It follows that the sequence $\{X_n\}$ is r-decreasing.

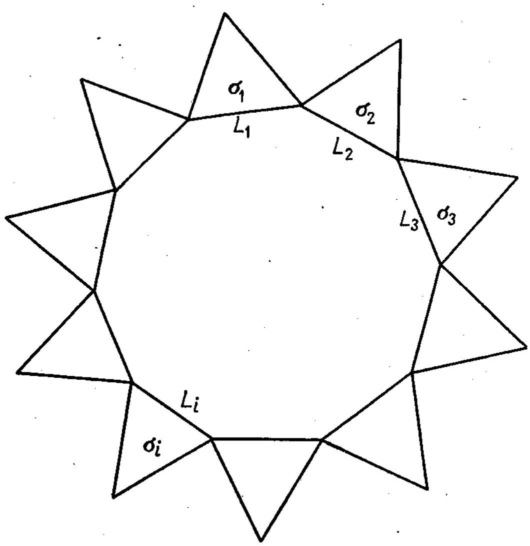


Fig. 9

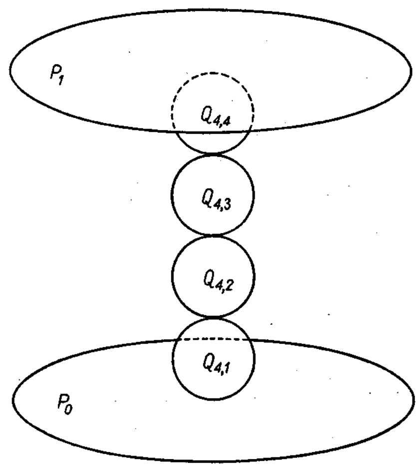


Fig. 10

One can show that there does not exist any r-decreasing sequence of plane AR-sets being polyhedra. On the other hand, there exists in the 3-dimensional Euclidean space E^3 an r-decreasing sequence of 2-dimensional polyhedra being AR-sets. Let us give here its construction, leaving to the reader the proof that it is r-decreasing.

Let (x, y, z) denote a point of E^3 with orthogonal coordinates x, y, z. Let us set:

$$P_0 = \{(x, y, z);\, x = 0,\, y^2+z^2 \leqslant 1\}, \qquad P_1 = \{(x, y, z);\, x = 1,\, y^2+z^2 \leqslant 1\},$$

$$Q_{n,k} = \left\{(x, y, z);\, z = 0,\, \left(x - \frac{2k-1}{2n}\right)^2 + y^2 \leqslant \frac{1}{4n^2}\right\}$$

for $k = 1, 2, \ldots, n$; $n = 1, 2, \ldots$

One readily sees that the set

$$A_n = P_0 \cup P_1 \cup \bigcup_{k=1}^{n} Q_{n,k}$$

is a 2-dimensional curvilinear polyhedron, being an AR-space (Fig. 10). It is easy to show that A_{n+1} is an r-image of A_n and that the converse is false, which means that the sequence $\{A_n\}$ is r-decreasing.

4. Power of the class of all r-types of compacta. Since every compactum is topologically contained in the Hilbert cube Q^ω, and since Q^ω contains only $2^{\aleph_0}$ distinct closed subsets, we infer that there exist at most $2^{\aleph_0}$ different r-types of compacta. On the other hand, let us observe that already among r-types of 2-dimensional AR-spaces there exists a family consisting of $2^{\aleph_0}$ different r-types. In fact, by the Second Theorem on Families of ANR-sets (VI, (6.2)) there exist in E^3 a family consisting of $2^{\aleph_0}$ locally r-incomparable 2-dimensional AR-sets. None of these sets is topologically contained in another, and consequently their r-types are different. Thus we infer that the power of the class of all r-types of compacta is equal to $2^{\aleph_0}$. It follows also that the power of the class of all r-types of AR-sets (and also of all ANR-sets) is equal to $2^{\aleph_0}$. Let us mention that it has been shown by K. Sieklucki [258] that there exists even a family consisting of $2^{\aleph_0}$ r-incomparable 1-dimensional AR-sets (dendrites). This explains, to some extent, why among ANR-spaces one meets species with rather paradoxical properties quite different from the properties of polyhedra. In fact, as we have seen (VI, Section 6), the collection of all r-types of polyhedra is only countable. It follows that from the point of view of the theory of retracts, the wealth of the topological phenomena among ANR-sets is much greater than among polyhedra. The ANR-spaces with the r-type of polyhedra are only exceptions. Perhaps they merit a special attention.

5. r-neighbors. A space Z is said to be *r-between* two spaces X and Y if either $X \underset{r}{<} Z \underset{r}{<} Y$ or $Y \underset{r}{<} Z \underset{r}{<} X$. Two r-comparable and r-different spaces X and Y are said to be *r-neighbors* ([61], p. 328) if there is no space r-between X and Y. If X and Y are r-neighbors and $X \underset{r}{<} Y$, then X is said to be the *left r-neighbor of Y* and Y is said to be the *right r-neighbor of X*.

Manifestly, the relation of lying r-between and also the relation of being a left r-neighbor or right r-neighbor are in fact relations between r-types. Thus in the sequel of this chapter, we shall not distinguish between spaces of the same r-type and thus, speaking about different r-neighbors of a given space we shall always mean the spaces with different r-types. Let us consider some examples.

(5.1) EXAMPLE. It is easy to see that *the segment has only one left r-neighbor*, namely the one-point space. However, the segment has many right r-neighbors. For instance, the space consisting of a segment and of an isolated point, the circle and also the space consisting of three segments starting from one endpoint (a *triod*).

Also the segment without one endpoint is a right r-neighbor of the segment. Another example of a right r-neighbor of the segment is the closure of the diagram of the function $y = \sin(1/x)$ for $0 < x < 1$. Let us observe that *there exist also right r-neighbors of the segment with arbitrarily high dimension.* In order to get such an example, consider an n-dimensional continuum A which does not contain any simple arc. We can assume that A lies in a linear subspace H of the Hilbert space E^ω and that $H \neq E^\omega$. One easily sees that there exists in $E^\omega - H$ a set B, homeomorphic to the segment without one of its endpoints, such that the set $X = A \cup B$ is a compactum which coincides with the closure of B. It is easy to observe that X is an n-dimensional right r-neighbor for the segment.

(5.2) EXAMPLE. *The disk has as its right neighbors the 2-dimensional sphere and the projective plane.* Also the 2-dimensional umbrella (as defined in V, Section 16) is a right neighbor of the disk.

(5.3) EXAMPLE. Let X be a curve which is the union of a circle and of one of its radii. It is easy to see that *X has two left r-neighbors, the circle and the triod* (the union of three segments issuing from one point).

(5.4) EXAMPLE. Let us observe that *the disk Q has only one left r-neighbor, namely the universal dendrite D* ([281], p. 137).

In fact, every space X satisfying the condition $X \underset{r}{\leqslant} Q$ is a plane AR-set. If $\dim X = 2$, then the disk Q is an r-image of X and consequently $X \underset{r}{=} Q$. It follows that X is a plane 1-dimensional AR-set, i.e. it is a dendrite. It follows that $X \underset{r}{\leqslant} D \underset{r}{<} Q$, and if X is a left r-neighbor for Q, then $X \underset{r}{=} D$.

It follows by these examples that a space can have many r-neighbors as well on the left, as on the right. The problem whether there exists a non-empty space without a right r-neighbor remains open. Such a space cannot be an ANR-space, because if X is a non-empty ANR-set, then we see at once that the space Y, which we get by adding an isolated point to X, is an ANR-set and a right r-neighboor of X. However, there exist AR-spaces without left r-neighbors. It is evident that the space consisting of only one point has this property. Not so trivial is the following

(5.5) EXAMPLE. *The Euclidean n-dimensional cube Q^n, where $n > 2$, has no left r-neighbor* ([72], p. 296).

In order to prove this, consider the family $\{X_\mu\}$ consisting of $2^{\aleph_0}$ $(n-1)$-dimensional AR-sets lying in the Euclidean n-space E^n and such that for $\mu \neq \mu'$ none $(n-1)$-dimensional closed subset of X_μ is topologically contained in $X_{\mu'}$; the existence of such a family is secured by

Theorem VI, (6.2). Now suppose that there exists a left r-neighbor X for the cube Q^n. Then X is homeomorphic to a subset X' of Q^n. If $\dim X' = n$, then X' contains topologically Q^n and consequently $X \underset{r}{=} Q^n$, contrary to the hypothesis that $X \underset{r}{<} Q^n$. Hence $\dim X' \leqslant n-1$. Now let us observe that every space X_μ is an r-image of X. In fact, if Q' is an n-dimensional cube lying in the set $Q^n - X'$, then Q' contains a subset X'_μ homeomorphic to X_μ. Consider an arc $L \subset Q^n$ with one endpoint in X', the other endpoint in X'_μ and with the interior lying in the set $Q^n - X' - X'_\mu$. It follows by V, (2.9), that the set $Y_\mu = X' \cup L \cup X'_\mu$ is an AR-space such that $X \underset{r}{\leqslant} Y_\mu \underset{r}{\leqslant} Q^n$. Since X is a left r-neighbor of Q^n, we infer that $X \underset{r}{=} Y_\mu$ and therefore X contains topologically the set X_μ for every index μ. Thus we see that for every index μ there is a subset X'_μ of X homeomorphic to X_μ. But for $\mu \neq \mu'$ none closed $(n-1)$-dimensional subset of X'_μ is homeomorphic to a subset of X'_μ, and hence $\dim(X'_\mu \cap X'_{\mu'}) \leqslant n-2$. But this contradicts the First Theorem on Families of ANR-sets (V, (15.1)).

