Sigma Series in Pure Mathematics
Vol. 2

J. Nagata

Modern Dimension Theory

revised and extended edition

Heldermann Verlag Berlin

CONTENTS

PREFACE

Since the appearance of W. Hurewicz and H. Wallman's book "Dimension Theory" in 1941 there have been remarkable developments in dimension theory, especially in the theory for general metric spaces. Though the purpose of this book is to give a rather brief account of modern dimension theory as it has been developed since 1941, the principal results of the classical theory for separable metric spaces are also included. Presupposing only some elementary mathematical knowledge, especially of topology, we shall begin with a brief description of some necessary results in general topology, emphasizing the modern development. No knowledge of dimension theory is assumed, so that the beginning student will be able to read the book without difficulty. However, to the reader who only wishes to get a quick view of the theory, we recommend Chapters I-IV. The author wishes to express his warmest thanks to Prof. J. de Groot who suggested the writing of this book on this interesting theory and helped him in all respects, to Dr. and Mrs. H. de Vries who carefully read the manuscript, gave suggestions and corrected it, especially in its English expression, and to Profs. M. Atsuji, K. Nagami, Y. Kodama and Prof. H. Tamano who helped him in various respects. Without their valuable assistance, this book would never have been written.

Osaka
May, 1964

Jun-iti Nagata

PREFACE TO THE REVISED EDITION

This is a revised edition of Modern Dimension Theory published in 1964, from North Holland Publishing Co. Since the original edition appeared seventeen years ago, there have been remarkable developments in dimension theory, especially on non-metrizable spaces. Thus a large scale of revisions and additions was made on the original text. For examples, Chapter VII (Dimension of non-metrizable spaces) was wholly rewritten, and Chapter VI (Infinite-dimensional spaces) was greatly innovated by adding two new sections. A new section on Pontrjagin-Schnirelmann's theorem was added to Chapter IV, and sections IV-7 (dimension and ring) and V-3 (dimension and metric function) were also rewritten. Besides, many smaller revisions and additions both in the contents and descriptions would be found in the new edition.

The author wishes to conclude this preface with his heartfelt thanks to the publisher, Heldermann Verlag and to those who pointed out errors in the original edition and gave him various comments and suggestions.

May, 1981
Amsterdam

Jun-iti Nagata

CHAPTER I

INTRODUCTION

In this book we assume an elementary knowledge of general topology [1]. However, in this chapter we shall give a quick review, without proofs, of some results, especially from the theory of coverings. In recent years remarkable progress has been made in the last mentioned field, and the reader might not be so familiar with the terminologies and the theorems we shall need.

I. 1. Coverings

Let R be a topological space and $\mathcal{U}$ a collection of subsets of R. Throughout this book we shall merely call such a collection of subsets of R a *collection* in R. If $\mathcal{U}$ consists of finitely many (countably many, two) members, then $\mathcal{U}$ is called a *finite (countable, binary) collection*. If every point of R is covered by only finitely many members of $\mathcal{U}$, then $\mathcal{U}$ is called a *point-finite collection*. If every point p of R has a neighbourhood $U(p)$ which intersects only finitely many (countably many) members of $\mathcal{U}$, then $\mathcal{U}$ is called a *locally finite (locally countable) collection*. If every member U of $\mathcal{U}$ intersects only finitely many members of $\mathcal{U}$, then $\mathcal{U}$ is called a *star-finite collection*. If for every subcollection $\mathcal{V}$ of $\mathcal{U}$

$$\overline{\cup\{U \mid U \in \mathcal{V}\}} = \cup\{\overline{U} \mid U \in \mathcal{V}\}$$

[1] As books on general topology we may recommend, for example, P. Alexandroff - H. Hopf [1], C. Kuratowski [2], J. W. Tukey [1], J. L. Kelley [1], H. J. Kowalsky [2], W. Franz [1], D. Bushaw [1], J. Dugundji [1], R. Engelking [1], J. Nagata [8].

holds, then $\mathcal{U}$ is called a *closure-preserving collection,* where $\bar{U}$ denotes the closure of U. It is easy to see that every locally finite collection is closure-preserving. In this connection we often use the fact that if $\mathcal{F}$ is a locally finite collection of closed sets, then $\cup\{F \mid F \in \mathcal{F}\}$ is closed. A closure-preserving collection $\mathcal{U}$ is called *discrete* if the closures of the members of $\mathcal{U}$ are disjoint, i.e. if any two of them do not intersect. In other words $\mathcal{U}$ is discrete if and only if each point of R has a neighbourhood which intersects at most one member of $\mathcal{U}$.

If $\mathcal{U}$ can be decomposed as

$$\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$$

for locally finite (star-finite, discrete, etc.) collections $\mathcal{U}_i$, then $\mathcal{U}$ is called a σ - *locally finite (σ - star-finite, σ - discrete, etc.) collection.*

$\mathcal{U}$ is called an *open (closed) collection* if every member of $\mathcal{U}$ is an open (closed) set. We may use the symbol $\cup\mathcal{U}$ to denote the set $\cup\{U \mid U \in \mathcal{U}\}$.

Let $\mathcal{U}$ and $\mathcal{V}$ be two collections. If for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ for which $U \subset V$, then we denote this relation by $\mathcal{U} < \mathcal{V}$ and call $\mathcal{U}$ a *refinement* of $\mathcal{V}$.

A collection $\mathcal{U}$ is called a *covering* if $\cup\{U \mid U \in \mathcal{U}\} = R$. In this book we are often concerned with *open coverings*, coverings consisting of open sets. The other attributes for collections are of course applicable to coverings too.

Frequently a collection will be indexed, e.g. $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$. As far as possible it will be tacitly assumed that the indexing is faithful, i.e. that distinct indices denote distinct members of the collection. In construction starting from a faithfully indexed collection, however, new collections may arise for which the indexing is normally not faithful. For an indexed collection it is sometimes useful to look at the indices instead of considering the members of the collections themselves.

A) Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be a point-finite open covering of a normal space [2] R. Then there exists an open covering $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for every $\alpha \in A$. Generally a covering $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is said to *shrink* to (or to be shrunk to) a covering $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ if $V_\alpha \subset U_\alpha$ for all $\alpha \in A$. We may also say (less frequently) that $\mathcal{V}$ shrinks $\mathcal{U}$.

Definition I. 1. *Let R be a topological space. If for each open covering $\mathcal{U}$ of R there exists a locally finite open (covering) refinement, i.e. a locally finite open covering $\mathcal{V}$ satisfying $\mathcal{V} < \mathcal{U}$, then we call R a paracompact space.*

---

[2] In this book we mean by a normal space a normal T_1 -space i.e. a normal space each of whose points is closed. Similarly every regular space will be a regular T_1 - space in this book.

If each countable open covering of R has a locally finite open refinement, then it is called a countably paracompact space.

If each open covering of a topological space R has a star-finite open (covering) refinement, then we shall say that R has the star-finite property.

If every subspace of R is paracompact (normal, etc.), then R is called hereditarily paracompact (normal, etc.).

If every closed set of a normal space R is a G_δ-set, then R is called perfectly normal.

B) A T_1-space R is hereditarily normal if and only if every subset of R is normal. Equivalently, for any subsets A, B of R satisfying $A \cap \bar{B} = \emptyset$, $\bar{A} \cap B = \emptyset$ there exist open sets U and V such that $U \supset A$, $V \supset B$, $U \cap V = \emptyset$, where we can select U and V so that $\bar{U} \cap \bar{V} \subset \bar{A} \cap \bar{B}$.

Now, we shall fix various notations for collections $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}_\alpha$;

$$\bar{\mathcal{U}} = \{ \bar{U} \mid U \in \mathcal{U} \}, \quad B(\mathcal{U}) = \{ B(U) \mid U \in \mathcal{U} \},$$

where $B(U)$ denotes the boundary of U,

$$\mathcal{U} \wedge \mathcal{V} = \{ U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V} \},$$

$$\wedge \{ \mathcal{U}_\alpha \mid \alpha \in A \} = \{ \cap \{ U_\alpha \mid \alpha \in A \} \mid U_\alpha \in \mathcal{U}_\alpha \text{ for each } \alpha \in A \}.$$

If $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{U}_\alpha$ are coverings, then $\bar{\mathcal{U}}$, $\mathcal{U} \wedge \mathcal{V}$ and $\wedge \{ \mathcal{U}_\alpha \mid \alpha \in A \}$ are also coverings.

We need further notations to be used in the theory of coverings. Let p, P and $\mathcal{U}$ be a point, a set and a collection respectively, then

$$S(p,\mathcal{U}) = S^1(p,\mathcal{U}) = \bigcup \{ U \mid p \in U \in \mathcal{U} \} \cup \{ p \},$$

$$S(P,\mathcal{U}) = S^1(P,\mathcal{U}) = \bigcup \{ U \mid U \in \mathcal{U}, U \cap P \neq \emptyset \} \cup P,$$

$$S^n(p,\mathcal{U}) = S(S^{n-1}(p,\mathcal{U}), \mathcal{U}), \quad S^\infty(p,\mathcal{U}) = \bigcup_{n=1}^{\infty} S^n(p,\mathcal{U}),$$

$$S^n(P,\mathcal{U}) = S(S^{n-1}(P,\mathcal{U}), \mathcal{U}), \quad S^\infty(P,\mathcal{U}) = \bigcup_{n=1}^{\infty} S^n(P,\mathcal{U}),$$

$$\mathcal{U}^\Delta = \{ S(p,\mathcal{U}) \mid p \in R \}, \quad \mathcal{U}^{\Delta\Delta} = (\mathcal{U}^\Delta)^\Delta,$$

$$\mathcal{U}^* = \{ S(U,\mathcal{U}) \mid U \in \mathcal{U} \}, \quad \mathcal{U}^{**} = (\mathcal{U}^*)^*.$$

If $\mathcal{U}$ is a covering, then $\mathcal{U}^\Delta$ and $\mathcal{U}^*$ are also coverings. It is clear that $\mathcal{U} < \mathcal{U}^\Delta < \mathcal{U}^* < \mathcal{U}^{\Delta\Delta}$.

C) A T_1-space R is normal if and only if for every finite open covering $\mathcal{U}$ there exists a finite open covering $\mathcal{V}$ such that $\mathcal{V}^* < \mathcal{U}$.

Definition I. 2. *Let* R *be a* T_1*-space. If for every open covering* $\mathcal{U}$ *of* R *there exists an open covering* $\mathcal{V}$ *such that* $\mathcal{V}^* < \mathcal{U}$ *, then we call* R *a fully normal space.*

Now we can formulate

Theorem I. 1 (A. H. Stone's Theorem) [3]. *A* T_2*-space* R *is fully normal if and only if it is paracompact.*

Corollary. *Every metric space is paracompact.*

Practically, the following theorem is proved in A. H. Stone [1].

Theorem I.1'. *Let* $\mathcal{U}$, $\mathcal{U}_1$, $\mathcal{U}_2$,... *be a sequence of open coverings such that* $\mathcal{U} > \mathcal{U}_1{}^* > \mathcal{U}_1 > \mathcal{U}_2^* > \dots$ *; then* $\mathcal{U}$ *has a locally finite open refinement. If* $\mathcal{U}$ *is a locally finite open covering of a normal space, then there is a locally finite open covering* $\mathcal{V}$ *such that* $\mathcal{V}^* < \mathcal{U}$.

I. 2. Metrization

Since one of the main purposes of this book is to study the recent development in dimension theory for general metric spaces, in this section we shall give a brief account of the theory of metric spaces and of metrization. In its methods the latter has a close connection with modern dimension theory.

As is well-known a topological space R is called *metrizable* if one can introduce a topology-preserving metric in R . As for necessary and sufficient conditions for a topological space to be metrizable the following classical theorem is still fundamental.

Theorem I. 2 (Alexandroff-Urysohn's Metrization Theorem). *A* T_1*-space* R *is metrizable if and only if there exists a sequence* $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \dots$ *of open coverings* $\mathcal{U}_i$ *such that* $\{ S(p,\mathcal{U}_i) \mid i = 1,2,\dots \}$ *is a neighbourhood basis for each point* p *of* R .

It is generally agreed that an open collection $\mathcal{U}$ in a topological space R is called an *open basis* if for every neighbourhood $V(p)$ of every point p of R there exists an element U of $\mathcal{U}$ such that $p \in U \subset V(p)$.

In metric spaces σ - locally finite open bases play a major role as indicated by the following theorem.

[3] Concerning the proof see A. H. Stone [1], or J. Nagata [8].

Theorem I. 3 (Nagata-Smirnov's Metrization Theorem) [4]. *A regular space R is metrizable if and only if there exists a σ-locally finite open basis of R.*

Corollary (Urysohn's Metrization Theorem). *A second countable space is metrizable if and only if it is regular, where we mean by a second countable space a topological space which has a countable open basis.*

Theorem I. 4 (Bing's Metrization Theorem) [5]. *A regular space R is metrizable if and only if there exists a σ-discrete basis of R.*

From now on throughout this section let us denote by R a metric space with the metric $\rho(x,y)$. (Later we may use a symbol like $\langle R,\rho\rangle$ to emphasize that we consider the metric space R together with the fixed metric ρ, especially when more than one metric compatible with the topology of R are involved in the discussion.) An open covering $\mathcal{U}$ of R is called a *uniform covering* if there exists $\varepsilon > 0$ such that

$$\{ S_\varepsilon(p) \mid p \in R \} < \mathcal{U} ,$$

where $S_\varepsilon(p)$ denotes the spherical neighbourhood of p with radius ε, i.e.

$$S_\varepsilon(p) = \{ q \mid \rho(p,q) < \varepsilon \} .$$

A) Every open covering of a compact metric space is a uniform covering.

We denote by $\delta(U)$ the *diameter* of a subset U of R and by mesh $\mathcal{U}$ the number $\sup \{ \delta(U) \mid U \in \mathcal{U} \}$ for a collection $\mathcal{U}$. Let $\mathcal{F}$ be a *filter* of R, i.e. $\mathcal{F}$ is a nonvacuous collection which satisfies

$$(1)\ \emptyset \notin \mathcal{F},\quad (2)\ F,G \in \mathcal{F} \text{ implies } F \cap G \in \mathcal{F},\quad (3)\ G \supset F \in \mathcal{F} \text{ implies } G \in \mathcal{F} .$$

If for every $\varepsilon > 0$ there exists an element F of $\mathcal{F}$ with $\delta(F) < \varepsilon$, then we call $\mathcal{F}$ a *Cauchy filter*. Let $\mathcal{F}$ and $\mathcal{G}$ be two Cauchy filters of R. If for every $\varepsilon > 0$ there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\delta(F \cup G) < \varepsilon$, then we call $\mathcal{F}$ and $\mathcal{G}$ *equivalent*.

We can classify all Cauchy filters of R by this equivalence relation. Then we denote by R^* the set of all classes. Let a and b be two points of R^* and let $\mathcal{F} \in a$, $\mathcal{G} \in b$. Then one can easily see that

$$\rho^*(a,b) = \sup \{ \rho(F,G) \mid F \in \mathcal{F},\ G \in \mathcal{G} \}$$

is uniquely determined by a and b, where

$$\rho(F,G) = \inf \{ \rho(x,y) \mid x \in F,\ y \in G \} .$$

One can also easily see that $\rho^*(a,b)$ gives a metric of R^*. If we identify each point p of R with the class containing the Cauchy filter $\mathcal{P} = \{ F \mid p \in F \}$,

[4] Concerning the proof see J. Nagata [1], Yu. Smirnov [1] or J. Nagata [8].
[5] See R. H. Bing [1] or J. Nagata [8].

then R can be considered as a sub-metric space of the metric space R^* .

It is easy to see that R^* satisfies

i) R is dense in R^* ,

ii) R^* is a *complete metric space* i.e. every Cauchy filter of R^* converges,

iii) every uniformly continuous mapping f of R into a complete metric space S can be extended to a uniformly continuous mapping f^* of R^* into S , i.e.

$$f(x) = f^*(x) \text{ for } x \in R .$$

This metric space R^* is called the *completion* of the metric space R .

B) Let A be a subset of a metric space R .
We denote by $\bar{A}^*$ the closure of A in the completion R^* . Then $\bar{A}^*$ consists of all the classes which contain a filter F such that $A \in \mathrm{F}$.

C) In connection with B), let $\mathcal{U} = \{ U_\gamma \mid \gamma \in \Gamma \}$ be a uniform covering of R ; then $\mathcal{U}' = \{ R^* - \overline{R - U_\gamma}^* \mid \gamma \in \Gamma \}$ is a uniform covering of R^* .

D) The completion of a totally bounded metric space is compact.

A metric space R is called *totally bounded* if for every $\varepsilon > 0$ the covering $\{ S_\varepsilon(x) \mid x \in R \}$ has a finite subcovering. It is well known that a metric space is compact if and only if it is complete and totally bounded, which implies the above proposition. It is also well known that a metric space R is *separable* (i.e. it has a countable dense set) if and only if it is second countable if and only if one can introduce a totally bounded metric onto R . As for complete metric spaces the following well-known theorem is also applicable in dimension theory.

Theorem I. 5 (Baire's Theorem). *Let* U_n , $n = 1, 2, \ldots$ *be open dense subsets of a complete metric space* R . *Then* $\bigcap_{n=1}^{\infty} U_n$ *is also dense in* R .

The *Hilbert cube*

$$I^\omega = \{ (x_1, x_2, \ldots) \mid |x_i| \leqq 1/i , \ i = 1, 2, \ldots \}$$

is an important example of a compact metric space and accordingly of a separable metric space.

E) A metric space R is separable if and only if R is homeomorphic to a subspace of I^ω .

Let us give another important example of a metric space. We denote by Ω a given set and let $N(\Omega) = \{ (\alpha_1, \alpha_2, \ldots) \mid \alpha_k \in \Omega , \ k = 1, 2, \ldots \}$, $\rho((\alpha_1, \alpha_2, \ldots), (\beta_1, \beta_2, \ldots)) =$

$= 1/\min\{k \mid \alpha_k \neq \beta_k\}$ for $(\alpha_1, \alpha_2, \dots) \in N(\Omega)$, $(\beta_1, \beta_2, \dots) \in N(\Omega)$. We can easily see that ρ is a metric of $N(\Omega)$. This metric space $N(\Omega)$ is called a *generalized Baire zero-dimensional space.* We shall deal with this space again in Chapter III.

F) A metric space R has a σ - star-finite open basis if and only if R is homeomorphic to a subspace of a topological product $N(\Omega) \times I^{\omega}$ for a suitable Ω , where an open basis is called a *σ - star-finite open basis* if it is the union of countably many star-finite open *coverings* (due to K. Morita).

G) Every locally compact metric space has the star-finite property, and accordingly a σ-star-finite open basis.

I. 3. Mappings

The theory of mappings is also a powerful tool for dimension theory. Let g be a continuous mapping of a topological space R into a topological space S . Let f be a continuous mapping of a subset A of R into S such that $f(x) = g(x)$ for every $x \in A$. Then g is called a *continuous extension of* f *over* R , and f is called the *restriction of* g *to* A , being denoted as $f = g \mid A$. We denote by $f\colon A \to S$ a mapping f of A into S .

Theorem I. 6 (Urysohn's Lemma). *Let F and G be disjoint closed sets of a normal space R ; then there exists a real-valued continuous function f such that $f(F) = 0$, $f(G) = 1$, $0 \leq f \leq 1$.*

Throughout this book we shall mean by $f(F) = a$ that $f(p) = a$ for every point $p \in F$

Corollary. *Let F and G be disjoint closed sets of a metric space R; then there exists a real-valued continuous function f such that $f^{-1}(0) = F$, $f^{-1}(1) = G$, $0 \leq f \leq 1$*

Theorem I. 7 (Tietze's Extension Theorem). *Let f be a continuous mapping of a closed subset F of a normal space R into the n-dimensional Euclidean cube I^n. Then there exists a continuous extension g of f over R .*

Corollary. *Let f be a continuous mapping of a closed subset F of a normal space R into the n-sphere S^n. Then there exists a continuous extension of f over an open set U containing F .*

Theorem I. 8 (M. H. Stone-Gelfand-Silov's Approximation Theorem)[6]. *Let C be a ring of real-valued continuous functions of a compact T_2-space R . Let C satisfy* *i) every constant function belongs to* C ,

ii) for any distinct points p, q *of* R *there exists* $g \in C$ *such that* $g(p) \neq g(q)$. *Then for any real-valued continuous function* f *of* R *and for any* $\varepsilon > 0$ *there exists* $f_\varepsilon \in C$ *which satisfies* $|f(p) - f_\varepsilon(p)| < \varepsilon$ *for every* $p \in R$.

Theorem I. 9 (A. H. Stone-Morita-Hanai's Theorem)[6]. *Let* f *be a closed continuous mapping of a metric space* R *onto a topological space* S *(namely* f *maps every closed set of* R *to a closed set of* S *). Then* S *is metrizable if and only if* $B(f^{-1}(y))$ *is compact for every* $y \in S$.

I. 4. Dimension [7].

In their book published in 1941 Hurewicz and Wallman had to limit themselves to separable metric spaces because it seemed impossible at that time to establish a theory of dimension for more general spaces. However, such a more general theory has been made possible by the developments in general topology which took place since 1948.

In 1948, A. H. Stone proved Theorem I.1 which was a considerable step forward in the theory of open coverings especially of locally finite coverings in metric spaces. Furthermore Theorem I.3, which was developed from Theorem I.1, determined the importance of locally finite open coverings in general metric spaces. Thus Theorem I.1 made an epoch not only for modern general topology but for modern dimension theory. On the foundation of the developed covering theory for metric spaces, M. Katětov [2] in 1952 and K. Morita [4] in 1954 independently succeeded in extending the principal results of the classical dimension theory to general metric spaces and in proving $\operatorname{Ind} R = \dim R$ for every metric space R [8] . It is an interesting fact that some of the results which have been established in dimension theory for general metric spaces since Katětov-Morita's work are quite new even for separable metric spaces. We are now inclined to think that we have obtained the final answers to the major problems in dimension theory for general metric spaces, though a few questions remain, and dimension theory for non-metrizable spaces has also greatly developed in these years.

The most important dimension functions for general metric spaces are *covering* (or *Lebesgue) dimension* $\dim R$ and *strong inductive* (or *large inductive* or *Čech*) *dimension* $\operatorname{Ind} R$.

Definition I. 3. *Let* U *be a collection in a topological space* R *and* p *a point of* R . *Then we mean by the order of* U *at* p *the number of members of* U *which contain* p , *and we denote it by* $\operatorname{ord}_p U$. *If there exist infinitely many such members, then* $\operatorname{ord}_p U = +\infty$.

[6] For a proof see J. Nagata [8].
[7] As for historical review on concepts of dimension as well as the earlier developments of dimension theory see W. Hurewicz and H. Wallman [1], P. Alexandroff [6], and also S. P. Zervos [1]. For modern developments, see J. Nagata [9], [10], [11].
[8] The definitions of $\dim R$ and $\operatorname{Ind} R$ will be found in Definitions I.4 and I.5, respectively.

The order of $\mathfrak{U}$ *will be the supremum of* $\mathrm{ord}_p\,\mathfrak{U}$ *and be denoted by* $\mathrm{ord}\,\mathfrak{U}$,

$$\mathrm{ord}\,\mathfrak{U} = \sup\{\mathrm{ord}_p\,\mathfrak{U} \mid p \in R\} .$$

Definition I. 4. *If for any finite open covering* $\mathfrak{U}$ *of a topological space* R *there exists an open covering* $\mathfrak{V}$ *such that* $\mathfrak{V} < \mathfrak{U}$, $\mathrm{ord}\,\mathfrak{V} \leqq n + 1$, *then* R *has covering dimension* $\leqq n$, $\dim R \leqq n$. R *has covering dimension* n , $\dim R = n$ *if it is true that* $\dim R \leqq n$ *and it is false that* $\dim R \leqq n - 1$. *If* $\dim R \leqq n$ *is false for each integer* n , *then* $\dim R = +\infty$. *We define that* $\dim \emptyset = -1$.

Definition I. 4'. *If we replace 'open covering' in Definition I.4 with 'finite functionally open covering', then we obtain the definition of Katětov-Smirnov covering dimension (an open set* U *of* R *is called functionally open or a cozero set if for some real-valued continuous function* f *defined on* R $U = \{x \in R \mid f(x) \neq 0\}$ *). The complement of a functionally open set is called functionally closed or a zero set. It is easy to see that Katětov-Smirnov dimension coincides with Lebesgue dimension if the space is normal. In this book,* $\dim R$ *means Lebesgue dimension unless otherwise specified. In a Tychonoff space* X *the Katětov-Smirnov dimension of* X *is* $\dim \beta X$ *where* βX *denotes the Stone-Čech compactification of* X .

Definition I. 5. *i) A topological space* R *has strong inductive dimension* -1 , $\mathrm{Ind}\,R = -1$, *if* $R = \emptyset$.

ii) If for any disjoint closed sets F *and* G *of a topological space* R *there exists an open set* U *such that* $F \subset U \subset R - G$, $\mathrm{Ind}\,B(U) \leqq n - 1$, *then* R *has strong inductive dimension* $\leqq n$, $\mathrm{Ind}\,R \leqq n$.

If it is true that $\mathrm{Ind}\,R \leqq n$ *and it is false that* $\mathrm{Ind}\,R \leqq n - 1$, *then* $\mathrm{Ind}\,R = n$. *If* $\mathrm{Ind}\,R \leqq n$ *is false for each* n , *then* $\mathrm{Ind}\,R = +\infty$.

The following notion of *weak inductive* (or *small inductive* or *Urysohn-Menger) dimension* is no longer so important as the preceding two notions, because as proved by P. Roy, it is not equivalent to the preceding dimensions for general metric spaces (though it is for separable metric spaces), and because some basic properties required for dimension are not possessed by this dimension function.

Definition I. 6. *i) A topological space* R *has weak inductive dimension* -1 , $\mathrm{ind}\,R = -1$, *if* $R = \emptyset$.

ii) If for every neighbourhood $U(p)$ *of every point* p *of* R *there exists an open neighbourhood* V *such that* $p \in V \subset U(p)$, $\mathrm{ind}\,B(V) \leqq n - 1$, *then* R *has weak inductive dimension* $\leqq n$, $\mathrm{ind}\,R \leqq n$.

If it is true that $\mathrm{ind}\,R \leqq n$ *and it is false that* $\mathrm{ind}\,R \leqq n - 1$, *then* $\mathrm{ind}\,R = n$. *If* $\mathrm{ind}\,R \leqq n$ *is false for each* n , *then* $\mathrm{ind}\,R = +\infty$.

CHAPTER II

DIMENSION OF METRIC SPACES

Throughout chapter II - chapter VI all spaces are metric unless the contrary is explicitly stated. We shall begin this chapter with the sum theorem and shall deduce further fundamental theorems from it.

II. 1. Lemmas to sum theorem

A) Let F be a closed subset of a space R. Then $\operatorname{Ind} F \leq \operatorname{Ind} R$.

Proof. The proof is by induction on the dimension of R. If $\operatorname{Ind} R = -1$, then $R = \emptyset$, which implies $F = \emptyset$; hence $\operatorname{Ind} F = -1$.

Assume A) for R with $\operatorname{Ind} R \leq n-1$. If $\operatorname{Ind} R \leq n$, then for given disjoint closed sets G and H of the subspace F there exists an open set U of R such that

$$G \subset U \subset R - H \text{, and } \operatorname{Ind} B(U) \leq n-1 \text{,}$$

because G and H are also closed in R. Then $V = U \cap F$ is an open set of the subspace F and satisfies

$$G \subset V \subset F - H \text{, and } B_F(V) \subset B(U) \text{,}$$

where $B_F(V)$ denotes the boundary of V in the subspace F. Hence $\operatorname{Ind} B_F(V) \leq n-$ follows from $\operatorname{Ind} B(U) \leq n-1$ and the induction hypothesis. Thus we get $\operatorname{Ind} F \leq n$

B) Let C and D be subsets of a space R. If $(\bar{C} \cap D) \cup (C \cap \bar{D}) = \emptyset$, then there exist open sets M and N satisfying $C \subset M$, $D \subset N$, $M \cap N = \emptyset$. This assertion can be deduced from I.1 B) and the hereditary normality of R, but we shall directly prove it as follows:

Proof. We assign to each point $p \in C$ a positive number $\varepsilon(p)$ satisfying $S_{\varepsilon(p)}(p) \cap D = \emptyset$ and to each point $q \in D$ a positive number $\varepsilon(q)$ satisfying $S_{\varepsilon(q)}(q) \cap C = \emptyset$.

Putting

$$M = \bigcup \{ S_{\frac{1}{2}\varepsilon(p)}(p) \mid p \in C \} \text{ , and } N = \bigcup \{ S_{\frac{1}{2}\varepsilon(q)}(q) \mid q \in D \}$$

we get the desired open sets.

C) Let A and B be subsets of a space R such that

$$R = A \cup B \text{ , } \operatorname{Ind} A \leqq n \text{ , } \operatorname{Ind} B \leqq 0 \text{ ;}$$

then $\operatorname{Ind} R \leqq n + 1$.

Proof. Let F and G be given disjoint closed sets of R; then by virtue of the normality of R there exist open sets V and W satisfying

$$F \subset V \text{ , } G \subset W \text{ , and } \bar{V} \cap \bar{W} = \emptyset \text{ .}$$

Since $\operatorname{Ind} B \leqq 0$, we can find an open and closed set U of the subspace B such that

$$\bar{V} \cap B \subset U \subset B - (\bar{W} \cap B) \text{ .}$$

Since $F \cup U$ and $G \cup (B - U)$ satisfy the condition of B) [1], we can find open sets M and N such that

$$F \cup U \subset M \text{ , } G \cup (B - U) \subset N \text{ , and } M \cap N = \emptyset \text{ .}$$

$B(M) \subset A$ is obvious by the above property of N and by the fact that U is open and closed in B.

Hence $\operatorname{Ind} B(M) \leqq \operatorname{Ind} A \leqq n$ follows from A). Since M is an open set satisfying $F \subset M \subset R - G$, we have $\operatorname{Ind} R \leqq n + 1$.

D) If $\operatorname{Ind} R \leqq n$, then there exists a σ-locally finite open basis $\mathcal{V}$ of R such that $\operatorname{Ind} B(V) \leqq n - 1$ for every $V \in \mathcal{V}$.

[1] We mean by B) Proposition B) of the same section and chapter, by 1 B) Proposition B) of Section 1 of the same chapter and by I.1 B) Proposition B) of Section 1 of Chapter I.

Proof. By Theorem I.1 or by Theorem I.3 there exists a σ-locally finite open basis $\bigcup_{i=1}^{\infty} \mathcal{U}_i$ of R, where we may suppose each $\mathcal{U}_i$ to be a locally finite open covering of R such that $\lim_{i\to\infty}$ mesh $\mathcal{U}_i = 0$. Let $\mathcal{U}_i = \{U_\alpha \mid \alpha \in A_i\}$; then by I.1 A) there exists an open covering $\mathcal{W}_i = \{W_\alpha \mid \alpha \in A_i\}$ satisfying $\bar{W}_\alpha \subset U_\alpha$. Since $\operatorname{Ind} R \leqq n$, we can construct open sets V_α for $\alpha \in A_i$ such that

$$\bar{W}_\alpha \subset V_\alpha \subset U_\alpha , \quad \text{and} \quad \operatorname{Ind} B(V_\alpha) \leqq n-1 .$$

Putting $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, and $\mathcal{V}_i = \{V_\alpha \mid \alpha \in A_i\}$ we get the desired open basis.

E) If a space R has a σ-locally finite open basis $\mathcal{V}$ such that $B(V) = \emptyset$ for every $V \in \mathcal{V}$, then $\operatorname{Ind} R \leqq 0$.

Proof. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ for locally finite open collections $\mathcal{V}_i$. Let F and G be disjoint closed sets of R. Then for every i

$$U_i = R - \bigcup \{V \mid V \in \bigcup_{j=1}^{i} \mathcal{V}_j ,\ V \cap F = \emptyset\}$$

is an open and closed set containing F because $\bigcup_{j=1}^{i} \mathcal{V}_j$ is a locally finite collection of open and closed sets V. Thus we get a sequence $\{U_i \mid i = 1,2,\dots\}$ of open and closed sets such that

$$U_1 \supset U_2 \supset \dots \supset F , \quad \text{and} \quad \bigcap_{i=1}^{\infty} U_i = F .$$

In a similar way we can construct a sequence $\{W_i \mid i = 1,2,\dots\}$ of open and closed sets such that

$$W_1 \supset W_2 \supset \dots \supset G , \quad \text{and} \quad \bigcap_{i=1}^{\infty} W_i = G .$$

Now

$$U = \bigcup_{i=1}^{\infty} (U_i - W_i)$$

is an open and closed set and satisfies $F \subset U \subset R - G$. Therefore $\operatorname{Ind} R \leqq 0$.

F) Let $A \subset R$, $\operatorname{Ind} R \leqq 0$; then $\operatorname{Ind} A \leqq 0$.

Proof. F) is a direct consequence of D) and E).

II. 2. Sum theorem

A) Let $\{F_i \mid i = 1,2,\dots\}$ be a closed covering of a space R such that

$$\operatorname{Ind} F_i \leqq 0 \quad \text{for} \quad i = 1, 2, \ldots \, .$$

Then $\operatorname{Ind} R \leqq 0$.

Proof. Let G and H be given disjoint closed sets of R . Since $\operatorname{Ind} F_1 \leqq 0$, there exists an open and closed subset U_1 of F_1 such that

$$F_1 \cap H \subset U_1 \subset F_1 - G \, .$$

Since $H \cup U_1$ and $G \cup (F_1 - U_1)$ are disjoint closed sets, by the normality of R we can find open sets V_1 and W_1 such that

(1) $$H \cup U_1 \subset V_1 \, , \quad G \cup (F_1 - U_1) \subset W_1 \, , \quad \text{and} \quad \bar{V}_1 \cap \bar{W}_1 = \emptyset \, .$$

Since $\operatorname{Ind} F_2 \leqq 0$, there exists an open and closed subset U_2 of F_2 such that

$$F_2 \cap H \subset F_2 \cap \bar{V}_1 \subset U_2 \subset F_2 - \bar{W}_1 \subset F_2 - G \, .$$

Since $\bar{V}_1 \cup U_2$ and $\bar{W}_1 \cup (F_2 - U_2)$ are disjoint closed sets, we can find open sets V_2 and W_2 such that

$$\bar{V}_1 \cup U_2 \subset V_2 \, , \quad \bar{W}_1 \cup (F_2 - U_2) \subset W_2 \, , \quad \text{and} \quad \bar{V}_2 \cap \bar{W}_2 = \emptyset \, .$$

By repeating this process we get sequences

$$\{ U_i \mid i = 1, 2, \ldots \} \, , \quad \{ V_i \mid i = 1, 2, \ldots \} \, , \quad \{ W_i \mid i = 1, 2, \ldots \}$$

of sets such that

(2) $$F_i \cap H \subset U_i \subset F_i - G \, ,$$

each U_i is an open and closed subset of F_i ,

(3) $$\bar{V}_{i-1} \cup U_i \subset V_i \, , \quad \bar{W}_{i-1} \cup (F_i - U_i) \subset W_i \, , \quad i = 2, 3, \ldots \, ,$$

(4) $$\bar{V}_i \cap \bar{W}_i = \emptyset \, ,$$

V_i and W_i are open sets.

Now we put

$$V = \bigcup_{i=1}^{\infty} V_i \, , \quad W = \bigcup_{i=1}^{\infty} W_i \, ;$$

then from (1) it follows that V and W contain H and G respectively. Furthermore it follows from (3) and (4) that V and W are disjoint open sets. Since (1) and (3) imply

$$V_i \cup W_i \supset U_i \cup (F_i - U_i) = F_i \, , \quad i = 1, 2, \ldots \, ,$$

we get

$$V \cup W \supset \bigcup_{i=1}^{\infty} F_i = R .$$

Thus V is an open and closed set satisfying $H \subset V \subset R - G$, which yields Ind $R \leqq 0$.

B) Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed covering of a space R such that Ind $F_\gamma \leqq 0$ for every $\gamma \in \Gamma$.
Then Ind $R \leqq 0$.

Proof. For each positive integer i we take, using the corollary to Theorem I.1, a locally finite open covering $\mathcal{U}_i = \{U_\delta \mid \delta \in \Delta_i\}$ such that mesh $\mathcal{U}_i < 1/i$ and such that the closure $\bar{U}_\delta$ of each element of $\mathcal{U}_i$ intersects only finitely many elements of $\{F_\gamma \mid \gamma \in \Gamma\}$. Then

(1) Ind $\bar{U}_\delta \leqq 0$

follows from A). On the other hand, by I.1 A) there exists an open covering $\mathcal{V}_i = \{V_\delta \mid \delta \in \Delta_i\}$ with $\bar{V}_\delta \subset U_\delta$ for every $\delta \in \Delta_i$.
Hence, by virtue of (1) we can find an open and closed set W_δ of $\bar{U}_\delta$ such that

$$\bar{V}_\delta \subset W_\delta \subset \bar{U}_\delta - (R - U_\delta) .$$

Since this relation implies $W_\delta \subset U_\delta$, W_δ is clearly also open and closed in R.
Thus, putting

$$\mathcal{W}_i = \{W_\delta \mid \delta \in \Delta_i\} , \mathcal{W} = \bigcup_{i=1}^{\infty} \mathcal{W}_i$$

we get a σ-locally finite open basis $\mathcal{W}$ consisting of open and closed sets. Hence by I E) Ind $R \leqq 0$.

C) Let $\{F_i \mid i = 1,2,\ldots\}$ be a closed covering of a space R such that Ind $F_i \leqq n$, $i = 1,2,\ldots$.
Then Ind $R \leqq n$.

D) Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed covering of a space R such that Ind $F_\gamma \leqq n$ for every $\gamma \in \Gamma$.
Then Ind $R \leqq n$.

Proof. The validity of C) and D) is clear for $n = -1$. To prove C) and D) at the same time by induction we assume C) and D) for $n = m - 1$ ($m \geqq 0$). First let us show C) for $n = m$.
Since Ind $F_i \leqq m$, by I D) there exists a σ-locally finite open basis $\mathcal{V}_i = \bigcup_{k=1}^{\infty} \mathcal{V}_{i,k}$ of the subspace F_i such that

(1) $\quad \operatorname{Ind} B_{F_i}(V) \leq m - 1 \quad$ for $\quad V \in V_i$.

Since the σ - local finiteness of V_i implies the σ - local finiteness of

$$V_i' = \{ B_{F_i}(V) \mid V \in V_i \} ;$$

the closed collection $\bigcup_{i=1}^{\infty} V_i'$ is also σ - locally finite.

Let

$$H_i = \bigcup \{ B_{F_i}(V) \mid V \in V_i \} ;$$

then $\bigcup_{i=1}^{\infty} V_i$ is a σ - locally finite closed covering of $\bigcup_{i=1}^{\infty} H_i$. Hence from (1) and the induction hypothesis it follows that

(2) $$\operatorname{Ind} \bigcup_{i=1}^{\infty} H_i \leq m - 1 .$$

On the other hand $V_i = \bigcup_{k=1}^{\infty} V_{i,k}$ restricted to $F_i - H_i$ constitutes an open basis of $F_i - H_i$ satisfying the condition of I E). Hence we have $\operatorname{Ind} (F_i - H_i) \leq 0$. Hence, if we define G_i by $G_i = F_i - \bigcup_{j=1}^{\infty} H_j$, then by I F) $\operatorname{Ind} G_i \leq 0$. Since each G_i is closed in $\bigcup_{i=1}^{\infty} G_i$,

(3) $$\operatorname{Ind} \bigcup_{i=1}^{\infty} G_i \leq 0$$

follows from A).

Since

$$R = (\bigcup_{i=1}^{\infty} G_i) \cup (\bigcup_{i=1}^{\infty} H_i) ,$$

it follows from I C) combined with (2) and (3) that $\operatorname{Ind} R \leq (m - 1) + 1 = m$. This proves C) for $n = m$.

We can prove D) in a similar way by using I C), I D), I E), I F) and B) .

Theorem II. 1 (Sum Theorem) [2] . *Let* $\{ F_\gamma \mid \gamma \in \Gamma \}$ *be a locally countable closed covering of a space* R *such that* $\operatorname{Ind} F_\gamma \leq n$ *for every* $\gamma \in \Gamma$. *Then* $\operatorname{Ind} R \leq n$.

Proof. Let V be an open covering each of whose elements meets at most countably many members of $\{ F_\gamma \mid \gamma \in \Gamma \}$. By virtue of the paracompactness and regularity of R we can find a locally finite open covering U such that $\bar{U} < V$. Suppose $U = \{ U_\delta \mid \delta \in \Delta \}$. Then each $\bar{U}_\delta$ meets at most countably many members of

[2] The sum theorem in the present form is due to K. Morita [4] and essentially to K. Morita [2] and M. Katětov [2].