6. AR-space with an infinite number of r-neighbors. As we have already seen, a space can have many left r-neighbors and right r-neighbors. Even a space so simple as the segment has (by Example (5.1)) an infinite number of right r-neighbors though almost all of them are spaces with a rather complicated structure. If, however, we confine our attention to r-neighbors being ANR-spaces, then it is easy to show that the segment has only three right r-neighbors: the circle, the union of the segment and of one isolated point, and the triod. The question appears whether there exists an ANR-space with an infinite number of r-neighbors which are also ANR-sets. The answer is positive by the following ([68], p. 346)

(6.1) THEOREM. *There exists in E^3 a 2-dimensional* AR-*space X having an infinite number of left r-neighbors and $2^{\aleph_0}$ of right r-neighbors, which are r-distinct* AR-*sets.*

Proof. In order to obtain a space X with this property let us use the r-decreasing sequence $\{A_n\}$ of 2-dimensional polyhedra being AR-sets, as it was constructed in Section 3. Let us recall that A_n consists of $n+2$ geometric disks $D_0, D_1, \ldots, D_{n+1}$ with centers lying on a straight line $L \subset E^3$ such that two successive disks have only one point in common, while the non-consecutive disks are disjoint. Let a and b denote the endpoints of the segment $L \cap \bigcup_{i=1}^{n} D_i$. The disks D_0 and D_{n+1} lie in two planes in E^3 perpendicular to L, and a is the center of D_0, and b is the center of D_{n+1}. The disks D_0 and D_{n+1} are said to be the *lower base* and the *upper base* of the polyhedron A_n, respectively. It is easy to see that $m < n$ implies $A_n \underset{r}{<} A_m$.

Now let B_n denote the polyhedron in E^3 obtained from n disjoint copies of the polyhedron A_n by the identification of their lower bases. Thus we get a sequence $\{B_n\}$ of polyhedra such that none of them is homeomorphic to a subset of the other (Fig. 11). Manifestly, each of the polyhedra B_n is 2-dimensional at every of its points.

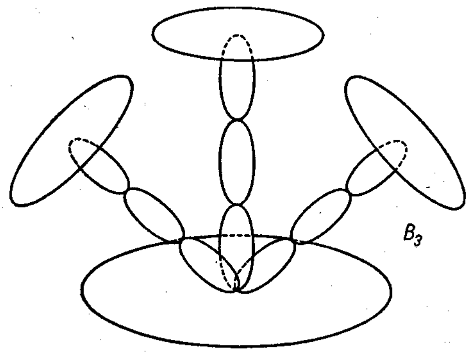


Fig. 11

Now let us consider in the space E^3 the segment J with endpoints $(0, 0, 0)$ and $(1, 0, 0)$ and let $\{w_n\}$ be a sequence consisting of all distinct rational numbers belonging to the open interval $(0, 1)$. Let J_n denote, for every $n = 1, 2, \ldots$, the segment with endpoints $(w_n, 1/n, 0)$ and $(w_n, -1/n, 0)$ and let Q_n denote a ball (in E^3) with center $(w_n, 1/n, 0)$ and radius ϱ_n, where $0 < \varrho_n < 1/n$ is chosen so that the sets $J_n \cup Q_n$ are mutually disjoint. It is clear that there exists a homeomorphism h_n mapping the set B_n onto a subset of the ball Q_n containing the point $(w_n, 1/n - \varrho_n, 0)$. One easily sees that the set

$$X = \bigcup_{n=1}^{\infty} [(J_n - Q_n) \cup h_n(B_n)] \cup J$$

is an AR-set.

Consider the subset Y of the space X made up of all points $y \in X$ at which X has dimension 2. Evidently

(6.2) *The components of Y coincide with the sets $h_n(B_n)$.*

(6.3) *For every point p of the segment J there exists an increasing sequence of indices $\{n_k\}$ such that $\lim_{k\to\infty} h_{n_k}(B_{n_k}) = (p)$.*

Now let us prove the following

(6.4) LEMMA. *If $M_1 = Y \cup N_1$ and $M_2 = Y \cup N_2$ are two compacta such that* $\dim N_1 \leqslant 1$ *and* $\dim N_2 \leqslant 1$ *and if h is a homeomorphism mapping M_1 into M_2, then*

(i) *$hh_n(B_n) \subset h_n(B_n)$ for every index $n = 1, 2, \ldots$,*

(ii) *$h(p) = p$ for every point $p \in J$.*

Proof. It follows by our hypotheses that Y coincides with the set of points at which M_1 has dimension 2, and also with the set of points at which M_2 has dimension 2. We infer that h maps every component of Y into a component of Y. It follows by (6.2) that for every $n = 1, 2, \ldots$ there exists a natural m such that $h(h_n(B_n)) \subset h_m(B_m)$. If we recall that none of the sets B_n is homeomorphic to a subset of another of these sets, we conclude that $n = m$. Hence

$$h(h_n(B_n)) \subset h_n(B_n) \quad \text{for every } n = 1, 2, \ldots$$

It remains to apply (6.3) in order to complete the proof of Lemma (6.4).

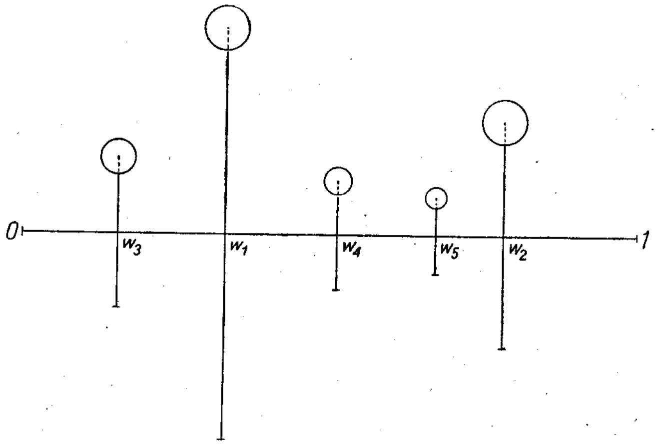


Fig. 12

Proof of Theorem (6.1). In order to prove that X has an infinite collection of left r-neighbors, let us denote by X_n, for every $n = 1, 2, \ldots$, the set which we get by removing from X the half-open segment consisting of all points $(w_n, t, 0)$ with $-1/n \leqslant t < 0$. It is clear that X_n is a retract of the space X; hence

(6.5) $$X_n \underset{r}{\leqslant} X \quad \text{for every } n = 1, 2, \ldots$$

On the other hand, the set X is not homeomorphic to any subset of the set X_n. In fact, if h were a homeomorphism of X into X_n, then we would infer by Lemma (6.4) (setting $M_1 = X$ and $M_2 = X_n$) that h maps the point $p_n = (w_n, 0, 0)$ onto itself. But this is impossible, because the order of this point in the space X is equal to 4, and in the space X_n, only to 3.

It follows by (6.5) that

(6.6) $$X_n \underset{r}{<} X \quad \text{for every } n = 1, 2, \ldots$$

By a quite analogous argument, we infer that

(6.7) *For $n \neq m$ the sets X_n and X_m are r-distinct.*

Now let us consider a space Z such that

$$X_n \underset{r}{\leqslant} Z \underset{r}{\leqslant} X. \tag{6.8}$$

We can assume that $Z \subset X$. The inequality $Z \underset{r}{\leqslant} X$ implies that Z is an AR-set. If $Z \subset X_n$, then we infer that $Z \underset{r}{=} X_n$. If, however, $Z - X_n \neq 0$, then $Z = (X_n \cap Z) \cup J'$, where $X_n \cap Z$ is an AR-set and J' is a segment made up of points $(w_n, t, 0)$ with $s_n \leqslant t \leqslant 0$, where s_n is a number satisfying the inequality $-1/n \leqslant s_n < 0$. Since $X_n \underset{r}{\leqslant} Z$, there exists a homeomorphism g mapping X_n into Z. Applying Lemma (6.4), we infer by (i) and (ii) that g maps every component of the set $X_n - J$ into itself and that $g(p) = p$ for every point p of the segment J.

It follows that g maps the set X_n into the set $Z \cap X_n$. If we set

$$g'(p) = g(p) \quad \text{for every point } p \in X_n,$$

$$g'(w_n, t, 0) = (w_n, -ns_n t, 0) \quad \text{for} \quad -\frac{1}{n} \leqslant t \leqslant 0,$$

we get a homeomorphism g' mapping X into Z. It follows that $X \underset{r}{=} Z$.

Thus we have shown that (6.8) implies that $X_n \underset{r}{=} Z$ or $X \underset{r}{=} Z$. According to (6.6) and (6.7), we infer that the sets $X_1, X_2, \ldots$ are r-distinct left r-neighbors of the set X.

It remains to prove that there exists a family consisting of $2^{\aleph_0}$ r-distinct AR-sets being right r-neighbors of the set X. Let t be an irrational number of the interval $\langle 0, 1\rangle$ and let J_t denote the segment made up of all points $(t, u, 0)$ with $-1 \leqslant u \leqslant 0$. It is evident that the set $X_t = X \cup J_t$ is an AR-set such that $X \underset{r}{\leqslant} X_t$. Using the same argument as in the first part of the proof, we show, step by step, that $X \underset{r}{<} X_t$ and that for $t_1 \neq t_2$ the spaces X_{t_1} and X_{t_2} are r-distinct. Also the proof that X_t is a right r-neighbor of X is quite analogous to the proof that X is a right r-neighbor of the space X_n. Since the parameter t runs through all irrational numbers of the interval $\langle 0, 1\rangle$, we conclude that the family $\{X_t\}$ consists of $2^{\aleph_0}$ r-distinct AR-sets which are right r-neighbors of X. Thus the proof of Theorem (6.1) is finished.

Let us remark that the question whether there exists an AR-set with $2^{\aleph_0}$ r-distinct left r-neighbors remains open.

7. Dimension of r-neighbors. As we have seen (5.1), there exist right r-neighbors of the segment with arbitrarily high dimension. Even among AR-spaces there exist r-neighbors with distinct dimensions. For instance, the segment and the space consisting of only one point, and also the disk and the universal dendrite. However the question remains open whether there exist two ANR-spaces X and Y which are r-neighbors such that

$$1 < \dim X < \dim Y,$$

and also the question whether there exist two ANR-spaces which are r-neighbors such that their dimensions differ more than by one.

We have only (compare [64], p. 461) the following

(7.1) THEOREM. *If X is an* ANR-*space containing the n-dimensional cube Q^n with $n > 2$, then the dimension of every r-neighbor Y of X is $\geqslant n$.*

Proof. If $Y \underset{r}{\geqslant} X$, then $\dim Y \geqslant \dim X \geqslant \dim Q^n = n$. Hence we can assume that Y is a retract of X and let r be a retraction of X to Y. Let X_1 be a component of X containing a subset Q homeomorphic to Q^n. If $X_1 \frown Y = 0$, then we select a point $x_1 \epsilon X_1$ and we see at once that the set $Z_1 = (x_1) \smile Y$ satisfies the condition

$$Y \underset{r}{<} Z_1 \underset{r}{<} X.$$

contrary to the hypothesis that Y is an r-neighbor of X. Hence $X_1 \frown Y \neq 0$. Now let us suppose that $\dim Y < n$. Then there exists a set $Q_1 \subset Q - Y$ homeomorphic to Q^n. By the Second Theorem on Families of ANR-sets (VI, (6.2)) there exists in Q_1 an uncountable family $\{Y_\mu\}$ of $(n-1)$-dimensional AR-sets such that for $\mu \neq \mu'$ none $(n-1)$-dimensional closed subset of Y_μ is topologically contained in $Y_{\mu'}$. Since the dimension of Y is less than n, we infer, by the First Theorem on Families of ANR-sets (V, (15.1)), that there is an index μ_0 such that the set Y_{μ_0} is not homeomorphic to any subset of the space Y.

Now let us consider an arc $L \subset X_1$ such that $L \frown Y$ consists of only one point a and $L \frown Y_{\mu_0}$ — of only one point b. By V, (2.9), the set $Z = L \smile Y_{\mu_0}$ is an AR and the set $Y \smile Z$ — an ANR. Moreover, Y is a retract of $Y \smile Z$, i.e. $Y \underset{r}{\leqslant} Y \smile Z$. Since $Z \epsilon \mathrm{AR}$, there exists a homotopy $\{\varphi_t\} \subset Z^Z$ such that φ_0 is the identity, φ_1 maps Z onto the point a and $\varphi_t(a) = a$ for $0 \leqslant t \leqslant 1$. Setting

$$\psi_t(x) = \begin{cases} \varphi_t(x) & \text{for every point } x \epsilon Z, \\ x & \text{for every point } x \epsilon Y, \end{cases}$$

we get a homotopy $\{\psi_t\} \subset (Y \cup Z)^{(Y \cup Z)}$ joining the identity ψ_0 with the map ψ_1. Setting

$$\psi_t'(x) = r\psi_{1-t}(x) \quad \text{for every point } x \in Y \cup Z \text{ and } 0 \leqslant t \leqslant 1,$$

we obtain a homotopy $\{\psi_t'\} \subset (Y \cup Z)^{(Y \cup Z)}$ joining the map $\psi_0' = \psi_1$ with the map ψ_1' satisfying the condition

$$\psi_1'(x) = r(x) \quad \text{for every point } x \in Y \cup Z.$$

It follows that ψ_0 and ψ_1' are homotopic and that ψ_1' has a continuous extension $\bar{\psi}_1 \in (Y \cup Z)^X$ given by the formula $\bar{\psi}_1(x) = r(x)$ for every point $x \in X$. Since $Y \cup Z \in \mathrm{ANR}$, we infer by IV, (8.3), that also ψ_0 has a continuous extension $\bar{\psi}_0 \in (Y \cup Z)^X$. Since ψ_0 is the identity, we infer that $\bar{\psi}_0$ is a retraction of X to $Y \cup Z$, whence $Y \cup Z \underset{r}{\leqslant} X$. Thus we have shown that

$$Y \underset{r}{\leqslant} Y \cup Z \underset{r}{\leqslant} X.$$

But the set $Y_{\mu_0} \subset Y \cup Z$ is not homeomorphic to any subset of the set Y, and consequently the sets Y and $Y \cup Z$ are r-distinct. Moreover, $\dim(Y \cup Z) = n-1 < \dim X$, and consequently the sets X and $Y \cup Z$ are r-distinct. Thus we see that the space $Y \cup Z$ lies r-between the sets X and Y and therefore X and Y are not r-neighbors, contrary to our hypothesis. Thus the proof of Theorem (7.1) is terminated.

Let us observe that Theorem (7.1) implies the following

(7.2) Corollary. *No space Y has a right r-neighbor which is a polyhedron of dimension greater than* 2 *and than* $\dim Y$.

8. Index of r-proximity. The notion of r-neighbors allows to introduce a notion, which may be usefull if one wants to measure how much the topological properties of one space X differ from the topological properties of another space Y. A natural number n is said to be the *index of r-proximity* ([61], p. 329) of two spaces X and Y with different r-types provided that there exists a system of $n+1$, but not less, spaces $X_0, X_1, \ldots, X_n$ such that $X_0 = X$, $X_n = Y$ and that X_i and X_{i+1} are r-neighbors for $i = 0, 1, \ldots, n-1$. If such a number n does not exist, then we say that the index of r-proximity of X and Y is *infinite*. In the case $X \underset{r}{=} Y$ we define the index of r-proximity of X and Y as zero. It is plain that the index of r-proximity is equal to 1 if and only if the spaces are r-neighbors. The index of r-proximity of two dimensional sphere and the projective plane is equal to 2, because one easily sees that the disk is a common r-neighbor of these spaces. Unfortunately the cases where the index of r-proximity is finite seem to be rather exceptional.

Among many open problems concerning the index of r-proximity, let us formulate the following ones:

(8.1) PROBLEM. *Is the index of r-proximity of the 2-dimensional sphere S^2 and of the torus $T = S^1 \times S^1$ finite?*

(8.2) PROBLEM. *Is the index of r-proximity of two* ANR-*sets, with different dimensions greater than* 1, *necessarily infinite?*

(8.3) PROBLEM. *Is it true that the index of r-proximity of two spaces $X, Y \in$* ANR *is* $\geqslant \sum_{k=0}^{\infty} |p_k(X) - p_k(Y)|$? *($p_k(Z)$ denotes the k-dimensional Betti number of Z).*

CHAPTER IX

A REVIEW OF VARIOUS RESULTS AND PROBLEMS OF THE THEORY OF RETRACTS

In Chapters I–VIII we intended to give an exposition of the theory of retracts with a special stress on its geometrical aspects (the theory of AR-spaces and of ANR-spaces). However, many important topics of the theory of retracts remained out of the scope. In the present chapter we give a short review of those topics, without proofs. Moreover, in order to give the more complete image of the present state of the theory of retracts, we shall formulate some open problems belonging to this theory. In conformity with the general trend of this book, we shall restrict our attention to the problems with simple formulations and with a clear geometric sense.

1. Modifications of the basic notions of the theory of retracts. Since the geometric point of view dominated in this book, many important notions concerning the extending of the theory of retracts to more general spaces were omitted. The importance of such generalizations appears, in the first line, in the theory of extension of maps and its applications, for instance, to the functional analysis. In particular, the notion of absolute retracts and absolute neighborhood retracts limited at beginning to compacta has been step by step generalized to metric separable spaces (by K. Kuratowski 1935, [196], p. 270), to the completely regular spaces (called also *Tychonoff spaces*) (by S. T. Hu 1947, [155], p. 1051), and studied by several authors, in particular by R. H. Fox [125]. The case of arbitrary Hausdorff spaces has been considered by W. C. Saalfrank 1949, [252] and the case of normal spaces by O. Hanner 1951, [142] and by C. H. Dowker [99]. In 1952, S. Saito has considered [255] the case of locally compact Hausdorff spaces and C. H. Dowker [99] the case of collectionwise normal spaces. In this book (Chapter IV) we have limited our considerations only to the generalization of the theory of absolute retracts and absolute neighborhood retracts to arbitrary metric spaces, given by E. Michael 1953, [228] and developed by J. Dugundji 1958, [103]. Let us mention that a systematical exposition of various generalizations of this theory is given in the paper of O. Hanner [144].

Recently J. R. Isbell ([167] and [168]) has given a remarkable generalization of the theory of retracts onto the case of uniform spaces. The retracts and neighborhood retracts of the Tychonoff cube have been studied by W. L. Strother and L. E. Ward, Jr., [272], and the retracts of totally disconnected compact spaces by R. Sikorski [264] and by R. Engelking [113]. Finally, let us mention a generalization of the notion of absolute neighborhood retract onto the case of perfectly normal spaces (i.e. spaces in which each closed set is a G_δ) given by B. H. McCandless [225] and [226].

In other direction go the studies of A. H. Kruse [192] concerning a theory including the theory of CW-complexes (in the sense of J. H. C. Whitehead [289]) and also the theory of absolute neighborhood retracts. The basic notion of this theory is the notion of a retraction of a map, which contains as a special case the usual notion of retraction of a space. Several generalizations of the notion of absolute retract and of absolute neighborhood retract for pairs of spaces were recently studied by M. Moszyńska [234].

There exist various notions intimately related to the notion of retract. First let us mention the notion of *homology absolute retracts* AHR and of *homology absolute neighborhood retract* ANHR introduced by S. Lefschetz [212]. These notions are useful in the theory of fixed points. Another notion of *homotopy absolute neighborhood retract* ANHR, introduced by A. F. Bartholomay [9] and generalized to compact Hausdorff spaces by C. W. Saalfrank [253] and [254], is strictly related to the notion of h-maps, as defined in I, Section 14. Finally, let us mention the notions of G-AR and G-ANR introduced by R. S. Polais in his paper [239] concerning the theory of G-spaces, i.e. of completely regular spaces X with a fixed action of a compact Lie groups G of homeomorphisms on X.