$\{F_\gamma \mid \gamma \in \Gamma\}$, and $\operatorname{Ind} F_\gamma \cap \bar{U}_\delta \leqq n$ follows from $\operatorname{Ind} F_\gamma \leqq n$ and I A). Hence by C) we have $\operatorname{Ind} \bar{U}_\delta \leqq n$. This implies $\operatorname{Ind} R \leqq n$ by virtue of D).

Corollary [3]. *Let $\{F_\gamma \mid \gamma < \tau\}$ be a covering of a space R such that* $\operatorname{Ind} F_\gamma \leqq n$ *for every $\gamma < \tau$ and such that $\cup\{F_\gamma \mid \gamma < \delta\}$ is closed for any $\delta < \tau$. Then* $\operatorname{Ind} R \leqq n$.

Proof. Let

$$G_{\delta,i} = F_\delta - S_{1/i}(\cup\{F_\gamma \mid \gamma < \delta\}) \quad \text{for} \quad \delta < \tau \ , \ i = 1,2,\ldots,$$

where $S_{1/i}(F)$ for a set F denotes the $1/i$-neighbourhood of F, i.e. the set $\{y \mid \rho(x,y) < 1/i$ for some $x \in F\}$. Then $\{G_{\delta,i} \mid \delta < \tau\}$ is discrete, and hence $\{G_{\delta,i} \mid \delta < \tau,\ i = 1,2,\ldots\}$ is a σ-discrete closed covering of R. Since by I A) $\operatorname{Ind} G_{\delta,i} \leqq n$, we have $\operatorname{Ind} R \leqq n$ by the sum theorem.

II. 3 Decomposition Theorem

Theorem II. 2. $\operatorname{Ind} R \leqq n$ *for* $n \geqq 0$ *if and only if there exists a σ-locally finite open basis $\mathcal{V}$ such that* $\operatorname{Ind} B(V) \leqq n-1$ *for every* $V \in \mathcal{V}$.

Proof. The "only if" part is proved in I D). The "if" part for $n=0$ is proved in I E). To prove the "if" part for any integer $n \geqq 0$ we put

$$A = \cup\{B(V) \mid V \in \mathcal{V}\}, \text{ and } B = R - A.$$

Since $\{B(V) \mid V \in \mathcal{V}\}$ is a σ-locally finite closed collection, from the sum theorem and $\operatorname{Ind} B(V) \leqq n-1$ for $V \in \mathcal{V}$ it follows that $\operatorname{Ind} A \leqq n-1$. Since $\mathcal{V}$ restricted to B is an open basis of B satisfying the condition of I E), we have $\operatorname{Ind} B \leqq 0$. Hence by virtue of I C)

$$\operatorname{Ind} R = \operatorname{Ind} A \cup B \leqq (n-1) + 1 = n.$$

Theorem II. 3. (Subspace Theorem). *For every subset A of a space R* $\operatorname{Ind} A \leqq \operatorname{Ind} R$.

Proof. This is an immediate consequence of Theorem II.2.

Theorem II. 4. (Decomposition Theorem). $\operatorname{Ind} R \leqq n$ *for* $n \geqq 0$ *if and only if* $R = \cup_{i=1}^{n+1} A_i$ *for some $n+1$ subsets A_i with* $\operatorname{Ind} A_i \leqq 0$, $i = 1,\ldots,n+1$.

Proof. The "if" part is directly deduced from I C). As for the "only if" part we saw in the proof of Theorem II.2 that R can be decomposed into two subsets A_1 and B_1 with $\operatorname{Ind} A_1 \leqq 0$, and $\operatorname{Ind} B_1 \leqq n-1$.

---

[3] Proved first by K. Nagami [1]. γ, τ, δ denote ordinal numbers.

We can decompose B_1 into two subsets A_2 and B_2 with $\operatorname{Ind} A_2 \leqq 0$, and $\operatorname{Ind} B_2 \leqq n - 2$.

By repeating this process R can be decomposed into $n + 1$ 0-dimensional subsets $A_1, \dots, A_{n+1}$.

Corollary. $\operatorname{Ind} A \cup B \leqq \operatorname{Ind} A + \operatorname{Ind} B + 1$ *for any subsets* A, B *of a space* R .

Proof. We can easily prove this corollary by decomposing A and B into 0-dimensional subsets by use of Theorem II.4.

II. 4. Product theorem

A) If V and W are subsets of spaces R and S respectively, then

$$B_{R \times S}(V \times W) = B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W) .$$

Proof. To prove

$$B_{R \times S}(V \times W) \subset B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W) ,$$

let

$$(p,q) \notin B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W) ,$$

where $p \in R$, $q \in S$. Then the following holds.

(1) $\qquad p \notin B_R(V)$ or $q \notin \bar{W}$

(2) $\qquad p \notin \bar{V}$ or $q \notin B_S(W)$.

Since $q \notin \bar{W}$ or $p \notin \bar{V}$ obviously implies $(p,q) \notin B_{R \times S}(V \times W)$, suppose $q \in \bar{W}$, $p \in \bar{V}$, which implies by (1) and (2) that $p \notin B_R(V)$, $q \notin B_S(W)$. Therefore $p \in \operatorname{Int}_R V$, $q \in \operatorname{Int}_S W$, where $\operatorname{Int}_R V$ for a subset V of R denotes the interior of V in R . This means $(p,q) \in \operatorname{Int}_{R \times S}(V \times W)$, and hence $(p,q) \notin B_{R \times S}(V \times W)$ Thus we have

$$B_{R \times S}(V \times W) \subset B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W) .$$

Since

$$B_{R \times S}(V \times W) \supset B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W)$$

is clear, we can conclude A).

Theorem II. 5. (Product Theorem)[4] . *Let* R *and* S *be spaces at least one of which is non-empty. Then* $\operatorname{Ind} R \times S \leqq \operatorname{Ind} R + \operatorname{Ind} S$.

---

[4] Proved first by M.Katětov [2] and K.Morita [4]. Recently E.Pol [2] proved the following theorem for infinite products: Let $\{R_\alpha \mid \alpha \in A\}$ be a collection of complete metric spaces such that $\dim R_{\alpha_1} \times \dots \times R_{\alpha_k} \leqq n$ for any $\alpha_1, \dots, \alpha_k \in A$; then $\dim \prod_{\alpha \in A} R_\alpha \leqq n$.

Proof. Let $\operatorname{Ind} R = n$, $\operatorname{Ind} S = m$. We shall prove this theorem by induction on the number $n + m$. If $n + m = -1$, this inequality clearly holds. Generally the theorem is obvious if either R or S is empty, so assume $n \geqq 0$ and $m \geqq 0$.

To complete the induction, note that by Theorem II.2 there exists a σ-locally finite open basis $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ of R such that

(1) $\operatorname{Ind} B_R(V) \leqq n - 1$ for every $V \in \mathcal{V}$

and a σ-locally finite open basis $\mathcal{W} = \bigcup_{j=1}^{\infty} \mathcal{W}_j$ of S such that

(2) $\operatorname{Ind} B_S(W) \leqq m - 1$ for every $W \in \mathcal{W}$,

where $\mathcal{V}_i$ and $\mathcal{W}_j$ are locally finite open collections of R and S respectively. Put

$$\mathcal{V}_i \times \mathcal{W}_j = \{ V \times W \mid V \in \mathcal{V}_i ,\ W \in \mathcal{W}_j \} ;$$

then $\mathcal{V}_i \times \mathcal{W}_j$ is a locally finite open collection of $R \times S$. Hence $\bigcup_{i,j=1}^{\infty} \mathcal{V}_i \times \mathcal{W}_j$ is a σ-locally finite open basis of $R \times S$.

On the other hand, it follows from (1) and (2) combined with the induction hypothesis, that

$$\operatorname{Ind} B_R(V) \times \bar{W} \leqq n + m - 1 , \quad \text{and} \quad \operatorname{Ind} \bar{V} \times B_S(W) \leqq n + m - 1$$

for every $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Since

$$B_{R \times S}(V \times W) = B_R(V) \times \bar{W} \cup \bar{V} \times B_S(W)$$

follows from A), we get, from the sum theorem,

$$\operatorname{Ind} B_{R \times S}(V \times W) \leqq n + m - 1$$

for every $V \times W \in \bigcup_{i,j=1}^{\infty} \mathcal{V}_i \times \mathcal{W}_j$.

Hence by Theorem II.2 we conclude $\operatorname{Ind} R \times S \leqq n + m$.

As the following examples shows [5], the equality $\operatorname{Ind} R \times S = \operatorname{Ind} R + \operatorname{Ind} S$ is not true in general even if R and S are separable.

Example II. 1 (P. Erdös [1]). Let R^{ω} be the set of points in Hilbert space all of whose coordinates are rational. Then $\operatorname{Ind} R^{\omega} = 1$ can be proved as follows. Since $R^{\omega} = R^{\omega} \times R^{\omega}$,

$$\operatorname{Ind} R^{\omega} \times R^{\omega} < \operatorname{Ind} R^{\omega} + \operatorname{Ind} R^{\omega} .$$

---

[5] W. Hurewicz [5] proved that the equality holds if R is a compact metric space with $\operatorname{Ind} R \geq 0$ and S is a separable metric space with $\operatorname{Ind} S = 1$. Recently K. Morita [7], [8] proved the equality $\dim R \times S = \dim R + \dim S$ under a much more general condition, which implies e.g. that $\operatorname{Ind} R \times S = \operatorname{Ind} R + \operatorname{Ind} S$ holds if R and S are nonempty metric spaces such that R is a countable sum of locally compact closed subsets and $\operatorname{Ind} S = 1$.

Now, let us show Ind $R^{\omega} = 1$. Since R^{ω} is a separable metric space, and, as will be proved later (Theorem IV.4), ind = Ind = dim holds for every separable metric space, it suffices to show ind $R^{\omega} = 1$. To prove ind $R^{\omega} \leqq 1$, we shall show that ind $B(S_{\varepsilon}(p_0)) \leqq 0$ for the point $p_0 = (0,0,\dots)$ of R^{ω} and every positive rational number ε, where $S_{\varepsilon}(p_0) = \{ p \in R^{\omega} \mid \rho(p,p_0) < \varepsilon \}$. Since each point of R^{ω} can be mapped to p_0 by a homeomorphism of R^{ω} onto itself, this would be sufficient. Let $p = (p_1,p_2,\dots) \in B(S_{\varepsilon}(p_0))$ and δ a given positive number such that δ^2 is irrational. Select n such that $\sum_{i=n+1}^{\infty} p_i^2 < \frac{\delta^2}{2}$. Put

$$U = \{ (x_1,x_2,\dots) \in R^{\omega} \mid \sum_{i=1}^{n} x_i p_i > \varepsilon^2 - \frac{\delta^2}{2} \} .$$

Then U is an open neighbourhood of p because $\sum_{i=1}^{\infty} p_i^2 = \varepsilon^2$ and accordingly $\sum_{i=1}^{n} p_i^2 > \varepsilon^2 - \frac{\delta^2}{2}$. Let $y = (y_1,y_2,\dots) \in B(U) \cap B(S_{\varepsilon}(p_0))$; then $\sum_{i=1}^{n} y_i p_i = \varepsilon^2 - \frac{\delta^2}{2}$, which cannot happen because y_i and p_i are all rational, and $\varepsilon^2 - \frac{\delta^2}{2}$ is irrational. Thus $B(U) \cap B(S_{\varepsilon}(p_0))$ and accordingly the boundary of $U \cap B(S_{\varepsilon}(p_0))$ in $B(S_{\varepsilon}(p_0))$ is empty.

To prove $U \cap B(S_{\varepsilon}(p_0)) \subset S_{\delta}(p)$, let $x = (x_1,x_2,\dots) \in U \cap B(S_{\varepsilon}(p_0))$. Then

$$\sum_{i=1}^{n} x_i p_i > \varepsilon^2 - \frac{\delta^2}{2}, \qquad \sum_{i=1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} p_i^2 = \varepsilon^2 .$$

Thus we obtain $\sum_{i=1}^{\infty} (x_i - p_i)^2 < \delta^2$, proving our claim. Hence ind $B(S_{\varepsilon}(p_0)) \leqq 0$, i.e. ind $R^{\omega} \leqq 1$.

To prove ind $R^{\omega} \geqq 1$, let U be a given open neighborhood of p_0 such that $U \subset S_1(p_0)$. Select a fixed point $q_0 = \left(\frac{1}{n_1}, \frac{1}{n_2}, \dots \right) \in U$ such that $n_1, n_2, \dots$ are natural numbers. Then, since $\left(1, \frac{1}{n_2}, \frac{1}{n_3}, \dots\right) \notin S_1(p_0) \supset U$, we can choose a natural number m_1 such that

$$a_1 = \left(\frac{1}{m_1}, \frac{1}{n_2}, \frac{1}{n_3}, \dots \right) \notin U, \text{ and } b_1 = \left(\frac{1}{m_1+1}, \frac{1}{n_2}, \frac{1}{n_3}, \dots \right) \in U .$$

Since $\left(\frac{1}{m_1+1}, 1, \frac{1}{n_3}, \dots \right) \notin S_1(p_0) \supset U$, looking at b_1, we can choose a natural number m_2 such that

$$a_2 = \left(\frac{1}{m_1+1}, \frac{1}{m_2}, \frac{1}{n_3}, \dots \right) \notin U, \text{ and } b_2 = \left(\frac{1}{m_1+1}, \frac{1}{m_2+1}, \frac{1}{n_3}, \dots \right) \in U .$$

Repeating the same process we get natural numbers $m_1, m_2, m_3, \dots$ such that

$$a_k = \left(\frac{1}{m_1+1}, \dots, \frac{1}{m_{k-1}+1}, \frac{1}{m_k}, \frac{1}{n_{k+1}}, \dots \right) \notin U, \text{ and}$$

$$b_k = \left(\frac{1}{m_1 + 1}, \dots, \frac{1}{m_{k-1} + 1}, \frac{1}{m_k + 1}, \frac{1}{n_{k+1}}, \dots \right) \in U .$$

Then both $\{a_k\}$ and $\{b_k\}$ converge to the point $q = \left(\frac{1}{m_1 + 1}, \frac{1}{m_2 + 1}, \dots \right) \in R^{\omega}$, from outside and inside of U, respectively. (Note that $\sum_{i=1}^{\infty} \left(\frac{1}{m_i + 1} \right)^2 \leqq 1$ follows from $b_k \in S_1(p_0)$, $k = 1, 2, \dots$.) Thus $q \in B(U)$, proving that $B(U) \neq \emptyset$. Hence $\operatorname{ind} R^{\omega} \geqq 1$.

II. 5. Strong inductive dimension and covering dimension

To prove the equivalence of the strong inductive dimension and the covering dimension, we shall prove the following theorem which was first proved by C. H. Dowker [1] for normal spaces and plays an important role in dimension theory for general metric spaces.

A) If F is a closed set of a space R, then $\dim F \leqq \dim R$.

Theorem II. 6 [6]. *Let* R *be a space; then* $\dim R \leqq n$ *if and only if for any locally finite open covering* $\mathcal{U}$ *of* R *there exists a locally finite open covering* $\mathcal{V}$ *with* $\operatorname{ord} \mathcal{V} \leqq n + 1$, $\mathcal{V} < \mathcal{U}$.

Proof. Since the "if" part is obvious, we shall prove the "only if" part. Let $\mathcal{U} = \{ U_\gamma \mid \gamma \in \Gamma \}$. Since $\mathcal{U}$ is locally finite, there exists an open covering $\mathcal{A}$ each of whose elements meets at most finitely many elements of $\mathcal{U}$.

By choosing a locally finite open covering $\mathcal{B}$ with $\bar{\mathcal{B}} < \mathcal{A}$ we get a locally finite closed covering $\mathcal{F} = \bar{\mathcal{B}} = \{ F_\nu \mid \nu < \tau \}$ such that

(1) each F_ν meets at most finitely many elements of $\mathcal{U}$.

Then in view of A) we have

(2) $\qquad \dim F_\nu \leqq n$.

We assume that $F_0 = \emptyset$ and $F_\mu \neq F_\nu$ whenever $\mu \neq \nu$

Now let us construct a transfinite sequence of open coverings $\{ U_{\nu,\gamma} \mid \gamma \in \Gamma \}$, $\nu < \tau$ such that $U_{0,\gamma} = U_\gamma$ and $U_{\mu,\gamma} \supset U_{\nu,\gamma}$ for $\mu < \nu$, and such that

(3) each point of F_ν is contained in at most $n + 1$ members of $\{ U_{\nu,\gamma} \mid \gamma \in \Gamma \}$, and

(4) $\qquad \cap \{ U_{\mu,\gamma} \mid \mu < \nu \} \cap (R - F_\nu) = U_{\nu,\gamma} \cap (R - F_\nu)$.

---

[6] This theorem was originally proved for every normal space by C. H. Dowker [1]; we owe the present version of the proof to H. de Vries.

The construction can be carried out by use of the induction on ν. Let $U_{\mu,\gamma}$ be determined for all $\mu < \nu$. Put

$$U^*_{\nu,\gamma} = \bigcap \{ U_{\mu,\gamma} \mid \mu < \nu \} .$$

Then we claim that $\{ U^*_{\nu,\gamma} \mid \gamma \in \Gamma \}$ is an open covering of R. First to prove that $U^*_{\nu,\gamma}$ is open, let $x \in U^*_{\nu,\gamma}$. Assume that

(5) $x \in F_{\mu_i}$, $\mu_i < \nu$, $i = 1,\dots,k$, and $x \notin F_\mu$ for all μ such that $\mu_k < \mu < \nu$.

Put $N = (R - \bigcup \{ F_\mu \mid \mu_k < \mu < \nu \}) \cap U_{\mu_k,\gamma}$. Then N is a nbd of x. Let y be an arbitrary point of N; then $y \in U_{\mu,\gamma}$ for all $\mu \leq \mu_k$. On the other hand $y \in U_{\mu,\gamma}$ for all $\mu > \mu_k$ with $\mu < \nu$ can be proved by induction on μ as follows. Assume $y \in U_{\mu',\gamma}$ for all $\mu' < \mu$. Then

$$y \in (\bigcap_{\mu' < \mu} U_{\mu',\gamma}) \cap (R - F_\mu) = U_{\mu,\gamma} \cap (R - F_\mu)$$

follows from the definition of N and the induction hypothesis (4). Hence $y \in U_{\mu,\gamma}$ for all $\mu < \nu$, i.e. $N \subset \bigcap_{\mu < \nu} U_{\mu,\nu} = U^*_{\nu,\gamma}$, which proves that $U^*_{\nu,\gamma}$ is an open set. To prove that $\{ U^*_{\nu,\gamma} \}$ covers R, let $x \in R$. Assume (5) again. Then $x \in U_{\mu_k,\gamma}$ for some γ. Then in view of the above argument on y we know $x \in U^*_{\nu,\gamma}$.

Now, restrict $\{ U^*_{\nu,\gamma} \mid \gamma \in \Gamma \}$ to F_ν and choose an open covering $\{ W_\gamma \mid \gamma \in \Gamma \}$ of F_ν with order $\leq n + 1$ such that $W_\gamma \subset U^*_{\nu,\gamma}$, $\gamma \in \Gamma$. (This is possible because of (1) and (2). Generally note that for a given finite open covering $\{ U_i \mid i = 1,\dots,k \}$ of a space with $\dim \leq n$, there is a finite open covering $\{ W_i \mid i = 1,\dots,k \}$ of order $\leq n + 1$ such that $W_i \subset U_i$, $i = 1,\dots,k$.) Put $U_{\nu,\gamma} = (U^*_{\nu,\gamma} - F_\nu) \cup W_\gamma$. At last $V = \{ V_\gamma \mid \gamma \in \Gamma \}$ meets our requirements if we put $V_\gamma = \bigcap \{ U_{\nu,\gamma} \mid \nu < \tau \}$. (The last claim is easy to check by use of an argument quite similar to the one used for $\{ U^*_{\nu,\gamma} \}$. Note that at any stage ν in the subsequent induction step, only pieces of F_ν were cut away from the elements of the preceding coverings.)

Corollary. *Let R be a space; then* $\dim R \leq n$ *if and only if for any open covering* $\mathfrak{U}$ *of* R *there exists a locally finite open refinement* $\mathfrak{V}$ *of* $\mathfrak{U}$ *with* $\operatorname{ord} \mathfrak{V} \leq n + 1$.

Proof. Use the corollary to Theorem I.1 and Theorem II.6.

B) Let $\{ U_\gamma \mid \gamma \in \Gamma \}$ be a locally finite open collection in a space R with $\dim R \leq n$ and $\{ F_\gamma \mid \gamma \in \Gamma \}$ a closed collection such that $F_\gamma \subset U_\gamma$. Then there exist open collections $\{ V_\gamma \mid \gamma \in \Gamma \}$ and $\{ W_\gamma \mid \gamma \in \Gamma \}$ such that

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset W_\gamma \subset U_\gamma , \quad \text{and} \quad \operatorname{ord} \{ W_\gamma - \bar{V}_\gamma \mid \gamma \in \Gamma \} \leq n .$$

Proof. We denote the binary covering $\{ U_\gamma , R - F_\gamma \}$ by $\mathfrak{U}_\gamma$. Then, since $\{ U_\gamma \mid \gamma \in \Gamma \}$ is locally finite, $\wedge \{ \mathfrak{U}_\gamma \mid \gamma \in \Gamma \}$ is a locally finite open covering. Hence we can find, by Theorem II.6, a locally finite open covering $\mathfrak{N} = \{ N_\delta \mid \delta \in \Delta \}$

such that

$$N < \wedge \{ U_\gamma \mid \gamma \in \Gamma \} \text{ , and ord } N \leqq n + 1 \text{ ,}$$

and such that every N_δ intersects at most finitely many members of $\{ F_\gamma \mid \gamma \in \Gamma \}$ (actually $N < \wedge \{ U_\gamma \mid \gamma \in \Gamma \}$ and the local finiteness of $\{ U_\gamma \}$ assures the property of N_δ). By I.1 A) we construct a closed covering $\{ K_\delta \mid \delta \in \Delta \}$ with $K_\delta \subset N_\delta$. To each N_δ and every F_γ intersecting N_δ , we assign an open set $N_\delta(F_\gamma)$ such that

$$(1) \qquad K_\delta \subset N_\delta(F_\gamma) \subset \overline{N_\delta(F_\gamma)} \subset N_\delta$$

and such that

$$(2) \qquad N_\delta(F_\gamma) \subset \overline{N_\delta(F_\gamma)} \subset N_\delta(F_{\gamma'})$$

or the inverse inclusion relation holds for $\gamma \neq \gamma'$. This is possible because there are at most finitely many F_γ's intersecting N_δ . Now, let

$$V_\gamma = \cup \{ N_\delta(F_\gamma) \mid \delta \in \Delta \} \text{ ;}$$

then from $N < U_\gamma$ it follows that

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset U_\gamma \text{ ,}$$

because for every $N_\delta(F_\gamma)$ constituting V_γ , $N_\delta \cap F_\gamma \neq \emptyset$ which implies $\overline{N_\delta(F_\gamma)} \subset N_\delta \subset U_\gamma$, and thus $\bar{V}_\gamma \subset U_\gamma$ follow since V_γ is a locally finite sum of $N_\delta(F_\gamma)$'s . On the other hand $F_\gamma \subset V_\gamma$ easily follows from $K_\delta \cap F_\gamma \subset N_\delta(F_\gamma)$. To show

$$\text{ord } \{ \bar{V}_\gamma - V_\gamma \mid \gamma \in \Gamma \} \leqq n$$

we assume the contrary, i.e.

$$\bigcap_{i=1}^{n+1} (\bar{V}_{\gamma_i} - V_{\gamma_i}) \neq \emptyset$$

for distinct γ_i , $i = 1,\ldots,n+1$. Then since each V_γ is a locally finite sum of $N_\delta(F_\gamma)$'s , there must be some $N_{\delta_i}(F_{\gamma_i})$, $i = 1,\ldots,n+1$ such that

$$(3) \qquad \bigcap_{i=1}^{n+1} (\overline{N_{\delta_i}(F_{\gamma_i})} - N_{\delta_i}(F_{\gamma_i})) \neq \emptyset \text{ .}$$

If $\delta_i = \delta_j$ and $i \neq j$, then from (2) we obtain

$$(\overline{N_{\delta_i}(F_{\gamma_i})} - N_{\delta_i}(F_{\gamma_i})) \cap (\overline{N_{\delta_j}(F_{\gamma_j})} - N_{\delta_j}(F_{\gamma_j})) = \emptyset$$

contradicting (3).

Therefore (3) implies that δ_i , $i = 1,\ldots,n+1$ are distinct from each other. Let

$$(4) \qquad p \in \bigcap_{i=1}^{n+1} (\overline{N_{\delta_i}(F_{\gamma_i})} - N_{\delta_i}(F_{\gamma_i}));$$

then from (1) it follows that

(5) $$p \notin \bigcup_{i=1}^{n+1} K_{\delta_i} .$$

Since $\{K_\delta \mid \delta \in \Delta\}$ covers R, there exists K_δ containing p. Then (5) implies $\delta \neq \delta_i$, $i = 1,\dots,n+1$. On the other hand, by (1) and (4) we have $p \in \bigcap_{i=1}^{n+1} N_{\delta_i}$. Hence

$$p \in K_\delta \cap [\bigcap_{i=1}^{n+1} N_{\delta_i}] \subset N_\delta \cap [\bigcap_{i=1}^{n+1} N_{\delta_i}] ,$$

which contradicts that $\operatorname{ord} N \leqq n+1$. Thus we can conclude

(6) $$\operatorname{ord} \{\bar V_\gamma - V_\gamma \mid \gamma \in \Gamma\} \leqq n .$$

Since $\{U_\gamma \mid \gamma \in \Gamma\}$ is locally finite, $\wedge \{U'_\gamma \mid \gamma \in \Gamma\}$ is a locally finite open covering, where $U'_\gamma = \{U_\gamma, R - B(V_\gamma)\}$. Hence there is an open covering M such that $M < \wedge \{U'_\gamma \mid \gamma \in \Gamma\}$.

Since, by virtue of the local finiteness of $\{U_\gamma\}$, $\{\bar V_\gamma - V_\gamma \mid \gamma \in \Gamma\}$ is a locally finite collection satisfying (6), we can construct M so that for each point p of R, $S(p,M)$ intersects at most n members of $\{\bar V_\gamma - V_\gamma \mid \gamma \in \Gamma\}$. Now, it is easy to see that putting

$$W_\gamma = V_\gamma \cup S(B(V_\gamma), M)$$

we get the desired open sets W_γ, because $W_\gamma - \bar V_\gamma \subset S(B(V_\gamma), M)$.

C) Let $\{U_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\dots$ be locally finite open collections in a space R with $\dim R \leqq n$, and $\{F_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\dots$ closed collections such that $F_\gamma \subset U_\gamma$. Then there exist open sets V_γ for $\gamma \in \bigcup_{i=1}^\infty \Gamma_i$ satisfying

$$F_\gamma \subset V_\gamma \subset \bar V_\gamma \subset U_\gamma , \quad \text{and} \quad \operatorname{ord} \{\bar V_\gamma - V_\gamma \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\} \leqq n .$$

Proof. By use of B) we define open sets

V_γ^1, W_γ^1 for every $\gamma \in \Gamma_1$,

V_γ^2, W_γ^2 for every $\gamma \in \Gamma_1 \cup \Gamma_2$,

....

V_γ^j, W_γ^j for every $\gamma \in \Gamma_1 \cup \dots \cup \Gamma_j$,

....

in such a way that

$$F_\gamma \subset \bar V_\gamma^j \subset W_\gamma^j \subset U_\gamma ,$$

$$\bar V_\gamma^j \subset V_\gamma^{j+1} , \qquad \bar W_\gamma^{j+1} \subset W_\gamma^j ,$$

$$\operatorname{ord} \{W_\gamma^j - \bar V_\gamma^j \mid \gamma \in \Gamma_1 \cup \dots \cup \Gamma_j\} \leqq n .$$

It can be seen without difficulty that $V_\gamma = \bigcup_{j=i}^{\infty} V_\gamma^j$ for $\gamma \in \Gamma_i$ are the desired open sets, because $\bar{V}_\gamma - V_\gamma \subset \bigcap \{ W_\gamma^j - \bar{V}_\gamma^j \mid j = i, i+1, \ldots \}$.

D) If there exists a σ-locally finite open basis V of a space R such that $\operatorname{ord} B(V) \leqq n$ $(n \geqq 0)$, then $\operatorname{Ind} R \leqq n$.

Proof. The proof will be carried out by use of the induction on n. In case $n = 0$ this proposition is implied by I E).

In case $n > 0$ let $V = \{ V_\gamma \mid \gamma \in \Gamma \}$. Then

$$V_\gamma = \{ B(V_\gamma) \cap V_{\gamma'} \mid \gamma' \in \Gamma \}$$

is a σ-locally finite open basis of $B(V_\gamma)$ such that

$$\operatorname{ord} \{ B_{B(V_\gamma)}(V) \mid V \in V_\gamma \} \leqq n - 1 .$$

Hence by use of the induction hypothesis we have $\operatorname{Ind} B(V_\gamma) \leqq n - 1$.
Now Theorem II.2 implies the assertion.

Theorem II. 7 (Coincidence Theorem) [7]. *For every space* R, $\dim R = \operatorname{Ind} R$.

Proof. Let $\operatorname{Ind} R \leqq n$; then by the decomposition theorem we can decompose R as $R = \bigcup_{i=1}^{n+1} A_i$, $\operatorname{Ind} A_i \leqq 0$.
Let $U = \{ U_j \mid j = 1, \ldots, k \}$ be a given open covering of R. We consider A_i for a fixed i. Then one can easily see that there exists an open covering $\{ V_j \mid j = 1, \ldots, k \}$ of A_i such that

(1) $\operatorname{ord} \{ V_j \mid j = 1, \ldots, k \} \leqq 1$, $V_j \subset U_j$.

For that purpose we select a covering $\{ W_j \mid j = 1, \ldots, k \}$ of A_i by open and closed sets of A_i such that $W_j \subset U_j$. Putting

$$V_j = W_j - \bigcup_{k=1}^{j-1} W_k ,$$

we get the desired covering $\{ V_j \mid j = 1, \ldots, k \}$ of A_i. For every point $x \in V_j$ we determine $\varepsilon(x) > 0$ such that

$$S_{\varepsilon(x)}(x) \cap A_i \subset V_j , \quad \text{and} \quad S_{\varepsilon(x)}(x) \subset U_j .$$

Let $W_j = \bigcup \{ S_{\frac{1}{2}\varepsilon(x)}(x) \mid x \in V_j \}$. Then $W_i = \{ W_j \mid j = 1, \ldots, k \}$ is an open collection which covers A_i.

---

[7] This important theorem was first proved for general metric spaces by M. Katětov [2] and K. Morita [4] independently. C. H. Dowker and W. Hurewicz [1] gave another proof.

We can easily see that $W_j \subset U_j$ and from (1) it follows that $\operatorname{ord} \mathcal{W}_i \leqq 1$. Hence $\mathcal{W} = \bigcup_{i=1}^{n+1} \mathcal{W}_i$ is an open refinement of $\mathcal{U}$ and has order $\leqq n+1$. Therefore $\dim R \leqq n$.

Conversely if $\dim R \leqq n$, then we choose, by use of the corollary to Theorem I.1, a sequence $\mathcal{U}_i = \{U_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\ldots$, of locally finite open coverings such that $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$. Then by I.1 A) we construct closed coverings $\mathcal{F}_i = \{F_\gamma \mid \gamma \in \Gamma_i\}$, $i = 1,2,\ldots$, satisfying $F_\gamma \subset U_\gamma$.

Now, by virtue of C) we obtain open sets

$$V_\gamma \quad \text{for} \quad \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i$$

such that

$$F_\gamma \subset V_\gamma \subset \bar{V}_\gamma \subset U_\gamma, \quad \text{and } \operatorname{ord} \{\bar{V}_\gamma - V_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\} \leqq n .$$

Since $\mathcal{V} = \{V_\gamma \mid \gamma \in \bigcup_{i=1}^{\infty} \Gamma_i\}$ is clearly a σ-locally finite open basis, by D) we can conclude $\operatorname{Ind} R \leqq n$. Thus the theorem is established. In view of this theorem we are entitled to use notations $\operatorname{Ind} R$ and $\dim R$ indiscriminately for every (metric) space R.

II. 6. Some theorems characterizing dimension

We can intuitively see that the dimensions of Euclidean spaces E^n, $n = 1,2,3$, are characterized by the following property:

Let $\{U_i \mid i = 1,\ldots,k\}$ and $\{F_i \mid i = 1,\ldots,k\}$ be an open collection and a closed collection in E^n respectively such that $F_i \subset U_i$; then there exists an open collection $\{V_i \mid i = 1,\ldots,k\}$ such that

$$F_i \subset V_i \subset U_i, \quad \text{and} \quad \operatorname{ord} \{B(V_i) \mid i = 1,\ldots,k\} \leqq n .$$

The possibility of such a characterization for general metric spaces is intimated by 5 B), C) too. The purpose of this section is to carry out an investigation along this idea.

Theorem II. 8. *A space R has dimension $\leqq n$ if and only if for every open collection $\{U_i \mid i = 1,\ldots,k\}$ and closed collection $\{F_i \mid i = 1,\ldots,k\}$ with $F_i \subset U_i$, there exists an open collection $\{V_i \mid i = 1,\ldots,k\}$ such that*

$$F_i \subset V_i \subset U_i,\ i = 1,\ldots,k, \text{ and } \operatorname{ord} \{B(V_i) \mid i = 1,\ldots,k\} \leqq n .$$

Proof. The "only if" part is implied by 5 B). To show the "if" part we consider a given finite open covering $\{U_i \mid i = 1,\ldots,k\}$. Then by I.1 A) there exists a closed covering $\{F_i \mid i = 1,\ldots,k\}$ with $F_i \subset U_i$.

Let $\{V_i \mid i = 1,\ldots,k\}$ be an open covering such that

(1) $\qquad F_i \subset V_i \subset \bar{V}_i \subset U_i$, and $\operatorname{ord}\{B(V_i) \mid i = 1,\ldots,k\} \leq n$.

(Because of the normality of R we can choose V_i satisfying $\bar{V}_i \subset U_i$ besides the condition of this theorem.)

Let $G_i = \bar{V}_i - \bigcup_{j=1}^{i-1} V_j$; then $\operatorname{ord}\{G_i \mid i = 1,\ldots,k\} \leq n + 1$. For, if p were a point such that

$$p \in G_{i_1} \cap G_{i_2} \cap \ldots \cap G_{i_{n+2}} ,$$

where $i_1 < i_2 < \ldots. < i_{n+2}$, then we had

$$p \in (\bar{V}_{i_1} - V_{i_1}) \cap (\bar{V}_{i_2} - V_{i_2}) \cap \ldots \cap (\bar{V}_{i_{n+1}} - V_{i_{n+1}})$$

contradicting (1). Hence $\{G_i \mid i = 1,\ldots,k\}$ is a closed covering with order $\leq n+1$ and $G_i \subset U_i$.

Therefore we can choose an open covering $\mathcal{U}$ such that for every point p of R , $S(p,\mathcal{U})$ intersects at most $n + 1$ members of $\{G_i \mid i = 1,\ldots,k\}$ and such that

(2) $\qquad \mathcal{U} < \bigwedge_{i=1}^{k} \mathcal{U}_i ,$

where $\mathcal{U}_i = \{U_i , R - G_i\}$. We let

(3) $\qquad W_i = S(G_i,\mathcal{U})$.

Then, from (2) it follows that $W_i \subset U_i$, $i = 1,\ldots,k$. Let $\mathcal{W} = \{W_i \mid i = 1,\ldots,k\}$; then from (3) it follows that $\mathcal{W}$ is an open covering with $\operatorname{ord} \mathcal{W} \leq n + 1$. Since $\{U_i \mid i = 1,\ldots,k\}$ is a given finite open covering of R , this proves $\dim R \leq n$.

Corollary [8] . *A space R has dimension $\leq n$ if and only if for every open collection $\{U_i \mid i = 1,\ldots,n + 1\}$ and closed collection $\{F_i \mid i = 1,\ldots,n + 1\}$ with $F_i \subset U_i$, there exists an open collection $\{V_i \mid i = 1,\ldots,n + 1\}$ such that*

$$F_i \subset V_i \subset U_i , \quad \text{and} \quad \bigcap_{i=1}^{n+1} B(V_i) = \emptyset .$$

Proof. Since the "only if" part is a direct consequence of the theorem, we shall prove only the "if" part. Let $\{U_i \mid i = 1,\ldots,k\}$ be a given finite open collection and $\{F_i \mid i = 1,\ldots,k\}$ a given finite closed collection such that $F_i \subset U_i$. Then we number all the combinations $C = \{i_1,\ldots,i_{n+1}\}$ of $n + 1$ numbers from

[8] This theorem was first proved for separable metric spaces by S. Eilenberg and E. Otto [1] and extended by E. Hemmingsen [1] and K. Morita [1] to normal spaces.

$\{1,\dots,k\}$ as $C_1, C_2, \dots, C_m$, where $m = \binom{k}{n+1}$. We can define open sets V_i^1 and W_i^1 for $i \in C_1$ such that

$$F_i \subset V_i^1 \subset \bar{V}_i^1 \subset W_i^1 \subset U_i\,, \quad \text{and} \quad \cap\{W_i^1 - \bar{V}_i^1 \mid i \in C_1\} = \emptyset\,.$$

Since C_1 consists of just $n+1$ numbers, we can choose V_i^1 to satisfy

$$F_i \subset V_i^1 \subset \bar{V}_i^1 \subset U_i \quad \text{and} \quad \cap\{B(V_i^1) \mid i \in C_1\} = \emptyset\,.$$

Then we obtain W_i^1 by use of the method in the proof of 5 B) or more precisely by the process which was applied there to construct W_γ from V_γ.

Suppose that open sets V_i^j and W_i^j, $i \in C_j$ have been defined for every $j < \ell$. Then we define open sets V_i^ℓ and W_i^ℓ for $i \in C_\ell$ so that

$$\text{if} \quad i \notin \bigcup_{j=1}^{\ell-1} C_j\,, \quad \text{then} \quad F_i \subset V_i^\ell \subset \bar{V}_i^\ell \subset W_i^\ell \subset U_i\,,$$

$$\text{if} \quad i \in \bigcup_{j=1}^{\ell-1} C_j\,, \quad \text{then} \quad F_i \subset V_i^j \subset \overline{V_i^j} \subset V_i^\ell \subset \overline{V_i^\ell} \subset W_i^\ell \subset W_i^j \subset U_i$$

for every j for which $1 \leq j \leq \ell - 1$, $i \in C_j$, and so that

$$(1) \qquad \cap\{W_i^\ell - \overline{V_i^\ell} \mid i \in C_\ell\} = \emptyset\,.$$

We can construct V_i^ℓ and W_i^ℓ by a similar application of the proof in 5 B) as in the construction of V_i^1 and W_i^1. Now, let $V_i' = \cup\{V_i^\ell \mid i \in C_\ell\}$. (Note that ℓ denotes a variable in the above formula.) Select open sets V_i such that $\overline{V_i'} \subset V_i \subset \cap\ \{W_i^\ell \mid i \in C_\ell\}$. Then one can easily see that $F_i \subset V_i \subset U_i$, $i = 1,\dots,k$, and $B(V_i) \subset W_i^\ell - \overline{V_i^\ell}$ for every ℓ satisfying $i \in C_\ell$. Hence for any combination C_ℓ of $n+1$ numbers from $\{1,\dots,k\}$

$$\cap\{B(V_i) \mid i \in C_\ell\} \subset \cap\{W_i^\ell - \overline{V_i^\ell} \mid i \in C_\ell\} = \emptyset$$

follows from (1). This means $\operatorname{ord}\{B(V_i) \mid i = 1,\dots,k\} \leq n$. Therefore by the theorem we conclude $\dim R \leq n$.

Theorem II. 9. *A space R has dimension $\leq n$ if and only if there exists a σ-locally finite open basis V such that* $\operatorname{ord} B(V) \leq n$.

Proof. The "if" part follows from 5 D) combined with Theorem II.7. To prove the "only if" part, we construct a σ-locally finite open basis $U = \bigcup_{i=1}^\infty U_i$ of R using Theorem I.3. We can suppose without loss of generality that each $U_i = \{U_\gamma \mid \gamma \in \Gamma_i\}$ is a locally finite open covering of R such that $\lim_{i\to\infty} \operatorname{mesh} U_i = 0$. Then by I.1 A) there exists a locally finite open covering

$\mathcal{W}_i = \{W_\gamma \mid \gamma \in \Gamma_i\}$ such that $\bar{W}_\gamma \subset U_\gamma$. By 5 C) we can construct open sets V_γ for $\gamma \in \bigcup_{i=1}^\infty \Gamma_i$ satisfying

$$\bar{W}_\gamma \subset V_\gamma \subset U_\gamma, \quad \text{and} \quad \operatorname{ord}\{B(V_\gamma) \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\} \leqq n.$$

Thus we get a σ-locally finite open basis $\mathcal{V} = \{V_\gamma \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\}$ with $\operatorname{ord} B(\mathcal{V}) \leqq n$.