If one requires the existence of continuous extensions for maps defined on closed subsets of some kind of spaces with values belonging to another given space, one obtains various notions, as the notion of a *solid* (N. Steenrod [268], p. 54, and O. Hanner [142], p. 375), of an *extension space* ES($\mathfrak{K}$) or of a *neighborhood extension space* NES($\mathfrak{K}$) for a given class of spaces $\mathfrak{K}$ (O. Hanner [144]) or the notion of an *absolute extensor* AE and of an *absolute neighborhood extensor* ANE for metric spaces (E. Michael [228] and Y. Kodama [181]). A systematic exposition of the theory of extensors is given in the book by S. T. Hu [162]. To such notions belongs also the notion of the relation τ introduced by A. D. Wallace [279]: Let X and Y be two spaces; then $X\tau Y$ denotes that for every closed subset A of X, every map $f\colon A \to Y$ has a continuous extension $\bar{f}\colon X \to Y$. This notion makes it possible, for instance, to characterize AR-spaces among all compacta Y of dimension $\leqslant n$ as spaces satisfying the relation $Q^{n+1}\tau Y$, where Q^{n+1} denotes the Euclidean $(n+1)$-ball. Some results concerning the relation τ are given in [199], [203] and in [171].

A generalization of the notion of retraction based on the multivalued maps and the corresponding notion of *M-retract* was studied by W. L. Strother [271]. Let us also mention the studies of R. J. Wille [295] concerning a notion of *weak retraction*, less restrictive than the usual one.

An other more geometric character has the generalization of the retraction given by H. Noguchi [237]. The aim of this generalization is not to extend the notions of the theory of retracts to a more general classes of spaces, or to consider instead of spaces some more general categories (as done by A. H. Kruse [192] or by M. Moszyńska [234]), but the replacing of the usual notion of retraction by a more general notion of the *approximative retraction*. A subset X_0 of a metric space X is said to be an *ε-retract* of X provided that for every positive number ε there exists a map $r_\varepsilon\colon X \to X_0$ satisfying the condition $\varrho(x, r_\varepsilon(x)) < \varepsilon$ for every point $x \in X_0$. This notion leads in a natural manner to the notions of the *approximative* AR-spaces and ANR-spaces. As it has been proved by H. Noguchi [237], an important part of theorems of the theory of retracts may be extended onto so modified notions. Another generalization of the notion of retract and of retraction has been studied recently by S. K. Hildebrand and D. E. Sanderson [145]. Instead of continuous retraction one considers more general class of connectedness retractions. Some invariants of these generalized retractions are given.

Finally, let us mention some researches concerning retractions restricted by some conditions, for instance the *open non-alternating* retractions, i.e. such open maps $f\colon X \to Y$ which are retractions and satisfy the condition that for every two points $y_1, y_2 \in Y$ the set $f^{-1}(y_1)$ does not separate X between any two points belonging to $f^{-1}(y_2)$. Such retractions have been studied by G. T. Whyburn [293]. In the case of vector spaces, the linear retractions are of a special interest. They have been studied by M. Pavel [246]. A more general class of retractions, satisfying some conditions analogous to the boundedness of the norm of linear transformations has been studied by N. Aronszajn and P. Panitchpakdi [8], and also by B. Grünbaum [141].

2. Embedding problems. As we have seen (V, (6.1)), every compactum can be completed to an AR-space by addition of an infinite (countable) polytope. This result belongs to the theory of embedding. Some other results belonging to this theory are: the construction of a 2-dimensional AR-space which does not contain topologically any disk ([17], p. 164), the existence of the universal dendrite [281], and the theorem that for $n > 1$ an universal n-dimensional AR-set does not exist (VI, Theorem (7.3)). Moreover, let us mention the following theorem on embedding proved recently by H. Bothe ([80], p. 210):

(2.1) THEOREM. *For every* $n = 0, 1, 2, \ldots$ *there exists* $(n+1)$*-dimensional* AR*-space which contains topologically every metric separable space of dimension* $\leqslant n$.

The proof of this important theorem depends upon the classical construction due to K. Menger [227] and G. Nöbeling [238] of an n-dimensional compactum X which contains topologically every metric separable space of dimension $\leqslant n$. It has been shown by H. Bothe [80] that there exists an infinite (countable) $(n+1)$-dimensional polytope P such that the set $X \cup P$ is an AR-space, and consequently it satisfies the required conditions.

An other recently obtained result, concerning the embedding, is the construction [15] of a 3-dimensional AR-space which does not contain any disk.

Many problems of the theory of retracts, concerning the embedding, remain open. Let us formulate some of them:

(2.2) PROBLEM ([15], p. 174). *Does there exist an* ANR*-space of a given dimension* $n > 3$, *which does not contain any disk?*

By VI, (6.2), there exists for every m-dimensional compactum X $(m > 1)$ an m-dimensional AR-set which does not contain topologically X. However, the following problems remain open:

(2.3) PROBLEM. *Given a compactum* X *of dimension* $m > 1$. *Does there exist an* ANR*-space of a given dimension* $n > m$ *which does not contain topologically the set* X?

(2.4) PROBLEM ([63], p. 52). *Is every* n*-dimensional* AR*-space topologically contained in the Euclidean* $2n$*-space?*

Let us notice that the answer to (2.4) is positive for $n = 1$, because 1-dimensional AR-spaces are the same as the dendrites, and every dendrite is topologically contained in the plane E^2.

(2.5) PROBLEM. *Does there exist, for every* $n = 1, 2, \ldots$, *an* n*-dimensional* AR*-space which is not contained topologically in the Euclidean* $(2n-1)$*-space?*

Let us notice that the example of the triod (a union of three segments with one common endpoint and with disjoint interiors) gives the positive answer to (2.5) in the case $n = 1$.

(2.6) PROBLEM ([60], p. 107, and [276], p. 55). *Is it true that every* n*-dimensional* ANR*-space contains at least one* $(n-1)$*-dimensional* ANR*-space?*

It is clear that for $n = 1, 2$ the answer is positive. However, it is plausible that the example constructed in [15] of a 3-dimensional AR-set which does not contain any disk gives a negative answer in the case $n = 3$.

(2.7) PROBLEM ([74], p. 98). *Does there exist an infinite-dimensional* AR-*space which does not contain any* ANR-*set of a finite dimension greater than* 1?

The problem of compactification belongs also to embedding problems. It has been studied for absolute retracts and absolute neighborhood retracts by R. Duda [101].

3. Problems on addition and matching. As it was shown (IV, (6.1)) the union of two ANR($\mathfrak{M}$)-sets (respectively of two AR($\mathfrak{M}$)-sets) with common part being an ANR($\mathfrak{M}$)-set (respectively, an AR($\mathfrak{M}$)-set) is also an ANR($\mathfrak{M}$)-set (respectively an AR($\mathfrak{M}$)-set). A generalization of this statement to the case of completely normal spaces is given by K. Iseki [169]. As it has been shown by O. Hanner ([144], p. 350), the analogous statement ceases to be true if we replace the class $\mathfrak{M}$ of all metric spaces by the class of all compact Hausdorff spaces. On the other hand, there exist theorems on addition of a much more general character. In particular, Y. Kodama (1956, [182], p. 122) has shown that if $\mathfrak{K}$ is the class of all collectionwise normal spaces (in the sense of R. H. Bing [13], p. 176), and if $\{X_\mu\}$ is a locally finite and star-finite covering of a space X such that for every finite system of indices $\mu_1, \mu_2, \ldots, \mu_k$ the set $\bigcap_{i=1}^{k} X_{\mu_i}$ is an ANR($\mathfrak{K}$)-set, then the space X is also an ANR($\mathfrak{K}$)-set. In the same paper, Y. Kodama proves a similar proposition concerning the case where $\mathfrak{K}$ is the class of spaces which satisfy both following conditions: complete normality and full normality. Some related questions are considered also in [181] and [182].

An operation more general than the addition is the operation of matching considered in V, Section 9. A far reaching generalization of Theorem (9.1) has been given recently by A. Lelek ([215], p. 225) for spaces of finite dimension. The most important case of the theorem of Lelek may be formulated as follows:

(3.1) THEOREM. *Let X be an* ANR-*space and $\{X_i\}$ a sequence of disjoint* AR-*sets lying in X. Let f be a map of X onto a finite-dimensional compactum Y such that the sets $f\left(X - \bigcup_{i=1}^{\infty} X_i\right), f(X_1), f(X_2), \ldots$ are disjoint and their diameters converge to zero. If the restriction $f\left|\left(X - \bigcup_{i=1}^{\infty} X_i\right)\right.$ is a homeomorphism and if $f(X_i) \in$ AR for $i = 1, 2, \ldots,$ then $Y \in$ ANR.*

The proof of this theorem is based on V, (12.1). Theorem (3.1), as well as Theorem V, (12.1), belongs to the category of theorems on the decompositions of ANR-spaces into ANR-sets. Some other theorems of this kind have been given by E. E. Floyd [118], [119] and [120].

The following problems remain open:

(3.2) PROBLEM. *Does Theorem* (3.1) *remain true without the hypothesis of the finite dimension?*

(3.3) PROBLEM. *Is it true that the decomposition space of an upper semi-continuous decomposition of a space* $X \in \mathrm{ANR}$ (*or of* $X \in \mathrm{AR}$) *into acyclic elements is an* ANR-*space* (*or an* AR-*space*)?

Problem (3.3) remains open even in the special case, where the decomposition contains only one non-degenerate element (i.e., an element consisting of many points).

There exist several theorems concerning upper semicontinuous decompositions of some special spaces. The oldest of them is a theorem of R. L. Moore (see, for instance, [203], p. 380) that the decomposition space of every upper semicontinuous decomposition of the Euclidean plane E^2 into acyclic compacta is homeomorphic with E^2. Let us mention here theorem of J. J. Andrews and M. L. Curtis [5], generalized by D. S. Gillman and J. M. Martin [138]; that the Cartesian product of E^1 by the decomposition space of every decomposition of E^n into a simple arc and individual points is homeomorphic to E^{n+1}, and an analogous theorem of K. W. Kwun and F. Raymond [206] that the Cartesian product of the disk Q^2 by the decomposition space of the decomposition of the Euclidean n-ball Q^n into a simple arc lying in the interior of Q^n and into individual points is homeomorphic to Q^{n+2}. Let us mention also a theorem of K. W. Kwun concerning decompositions of the n-dimensional sphere S^n into AR-sets satisfying some special conditions [205].