Theorem II. 10 [9]. *Let B be a subset of a space R and let $\dim B = n$. Then there exists a G_δ-set A of R such that $B \subset A$, $\dim A = n$.*

Proof. In case $n = 0$, as in the proof of Theorem II.9, we choose a σ-locally finite open basis $\mathcal{U} = \{U_\gamma \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\}$ and a σ-locally finite open collection $\mathcal{W} = \{W_\gamma \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\}$ such that $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$ and $\bar{W}_\gamma \subset U_\gamma$.
Since $\bar{W}_\gamma$ and $R - U_\gamma$ are disjoint closed sets of R, in the same way as we constructed M for F and G in the proof of I C), we can construct an open set V_γ for $\bar{W}_\gamma$ and $R - U_\gamma$ such that $\bar{W}_\gamma \subset V_\gamma \subset U_\gamma$, and $B(V_\gamma) \cap B = \emptyset$. Let $A = R - \bigcup\{B(V_\gamma) \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\}$; then A is a G_δ-set containing B because each $\bigcup\{B(V_\gamma) \mid \gamma \in \Gamma_i\}$ for a fixed i is a closed set by virtue of the local finiteness of $\{U_\gamma \mid \gamma \in \Gamma_i\}$. Since for any γ $B_A(V_\gamma \cap A) = \emptyset$, $\mathcal{V} = \{V_\gamma \cap A \mid \gamma \in \bigcup_{i=1}^\infty \Gamma_i\}$ is a σ-locally finite open basis of A which satisfies the condition of Theorem II.9 for $n = 0$. Hence $\dim A \leqq 0$, which combined with Theorem II.3 implies $\dim A = 0$.
In case $n > 0$, by use of the decomposition theorem we can decompose B into $n+1$ 0-dimensional subsets B_i, $i = 1,\dots,n+1$. By virtue of the preceding result we can find G_δ-sets A_i, $i = 1,\dots,n+1$ such that $B_i \subset A_i$, and $\dim A_i \leqq 0$. Let $A = \bigcup_{i=1}^{n+1} A_i$; then again by the decomposition theorem and Theorem II.3 A is the desired G_δ-set.

II. 7. The rank of a covering

As implied by Theorem I.3, the σ-local finiteness of open bases of a metric space is important. In fact we cannot delete the σ-local finiteness from Theorem II.2 or Theorem II.9.
We can, however, establish a simple characterization theorem of dimension in terms of open bases without σ-local finiteness or any similar property. To this purpose we shall introduce a new notion "rank of a collection". Let us begin with some lemmas.

---

[9] This theorem was first proved for separable metric spaces by L. Tumarkin [2].

A) Let S be a space of $\dim S \leqq 0$. Then there are locally finite open coverings $\mathcal{U}_i = \{ U(\alpha_1,\ldots,\alpha_i) \mid \alpha_1,\ldots,\alpha_i \in A \}$, $i = 1,2,\ldots$ of S such that $U(\alpha_1,\ldots,\alpha_i) \supset U(\alpha_1,\ldots,\alpha_i, \alpha_{i+1})$, $U(\alpha_1,\ldots,\alpha_i) \cap U(\alpha_1,\ldots,\alpha_i') = \emptyset$ if $\alpha_i \neq \alpha_i'$, and $\lim_{i\to\infty} \operatorname{mesh} \mathcal{U}_i = 0$.

Proof. Let $\mathcal{V}_i$ be a locally finite open covering of S such that $\operatorname{mesh} \mathcal{V}_i < 1/i$. Find, by use of Theorem II.6, a locally finite open refinement $\mathcal{V}_i'$ of $\mathcal{V}_i$ satisfying $\operatorname{ord} \mathcal{V}_i' \leqq 1$. Put $\mathcal{U}_i = \mathcal{V}_1' \wedge \ldots \wedge \mathcal{V}_i'$. Then $\mathcal{U}_i$ is a locally finite open covering of order $\leqq 1$ and with mesh $\leqq 1/i$. Note that each (non-empty) element of $\mathcal{U}_{i+1}$ is contained in one and only one element of $\mathcal{U}_i$, and each element of $\mathcal{U}_i$ is a sum of elements of $\mathcal{U}_{i+1}$. Thus by choosing appropriate subindices and adding as many empty sets as necessary, we can let $\{ \mathcal{U}_i \}$ satisfy the desired conditions.

B) Let S be a subspace of R with $\dim S \leqq 0$. Assume that $\mathcal{U}_i$, $i = 1,2,\ldots$ are open coverings of S satisfying the conditions of A). Then we can extend $\mathcal{U}_i$ to a collection $\mathcal{W}_i$ of open sets in R which satisfies the same conditions as $\mathcal{U}_i$, except that $\mathcal{W}_i$ covers only S (but not necessarily R) and $\mathcal{W}_i$ is not necessarily locally finite on R.

Proof. Let i be fixed. Then to each $x \in U(\alpha_1,\ldots,\alpha_i)$ we assign $\varepsilon_i(x) > 0$ such that $\varepsilon_i(x) < 1/i$ and $S_{\varepsilon_i(x)}(x) \cap S \subset U(\alpha_1,\ldots,\alpha_i)$. Now put

$$W(\alpha_1,\ldots,\alpha_i) = \bigcup \{ S_{\varepsilon_i(x)/2}(x) \mid x \in U(\alpha_1,\ldots,\alpha_i) \} .$$

Then $\mathcal{W}_i = \{ W(\alpha_1,\ldots,\alpha_i) \mid \alpha_1,\ldots,\alpha_i \in A \}$ satisfies the desired conditions if we choose $\{ \varepsilon_i(x) \}$ satisfying $\varepsilon_i(x) \geqq \varepsilon_{i+1}(x)$.

Definition II. 1. *Two subsets C and D of a space R are called independent if $C \not\subset D$ and $D \not\subset C$. A finite number of subsets $C_1,\ldots,C_k$ are called independent if any two of them are independent.*

Definition II. 2. *Let $\mathcal{U}$ be a collection in a space R and p a point of R. Then we mean by the rank of $\mathcal{U}$ at p the largest integer n such that there are independent n members of $\mathcal{U}$ containing p, and denote it by* $\operatorname{rank}_p \mathcal{U}$. *If there exist arbitrarily many independent members of $\mathcal{U}$ containing p, then* $\operatorname{rank}_p \mathcal{U} = +\infty$.
We mean by the rank of $\mathcal{U}$ the supremum of $\operatorname{rank}_p \mathcal{U}$ *for p ranging through R and denote it by* $\operatorname{rank} \mathcal{U}$, *i.e.* $\operatorname{rank} \mathcal{U} = \sup \{ \operatorname{rank}_p \mathcal{U} \mid p \in R \}$.

In view of this definition we clearly see $\operatorname{rank}_p \mathcal{U} \leqq \operatorname{ord}_p \mathcal{U}$ for any point p and collection $\mathcal{U}$, and accordingly $\operatorname{rank} \mathcal{U} \leqq \operatorname{ord} \mathcal{U}$.

Theorem II. 11 [10]. *A space* R *has dimension* $\leqq n$, *if and only if it has an open basis* $\mathfrak{U}$ *with* rank $\mathfrak{U} \leqq n+1$.

Proof. To begin with let us prove the "if" part by induction. Let $n = 0$, i.e. we suppose $\mathfrak{U}$ is an open basis with rank $\mathfrak{U} \leqq 1$. Let F and G be given disjoint closed sets of R . Then we put

$$U = \bigcup \{ U' \mid U' \in \mathfrak{U} , U' \cap F \neq \emptyset , U' \cap G = \emptyset \} .$$

Since $\mathfrak{U}$ is an open basis of R , U is an open set satisfying $F \subset U \subset R - G$. If $p \notin U$, then there exists $U' \in \mathfrak{U}$ such that $p \in U' \subset R - F$. If we assume $U' \cap U \neq 0$, then $U' \cap U'' = \emptyset$ for some $U'' \in \mathfrak{U}$ such that $U'' \cap F \neq \emptyset$ and $U'' \subset U$. Since U' and U'' are clearly independent, we reach a contradiction to rank $\mathfrak{U} \leqq 1$. Hence $U' \cap U = \emptyset$, which means that the open set U is closed in R . Thus we conclude $\dim R \leqq 0$.

Suppose we have proved that the existence of an open basis with rank $\leqq n$ implies $\dim R \leqq n - 1$. Now we let $\mathfrak{U}$ be an open basis with rank $\leqq n + 1$. Let F and G be given disjoint closed sets of R . Then we define an open set U by

$$U = \bigcup \{ U' \mid U' \in \mathfrak{U} , U' \cap F \neq \emptyset . U' \cap G = \emptyset \} .$$

U clearly satisfies $F \subset U \subset R - G$. We assert that $\mathfrak{U}' = \{ U' \mid U' \in \mathfrak{U} , U' \cap F = \emptyset \}$ restricted to $B(U)$ is an open basis of $B(U)$ satisfying rank $\mathfrak{U}' \leqq n$.

It is evident that $\mathfrak{U}'$ is an open basis of $B(U)$ if restricted to $B(U)$. Hence all we have to show is that $\text{rank}_p \mathfrak{U}' \leqq n$ for a given point $p \in B(U)$.

Assume the contrary, i.e. we suppose $U_1, \dots, U_{n+1}$ are independent sets of $\mathfrak{U}'$ which contain p . Since $p \in B(U)$, we get

$$q \in U_1 \cap \dots \cap U_{n+1} \cap U \neq \emptyset .$$

Then

$$q \in U_1 \cap \dots \cap U_{n+1} \cap U'$$

for some $U' \in \mathfrak{U}$ such that $U' \cap F \neq \emptyset$. $U' \subset U$. Since $p \in U_i \cap (R - U') \neq \emptyset$, and by the definition of $\mathfrak{U}'$ $U_i \cap F = \emptyset$, $U_1, \dots, U_{n+1}$ and U' are independent. Hence $\text{rank}_q \mathfrak{U} \geqq n + 2$, which contradicts rank $\mathfrak{U} \leqq n + 1$.

Thus we conclude that $\text{rank}_p \mathfrak{U}' \leqq n$ for any point $p \in B(U)$. and hence from the induction hypothesis it follows that $\dim B(U) \leqq n - 1$. Therefore $\dim R \leqq n$ is proved.

To prove the "only if" part we suppose R is a space with $\dim R \leqq n$. Note that if $n = 0$, then the assertion follows directly from A). because $\bigcup_{i=1}^{\infty} \mathfrak{U}_i$ in A) is an open basis of rank $\leqq 1$. In the general case, decompose R . by use of Theorem II.4, as $R = \bigcup_{k=1}^{n+1} A_k$, where $\dim A_k \leqq 0$. Then by B), there is a sequence $\mathfrak{W}_i^k$,

[10] Proved first by J. Nagata [5].

$i = 1,2,\ldots$, of open collections which cover A_k and satisfy rank $\bigcup_{i=1}^{\infty} \mathcal{W}_i^k \leq 1$ and $\lim_{i\to\infty}$ mesh $\mathcal{W}_i^k = 0$. Now it is easy to see that $\mathcal{U} = \bigcup_{i=1}^{\infty} \bigcup_{k=1}^{n+1} \mathcal{W}_i^k$ is an open basis for R with rank $\mathcal{U} \leq n+1$ [11].

II. 8. Normal families

The theory of normal families was first established by W. Hurewicz [1] to deduce systematically fundamental theorems of dimension theory for separable metric spaces and was extended by K. Morita [4] to nonseparable metric spaces. The purpose of this section is to give a short skètch of this theory to re-establish the principal results of dimension theory.

Definition II. 3. *A family N of (metric) spaces is called a normal family if it satisfies.*

i) if $S \subset R$ and $R \in N$, *then* $S \in N$,

ii) if $\{F_\gamma \mid \gamma \in \Gamma\}$ *is a locally countable closed covering of a space* R *such that* $F_\gamma \in N$ *for every* $\gamma \in \Gamma$, *then* $R \in N$.

A) A family N of spaces is a normal family if and only if it satisfies i), iii) if $\{F_i \mid i = 1,2,\ldots\}$ is a closed covering of a space R such that $F_i \in N$, $i = 1,2,\ldots$, then $R \in N$, and iv) if $\{F_\gamma \mid \gamma \in \Gamma\}$ is a locally finite closed covering of a space R such that $F_\gamma \in N$ for every $\gamma \in \Gamma$, then $R \in N$.

Proof. The "only if" part is evident. Conversely, let N be a family satisfying i), iii) and iv). If $\{F_\gamma \mid \gamma \in \Gamma\}$ is a locally countable closed covering of R such that $F_\gamma \in N$, then we can construct a locally finite closed covering $\{G_\delta \mid \delta \in \Delta\}$ such that each G_δ intersects at most countably many elements of $\{F_\gamma \mid \gamma \in \Gamma\}$. It follows from i) and iii) that $G_\delta \in N$, and hence from iv) we obtain $R \in N$. This means that $\{F_\gamma \mid \gamma \in \Gamma\}$ satisfies ii), i.e. N is a normal family.

Definition II. 4. *For any normal family N we define a family N' of spaces by* $N' = \{R \mid$ *for any disjoint closed sets* F *and* G *there exists an open set* U *of* R *such that* $F \subset U \subset R - G$, $B(U) \in N\}$.

[11] Bases with finite rank were extensively studied not only in dimension theory but also in metrization theory by A. V. Arhangelskii [1], A. V. Arhangelskii - V. V. Filippov [1], G. Gruenhage-P. Nyikos [1] and others. For example, it was proved in the last paper that every compact Hausdorff space with an open basis of finite rank is metrizable. It should also be noted that A. V. Arhangelskii [1] proved that a normal space has $\dim \leq n$ if and only if for every finite open covering $\mathcal{U}$ there is an open refinement $\mathcal{V}$ of rank $\leq n+1$.

To re-establish the principal results of dimension theory by use of normal families we need the following four propositions B) - E) on 0-dimensional spaces which have been proved once in Sections 1 and 2 and are easily deduced by some elementary knowledge of general topology.

B) Ind $R \leqq 0$ if and only if there exists a σ-locally finite open basis V of R consisting of open and closed sets.

C) If $S \subset R$ and Ind $R \leqq 0$, then Ind $S \leqq 0$.

D) Let $\{F_i \mid i = 1,2,\dots\}$ be a closed covering of R such that Ind $F_i \leqq 0$, $i = 1,2,\dots$; then Ind $R \leqq 0$.

E) Let $\{F_\gamma \mid \gamma \in \Gamma\}$ be a locally finite closed covering of R such that Ind $F_\gamma \leqq 0$ for every $\gamma \in \Gamma$; then Ind $R \leqq 0$.

By A), C), D) and E) we conclude that

F) All the (metric) spaces with Ind $\leqq 0$ form a normal family.

Theorem II. 12. *Let N be a normal family and R a space. Then $R \in N'$ if and only if there exist two subspaces A and B such that $R = A \cup B$, $A \in N$, and Ind $B \leqq 0$.*

Proof. To see the "only if" part let $R \in N'$. By I.1 A) and the Corollary to Theorem I.1 we can construct σ-locally finite open bases $U = \{U_\gamma \mid \gamma \in \Gamma\}$ and $W = \{W_\gamma \mid \gamma \in \Gamma\}$ of R such that $\bar{W}_\gamma \subset U_\gamma$. Since $R \in N'$, there exist open sets V_γ for $\gamma \in \Gamma$ satisfying $\bar{W}_\gamma \subset V_\gamma \subset U_\gamma$, and $B(V_\gamma) \in N$. Put $A = \cup\{B(V_\gamma) \mid \gamma \in \Gamma\}$ and $B = R - A$. It follows from the conditions iii) and iv) in A) that $A \in N$ because $\{B(V_\gamma) \mid \gamma \in \Gamma\}$ is a σ-locally finite closed collection. Since $\{V_\gamma \mid \gamma \in \Gamma\}$ restricted to B is a σ-locally finite open basis consisting of open and closed subsets of B, using B) we get Ind $B \leqq 0$.

The proof of the "if" part is as follows. Let F and G be disjoint closed sets of R. Since Ind $B \leqq 0$, we can construct an open set M with $F \subset M \subset R - G$, $B(M) \subset A$ in the same way as in the proof of 1 C). Since $A \in N$, from the condition i) for N it follows that $B(M) \in N$. This proves $R \in N'$.

Theorem II. 13. *If N is a normal family, then N' is also a normal family.*

Proof. We assert that N' satisfies i), iii) and iv). Let $S \subset R \in N'$. Then by Theorem II.12 $R = A \cup B$, $A \in N$, and Ind $B \leqq 0$. This implies $S = (A \cap S) \cup (B \cap S)$. Since N is normal, from i) we obtain $A \cap S \in N$. Moreover, from C) we obtain

Ind $(B \cap S) \leq 0$. Thus, using Theorem II.12 once more, we conclude that $S \in N'$. Therefore N' satisfies i).

Let $\{ F_\nu \mid \nu < \tau \}$ be a locally finite closed covering of a space R such that

(1) $\qquad F_\nu \in N'$.

Then

(2) $\qquad G_\nu = F_\nu - \cup \{ F_\mu \mid \mu < \nu \}$

is an F_σ-set, because $\{ F_\mu \mid \mu < \nu \}$ is a locally finite closed collection, and in consequence its sum is a closed set. Since (1) implies $G_\nu \in N'$, by Theorem II.12 we can decompose G_ν as $G_\nu = A_\nu \cup B_\nu$, $A_\nu \in N$, with Ind $B_\nu \leq 0$. By (2) G_ν and G_μ are disjoint if $\nu \neq \mu$, and hence A_ν is an F_σ-set in $A = \cup \{ A_\nu \mid \nu < \tau \}$, and B_ν is also an F_σ-set in $B = \cup \{ B_\nu \mid \nu < \tau \}$. Thus $\{ A_\nu \mid \nu < \tau \}$ is a locally finite covering of A consisting of F_σ-sets A_ν with $A_\nu \in N$.

Since N is a normal family, it follows from i), ii) that $A \in N$. Moreover, from C), D) and E) it follows that Ind $B \leq 0$. Since $R = A \cup B$, by Theorem II.12 we can conclude that $R \in N'$. Therefore N' satisfies iv). In a similar way we can also show that N' satisfies iii). Thus N' is a normal family by A).

Denote by $N^{(-1)}$ the normal family consisting of the empty set alone. Then, let

$$N^{(k+1)} = (N^{(k)})' , \; k = -1,0,1,\dots .$$

Then $N^{(n)}$ is the family consisting of all the spaces with Ind $\leq n$. On the other hand, by Theorem II.13, $N^{(n)}$ is a normal family. Hence we can again deduce Theorem II.3, the sum theorem and the decomposition theorem from i), ii) and Theorem II.12 respectively.

We can apply the theory of normal families to dimension theory in other ways. For example we may let $N^{(0)} = \{ R \mid R$ is a countable sum of closed subspaces R_i , $i = 1,2,\dots$ such that any point p of R has a neighbourhood containing at most countably many points of $R_i \}$. One can easily see that $N^{(0)}$ is a normal family; hence for every positive integer n $N^{(n)}$ is also a normal family. A space R is said to have *rational dimension* $\leq n$ if and only if $R \in N^{(n)}$.

Thus we can apply the theory of normal families to establish another dimension theory on this new concept of dimension, but the details are left to the reader.

More generally, let P be a (not necessarily normal) family of metric spaces. Then we can define $P^{(k)}$, $k = -1,0,1,2,\dots$, in the same way as we did for a normal family. We can also modify Definition II.4 as follows: $'P = \{ R \mid$ for any point $p \in R$ and any neighbourhood U of p there is an open neighbourhood V of p such that $V \subset U$ and $B(V) \in P \}$. Then ${}^{(k)}P$ can be defined in a similar way as $P^{(k)}$. J. de Groot defined *compactness degree* of a separable metric space R (notation: comp R) by comp $R \leq n$ if and only if $R \in {}^{(n)}P$, where P is the family of all compact (metric) spaces. He also defined *deficiency* of a separable metric space R (notation: def R) by def R = the least n such that there is a compact (metric) space R' which contains R as a (topological) subspace satisfying $\dim (R' - R) = n$ and asked the question 'comp $R \leqq n$ if and only if def $R \leqq n$?' This question is still unsolved though it is partially answered in the positive by J. de Groot - T. Nishiura [1]. (The answer is positive e.g. in case of $n = 0$. It is generally true that comp $R \leqq$ def $R \leqq \dim R$.) J. M. Aarts [1] defined *completeness degree* and *complete deficiency* in a similar way, but by use of 'completely metrizable spaces' in place of 'compact spaces' and $P^{(n)}$ in place of ${}^{(n)}P$. Thus he proved that the completeness degree coincides with the complete deficiency for every metric space [12] .

[12] In a similar aspect of dimension theory Yu. Smirnov [9] studied $\dim (\beta X - X)$ for a normal space X .

CHAPTER III

MAPPINGS AND DIMENSION

In the preceding chapter our investigations were chiefly conducted on the basis of the theory of coverings. In this chapter we shall concern ourselves with mappings, another powerful tool for dimension theory. In Section 7 we shall study the theory of uniformly 0-dimensional mappings which has been recently developed by M. Katětov while in the other sections the main object will be the extension of classical theorems to general metric spaces.

III. 1. Stable value

Definition III. 1. *Let f be a continuous mapping of a space R into a space S A point q of $f(R)$ is called an unstable value of f if for every $\varepsilon > 0$ there exists a continuous mapping g of R into S such that $\rho(f(p), g(p)) < \varepsilon$ for every $p \in R$, $g(R) \subset S - \{q\}$, where we denote by ρ the metric of S. A point q of $f(R)$ is called a stable value if it is not unstable.*

Example III.1. We consider a continuous mapping $f(x) = x^2$ of the 1-dimensional Euclidean space E^1 into itself. Then all the values $y > 0$ are stable, while 0 is unstable. On the other hand the mapping $F(x) = (x, f(x))$ of E^1 in E^2 has only unstable values.

A) Let f be a continuous mapping of a (metric) space R into the $n+1$-dimensional cube $I^{n+1} = \{(x_1, \dots, x_{n+1}) \mid |x_i| \leqq 1,\ i = 1, \dots, n+1\}$. If $\dim R \leqq n$, then all values of f are unstable.

Proof. Let $f(p)$ have the coordinates $f_1(p),\ldots,f_{n+1}(p)$ in I^{n+1}. To begin with, let us show that every point of $B(I^{n+1})$ is unstable. For any given $\varepsilon > 0$, the mapping

$$g(p) = (g_1(p),\ldots,g_{n+1}(p)),\ g_i(p) = (1-\varepsilon)f_i(p),\ i = 1,\ldots,n+1$$

is a mapping whose image contains no boundary point of I^{n+1}. Since we can easily see that $\rho(f(p), g(p)) \leqq \varepsilon\sqrt{n+1}$ for every $p \in R$, our assertion is true.

Now, let us prove that any interior point q of I^{n+1} is unstable. To this end we may assume q to be the origin $(0,\ldots,0)$. Let ε be a given positive number. We put

$$F_i = \{ p \mid f_i(p) \geqq \varepsilon \},\ G_i = \{ p \mid f_i(p) \leqq -\varepsilon \},\ i = 1,\ldots,n+1 .$$

Since F_i is a closed set contained in the open set $R - G_i$, by the corollary of Theorem 11.8 there exist open sets V_i, $i = 1,\ldots,n+1$ such that

(1) $$F_i \subset V_i \subset R - G_i \quad \text{and} \quad \bigcap_{i=1}^{n+1} B(V_i) = \emptyset .$$

By repeated application of the corollary to Theorem 1.6, we can construct a continuous real valued function φ_i such that $|\varphi_i| \leqq \varepsilon$, $\{ p \mid \varphi_i(p) = 0\} = B(V_i)$, $\{ p \mid \varphi_i(p) = \varepsilon \} = F_i$, $\{ p \mid \varphi_i(p) = -\varepsilon \} = G_i$. Defining g_i by $g_i(p) = f_i(p)$ for $p \in F_i \cup G_i$, and $g_i(p) = \varphi_i(p)$ for $p \in R - F_i \cup G_i$, we obtain a continuous function $g_i(p)$, which, as easily seen, satisfies

(2) $$| f_i(p) - g_i(p) | < 2\varepsilon \quad \text{for every } p \in R, \text{ and}$$

(3) $$\{ p \mid g_i(p) = 0 \} = B(V_i) .$$

Let $g(p) = (g_1(p),\ldots,g_{n+1}(p))$; then g is a continuous mapping of R in I^{n+} and satisfies $\rho(f(p), g(p)) < 2\varepsilon\sqrt{n+1}$ for every $p \in R$ by virtue of (2). From (1) and (3) we obtain $(0,\ldots,0) \notin g(R)$. This means that the origin $(0,\ldots,0)$ is an unstable value.

B) If all values of every continuous mapping f of a space R into I^{n+1} are unstable, then $\dim R \leqq n$.

Proof. Let $\{ U_i \mid i = 1,\ldots,n+1 \}$ be an open collection and $\{ F_i \mid i = 1,\ldots,n+1 \}$ a closed collection such that $F_i \subset U_i$, $i = 1,\ldots,n+1$. By Theorem 1.6 we construct continuous functions f_i, $i = 1,\ldots,k$ defined over R such that

(1) $$\begin{cases} -1 \leqq f_i \leqq 1 , \\ f_i(F_i) = -1 , \\ f_i(R - U_i) = 1 , \quad i = 1,\ldots,n+1 . \end{cases}$$

Since $f = (f_1, \dots, f_{n+1})$ is a continuous mapping of R into I^{n+1}, the origin $q = (0, \dots, 0)$ of I^{n+1} is unstable. This means that there exists a continuous mapping $g = (g_1, \dots, g_{n+1})$ of R into I^{n+1} such that

(2) $q \notin g(R)$, and

(3) $\rho(f(p), g(p)) < \frac{1}{2}$ for every $p \in R$.

Then we define open sets V_i by $V_i = \{ p \mid g_i(p) < 0 \}$. From (1) and (3) it follows that $F_i \subset V_i \subset U_i$ and $B(V_i) \subset \{ p \mid g_i(p) = 0 \}$. Since (2) implies $\bigcap_{i=1}^{n+1} \{ p \mid g_i(p) = 0 \} = \emptyset$, we get $\bigcap_{i=1}^{n+1} B(V_i) = \emptyset$. Hence from the corollary to Theorem II.8 we obtain $\dim R \leqq n$.

By combining B) with A) we get

Theorem III. 1. *A space R has dimension $\leq n$ if and only if all values of every continuous mapping of R into I^{n+1} are unstable.*

III. 2. Extensions of mappings

A) Let f be a continuous mapping of a space R into I^{n+1} such that the center q of I^{n+1} is an unstable value of f. Then there exists a continuous mapping g of R into I^{n+1} such that $q \notin g(R)$ and $g(p) = f(p)$ for every point p whose image is contained in $B(I^{n+1})$.

Proof. We may assume the boundary $B(I^{n+1})$ of I^{n+1} to be the n-sphere $S^n = \{ (x_1, \dots, x_{n+1}) \mid x_1^2 + \dots + x_{n+1}^2 = 1 \}$. Since the center q is an unstable value of $f(R)$, there exists a continuous mapping g' of R into I^{n+1} such that, in vectorial notation,

(1) $\rho(f(p), g'(p)) = | f(p) - g'(p) | < \frac{1}{3}$ for every $p \in R$, and $q \notin g'(R)$.

Then, let $F = \{ p \mid | f(p) | \leqq \frac{1}{3} \}$ and $G = \{ p \mid | f(p) | \geqq \frac{2}{3} \}$ to construct by Theorem I.6 a real valued continuous function φ over R such that $\varphi(F) = 0$, $\varphi(G) = 1$, and $0 \leqq \varphi \leqq 1$. Using this function we define a mapping g of R by

$$g(p) = g'(p) \quad \text{for } p \in F,$$
$$g(p) = \varphi(p) f(p) + (1 - \varphi(p)) g'(p) \quad \text{for } p \in R - F \cup G,$$
$$g(p) = f(p) \quad \text{for } p \in G.$$

We can easily see that g is a continuous mapping of R into I^{n+1} and that $g(F) \cup g(G) \subset I^{n+1} - \{ q \}$, and $g(p) = f(p)$ if $f(p) \in S^n (= B(I^{n+1}))$. There only remains to prove

(2) $g(R - F \cup G) \subset I^{n+1} - \{ q \}$.

Let p be a given point of $R - F \cup G$; then $\frac{1}{3} < | f(p) | < \frac{2}{3}$, and hence, from (1) and

$$| g(p) - f(p) | = (1 - \varphi(p)) \, | g'(p) - f(p) | ,$$

we obtain

$$| g(p) | \leq (1 - \varphi(p)) \, | g'(p) - f(p) | + | f(p) | < 1 ,$$
$$| g(p) | \geq | f(p) | - (1 - \varphi(p)) \, | g'(p) - f(p) | > 0 ,$$

which proves (2). Thus g is the desired mapping.

Theorem III. 2. *A space R has dimension $\leq n$ if and only if for every closed set C of R and every continuous mapping f of C into S^n there exists a continuous extension of f over R.*

Proof. To prove the necessity we regard S^n as the boundary of I^{n+1}. Then we can regard f as a mapping of C into I^{n+1}. Hence by Theorem I.7 there exists a continuous mapping f' of R into I^{n+1} such that

(1) $\qquad f'(p) = f(p)$ for $p \in C$.

Since $\dim R \leq n$, by Theorem III.1 the center q of I^{n+1} is an unstable value of f'. Hence in view of A) we can construct a continuous mapping g of R into I^{n+1} such that

(2) $\qquad q \notin g(R)$, and $g(p) = f'(p)$ if $f'(p) \in S^n$.

Since $p \in C$ implies by (1) $f'(p) = f(p) \in S^n$, we obtain $g(p) = f(p)$ if $p \in C$. Let $\varphi(p)$ be the projection of the point $g(p)$ on the boundary S^n of I^{n+1} from the center. Such a projection is possible by virtue of (2). Now it is clear that φ is the desired extension of $f \mid C \to S^n$ over R.

To show the sufficiency it is enough, as Theorem III.1 asserts, to prove that every continuous mapping f of R into I^{n+1} has unstable values only. As seen in the proof of I A), any boundary point of I^{n+1} is unstable, and hence we shall prove only the unstability of the center q of I^{n+1}. For a given $\varepsilon > 0$ we consider a spherical neighbourhood U of q with radius $\frac{1}{2}\varepsilon$ and with boundary S^n. Let $C = f^{-1}(S^n)$; then by the hypothesis there exists a continuous mapping φ of R into S^n such that $\varphi(p) = f(p)$ for $p \in C$.

We define a new continuous mapping g of R into I^{n+1} as follows:

$$g(p) = f(p) \quad \text{if} \quad f(p) \notin U ,$$
$$g(p) = \varphi(p) \quad \text{if} \quad f(p) \in U .$$

Then we can easily see that g is a continuous mapping of R into $I^{n+1} - U$ such that $\rho(f(p), g(p)) < \varepsilon$, for every $p \in R$. This proves the unstability of q. Therefore from Theorem III.1 it follows that $\dim R \leq n$.

Corollary. *Let C be a closed set of a space R. If $\dim (R - C) \leq n$, then every continuous mapping f of C into S^n can be continuously extended over R.*

Proof. By the corollary to Theorem I.7 there is an open set U containing C and a continuous extension g of f over U. We consider an open set V such that $C \subset V \subset \bar{V} \subset U$. Then we denote by g_0 the restriction of g to $\bar{V} - C$. Since $\bar{V} - C$ is a closed subset of $R - C$ with $\dim (R - C) \leqq n$, g_0 can be continuously extended over $R - C$ by Theorem III.2. We denote this continuous extension by h. Letting $\varphi(p) = f(p)$ for $p \in C$, and $\varphi(p) = h(p)$ for $p \in R - C$, we obtain the desired continuous extension φ of f over R.

III. 3. Essential mappings

Definition III. 2. *A continuous mapping f of a space R into a space S is called homotopic to a continuous mapping g of R into S if there exists a continuous mapping $f(p,t)$ of $R \times I$, the topological product of R and the unit segment $I = \{x \mid 0 \leqq x \leqq 1\}$, into S such that $f(p,0) = f(p)$ $f(p,1) = g(p)$.*

Roughly speaking, f is homotopic to g if f can be continuously deformed into g.

Definition III. 3. *A continuous mapping f of a space R into the n-sphere S^n is called essential if any continuous mapping g of R into S^n which is homotopic to f satisfies $g(R) = S^n$. If f is not essential, then it is called inessential.*

Example III. 2. Let f and g be continuous mappings of a space R into S^n with diameter d. If $\rho(f(p), g(p)) < d$ for every $p \in R$, then f is homotopic to g. To see this, for each $p \in R$ we consider the minor arc of the great circle joining $f(p)$ with $g(p)$. For $p \in R$, $0 \leqq t \leqq 1$, we denote by $f(p,t)$ the point dividing this arc in the ratio $t : 1 - t$. Then $f(p,t)$ is a mapping of $R \times I$ into S^n and satisfies the condition of Definition III.2. Hence f is homotopic to g. It follows from this assertion that

A) a continuous mapping f from R into S^n is inessential if and only if it is homotopic to a constant mapping.

B) Let A be a subset of a space R and I the unit segment. If U is an open set of the topological product $R \times I$ such that $U \supset A \times I$, then there exists an open set V of R such that $A \subset V$ and $V \times I \subset U$.

Proof. Let p be a given point of A and $V_i(p)$, $i = 1,2,\dots$ an open neighbourhood basis of p. If we suppose $V_i(p) \times I \not\subset U$, $i = 1,2,\dots$, then we get a sequence $\{(p_i, x_i) \mid i = 1,2,\dots\}$ of points of $R \times I$ such that $(p_i, x_i) \in (V_i(p) \times I) - U$, $i = 1,2,\dots$. Since I is compact, we can choose a converging subsequence $\{x_{i_k} \mid k =$

$= 1,2,\ldots]$ of $\{x_i\}$. Let $x_{i_k} \to x$; then

$$(R\times I) - U \ni (p_{i_k}, x_{i_k}) \to (p,x) \in A\times I ,$$

which contradicts the fact that U is an open set containing $A\times I$. Hence $V(p)\times I \subset U$ for some open neighbourhood $V(p)$ of p . Now for each point p of A we choose such an open neighbourhood $V(p)$, and put $V = \cup\{V(p) \mid p\in A\}$. Then V is the desired open set.

Theorem III. 3 [1]. *Let C be a closed subset of a space R and f and g homotopic continuous mappings of C into S^n . If f admits a continuous extension φ over R , then g also admits a continuous extension ψ over R which is homotopic to φ .*

Proof. Since f and g are homotopic, there exists a mapping $f(p,t)$ of $C\times I$ into S^n such that

$$(1) \qquad \begin{cases} f(p,0) = f(p) \\ f(p,1) = g(p) \ , \end{cases}$$

where we denote by I the unit segment. Let us define a closed set F of $R\times I$ by

$$F = \{(p,0) \mid p\in R\} \cup \{(p,t) \mid p\in C\ ,\ t\in I\} .$$

We define a continuous mapping $\varphi(p,t)$ of F into S^n by

$$(2) \qquad \begin{cases} \varphi(p,0) = \varphi(p) & \text{for } p\in R \\ \varphi(p,t) = f(p,t) & \text{for } p\in C\ ,\ t\in I . \end{cases}$$

By the corollary to Theorem 1.7 we can continuously extend $\varphi(p,t)$ over an open set U containing F in $R\times I$. We denote this extension by $\varphi_1(p,t)$. By B) there exists an open set V of R such that $C\subset V$, $V\times I\subset U$. Using Theorem 1.6 we can construct a real valued continuous function $u(x)$ over R satisfying

(3) $\quad u(C) = 1$,

(4) $\quad u(R-V) = 0$, and $0 \leqq u \leqq 1$.

Now, we set

(5) $\quad \psi(p,t) = \varphi_1(p, tu(p)),\ p\in R\ ,\ t\in I$.

Then ψ is a continuous mapping of $R\times I$ into S^n . For, if $p\in R-V$, then by (4) $u(p) = 0$, and hence $\psi(p,t) = \varphi_1(p,0) = \varphi(p,0)$ is defined for every t . If $p\in V$, then $(p, tu(p))\in V\times I$, and hence $\varphi_1(p,tu(p))$ is defined for every t . Thus $\psi(p,t)$ is defined for every $(p,t)\in R\times I$. We define a continuous mapping ψ of R into S^n by $\psi(p) = \psi(p,1)$.

[1] This theorem is due to K. Borsuk [1].

Let p be a given point of C ; then by (5), (3), (2), (1) $\psi(p) = \psi(p,1) = \varphi_1(p,1) = f(p,1) = g(p)$. Hence ψ is a continuous extension of g over R . On the other hand from (5) and (2) it follows that $\psi(p,0) = \varphi_1(p,0) = \varphi(p)$. Therefore φ and ψ are homotopic.

C) Let f and g be two continuous mappings of a space R into S^n such that $\dim\{p \mid f(p) \neq g(p)\} \leqq n-1$. Then f and g are homotopic.

Proof. Let $D = \{p \mid f(p) \neq g(p)\}$; then D is obviously an open set of R . We define a closed set F of the topological product $R \times I$ of R with the unit segment I by

$$F = R \times \{0\} \cup R \times \{1\} \cup (R-D) \times I \ .$$

Then we define a continuous mapping $f(p,t)$ of F into S^n by

$$f(p,t) = f(p) = g(p) \quad \text{for} \quad p \in R-D \ ,$$
$$f(p,0) = f(p) \ ,$$
$$f(p,1) = g(p) \ .$$

Now, it suffices to show that $f(p,t)$ can be continuously extended over $R \times I$. It follows from the product theorem that $\dim D \times I \leqq n$, because $\dim I \leqq 1$ is clear. Since $(R \times I) - F \subset D \times I$, we get $\dim((R \times I) - F) \leqq n$.

Hence by the corollary to Theorem III.2 we can continuously extend $f(p,t)$ over $R \times I$. Thus f and g are homotopic.

The following is a direct consequence of C).

Theorem III. 4. *If* $\dim R \leqq n-1$, *then all continuous mappings of* R *into* S^n *are homotopic, and hence inessential.*

D) Let a space R be the sum of two closed subsets F and G . Let f and g be continuous mappings of F and G respectively into S^n . If

$$\dim\{p \mid f(p) \neq g(p)\ ,\ p \in F \cap G\} \leqq n-1 \ ,$$

then f can be continuously extended over R .

Proof. The restriction f_0 and g_0 of f and g respectively to $F \cap G$ are homotopic by virtue of C). Since g_0 can be continuously extended to the mapping g over G , by Theorem III.3 there exists a continuous extension f_1 of f_0 over G . Putting $\varphi(p) = f(p)$ for $p \in F$, and $\varphi(p) = f_1(p)$ for $p \in G$, we obtain a desired extension φ of f .

Definition III. 4. *Let* f *and* g *be continuous mappings of a space* R *into a space* S . *Let* A *be a subset of* R ; *then* f *and* g *are called homotopic relative to* A *if there exists a continuous mapping* $f(p,t)$ *of* $R \times I$ *into* S *such that*

$f(p,0) = f(p)$, $f(p,1) = g(p)$, and $f(p,t) = f(p) = g(p)$ for $p \in A$, $t \in I$, where the unit segment is denoted by I .

Definition III. 5. *Let f be a continuous mapping of a space R into the n-dimensional simplex T^n or equivalently into the n-dimensional cube I^n . If any continuous mapping g of R into T^n which is homotopic to f relative to $f^{-1}(S^{n-1})$ satisfies $g(R) = I^n$, then f is called essential, where S^{n-1} denotes the $n-1$-sphere which is regarded as the boundary of T^n in the n-dimensional Euclidean space E^n . If f is not essential, then it is called inessential.*

E) A continuous mapping f of R into I^n is essential if and only if every continuous mapping g of R into I^n which coincides with f on $f^{-1}(S^{n-1})$ satisfies $g(R) = I^n$, where we regard S^{n-1} as the boundary of I^n .

Proof. Suppose f is a continuous mapping of R into I^n . If a continuous mapping g of R into I^n satisfies $f(p) = g(p)$ for every $p \in f^{-1}(S^{n-1})$, then for each point p of R we consider the segment joining $f(p)$ and $g(p)$ which lies in I^n . For $p \in R$, $0 \leq t \leq 1$ we denote by $f(p,t)$ the point dividing this segment in the ratio $t : 1 - t$. Then $f(p,t)$ is a continuous mapping of $R \times I$ into I^n satisfying the condition of Definition III.4 for $S = I^n$, $A = f^{-1}(S^{n-1})$. Hence g is homotopic to f relative to $f^{-1}(S^{n-1})$. Therefore E) is proved.

Theorem III. 5. *A space R has* $\dim \leq n$ *if and only if every continuous mapping of R into I^{n+1} is inessential.*

Proof. Let $\dim R \geq n+1$; then by Theorem III.1 there exists a continuous mapping f of R into I^{n+1} with a stable value $f(p) = p'$. Let us suppose that ε is a positive number for which

(1) $\rho(f(x), g(x)) < \varepsilon$ for every $x \in R$

implies $p' \in g(R)$. If $p' \in B(f(R))$, then we can choose a point

$$q \in S_{\frac{1}{2}\varepsilon}(p') \cap (I^{n+1} - f(R))$$

and consider the projection $\varphi(y)$ of $S_{\frac{1}{2}\varepsilon}(p') \cap f(R)$ on the boundary $B(S_{\frac{1}{2}\varepsilon}(p'))$ from q . We define a continuous mapping g of R into I^{n+1} by

$$g(x) = \varphi f(x) \quad \text{if} \quad f(x) \in S_{\frac{1}{2}\varepsilon}(p') \cap f(R) ,$$
$$g(x) = f(x) \quad \text{if} \quad f(x) \in f(R) - S_{\frac{1}{2}\varepsilon}(p') .$$

Then one can easily see that $p' \notin g(R)$, and $\rho(f(x), g(x)) < \varepsilon$ for every $x \in R$, which contradicts the fact that p' is a stable value. Hence p' must be an interior point of $f(R)$. Thus we can choose a positive number δ such that $\delta < \frac{1}{2}\varepsilon$,

$\overline{S_\delta(p')} \subset f(R)$. We denote by $\psi(y)$ the projection of $f(R) - S_\delta(p')$ on $B(S_\delta(p'))$ from p' and define a continuous mapping h of R onto $\overline{S_\delta(p')}$ by

$$h(x) = \psi f(x) \quad \text{if} \quad f(x) \in f(R) - S_\delta(p') ,$$

$$h(x) = f(x) \quad \text{if} \quad f(x) \in S_\delta(p') .$$

It easily follows from (1) that h is an essential mapping from R into $\overline{S_\delta(p')}$ which is homeomorphic to I^{n+1} .

Conversely, suppose f is an essential mapping of R onto I^{n+1} . Then, by E) and 2 A), the center of I^{n+1} is a stable value of f . Hence from Theorem III.1 it follows that $\dim R \geqq n+1$, which proves this theorem.

Remark. It is easy to see that the converse of Theorem III.4 is also true if $n > 0$. Because if all continuous mappings from R to S^n are inessential, then for every continuous mapping $f\colon R \to I^n$ the composite mapping $gf\colon R \to S^n$ of f with the quotient mapping $g\colon I^n \to S^n$ is inessential. Thus f is inessential.

It is easy to construct examples of inessential mappings while the identity mapping from I^n (S^n) onto itself is essential as proved later on (IV. 3 A)).

III. 4. Some lemmas

In the present section we are going to prove some propositions which will be used in the next section and also later in the book.

A) Let $\{U_\alpha \mid \alpha < \tau\}$ be an open collection in a space R such that $\{U_\beta \mid \beta < \alpha\}$ is locally finite for every $\alpha < \tau$. Let $\{F_\alpha \mid \alpha < \tau\}$ be a closed collection in R such that $F_\alpha \subset U_\alpha$ for all α . If A_n , $n = 1, 2, \ldots$ are subsets of R with $\dim A_n \leqq 0$, then there exists for every $\alpha < \tau$ an open set V_α such that $F_\alpha \subset V_\alpha \subset U_\alpha$ and $\operatorname{ord}_p \{B(V_\alpha) \mid \alpha < \tau\} \leqq n - 1$ for every point $p \in A_n$.

Proof. We shall define, by induction on the ordinal α , V_α satisfying

a)$_\alpha$ $\quad F_\alpha \subset V_\alpha \subset U_\alpha$,

b)$_\alpha$ $\quad \operatorname{ord}_p \{B(V_\beta) \mid \beta \leqq \alpha\} \leqq n - 1$ for every $p \in A_n$.

First, by virtue of the normality of R there exist open sets P and Q such that $P \supset F_0$, $Q \supset R - U_0$ and $\bar{P} \cap \bar{Q} = \emptyset$. Since $\dim A_1 \leqq 0$, there exists an open and closed set N of A_1 satisfying $\bar{P} \cap A_1 \subset N \subset (R - \bar{Q}) \cap A_1$. If we let

(1) $\quad B = N \cup F_0 , \quad C = (A_1 - N) \cup (R - U_0) ,$

then $(\bar{B}\cap C)\cup(B\cap\bar{C})=\emptyset$. Hence by I B) there exists an open set V_0 such that

(2) $$B\subset V_0\subset\bar{V}_0\subset R-C\ .$$

Then from (1) and (2) it follows that $B(\bar{V}_0)\cap A_1=\emptyset$, because N is open and closed in A_1 .

Thus V_0 satisfies $a)_\alpha$ and $b)_\alpha$ for $\alpha=0$. Suppose that V_β has been constructed for every $\beta<\alpha\ (<\tau)$. Then we let $H_1=A_1$,

(3) $$H_n=\cup\{B(V_{\beta_1})\cap\ldots\cap B(V_{\beta_{n-1}})\cap A_n\mid \beta_1,\ldots,\beta_{n-1}<\alpha\ ,\ \beta_i\neq\beta_j \text{ if } i\neq j\},\ n=2,3,\ldots\ .$$

(4) $$K_\alpha=\bigcup_{n=1}^{\infty}H_n\ .$$

Since $H_n\subset A_n$, it follows from $\dim A_n\leqq 0$ and Theorem II.3 that

(5) $$\dim H_n\leqq 0\ .$$

We can easily see that for every n $\bigcup_{i=1}^{n}H_i$ is open in K_α . For let $p\in\bigcup_{i=1}^{n}H_i$; then $p\in\bigcup_{i=1}^{n}A_i$. Hence from the induction hypothesis $b)_\beta$ for $\beta<\alpha$ we obtain $\operatorname{ord}_p\{B(V_\beta)\mid\beta<\alpha\}\leqq n-1$. Therefore we can choose, from every collection $\{B(V_{\beta_1}),\ldots,B(V_{\beta_j})\}$ which satisfies $\beta_1,\ldots,\beta_j<\alpha$, $j\geqq n$ and $\beta_i\neq\beta_j$ if $i\neq j$, $B(V_{\beta_i})$ such that $p\notin B(V_{\beta_i})$.

Hence, if we let $U(p)=\bigcap\{R-B(V_\beta)\mid\beta<\alpha\ ,\ p\notin B(V_\beta)\}$, then from (3) it follows that

$$U(p)\cap(\bigcup_{i=n+1}^{\infty}H_i)=\emptyset\ .$$

Since $\{B(V_\beta)\mid\beta<\alpha\}$ is locally finite by virtue of the local finiteness of $\{U_\beta\mid\beta<\alpha\}$ and $a)_\beta$ for $\beta<\alpha$, $U(p)$ is an open neighbourhood of p . Hence $\bigcup_{i=1}^{n}H_i$ is open in K_α . Thus, for every n

$$I_n=H_n-\bigcup_{i=1}^{n-1}H_i=\bigcup_{i=1}^{n}H_i-\bigcup_{i=1}^{n-1}H_i$$

is an F_σ-set in K_α . Hence $I_n=\bigcup_{m=1}^{\infty}I_{nm}$ for certain closed sets I_{nm} of K_α . Since $I_{nm}\subset I_n\subset H_n$, by (5) and Theorem II.3 we obtain $\dim I_{nm}\leqq 0$. Since

$$K_\alpha=\bigcup_{n=1}^{\infty}I_n=\bigcup_{n,m=1}^{\infty}I_{nm}\ ,$$

we have $\dim K_\alpha\leqq 0$ by the sum theorem. Therefore we can define, in the same way as for the case $\alpha=0$, an open set V_α such that

(6) $$F_\alpha\subset V_\alpha\subset U_\alpha\ ,$$

(7) $\quad B(V_\alpha) \cap K_\alpha = \emptyset$.

The condition (6) is equal to a)$_\alpha$. In view of (3) and (4) we easily see that (7) implies b)$_\alpha$. Thus the proof of this proposition is complete.

In view of the nature of the proof of this proposition we can modify it as follows.

B) Let $\{ U_\alpha \mid \alpha < \tau \}$, $\{ F_\alpha \mid \alpha < \tau \}$ and $\{ A_n \mid n = 1, 2, \dots \}$ satisfy the same conditions as in A). Let $\mathcal{U}_1$ be a locally finite open collection such that $\operatorname{ord}_p B(\mathcal{U}_1) \leqq n-1$ for every $p \in A_n$. Then there exists an open collection $\mathcal{U}_2 = \{ U'_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset U'_\alpha \subset U_\alpha$, and $\operatorname{ord}_p B(\mathcal{U}_1 \cup \mathcal{U}_2) \leqq n-1$ for every $p \in A_n$.

C) Let $\{ U_\gamma \mid \gamma \in \Gamma \}$ be an open collection and $\mathcal{F} = \{ F_\gamma \mid \gamma \in \Gamma \}$ a locally finite closed collection such that $F_\gamma \subset U_\gamma$ for all $\gamma \in \Gamma$ and $\operatorname{ord} \mathcal{F} \leqq n$. Then there exists a locally finite open collection $\mathcal{V} = \{ V_\gamma \mid \gamma \in \Gamma \}$ such that $F_\gamma \subset V_\gamma \subset U_\gamma$ for all $\gamma \in \Gamma$ and $\operatorname{ord} \mathcal{V} \leqq n$. Thus $\dim R \leqq n$ holds if every finite open covering of R has a locally finite closed refinement of order $\leqq n+1$.

Proof. We, of course, restrict ourselves to a (metric) space R . Since $\mathcal{F}$ is a locally finite closed collection of order $\leqq n$, there exists an open covering $\mathcal{P}$ such that each element of $\mathcal{P}$ intersects at most n elements of $\mathcal{F}$. Since R is fully normal, we can choose an open covering $\mathcal{Q}$ such that $\mathcal{Q}^* < \mathcal{P}$. Let $V_\gamma = S(F_\gamma, \mathcal{Q}) \cap U_\gamma$. Then as is easily seen, $\mathcal{V} = \{ V_\gamma \mid \gamma \in \Gamma \}$ is the desired open collection.

D) Let F be a closed set of R with $\dim F \leqq n$. Let F_α and U_α , $\alpha < \tau$, be closed and open sets, respectively such that $F_\alpha \subset U_\alpha$, and $\{ U_\alpha \mid \alpha < \tau \}$ is locally finite. Then there exist open sets V_α satisfying $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, $\dim B_k \leqq n-k$, $k = 1, \dots, n+1$, where $B_k = \{ p \mid p \in F, \operatorname{ord}_p B(\mathcal{V}) \geqq k \}$, and $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$.

Proof. By the decomposition theorem we can decompose F into the sum of $n+1$ subsets A_k , $k = 1, \dots, n+1$ with $\dim A_k \leqq 0$. Hence by II.7 A) we can construct open sets V_α for $\alpha < \tau$ such that $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\operatorname{ord}_p B(\mathcal{V}) \leqq k-1$ for $p \in A_k$, where we put $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$.

Hence $B_k \subset A_{k+1} \cup \dots \cup A_{n+1}$, which implies $\dim B_k \leqq n-k$ by virtue of the decomposition theorem.

E) Let F , F_α and U_α satisfy the same condition as in D). Then there exist open sets V_α , W_α , $\alpha < \tau$ satisfying $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset W_\alpha \subset U_\alpha$, and $\operatorname{ord}_p \{ W_\alpha - \bar{V}_\alpha \mid \alpha < \tau \} \leqq n$ for every $p \in F$.

Proof. By II.5 B) there exist open sets V'_α and W'_α of the subspace F such that $F_\alpha \cap F \subset V'_\alpha \subset \bar{V}'_\alpha \subset W'_\alpha \subset U_\alpha \cap F$ and $\operatorname{ord} \{ W'_\alpha - \bar{V}'_\alpha \mid \alpha < \tau \} \leqq n$.

We can easily construct open sets V_α and W_α of R such that $V_\alpha \cap F = V'_\alpha$, $\bar{V}_\alpha \cap F = \bar{V}'_\alpha$, $W_\alpha \cap F = W'_\alpha$, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset W_\alpha \subset U_\alpha$. Then $(W_\alpha - \bar{V}_\alpha) \cap F = W'_\alpha - \bar{V}'_\alpha$. Hence $\operatorname{ord}_p \{ W_\alpha - \bar{V}_\alpha \mid \alpha < \tau \} \leq n$ for every $p \in F$.

F) Let G_k, $k = 0, \ldots, n$ be closed sets with $\dim G_k \leq n - k$ of a space R. Let $\{ F_\alpha \mid \alpha < \tau \}$ be a closed collection and $\{ U_\alpha \mid \alpha < \tau \}$ a locally finite open collection such that $F_\alpha \subset U_\alpha$. Then there exists an open collection $V = \{ V_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$ and $\operatorname{ord}_p B(V) \leq n - k$ for every $p \in G_k$.

Proof. We can easily prove this proposition by using E). In fact, by E) we can construct open sets V^0_α, W^0_α such that $F_\alpha \subset V^0_\alpha \subset \overline{V^0_\alpha} \subset W^0_\alpha \subset U_\alpha$ and $\operatorname{ord}_p \{ W^0_\alpha - \overline{V^0_\alpha} \mid \alpha < \tau \} \leq n$ for every $p \in G_0$. Now, again by use of E), we can construct open sets V^1_α, W^1_α such that $\overline{V^0_\alpha} \subset V^1_\alpha \subset \overline{V^1_\alpha} \subset W^1_\alpha \subset W^0_\alpha$, and $\operatorname{ord}_p \{ W^1_\alpha - \overline{V^1_\alpha} \mid \alpha < \tau \} \leq n - 1$ for every $p \in G_1$.

By repeating this process we get open sets V^k_α, W^k_α, $k = 0, \ldots, n$, such that $\overline{V^{k-1}_\alpha} \subset V^k_\alpha \subset \overline{V^k_\alpha} \subset W^k_\alpha \subset W^{k-1}_\alpha$, and $\operatorname{ord}_p \{ W^k_\alpha - \overline{V^k_\alpha} \mid \alpha < \tau \} \leq n - k$ for every $p \in G_k$. Letting $V_\alpha = \bigcup_{k=0}^n V^k_\alpha$ we obtain the desired open sets V_α for $\alpha < \tau$.

III. 5. Continuous mappings which lower dimension

The projection of E^2 on E^1 lowers the dimension of E^2 by one, the dimension of the inverse image of each point of E^1. (Precisely speaking, $\dim E^2 = 2$ is not yet proved while $\dim E^1 = 1$ is easy to see.) Generally we can prove the following Theorem III.6 as a result of investigations in this direction.

Definition III. 6. *A mapping f of a space R into a space S is called a closed (open) mapping if the image of every closed (open) set of R is closed (open) in S. A closed continuous mapping f is called a perfect mapping if $f^{-1}(q)$ is compact for every $q \in S$.*

Theorem III. 6 [2]. *Let f be a closed continuous mapping of a space R onto a space S such that $\dim f^{-1}(q) \leq k$ for every $q \in S$. Then $\dim R \leq \dim S + k$.*

Proof. Since the validity of this proposition is clear if $\dim S = -1$, we shall assume this theorem for every S with $\dim S < n$ $(n \geq 0)$ to prove it by induction for S with $\dim S = n$. Let $U = \{ U_i \mid i = 1, \ldots, s \}$ be a given open covering of R.

[2] This theorem was first proved for separable metric spaces by W. Hurewicz and extended by K. Morita [6] and K. Nagami [1] to general types of spaces which include metric spaces as a special case. See Theorem VII. 7.

Since $\dim f^{-1}(q) \leqq k$ for each point $q \in S$, by I.1 A) and III.4 C) there exist open sets V_i, $i = 1,\dots,s$ such that $f^{-1}(q) \subset \bigcup_{i=1}^{s} V_i$, $V_i \subset U_i$, $i = 1,\dots,s$, and

(1) $\qquad \operatorname{ord} \{ V_i \mid i = 1,\dots,s \} \leqq k+1$.

Let

(2) $\qquad V'(q) = \bigcup_{i=1}^{s} V_i$;

then $\{ V'(q) \mid q \in S \}$ is an open covering of R satisfying $V'(q) \supset f^{-1}(q)$. We choose for each q an open set $V(q)$ satisfying

(3) $\qquad V'(q) \supset \overline{V(q)} \supset V(q) \supset f^{-1}(q)$.

It follows from $q \in S - f(R - V(q))$ and the closedness of f, that $\mathcal{V} = \{ S - f(R - V(q)) \mid q \in S \}$ is an open covering of S. Since $\dim S \leqq n$, by use of III. 4 D) combined with the paracompactness of S we can construct a locally finite open covering $\mathcal{W}$ of S such that $\mathcal{W} < \mathcal{V}$ and

(4) $\qquad \dim B_m \leqq n-m$, $m = 1,\dots,n+1$,

where $B_m = \{ p \mid \operatorname{ord}_p B(\mathcal{W}) \geqq m \}$. Let $W_i \in \mathcal{W}$, $i = 1,\dots,m$; then since

$$ff^{-1}(B(W_1) \cap \dots \cap B(W_m)) = B(W_1) \cap \dots \cap B(W_m),$$

f is clearly a closed continuous mapping of $f^{-1}(B(W_1) \cap \dots \cap B(W_m))$ onto $B(W_1) \cap \dots \cap B(W_m)$. Since $B(W_1) \cap \dots \cap B(W_m) \subset B_m$, it follows from (4) that $\dim (B(W_1) \cap \dots \cap B(W_m)) \leqq n-m$.

Hence we obtain from the induction hypothesis

$$\dim f^{-1}(B(W_1) \cap \dots \cap B(W_m)) \leqq n-m+k, \quad m = 1,\dots,n+1.$$

Hence, in view of

$$B(f^{-1}(W_1)) \cap \dots \cap B(f^{-1}(W_m)) \subset f^{-1}(B(W_1) \cap \dots \cap B(W_m)),$$

we conclude that

$$\dim B(f^{-1}(W_1)) \cap \dots \cap B(f^{-1}(W_m)) \leqq n-m+k.$$

Thus, by virtue of the sum theorem, $f^{-1}(\mathcal{W}) = \{ f^{-1}(W) \mid W \in \mathcal{W} \}$ is a locally finite open covering of R such that

(5) $\qquad f^{-1}(\mathcal{W}) < \{ V(q) \mid q \in S \}$, and

(5') $\qquad \dim A_m \leqq n-m+k$, $m = 1,\dots,n+1$,

for $A_m = \{ p \mid \operatorname{ord}_p B(f^{-1}(\mathcal{W})) \geqq m \}$, because A_m is a locally finite sum of closed sets like $B(f^{-1}(W_1)) \cap \dots \cap B(f^{-1}(W_m))$.

We well-order the elements of $f^{-1}(W)$ as $f^{-1}(W) = \{ M_\alpha \mid \alpha < \tau \}$ to define a new locally finite open collection $N = \{ N_\alpha \mid \alpha < \tau \}$ by

$$N_0 = M_0 \ , \quad N_\alpha = M_\alpha - \cup \{ \bar{M}_\beta \mid \beta < \alpha \} \ , \ 0 < \alpha < \tau \ .$$

Then $C_m = \{ p \mid \mathrm{ord}_p \bar{N} \geq m \} \subset A_{m-1}$, $m = 2, \ldots, n+1$, and hence, from (5)', it follows that

(6) $\dim C_m \leq n - m + k + 1$, $m = 2, \ldots, n+1$.

Since by (3) and (5) $\bar{N}_\alpha \subset V'(q)$ for every α and for some q , from (1), (2) and I.1 A) it follows that there exists a closed covering $\{ K_i^\alpha \mid i = 1, \ldots, s \}$ of $\bar{N}_\alpha$ satisfying $K_i^\alpha \subset U_i$, $i = 1, \ldots, s$, and $\mathrm{ord} \{ K_i^\alpha \mid i = 1, \ldots, s \} \leq k + 1$.

By use of 4 C) we construct open sets P_i^α such that $K_i^\alpha \subset P_i^\alpha \subset \bar{P}_i^\alpha \subset U_i$, and

(7) $\mathrm{ord} \{ P_i^\alpha \mid i = 1, \ldots, s \} \leq k + 1$.

Note that we choose P_i^α such that $\{ P_i^\alpha \mid \alpha < \tau , i = 1, \ldots, s \}$ is locally finite, because $\{ K_i^\alpha \mid \alpha < \tau , i = 1, \ldots, s \}$ is locally finite.

Now, in view of (6), we apply F) to K_i^α, P_i^α to get open sets Q_i^α, $\alpha < \tau$, $i = 1, \ldots, s$ such that $K_i^\alpha \subset Q_i^\alpha \subset \bar{Q}_i^\alpha \subset P_i^\alpha$, and

(8) $\mathrm{ord}_p B(Q) \leq n - m + k + 1$ for $p \in C_m$,

where $Q = \{ Q_i^\alpha \mid \alpha < \tau , i = 1, \ldots, s \}$. Let

(9) $S_i^\alpha = \bar{N}_\alpha \cap \overline{Q_i^\alpha - (Q_1^\alpha \cup \ldots \cup Q_{i-1}^\alpha)}$.

We shall show that $S = \{ S_i^\alpha \mid \alpha < \tau , i = 1, \ldots, s \}$ is a closed covering of order $\leq n + k + 1$ and a refinement of U . Suppose

(10) $p \in \bigcap_{\nu=1}^{s_1} S_{i(1,\nu)}^{\alpha_1} \bigcap_{\nu=1}^{s_2} S_{i(2,\nu)}^{\alpha_2} \cdots \bigcap_{\nu=1}^{s_m} S_{i(m,\nu)}^{\alpha_m} \neq \emptyset$ [3] .

If $m = 1$, then from the definition of S_i^α it follows that

$$\bigcap_{\nu=1}^{s_1} \overline{Q}_{i(1,\nu)}^{\alpha_1} \neq \emptyset \ ,$$

which combined with the relation $\bar{Q}_i^\alpha \subset P_i^\alpha$ implies

$$\bigcap_{\nu=1}^{s_1} P_{i(1,\nu)}^{\alpha_1} \neq \emptyset \ ,$$

[3] For brevity we abridge some symbols $\cap$ in this formula, but the reader will easily understand the real meaning.

and hence by (7) $s_1 \leqq k+1 \leqq n+k+1$. If $m \geqq 2$. then by assuming $i(l,1) < i(l,2) < \dots < i(l,s_l)$, $l = 1,\dots,m$, we get from (9) and (10) that

$$p \in \bigcap_{\nu=1}^{s_1} S^{\alpha_1}_{i(1,\nu)} \cap \dots \bigcap_{\nu=1}^{s_m} S^{\alpha_m}_{i(m,\nu)}$$
$$\subset \bigcap_{\nu=1}^{s_1-1} B(Q^{\alpha_1}_{i(1,\nu)}) \cap \dots \bigcap_{\nu=1}^{s_m-1} B(Q^{\alpha_m}_{i(m,\nu)}) \bigcap_{i=1}^{m} \bar{N}_{\alpha_i} .$$

Since $p \in \bigcap_{i=1}^{m} \bar{N}_{\alpha_i} \subset C_m$, it follows from (8) that $(s_1-1)+(s_2-1)+\dots+(s_m-1) \leqq n-m+k+1$, and hence $s_1+s_2+\dots+s_m \leqq n+k+1$. Thus in any case we get $\operatorname{ord}_p S \leqq n+k+1$.

Now we note that $\{M_\alpha \mid \alpha < \tau\}$ covers R , and hence by the definition of N , $\bar{N}$ also covers R . On the other hand, since $\{K^\alpha_i \mid i=1,\dots,s\}$ and accordingly $\{Q^\alpha_i \mid i=1,\dots,s\}$ covers $\bar{N}_\alpha$, we have $\bar{N}_\alpha = \bigcup_{i=1}^{s} S^\alpha_i$. Therefore S covers R . Since by (9) and the definitions of Q^α_i and P^α_i we have $S^\alpha_i \subset \bar{Q}^\alpha_i \subset \bar{P}^\alpha_i \subset U_i$, S is a locally finite closed refinement of U . Thus by 4 C) $\dim R \leqq n+k$.

III. 6. Continuous mappings which raise dimension

As shown by Peano, a closed segment can be continuously mapped on a closed square. This example shows that a closed continuous mapping can raise dimension. As for the raising of dimension by a closed continuous mapping we have the following theorem.

Theorem III. 7 [4] . *Let f be a closed continuous mapping of a space R onto a space S such that for each point q of S , $B(f^{-1}(q))$ contains at most $m+1$ points $(m \geqq 0)$; then* $\dim S \leqq \dim R + m$.

Proof. We shall begin the proof with the special case that every $f^{-1}(q)$ contains at most $m+1$ points. Let $\dim R \leqq n$; then we shall carry out the proof of $\dim S \leqq n+m$ by induction on the numbers $n \geqq -1$ and $m \geqq 0$.

i) This assertion is clearly true for every $m \geqq 0$ if $n = -1$.

ii) Let us show this theorem for every $n > -1$ in case $m = 0$. To use induction on n , assume that the theorem is true if $\dim R \leqq n-1$. Assume G_1 and G_2 are

[4] W. Hurewicz [3] first proved this theorem for separable metric spaces considering $f^{-1}(q)$ in place of $B(f^{-1}(q))$. With respect to this theorem K. Nagami [2] solved a problem posed by W. Hurewicz as follows: Let S be an n-dimensional space and $0 < m < n$; then there exists an m-dimensional space R and a closed continuous mapping f of R onto S such that $f^{-1}(q)$ contains at most $n-m+1$ points for each $q \in S$.

given closed sets of S such that $G_1 \cap G_2 = \emptyset$. Then $F_1 = f^{-1}(G_1)$ and $F_2 = f^{-1}(G_2)$ are disjoint closed sets of R. Since $\dim R \leqq n$, there exists an open set U satisfying $F_1 \subset U \subset R - F_2$, and

(1) $\dim B(U) \leqq n-1$.

Since f is a closed mapping, $V = S - f(R-U)$ is an open set of S. It is obvious that

(2) $G_1 \subset V \subset S - G_2$.

Then

(3) $B(V) \subset f(B(U))$.

Because for each point $q \in B(V)$, $f^{-1}(q) = \{p\}$. If $p \in U$, then $q \in V$. If $p \notin \bar{U}$, then $q \notin \bar{V}$. Since both are impossible, $p \in B(U)$, and hence $q = f(p) \in f(B(U))$ proving the assertion. Since f is a one-to-one closed continuous mapping even if restricted to $B(V)$, by the induction hypothesis and (1) we obtain $\dim f(B(U)) \leqq n-1$. Thus $\dim B(V) \leqq n-1$ follows from (3). Hence $\dim S \leqq n$ is proved.

iii) Now we assume the proposition for the case in which $\dim R \leqq n-1$, and every $f^{-1}(q)$ contains at most $m+1$ points, and for the case in which $\dim R \leqq n$, and every $f^{-1}(q)$ contains at most m points.

Then we shall prove it for the case in which $\dim R \leqq n$, and every $f^{-1}(q)$ contains at most $m+1$ points. Let G_1 and G_2 be given disjoint closed sets of S. Then we can define F_1, F_2, U and V in the same way as in the proof ii). Let $H = f(B(U))$. The restriction of f to $B(U)$ is a closed continuous mapping of $B(U)$ onto H. Therefore it follows from $\dim B(U) \leqq n-1$ and the induction hypothesis that

(4) $\dim H \leqq n+m-1$.

$B(V) - H$ is a G_δ-set of S. Therefore, let

(5) $B(V) - H = \bigcup_{k=1}^{\infty} H_k$ for closed sets H_k, $k = 1, 2, \ldots$.

Note that for each $q \in B(V) - H$, $f^{-1}(q) \cap B(U) = \emptyset$ holds. Then put

$$f^{-1}(q) \cap U = f^{-1}(q)_1 \text{ , and } f^{-1}(q) \cap (R - \bar{U}) = f^{-1}(q)_2 .$$

Further put

$$E_k = f^{-1}(H_k) \cap (R-U) = f^{-1}(H_k) \cap (R - \bar{U}) = \cup \{ f^{-1}(q)_2 \mid q \in H_k \} .$$

Then E_k is a closed set of R. Note that for each $q \in H_k$, $f^{-1}(q)_2 \neq \emptyset$ holds, because otherwise $q \in V$ would follow, contradicting $q \in B(V)$. Thus $f(E_k) = H_k$. Namely $f \mid E_k = f_k$ is a closed continuous mapping from E_k onto H_k.

On the other hand $f^{-1}(q)_1 \neq \emptyset$ for each $q \in H_k$, because otherwise $q \notin \bar{V}$ would follow. Hence $f_k^{-1}(q) = f^{-1}(q)_2$ consists of at most m points for each $q \in H_k$.

Since $\dim E_k \leq n$ is obvious we obtain from the induction hypothesis $\dim H_k \leq n+m-1$. Thus by (4) and the sum theorem we can conclude that

$$\dim (H \cup (\bigcup_{k=1}^{\infty} H_k)) \leq n+m-1 ,$$

because H and H_k, $k=1,2,\dots$ are closed sets.

Note that (5) implies $B(V) \subset H \cup (\bigcup_{k=1}^{\infty} H_k)$. Therefore $\dim B(V) \leq n+m-1$, which proves $\dim S \leq n+m$.

Now, let us turn to the most general case in which each $B(f^{-1}(q))$ contains at most $m+1$ points. To each point $q \in S$ with $B(f^{-1}(q)) = \emptyset$ we assign a point $p(q) \in f^{-1}(q)$ and put $C = \{p(q) \mid q \in S ,\ B(f^{-1}(q)) = \emptyset\}$. Let

$$B = \cup \{B(f^{-1}(q)) \mid q \in S\} \cup C .$$

Then B is a closed set of R with $\dim B \leq n$, and $f \mid B$ is a closed continuous mapping from B onto S such that each inverse image of a point of S contains at most $m+1$ points. Hence from the above argument of the special case it follows that $\dim S \leq n+m$ [5].

III. 7. Baire's zero-dimensional spaces

Definition III. 7. *Let Ω be a given set. We denote by $N(\Omega)$ the set of all sequences of elements of Ω. Defining the distance of two points $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ of $N(\Omega)$ by*

$$\rho(\alpha, \beta) = \frac{1}{\min\{k \mid \alpha_k \neq \beta_k\}} ,$$

we get a 0-dimensional metric space $N(\Omega)$. We call this space a generalized Baire's 0-dimensional space.

As easily seen, $N(\Omega)$ is topologically the countable product of discrete spaces Ω. We characterize n-dimensional metric space in terms of $N(\Omega)$, that is to say we can get every n-dimensional metric space as the image of a subset of $N(\Omega)$ by a closed continuous mapping which satisfies the condition of Theorem III.7 for $m=n$.

---

[5] In relation to this theorem J. Suzuki [1] proved that if f is a closed continuous mapping of a space R onto a space S such that each $f^{-1}(q)$ consists of exactly k points, then $\dim R = \dim S$. As for open mappings K. Nagami [2] proved that if f is an open continuous mapping of R onto S such that each $f^{-1}(q)$ consists of finitely many points, then $\dim R = \dim S$. R. Hodel [1] proved the same for a countable-to-one open mapping but assuming that R is locally compact.
Yu. A. Rozanskaya [1] proved that if $m < n$, then there exists no open continuous mapping which maps I^m onto I^n. See also J. Keesling [1] for results in this aspect.

A) Let A_n, $n = 1, 2, \dots$ be a sequence of 0-dimensional sets of a space R. Let $\mathcal{U} = \{ U_\alpha \mid \alpha < \tau \}$ be a locally finite open covering. Then there exists a closed covering $\mathcal{F} = \{ F_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset U_\alpha$ and $\operatorname{ord}_p \mathcal{F} \leqq n$ for every $p \in A_n$.

Proof. By 4.A) combined with I.1 A) there exists a locally finite open covering $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$ such that $\bar{V}_\alpha \subset U_\alpha$ and $\operatorname{ord}_p B(\mathcal{V}) \leqq n-1$ for $p \in A_n$. Let

$$F_0 = \bar{V}_0 \,, \text{ and } F_\alpha = \overline{V_\alpha - \cup \{ \bar{V}_\beta \mid \beta < \alpha \}} \quad \text{for } \alpha < \tau \,.$$

Then it is easy to see that $\mathcal{F} = \{ F_\alpha \mid \alpha < \tau \}$ is the desired covering. For, let p be a given point of A_n. We suppose $p \in V_\alpha$ and $p \notin V_\beta$ for every $\beta < \alpha$. Then from the definition of F_α we have $p \notin F_\gamma$ for every $\gamma > \alpha$. If $p \in F_\beta$ for some $\beta < \alpha$, then from $p \notin V_\beta$ it follows that $p \in \bar{V}_\beta - V_\beta = B(V_\beta)$. Hence from $\operatorname{ord}_p B(\mathcal{V}) \leqq n-1$ we obtain $\operatorname{ord}_p \mathcal{F} \leqq n$.

B) Let A_n, $n = 1, 2, \dots$ be a countable number of 0-dimensional sets of a space R. Then there exists a sequence $\{ \mathcal{F}_i \mid i = 1, 2, \dots \}$ of locally finite closed coverings of R satisfying

i) for every neighbourhood $U(p)$ of every point p of R there exists some i with $S(p, \mathcal{F}_i) \subset U(p)$,

ii) $\mathcal{F}_i = \{ F(\alpha_1, \dots, \alpha_i) \mid \alpha_k \in \Omega,\ k = 1, 2, \dots, i \}$, where $F(\alpha_1, \dots, \alpha_i)$ may be empty,

iii) $F(\alpha_1, \dots, \alpha_{i-1}) = \cup \{ F(\alpha_1, \dots, \alpha_{i-1}, \beta) \mid \beta \in \Omega \}$,

iv) $\operatorname{ord}_p \mathcal{F}_i \leqq n$ for each point $p \in A_n$ and for each $i \in \{ 1, 2, \dots \}$.

Proof. First we define $\mathcal{S}_i = \{ S_{1/i}(p) \mid p \in R \}$, $i = 1, 2, \dots$. Then we shall define $\mathcal{F}_i$ satisfying ii), iii), iv) and $\mathcal{F}_i < \mathcal{S}_i$. For $i = 1$ we can define, by use of A) and the corollary to Theorem I.1, a locally finite closed covering $\mathcal{F}_1$ such that $\mathcal{F}_1 < \mathcal{S}_1$ and $\operatorname{ord}_p \mathcal{F}_1 \leqq n$ for every $p \in A_n$. We shall use induction and accordingly assume that $\mathcal{F}_k$ has been defined for every $k < i$. Now we shall define $\mathcal{F}_i$.

Let us well-order the elements of $\mathcal{F}_{i-1}$ as $\mathcal{F}_{i-1} = \{ F_\alpha \mid \alpha < \tau \}$. To obtain $\mathcal{F}_i$ we shall define closed sets $F_{\alpha\beta}$ for $\alpha < \tau$ and $\beta \in \Omega$ such that

a)$_\alpha$ $\qquad F_\alpha = \cup \{ F_{\alpha\beta} \mid \beta \in \Omega \}$, $\{ F_{\alpha\beta} \mid \beta \in \Omega \} < \mathcal{S}_i$,

b)$_\alpha$ $\qquad \mathcal{G}_\alpha = \{ F_{\alpha'\beta} \mid \alpha' \leqq \alpha\,,\ \beta \in \Omega \} \cup \{ F_{\alpha'} \mid \alpha' > \alpha \}$

is locally finite for every $\alpha < \tau$,

c)$_\alpha$ $\qquad \operatorname{ord}_p \mathcal{G}_\alpha \leqq n$ for every $\alpha < \tau$ and for every point $p \in A_n$.

First we define $F_{0\beta}$ for $\beta \in \Omega$ as follows. We let

$$H_{10} = \{p \mid p \in F_0 \cap A_1\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = 0\},$$
$$H_{21} = \{p \mid p \in F_0 \cap A_2\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = 1\},$$
$$H_{32} = \{p \mid p \in F_0 \cap A_3\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = 2\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$K_1 = H_{10} \cup H_{21} \cup H_{32} \cup \ldots$$

and

$$H_{20} = \{p \mid p \in F_0 \cap A_2\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = 0\},$$
$$H_{31} = \{p \mid p \in F_0 \cap A_3\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = 1\},$$
$$H_{42} = \{p \mid p \in F_0 \cap A_4\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = 2\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$K_2 = H_{20} \cup H_{31} \cup H_{42} \cup \ldots$$

and generally

(1) $$H_{r+ss} = \{p \mid p \in F_0 \cap A_{r+s}\,,\ \mathrm{ord}_p\{F_\alpha \mid 0<\alpha<\tau\} = s\},$$

(2) $$K_r = \bigcup_{s=0}^{\infty} H_{r+ss}\,.$$

If $p \in H_{r+ss}$, then $p \in F_{\alpha_1} \cap \ldots \cap F_{\alpha_s}$ for some different $\alpha_1, \ldots, \alpha_s > 0$, and $p \notin F_{\alpha_1} \cap \ldots \cap F_{\alpha_{s+1}}$ for any different $\alpha_1, \ldots, \alpha_{s+1} > 0$. Hence the open neighbourhood $U(p) = \bigcap\{R - F_\alpha \mid p \notin F_\alpha\,,\ 0<\alpha<\tau\}$ of p satisfies $U(p) \cap H_{r+tt} = \emptyset$ for every $t > s$. Hence for every s' , $\bigcup_{s=0}^{s'} H_{r+ss}$ is open in K_r . Since (1) implies $H_{r+ss} \subset A_{r+s}$, from $\dim A_{r+s} \leqq 0$ it follows that $\dim H_{r+ss} \leqq 0$ for every $s \geqq 0$.

Since any open set is an F_σ-set, each H_{r+ss} is an F_σ-set in K_r . Hence from the sum theorem we obtain $\dim K_r \leqq 0$. Therefore by A) we can define a locally finite closed covering $\mathcal{G}'_0 = \{F_{0\beta} \mid \beta \in \Omega\}$ of F_0 such that $\mathcal{G}'_0 < \mathcal{S}_i$ and

(3) $$\mathrm{ord}_p \mathcal{G}'_0 \leqq n \quad \text{for every} \quad p \in K_n\,.$$

We put $\mathcal{G}_0 = \mathcal{G}'_0 \cup \{F_\alpha \mid \alpha > 0\}$. To show $\mathrm{ord}_p \mathcal{G}_0 \leqq n$ for every $p \in A_n$ we suppose $\mathrm{ord}_p \mathcal{F}_{i-1} = s+1$ and

(4) $$p \in F_0 \cap F_{\alpha_1} \cap \ldots \cap F_{\alpha_s}$$

for some different $\alpha_1, \ldots, \alpha_s > 0$. Then $0 \leqq s \leqq n-1$ because by the induction hypothesis $\mathrm{ord}_p \mathcal{F}_{i-1} \leqq n$. Thus from (1) and (2) we deduce that $p \in H_{ns} \subset K_{n-s}$. Hence by (3) $\mathrm{ord}_p \mathcal{G}'_0 \leqq n-s$, which combined with (4) proves c)$_0$ $\mathrm{ord}_p \mathcal{G}_0 \leqq n$.

Furthermore a)$_0$ and b)$_0$ are clearly satisfied by $F_{0\beta}$ and $\mathcal{G}_0$.

Suppose we have defined $F_{\alpha'\beta}$ for every $\alpha' < \alpha$; then we can define $F_{\alpha\beta}$ for every $\beta \in \Omega$ satisfying a)$_\alpha$, b)$_\alpha$, c)$_\alpha$. The definition of $F_{\alpha\beta}$ is quite analogous to that of $F_{0\beta}$ except that we adopt F_α and $\{F_{\alpha'\beta} \mid \alpha' < \alpha\,,\ \beta \in \Omega\} \cup \{F_{\alpha'} \mid \alpha' > \alpha\}$ in place of F_0 and $\{F_\alpha \mid \alpha > 0\}$ respectively, so the proof is left to the reader.

Thus we can define the desired covering $\mathfrak{F}_i = \{ F_{\alpha\beta} \mid \alpha < \tau, \beta \in \Omega \}$. It easily follows from a)$_\alpha$ and c)$_\alpha$ that $\mathfrak{F}_i$ satisfies ii), iii), iv) and $\mathfrak{F}_i < \mathfrak{S}_i$. From b)$_\alpha$ combined with the local finiteness of $\mathfrak{F}_{i-1}$ it follows that $\mathfrak{F}_i$ is locally finite. (To assure that $\mathfrak{F}_i$ is subindexed as in ii), we select a sufficiently large Ω at the beginning of the induction and at the end of each stage of the induction we put $F(\alpha_1,\ldots,\alpha_i) = \emptyset$ for those $(\alpha_1,\ldots,\alpha_i)$ for which F has not been defined.)

C) If a space R has dimension $\leqq n$, then there exists a sequence $\{ \mathfrak{F}_i \mid i = 1,2,\ldots \}$ of locally finite closed coverings of R which satisfies

i) for every neighbourhood $U(p)$ of every point p of R there exists some i for which $S(p,\mathfrak{F}_i) \subset U(p)$,

ii) $\mathfrak{F}_i = \{ F(\alpha_1,\ldots,\alpha_i) \mid \alpha_k \in \Omega,\ k = 1,\ldots,i \}$, where $F(\alpha_1,\ldots,\alpha_i)$ may be empty,

iii) $F(\alpha_1,\ldots,\alpha_{i-1}) = \bigcup \{ F(\alpha_1,\ldots,\alpha_{i-1},\beta) \mid \beta \in \Omega \}$,

iv) $\operatorname{ord} \mathfrak{F}_i \leqq n+1$.