The upper semicontinuous decompositions in which each element contains at most two points have been studied by several authors [78], [172], [259]. If the diameters of all non-degenerated elements of such a decomposition are greater than a positive constant, then the decomposition space obtained that way from an ANR-space is necessarily an ANR-space. Some more precise results are given in [172].

4. Various operations on AR's and ANR's. We have already considered several operations on the absolute retracts and on the absolute neighborhood retracts: the addition (IV, Section 6), the Cartesian product (IV, Section 7), the bundle space (IV, Section 10), the matching (V, Section 9), the upper semicontinuous decomposition (V, Section 12) and the functional space (IV, Section 5). Now let us mention the operation which assigns to every metric space X the space 2^X consisting of all non-empty compacta lying in X with the distance given by the formula of Hausdorff

$$\varrho(A, B) = \mathrm{Max}\,[\sup_{x \in A} \varrho(x, B); \sup_{x \in B} \varrho(x, A)].$$

Let us mention the interesting result of M. Wojdysławski ([296], p. 190) that in the case where X is a locally connected continuum 2^X is an AR-set. Some results in this direction are obtained also by L. F. Foulis [121].

(4.1) PROBLEM. *Is it true that for every metric complete, connected and locally connected space X the set 2^X is an* $\mathrm{AR}(\mathfrak{M})$?

Given a natural number n, let us denote by $X^{[n]}$ the subset of the space 2^X consisting of all non-empty subsets of X containing at most n points. The sets $X^{[n]}$, called also *symmetric n potences* of the space X, have been studied by several authors ([79], [81], [134], [229]). In particular, it was shown ([134], p. 315) that if $X \in \mathrm{ANR}$, then $X^{[n]} \in \mathrm{ANR}$ and if $X \in \mathrm{AR}$, then $X^{[n]} \in \mathrm{AR}$.

(4.2) PROBLEM. *Is it true that $X \in \mathrm{ANR}(\mathfrak{M})$ implies that $X^{[n]} \in \mathrm{ANR}(\mathfrak{M})$ and $X \in \mathrm{AR}(\mathfrak{M})$ implies that $X^{[n]} \in \mathrm{AR}(\mathfrak{M})$?*

5. Dimension of ANR-**spaces.** As it was shown (VII, Section 9), the modular dimensions of compacta satisfying Condition (Δ) are equal to the usual one. On the other hand, we know that there exist AR-spaces with singularity of Peano, and we know (VII, (3.7)) that such spaces lack property (Δ). Thus it arises the following

(5.1) PROBLEM. *Is it true that for* ANR-*spaces the modular dimensions are equal to the usual one?*

As yet it is only known ([27], p. 376) that for every n-dimensional ANR-space X there exists a natural $m \geqslant 2$ such that the dimension of X modulo m is equal to n.

As we have already noticed, the dimension modulo a prime number p of the Cartesian product of two compacta is equal to the sum of dimensions modulo p of these compacta ([248], p. 1106). Consequently the affirmative answer to Problem (5.1) would give also the affirmative answer to the following

(5.2) PROBLEM. *Is it true that the equality* $\dim X \times Y = \dim X + \dim Y$ *holds for spaces X, $Y \in \mathrm{ANR}$?*

Only some partial results concerning this problem have been obtained by Y. Kodama [185] and [186], but the general problem remains still open.

In other direction go the studies concerning the dimension of images of ANR-sets by continuous maps with inverse images of points consisting of at most two points. In particular, it has been proved by K. Sieklucki (not published yet) that:

(5.3) *If f is a map of a space $X \in$ ANR of dimension less than 2 onto a compactum Y and if $f^{-1}(y)$ contains at most two points for every point $y \in Y$, then* $\dim f(X) = \dim X$.

(5.4) *There exists a map f of a space $X \in$ AR of dimension 2 such that $f^{-1}(y)$ contains at most two points for every point $y \in f(X)$ and that* $\dim f(X) = 3$.

6. AR-sets and ANR-sets in Euclidean spaces. As it was shown (V, (2.20), and (2.21)), an ANR-set X lying in the Euclidean n-space E^n can decompose this space only into a finite number of components, and if $X \in$ AR and $n > 1$, then $E^n - X$ is connected. Moreover, it was shown (V, (14.1) and (13.1)) that for $n = 2$ the first condition characterizes ANR-sets among all locally connected compacta lying in E^2, and the second condition characterizes AR-sets among all non-empty locally connected continua lying in E^2. It is evident that the situation in Euclidean spaces of higher dimensions is more complicated. In particular, the following problem remains open:

(6.1) PROBLEM ([63], p. 52). *Is it true that* AR*-sets lying in E^3 are identical with the acyclic and locally contractible compacta lying in E^3?*

The necessity of these conditions is obvious. It remains to show that they are also sufficient. Let us notice that this is true in the case of polyhedra ([26], p. 73). It seems also interesting to study whether the homotopy condition of the local contractibility of the set $X \subset E^3$ can be replaced here by the homology condition that every infinite cycle having a carrier $X_0 \subset X$ with a diameter sufficiently small is homologous to zero in a subset of X with arbitrarily small diameter.

The homology properties of the complement $E^n - X$ of an ANR-set lying in the space E^n have been studied by G. Chogoshvili [86]. Also the notion of the transverse sets belong to the domain of this set of problems. Let A and B be two compact, disjoint subsets of E^n. The set B is said to be *transverse to A* provided that A is a retract of $E^n - B$. Some results concerning this notion are given in [75] and [42]. Also the more delicate question, when A is a deformation retract of $E^n - B$, has been studied in [75] in the special case where A and B are simple closed polygons lying in E^3. See also [5[a]], [88], p. 62, [108] and [110]. The deformation retracts of some plane sets have been studied also by M. Wojdysławski [297]. Finally, let us mention an interesting result of S. Łojasiewicz [221] that every subset of E^n given by an equation $f(x) = 0$, where f is an analytic function defined in an open subset of E^n, is a deformation retract of some of its neighborhoods. A generalization of the notion of deformation retract is given by M. C. McCard [226[a]].

(6.2) PROBLEM. *Is it true that for every* ANR*-set $X \subset E^n$ there exists an* ANR*-set $Y \subset E^n$ such that each of the X, Y is transverse to the other?*

(6.3) PROBLEM. *Is it true that every curvilinear polyhedron $A \subset E^n$ having arbitrary small neighborhoods for which it is a deformation retract is tame* (i.e. that there exists a homeomorphism mapping E^n onto itself, and A onto a rectilinear polyhedron)?

To the domain of this set of problems belong also some theorems concerning some analogies between the properties of polyhedra and of ANR-sets lying in Euclidean spaces [21]. Actually, if $X \in \mathrm{ANR}$ lies in the space E^n, then its boundary $Y = X \cap \overline{E^n - X}$ is locally connected, and for every point y of Y lying on the boundary of a component G of the set $E^n - X$ there exists a simple arc L such that y is one of its endpoints and $L-(y) \subset G$.

7. Some homotopy invariants. By the *category* of a space X one understands [123] the least number of open sets which are contractible in X and constitute a covering of X. The theory of this invariant (and also of some invariants of a similar character, as *strong category*, *n-dimensional category*, *homology category*), introduced by L. Lusternik and L. Schnirelmann for manifolds in connection with their studies on calculus of variation in the large, has been carried over to the case of ANR-spaces [36] and developped largely by R. H. Fox [122], [123], and [124]. Some special results belonging to this theory are given also in [25], [29], [31], [32], [40], [106], [107], [108].

8. Decompositions of ANR-spaces into Cartesian products. A topological property (α) is said to be *multiplicative* (compare VII, Section 5) provided that for every two spaces X_1, X_2 with the property (α) their Cartesian product $X_1 \times X_2$ has also this property. A topological property (α) is said to be *factorisable*, when both spaces X_1 and X_2 have the property (α) if $X_1 \times X_2$ has it.

As it was shown in IV, Section 7, the properties $\mathrm{AR}(\mathfrak{M})$ and $\mathrm{ANR}(\mathfrak{M})$ are multiplicative and factorisable, and in V, Section 2, an analogous proposition was proved for properties AR and ANR. Moreover, in VII, Section 5, it was proved that condition (Δ) is multiplicative, and in VII, Section 10, an analogous statement was proved concerning Condition (Γ). The following problems remain open:

(8.1) PROBLEM. *Is condition* (Δ) *factorisable?*

(8.2) PROBLEM. *Is condition* (Γ) *factorisable?*

(8.3) PROBLEM. *Are the singularities of Peano, of Brouwer and of Mazurkiewicz multiplicative?*

As we have already defined in I, Section 1, a system $\{X_\mu\}$, $\mu \in M$, of spaces is said to be a Cartesian decomposition of a space X provided that X is homeomorphic to the Cartesian product $\underset{\mu \in M}{P} X_\mu$. A space X is said to be *prime* if it contains at least two points and if there does not exist a Cartesian decomposition of it with at least two factors, each containing at least two points. Two Cartesian decompositions $\{X_\mu\}$, $\mu \in M$, and $\{Y_\nu\}$, $\nu \in N$, of a space X are said to be *equivalent* provided that there exists a one-to-one function φ transforming M onto N and such that for every $\mu \in M$ the space X_μ is homeomorphic to the space $Y_{\varphi(\mu)}$.

There exist various examples of ANR-spaces with many non-equivalent Cartesian decompositions into prime factors [286], p. 827, [130], [53], [14]. As it was proved by V. Poenaru [247], there exist also many non-equivalent Cartesian decompositions into prime factors of the Euclidean 5-dimensional cube Q^5. Moreover, there exist also factors of cube Q^n (for $n > 4$) which are not polyhedra [206], but each one- or two-dimensional factor of a polyhedron is a polyhedron ([188], p. 325). Moreover, it is known that all Cartesian decompositions of a polyhedron into prime factors of dimension $\leqslant 1$ are equivalent ([44], p. 139). Also all Cartesian decompositions of a compact manifold into one-dimensional and two-dimensional prime factors are equivalent ([46], p. 296) and so are all decompositions of Euclidean cubes of dimension 3 [273] and 4 [14]. Recently H. Patkowska has shown [240], [241], and [242] that all Cartesian decompositions of ANR-spaces into prime factors of dimensions less than 2 are equivalent. Some other results concerning these questions are given in [54], [130] and [131].