Proof. We can easily prove this proposition by applying to R the decomposition theorem and B), putting $A_i = \emptyset$ for $i > n+1$. (Note that the condition iv) should be understood in the strong sense that each point of R is contained in $F(\alpha_1,\ldots,\alpha_i)$ for at most $n+1$ distinct subindices $(\alpha_1,\ldots,\alpha_i)$.)

Theorem III.8 [6]. *A space R has dimension $\leqq n$ if and only if there exists a subspace P of $N(\Omega)$ for suitable Ω and a closed continuous mapping f of P onto R such that for each point q of R, $f^{-1}(q)$ consists of at most $n+1$ points.*

Proof. Since $\dim N(\Omega) = 0$, the "if" part follows from Theorem III.7. To show the "only if" part we construct closed coverings $\mathfrak{F}_i$, $i = 1,2,\ldots$ satisfying the conditions of C). Then we define a subset P of $N(\Omega)$ by

$$P = \{ \alpha \mid \alpha = (\alpha_1,\alpha_2,\ldots) \in N(\Omega),\ \bigcap_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i) \neq \emptyset \} .$$

Furthermore, we define a mapping f of P onto R by

$$f(\alpha) = \bigcap_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i) \quad \text{for } \alpha = (\alpha_1,\alpha_2,\ldots) \in P .$$

[6] First proved by W. Hurewicz [2] and C. Kuratowski [1] in the separable case. This theorem for general metric space and Theorem III.9 are due to K. Morita [5]. Yu. Smirnov [4], V. Ponomarev [1] and K. Nagami [3] obtained a wider category of topological spaces as images of zero-dimensional metric spaces by continuous mappings satisfying some conditions. For example, the last author showed that every non-empty metric space S is the image of a metric space R with $\dim R = 0$ by an open continuous mapping f such that $f^{-1}(q)$ is compact for every point q of S.

By virtue of the condition i) of C) f is continuous. From iv) it follows that for every $q \in R$, $f^{-1}(q)$ consists of at most $n+1$ points. (Recall the note in the proof of C).)

To show that f is a closed mapping, for each collection $(\alpha_1, \ldots, \alpha_i)$ of elements of Ω we define an open set $N(\alpha_1, \ldots, \alpha_i)$ of P by

$$N(\alpha_1, \ldots, \alpha_i) = \{ (\alpha'_1, \alpha'_2, \ldots) \mid (\alpha'_1, \alpha'_2, \ldots) \in P;\ \alpha'_k = \alpha_k,\ k = 1, \ldots, i \}$$

and an open covering N_i of P by

$$N_i = \{ N(\alpha_1, \ldots, \alpha_i) \mid \alpha_k \in \Omega,\ k = 1, \ldots, i \}.$$

Let K be a closed subset of P and choose $q \notin f(K)$ in R. Then it follows from the finiteness of $f^{-1}(q)$ that $S(K, N_i) \cap f^{-1}(q) = \emptyset$ for some i. Hence $H = \cup \{ f(N) \mid N \in N_i,\ N \cap K \neq \emptyset \}$ satisfies $q \notin H \supset f(K)$. Since by iii) $f(N(\alpha_1, \ldots, \alpha_i)) = F(\alpha_1, \ldots, \alpha_i)$, the local finiteness of F_i implies that H is a closed set of R. Hence putting $U(q) = R - H$ we obtain a neighbourhood of q satisfying $U(q) \cap f(K) = \emptyset$, which shows that $f(K)$ is a closed set of R. Therefore f is a closed mapping.

Thus the proof of this theorem is complete.

Theorem III. 9. *A space R has dimension $\leq n$ if and only if there exists a sequence $\{ F_i \mid i = 1, 2, \ldots \}$ of locally finite closed coverings of R which satisfies the conditions i) - iv) of C)* [7].

Proof. The "only if" part is proved by C). Conversely, if there exists such a sequence $\{ F_i \mid i = 1, 2, \ldots \}$, then by use of the proof of Theorem III.8 we can construct a subspace P of $N(\Omega)$ and a closed continuous mapping f of P onto R such that $f^{-1}(q)$ consists of at most $n+1$ points for each $q \in R$. Hence $\dim R \leq n$ is deduced from Theorem III.8.

III. 8. Uniformly zero-dimensional mappings

Closely related to the diameter $\delta(S)$ of a subset S of a space R is the following notion.

Definition III.3. *Let U and S be an open covering and a subset respectively of a space R. We denote by $\tilde{\delta}(S) < U$ the fact that there exists an open collection V satisfying $V < U$, $\cup \{ V \mid V \in V \} \supset S$, and* ord $V \leq 1$.

[7] Here iv) should be understood in the strong sense as noted in the proof of C).

Definition III. 9. *Let* f *be a mapping of a space* R *into a space* S. *Suppose for every* $\varepsilon>0$ *there exists* $\eta>0$ *such that* $\delta(T)\leq\eta$ *for* $T\subset S$ *implies*

$$\tilde{\delta}(f^{-1}(T))<\mathcal{U}_\varepsilon=\{U \mid U \text{ is an open set of } R \text{ with } \delta(U)\leq\varepsilon\}.$$

Then we call f *a uniformly 0-dimensional mapping.*

A) If there is a uniformly 0-dimensional continuous mapping f of a space R onto S, then $\dim R\leq\dim S$.

Proof. Suppose $\dim S\leq n$. By use of the decomposition theorem we decompose S into $n+1$ 0-dimensional spaces S_i, $i=1,\dots,n+1$. The mapping f is clearly a uniformly 0-dimensional continuous mapping of $R_i=f^{-1}(S_i)$ onto S_i if it is restricted to R_i. Hence, again in view of the decomposition theorem it suffices to prove $\dim R_i\leq 0$. Namely, all we have to do is to prove this proposition in case $\dim S=0$.

We define $\mathcal{S}_i=\{S_{1/i}(p)\mid p\in R\}$. Since f is uniformly 0-dimensional, there exists $\eta>0$ such that if $\delta(T)\leq\eta$ for a set T of S, then $\tilde{\delta}(f^{-1}(T))<\mathcal{S}_i$.

Since S has dimension ≤ 0, there exists an open covering $\mathcal{V}_i$ of S such that mesh $\mathcal{V}_i\leq\eta$, i.e. $\delta(V)\leq\eta$ for every $V\in\mathcal{V}_i$, and ord $\mathcal{V}_i\leq 1$. Then every element V of $\mathcal{V}_i$ satisfies $\tilde{\delta}(f^{-1}(V))<\mathcal{S}_i$. Hence for every $V\in\mathcal{V}_i$ there exists an open covering $\mathcal{U}_i(V)$ of $f^{-1}(V)$ with ord $\mathcal{U}_i(V)\leq 1$ and $\mathcal{U}_i(V)<\mathcal{S}_i$. Thus $\mathcal{W}_i=\cup\{\mathcal{U}_i(V)\mid V\in\mathcal{V}_i\}$ is an open covering of R such that ord $\mathcal{W}_i\leq 1$ and $\mathcal{W}_i<\mathcal{S}_i$.

Therefore we can easily see that $\cup_{i=1}^{\infty}\mathcal{W}_i$ is a σ-locally finite open basis of R consisting of sets which are both open and closed. Hence by Theorem II.2 we conclude $\dim R\leq 0$.

B) We denote by $C(R,I^n)$ the set of all continuous mappings of a space R into a Euclidean n-cube I^n. We regard $C(R,I^n)$ as a metric space by assigning a metric ρ to it by

$$\rho(f,g)=\sup\{d(f(x),\ g(x))\mid x\in R\} \quad \text{for } f,g\in C(R,I^n),$$

where $d(a,b)$ denotes the metric of I^n. Moreover, for $\varepsilon>0$ we put $D_\varepsilon(R,I^n)=\{f\mid f\in C(R,I^n)$, there exists $\eta>0$ such that for every $T\subset I^n$ with $\delta(T)\leq\eta$, $\tilde{\delta}(f^{-1}(T))<\mathcal{U}_\varepsilon$ holds$\}$, where $\mathcal{U}_\varepsilon$ denotes the same covering as in Definition III.9. Then $D_\varepsilon(R,I^n)$ is an open subset of $C(R,I^n)$.

Proof. Let f be a point of $D_\varepsilon(R,I^n)$ and η a positive number such that

(1) $\qquad \delta(T)\leq\eta$ implies $\tilde{\delta}(f^{-1}(T))<\mathcal{U}_\varepsilon$ for all $T\subset I^n$.

Suppose g to be a point of $C(R,I^n)$ satisfying $\rho(f,g)<\frac{1}{3}\eta$. Consider $T\subset I^n$ with $\delta(T)\leq\frac{1}{3}\eta$. Then a', $b'\in fg^{-1}(T)$ implies $a'=f(a)$ and $b'=f(b)$ for some a, $b\in R$ for which $g(a)\in T$ and $g(b)\in T$.

Therefore

$$d(a',b') = d(f(a),\ f(b))$$
$$\leqq d(f(a),\ g(a)) + d(g(a),\ g(b)) + d(g(b),\ f(b)) < \eta\ .$$

Hence $\delta(fg^{-1}(T)) \leqq \eta$. Thus it follows from (1) that $\tilde{\delta}(f^{-1}fg^{-1}(T)) < U_\varepsilon$, which combined with $f^{-1}fg^{-1}(T) \supset g^{-1}(T)$ implies $\tilde{\delta}(g^{-1}(T)) < U_\varepsilon$. This means $g \in D_\varepsilon(R, I^n)$. In consequence the neighbourhood $S_{\frac{1}{3}\eta}(f)$ of f is contained in $D_\varepsilon(R, I^n)$, which proves that $D_\varepsilon(R, I^n)$ is an open set of $C(R, I^n)$.

C) Using the same notation as in B), for any $\varepsilon > 0$ and any 0-dimensional space R , $D_\varepsilon(R, I^0)$ is dense in $C(R, I^0)$, where I^0 is the space of just one point p_0 .

Proof. As a matter of fact, $D_\varepsilon(R, I^0)$ coincides with $C(R, I^0)$ itself which consists of just one point f: $f(R) = p_0$.

D) Let F be a closed set of a space R . If $h' \in D_\varepsilon(F, I^k)$ and h is a continuous extension of h' over R , then there exists an open set U containing F such that the restriction of h to U belongs to $D_\varepsilon(U, I^k)$, i.e. for some $\eta' > 0$, $\delta(T) \leqq \eta'$ and $T \subset I^k$ imply $\tilde{\delta}(h^{-1}(T) \cap U) < U_\varepsilon$, where U_ε is the same covering as in Definition III.9.

Proof. Suppose $\delta(T) \leqq \eta$ and $T \subset F$ imply $\tilde{\delta}(h'^{-1}(T)) < U_\varepsilon$. Since I^k is compact, we can cover it with finitely many open balls with radius $\frac{1}{4}\eta$, which we denote by $V_i = S_{\frac{1}{4}\eta}(q_i)$, $i = 1, \dots, s$. Define

(1) $$S_i = S_{\frac{1}{2}\eta}(q_i)\ ,\quad i = 1, \dots, s\ .$$

Then there exists an open covering N_i of $h'^{-1}(S_i)$ with $\operatorname{ord} N_i \leqq 1$ and $N_i < U_\varepsilon$. Let $N_i = \{N_\gamma \mid \gamma \in \Gamma_i\}$; then we can assign to every N_γ an element $A_\gamma \in U_\varepsilon$ such that $N_\gamma \subset A_\gamma$.

For each $p \in h'^{-1}(S_i)$ there exists a unique $N_{\gamma(p)} \in N_i$ such that $p \in N_{\gamma(p)}$. Hence to each point p of $h'^{-1}(S_i)$ we can assign a positive number $\varepsilon_i(p)$ satisfying

i) $F \cap S_{\varepsilon_i(p)}(p) \subset N_{\gamma(p)} \in N_i$,

ii) $S_{\varepsilon_i(p)}(p) \subset A_{\gamma(p)} \in U_\varepsilon$,

iii) $d(h(q),\ h(p)) < \frac{1}{4}\eta$ for every $q \in S_{\varepsilon_i(p)}(p)$.

We let

$$(2)\quad \begin{cases} \varepsilon(p) = \min \{ \frac{1}{2}\varepsilon_i(p) \mid h'^{-1}(S_i) \ni p \} , \\ U_\gamma = \cup \{ S_{\varepsilon(p)}(p) \mid p \in N_\gamma \} \quad \text{for every } \gamma \in \Gamma_i , \\ U_i = \cup \{ U_\gamma \mid \gamma \in \Gamma_i \} , \ i = 1,\dots,s, \\ U = \bigcup_{i=1}^{s} U_i . \end{cases}$$

Then U is an open set containing F.

Now, to show

$$(3)\qquad h^{-1}(V_i) \cap U = U_i$$

we assume the contrary, i.e. $x \in h^{-1}(V_i) \cap U - U_i \neq \emptyset$. Then by (2) $x \notin S_{\varepsilon(p)}(p)$ for every $p \in \cup\{N_\gamma \mid \gamma \in \Gamma_i\}$. On the other hand, since $x \in U$, then necessarily, by (2) $x \in S_{\varepsilon(p)}(p)$ for some $p \in F$. Hence we know that $x \in S_{\varepsilon(p)}(p)$ for some p with

$$p \in F - \cup\{N_\gamma \mid \gamma \in \Gamma_i\} = F - h'^{-1}(S_i) = F - h^{-1}(S_i) .$$

This implies, by virtue of (1) and iii) $d(h(p),q_i) \geq \frac{1}{2}\eta$ and then $d(h(x),q_i) > \frac{1}{4}\eta$. On the other hand, it follows from $x \in h^{-1}(V_i)$ that $h(x) \in V_i$, i.e. $d(h(x),q_i) < \frac{1}{4}\eta$.

This contradiction proves (3).

By (2) $U_i' = \{U_\gamma \mid \gamma \in \Gamma_i\}$ is an open covering of U_i. ii) and (2) imply that $U_\gamma \subset A_\gamma$ for every $\gamma \in \Gamma_i$ and accordingly $U_i' < U_\varepsilon$. Since N_i has order ≤ 1, i) and (2) imply that ord $U_i' \leq 1$. Hence $\delta(U_i) < U_\varepsilon$. By virtue of (3) this proves $\delta(h^{-1}(V_i) \cap U) < U_\varepsilon$.

Since I^k is compact, by 1.2 A) we can find $\eta' > 0$ such that if a subset T of I^k has diameter $\leq \eta'$, then $T \subset V_i$ for some i. Then $\delta(T) \leq \eta'$ implies $\delta(h^{-1}(T) \cap U) < U_\varepsilon$.

Thus the proof of this assertion is complete.

E) For any $\varepsilon > 0$ and any space R with $\dim R \leq n$, $D_\varepsilon(R,I^n)$ is dense in $C(R,I^n)$

Proof. Since for $n = 0$ this assertion is proved in C), to use induction on n we assume the assertion valid for $n-1$.

Let Φ be a given point of $C(R,I^n)$ and ε' a given positive number. Suppose

$$\Phi(p) = (g(p),\varphi(p)),\quad g \in C(R,I^{n-1}) ,\quad \varphi \in C(R,I^1) .$$

Then, since $\dim R \leq n$, in view of the proof of 7 A), we can easily construct an open collection $M = \{M_\gamma \mid \gamma \in \Gamma\}$ in R such that

i) M is a locally finite open collection with ord $M \leq 1$,

ii) $\overline{\cup\{M_\gamma \mid \gamma \in \Gamma\}} = R$.

iii) $M < \mathcal{U}_\varepsilon$ for the covering $\mathcal{U}_\varepsilon$ in Definition III.9,

iv) $\dim \bigcup \{ B(M_\gamma) \mid \gamma \in \Gamma \} \leq n-1$,

v) $| \varphi(a) - \varphi(b) | < \frac{1}{3}\varepsilon'$ if $a, b \in \bar{M}_\gamma$.

(Choose a sufficiently small $\mathcal{U}$ in the said proof and $\mathcal{V}$ satisfying $\dim B(V_\alpha) \leq n-1$ for every α ; then put $M_\alpha = V_\alpha - \bigcup \{ \bar{V}_\beta \mid \beta < \alpha \}$.) By applying the induction hypothesis to $F = \bigcup \{ (B(M_\gamma) \mid \gamma \in \Gamma \}$ we get a mapping $h' \in D_\varepsilon(F, I^{n-1})$ such that $d(h'(p), g(p)) \leq \delta < \frac{1}{6}\varepsilon'$ for every $p \in F$. By Theorem I.7 we can continuously extend h' to a mapping h_0 of R into I^{n-1} . Then, for each $p \in F$ we choose a neighbourhood $N(p)$ such that $q \in N(p)$ implies $d(h_0(p), h_0(q)) < \delta$ and $d(g(p), g(q)) < \delta$

Define $N = \bigcup \{ N(p) \mid p \in F \}$. Then we construct a real valued continuous function α on R satisfying $\alpha(F) = 0$, $\alpha(R - N) = 1$, and $0 \leq \alpha \leq 1$.

Putting $h = \alpha g + (1-\alpha)h_0$ in vectorial notation, we have a continuous extension h of h' over R .

Since $d(h_0(p), g(p)) \leq \delta$ for every $p \in F$, it follows from the definition of N that $d(h(q), g(q)) \leq 3\delta$ for every $q \in R$, and hence

(1) $\rho(h, g) \leq 3\delta < \frac{1}{2}\varepsilon'$

where ρ denotes the metric in $C(R, I^{n-1})$. Then, by use of D) we can construct an open set U such that $U \supset F$, and

(2) for some $\eta' > 0$, $\delta(T) \leq \eta'$ and $T \subset I^{n-1}$ imply $\tilde{\delta}(U \cap h^{-1}(T)) < \mathcal{U}_\varepsilon$.

Now, we consider a real valued continuous function σ on R such that $\sigma(R-U) = 0$ $\sigma(F) = 1$ and $0 \leq \sigma \leq 1$, and put $\psi = \varphi + \frac{1}{2}\varepsilon'\sigma$. Then $\Psi(p) = (h(p), \psi(p))$ is a mapping which satisfies $\Psi \in C(R, I^n)$ and $\rho(\Psi, \Phi) < \varepsilon'$.

We can easily prove this assertion by use of (1) and the definition of ψ . Thus to complete this proof it remains only to prove that $\Psi \in D_\varepsilon(R, I^n)$.

For this purpose we shall show that if

(3) $\delta(T) \leq \min(\eta', \frac{1}{6}\varepsilon')$

for a closed set T of I^n , then

(4) either $\bar{M}_\gamma \cap \Psi^{-1}(T) \subset R - F$ or $\bar{M}_\gamma \cap \Psi^{-1}(T) \subset U$

for each $\gamma \in \Gamma$.

Let us assume the contrary: then for some γ we get two points a and b such that $a \in F \cap \bar{M}_\gamma \cap \Psi^{-1}(T) \neq \emptyset$, which implies $\psi(a) = \varphi(a) + \frac{1}{2}\varepsilon'\sigma(a) = \varphi(a) + \frac{1}{2}\varepsilon'$ and $b \in (R-U) \cap \bar{M}_\gamma \cap \Psi^{-1}(T) \neq \emptyset$, which implies $\psi(b) = \varphi(b) + \frac{1}{2}\varepsilon'\sigma(b) = \varphi(b)$.

Since $\psi(a)$ and $\psi(b) \in T$, in view of (3) we know that

$$| \varphi(a) - \varphi(b) + \tfrac{1}{2}\varepsilon' | = | \psi(a) - \psi(b) | \leq \tfrac{1}{6}\varepsilon' .$$

Hence $| \varphi(a) - \varphi(b) | \geq \frac{1}{3}\varepsilon'$. This, however, contradicts v) and accordingly proves (4)

We let

$$\Gamma' = \{\gamma \mid \gamma \in \Gamma,\ \bar{M}_\gamma \cap \Psi^{-1}(T) \subset R - F\} \quad \text{, and} \quad \Gamma'' = \{\gamma \mid \gamma \in \Gamma,\ \bar{M}_\gamma \cap \Psi^{-1}(T) \subset U\}:$$

then by (4) $\Gamma = \Gamma' \cup \Gamma''$. Suppose T is a given closed set of I^n which satisfies (3). If $\gamma \in \Gamma'$, then by the definition of F

$$\bar{M}_\gamma \cap \Psi^{-1}(T) \subset \bar{M}_\gamma \cap \Psi^{-1}(T) \cap (R - F)$$
$$\subset \bar{M}_\gamma \cap \Psi^{-1}(T) \cap (R - B(M_\gamma)) \subset M_\gamma \text{ , i.e.}$$

(5) $\bar{M}_\gamma \cap \Psi^{-1}(T) \subset M_\gamma$ for every $\gamma \in \Gamma'$.

If $\gamma \in \Gamma''$, then note that $\Psi^{-1}(T) \subset h^{-1}(T')$, where T' denotes the projection of T on I^{n-1} . Therefore

$$\bar{M}_\gamma \cap \Psi^{-1}(T) \subset U \cap \bar{M}_\gamma \cap \Psi^{-1}(T) \subset U \cap h^{-1}(T) \quad \text{for every} \quad \gamma \in \Gamma'' .$$

Hence

(6) $A = \cup\{\bar{M}_\gamma \mid \gamma \in \Gamma''\} \cap \Psi^{-1}(T) \subset U \cap h^{-1}(T')$.

On the other hand, we can deduce from (2), (3) and $\delta(T') \leqq \delta(T)$ that $\tilde{\delta}(U \cap h^{-1}(T')) < \mathcal{U}_\varepsilon$. In consequence, (6) implies

(7) $\delta(A) < \mathcal{U}_\varepsilon$.

From the closedness of T and i) it follows that A is closed. Therefore

$$\mathcal{K} = \{A, \bar{M}_\gamma \cap \Psi^{-1}(T) \mid \gamma \in \Gamma' - \Gamma''\}$$

is a discrete closed collection by virtue of (5) and i).

Now we choose an open set W of R such that $W \supset A$ and $\bar{W} \cap \bar{M}_\gamma \cap \Psi^{-1}(T) = \emptyset$ for all $\gamma \in \Gamma' - \Gamma''$.

Because of (7) there is an open collection $\mathcal{W}$ of order $\leqq 1$ which covers A and refines $\mathcal{U}_\varepsilon$. We may assume that each member of $\mathcal{W}$ is contained in W . Then $\mathcal{W} \cup \{M_\gamma \cap (R - \bar{W}) \mid \gamma \in \Gamma' - \Gamma''\}$ is an open collection of order $\leqq 1$ which refines $\mathcal{U}_\varepsilon$ and covers $\cup\{K \mid K \in \mathcal{K}\} = \Psi^{-1}(T)$. (Recall i), ii), iii) and (5).)
This implies $\tilde{\delta}(\Psi^{-1}(T)) < \mathcal{U}_\varepsilon$, and thus $\Psi \in D_\varepsilon(R, I^n)$ is proved, and hence $D_\varepsilon(R, I^n)$ is dense in $C(R, I^n)$.

Theorem III. 10 [3] . *A space R has $\dim \leqq n$ $(n \geqq 0)$ if and only if it satisfies*
1) for any metrization of R there exists a uniformly 0-dimensional continuous mapping of R into I^n or

[3] Proved first by M. Katětov [2].

2) *for a suitable metrization of* R *there exists a uniformly 0-dimensional continuous mapping of* R *into* I^n .

Proof. Let $\dim R \leq n$; then from B) and E) it follows that $D_{1/i}(R, I^n)$, $i = 1, 2, \ldots$ are open and dense in $C(R, I^n)$, for any metric of R . Since $C(R, I^n)$ is a complete metric space, by Theorem I.5 we know that $\bigcap_{i=1}^{\infty} D_{1/i}(R, I^n) \neq \emptyset$. Thus $f \in \bigcap_{i=1}^{\infty} D_{1/i}(R, I^n)$ is the desired mapping in 1).

1) obviously implies 2). Finally, by A) we can deduce $\dim R \leq n$ from 2) because $\dim I^n \leq n$.

CHAPTER IV

DIMENSION OF SEPARABLE METRIC SPACES

Although we are now in a position to establish most of the main theorems of dimension theory for general metric spaces, there are yet some theorems which make it preferable to devote one chapter to separable metric spaces. We can extend some of them to more general spaces than separable metric spaces, but it is often true that their best form is achieved only in separable metric spaces. We can extend, for example, Theorem IV.9 about relations between dimension and measure to non-metrizable spaces in a modified form. As for the imbedding theorem we can extend it to general metric spaces [1], but imbedding into Euclidean spaces is possible only for separable metric spaces.

The purpose of this chapter is to discuss theorems in which separability has its own meaning and to give an account of classical theorems in Euclidean spaces.

IV. 1. Cantor manifolds

Theorem IV. 1. $\operatorname{Ind} R = \operatorname{ind} R = \dim R$ *for every separable metric space* R .

Proof. Since $\operatorname{Ind} R \geq \operatorname{ind} R$ is clear, let us show $\operatorname{Ind} R \leq \operatorname{ind} R$.

If $\operatorname{ind} R = -1$, then this assertion is obvious. Suppose that $\operatorname{Ind} R \leq \operatorname{ind} R$ has been established for every R with $\operatorname{ind} R \leq n-1$. Now, assume $\operatorname{ind} R = n$ for a separable metric space R with a countable open basis $\{U_i \mid i = 1,2,\dots\}$. We define open sets W_{ij}, $i,j = 1,2,\dots$ as follows. If there exist open sets W satisfying $\bar{U}_i \subset W \subset U_j$ and $\operatorname{ind} B(W) \leq n-1$, i.e. $\operatorname{Ind} B(W) \leq n-1$ by virtue of the induction hypothesis, then we put $W_{ij} = W$ for one of those W . If there exists no open set satisfying this condition, then we put $W_{ij} = \emptyset$ . .

[1] See Chapter VI.6.

It follows from $\operatorname{ind} R = n$ that $\{ W_{ij} \mid i, j = 1, 2, \dots \}$ is an open basis with $\operatorname{Ind} B(W_{ij}) \leqq n-1$. Hence R satisfies the condition of Theorem II.2, and hence $\operatorname{Ind} R \leqq n$. This means $\operatorname{Ind} R \leqq \operatorname{ind} R$.

Since $\operatorname{Ind} R = \dim R$ is established for every metric space in Theorem II.7, we can conclude $\operatorname{Ind} R = \operatorname{ind} R = \dim R$ for every separable metric space R.

By virtue of this theorem $\operatorname{Ind} R$, $\operatorname{ind} R$ and $\dim R$ have the same meaning throughout this chapter.

It is known that $\operatorname{ind} R = \dim R$ for any space R which can be expressed as a countable sum of closed subspaces with the star-finite property (or, more generally, which has a σ-star-finite open basis)[2]. On the other hand P. Roy [1] gave an example of a metric space R such that $\operatorname{ind} R = 0$ and $\operatorname{Ind} R = 1$ and thus negatively solved the famous problem on the equivalence between ind and Ind for general metric spaces [3].

---

[2] See K. Morita [4] and A. Zarelua [1].

[3] We shall give here only a description of his example, a complete metric space R such that $\operatorname{ind} R = 0$ but $\operatorname{Ind} R = 1$. For that purpose we introduce the following notations. We mean by a sequence a function defined on either the set of non-negative integers or the set of positive integers or any initial segment of either of them.

X = the set of all finite sequences $\{ x_i \}$ of real numbers defined on an initial segment of the non-negative integers such that $x_i = 0$ only in case $i = 0$. If $x = \{ x_i \mid i = 0, \dots, n \}$, then we denote by $|x|$ the integer n. Y = the set of all sequences $\{ y_i \}$ of positive numbers defined on the set of all positive integers such that $y_i \neq y_j$ if $i \neq j$. If r is a positive number, then Y_r denotes the set of all members of Y which take the value r and we denote by F_r a one-to-one mapping from the positive numbers onto Y_r. Z = the set of all sequences $\{ z_i \}$ of positive numbers defined on the set of all positive integers.

Now we define the desired space R as follows: $R = R_1 \cup R_2$, where R_1 = the set of all sequences of non-zero real numbers defined on the set of all positive integers. $R_2 = X \times Y \times Z$ with $p(X)$, $p(Y)$ and $p(Z)$ denoting the coordinates of a point p of R_2. The topology of R is defined by the open basis $\mathcal{U}$ consisting of two types of subsets U_x and $U_{p,n}$ of R. Let $x = \{ x_i \} \in X$; then we define $U_x = U^1 \cup U^2$ as follows:

$$U^1 = \{ p \in R_1 \mid p_i = x_i \text{ for } i = 1, \dots, |x|, \text{ if } |x| > 0 \},$$

$$U^2 = \{ p \in R_2 \mid |p(X)| \geqq |x| ;\ p(X)_i = x_i \text{ for } i = 1, \dots, |x|, \text{ if } |x| > 0 \}.$$

where we denote by $p(X)_i$ the i-th term of the sequence $p(X)$.

Let n be a positive integer and p a point of R_2; then we define $U_{n,p} = U^0 \cup U^+ \cup U^-$ as follows:

$$U^0 = \{ q \in R_2 \mid q(X) = p(X) ;\ q(Y) = p(Y) ;\ q(Z)_i = p(Z)_i \text{ for } i = 1, \dots, n-1, \text{ if } n > 1 \},$$

$$U^+ = \bigcup_{j=1}^{\infty} U_{\gamma(p,n,+,j)}, \quad U^- = \bigcup_{j=1}^{\infty} U_{\gamma(p,n,-,j)},$$

where $\{ \gamma(p,n,+,j) \mid j = 1, 2, \dots \}$ and $\{ \gamma(p,n,-,j) \mid j = 1, 2, \dots \}$ are two infinite sequences of members of X such that

$$|\gamma(p,n,\pm,j)| = |p(X)| + n + 1,$$

$$\gamma(p,n,\pm j)_i = \begin{cases} p(X)_i & \text{for } i = 0, \dots, |p(X)|, \\ \pm p(Y)_{n+j-1} & \text{for } i = |p(X)| + 1, \\ \mp F_r^{-1}(p(Y)) \text{ with } r = p(Y)_{n+j-1} & \text{for } i = |p(X)| + 2, \end{cases}$$

and if $n > 1$, then $\gamma(p,n,\pm,j)_i = \mp p(Z)_{i'}$ with $i' = i - |p(X)| - 2$ for $i = |p(X)| + 3, \dots, |p(X)| + n + 1$.

Definition IV. 1. *For* $n>0$ *a compact n-dimensional metric space* R *is called an n-dimensional Cantor manifold if* $R-A$ *is connected for every at most* $(n-2)$*-dimensional subset* A *of* R *.*

Note that the above condition preceded by 'if' implies $\dim R \geq n$ if R contains at least two points.

A) Let R be a compact metric space and f a continuous mapping of a closed subset F into S^n. If f cannot be extended over R, then there exists a closed subset G of R such that

i) f cannot be extended over $F \cup G$, but

ii) f can be extended over $F \cup H$ for every proper closed subset H of G.

Proof. By use of Zorn's lemma [4] we obtain a maximal family $\{G_\alpha \mid \alpha \in A\}$ of closed sets such that $\{G_\alpha\}$ is totally ordered with respect to the order $\subset$, i.e. either $G_\alpha \subset G_\beta$ or $G_\alpha \supset G_\beta$ for every $\alpha, \beta \in A$, and

(1) f cannot be extended over $F \cup G_\alpha$ for every $\alpha \in A$.

If we let $G = \cap \{G_\alpha \mid \alpha \in A\}$, then G is a closed subset of R. We can see that f cannot be extended over $F \cup G$. For suppose f were extended over $F \cup G$, then there would be, by the corollary to Theorem I.7, an open set U containing $F \cup G$ over which f could be extended. By virtue of the compactness of R there would be G_α satisfying $G_\alpha \subset U$, and hence $F \cup G_\alpha \subset U$. But this contradicts (1).

If follows from the maximality condition of $\{G_\alpha\}$ that f can be extended over $F \cup H$ for every proper closed subset H of G.

Theorem IV. 2. *Any compact n-dimensional space* R *contains an n-dimensional Cantor manifold as a subset* $(n>0)$ *.*

Proof. Since $\dim R = n$, there exists, by Theorem III.2, a closed subset F of R and a mapping f of R into S^{n-1} which cannot be extended over R. We get, by use of A), a closed set G satisfying i) and ii) there.

Let us show that G is an n-dimensional Cantor manifold. For if we assume the contrary, then there exist proper closed sets G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $\dim G_1 \cap G_2 \leq n-2$. It follows from ii) that f can be extended to mappings f_1 and f_2 defined over $F \cup G_1$ and $F \cup G_2$ respectively. By virtue of

[4] By Zorn's lemma we mean the following: A partially ordered system P each of whose totally ordered subsystems has an upper bound, contains a maximal element. In the present case we consider the system of all the families of closed sets satisfying the conditions concerned.

III.3.D) f_1 can be extended to a mapping of $F \cup G$ into S^{n-1} which is also an extension of f over $F \cup G$. But this contradicts i) of A). (Note that $\dim G \geqq n$ automatically follows as remarked right after Definition IV.1.).

IV. 2. Dimension of E^n

A) Let $\{S_i \mid i = 0, \ldots, n\}$ be the faces of an n-simplex T^n and $\{F_i \mid i = 0, \ldots, n\}$ a covering of T^n for which $F_i \subset T^n - S_i$. If K is a given geometrical complex triangulating T^n, then K contains an n-simplex which intersects every member of $\{F_i \mid i = 0, \ldots, n\}$ [5].

Proof. We assign to each vertex p of K a number $i(p)$ for which $p \in F_{i(p)}$. To each r-simplex $T = [p_0, \ldots, p_r]$ of K we assign the set $\{i(p_0), \ldots, i(p_r)\}$ of numbers, which is denoted by $i(T) = \{i(p_0), \ldots, i(p_r)\}$. Suppose $\{T_j \mid j = 1, \ldots, k\}$ are all the n-simplexes belonging to K. Denote by s the number of T_j's such that $i(T_j) = \{0, 1, \ldots, n\}$. Then we shall prove by induction on n that s is odd.

In case of $n = 1$ this assertion is clearly true, because 0 is assigned to a vertex of T^1 and 1 to another vertex.

Now we assume this assertion for $(n-1)$-simplexes to prove it for the n-simplex T^n. Let s_j be the number of the $(n-1)$-dimensional faces T' of T_j such that $i(T') = \{1, \ldots, n\}$. Then it is easily seen that $s_j = 0$, 1 or 2. For, if $s_j \neq 0$, then there exists a face $T' = [p_1, \ldots, p_n]$ of $T_j = [p_0, p_1, \ldots, p_n]$ for which $i(T') = \{1, \ldots, n\}$. If $i(p_0) = 0$, then T' is the only face satisfying this condition, and hence $s_j = 1$. In this case we note that $i(T_j) = \{0, 1, \ldots, n\}$.

If $1 \leqq i(p_0) \leqq n$, then clearly $s_j = 2$. We denote by t the number of such T_j's. Thus we obtain

$$(1) \qquad \sum_{j=1}^{k} s_j = s + 2t .$$

On the other hand, we denote by u the number of $(n-1)$-simplexes T' of K for which

$$(2) \qquad \begin{cases} i(T') = \{1, \ldots, n\}, \\ T' \text{ is contained in } \bigcup_{i=0}^{n} S_i , \end{cases}$$

[5] This proposition is due to E. Sperner [1] and called Sperner's lemma. As for elementary knowledge of combinatorial topology see P. Alexandroff - H. Hopf [1] or J. Nagata - Y. Kodama [1]. See also the introductory portion of VIII.1 especially the footnote to Example VIII.1 for the definition of a complex.

and by v the number of the $n-1$-simplexes T'' of K such that $i(T'') = \{1,\dots,n\}$, and T'' is not contained in $\bigcup_{i=0}^{n} S_i$. The former type of simplex faces just one of $\{T_j \mid j=1,\dots,k\}$, and the latter type faces just two of them. Therefore we obtain $\sum_{j=1}^{k} s_j = u+2v$, which combined with (1) implies $s+2t=u+2v$.

Thus, to prove that s is odd it suffices to show that u is odd. We note that if T' is an $(n-1)$-simplex of K which satisfies (2), then T' must be contained in S_0. For, if T' is contained in S_i for some i different from 0, then from $F_i \subset T^n - S_i$ it follows that $i(T')$ cannot contain i contradicting (2). Hence u is the number of the $(n-1)$-simplexes T' of K contained in S_0 with $i(T') = \{1,\dots,n\}$. Applying the induction hypothesis to the covering $\{F_i \cap S_0 \mid i = 1,\dots,n\}$ of the $(n-1)$-simplex S_0, we conclude that u is odd. This completes the proof that s is odd, and hence K contains at least one n-simplex T_j with $i(T_j) = \{0,1,\dots,n\}$, i.e. a simplex which intersects each of $\{F_i \mid i=0,\dots,n\}$.

Now we are in a position to prove

Theorem IV. 3. $\dim T^n = n$.

Proof. Since $\dim E^n \leqq n$ is deduced by the product theorem from $\dim E^1 \leqq 1$, in view of $T^n \subset E^n$ we obtain $\dim T^n \leqq n$. Hence it suffices to show that $\dim T^n \geqq n$.

Let $\{S_i \mid i=0,\dots,n\}$ be the faces of T^n. Then we consider the open covering $\{T^n - S_i \mid i=0,\dots,n\}$ of T^n. Suppose $V = \{V_i \mid i=0,\dots,n\}$ is a given open covering of T^n for which $V_i \subset T^n - S_i$. Then there exists a closed covering $\{F_i \mid i=0,\dots,n\}$ of T^n such that $F_i \subset V_i$.

Now we shall show that $\bigcap_{i=0}^{n} F_i \neq \emptyset$.

If the contrary is true, then since T^n is compact, $\{T^n - F_i \mid i=0,\dots,n\}$ is a uniform covering of T^n. Hence there exists $\varepsilon > 0$ such that for every $p \in T^n$ $S_\varepsilon(p) \subset T^n - F_i$ for some i.

We construct a triangulation K of T^n each of whose simplexes has diameter $< \varepsilon$. Then every simplex of K is contained in some $T^n - F_i$, but this contradicts A). Thus we obtain $\bigcap_{i=0}^{n} F_i \neq \emptyset$, which implies $\bigcap_{i=0}^{n} V_i \neq \emptyset$. This means $\operatorname{ord} V \geqq n+1$.

Therefore we conclude that $\dim T^n \geqq n$, which completes the proof.

Theorem IV,3 implies $\dim I^n = n$, $\dim S^n = n$ and especially

Theorem IV. 4. $\dim E^n = n$.

B) Let A and B be countable subsets which are dense in E^n. Then there exists a topological mapping of E^n onto itself which maps A onto B.

Proof. In case $n=1$ we write $A = \{a_i \mid i=1,2,\dots\}$ and $B = \{b_i \mid i=1,2,\dots\}$ to establish a one-to-one mapping f between A and B. First we define $f(a_1) = b_1$.

Suppose we have defined $f(a_i) = b_{m_i}$, $i=1,\dots,k-1$, so that f preserves the natural order, i.e. $a_i < a_j$ implies $f(a_i) < f(a_j)$. Then we define $f(a_k)$ as follows. We denote by m_k the first number of $N_k = \{ m \mid m \neq m_1, \dots, m_{k-1}$; if $i \leq k-1$, then $a_i < a_k$ or $a_i > a_k$ implies $b_{m_i} < b_m$ or $b_{m_i} > b_m$ respectively $\}$. Note that N_k is not empty because B is dense in E^1. Then we put $f(a_k) = b_{m_k}$.

It is easy to see that f is a one-to-one mapping between A and B preserving the natural order. To prove by induction that the mapping is onto, assume that $\{ b_1, \dots, b_{i-1} \} \subset f(A)$. Then $\{ 1, \dots, i-1 \} \subset \{ m_1, \dots, m_k \}$ for some k. Assume $i \notin \{ m_1, \dots, m_k \}$ and let e.g.

$$b_{m_1} < b_{m_2} < \dots < b_{m_{j-1}} < b_i < b_{m_j} < \dots < b_{m_k} .$$

Then, since A is dense, we obtain $a_{j-1} < a_\omega < a_j$ for some $\omega > k$. If ω is the first number satisfying this condition, then $m_\omega = i$, i.e. $f(a_\omega) = b_i \in f(A)$.

We can extend f to a topological mapping from E^1 onto itself as follows. For a given point x of $E^1 - A$, we put $A_1 = \{ a \mid a < x,\ a \in A \}$ and $A_2 = \{ a \mid a > x,\ a \in A \}$. Then A_1 and A_2 are subsets of A such that $B = f(A_1) \cup f(A_2)$ and such that $b < b'$ whenever $b \in f(A_1)$ and $b' \in f(A_2)$. Note that $f(A_1)$ has no maximum, and $f(A_2)$ has no minimum. Since $\bar{B} = E^1$, there is one and only one number y such that $b \in f(A_1)$ or $b \in f(A_2)$ implies $b < y$ or $b > y$, respectively. Setting $f(x) = y$, we obtain a topological mapping (= homeomorphism) f from E^1 onto itself which maps A onto B.

In case $n = 2$ we suppose $A = \{ a_i = (a_i^1, a_i^2) \mid i = 1, 2, \dots \}$ and $B = \{ b_i = (b_i^1, b_i^2) \mid i = 1, 2, \dots \}$.

We may assume that A satisfies $a_i^1 \neq a_j^1$ and $a_i^2 \neq a_j^2$ whenever $i \neq j$, because, using the previous argument on E^1, we can map A to a dense subset of $Q \times Q$ by a homeomorphism φ from E^2 onto itself, where Q denotes the set of all rational numbers. Then consider the rotation θ of E^2 around the origin by $\pi/6$. Now $\theta \circ \varphi$ is a homeomorphism of E^2 onto itself which maps A to a dense subset satisfying the above condition. We assume the same condition for B, too.

To define a one-to-one, order-preserving mapping f from A onto B, first let $f(a_1) = b_1$. (The following definition of N_k will indicate what we mean by 'order-preserving'.) Assume that we have defined $f(a_i) = b_{m_i}$ for $i \leq k-1$. Define $N_k = \{ m \mid m \neq m_1, \dots, m_{k-1}$; if $i \leq k-1$, then $a_i^1 < a_k^1$ $(a_i^1 > a_k^1)$ implies $b_{m_i}^1 < b_m^1$ $(b_{m_i}^1 > b_m^1)$, and $a_i^2 < a_k^2$ $(a_i^2 > a_k^2)$ implies $b_{m_i}^2 < b_m^2$ $(b_{m_i}^2 > b_m^2)$ $\}$. Denote by m_k the first number of N_k and define $f(a_k) = b_{m_k}$.

$f(A) = B$ can be proved analogously to the one-dimensional case. Then it is also easy to extend f to a homeomorphism from E^2 onto itself. We can prove the assertion for E^n in a similar way, but the details are left to the reader.

Theorem IV. 5. *Let S be a subset of E^n. Then* $\dim S = n$ *if and only if S contains an open set of E^n.*

Proof. Since the "if" part is clear, we suppose S contains no open set to show $\dim S \leqq n-1$. Then $E^n - S$ is dense in E^n, and hence we can choose a countable subset A of $E^n - S$ such that A is dense in E^n. By E^n_k we denote the set of all points of E^n exactly k of whose coordinates are irrational. Note that $\dim E^n_k = 0$ for each k follows from the sum theorem. Then E^n_0 is a countable set which is dense in E^n. Hence, by use of B) we can construct a topological mapping f of E^n onto itself which maps A onto E^n_0. Hence S is mapped into $E^n - E^n_0$ by f. Since $E^n - E^n_0 = \bigcup_{i=1}^{n} E^n_i$, and $\dim E^n_i = 0$, $i = 1, \dots, n$, it follows from the decomposition theorem that $\dim (E^n - E^n_0) \leqq n-1$. Therefore $\dim S \leqq n-1$, which proves the "only if" part of this theorem.

IV. 3. Some theorems in Euclidean space

In this section we shall give an account of some classical theorems due to Brouwer as examples of interesting applications of dimension theory.

A) The identity mapping f from I^n onto itself is essential.

Proof. If we assume f to be inessential, then by use of a projection to the boundary S^{n-1} of I^n from an interior point of I^n we can construct a continuous mapping g from I^n into S^{n-1} such that $g(q) = q$ for every $q \in S^{n-1}$. Then, since $\dim I^n = n$, by Theorem III.5 there is an essential mapping h from I^n onto itself. Now $g \cdot h$ is a continuous mapping from I^n into $S^{n-1} \subsetneqq I^n$. If $p \in h^{-1}(S^{n-1})$, then $g \cdot h(p) = h(p)$, which contradicts the fact that h is essential. Thus f must be essential.

In a similar way we can also prove that the identity mapping from S^n onto itself is essential.

Theorem IV. 6 (Brouwer's fixed-point theorem). *Every continuous mapping f of I^n into itself has a fixed point, i.e. a point $p \in I^n$ such that $f(p) = p$.*

Proof. Let us assume the contrary; then there exists a continuous mapping f of I^n into itself such that $f(x) \neq x$ for every point $x \in I^n$. We regard S^{n-1} as the boundary of I^n. Then for $x \in I^n$ we denote by $g(x)$ the intersection of S^{n-1} with the ray beginning at $f(x)$ and passing through x. One can easily see that g is a continuous mapping of I^n into S^{n-1} and leaves each point of S^{n-1} fixed. This contradicts A).

B) Let S be a closed subspace of E^n. Then a point p of S is contained in the boundary $B(S)$ of S with respect to E^n if and only if for every neighbourhood $V(p)$ of p in S there exists a neighbourhood $U(p)$ of p in S such that

$U(p) \subset V(p)$ and such that every continuous mapping of $S - U(p)$ into S^{n-1} can be continuously extended over S.

Proof. To show the "only if" part, for the given neighbourhood $V(p)$ of a point $p \in S$ we choose a spherical neighbourhood $S(p)$ in E^n such that $U(p) = S(p) \cap S \subset V(p)$.

Let f be a given continuous mapping of $S - U(p)$ into S^{n-1}. Since $B(S(p)) = S^{n-1}$, we obtain $\dim B(S(p)) \leqq n-1$. Hence by use of the corollary to Theorem III.2 we can extend f to a mapping g of $(S - U(p)) \cup B(S(p))$ into S^{n-1}. Now, choose a point $q \in S(p) - S$ and define φ as the projection from q on $B(S(x))$. Then we get an extension h of f over S by putting

$$h(x) = g(\varphi(x)) \quad \text{for every } x \in U \cap S\,,$$
$$h(x) = f(x) \quad \text{for every } x \in S - U\,.$$

To show the "if" part we suppose p is an inner point of S. Then there exists a spherical neighbourhood $S(p)$ of p satisfying $\overline{S(p)} \subset S$. Let $U(p)$ be an arbitrary neighbourhood of p contained in $S(p)$. We denote by f the projection of $S - U(p)$ from p onto S^{n-1} which we regard as the boundary of $S(p)$. Then f cannot be extended over S. To show this we suppose g is a continuous extension of f over S. By restricting g to $\overline{S(p)} = I^n$ we get a continuous mapping g' of I^n onto S^{n-1} which fixes every point of S^{n-1}. But this contradicts A).

Let f be a topological mapping of a space R onto a space S; then f maps each open set A of R to an open set B of S. On the other hand if f is a topological mapping of an open set A of R onto a subset B of S, then B may not be open in S (e.g. let $R = A = E^1$, $S = E^2$, $B = \{(x,y) \in E^2 \mid y = 0\}$). However, if $R = S = E^n$, then B must be open as the following theorem asserts.

Theorem IV. 7. *Let f be a topological mapping of a subset A of E^n onto a subset B of E^n. If $p \in A$ is an inner point of A, then $f(p)$ is also an inner point of B. In particular, if A is open, then B is also open.*

Proof. This theorem is a direct consequence of B) because B) characterizes the inner points of a closed subset S of E^n only by the topological properties of S itself. To be more precise, choose $p \in \operatorname{Int} A$; then there is $\varepsilon > 0$ for which $\overline{S_\varepsilon(p)} \subset A$. Note that $\overline{S_\varepsilon(p)}$ and $f(\overline{S_\varepsilon(p)})$ are both closed sets of E^n. Hence by B) we obtain that $f(p) \in \operatorname{Int} f(\overline{S_\varepsilon(p)}) \subset \operatorname{Int} B$.

IV. 4. Imbedding

The purpose of this section is to topologically imbed a given separable metric space R of dimension $\leqq n$ in a $2n+1$-dimensional Euclidean space. More precisely, we can construct a universal n-dimensional set in E^{2n+1}, i.e. a set in which every separable metric space of dimension $\leq n$ can be imbedded, but in which no space of dimension $> n$ can be imbedded. Generally speaking, we cannot imbed R in a Euclidean space of lower dimension as will be shown in the following.

Example IV. 1. We consider a tetrahedron $[a_0, a_1, a_2, a_3]$ in E^3. We take two points b, c on the edges $[a_0, a_3]$ and $[a_1, a_2]$ respectively such that b and c do not coincide with any vertex. Then the six edges of the tetrahedron plus the segment $[b,c]$ are intuitively seen to form a one-dimensional space which cannot be imbedded in E^2.

Definition IV. 2. *Let f be a continuous mapping of a space R into a space S and V a covering of R. If for every point y of S there exists a neighbourhood $U(y)$ of y such that $f^{-1}(u(y)) \subset V$ for some $V \in V$, then f is called a V-mapping.*

A) Let us denote the metric space of all continuous mappings of a separable metric space R into I^{2n+1} by $C(R, I^{2n+1})$. Let L be an n-dimensional plane of I^{2n+1} and V a finite open covering of R. Then

$$D(L,V) = \{ f \mid f \in C(R, I^{2n+1}),\ \overline{f(R)} \cap L = \emptyset\ ,\ f \text{ is a } V\text{-mapping} \}$$

is an open subset of $C(R, I^{2n+1})$.

Proof. To prove A) it suffices to show that a given point f of $D(L,V)$ is an interior point of it.

For every point x of I^{2n+1} we fix an open neighbourhood $U(x)$ of x such that $f^{-1}(U(x)) \subset V$ for some $V \in V$. Since I^{2n+1} is compact, there exists a finite subcovering $\{ U(x_j) \mid j = 1, \ldots, s \}$ of $\{ U(x) \mid x \in I^{2n+1} \}$, i.e.

$$I^{2n+1} = \bigcup_{j=1}^{s} U(x_j)\ .$$

By virtue of the compactness of I^{2n+1} there exists a positive number δ such that for every $x \in I^{2n+1}$ and for some j

(1) $\quad S_\delta(x) \subset U(x_j)$.

Since $\overline{f(R)}$ and L are compact because of the compactness of I^{2n+1}, for the metric ρ of I^{2n+1} we have $\rho(\overline{f(R)}, L) > 0$. Hence we can choose δ such that

(2) $\quad \delta < \rho(\overline{f(R)}, L)$.

Now, let g be any mapping satisfying

(3) $\rho'(f,g) < \frac{1}{6}\delta$,

where we denote by ρ' the metric of $C(R, I^{2n+1})$. Let x be a given point of I^{2n+1} and put $N = g^{-1}(S_{\frac{1}{6}\delta}(x))$. We denote by y a fixed point of N . Then we assert that

(4) $f(N) \subset S_\delta(f(y)) \subset U(x_j)$

for some j . For, if $z \in N$, then $g(y) \in g(N) \subset S_{\frac{1}{6}\delta}(x)$, and $g(z) \in g(N) \subset S_{\frac{1}{6}\delta}(x)$. This implies $\rho(g(y), g(z)) < \frac{1}{3}\delta$.

On the other hand, from (3) it follows that $\rho(f(y), g(y)) < \frac{1}{6}\delta$, and $\rho(g(z), f(z)) < \frac{1}{6}\delta$.

Therefore we get $\rho(f(y), f(z)) < \frac{2}{3}\delta < \delta$, which means $f(z) \in S_\delta(f(y))$. Since z is a given point of N , this means $f(N) \subset S_\delta(f(y))$. The latter half of the relation (4) follows from (1). Thus from (4) it follows that

$$N = g^{-1}(S_{\frac{1}{6}\delta}(x)) \subset f^{-1}(U(x_j)) \subset V$$

for some $V \in \mathcal{V}$. The last part of this relation is deduced from the definition of $U(x)$. Hence g is a $\mathcal{V}$-mapping.

Furthermore, we can deduce from (2) and (3) that $\overline{g(R)} \cap L = \emptyset$. Consequently $g \in D(L, \mathcal{V})$, which proves $S_{\frac{1}{6}\delta}(f) \subset D(L, \mathcal{V})$. Therefore f is an interior point of $D(L, \mathcal{V})$.

B) Under the same assumption as in A), $D(L, \mathcal{V})$ is dense in $C(R, I^{2n+1})$ if $\dim R \leq n$.

Proof. Let f be a given point of $C(R, I^{2n+1})$ and δ a given positive number. We shall construct g such that $\rho'(f,g) < \delta$, $g \in D(L, \mathcal{V})$. First we let

(1) $I^{2n+1} = \bigcup_{j=1}^{m} S_{\frac{1}{4}\delta}(x_j)$, and $\mathcal{W} = \{ f^{-1}(S_{\frac{1}{4}\delta}(x_j)) \mid j = 1, \dots, m \}$.

Then, since $\dim R \leq n$, we can choose a finite open covering $\mathcal{N}$ of R such that

(2) $\mathcal{N}^\Delta < \mathcal{V} \wedge \mathcal{W}$,

(3) $\operatorname{ord} \mathcal{N} \leq n + 1$.

Suppose $\mathcal{N} = \{ N_i \mid i = 1, \dots, s \}$; then by use of the property of I^{2n+1} we can choose vertices $x(N_i)$ in I^{2n+1} and $p_1, \dots, p_{n+1}$ in L such that

(4) $\rho(f(N_i), x(N_i)) < \frac{1}{4}\delta$, and

(5) the vertices $x(N_1),\dots,x(N_s)$, $p_1,\dots,p_{n+1}$, are in a *general position*, i.e. if $0 \leqq m \leqq 2n$, then any $m+2$ vertices do not lie in an m-dimensional plane of E^{2n+1}.

(Note that L is spanned by $p_1,\dots,p_{n+1}$.)

Now we define a *Kuratowski mapping* $g \in C(R, I^{2n+1})$ by

$$(6)\qquad g(x) = \frac{\sum_{i=1}^{s} \rho(x, R - N_i)x(N_i)}{\sum_{i=1}^{s} \rho(x, R - N_i)},$$

where each $x(N_i)$ should be regarded as a position vector. To show $\rho'(f,g) < \delta$, we suppose x is a given point of R. If $x \notin N_i$, then $\rho(x, R - N_i) = 0$. If $x \in N_i$ then we obtain from (1) and (2) that $\rho(f(x), f(y)) < \frac{1}{2}\delta$ for every $y \in N_i$. Hence from (4) it follows that $\rho(f(x), x(N_i)) < \frac{3}{4}\delta$. Hence by (6) $\rho(f(x), g(x)) < \frac{3}{4}\delta$ holds for every $x \in R$. This proves $\rho'(f,g) \leqq \frac{3}{4}\delta < \delta$.

It only remains to prove $g \in D(L, V)$.

Suppose $N_{i_1},\dots,N_{i_t}$ are all the members of N which contain a given point p of R. Then we denote by $L(p)$ the $(t-1)$-dimensional plane spanned by $x(N_{i_1}),\dots,x(N_{i_t})$. Since there is only a finite number of distinct planes $L(p)$ for p running through R,

$$\eta = \min\{\rho(L(p), L(p')) \mid L(p) \cap L(p') = \emptyset\} > 0 .$$

Then any two of those planes $L(p)$ and $L(p')$ either meet or else are at a distance $\geqq \eta$ from each other.

We consider a given point x of I^{2n+1}. Let $p, p' \in g^{-1}(S_{\frac{1}{2}\eta}(x))$; then $\rho(g(p), g(p')) < \eta$. Since $g(p) \in L(p)$ and $g(p') \in L(p')$ follows from (6), we have $\rho(L(p), L(p')) < \eta$. Therefore from the property of η we obtain

$$(7)\qquad L(p) \cap L(p') \neq \emptyset .$$

Suppose $L(p)$ is spanned by $x(N_{i_1}),\dots,x(N_{i_t})$ and $L(p')$ is spanned by $x(N_{j_1}),\dots,x(N_{j_u})$. Then by (7) $x(N_{i_1}),\dots,x(N_{i_t})$, $x(N_{j_1}),\dots,x(N_{j_u})$ lie on a $(t+u-2)$-dimensional plane. On the other hand, it follows from (3) that $t \leqq n+1$, $u \leqq n+1$; hence $t+u \leqq 2n+2$. Thus, in view of (5) we conclude that at least one of the $x(N_{i_1}),\dots,x(N_{i_t})$ coincides with one of the $x(N_{j_1}),\dots,x(N_{j_u})$. Suppose, for example $x(N_{i_1}) = x(N_{j_1})$, i.e. $N_{i_1} = N_{j_1}$. Then this implies $p, p' \in N_{i_1}$, and hence $p' \in S(p, N)$ for every p, p' with $p, p' \in g^{-1}(S_{\frac{1}{2}\eta}(x))$.

Therefore, by (2) $g^{-1}(S_{\frac{1}{2}\eta}(x)) \subset S(p, N) \subset V$ for some $V \in V$. Hence g is a V-mapping.

Now, let p be a given point of R and $N_{i_1},\dots,N_{i_t}$ all the members of N which contain p. Since $x(N_{i_1}),\dots,x(N_{i_t})$, $p_1,\dots,p_{n+1}$ are in a general position by (5), we obtain $\rho(L(p), L) > 0$. Since there is only a finite number of distinct $L(p)$,

$\eta' = \min \{ \rho(L(p),L) \mid p \in R \} > 0$.

Then $\rho(L(p),L) \geqq \eta'$ for every $p \in R$. Since $g(p) \in L(p)$, we have $\rho(g(p),L) \geqq \eta'$ for every $p \in R$. Therefore we can conclude that $\overline{g(R)} \cap L = \emptyset$, which proves $g \in D(L,V)$. Thus the proof of this assertion is complete.

We should note that this proof implies the following assertion, which will be used later, because the mapping (6) maps R into an n-dimensional polyhedron in I^{2n+1} whose vertices are $x(N_1),\dots,x(N_s)$.

C) The set of all mappings in $C(R,I^{2n+1})$ which map R into an n-dimensional polyhedron in I^{2n+1} is dense in $C(R,I^{2n+1})$ if $\dim R \leqq n$.

Theorem IV. 8 (Imbedding theorem) [6]. *A separable metric space R has dimension $\leqq n$ if and only if R is homeomorphic to a subset of*

$$S = I^{2n+1} \cap (E^{2n+1}_{n+1} \cup E^{2n+1}_{n+2} \cup \dots \cup E^{2n+1}_{2n+1}),$$

where we denote by E^{2n+1}_i the set of points in E^{2n+1} exactly i of whose coordinates are irrational.

Proof. To begin with, from the product theorem and the sum theorem we can deduce $\dim E^{2n+1}_i = 0$, $i = n+1,\dots,2n+1$. Hence by use of the decompositon theorem we obtain $\dim S \leqq n$, which proves the "if" part of this theorem.

Conversely, let $\dim R \leqq n$ to prove the "only if" part. For every subset $\{ i_1,\dots,i_{n+1} \}$ of $\{ 1,\dots,2n+1 \}$ and every set $r_1,\dots,r_{n+1}$ of $n+1$ rational numbers, we define an n-dimensional plane

$$L = \{ (x_1,\dots,x_{2n+1}) \mid x_{i_1} = r_1,\dots,x_{i_{n+1}} = r_{n+1} \}.$$

Then we denote by $L_1, L_2, \dots$ the totality of those planes. We note that

$$S = I^{2n+1} - \bigcup_{i=1}^{\infty} L_i .$$

Moreover, we take a countable basis $\{ V_1, V_2, \dots \}$ of R and construct, for any pair V_i, V_j with $\bar{V}_i \subset V_j$, an open covering $\{ V_j, R - \bar{V}_i \}$. Then we denote by $\mathcal{V}_1, \mathcal{V}_2, \dots$ the totality of those open coverings. Now, we denote by $D(L_k, \mathcal{V}_k)$ the subspace of $C(R,I^{2n+1})$ defined by

$$D(L_k,\mathcal{V}_k) = \{ f \mid f \in C(R,I^{2n+1}),\ \overline{f(R)} \cap L_k = \emptyset ,\ f \text{ is a } \mathcal{V}_k\text{-mapping} \}.$$

[6] This theorem is due to K. Menger [1], G. Nöbeling [1], L. Pontrjagin - G. Tolstowa [1], and W. Hurewicz [4]. The present proof is due to C. Kuratowski [3].

Since by A) and B) each $D(L_k, V_k)$ is open and dense in the complete metric space $C(R, I^{2n+1})$, from Theorem I.5 we obtain $\bigcap_{k=1}^{\infty} D(L_k, V_k) \neq \emptyset$.

Let $f \in \bigcap_{k=1}^{\infty} D(L_k, V_k)$; then f is clearly a continuous mapping of R into S.

Let $W(p)$ be a given neighbourhood of a point p of R. Then there exist open sets V_i and V_j with $p \in V_i \subset \bar{V}_i \subset V_j \subset W(p)$. Since f is a V_k-mapping for every k, there exists a neighbourhood $U(f(p))$ of $f(p)$ such that $f^{-1}(U(f(p))) \subset V_j$ or $f^{-1}(U(f(p))) \subset R - \bar{V}_i$. Since from $p \in V_i$ it follows that

$$p \in f^{-1}(U(f(p))) \cap V_i \neq \emptyset,$$

we conclude $f^{-1}(U(f(p))) \subset V_j$. Hence, if $q \notin W(p)$ $(\supset V_j)$, then $f(q) \notin U(f(p))$. This means that f is a homeomorphism between R and a subset $f(R)$ of S.

We shall see in the following that the same technique of imbedding produces various interesting results.

IV. 5. ε-mappings

It is natural to try to approximate n-dimensional compact sets in Euclidean space by elementary figures like n-dimensional polyhedra. In the present section we shall give an account of Alexandroff's theory [7] in this area which is also of great importance in the history of topology itself.

Definition IV. 3. *A continuous mapping f of a space R into a space S is called an ε-mapping if the inverse image $f^{-1}(q)$ of each point q of S has diameter $< \varepsilon$.*

A) Let R be a compact metric space with $\dim R \geq n$. Then there exists a positive number ε such that R cannot be mapped by any ε-mapping onto any metric space S of dimension $\leq n-1$.

Proof. Since $\dim R \geq n$, there exists such a positive number ε that any open refinement $\mathcal{U}$ of $S_\varepsilon = \{S_\varepsilon(x) \mid x \in R\}$ has order $\geq n+1$. Suppose f is an ε-mapping which maps R onto a space S of dimension $\leq n-1$. Then by virtue of the compactness of R, we can choose $\delta(\varepsilon) > 0$ such that $\rho(x,y) \geq \varepsilon$ in R implies $\rho'(f(x), f(y)) \geq \delta(\varepsilon)$ in S.

[7] See P. Alexandroff [1]. In this connection there is another interesting aspect of investigations using the concept of inverse limit. H. Freudenthal [1] proved that a compact metric space R has $\dim \leq n$ if and only if it is homeomorphic with the inverse limit of an inverse sequence of compact polyhedra P_i of $\dim \leq n$. S. Mardešić [1] extended Freudenthal's result to compact T_2-spaces. See also K. Nagami [3], [4], B. Pasynkov [1], [3], and V. Kljušin [1].

In view of $\dim S \leqq n-1$ we can choose an open covering $\mathcal{V}$ of S satisfying

(1) $\quad \operatorname{ord} \mathcal{V} \leqq n$ and $\mathcal{V} < \mathcal{S} = \{ S_{\delta(\varepsilon)}(y) \mid y \in S \}$

Let V be a given element of $\mathcal{V}$; then $V \subset S_{\delta(\varepsilon)}(y)$ for some $y \in S$. Choose a fixed point $x \in f^{-1}(y)$; then by the property of $\delta(\varepsilon)$ $f^{-1}(V) \subset S_{\varepsilon}(x)$.

Hence $\{ f^{-1}(V) \mid V \in \mathcal{V} \}$ is an open refinement of S_{ε} and has order $\leqq n$ by virtue of (1). But this contradicts the property of S_{ε} .

B) Let R be an n-dimensional compact set in a Euclidean space E^m . Then for a given positive number ε , R can be mapped onto a polyhedron of dimension $\leqq n$ by such a continuous mapping f that $\rho(x, f(x)) < \varepsilon$ for every $x \in R$.

Proof. The first half of the proof is quite analogous to that of the imbedding theorem. For a given $\varepsilon > 0$ we denote by $\mathcal{V} = \{ V_i \mid i = 1, \dots, s \}$ an open covering of R such that

(1) $\quad \mathcal{V} < \mathcal{S} = \{ S_{\frac{1}{8}\varepsilon}(p) \mid p \in R \}$,

(2) $\quad \operatorname{ord} \mathcal{V} \leqq n+1$.

With any member V_i of $\mathcal{V}$ we associate a vertex $x(V_i)$ satisfying

(3) $\quad \rho(x(V_i), V_i) < \frac{1}{4}\varepsilon$

such that $n+1$ of the $x(V_i)$, $i = 1, \dots, s$, never lie on an $(n-1)$-dimensional plane. Then we define a Kuratowski mapping g by

$$g(x) = \frac{\sum_{i=1}^{s} \rho(x, R - V_i) x(V_i)}{\sum_{i=1}^{s} \rho(x, R - V_i)} .$$

Let $V_{i_1}, \dots, V_{i_t}$ be all the members of $\mathcal{V}$ which contain a given point x of R . Then $g(x)$ is contained in the simplex $[x(V_{i_1}), \dots, x(V_{i_t})]$. Since from (1), (3) and $x \in V_{i_k}$ it follows that $\rho(x, x(V_{i_k})) < \frac{1}{2}\varepsilon$, we obtain

(4) $\quad \rho(x, g(x)) < \frac{1}{2}\varepsilon$ for every $x \in R$.

Thus g is a continuous mapping which maps R into the polyhedron P composed of such simplexes $[x(V_{i_1}), \dots, x(V_{i_t})]$.

Now, we construct a triangulation of P into simplexes $T_1, \dots, T_r$ with diameters less than $\frac{1}{2}\varepsilon$. We assume

$$\dim T_1 \geqq \dim T_2 \geqq \dots \geqq \dim T_r .$$

If $T_1 \not\subset g(R)$, then by the compactness of $g(R)$, $T_1 - g(R)$ contains an interior point p of T_1. Hence we denote by g_1 the mapping which projects $T_1 \cap g(R)$ onto the faces of T_1 from p while it leaves the other points of $g(R)$ fixed. If $T_1 \subset g(R)$, then we regard g_1 as the identity mapping. We define $g_2, \ldots, g_r$ for $T_2, \ldots, T_r$ in the same way. Then $f = g_r \cdots g_1 g$ is obviously a continuous mapping of R onto a polyhedron P' contained in P.

Since (2) implies $t \leqq n+1$ and accordingly $\dim P \leqq n$, we have $\dim P' \leqq n$. On the other hand, since

$$\delta(T_j) = \text{diameter } T_j < \tfrac{1}{2}\varepsilon, \quad j = 1, \ldots, r,$$

we easily see that $\rho(g(x), f(x)) < \frac{1}{2}\varepsilon$. This combined with (4) proves that $\rho(x, f(x)) < \varepsilon$ for every $x \in R$.

Theorem IV. 9. *A compact set R in a Euclidean space has dimension $\leqq n$ if and only if for every $\varepsilon > 0$ it can be mapped onto a polyhedron of dimension $\leqq n$ by such a continuous mapping f that $\rho(x, f(x)) < \varepsilon$ for every $x \in R$* [8].

Proof. The "only if" part is a direct consequence of B). On the other hand, a mapping f satisfying this condition is clearly a 2ε-mapping. Therefore the "if" part is a direct consequence of A).

Theorem IV. 10. *A compact metric space R has dimension $\leqq n$ if and only if for every $\varepsilon > 0$ it can be mapped onto a polyhedron of dimension $\leqq n$ by an ε-mapping.*

Proof. Let $\dim R \leqq n$; then by the imbedding theorem R can be mapped by a topological mapping g onto a compact subset S of E^{2n+1}. Since R is compact, there exists $\delta > 0$ such that $g^{-1}(S_\delta(y)) \subset S_{\frac{1}{3}\varepsilon}(x)$ for every $y \in E^{2n+1}$ and some $x \in R$. Then by use of B) we can map S onto a polyhedron P of dimension $\leqq n$ by a continuous mapping f which satisfies $\rho(y, f(y)) < \delta$ for every $y \in S$. Thus f satisfies $f^{-1}(z) \subset S_\delta(z)$ for every $z \in P$. Hence we obtain $g^{-1}f^{-1}(z) \subset S_{\frac{1}{3}\varepsilon}(x)$ for every $z \in P$ and for some $x \in R$. Thus $f \cdot g$ is an ε-mapping of R onto P.

The "if" part is a direct consequence of A).

IV. 6. Pontrjagin-Schnirelmann's theorem

In the following we describe another good example of an application of the imbedding technique.

---

[8] It is easy to see that this theorem is valid for every compact set R in a Hilbert space, too.

Definition IV. 4. *Let* R *be a metric space with metric* ρ *and let* $\varepsilon > 0$. *Then* $N(\varepsilon, R, \rho) = \inf \{ |a| \mid a$ *is a finite covering of* R *with* mesh $a \leq \varepsilon \}$, *where* $|a|$ *denotes the cardinality of* a .

The following proposition is easy to prove, so that the proof can be left to the reader.

A) Let P be an n-dimensional polyhedron with Euclidean metric ρ . Then

$$c/\varepsilon^n \leq N(\varepsilon, P, \rho) \leq c'/\varepsilon^n$$

for some positive constants c and c' if $\varepsilon > 0$ is sufficiently small.

The above proposition implies that

$$n = \dim P = \lim_{\varepsilon \to 0} \frac{-\log N(\varepsilon, P, \rho)}{\log \varepsilon} .$$

Generally we can characterize dimension of separable metric spaces by use of the function $N(\varepsilon, R, \rho)$.

B) Let S be a compact subset of E^k with $\dim S \geq n$, where $k \geq n$. Then for some n-dimensional coordinate plane E^n of E^k $\dim \pi(S) = n$ holds, where π denotes the projection from E^k onto E^n .

Proof. Let L be a covering of E^k by congruent k-dimensional closed cubes with faces parallel to the $(k-1)$-dimensional coordinate planes such that the interiors of any two elements of L do not intersect and such that ord $L = k+1$. Such a covering may be called a *Lebesgue brick covering*, and an example is the covering of E^2 by the squares whose vertices are $(2m, 4n)$, $(2m+2, 4n)$, $(2m+2, 4n+2)$, $(2m, 4n+2)$ or $(2m+1, 4n+2)$, $(2m+3, 4n+2)$, $(2m+3, 4n+4)$, $(2m+1, 4n+4)$, where m and n are arbitrary integers. Obviously we can construct a Lebesgue brick covering of arbitrarily small mesh.

Now, denote by $E_1, \dots, E_\ell$ all n-dimensional coordinate planes in E^k and by π_i the projection from E^k onto E_i . Assume that $\dim \pi_i(S) \leq n-1$, $i = 1, \dots, \ell$. Denote by $F_1, F_2, \dots$ the collection of the intersections of $n+1$ elements of L . Since S is compact, only finitely many of them, say $F_1, \dots, F_s$ intersect S , and $\rho(S, \bigcup_{j=s+1}^{\infty} F_j) > 0$.

Now, F_1 is contained in a $(k-n)$-dimensional Euclidean space E^{k-n} which is perpendicular to some E_i . Thus $\pi_i(F_1)$ is a single point of E_i . Since $\dim \pi_i(S) \leq n-1$, $E_i - \pi_i(S)$ is dense in E_i . Hence we can map S to $t_1(S) \subset E^k$ by an arbitrarily small translation t_1 parallel to E_i in such a way that

$$t_1(S) \cap F_1 = \emptyset \text{ and } \rho(t_1(S), \bigcup_{j=s+1}^{\infty} F_j) > 0 .$$

By a similar argument we can find a translation t_2 such that

$$t_2 t_1(S) \cap F_2 = \emptyset \text{ and } \rho(t_2 t_1(S), F_1 \cup \bigcup_{j=s+1}^{\infty} F_j) > 0 .$$

Continuing this process, we obtain a translation t such that $t(S) \cap (\bigcup_{j=1}^{\infty} F_j) = \emptyset$. Then the restriction L' of L to $t(S)$ is a closed covering with order $\leq n$. It is easy to see that L can be slightly swelled to an open covering U of $t(S)$ with order $\leq n$. Since mesh U can be arbitrarily small and $t(S)$ is compact, this implies that $\dim t(S) \leq n-1$, which is a contradiction. Thus the proposition is proved.

C) Let R be a compact metric space with $\dim R \geq n$. Then there is a continuous mapping ψ from R onto an n-dimensional subset R' of E^n such that $\rho(x,y) \geq \rho'(\psi(x), \psi(y))$ for every $x,y \in R$, where ρ and ρ' denote the metric of R and E^n, respectively.

Proof. Let $a_1,\dots,a_k \in R$ be fixed points. Then define $\varphi : R \to E^k$ by

$$\varphi(x) = \left(\frac{\rho(a_1,x)}{\sqrt{k}}, \dots, \frac{\rho(a_k,x)}{\sqrt{k}} \right) .$$

It is easy to see that φ satisfies $\rho(x,y) \geq \rho'(\varphi(x), \varphi(y))$. Now we can select $a_1,\dots,a_k$ as follows to make φ satisfy another condition, namely $\dim \varphi(R) \geq n$.

Let ε be a positive number which satisfies the condition of 5 A). Select $a_1,\dots,a_k$ in such a way that for each $x \in R$ there is a_i satisfying $\rho(a_i,x) < \varepsilon/2$, which is possible because R is compact. Now, let x and y be two points of R such that $\rho(x,y) > \varepsilon$. Then choose a_i satisfying the above condition, which implies that

$$\rho(a_i,y) \geq \rho(x,y) - \rho(a_i,x) > \varepsilon/2 ,$$

and accordingly $\rho(a_i,x) \neq \rho(a_i,y)$. Thus $\varphi(x) \neq \varphi(y)$. This proves that φ is an ε-mapping. Hence by 5 A) we get $\dim \varphi(R) \geq n$. By B) there is an n-dimensional coordinate plane E^n such that the projection $\pi(\varphi(R)) = R'$ of $\varphi(R)$ in E^n has $\dim \geq n$. Now the mapping $\psi = \pi \circ \varphi$ satisfies the desired condition.

D) Let $\langle R, \rho \rangle$ be a compact metric space with $\dim R \geq n$. Then $N(\varepsilon, R, \rho) \geq c/\varepsilon^n$ holds for a positive constant c and sufficiently small $\varepsilon > 0$.

Proof. Let φ be a continuous mapping of R onto $R' \subset E^n$ satisfying the condition of C). Then since $\dim R' \geqq n$, by Theorem IV.5 R' contains an n-dimensional simplex. Hence it follows from A) that $N(\varepsilon,R,\rho) \geqq N(\varepsilon,R',\rho') \geqq c/\varepsilon^n$ for some $c>0$.

E) Let R be a compact metric space with $\dim R \leqq n$. Then there is a topological mapping g from R into I^{2n+1} such that

$$N(\varepsilon_i, g(R), \rho) \leqq 1/\varepsilon_i^{n+\delta_i}, \quad i = 1,2,\ldots$$

for some monotone decreasing sequences $\{\varepsilon_i\}$ and $\{\delta_i\}$ of positive numbers, both converging to 0, where ρ denotes the Euclidean metric of I^{2n+1}.

Proof. The proof will be carried out by repeated use of 4 A) and 4 B) (actually the method used to prove 4 B)).

Let $\mathcal{V}_1 > \mathcal{V}_2 > \ldots$ be a sequence of finite open coverings of I^{2n+1} with mesh $\mathcal{V}_i \to 0$ and ord $\mathcal{V}_i \leqq n+1$. Denote by $D(\mathcal{V}_i)$ the set of all $\mathcal{V}_i$-mappings in $C(R, I^{2n+1})$. First select a Kuratowski mapping $g_1 \in D(\mathcal{V}_1)$ (see the proof of 4 B)). Then $g_1(R)$ is a compact subset of an n-dimensional polyhedron in I^{2n+1}. Hence by A) $N(\varepsilon, g_1(R), \rho) \leqq c/\varepsilon^n$ holds for a constant c and sufficiently small $\varepsilon > 0$. Thus we can select ε_1 and δ_1 such that

$$0 < \varepsilon_1 < 1, \quad 0 < \delta_1 < 1, \quad \text{and} \quad N(\varepsilon_1, g_1(R), \rho) < 1/\varepsilon_1^{n+\delta_1}$$

and a finite collection $\mathcal{U}_1$ of open sets of I^{2n+1} which covers $g_1(R)$ satisfying mesh $\mathcal{U}_1 \leqq 2\varepsilon_1$ and $|\mathcal{U}_1| = N(\varepsilon_1, g_1(R), \rho)$. Put

$$\eta_1 = \rho(g_1(R), I^{2n+1} - \cup\, \mathcal{U}_1) > 0.$$

Choose $\xi_1 > 0$ such that

$$\xi_1 < \min\{1/2, \eta_1\}, \quad \text{and} \quad \overline{S_{\xi_1}(g_1)} \subset D(\mathcal{V}_1) \quad \text{(see 4 A)).}$$

Next select a Kuratowski mapping $g_2 \in D(\mathcal{V}_2) \cap S_{\xi_1}(g_1)$. Then $g_2(R)$ is a compact subset of the intersection of an n-dimensional polyhedron and $S_{\eta_1}(g_1(R)) \subset \cup\, \mathcal{U}_1$. Thus we can find ε_2 and δ_2 such that

$$0 < \varepsilon_2 < \min\{1/2, \varepsilon_1\}, \quad 0 < \delta_2 < \min\{1/2, \delta_1\}, \quad \text{and}$$

$$N(\varepsilon_2, g_2(R), \rho) < 1/\varepsilon_2^{n+\delta_2}$$

and a finite open collection $\mathcal{U}_2$ in I^{2n+1} which covers $g_2(R)$ satisfying $\bar{\mathcal{U}}_2 < \mathcal{U}_1$, mesh $\mathcal{U}_2 \leqq 2\varepsilon_2$, and $|\mathcal{U}_2| = N(\varepsilon_2, g_2(R), \rho)$. Put $\eta_2 = \rho(g_2(R), I^{2n+1} - \cup\, \mathcal{U}_2)$ and choose $\xi_2 > 0$ satisfying

$$\xi_2 < \min\{1/2^2, \eta_2\}, \text{ and } \overline{S_{\xi_2}(g_2)} \subset D(V_2) \cap S_{\xi_1}(g_1) .$$

Continuing this process we obtain monotone decreasing positive sequences $\{\varepsilon_i\}$ and $\{\delta_i\}$, a positive sequence $\{\xi_i\}$, $g_i \in D(V_i)$, $i = 1,2,\ldots$ and finite open collections $\mathcal{U}_i$, $i = 1,2,\ldots$ in I^{2n+1} such that

(1) $\lim \varepsilon_i = \lim \delta_i = 0$,

(2) $\xi_i < 1/2^i$,

(3) $\overline{S_{\xi_i}(g_i)} \subset D(V_i) \cap S_{\xi_{i-1}}(g_{i-1})$,

(4) $g_i(R) \subset \cup\, \mathcal{U}_i$,

(5) $\bar{\mathcal{U}}_{i+1} < \cup\, \mathcal{U}_i$,

(6) $\operatorname{mesh} \mathcal{U}_i \leqq 2\varepsilon_i$,

(7) $|\mathcal{U}_i| = N(\varepsilon_i, g_i(R), \rho) < 1/\varepsilon_i^{n+\delta_i}$

Then by (2) and (3) $\{g_i\}$ converges to g in $C(R, I^{2n+1})$. It follows from (3) that $g \in \bigcap_{i=1}^{\infty} D(V_i)$. Hence g is a topological mapping from R into I^{2n+1}. From (4) and (5) it follows that $g(R)$ is covered by every $\mathcal{U}_i$, $i = 1,2,\ldots$. By (6) and (7) we get

$$N(2\varepsilon_i, g(R), \rho) < 1/\varepsilon_i^{n+\delta_i}, \quad i = 1,2,\ldots$$

By modifying the values of ε_i and δ_i slightly, we obtain the validity of E).

Definition IV. 5. *Let* $\langle R, \rho \rangle$ *be a metric space. Then* $k(R,\rho) = \sup\{\inf\{-\frac{\log N(\varepsilon, R, \rho)}{\log \varepsilon} \mid 0 < \varepsilon < \varepsilon_0\} \mid \varepsilon_0 > 0\}$.

The following theorem, which we owe to L. Pontrjagin-L. Schnirelmann [1], is interesting in the sense that dimension is characterized in terms of the global cardinality of coverings while it was originally defined in terms of local cardinality, order of covering.

Theorem IV. 11. *If* R *is a compact metrizable space, then*

$$\dim R = \inf\{k(R,\rho) \mid \rho \text{ is a metric for } R\}.$$

Proof. The inequality $\inf\{k(R,\rho)\} \geqq \dim R$ follows from D), and so does $\inf\{k(R,\rho)\} \leqq \dim R$ from E).

Corollary [9]. *If* R *is a separable metrizable space, then*

$$\dim R = \inf \{ k(R,\rho) \mid \rho \text{ is a totally bounded metric for } R \} .$$

Proof. Generally, let $\langle X',\rho' \rangle$ be a totally bounded metric space and X a dense subset of X'. Then it is almost obvious that $N(\varepsilon,X',\rho') = N(\varepsilon,X,\rho)$ holds for every $\varepsilon > 0$, where ρ is the restriction of ρ' to X. Thus $k(X',\rho') = k(X,\rho)$ follows.