A natural generalization of the decomposition into Cartesian product is the notion of fibering. As it was shown by M. Pavel [244], the base of a fibering of an ANR-space is necessarily an ANR-space. The theory of fiberings, in which all fibers are ANR-spaces, has been developped by W. Hurewicz and N. E. Steenrod [165], by E. H. Spanier and J. H. C. Whitehead [267], and by E. Eckmann [105]. See also R. H. Fox [127] and [128], S. T. Hu [159], and E. R. Fadell [116].

9. ANR-spaces and polyhedra. In V, Section 4, we established a theorem concerning a relation between the notion of the ANR-space (belonging to the general topology) and the notion of polyhedron (belonging to the elementary geometry). As a corollary to this theorem, it was shown that for every ANR-space X there exists a polyhedron P which dominates homotopically over X. Let us formulate the following

(9.1) PROBLEM. *Does there exist for every* ANR-*space* X *a polyhedron* P *which has the same homotopy type as* X?

We have already mentioned (VII, Section 14) that this problem has been solved affirmatively for simply connected spaces by C. B. De Lyra [96].

Let us notice that for a given ANR-space X there needs not exist a polyhedron having the same r-type as X. For instance, there exists for a dendrite D a polyhedron of the same r-type only if D itself is a polyhedron (a tree). The class of all ANR-spaces with r-types of polyhedra has never been systematically studied.

10. Problems of metrization. As it was shown by K. Kuratowski [198], a closed subset A of a metric space X is a retract of X if and only if there exists such a metrization of X that for every point $x \in X$ there is exactly one point $f(x) \in A$, so that $\varrho(x, f(x)) = \varrho(x, A)$. See also W. Nitka [235].

The theory of absolute retracts and absolute neighborhood retracts is related to various other problems of metrization. As we have shown in IV, Section 2 and Section 3, the AR$(\mathfrak{M})$-spaces are identical with r-images of convex subsets of linear, normed spaces, and the ANR$(\mathfrak{M})$-spaces are identical with r-images of open subsets of convex subsets of linear, normed spaces.

The notion of convex subset of the Hilbert space can be generalized as follows: a metric ϱ defined in a space X is said to be *strongly convex* provided that for every two points x_1, x_2 of X there exists only one *center*, i.e. only one point $x_0 \in X$ such that $\varrho(x_0, x_1) = \varrho(x_0, x_2) = \frac{1}{2}\varrho(x_1, x_2)$. It is well known that if X is a strongly convex compactum, then for every two points $x_1, x_2 \in X$ and for every $0 \leqslant t \leqslant 1$ there is in X only one point $\varphi(x_1, x_2, t)$ such that

$$\varrho(x_1, \varphi(x_1, x_2, t)) = (1-t)\cdot\varrho(x_1, x_2),$$
$$\varrho(x_2, \varphi(x_1, x_2, t)) = t\cdot\varrho(x_1, x_2),$$

and that the point $\varphi(x_1, x_2, t)$ depends continuously on x_1, x_2 and t. If we select a fixed point $x_1 \in X$ then setting $f_t(x) = \varphi(x_1, x, t)$ for every point $x \in X$, and for $0 \leqslant t \leqslant 1$, we get a homotopy $\{f_t\}$ which contracts the space X to the point x_1, so that every sufficiently small neighborhood of x_1 is contracted to x_1 in an arbitrarily given neighborhood of x_1. Consequently, a compactum which can be metrized in a strongly convex manner is contractible in itself and locally contractible. It follows by V, (10.5), that finite-dimensional compacta, metrizable in a strongly convex manner are AR-sets.

A metric ϱ defined in a space X is said to be *locally strongly convex* provided that every point $x_0 \in X$ has a neighborhood U such that, for every pair of points $x_1, x_2 \in U$, there exists in X only one center of the pair x_1, x_2. We see at once that a compactum metrizable in a locally strongly convex manner is locally contractible, and if its dimension is finite, it is an ANR-space.

(10.1) PROBLEM. *Is it true that every strongly convex compactum is an* AR-*space and every locally strongly convex compactum is an* ANR-*space?*

Let us observe that there exist ANR-spaces which cannot be metrized in a locally strongly convex manner. In fact, one sees readily that a locally strongly convex compactum can be represented as the union of a finite number of arbitrarily small contractible compacta (actually, of closed balls with sufficiently small radii) and there exist, as we have seen in VI, Section 4, 2-dimensional AR-spaces for which a such decomposition is impossible. Moreover, there exist already 2-dimensional polyhedra, that are AR-sets, for which a strongly convex metrization is impossible [260]. However, for every 2-dimensional polyhedron there exists a locally strongly convex metrization [62].

(10.2) PROBLEM. *Is it true that every polyhedron has a locally strongly convex metrization?*

11. Hyperspaces of ANR-spaces. The collection of all non-empty compacta lying in a space X can be considered as a space 2^X metrized by the known formula of Hausdorff. It is clear that topological properties of compacta belonging to 2^X and being arbitrarily close may be quite different. It arises the problem to introduce a metric in the collection of compacta lying in a given space X such that the compacta, which are within a sufficiently small distance of each other, have close topological properties, or — more exactly — that the topological properties of compacta belonging to a convergent sequence would allow to determine the topological properties of the limit. This, and similar problems have been considered by several authors (S. Mazurkiewicz 1935 [223], G. T. Whyburn 1935 [291] and [292], E. G. Begle 1944 [11], P. A. White 1944 [284], M. L. Curtis 1953 [89]). In particular, the problem to define the distance of ANR-sets in such a manner that the homotopy properties of sufficiently close sets are similar, has been considered in [55]. Actually a metric (called *homotopy metric* ϱ_h) was defined in the collection of all non-empty ANR-sets lying in a given compactum X so that the space 2_h^X obtained is complete and the homotopy type of ANR-sets is invariant under the limit operation. If X is a finitely dimensional compactum, then the topology of the space 2_h^X can be characterized by the condition that a sequence A_n of ANR-sets lying in X converges to an ANR-set $A \subset X$ if and only if the Hausdorff distances $\varrho(A_n, A)$ converge to zero and, moreover, the sets A_n are *equally locally contractible*, which means that for every positive number ε there exists a positive number η such that every subset Z of A_n, with diameter less than η, is contractible in a subset of A_n with diameter less than ε for every $n = 1, 2, \ldots$

Since the space 2_h^X is complete, one may expect that it can be usefull in proving the existence of some types of ANR-sets by means of the classical theorems of Baire on the sets of the first category in complete spaces. However, heretofore no such application of the space 2_h^X has been given.

Another metric for the class of all non-empty compacta lying in a metric space X is the *metric of continuity* ϱ_c [55]. The distance $\varrho_c(A, B)$ of two compacta $A, B \subset X$ is defined as the greatest lower bound of the numbers d such that there exist a map $\varphi: A \to B$ and a map $\psi: B \to A$ such that

$$\varrho(x, \varphi(x)) \leqslant d \quad \text{for every point } x \in A,$$
$$\varrho(x, \psi(x)) \leqslant d \quad \text{for every point } x \in B.$$

This way we get a metric space which we denote by 2_c^X. Some results concerning this space are obtained by J. De Groot [140] and T. Ganea [135]. To this scope belongs also a theorem of M. Pavel that an ANR-space which can be mapped onto an AR-set by a map with arbitrarily small diameters of inverse images of points is necessarily an AR-space [243]. A systematical study of relations between properties of two *quasi-homeomorphic* spaces (it means, such spaces that for every $\varepsilon > 0$ each of them can be mapped onto another with diameters of inverse images of points $< \varepsilon$) was started in 1933 by Kuratowski and Ulam [204]. In particular, the case of spaces quasi-homeomorphic with ANR's has been studied in [45], [136] and [137]. Let us mention here a remarkable theorem of S. Eilenberg [109] that, for every $X \in$ ANR and every $\varepsilon > 0$, there exists an $\eta > 0$ such that, for every map f of X onto another space Y with diameters of inverse images $f^{-1}(y)$ less than η, there exists a map $\varphi: Y \to X$ satisfying the condition $\varrho(\varphi f(x), x) < \varepsilon$ for every point $x \in X$.

Some other metrizations of the class of subsets of a given space, concerning instead of ANR-spaces the **LC**^{n}-spaces, have been introduced by K. Kuratowski [201].

Among many questions concerning the space 2_h^X, let us formulate two following problems:

(11.1) PROBLEM. *Let X be a polyhedron. Is it true that the class of all polyhedra lying in X is dense in 2_h^X? What is the category (in the sense of Baire) of this class?*

(11.2) PROBLEM. *Let $X = Q^n$ be an n-dimensional Euclidean cube. What is the category (in the sense of Baire) in the space 2_h^X, of the collection of all* ANR-*sets lying in Q^n and having the singularity of Peano, the singularity of Brouwer or of Mazurkiewicz, respectively?*

12. Problems of classification of ANR-**spaces.** A basic problem in the theory of r-types is the following

(12.1) PROBLEM. *Is it true that two r-equal* ANR-*spaces are necessarily of the same homotopy type?*

It should be mentioned that the analogous question for ANR($\mathfrak{M}$)-spaces has been negatively solved by T. E. Stewart [269].

The partial order of r-types given by the relation $\underset{r}{\lessgtr}$ has been not much studied. Some results concerning this partial ordering were given in Chapter VIII. Let us mention a result due to K. Sieklucki [261], that there exists a family consisting of $2^{\aleph_0}$ dendrites ordered by the relation $\underset{r}{\lessgtr}$ similarly to the segment $\langle 0, 1\rangle$.

The notion of r-neighbors leads to many open problems. Let us formulate several of them:

(12.2) PROBLEM. *Is it true that the number of different r-types of left r-neighbors of a given polyhedron is always finite?*

(12.3) PROBLEM. *Is it true that the number of different r-types among* ANR*-sets being right r-neighbors of a given polyhedron is always finite?*

(12.4) PROBLEM. *Does there exist for every polyhedron P a polyhedron of the same homotopy type which is a right r-neighbor of P?*

(12.5) PROBLEM. *Does there exist for every* ANR*-set X an* ANR*-set of the same homotopy type, which is a right r-neighbor of X?*

The following notion, intimately related to the notion of r-neighbor, has been studied by A. Trybulec [275]. Let $\{X_k\}$ be an r-increasing sequence of spaces and X a space satisfying two conditions:

(i) $X_k \underset{r}{<} X$ for $k = 1, 2, \ldots$

(ii) There is no space Y such that $X_k \underset{r}{<} Y \underset{r}{<} X$ for $k = 1, 2, \ldots$

Then the space X is said to be *r-accessible from the left* by the sequence $\{X_k\}$. In a similar manner one introduces the notion of a space *r-accessible from the right* by an r-decreasing sequence of spaces. There exist r-different spaces r-accessible from the left (or from the right) by the same r-increasing (or r-decreasing) sequence (see [275]). Among many open problems concerning these notions, let us formulate the following one:

(12.6) PROBLEM. *Does there exists a polyhedron r-accessible from the left (or from the right) by a sequence of polyhedra?*

As the notion of the r-domination leads to the r-classification of ANR-spaces, so the notion of the h-domination (I, Section 14) induces a classification of ANR-spaces into *h-types*, where two ANR-spaces belong to the same h-type if and only if each of them h-dominates the other (compare [9] and [253]). Evidently, two r-equal spaces are necessarily of the same h-type, and also two spaces of the same homotopy type (in the sense of Hurewicz) are necessarily of the same h-type. Let us recall that two spaces X and Y are said to be of the same homotopy type (I, Section 14) if there exist two maps

$$f\colon X \to Y \quad \text{and} \quad g\colon Y \to X,$$

called *homotopy equivalences*, such that both maps $gf\colon X \to X$ and $fg\colon Y \to Y$ are homotopic to the identities.

As we know (II, Section 7), a homotopy equivalence f induces an isomorphism f_* of the homotopy group $\pi_n(X)$ onto $\pi_n(Y)$ for every $n = 1, 2, \ldots$ By a result of J. H. C. Whitehead ([289], p. 1133), if X and Y are connected ANR-spaces, then the converse is also true. Moreover, if for connected ANR-spaces X and Y the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ are trivial, then a map $f\colon X \to Y$ is a homotopy equivalence if and only if it induces an isomorphism of the Betti group $H_n(X)$ onto $H_n(Y)$ for every $n = 2, 3, \ldots$ ([288], p. 1135). The homotopy type of CW-complexes was studied also by J. H. C. Whitehead ([285], p. 277).

The following problem remains open:

(12.7) PROBLEM. *Is it true that two* ANR-*spaces of the same h-type are necessarily of the same homotopy type?*

This problem is intimately related to the theory of homotopy types as developped by M. M. Postnikov [249].

If Y h-dominates X but the converse is false, then we write $X \underset{h}{<} Y$ and thus we get a relation $\underset{h}{<}$ which partially orders the collection of all spaces, and also the collection of all h-types. This relation leads in a natural way to many notions such as: *h-minorant*, *h-majorant*, *left h-neighbor* and *right h-neighbor*, *index of h-proximity* and so on. Among many open problems concerning these notions, let us mention the following one:

(12.8) PROBLEM. *Do there exist two* ANR-*spaces for which the index of h-proximity is infinite?*

13. Problem of elimination of singularities. The notion of ANR-spaces is based only on notions of general topology (metrizability, compactness, extension of maps). The class of polyhedra, defined in a constructive way by means of the elementary geometry, is contained in the class of ANR-spaces. We know that the ANR-spaces show much similarity to polyhedra; however, as it was shown in Chapter VI, among ANR-spaces there exist topological phenomena which are impossible for polyhedra. Thus, between the notion of ANR-spaces and the notion of polyhedra there is a gap difficult to bridge. Some light onto the nature of these difficulties is thrown by remarks given in VI, Section 6, from which it follows that the polyhedra constitute a rather exceptional kind of ANR-spaces. The problem of a characterization of polyhedra among all ANR-spaces by topological means has been solved only in the 2-dimensional case by A. Kosiński ([187] and [190]). In the case of higher dimension, this problem seems to be very difficult although not very important, because the class of curvilinear polyhedra seems to be too narrow, even from the point of view of geometric topology.

Perhaps it would be more important to distinguish among all ANR-spaces, by topological means, a special class of spaces sufficiently general to include all polyhedra, but special enough to eliminate any space with much too paradoxical properties (compare [50], [51], and [56]). In Chapter VII two conditions have been given (condition (Δ) and condition (Γ)) such that if we add them to the conditions characterizing the ANR-spaces, then we obtain a class of spaces containing all polyhedra but free from several singularities. We have established in Chapter VII some theorems concerning the topological properties of ANR-spaces satisfying condition (Δ) or condition (Γ). Now let us mention one more consequence of condition (Δ), namely the existence in every ANR-space X satisfying (Δ), for every $k = 0, 1, \ldots,$ of a *homotopy k-skeleton* [65], i.e. of a compactum X^k of dimension $\leqslant k$ such that every compactum $A \subset X$ of dimension $\leqslant k$ can be deformed, by a homotopy in X, onto a subset of X^k. The existence of homotopy k-skeletons makes it possible to introduce for such spaces a kind of groups, called *generalized cohomotopy groups* (see [66], [173] and [174]).

However, conditions (Δ) and (Γ) cannot be considered as final solution of the *problem of the elimination of singularities*, because they are rather "ad hoc" conditions, eliminating only singularities of a certain kind: (Δ) eliminates the singularity of Peano, and (Γ) the singularity of Mazurkiewicz and the relations between these conditions and other singularities remain obscure. In particular, the following problem remains open:

(13.1) PROBLEM. *Does there exist an* ANR-*space with the singularity of Brouwer, satisfying both conditions* (Δ) *and* (Γ)?

It should be mentioned that the notion of ANR-space is invariant under r-maps, but no class of maps, more general than the class of homeomorphisms, is known, under which conditions (Γ) and (Δ) are invariant. The problem of elimination of singularities may be formulated as follows:

(13.2) PROBLEM. *Define a class* Φ *of maps such that:*

(a) Φ *contains all homeomorphisms and is included in the class of all r-maps.*

(b) *If the maps* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *belong to* Φ, *then the map* $gf\colon X \to Z$ *belongs to* Φ.

And define a class of spaces $\mathfrak{K}$ *such that:*

(c) $\mathfrak{K}$ *contains all polyhedra and is included in the class of all* ANR-*spaces.*

(d) $\mathfrak{K}$ *is multiplicative.*

(e) *Every space belonging to $\mathfrak{K}$ is free of the singularities of Peano, Brouwer and Mazurkiewicz together.*

(f) *$\mathfrak{K}$ coincides with the class of all images of the infinite prisms under the maps belonging to Φ.*

There exist also other ways which make it perhaps possible to eliminate reasonably the singularities. For instance, one can expect that the closure of the subset of the space 2^{Q^ω} consisting of all polyhedra is a class of compacta with topological properties close to the properties of polyhedra. But it is also possible that this class coincides with the class of all ANR-spaces. An other class of spaces, which seems to be interesting from the point of view of the elimination of singularities, is the class of such compacta, which are metrizable in a locally strongly convex manner.

14. Fixed points in ANR**-spaces.** One of the most remarkable theorems of the topology is the fixed point theorem of S. Lefschetz and H. Hopf. In order to formulate it let us recall the notion of the trace of an endomorphism.

An Abelian group $\mathfrak{A}$ is called a *group with division* provided that for every element $x \in \mathfrak{A}$ and for every natural number m there is exactly one element $y \in \mathfrak{A}$ such that $my = x$. One denotes this element y by $\frac{1}{m}x$, and its multiple by an integer k, by $\frac{k}{m}x$. Thus, the linear combination

$$w_1x_1 + w_2x_2 + \ldots + w_nx_n \tag{14.1}$$

is uniquely defined for every system of elements $x_1, x_2, \ldots, x_n \in \mathfrak{A}$ and for every system of rationals $w_1, w_2, \ldots, w_n$.

Now let us assume that $\mathfrak{A}$ has a finite rational basis, i.e. a finite system of elements $x_1, x_2, \ldots, x_n$ such that every element $x \in \mathfrak{A}$ can be uniquely represented as a linear combination of the form (14.1). If

$$f\colon \mathfrak{A} \to \mathfrak{A}$$

is an *endomorphism* (i.e. a homomorphism of $\mathfrak{A}$ into itself), then

$$f(x_i) = w_{i,1}x_1 + w_{i,2}x_2 + \ldots + w_{i,n}x_n \quad \text{for} \quad i = 1, 2, \ldots, n.$$

One easily proves that the number

$$\mathrm{Trace}\,(f) = \sum_{i=1}^{n} w_{i,i}$$

does not depend on the choice of the rational basis $x_1, x_2, \ldots, x_n$. This number is called the *trace of the endomorphism f*.

Now let us consider an ANR-space X and a map

$$f\colon X \to X,$$

and let f_i denote the endomorphism of the homology group $H_i(X, \mathfrak{R})$ (where $\mathfrak{R}$ is the group of rational numbers) induced by f. Applying V, (4.6), we see at once that each of the groups $H_i(X, \mathfrak{R})$ has a finite rational base and that for almost all i the group $H_i(X, \mathfrak{R})$ is trivial. It follows that all numbers Trace(f_i) are defined and almost all of them vanish. This makes it possible to assign to the map f a number $\Lambda(f)$ (called the *Lefschetz number* of the map f), given by the formula

$$\Lambda(f) = \sum_{i=-1}^{\infty} (-1)^i \operatorname{Trace}(f_i).$$

The famous theorem of Lefschetz-Hopf (proved first by S. Lefschetz [208] for manifolds, and by H. Hopf [152] for polyhedra, and generalized by S. Lefschetz [210] to all ANR-spaces) asserts that for every ANR-space X and for every map $f\colon X \to X$ the condition $\Lambda(f) \neq -1$ implies that f has a fixed point, i.e. a point $x_0 \in X$ such that $f(x_0) = x_0$.