By Theorem V.2 (to be proved later) there is a completion R^* of R (with an appropriate totally bounded metric) such that $\dim R^* = \dim R$. Then by I.2 D) R^* is compact. Hence by Theorem IV.11 there is a metric ρ^* for R^* such that $\dim R^* = k(R^*,\rho^*)$. Thus $\dim R = k(R,\rho)$, where ρ is the restriction of ρ^* to R, and hence $\dim R \geqq \inf \{ k(R,\rho) \}$.

On the other hand, let ρ be a given totally bounded metric for R. Then the completion $\langle R',\rho' \rangle$ of $\langle R,\rho \rangle$ is compact. Hence by the preceding Theorem

$$\dim R \leqq \dim R' \leqq k(R',\rho') = k(R,\rho) ,$$

which implies that $\dim R \leqq \inf \{ k(R,\rho) \}$. Thus we obtain the desired equality.

IV. 7. Dimension and measure

There is a remarkable connection between the concept of dimension and the concept of measure. Measure is not a topological concept, but a separable metric space of dimension $< n$ is characterized as a space which is homeomorphic to a space with n-dimensional measure zero (defined in the following). The purpose of this section is to show this connection and apply it to an interesting metrization of n-dimensional separable metric spaces.

Definition IV. 6. *Let* R *be a separable metric space and* p *a non-negative number. Define*

$$m_p^{\varepsilon}(R) = \inf \{ \sum_{i=1}^{\infty} [\delta(A_i)]^p \mid R = \bigcup_{i=1}^{\infty} A_i,\ \delta(A_i) < \varepsilon ,\ i = 1,2,\dots \} , \text{ and}$$

$$m_p(R) = \sup \{ m_p^{\varepsilon}(R) \mid \varepsilon > 0 \} ,$$

where we denote by $\delta(A)$ *the diameter of* A *and put* $[\delta(A)]^0 = 0$ *if* $A = \emptyset$, *and* $[\delta(A)]^0 = 1$ *if* $A \neq \emptyset$. *Then we call* $m_p(R)$ *the p-dimensional measure of* R .

[9] This generalization of Pontrjagin-Schnirelmann's theorem is due to J. Bruijning [1].

We can easily verify that this measure is an outer measure and that for a subset R of E^n $m_n(R)=0$ holds if and only if the Lebesgue outer measure $m^*(R)$ of R vanishes [10]. The verifications of the following assertions A) and B) are left to the reader.

A) $m_0(R)=n$ if R is a set of n points.

B) An n-dimensional polyhedron has $n+1$-dimensional measure zero.

C) Let $m_{p+1}(R)=0$ for a separable metric space R and x a given point of R. Then $m_p(B(S_r(x)))=0$ holds for almost every positive number r, i.e. $m^*(A)=0$ for $A=\{r \mid r>0,\ m_p(B(S_r(x)))>0\}$.

Proof. Since $m_{p+1}(R)=0$, there exist coverings $\{A_i^m \mid i=1,2,\dots\}$, $m=1,2,\dots$ of R such that $\delta(A_i^m)<1/m$, and

$$(1) \qquad \lim_{m\to\infty}\sum_{i=1}^{\infty}[\delta(A_i^m)]^{p+1}=0$$

Define $a_{mi}=\sup\{\rho(x,y) \mid y\in A_i^m\}$ and $b_{mi}=\inf\{\rho(x,y) \mid y\in A_i^m\}$. Then it is clear that

$$(2) \qquad \delta(A_i^m)\geqq a_{mi}-b_{mi}$$

We define functions d_{mi} and d_m on $[0,\infty)$ as follows:

$$(3) \qquad d_{mi}(r)=\begin{cases}0 & \text{if } 0\leqq r<b_{mi} \text{ or } r>a_{mi}, \\ [\delta(A_i^m)]^p & \text{if } b_{mi}\leqq r\leqq a_{mi},\end{cases}$$

$$d_m(r)=\sum_{i=1}^{\infty}d_{mi}(r).$$

From (2) and (3) it follows that for any $k>0$

$$\int_0^k d_{mi}(r)dr\leqq[\delta(A_i^m)]^{p+1}$$

which combined with (1) implies

$$\lim_{m\to\infty}\sum_{i=1}^{\infty}\int_0^k d_{mi}(r)dr=0.$$

---

[10] As for measure theory, see for example S. Saks [1] or H. Royden [1].

Since $d_{mi}(r) \geqq 0$, we can interchange integration and summation in the preceding formula and thus obtain

$$\lim_{m\to\infty} \int_0^k \sum_{i=1}^{\infty} d_{mi}(r)dr = \lim_{m\to\infty} \int_0^k d_m(r)dr = 0 .$$

Hence we can find a subsequence $\{m_j \mid j = 1,2,\dots\}$ such that

(4) $$\lim_{j\to\infty} d_{m_j}(r) = 0 \text{ for almost all } r .$$

The definition (3) of d_{mi} implies

$$[\delta(A_i^{m_j} \cap B(S_r(x)))]^p \leqq d_{m_j i}(r) \quad \text{for every } r ,$$

and hence from (4) we can deduce

$$\lim_{j\to\infty} \sum_{i=1}^{\infty} [\delta(A_i^{m_j} \cap B(S_r(x)))]^p = 0 \text{ for almost all } r .$$

Since $\{A_i^{m_j} \cap B(S_r(x)) \mid i = 1,2,\dots\}$, $j = 1,2,\dots$ are coverings of $B(S_r(x))$ with $\delta(A_i^{m_j} \cap B(S_r(x))) < 1/m_j$ we get $m_p(B(S_r(x))) = 0$ for almost all r.

D) If $m_{n+1}(R) = 0$ for a separable metric space R, then $\dim R \leqq n$.

Proof. If $n = -1$, then this assertion is obviously true. We assume D) for every space R' with $m_n(R') = 0$ in an inductive proof. Let $m_{n+1}(R) = 0$; then from C) it follows that for every neighbourhood $U(x)$ of every point x of R there exists a spherical neighbourhood $S_r(x)$ satisfying $S_r(x) \subset U(x)$ and $m_n(B(S_r(x))) = 0$. By use of the induction hypothesis we obtain $\dim B(S_r(x)) \leqq n-1$. This means $\operatorname{ind} R \leqq n$. Since R is a separable metric space, we conclude $\dim R \leqq n$.

E) A separable metric space R of dimension $\leqq n$ is homeomorphic to a subset S of I^{2n+1} such that $m_{n+1}(S) = 0$.

Proof. We still denote by $C(R, I^{2n+1})$ the complete metric space of all continuous mappings of R into I^{2n+1} as in the proof of the imbedding theorem. Moreover, let

$$D_k = \{f \mid f \in C(R, I^{2n+1}),\ m_{n+1}^{1/k}(\overline{f(R)}) < 1/k\} .$$

Then every D_k contains the set D consisting of all continuous mappings f such that $m_{n+1}(\overline{f(R)}) = 0$. Since by B) any n-dimensional polyhedron has $n+1$-dimensional measure zero, by 4 C) D_k is dense in $C(R, I^{2n+1})$.

To show that D_k is open, we suppose f is a given point of D_k. Then we can choose ε satisfying

$$m_{n+1}^{1/k}(\overline{f(R)}) < \varepsilon < 1/k .$$

Thus we can construct a covering $\{A_i \mid i = 1,2,\dots\}$ of $\overline{f(R)}$ such that $\delta(A_i) < 1/k$ and

$$(1) \qquad \sum_{i=1}^{\infty} [\delta(A_i)]^{n+1} < \varepsilon .$$

Now it is easy to see that we can construct open sets U_i, $i = 1,2,\dots$ in I^{2n+1} such that

$$(2) \qquad A_i \subset U_i , \quad \delta(U_i) < 1/k , \text{ and } [\delta(U_i)]^{n+1} < [\delta(A_i)]^{n+1} + (\tfrac{1}{k} - \varepsilon)/2^i$$

which combined with (1) implies

$$(3) \qquad \sum_{i=1}^{\infty} [\delta(U_i)]^{n+1} < 1/k .$$

Since $\overline{f(R)}$ is compact

$$\rho(\overline{f(R)},\ I^{2n+1} - \bigcup_{i=1}^{\infty} U_i) = \eta > 0 .$$

Let g be an arbitrary point of $C(R, I^{2n+1})$ satisfying $\rho'(f,g) < \frac{1}{2}\eta$, where we denote by ρ' the metric of $C(R, I^{2n+1})$. Then we can easily verify $\overline{g(R)} \subset \bigcup_{i=1}^{\infty} U_i$. Hence from (2) and (3) it follows that

$$m_{n+1}^{1/k}(\overline{g(R)}) < 1/k .$$

This proves $g \in D_k$. In consequence $S_{\frac{1}{2}\eta}(f) \subset D_k$. Thus each D_k is an open dense set of $C(R, I^{2n+1})$. Now we denote by $D(V_k)$ the set of all V_k-mappings with V_k as defined in the proof of 6 E). Then we obtain, by use of Theorem I.5,

$$\bigcap_{k=1}^{\infty} [D_k \cap D(V_k)] \neq \emptyset .$$

Any mapping contained in this intersection is a homeomorphism of R into I^{2n+1} which satisfies $m_{n+1}(f(R)) = 0$.

Combining E) with D), we obtain

Theorem IV. 12. [11] *A separable metric space R has dimension $\leq n$ if and only if R is homeomorphic to a subset S of I^{2n+1} with $m_{n+1}(S) = 0$.*

[11] This theorem is due to E. Szpilrajn [1]. Szpilrajn's theorem and Pontrjagin-Schnirelmann's theorem are based on somewhat similar ideas. Let $\langle R, \rho \rangle$ be a separable metric space; then $hd(R,\rho) = \sup\{p \mid m_p(R) > 0\}$ is called the *Hausdorff dimension* of $\langle R, \rho \rangle$. J. Hawkes [1] proved $hd(R,\rho) \leq k(R,\rho)$ for every compact metric space. But they do not coincide in general.

Corollary. *A separable metric space R has dimension $\leqq n$ if and only if one can introduce a metric ρ in R such that for almost all $r>0$ the boundary $B(S_r(x))$ of the spherical neighbourhood $S_r(x)$ of each point x has dimension $\leqq n-1$.*

Proof. This statement is directly deduced from Theorem IV.12 in combination with C) and Theorem IV.1.

Another interesting application of the imbedding technique can be found in L. Janos' theory to characterize dimension by use of "bisectors". A metric d for X is called *strongly rigid* if $d(x,y) \neq d(u,v)$ whenever (x,y) and (u,v) are different pairs of different points. L. Janos [1] proved that: $\dim X = 0$ *holds for a non-empty separable metrizable space X if and only if X admits a compatible strongly rigid metric.*

Let Y and Z be subsets of a metric space $\langle X,d\rangle$. Then Z is called a *bisector* of Y (denoted by $Y \triangleright Z$) if $Y \supset Z$ and there are distinct points $y_1, y_2 \in Y$ such that $Z = \{ z \in Y \mid d(z,y_1) = d(z,y_2)\}$. A chain $X = X_0 \triangleright X_1 \triangleright X_2 \triangleright \dots \triangleright X_{n-1} \triangleright X_n$ of subsets of X is called a *reduced bisector chain* if $\dim X_{n-1} > 0$ and $\dim X_n \leqq 0$, and n is called the *length* of the chain. Denote by $r(X,d)$ the maximal length of reduced bisector chains in $\langle X,d\rangle$, and let $r(X) = \min\{r(X,d) \mid d$ is a metric for $X\}$. Then L. Janos [2] proved:

$r(X) = \dim X$ *if X is a non-empty compact metrizable space.*

An important role in his theory is played by J. H. Roberts' [1] theorem:

If X is a separable metric space with $\dim \leqq n$, *then there is a topological imbedding $f: X \to E^{2n+1}$ such that* $\dim(f(X) \cap Y) \leqq 0$ *for every $(n+1)$-dimensional plane Y of E^{2n+1}.*

The first theorem of Janos was extended by H. Martin [1] to more general metric spaces, and the second was extended by L. Janos - H. Martin [1] to separable metrizable spaces.

J. H. Roberts [1] showed that his theorem above practically implies the following stronger theorem:

If X is a separable metric space with $\dim \leqq n$, *then there is a topological imbedding $f: X \to E^{2n+1}$ such that for every k-dimensional plane Y of E^{2n+1} we have* $\dim (f(X) \cap Y) \leqq k-n-1$.

This theorem implies the following theorem which improves the corollary of Theorem IV.12:

A separable metric space R has dimension $\leqq n$ if and only if one can introduce a metric ρ in R such that for every $r>0$ the boundary $B(S_r(x))$ of the spherical neighbourhood $S_r(x)$ of each point x has $\dim \leqq n-1$.

Another interesting imbedding theorem for n-dimensional compact metric spaces can be found in M. A. Štanko [1].

IV. 8. Dimension and the ring of continuous functions

Let $C(R)$ denote the ring of all real-valued continuous functions defined on R. It is well-known that the topology of R is completely characterized by the ring structure of $C(R)$ if R is a compact Hausdorff space. In this respect it is an interesting problem to study relations between the dimension of R and the algebraic structure of $C(R)$.

M. Katětov [1] defined dimension of $C(R)$ and established a remarkable connection between the dimension of $C(R)$ and that of R. The purpose of this section is to give a quick review of his theory in the most interesting case in which R is a compact metric space.

Note that in the following discussion in the present section *all spaces are assumed to be compact metric,* although some propositions are true for more general spaces.

A) Let $\mathcal{U}$ be an open covering of a space R and T a closed set of R. Then $\tilde{\delta}(T) < \mathcal{U}$ if and only if the decomposition of T into its components is a refinement of $\mathcal{U}$. (Recall Definition III.8.)

Proof. The 'only if' follows directly from Definition III.8.

To prove the 'if' part, let us denote by $K(x)$ the component of T which contains $x \in T$. Then $K(x)$ will be proved to be an intersection of sets which are closed and open in T. Put $K' = \cap\{F \mid F$ is a closed and open set in T such that $F \supset K(x)\}$.

To prove that K' is connected, assume the contrary. Then $K' = K_1 \cup K_2$ where $K_1 \cap K_2 = \emptyset$ for some non-empty closed sets K_1 and K_2 in R. There are open sets U_1 and U_2 in R such that $U_1 \supset K_1$, $U_2 \supset K_2$, $U_1 \cap U_2 = \emptyset$. Since $U_1 \cup U_2 \supset K'$ and since R is compact, there is a set F closed and open in T satisfying $K(x) \subset K' \subset F \subset U_1 \cup U_2$.

Assume e.g. $x \in F \cap U_1 = F \cap (R - U_2)$. Then $F \cap U_1$ is closed and open in T and accordingly contains $K(x)$, because $K(x)$ is connected. Hence $F \cap U_1 \supset K'$, which implies $K_2 \cap K' = \emptyset$, and thus $K_2 = K_2 \cap K' = \emptyset$, which is a contradiction. Therefore K' is connected, and hence $K' = K(x)$ follows because $K(x)$ is a component.

Now recall that we have assumed the condition $\{K(x) \mid x \in T\} < \mathcal{U}$. Because of the above argument we can select a closed and open set $F(x)$ in T and $U(x) \in \mathcal{U}$ such that $K(x) \subset F(x) \subset U(x)$ for each $x \in T$. Since T is compact we can cover it by finitely many of the $F(x)$'s, say $T = \cup_{j=1}^{k} F(x_j)$. Put

$$F_j = F(x_j) - F(x_1) \cup \dots \cup F(x_{j-1}) ;$$

then each F_j is a closed set in R satisfying $F_j \subset U(x_j)$, $T = \bigcup_{j=1}^{k} F_j$, and $F_j \cap F_l = \emptyset$ if $j \neq l$.
We can find open sets V_j , $j = 1,\dots,k$ in R such that $F_j \subset V_j \subset U(x_j)$. Then $\{ V_j \mid j = 1,\dots,k \}$ is an open collection which refines U , covers T and has order 1 . Thus $\tilde{\delta}(T) < U$.

B) A space R is totally disconnected (i.e. each component is a singleton) if and only if $\dim R \leqq 0$. (This proposition is generally true for every compact Hausdorff space).

Proof. The proof easily follows from the fact that each component of R is an intersection of closed and open sets, which was proved in the proof of A), but the detail is left to the reader.

C) Let U be an open covering of a space R and f a continuous mapping from R into another space S such that $\tilde{\delta}(f^{-1}(y)) < U$ for $y \in Y$. Then there is an open neighbourhood W of y in S such that $\tilde{\delta}(f^{-1}(W)) < U$.

Proof. Let V be an open collection which satisfies $V < U$, $f^{-1}(y) \subset \bigcup V$, and ord $V \leqq 1$. Then $W = Y - f(X - \bigcup V)$ satisfies the desired condition.

Definition IV. 7. *A subset C_1 of $C(R)$ is called an analytical subring if it satisfies*

i) *C_1 is a subring of $C(R)$,*
ii) *every constant belongs to C_1 ,*
iii) *if $f \in C(R)$ and $f^2 \in C_1$, then $f \in C_1$,*
iv) *C_1 is closed in the metric space $C(R)$ with the metric*
$\rho(f,g) = \sup\{ |f(x) - g(x)| \mid x \in R \}$.

A subset D of $C(R)$ is called an analytical base if every analytical subring containing D is equal to $C(R)$. Finally, the analytical dimension of $C(R)$, a - dim $C(R)$ *, is defined as the least cardinality of analytical bases of $C(R)$.*

Note that a - dim $C(R) = 0$ means that $\emptyset$ is an analytical base of $C(R)$, i.e. $C(R)$ is the only analytical subring of $C(R)$.

D) Let C_1 be an analytical subring of $C(R)$. If $g \in C(R)$ takes on only finitely many distinct values, then $g \in C_1$.

Proof. Assume that g takes on ℓ distinct values.

1) In case of $\ell = 1$ the assertion is obviously true.

2) Let $\ell = 2$, and $g(R) = \{ a,b \}$. Then

$$(g - \frac{a+b}{2})^2 = (\frac{a-b}{2})^2 \in C_1 .$$

Hence from condition iii) it follows that $g-(a+b)/2 \in C_1$. Therefore $g \in C_1$ by the condition ii).

3) To prove the general case, assume that the assertion is true for every function which takes on at most $\ell - 1$ distinct values. Let $g \in C(R)$ take on ℓ values $a_1, \dots, a_\ell$. Then define $h \in C(R)$ by

$$h(x) = a_i \text{ if } x \in g^{-1}(a_i) \text{ for some } i \text{ satisfying } 1 \leqq i \leqq \ell - 1 ,$$
$$h(x) = a_{\ell - 1} \text{ if } x \in g^{-1}(a_\ell) .$$

Then $h \in C_1$ follows from the induction hypothesis. Note that $g - h$ is identical to 0 or $a_\ell - a_{\ell - 1}$. Thus from the case 2) it follows that $g - h \in C_1$. This yields $g \in C_1$.

E) Let R be a non-empty space; then $\text{a-dim}\, C(R) = 0$ if and only if $\dim R = 0$.

Proof. Assume $\dim R = 0$. We consider an arbitrary analytical subring C_1 of $C(R)$, $f \in C(R)$, and $\varepsilon > 0$. Since R is compact, $\{ f^{-1}(S_{\frac{\varepsilon}{2}}(y)) \mid y \in R\}$ has a finite subcovering, say $U = \{ f^{-1}(S_{\frac{\varepsilon}{2}}(y_i)) \mid i = 1, \dots, k \}$. Choose an open covering $W = \{ W_1, \dots, W_k \}$ such that $W < U$ and $\operatorname{ord} W = 1$. Select $x_j \in W_j$, $j = 1, \dots, k$ to define $g \in C(R)$ by $g(x) = f(x_j)$ if $x \in W_j$. Then it is easy to see that $\rho(f,g) < \varepsilon$ in the metric space $C(R)$. Besides $g \in C_1$ follows from D). Thus $f \in \bar{C}_1 = C$ is obtained by the condition iv). This proves that $C_1 = C(R)$. Therefore $\text{a-dim}\, C(R) = 0$.

Conversely, let us assume that $\text{a-dim}\, C(R) = 0$, and let K be a component of R . Now $C_1 = \{ f \in C(R) \mid f$ is constant on $K \}$ is obviously an analytical subring of $C(R)$. Since $\text{a-dim}\, C(R) = 0$ implies $C(R) = C_1$, K is a singleton. Hence R is totally disconnected. Thus by B) we conclude that $\dim R = 0$.

F) Let $\dim R \leqq n$ $(1 \leqq n \leqq \infty)$. Then $\text{a-dim}\, C(R) \leqq n$.

Proof. By Theorem III.10 there is a uniformly 0-dimensional continuous mapping f from R into I^n . Put $f = (f_1, \dots, f_n)$, where $f_i \in C(R)$, $i = 1, \dots, n$. Then we claim that $\{ f_1, \dots, f_n \}$ is an analytical base for $C(R)$. As will be seen in the following, we can prove our claim by use of Theorem I.8. Let C be an analytical subring which contains $f_1, \dots, f_n$. Suppose p and q are distinct points of R . Then we can find $h \in C_1$ satisfying $h(p) \neq h(q)$, as follows. Let

(1) $\rho(p,q) > \varepsilon > 0$.

Then we choose $\eta > 0$ such that for every subset T of I^n with $\delta(T) \leqq \eta$, the following holds:

(2) $\tilde{\delta}(f^{-1}(T)) < \mathcal{U}_\varepsilon = \{ U | U$ is an open set of R with $\delta(U) \leqq \varepsilon \}$.

Now we choose $\xi > 0$ such that

(3) $$\delta[\prod_{i=1}^{n} (r_i - \xi, r_i + \xi)] \leqq \eta \text{ for every } (r_1,\dots,r_n) \in I^n ,$$

where $(r_i - \xi , r_i + \xi)$ denotes the open segment $\{ x | r_i - \xi < x < r_i + \xi \}$. Put $f(p) = (r_1,\dots,r_n)$, i.e. $f_i(p) = r_i$, $i = 1,\dots,n$, $J = \prod_{i=1}^{n} (r_i - \xi , r_i + \xi)$, and $U = f^{-1}(J)$. Then U is an open neighbourhood of p.

Now we define $g_i \in C(R)$, $i = 1,\dots,n$ by

$$g_i(x) = \frac{1}{\xi} (\xi - | f_i(x) - r_i |), \ x \in R .$$

Since $| f_i - r_i |^2 = (f_i - r_i)^2 \in C_1$, it follows from iii) of Definition IV.7 that $| f_i - r_i | \in C_1$. Therefore $g_i \in C_1$. Define $g \in C(R)$ by $g = \pi_{i=1}^{n} g_i$. Then $g \in C_1$. It follows from the definitions of g and g_i that

(4) $g(p) = 1$,

(5) $g(x) = 0$ for every $x \in B(U)$.

Note that $\delta(J) \leqq \eta$ is implied by (3). Hence we can choose, in view of (2), an open covering $\mathcal{V}$ of $U = f^{-1}(J)$ with mesh $\mathcal{V} \leqq \varepsilon$ and ord $\mathcal{V} \leqq 1$. Put $V = S(p,\mathcal{V})$; then V is an open neighbourhood of p satisfying

(6) $p \in V$ and $q \notin V$ (see (1)).

Note that $B(V) \subset B(U)$. Finally, we define $h \in C(R)$ by

(7) $h(x) = g(x)$ for $x \in V$, and $h(x) = 0$ for $x \notin V$.

(Recall (5) to verify $h \in C(R)$.) Now it is obvious that $(h - (g/2))^2 = (g/2)^2 \in C_1$, which implies $h - (g/2) \in C_1$ and hence $h \in C$. By (4), (6) and (7) we obtain $h(p) = g(p) = 1$, and by (6) and (7) $h(q) = 0$. Thus we can apply Theorem I.8 to conclude that $\bar{C}_1 = C(R)$, which implies $C_1 = C(R)$ because of iv) of Definition IV.7. This proves that $\{ f_1,\dots,f_n \}$ is an analytical base for $C(R)$. Therefore a-dim $C(R) \leqq n$.

G) Let $\{ f_1,\dots,f_n \}$ be an analytical base for $C(R)$. Then the mapping $f = (f_1,\dots,f_n) : R \to E^n$ satisfies: For every binary open covering $\mathcal{U}$ of R and for every $y \in f(R)$, $\tilde{\delta}(f^{-1}(y)) < \mathcal{U}$.

Proof. Note that $C_1 = \{ g \in C(R) | g$ is constant on each component of $f^{-1}(y) \}$ is an analytical subring of $C(R)$ such that $f_1,\dots,f_n \in C_1$. Hence $C_1 = C(R)$.

Let $\mathcal{U} = \{U_1, U_2\}$ and put $F_1 = R - U_1$ and $F_2 = R - U_2$. Then there is a continuous function $h: R \to [0,1]$ such that $h(F_1) = 0$ and $h(F_2) = 1$. Since $h \in C_1$ follows from the above note, each component of $f^{-1}(y)$ cannot hit both F_1 and F_2 at the same time, i.e. it is contained either in U_1 or in U_2. Hence by A) we obtain $\tilde{\delta}(f^{-1}(y)) < \mathcal{U}$.

H) If a-dim $C(R) \leqq n$, then $\dim R \leqq n$. $(1 \leqq n \leqq \infty)$.

Proof. Assume that $\{f_1, \ldots, f_n\}$ is an analytical base for $C(R)$. Then $f = (f_1, \ldots, f_n)$ is a function satisfying the condition of G). Since $f(R)$ is a compact subset of E^n, we may assume $f(R) \subset I^n$. Decompose I^n by use of the decomposition theorem as $I^n = Y_0 \cup \ldots \cup Y_n$ with $\dim Y_i = 0$. Put $X_i = f^{-1}(Y_i)$, $i = 0, \ldots, n$; then $R = \bigcup_{i=0}^{n} X_i$.

Consider $x \in X_i$ and an open neighbourhood U of x in X_i. Find an open set V in R such that $V \cap X_i = U$. Then put $\mathcal{U} = \{R - \{x\}, V\}$. By the above mentioned property of f we obtain $\tilde{\delta}(f^{-1}(y)) < \mathcal{U}$, where $y = f(x)$. By C) there is an open neighbourhood W of y in I^n such that

$$(1) \qquad \tilde{\delta}(f^{-1}(W)) < \mathcal{U}.$$

Since $\dim Y_i = 0$, we may assume that $W \cap Y_i$ is closed and open in Y_i. Hence $f^{-1}(W) \cap X_i$ is closed and open in X_i. By (1) there is an open covering $\mathcal{P}$ of $f^{-1}(W)$ such that

$$(2) \qquad \mathcal{P} < \mathcal{U} \text{ and } \operatorname{ord} \mathcal{P} \leqq 1.$$

Let P be the element of $\mathcal{P}$ which contains x. Then P is a closed and open neighbourhood of x in X_i such that $P \subset V \cap X_i = U$ because of (2). This proves $\operatorname{ind} X_i \leqq 0$, and accordingly $\dim X_i \leqq 0$ follows. Hence again by use of the decomposition theorem we conclude that $\dim R \leqq n$.

Theorem IV. 13. *If R is a non-empty compact metric space, then either* $\dim R$ *and* a-dim $C(R)$ *are both finite and equal, or they are both infinite* [12].

Proof. Combine E), F) and H).

---

[12] Extensions to more general spaces of this theorem of Katětov were proposed in a number of articles. We just mention J. Hejcman [1] who extended it to uniform spaces. We shall discuss later (Chapter VI) a completely different definition of dim $C(R)$ to characterize the dimension of non-metrizable spaces. N. T. Peck [1] found a partial but interesting characterization of $\dim R$ of a compact metric space R in terms of a Banach space property of $C(R, E^n)$.

CHAPTER V

DIMENSION AND METRIZATION

It is interesting to find in the following list a remarkable analogy between dimension theory and metrization theory.

Dimension theory

Theorem II. 2. *A metric space* R *has* $\dim \leq n$ *if and only if there exists a* σ*-locally finite open basis* $\mathcal{V}$ *such that* $\operatorname{Ind} B(V) \leq n-1$ *for every* $V \in \mathcal{V}$.

Theorem III. 9. *A metric space* R *has* $\dim \leq n$ *if and only if there exists a sequence* $\{\mathcal{F}_i \mid i = 1, \dots\}$ *of locally finite closed coverings of* R *which satisfies the following conditions:*

i) *for every neighbourhood* $U(p)$ *of every point* p *of* R *, there exists some* i *with* $S(p, \mathcal{F}_i) \subset U(p)$,

ii) $\mathcal{F}_i = \{F(\alpha_1, \dots, \alpha_i) \mid \alpha_k \in \Omega$, $k = 1, \dots, i\}$ *, where* $F(\alpha_1, \dots, \alpha_i)$ *may be empty,*

Metrization theory

Theorem I. 3 (Nagata-Smirnov's Metrization Theorem). *A regular space* R *is metrizable if and only if there exists a* σ*-locally finite open basis* $\mathcal{V}$.

A T_1*-space* R *is metrizable if and only if there exists a sequence* $\{\mathcal{F}_i \mid i = 1, 2, \dots\}$ *of locally finite closed coverings of* R *which satisfies the condition i) on the left* [1] .

[1] First proved by K. Morita [5].

iii) $F(\alpha_1,\ldots,\alpha_{i-1}) = \cup\{F(\alpha_1,\ldots,\alpha_{i-1},\beta) \mid \beta\in\Omega\}$,

iv) ord $F_i \leq n+1$.

Theorem II. 1 (The Sum Theorem). *Let* $\{F_\gamma \mid \gamma\in\Gamma\}$ *be a locally countable closed covering of a metric space* R *such that* $\dim F_\gamma \leq n$ *for* $\gamma\in\Gamma$. *Then* $\dim R \leq n$.

Let $\{F_\gamma \mid \gamma\in\Gamma\}$ *be a locally finite closed covering of a topological space* R *such that each* F_γ *is metrizable. Then* R *is metrizable* [2].

Let f *be a closed continuous mapping of a metric space* R *of* $\dim \leq n$ *onto a metric space* S *such that for each point* $q\in S$ *the set* $B(f^{-1}(q))$ *consists of at most one point. Then* $\dim S \leq n$ [3].

Let f *be a closed continuous mapping of a metric space* R *onto a topological space* S *such that for each point* $q\in S$ *the set* $B(f^{-1}(q))$ *is compact. Then* S *is metrizable* [4].

Theorem I. 2 (Alexandroff-Urysohn's Metrization Theorem). *A* T_1*-space* R *is metrizable if and only if there exists a sequence* $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \ldots$ *of open coverings* $\mathcal{U}_i$ *such that* $\{S(p,\mathcal{U}_i) \mid i=1,2,\ldots\}$ *is a neighbourhood basis for each point* p *of* R.

To complete the list we can also find theorems in dimension theory (Corollary to Theorem V.1, Theorem V.2) which correspond to Alexandroff-Urysohn's metrization theorem. In this chapter we shall first establish Theorem V.1 and related theorems as the foundation of further investigations. Then by applying them we shall investigate relations between dimensions and metric functions of metric spaces.

---

[2] First proved by J. Nagata [1]. Yu. Smirnov [2] and A. H. Stone [3] proved, under some additional conditions, that the union of countably many closed metrizable spaces is metrizable.

[3] This statement is the special case $m=0$ of Theorem III.7.

[4] This theorem, a part of Theorem I.9, is due to A. H. Stone [2] and to K. Morita - S. Hanai [1]. To this list we may also add Theorem II.9 corresponding to Theorem I.3. Incidentally, it will be interesting to see the corollaries to Theorem II.6 and Theorem III.2 in comparison with the conditions for paracompactness and normality, respectively.

V. 1. Characterization of dimension by a sequence of coverings

Theorem V. 1 [5]. *A (metric) space R has dimension $\leq n$ if and only if there exists a sequence $\{\mathcal{U}_k \mid k=1,2,\dots\}$ of open coverings such that*

i) $\mathcal{U}_1 > \mathcal{U}_2 > \mathcal{U}_3, \dots,$

ii) ord $\mathcal{U}_k \leq n+1$, $k=1,2,\dots,$

iii) mesh $\mathcal{U}_k \to 0$ as $k \to \infty$.

Proof. Since the necessity of these conditions is clear, we shall prove only the sufficiency. We can assume without loss of generality that mesh $\mathcal{U}_k < 2^{-k}$. Let

$$\mathcal{U}_k = \{ U_{k\alpha} \mid \alpha \in A_k \} , \quad N = \{ (k,\alpha) \mid k=1,2,\dots, \alpha \in A_k \} .$$

With each $\alpha \in A_k$ $(k \geq 2)$ we associate by use of i), an index $\beta = f(\alpha) \in A_{k-1}$ such that $U_{k\alpha} \subset U_{k-1\beta}$. Suppose $\mathcal{Q} = \{ Q_\gamma \mid \gamma \in \Gamma \}$ is a given open covering of R . Then we let

$$V_k = \{ x \mid \rho(x, R - Q_\gamma) > 2^{-k} \text{ for some } \gamma \in \Gamma \} , \ k=1,2,\dots, \ V_{-1} = V_0 = \emptyset .$$

It is clear that V_k is open and satisfies $V_k \subset V_{k+1}$. For any $(k,\alpha) \in N$ we obtain

$$(1) \qquad U_{k\alpha} - V_k = \emptyset \ \text{ or } \ U_{k\alpha} \cap V_{k-1} = \emptyset ,$$

because the contrary would imply

$$\delta(U_{k\alpha}) = \text{diameter } U_{k\alpha} > 2^{-k}$$

---

[5] This theorem was first proved by P. Vopenka [2] after C. H. Dowker - W. Hurewicz [1] proved that the existence of a sequence $\{\mathcal{U}_k\}$ of locally finite open coverings satisfying i)' $\mathcal{U}_1 > \bar{\mathcal{U}}_2 > \mathcal{U}_2 > \bar{\mathcal{U}}_3, \dots,$ ii), iii) implies the n-dimensionality of R and J. Nagata [2] proved the corollary to this theorem.

"Is the existence of a sequence $\{\mathcal{U}_k\}$ of open coverings satisfying only ii) and iii) sufficient for a space R to be of dimension $\leq n$?" had been a major problem posed by P. Alexandroff until K. Sitnikov [1] gave an example of a 2-dimensional separable metric space which has a sequence $\{\mathcal{U}_k\}$ satisfying ii) for $n=1$ and iii).

We define the *metric dimension* $\mu \dim R$ of a space as the smallest number n such that there exists a sequence $\{\mathcal{U}_k \mid k=1,2,\dots\}$ of open coverings satisfying ii) and iii). Yu. Smirnov [5], V. Egorov [1] and M. Katětov [4] investigated metric dimension. In particular the last paper established for every space R the following relation between $\mu \dim R$ and $\dim R$: $\mu \dim R \leq \dim R \leq 2 \mu \dim R$.

Various other kinds of metric-dependent dimension functions were studied by K. Nagami, J. H. Roberts and his school; see e.g. K. Nagami - J. H. Roberts [1], J. H. Roberts [2], and J. C. Nicholas - J. C. Smith [1]; the last showed that the finite sum theorem does not hold for μ-dim and other metric-dependent dimensions. J. R. Isbell [1] and Yu. Smirnov [8] studies uniform dimension and proximity dimension, respectively. For further developments in this aspect, see H. Herrlich [1], H. Pust [1].

which contradicts our assumption. Now, we define open sets $W_{k\alpha}$ by $W_{k\alpha} = U_{k\alpha} \cap (V_k - \bar{V}_{k-2})$ for $(k,\alpha) \in N$. Then we can easily see that

(2) $\{ W_{k\alpha} \mid (k,\alpha) \in N \}$ covers R,

because for every $x \in R$ there exists $k \geq 1$ satisfying $x \in V_k - \bar{V}_{k-2}$ and $(k,\alpha) \in N$ satisfying $x \in U_{k\alpha}$. We divide $N_0 = \{ (k,\alpha) \mid (k,\alpha) \in N ,\ W_{k\alpha} \neq \emptyset \}$ into three disjoint parts A, B and C as follows:

$$A = \{(k,\alpha) \mid (k,\alpha) \in N_0 ,\ W_{k\alpha} \cap V_{k-1} = \emptyset \},$$
$$B = \{(k,\alpha) \mid (k,\alpha) \in N_0 ,\ W_{k\alpha} \subset V_{k-1} \},$$
$$C = \{(k,\alpha) \mid (k,\alpha) \in N_0 ,\ W_{k\alpha} - V_{k-1} \neq \emptyset ,\ W_{k\alpha} \cap V_{k-1} \neq \emptyset \}.$$

Furthermore we let

$$\mathcal{A} = \{ W_{k\alpha} \mid (k,\alpha) \in A \} ,$$
$$\mathcal{B} = \{ W_{k\alpha} \mid (k,\alpha) \in B \} ,$$
$$\mathcal{C} = \{ W_{k\alpha} \mid (k,\alpha) \in C \} .$$

Then we can prove the following

(3) If $\alpha \in A_1$ and $W_{1\alpha} \neq \emptyset$, then $(1,\alpha) \in A$.

(4) If $(k,\alpha) \in B$, then $(k-1 , f(\alpha)) \in A \cup C$ and $W_{k\alpha} \subset W_{k-1\, f(\alpha)}$.

(5) If $(k,\alpha) \in C$, then $(k-1 , f(\alpha)) \in A$.

Since $V_0 = \emptyset$, (3) is obvious. If $(k,\alpha) \in B$, then $k > 1$, $W_{k\alpha} \subset V_{k-1}$, and hence

$$U_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) = W_{k\alpha} \neq \emptyset .$$

This implies

$$U_{k-1\, f(\alpha)} \cap (V_{k-1} - \bar{V}_{k-2}) \neq \emptyset ,$$

because $U_{k-1\, f(\alpha)} \supset U_{k\alpha}$. So $W_{k-1\, f(\alpha)} - \bar{V}_{k-2} \neq \emptyset$; thus $(k-1 , f(\alpha)) \in A \cup C$, and

$$W_{k\alpha} = U_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) \subset U_{k-1\, f(\alpha)} \cap (V_{k-1} - \bar{V}_{k-2}) \subset W_{k-1\, f(\alpha)} .$$

Now, let us turn to the proof of (5). First note that $(k,\alpha) \in C$ implies $W_{k\alpha} \cap V_{k-1} \neq \emptyset$. Since by the definition of $W_{k\alpha}$, $W_{k\alpha} \cap \bar{V}_{k-2} = \emptyset$, we obtain

$$W_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) \neq \emptyset .$$

Therefore

$$U_{k\alpha} \cap (V_{k-1} - \bar{V}_{k-2}) \neq \emptyset ,$$

and hence

$$U_{k-1\,f(\alpha)} \cap (V_{k-1} - \bar{V}_{k-3}) \neq \emptyset ,$$

which means $W_{k-1\,f(\alpha)} \neq \emptyset$. Next, we note that $(k,\alpha) \in C$ implies $W_{k\alpha} \cap (R - V_{k-1}) \neq \emptyset$ Since by the definition of $W_{k\alpha}$, $W_{k\alpha} \subset V_k$, we obtain

$$W_{k\alpha} \cap (V_k - V_{k-1}) \neq \emptyset .$$

On the other hand

$$W_{k\alpha} \cap (V_k - V_{k-1}) \subset U_{k\alpha} - V_{k-1}$$

follows from $W_{k\alpha} \subset U_{k\alpha}$. Thus we obtain $U_{k\alpha} - V_{k-1} \neq \emptyset$. which implies $U_{k-1\,f(\alpha)} - V_{k-1} \neq \emptyset$.

Hence it follows from (1) that $U_{k-1\,f(\alpha)} \cap V_{k-2} = \emptyset$. Thus we get $(k-1, f(\alpha)) \in A$ proving (5).

We define open sets $H_{k\beta}$ by

$$H_{k\beta} = W_{k\beta} \cup [\, \cup \{ W_{k+1\,\alpha} \mid (k+1, \alpha) \in C ,\ f(\alpha) = \beta \}]$$

for $(k,\beta) \in A$. Then it follows from the definitions of $H_{k\beta}$, $W_{k\beta}$ and A that

(6) $\quad H_{k\beta} \subset U_{k\beta}$, i.e. $\delta(H_{k\beta}) < 2^{-k}$,

(7) $\quad H_{k\beta} \subset V_{k+1} - \bar{V}_{k-1}$.

To see (7) we should note that $(k,\beta) \in A$ implies $W_{k\beta} \cap V_{k-1} = \emptyset$ and accordingly $W_{k\beta} \cap \bar{V}_{k-1} = \emptyset$. Furthermore we claim that

(8) $\quad H_{k\beta} \cap V_k \subset W_{k\beta}$.

For, if $f(\alpha) = \beta$, then

$$W_{k+1\,\alpha} \cap V_k = U_{k+1\,\alpha} \cap (V_k - \bar{V}_{k-1}) \subset U_{k\beta} \cap (V_k - \bar{V}_{k-2}) = W_{k\beta} .$$

Hence (8) follows from the definition of $H_{k\beta}$.