In particular, if all Betti numbers of an ANR-space X vanish, then all groups $H_n(X, \mathfrak{R})$ are trivial and consequently X has the fixed point property (as defined in I, Section 7), because for every map $f\colon X \to X$ the Lefschetz number $\Lambda(f)$ vanishes. Let us mention that the analogous proposition for arbitrary compacta (even for locally connected continua) fails, because there exist, already in the space E^3, locally connected and acyclic continua without the fixed point property [28] (see also [178]).

Thus the application of the Lefschetz number to the theory of fixed points cannot be extended from ANR-spaces to arbitrary compacta. Let us mention that there exist also other theorems on fixed points, in which the hypothesis that the considered spaces are ANR's seems to be essential ([82], [114], [115], and [177]). In some cases the theory of fixed points for ANR-spaces implies theorems on fixed points for some special classes of spaces which are not ANR's, but which are intimately related to them (see [84], [85], [92], [93], [94], [145], and [224]).

15. Homotopy classification of maps into ANR-spaces. The classical theorem of H. Hopf (II, (2.5)) permits to classify the homotopy classes of maps of an n-dimensional compactum X into the n-dimensional Euclidean sphere S^n by the homology properties of these maps. The more general problem of classification of the homotopy classes of maps of a given compactum X into a given ANR-space Y is far more difficult. It is strictly related to the general theory of homotopy invariants developped by M. M. Postnikov [250], but it contains various special questions which remain open. An important step in this direction was made by S. T. Hu [158], who proved a generalization of Hopf theorem on the case where X is a compactum of dimension $\leqslant n$ and Y is an ANR-space with trivial homotopy groups $\pi_k(Y)$ for $k = 0, 1, \ldots, n-1$. See also the paper [179] of Y. Kodama.

16. Colocalization of topological properties. Spheroidal spaces and *r*-spaces. Given a topological property (α), let us denote by $\mathscr{I}(\alpha)$ the property defined as follows:

(16.1) *A space X has the property $\mathscr{I}(\alpha)$ at a point $x \in X$ provided that every neighborhood U of x contains a neighborhood V of x such that $X-V$ has property (α). If X has the property $\mathscr{I}(\alpha)$ at every of its points, then X is said simply to have the property $\mathscr{I}(\alpha)$.*

We say that property $\mathscr{I}(\alpha)$ is the *colocalization* of property (α).

In some sense the colocalization is an operation opposite to the localization of topological properties. It is clear that if property (α) has a local character (that is, it depends only on the properties of space in arbitrary, small neighborhood of its points), then property $\mathscr{I}(\alpha)$ (for spaces consisting of many points) implies property (α), but the converse is false. Thus, if we set $(\alpha) = \text{ANR}$, then every space with property $\mathscr{I}(\text{ANR})$ is an ANR-space, but not every ANR-space has property $\mathscr{I}(\text{ANR})$. For instance, the property $\mathscr{I}(\text{ANR})$ fails for 2-dimensional indecomposable ANR-space, as it was mentioned at the end of Section 4, Chapter VI.

(16.2) PROBLEM. *Is it true that the property $\mathscr{I}(\text{ANR})$ implies condition* (Γ)?

Spaces with property $\mathscr{I}(\text{AR})$ are said to be *spheroidal spaces* ([24] and [41]). Besides the Euclidean spheres, the Hilbert cube Q^{ω} belongs to them, as it follows by the topological homogeneity of Q^{ω} proved by O. H. Keller [175]. It is proved [41] that various topological properties of spheres hold for all spheroidal spaces, and that spheroidal spaces of dimensions 0, 1, 2 are the same as Euclidean spheres of corresponding dimensions. The following problem remains open:

(16.3) PROBLEM. *Does there exist a 3-dimensional spheroidal space which is not a sphere?*

By a remark due to T. Ganea, an analogous problem for 4-dimensional spheres has a positive answer. Actually, the space X defined as the suspension ([148], p. 336) of a Poincaré sphere P (that is of a 3-dimensional manifold P with non-trivial fundamental group, and with all homology groups isomorphic to the corresponding groups of S^3) is spheroidal, but it is not a manifold.

The theory of spheroidal spaces is closely related to the theory of r-spaces introduced by A. Kosiński ([189] and [191]). A finitely dimensional compactum X is said to be *r-space* provided that every point $x \in X$ has arbitrarily small neighborhoods U with the boundary $\overline{U} \cap \overline{X-U}$, which is a deformation retract of the set $\overline{U}-(y)$ for each point $y \in U$.

Manifestly, every manifold (in the classical sense) is an r-space. As it has been shown by A. Kosiński ([189], p. 116), every connected r-space of dimension $\leqslant 2$ is a manifold, and the r-spaces of dimensions greater than 2 show a far reaching similarity with manifolds. Many problems concerning r-spaces are given in the papers of A. Kosiński [189] and [191]. See also [207]. Several similar questions have been treated also by M. L. Curtis [90].

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LIST OF SPECIAL SYMBOLS

The number gives the page on which the symbol first appears. The symbols which are in common use, are not listed.

$\underset{r}{\leqslant}$	r-inequality, p. 8.
$\underset{\mu\in M}{P} X_\mu$	The Cartesian product of spaces X_μ, $\mu \in M$, p. 8.
$\langle \alpha, \beta \rangle$	The set of all real numbers t with $\alpha \leqslant t \leqslant \beta$, p. 9.
E^ω	The Hilbert space, p. 10.
Q^ω	The Hilbert cube, p. 10.
E^n	The Euclidean n-space, p. 11.
$\|\|x\|\|$	The distance of a point $x \in E^\omega$ from 0, p. 11.
$\|a, b\|$	The segment with endpoints a and b, p. 11.
$\|x_0, x_1, \ldots, x_m\|$	The geometric simplex with vertices $x_0, x_1, \ldots, x_m$, p. 11.
$\underset{r}{=}$	r-equality, p. 17.
$\underset{r}{<}$	r-inequality, p. 17.
Y^X	The space of all maps $f\colon X \to Y$, p. 20.
Φ_f	The operation assigning to a map φ the map $f\varphi$, p. 20.
Φ^f	The operation assigning to a map φ the map φf, p. 20.
$(Y, Y_0)^{(X, X_0)}$	The space of all maps $f\colon (X, X_0) \to (Y, Y_0)$, p. 22.
$[Y, Y_0]^{[X, X_0]}$	The set of all homotopy classes of maps $f\colon (X, X_0) \to (Y, Y_0)$, p. 23.
$\mathbf{C}$	The class of all contractible spaces, p. 26.
$[f]$	The homotopy class of a map f, p. 26.
$X^\bullet$	The boundary of X, p. 25.
$\underset{h}{=}$	h-equality, p. 27.
$\underset{h}{<}, \underset{h}{\leqslant}$	h-inequalities, p. 27.
$\underset{h}{\simeq}$	h-equivalence, p. 27.
$\mathbf{LC}$	The class of all locally contractible spaces, p. 28.
$\mathbf{C}^n$	The class of all spaces k-connected for $k = 0, 1, \ldots, n$, p. 30.
$\mathbf{C}^\infty$	The class of all spaces k-connected for $k = 0, 1, \ldots,$ p. 30.
$\mathbf{LC}^n$	The class of all spaces locally k-connected for $k = 0, 1, \ldots, n$, p. 30.
$\mathbf{LC}^\infty$	The class of all spaces locally k-connected for $k = 0, 1, \ldots,$ p. 30.

$\mathrm{Ker}(\varphi)$	The kernel of a homomorphism φ, p. 33.
$\mathfrak{R}$	The additive group of rational numbers, p. 35.
$\mathfrak{N}$	The additive group of integers, p. 35.
$H_n(X, X_0; \mathfrak{A})$	The nth homology group of the pair (X, X_0) over the coefficient group $\mathfrak{A}$, p. 35.
$H^n(X, X_0; \mathfrak{A})$	The nth cohomology group of the pair (X, X_0) over the coefficient group $\mathfrak{A}$, p. 35.
$p_n(X)$	The nth Betti number of X, p. 35.
$\partial \varkappa$	The boundary of a chain $\varkappa$, p. 38.
$\varphi \cdot \psi$	A join of maps φ and ψ, p. 44.
$[f_1]_{G_1 \cdot G_2}[f_2]$	The homotopy join of $[f_1]$ and $[f_2]$ with respect to G_1 and G_2, p. 47.
$\pi_n(Y, y_0)$	The nth homotopy group of Y with base point y_0, p. 50.
$\pi^n(X, x_0)$	The nth cohomotopy group of (X, x_0), p. 62.
$\omega(M)$	The set of all maps $\varphi: (X, X_0) \to (Y, Y_0)$ homotopically dependent on a set M of maps, p. 64.
$\boldsymbol{M}$	The set of all homotopy classes of maps belonging to a set M, p. 65.
$\lambda(\boldsymbol{M})$	The set of all homotopy classes of maps dependent on a set $\boldsymbol{M}$ of homotopy classes, p. 65.
$\mathrm{St}_{\mathscr{T}}(x)$	The star of a vertex x relative to a triangulation $\mathscr{T}$, p. 72.
$N(G)$	The nerve of a covering G, p. 76.
$\mathfrak{M}$	The class of all metrizable spaces, p. 85.
$\mathrm{AR}(\mathfrak{M})$	Absolute retract for metric spaces, p. 85.
$\mathrm{ANR}(\mathfrak{M})$	Absolute neighborhood retract for metrizable spaces, p. 85.
AR	Absolute retract (compact), p. 100.
ANR	Absolute neighborhood retract (compact), p. 100.
$X_1 \overset{\varphi}{\cup} X_2$	The matching of X_1 and X_2 by a map φ, p. 116.
ϱ_h	The homotopy metric, p. 220.
ϱ_c	The metric of continuity, p. 221.
Trace (f)	The trace of an endomorphism f, p. 225.
$\Lambda(f)$	The Lefschetz number of a map f, p. 226.
$\mathscr{I}(\alpha)$	The colocalization of a property (α), p. 227.

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