We know, in view of (3) and (4), that $\mathcal{B} < \mathcal{A} \cup \mathcal{C}$. On the other hand, the definition of $H_{k\beta}$, combined with (3) and (5), implies that $\mathcal{A} \cup \mathcal{C} < \{ H_{k\beta} \mid (k,\beta) \in A \}$. Hence, by (2) $\{H_{k\beta}\}$ covers R. Let us denote this open covering by H.

Now, let $H_{k\beta}$ be a given element of H. Then we choose a point $x \in W_{k\beta}$. Evidently $x \in V_k$, and hence

$$\rho(x, R - Q_\gamma) > 2^{-k} \quad \text{for some} \quad \gamma \in \Gamma .$$

Since by the definition of $H_{k\beta}$ we have $x \in H_{k\beta}$, we obtain from (6) $H_{k\beta} \subset Q_\gamma$. Therefore we conclude that $H < \mathcal{Q} = \{ Q_\gamma \mid \gamma \in \Gamma \}$.

Finally, let us prove that ord $H \leqq n+1$.

Let x be a given point of R; then we choose r such that $x \in V_r - V_{r-1}$. If $x \in H_{k\beta}$, then from (7) we obtain either $k = r$ or $k = r-1$. We suppose

(9) $\quad x \in H_{r-1\,\beta}$ for p distinct indices β,

(10) $\quad x \in H_{r\alpha'}$ for q distinct indices α',

It follows from $W_{r-1\,\beta} \subset V_{r-1}$ that $x \notin W_{r-1\,\beta}$. Hence by virtue of (9) and the definition of $H_{k\beta}$ we have $x \in W_{r\alpha}$ for at least p distinct indices α with $(r, \alpha) \in C$, because the mapping $f(\alpha) = \beta$ is unique. On the other hand, by (10) and (8) we have $x \in W_{r\alpha'}$ for at least q distinct indices α' with $(r, \alpha') \in A$, because $x \in V_r$. Therefore $x \in W_{r\alpha} \subset U_{r\alpha}$ for at least $p+q$ distinct indices α.

Since ord $U_r \leqq n+1$, this implies $p+q \leqq n+1$.

Therefore $\operatorname{ord}_x H \leqq n+1$, i.e. ord $H \leqq n+1$. Thus we conclude $\dim R \leqq n$.

Corollary. *A T_1-space R is a metrizable space of dimension $\leqq n$ if and only if there exists a sequence*

$$\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \dots$$

of open coverings $\mathcal{U}_i$ such that $\{ S(p, \mathcal{U}_i) \mid i = 1, 2, \dots \}$ is a neighbourhood basis of each point p of R, ord $\mathcal{U}_i \leqq n+1$, $i = 1, 2, \dots$.

Proof. Combine Theorem I.2 with Theorem V.1.

We shall need later the following proposition which is a slight modification of the "only if" part of the above corollary.

A) A space R of dimension $\leqq n$ has a sequence

$$\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \dots$$

of open coverings such that $\{ S(p, \mathcal{U}_i) \mid i = 1, 2, \dots \}$ is a neighbourhood basis of each point p of R, and such that each element of $\mathcal{U}_{i+1}$ meets at most $n+1$ elements of $\mathcal{U}_i$.

Proof. Since R is a metric space of dimension $\leqq n$, by use of the paracompactness of R we can construct a locally finite open covering $\mathcal{U}'_1$ with mesh $\mathcal{U}'_1 < 1$ and ord $\mathcal{U}'_1 \leqq n+1$.

Suppose $\mathcal{U}'_1 = \{U'_\gamma \mid \gamma \in \Gamma\}$; then by I.1 A) we can find an open covering $\mathcal{U}_1 = \{U_\gamma \mid \gamma \in \Gamma\}$ with $\bar{U}_\gamma \subset U'_\gamma$. Since $\bar{\mathcal{U}}_1$ is a locally finite closed covering with ord $\bar{\mathcal{U}}_1 \leqq n+1$, we obtain from the corollary to Theorem II.6 a locally finite open covering $\mathcal{U}'_2$ such that each element of $\mathcal{U}'_2$ meets at most $n+1$ elements of $\mathcal{U}_1$ and such that

$$\mathcal{U}'^*_2 < \mathcal{U}_1 \text{ , mesh } \mathcal{U}'_2 < \tfrac{1}{2} \text{ , and ord } \mathcal{U}'_2 \leqq n+1 .$$

Then we construct a locally finite open covering $\mathcal{U}_2$ satisfying $\bar{\mathcal{U}}_2 < \mathcal{U}'_2$. $\bar{\mathcal{U}}_2$ is a locally finite closed covering with

$$\text{mesh } \mathcal{U}_2 < \tfrac{1}{2} \text{ , ord } \bar{\mathcal{U}}_2 \leqq n+1 \text{ , and } \mathcal{U}^*_2 < \mathcal{U}_1 ,$$

such that each element of $\mathcal{U}_2$ meets at most $n+1$ elements of $\mathcal{U}_1$. By repeating this process we get the desired sequence $\mathcal{U}_1 > \mathcal{U}^*_2 > \dots$ of open coverings.

Theorem V. 2. *Let R be a metrizable space. Then there is a metric ρ of R such that the completion $\langle R^*, \rho^* \rangle$ of $\langle R, \rho \rangle$ satisfies* $\dim R^* = \dim R$. *If moreover R is separable, then we can choose as ρ a totally bounded metric.*

Proof. Assume that $\dim R \leqq n$. Then it is easy to see that we can introduce a metric ρ (a totally bounded metric ρ if R is separable) such that there is a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \dots$ of uniform coverings of $\langle R, \rho \rangle$ with

$$\text{mesh } \mathcal{U}_i \to 0 \text{ as } i \to \infty \text{ , and ord } \mathcal{U}_i \leqq n+1 .$$

Construct the completion $\langle R^*, \rho^* \rangle$ of $\langle R, \rho \rangle$ (see I.2.). Now, suppose $\mathcal{U}_i = \{U_\gamma \mid \gamma \in \Gamma_i\}$, and put

$$\mathcal{V}_i = \{ R^* - \overline{R - U_\gamma}^* \mid \gamma \in \Gamma_i \} ,$$

where the symbol $\overline{}^*$ denotes closure in R^*.

Then by I.2 C) each $\mathcal{V}_i$ is a uniform covering of $\langle R^*, \rho^* \rangle$. By use of I.2 B) and the definition of ρ^* we can easily prove that $\{\mathcal{V}_i\}$ satisfies

$$\mathcal{V}_1 > \mathcal{V}_2 > \dots \text{ , ord } \mathcal{V}_i \leqq n+1 \text{ , and mesh } \mathcal{V}_i = \text{mesh } \mathcal{U}_i .$$

The last equality implies that mesh $V_i \to 0$ as $i \to \infty$. Hence from Theorem V.1 it follows that $\dim R^* \leqq n$.

On the other hand $\dim R \leqq \dim R^*$ is obviously true, and thus we obtain $\dim R^* = \dim R$.

V. 2. Length of coverings

We shall apply the result of Section 1 to the notion of the "length of a multiplicative covering" (due to P. Alexandroff and A. Kolmogoroff) to establish another theorem for n-dimensionality.

Definition V. 1. *We call a covering* $\mathcal{U}$ *a multiplicative covering if* $\emptyset \notin \mathcal{U}$ *and if every non-empty intersection* $\cap_{i=1}^{k} U_i$ *of elements* U_i, $i = 1,\ldots,k$, *of* $\mathcal{U}$ *is also an element of* $\mathcal{U}$.

Definition V. 2. *Let* $\mathcal{U}$ *be a multiplicative covering. We mean by the length of an element* U *of* $\mathcal{U}$ *the greatest number* r *such that there exists a sequence* $U = U_1 \supsetneqq U_2 \supsetneqq \ldots \supsetneqq U_r$ *of elements of* $\mathcal{U}$. *We mean by the length of* $\mathcal{U}$ *the greatest length of elements of* $\mathcal{U}$, *i.e.* length $\mathcal{U} = \max\{\text{length } U \mid U \in \mathcal{U}\}$. *It is easy to see that for every multiplicative covering* $\mathcal{U}$ length $\mathcal{U} \leqq \operatorname{ord} \mathcal{U}$. *We can easily construct examples of multiplicative coverings* $\mathcal{U}$ *for which length* $\mathcal{U} < \operatorname{rank} \mathcal{U}$, $\operatorname{rank} \mathcal{U} <$ length $\mathcal{U}$ *and* length $\mathcal{U} = \operatorname{rank} \mathcal{U}$ *respectively hold.*

Theorem V. 3 [6]. *A* T_1*-space* R *is a metrizable space of dimension* $\leqq n$ *if and only if there exists a sequence*

$$\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \ldots$$

of multiplicative open coverings of length $\leqq n+1$ *such that* $\{S(p,\mathcal{U}_i) \mid i = 1,2,\ldots\}$ *is a neighbourhood basis of each point* p *of* R $(n \geqq 0)$.

Proof. If $\dim R \leqq n$, then by the corollary to Theorem V.1 we can construct a sequence $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \ldots$ of open coverings of order $\leqq n+1$ such that

---

[6] It remains open whether we can replace the condition ii) in Theorem V.1 by length $\mathcal{U}_k \leqq n+1$ to improve Theorem V.3. K. Nagami [2] generalized this theorem as follows: A T_1-space R is a metrizable space of dimension $\leqq n$ if and only if there exists a sequence $\{\mathcal{F}_i \mid i = 1,2,\ldots\}$ of locally finite multiplicative coverings such that $\mathcal{F}_{i+1} < \mathcal{F}_i$, length $\mathcal{F}_i \leqq n+1$, for every neighbourhood $U(p)$ of every point p of R there exists i with $S(p,\mathcal{F}_i) \subset U(p)$.

$\{S(p,U_i) \mid i=1,2,\ldots\}$ is a neighbourhood basis of p. Since U_i plus all the intersections of a finite number of elements of U_i is obviously a multiplicative covering of length $\leq n+1$, the necessity is proved.

Let us assume the existence of a sequence satisfying the condition of the proposition. First we shall construct an open covering M_1 such that

$$U_{n+2} < M_1 < U_1 \text{ , ord } M_1 \leqq n+1 \text{ .}$$

As a matter of fact, this is the principal part of the proof. We denote by $\{U_{r\alpha} \mid \alpha \in A_r\}$ all the elements of U_1 of length r. Then

$$U_1 = \{U_{r\alpha} \mid \alpha \in A_r,\ r=1,\ldots,n+1\}$$

because length $U_1 \leqq n+1$. We define open sets $V_{r\alpha}^{(i)}$, $i=1,\ldots,n+1$ by

$$V_{r\alpha}^{(1)} = U_{r\alpha}\ ,\quad V_{r\alpha}^{(i)} = \operatorname{Int}\ \{x \mid S(x,U_i) \subset V_{r\alpha}^{(i-1)}\},\quad i=2,\ldots,n+1\ ,$$

where we denote by $\operatorname{Int} S$ the interior of the set S. It follows directly from the above definition and $U_i^* < U_{i-1}$ that

(1) $$V_{r\alpha}^{(n+1)} \subset \ldots \subset V_{r\alpha}^{(2)} \subset V_{r\alpha}^{(1)} = U_{r\alpha}\ ,$$

(2) $$U_i < \{V_{r\alpha}^{(i)} \mid \alpha \in A_r\ ,\ r=1,\ldots,n+1\},\ i=1,\ldots,n+1\ ,$$

(3) $$S(V_{r\alpha}^{(i)},U_i) \subset V_{r\alpha}^{(i-1)},\ i=2,\ldots,n+1\ .$$

Next we define open sets $M_{r\alpha}$, $\alpha \in A_r$, $r=1,\ldots,n+1$ by

$$M_{1\alpha} = V_{1\alpha}^{(1)} = U_{1\alpha}\ ,$$

(4) $$M_{r\alpha} = V_{r\alpha}^{(r)} - \overline{[\cup\{S(V_{1\alpha}^{(r)},\ U_{n+2}) \mid \alpha \in A_1\}]} \cup \overline{[\cup\{S(V_{2\alpha}^{(r)},\ U_{n+2}) \mid \alpha \in A_2\}]} \cup \ldots \cup \overline{[\cup\{S(V_{r-1\,\alpha}^{(r)},\ U_{n+2}) \mid \alpha \in A_{r-1}\}]},\quad r=2,\ldots,n+1\ .$$

Let us show that

(5) $$U_{n+2} < M_1 = \{M_{r\alpha} \mid \alpha \in A_r,\ r=1,\ldots,n+1\}\ .$$

Let U be a given element of $\mathcal{U}_{n+2}$; then by using (2) for $i = n+1$, we can find $V_{r\alpha}^{(n+1)}$ with $U \subset V_{r\alpha}^{(n+1)}$. Hence from (1) we can deduce that $U \subset V_{r\alpha}^{(r)}$, which implies

$$\mathcal{U}_{n+2} < \{ V_{r\alpha}^{(r)} \mid \alpha \in A_r \ , \ r = 1, \dots, n+1 \} .$$

Therefore for every $U \in \mathcal{U}_{n+2}$ we can find the least number r such that

(6) $$U \subset V_{r\alpha}^{(r)} .$$

To prove (5) we shall show that

(7) $$U \cap S(V_{k\alpha}^{(r)}, \mathcal{U}_{n+2}) = \emptyset$$

for this r and every k with $1 \leqq k \leqq r-1$ and for every $\alpha \in A_k$. If we assume the contrary:

$$U \cap S(V_{k\alpha}^{(r)}, \mathcal{U}_{n+2}) \neq \emptyset$$

for some k , α with $1 \leqq k \leqq r-1$, $\alpha \in A_k$, then from $\mathcal{U}_{n+2}^{*} < \mathcal{U}_r$, (1) and (3), it follows that

$$U \subset S(V_{k\alpha}^{(r)}, \mathcal{U}_r) \subset V_{k\alpha}^{(r-1)} \subset V_{k\alpha}^{(k)} .$$

This contradicts the definition of r because $k < r$. Hence we obtain (7). This combined with (6) and (4) implies $U \subset M_{r\alpha}$, which proves (5).

Now, to show that

(8) $$\operatorname{ord} \mathcal{M}_1 \leqq n+1 ,$$

we prove

(9) $$M_{r\alpha} \cap M_{r\beta} = \emptyset \quad \text{for} \quad \alpha \neq \beta \quad \text{and for every} \quad r .$$

In case $\alpha, \beta \in A_1$, $\alpha \neq \beta$ clearly implies

$$M_{1\alpha} \cap M_{1\beta} = U_{1\alpha} \cap U_{1\beta} = \emptyset$$

because length $U_{1\alpha}$ = length $U_{1\beta} = 1$. To prove the same assertion in case $r > 1$, we show that

(10) $$U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma} \quad \text{for} \quad \alpha, \beta \in A_r \quad \text{and} \quad \gamma \in A_{r'},$$

implies

$$V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} = V^{(r)}_{r'\gamma} .$$

First, by the definition of $V^{(i)}_{r\alpha}$, it is obvious that

$$V^{(2)}_{r'\gamma} \subset V^{(2)}_{r\alpha} \cap V^{(2)}_{r\beta} .$$

Conversely, suppose $x \in V^{(2)}_{r\alpha} \cap V^{(2)}_{r\beta}$; then there exist neighbourhoods $P(x)$ and $Q(x)$ of x such that

$$S(P(x), U_2) \subset U_{r\alpha} , \quad S(Q(x), U_2) \subset U_{r\beta} .$$

Hence

$$S(P(x) \cap Q(x), U_2) \subset U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma} .$$

This means that $x \in V^{(2)}_{r'\gamma}$, proving

$$V^{(2)}_{r'\gamma} = V^{(2)}_{r\alpha} \cap V^{(2)}_{r\beta} .$$

Repeating this process, we conclude that

$$V^{(r)}_{r'\gamma} = V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} \text{ for every } r .$$

We now turn to the proof of (9). By use of (10) and (4) we obtain

$$M_{r\alpha} \cap M_{r\beta} \subset V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} - \overline{[\cup\{S(V^{(r)}_{1\alpha}, U_{n+2}) \mid \alpha \in A_1\}] \cup \ldots \cup}$$
$$\overline{[\cup\{S(V^{(r)}_{r-1\alpha}, U_{n+2}) \mid \alpha \in A_{r-1}\}]} \subset V^{(r)}_{r\alpha} \cap V^{(r)}_{r\beta} - S(V^{(r)}_{r'\gamma}, U_{n+2}) = \emptyset$$

for r' determined by $U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma}$, because $r' < r$ and in consequence

$$S(V^{(r)}_{r'\gamma}, U_{n+2}) \subset [\cup\{S(V^{(r)}_{1\alpha}, U_{n+2}) \mid \alpha \in A_1\}] \cup \ldots \cup$$
$$[\cup\{S(V^{(r)}_{r-1\alpha}, U_{n+2}) \mid \alpha \in A_{r-1}\}]$$

holds. Thus (9) is proved for $r = 1, \ldots, n+1$.

Since

$$M_1 = \{M_{r\alpha} \mid \alpha \in A_r, \ r = 1, \ldots, n+1\},$$

the assertion (8) follows directly from (9). Since $M_1 < U_1$ is clear, from (5) combined with (8) we obtain the desired open covering M_1 satisfying

$$\mathcal{U}_{n+2} < \mathcal{M}_1 < \mathcal{U}_1 \ , \ \text{ord}\ \mathcal{M}_1 \leqq n+1 \ .$$

Repeating this process, we get a sequence $\mathcal{M}_i$, $i = 1,2,\ldots$ of open coverings such that

$$\text{ord}\ \mathcal{M}_i \leqq n+1 \ , \ \text{and} \ \ \mathcal{U}_{1+i(n+1)} < \mathcal{M}_i < \mathcal{U}_{1+(i-1)(n+1)} \ .$$

Therefore from the corollary to Theorem V.1 we can deduce $\dim R \leqq n$ and the metrizability of R .

Corollary [7]. *A space R has dimension $\leqq n$ if and only if for every open covering $\mathcal{U}$ of R there exists a multiplicative open covering $\mathcal{V}$ of R such that* $\mathcal{V} < \mathcal{U}$, length $\mathcal{V} \leqq n+1$.

V. 3. Dimension and metric function

It is an interesting problem to characterize the dimension of a metric space by the property of its metric function. We know that if $\rho(x,y_i) < \varepsilon$, $i = 1,2,3$ on the 1-dimensional Euclidean space E^1 , then $\rho(y_i,y_j) < \varepsilon$ for some two points y_i , y_j of the three points y_1 , y_2 , y_3 and the same is also true for seven points y_i , $i = 1,\ldots,7$ and a point x of E^2 . The number of points y_i having such a property will increase with the dimension number n of E^n . This example leads us to a new idea to characterize the dimension of a metric space as follows.

The main theorem (Theorem V.4) of this section was first proved by J. Nagata [7] and P. A. Ostrand [2] independently. The proof to be presented here is a modification of Ostrand's proof.

A) If $\dim R \leqq n$, then for each open covering $\mathcal{U}$ of R , there is an open covering $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and $\mathcal{V} = \bigcup_{i=1}^{n+1} \mathcal{V}_i$, where each $\mathcal{V}_i$ is a discrete open collection [8].

Proof. By use of the method of the proof of II.7 B) we can construct an open covering $\mathcal{W}$ such that $\mathcal{W} < \mathcal{U}$ and $\mathcal{W} = \bigcup_{i=1}^{n+1} \mathcal{W}_i$ with open collections $\mathcal{W}_i$ of order $\leqq 1$. Let $\mathcal{W}_i = \{ W_{i\alpha} \mid \alpha \in A_i \}$. Select an open covering $\mathcal{P}$ such that $\mathcal{P}^* < \mathcal{W}$ and put

$$V_{i\alpha} = \bigcup \{ P \mid P \in \mathcal{P} ,\ S(P,\mathcal{P}) \subset W_{i\alpha} \} ,$$

---

[7] P. Alexandroff - A. Kolmogoroff [1] proved this theorem for a normal space R and for finite open coverings $\mathcal{U}$, $\mathcal{V}$.

$$V_i = \{ V_{i\alpha} \mid \alpha \in A_i \} , \quad V = \bigcup_{i=1}^{n+1} V_i .$$

Then V is obviously an open refinement of U because $V_{i\alpha} \subset W_{i\alpha}$ and $W < U$. Each V_i is discrete, because each member of P meets at most one member of V_i.

Definition V. 3. *Let U be a covering of a space R. Then for each non-negative integer n we define*

$$[U]^n = \{ S^n(U,U) \mid U \in U \} ,$$

where $S^0(U,U) = U$.

B) Let $\dim R \leqq n$. Then there is a sequence $U_1, U_2, \ldots$ of open coverings of R such that

i) $U_j = \bigcup_{i=1}^{n+1} U_j^i$, where each U_j^i is a discrete collection,

ii) mesh $U_j < 1/j$,

iii) $[U_{j+1}]^{20} < U_j$,

iv) if $j < k$ and $1 \leqq i \leqq n+1$, then each member of $[U_k]^{20}$ meets at most one member of U_j^i.

---

[8] Obviously the converse of this proposition is also true. P. A. Ostrand [1] obtained a similar characterization of dimension as follows:

A metric space R has $\dim \leqq n$ if and only if for each open covering U of R and each integer $k \geqq n+1$ there are k discrete open collections $V_1, \ldots, V_k$ such that $\bigcup_{i=1}^{k} V_i < U$, and the union of any $n+1$ of the V_i is a covering of R.

He applied this theorem to prove the following theorem concerning the representation of a function of many variables as a composite of functions of one variable, which gives a good example of an application of dimension theory. (Another such example was presented in IV.3.):

Let X^p, $p = 1, \ldots, m$ be compact metric spaces such that $\dim X^p = d_p < \infty$ and let $n = \sum_{p=1}^{m} d_p$. Then there are continuous functions $\psi^{pq} : X^p \to [0,1]$, $p = 1, \ldots, m$, $q = 1, \ldots, 2n+1$ such that every real-valued continuous function f defined on $\prod_{p=1}^{m} X^p$ is representable in the form

$$f(x_1, \ldots, x_m) = \sum_{q=1}^{2n+1} \varphi_q \left(\sum_{p=1}^{m} \psi^{pq}(x_p) \right) ,$$

where φ_q, $q = 1, \ldots, 2n+1$ are real-valued continuous functions with a real variable.

E. Ščepin [2] and R. Mañé [1] also obtained interesting results in dimension theory related with other fields: The former proved that every finite-dimensional compact ANR (see J. Nagata [8] for the definition) is metrizable, and the latter proved that if a compact metric space R has a homeomorphism $f : R \to R$ and there exists $\varepsilon > 0$ such that $\rho(f^i(x), f^i(y)) < \varepsilon$, $i = 1, 2, \ldots$ implies $x = y$, then $\dim R < \infty$.

Proof. By use of A) we find an open covering $\mathcal{U}$ satisfying i) and ii) for $j=1$. Then each point x of R has a neighbourhood $P(x)$ which meets at most one member of $\mathcal{U}_1^i$ for each i and is contained in some member of $\mathcal{U}_1$. Select an open covering $\mathcal{Q}$ such that $[\mathcal{Q}]^{20} < \{P(x) \mid x \in R\}$. By A) there is an open covering $\mathcal{U}_2$ such that $\mathcal{U}_2 < \mathcal{Q}$, mesh $\mathcal{U}_2 < 1/2$ and $\mathcal{U}_2 = \cup_{i=1}^{n+1} \mathcal{U}_2^i$ for discrete collections $\mathcal{U}_2^i$, $i=1,\ldots,n+1$.

Then $\{\mathcal{U}_1, \mathcal{U}_2\}$ satisfies i) - iv). Continue the same process to get a sequence $\{\mathcal{U}_1, \mathcal{U}_2, \ldots\}$ satisfying the desired conditions.

C) Let Q^* denote the set of all rationals of the form $2^{-m_1} + 2^{-m_2} + \ldots + 2^{-m_t}$, where m_i are natural numbers satisfying $1 \leqq m_1 < m_2 < \ldots < m_t$. If $\dim R \leqq n$, then we can define open coverings $S(m)$ for all $m \in Q^*$ such that

i) $S(m) = \cup_{i=1}^{n+1} S^i(m)$, where each $S^i(m)$ is a discrete collection,

ii) $\{S(x, S(m)) \mid m \in Q^*\}$ is a neighbourhood base at each $x \in R$,

iii) if $m, p \in Q^*$ satisfy $m < p$, then $S(m) < S(p)$,

iv) if $m, p \in Q^*$, $m < p$, $1 \leqq i \leqq n+1$, and $S_1 \in S^i(m)$, $S_2 \in S^i(p)$, then either $S_1 \subset S_2$ or $S_1 \cap S_2 = \emptyset$,

v) if $m, p \in Q^*$ satisfy $m + p < 1$, and if $S_1 \in S(m)$, $S_2 \in S(p)$ satisfy $S_1 \cap S_2 \neq \emptyset$, then $S_1 \cup S_2 \subset S_3$ for some $S_3 \in S(m+p)$.

Proof. The proof given here is somewhat sketchy to leave the detail to the reader. To begin with, construct open coverings $\mathcal{U}_1, \mathcal{U}_2, \ldots$ which satisfy the four conditions of B). (We assume that each $\mathcal{U}_i$ contains no empty set.) Then define open collections by

$$(1) \qquad \mathcal{V}_j^i = \{S^{20}(U, \mathcal{U}_{j+1}) \mid U \in \mathcal{U}_j^i\}, \quad 1 \leqq i \leqq n+1, \quad j = 1, 2, \ldots$$

$$\mathcal{V}_j = \bigcup_{i=1}^{n+1} \mathcal{V}_j^i .$$

For $A \subset R$, $1 \leqq i \leqq n+1$, $j \geqq 1$ and $k \geqq 0$, we define sets $T^k(A, i, j)$ and $T(A, i, j)$ as follows

$$(2) \qquad T^0(A, i, j) = S(A, \mathcal{V}_j^i),$$

$$T^k(A, i, j) = S(T^{k-1}(A, i, j), \mathcal{V}_{j+k}^i) \quad \text{for } k \geqq 1,$$

$$T(A, i, j) = \bigcup_{k=0}^{\infty} T^k(A, i, j).$$

Let $m = 2^{-m_1} + \ldots + 2^{-m_t} \in Q^*$, where $1 \leqq m_1 < \ldots < m_t$, and $1 \leqq i \leqq n+1$; then for each $A \subset R$ we define an open set $S(A, i, m)$ as follows.

(3) $S(A,i,m) = T(A,i,m_1+1)$ if $t=1$,

$S(A,i,m) = T(S(S(A,i,m'),V_{m_t}),i,m_t)$ if $t>1$,

where $m' = 2^{-m_1} + \dots + 2^{-m_{t-1}} = m - 2^{-m_t}$.

Now we define open collections $S(m)$ for $m = 2^{-m_1} + \dots + 2^{-m_t} \in Q^*$ by

$$S^i(m) = \{S(U,i,m) \mid U \in \mathcal{U}^i_{m_1}\} , \quad 1 \leqq i \leqq n+1 , \quad S(m) = \bigcup_{i=1}^{n+1} S^i(m) .$$

In the rest of the proof we denote by m and p members of Q^* such that $m = 2^{-m_1} + \dots + 2^{-m_t}$, $p = 2^{-p_1} + \dots + 2^{-p_u}$, $1 \leqq m_1 < \dots < m_t$, and $1 \leqq p_1 < \dots < p_u$.

Since $\mathcal{U}^i_{m_1} < S^i(m)$ and since $\mathcal{U}_{m_1}$ covers R , $S(m)$ is a covering of R . From iii) of B) and (1) it follows that $V_j < \mathcal{U}^*_j$, and accordingly by (2) we obtain $T(A,i,j) \subset S^4(A,\mathcal{U}_j)$. Thus from (3) it follows that

(4) $S(U,i,m) \subset S^4(U,\mathcal{U}_{m_1+1})$ if $t=1$.

Hence by iv) of B), each member of $\mathcal{U}_{m_1+1}$ meets at most one member of $S^i(m)$.

In the general case $t>1$, we can prove without much difficulty that

(5) $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1})$,

and thus each member of $\mathcal{U}_{m_1+1}$ meets at most one member of $S^i(m)$ because of iv) of B). Therefore each $S^i(m)$ is discrete.

From (4) and iii) of B) it follows that $S(x,S(m)) \subset S(x,\mathcal{U}_{m_1-1})$ if $m = 2^{-m_1}$. Thus ii) of C) follows from ii) of B).

Now, to prove iii) of C), suppose $m<p$.

Case a): $m_1 = p_1$, $m_2 = p_2$, ..., $m_t = p_t$, and $t<u$.

Then $S(m) < S(p)$ is obvious because

(6) $S(U,i,m) \subset S(U,i,p)$ for each i and $U \in \mathcal{U}^i_{m_1} = \mathcal{U}^i_{p_1}$.

Case b): $m_1 = p_1, \dots, m_{s-1} = p_{s-1}$, $m_s > p_s$ for some $s>1$.

Then put

$$p' = 2^{-p_1} + \dots + 2^{-p_s} ,$$

$$m' = 2^{-m_1} + \dots + 2^{-m_{s-1}} = 2^{-p_1} + \dots + 2^{-p_{s-1}} .$$

Now, by use of iii) of B), we obtain

$$S(U,i,m)\subset S^6(S(U,i,m'),U_{m_s})\subset S(S(U,i,m'),U_{p_s}) .$$

On the other hand by (3) we get

$$S(U,i,p')\supset S(S(U,i,m'),V_{p_s})\supset S(S(U,i,m'),U_{p_s}) .$$

Thus

(7) $\qquad S(U,i,m)\subset S(U,i,p')\subset S(U,i,p)$

holds for each $U\in U^i_{m_1}=U^i_{p_1}$, and we have proved $S(m)<S(p)$.

Case c): $m_1>p_1$.

Let $U\in U^i_{m_1}$; then by (4) and (5) $S(U,i,m)\subset S^{10}(U,U_{m_1+1})$. Hence by iii) of B) $S(U,i,m)\subset U'$ for some $U'\in U_{p_1}$.

Suppose $U'\in U^j_{p_1}$; then

$$S(U,i,m)\subset U'\subset S(U',j,p)\in S^j(p)\subset S(p) .$$

This implies $S(m)<S(p)$.

To prove iv), we first note the following fact.

(8) $\qquad$ If a) or b) (defined in the above) is the case, then $S^i(m)<S^i(p)$ for each i .

Because $m_1=p_1$ holds in this case, and, as shown in the above ((6) and (7)), for each $U\in U^i_{m_1}=U^i_{p_1}$, we have $S(U,i,m)\subset S(U,i,p)$. Now iv) follows from (8) and the discreteness of $S^i(p)$ if a) or b) is the case.

Assume the case c) and also that $p_s\leqq m_1<p_{s+1}$, $m_1=p_s+k$ (where $s\geqq 1$, $k\geqq 0$) .

Suppose that $S(U,i,m)\in S^i(m)$ and $S(U',i,p)\in S^i(p)$ meet each other, where $U\in U^i_{m_1}$, $U'\in U^i_{p_1}$.

Assume $s>1$. Then by (3) there are sequences $\sigma_1,\dots,\sigma_n$ of open sets such that

$$\begin{aligned}
\sigma_1 &= \{U',S_{11},S_{12},\dots\dots\} ,\\
\sigma_2 &= \{V_2,S_{20},S_{21},\dots\dots\} ,\\
&\dots\dots\dots\\
\sigma_s &= \{V_s,S_{s0},S_{s1},\dots,S_{sk},\dots\} ,\\
&\dots\dots\dots\\
\sigma_u &= \{V_u,S_{u0},S_{u1},\dots\dots\} ,
\end{aligned}$$

where $V_j \in \mathcal{V}_{p_j}$, $S_{jl} \in \mathcal{V}^i_{p_j+l}$ or $S_{jl} = \emptyset$, each non-empty member of a sequence meets its non-empty successor in the same sequence, the sum X_j of the members of σ_j meets V_{j+1}, and $\bigcup_{j=1}^{u} X_j$ meets $S(U,i,m)$.

Case α): $S(U,i,m) \cap X_j \neq \emptyset$ for some $j < s$ or $S(U,i,m) \cap V_s \neq \emptyset$.

Then we may assume $S(U,i,m) \cap V_s \neq \emptyset$, because $\mathcal{V}_{p_s}$ is a covering. By (4) and (5)

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) \subset S^{20}(U,\mathcal{U}_{p_s+k+1}) \in \mathcal{V}^i_{p_s+k},$$

because $U \in \mathcal{U}^i_{p_s+k}$. Since $S^{20}(U,\mathcal{U}_{p_s+k+1}) \cap V_s \neq \emptyset$,

$$S(U,i,m) \subset S^{20}(U,\mathcal{U}_{p_s+k+1}) \subset S(U',i,p) .$$

Case β): $S(U,i,m) \cap S_{sl} \neq \emptyset$ for some l with $0 \leq l \leq k-1$.

By an argument similar to the above $S(U,i,m) \subset S(U',i,p)$ can be proved.

Case γ): $S(U,i,m) \cap S_{sk} \neq \emptyset$.

Let

$$S_{sk} = S^{20}(U_0,\mathcal{U}_{p_s+k+1}),\ U_0 \in \mathcal{U}^i_{p_s+k} .$$

Since $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{p_s+k+1})$, there is a member of $[\mathcal{U}_{p_s+k+1}]^{20}$ which meets both U and U_0. Thus by iv) of B) $U = U_0$. Hence

$$S(U,i,m) \subset S_{sk} \subset S(U',i,p) .$$

Case δ): $S_{sk} \neq \emptyset$, and $S(U,i,m) \cap S_{sl} \neq \emptyset$ for some $l > k$
or $S(U,i,m) \cap X_j \neq \emptyset$ for some $j > s$.

Let

$$S_{sk} = S^{20}(U_0,\mathcal{U}_{p_s+k+1}),\ U_0 \in \mathcal{U}^i_{p_s+k} .$$

Then there is a chain of ten members of $\mathcal{U}_{p_s+k+1}$ which connects S_{sk} and $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{p_s+k+1})$, where we mean by a chain a finite sequence of sets each member of which meets its successor. Hence there is a member of $[\mathcal{U}_{p_s+k+1}]^{20}$ which meets both U and U_0. Thus $U = U_0$, and $S(U,i,m) \subset S(U',i,p)$.

Case ε): $S_{sk} = \emptyset$, and $S(U,i,m) \cap S_{sl} \neq \emptyset$ for some $l > k$,
or $S(U,i,m) \cap X_j \neq \emptyset$ for some $j > s$.

Assume that S_{sk} is the last non-empty member of the sequence

$$S_{s0}, S_{s1}, \dots, S_{sk-1} .$$

Then there is a chain of ten members of $\mathcal{U}_{p_s+k+1}$ which connects S_{sk} with $S(U,i,m) \subset S^{10}(U,\mathcal{U}_{p_s+k+1})$. Thus

$$S^{20}(U,\mathcal{U}_{p_s+k+1}) \in \mathcal{V}^i_{p_s+k} \text{, and } S^{20}(U,\mathcal{U}_{p_s+k+1}) \cap S_{sk} \neq \emptyset .$$

Hence

$$S(U,i,m) \subset S^{20}(U,\mathcal{U}_{p_s+k+1}) \subset S(U',i,p) .$$

If $S_{s0},\dots,S_{sk-1}$ are all empty, we can prove the same by use of V_s in place of S_{sk} . In the above argument we have assumed $s > 1$. However a similar argument is valid in the special case $s = 1$. Thus iv) of C) is established in all cases.

Finally, to prove v), let $m + p < 1$. Suppose $S(U,i,m) \cap S(U',j,p) \neq \emptyset$, where $U \in \mathcal{U}^i_{m_1}$, $U' \in \mathcal{U}^j_{p_1}$.

Case 1: $m_1 = p_1$.

It follows from (4) and (5) that

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) ,$$

$$S(U',j,p) \subset S^{10}(U',\mathcal{U}_{m_1+1}) ,$$

$$S^{10}(U,\mathcal{U}_{m_1+1}) \cap S^{10}(U',\mathcal{U}_{m_1+1}) \neq \emptyset .$$

Thus by iii) of B)

$$S(U,i,m) \cup S(U',j,p) \subset W \subset S_0$$

for some $W \in \mathcal{U}_{m_1-1}$ and $S_0 \in S(m+p)$, because $m + p = 2^{-(m_1-1)} + \dots$.

Case 2: $m_1 > p_1$ (and thus $m < p$) .

Assume that $p_s < m_1 \leq p_{s+1}$ $(s \geq 1)$.

Case 2.1: $m_1 < p_{s+1}$.

Assume e.g. $s = 1$. It follows from (4) and (5) that

$$S^{10}(U,\mathcal{U}_{m_1+1}) \cap S(U',j,p) \neq \emptyset .$$

Since $m_1 + 1 \leqq p_2$, by (3) there is a chain of six members of $\mathcal{U}_{m_1+1}$ which connects $T(U',j,p_1+1)$ with $S^{10}(U,\mathcal{U}_{m_1+1})$. Thus

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) \subset S^{20}(U,\mathcal{U}_{m_1+1}) \in \mathcal{V}_{m_1} ,$$

and

$$S^{20}(U,\mathcal{U}_{m_1+1}) \cap T(U',j,p_1+1) \neq \emptyset .$$

Hence

$$S(U,i,m) \subset S(T(U',j,p_1+1) , \mathcal{V}_{m_1}) \subset S(U',j,q) ,$$

where $q = 2^{-p_1} + 2^{-m_1} + 2^{-p_2} + \dots + 2^{-p_n}$. On the other hand we obtain $S(U',j,p) \subset S(U',j,q)$ from (7). Thus

$$S(U,i,m) \cup S(U',j,p) \subset S_0 \quad \text{for some} \quad S_0 \in \mathcal{S}(p+m) ,$$

because $q \leqq p+m$.

In case of $s > 1$, too, we can prove the same by a similar argument. Also note that a similar argument can be applied to the special case $p_u < m_1$ if (6) instead of (7) is used.

Case 2.2: $m_1 = p_{s+1}$.

Denote by k the number satisfying $0 \leqq k < s$, $p_{s+1} - 1 = p_s$, $p_s - 1 = p_{s-1}, \dots, p_{s-k+2} - 1 = p_{s-k+1}$, and $p_{s-k+1} - 1 > p_{s-k}$. (Note that $k=0$ means $p_{s+1} - 1 > p_s$, and $k=s$ means that $p_{j+1} - 1 = p_j$ for all $j = 1,2,\dots,s$.)

Case 2.2.1: $k = s$.

Then note that

$$2^{-p_1} + \dots + 2^{-p_{s+1}} + 2^{-m_1} = 2^{-(p_1 - 1)} \leqq m + p < 1 .$$

Since $S(U',j,p) \subset S^{10}(U',\mathcal{U}_{p_1+1})$, and

$$S(U,i,m) \subset S^{10}(U,\mathcal{U}_{m_1+1}) \subset S^{10}(U,\mathcal{U}_{p_1+1}) ,$$

it is true that

$$S^{10}(U',\mathcal{U}_{p_1+1}) \cap S^{10}(U,\mathcal{U}_{p_1+1}) \neq \emptyset .$$

Hence

$$S(U',j,p) \cup S(U,i,m) \subset U_0 \subset S_0$$

for some $U_0 \in \mathcal{U}_{p_1 - 1}$ and $S_0 \in S(m+p)$.

Case 2.2.2: $k < s$.

Put

$$p' = 2^{-p_1} + \ldots + 2^{-p_{s-k}} \quad \text{and} \quad p'' = p' + 2^{-(p_{s-k+1} - 1)} .$$

(Note that $p'' \leq m + p$) . Then it follows from (4) and (5) that $S(U,i,m) \subset S^{10}(U, \mathcal{U}_{m_1+1})$, and there is a chain of six members of $\mathcal{U}_{p_{s-k+1}}$ which connects $S(U',j,p')$ with $S^{10}(U,\mathcal{U}_{m_1+1})$. Denote by C the sum of the members of this chain. Then there is $U_0 \in \mathcal{U}_{p_{s+k+1}-1}$ such that

$$S(U,i,m) \cup C \subset S^{10}(U,\mathcal{U}_{m_1+1}) \cup C \subset U_0$$

because $p_{s-k+1} \leq m_1$. Since

$$U_0 \cap S(U',j,p') \neq \emptyset \quad \text{and} \quad S^{20}(U_0, \mathcal{V}_{p_{s-k+1}}) \in \mathcal{V}_{p_{s-k+1}-1} ,$$

we obtain

$$S(U,i,m) \subset U_0 \subset S^{20}(U_0, \mathcal{V}_{p_{s-k+1}}) \subset S(U',j,p'') .$$

On the other hand $S(U',j,p) \subset S(U',j,p'')$ follows from (7). Hence

$$S(U,i,m) \cup S(U',j,p) \subset S_0 \quad \text{for some} \quad S_0 \in S(m+p) ,$$

because $p'' \leq m + p$. Thus all conditions for $S(m)$ are verified.

D) Let $\dim R \leq n$ $(n \geq 0)$. Then there is a topology-preserving metric ρ of R such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R there is a pair of indices i,j such that $i \neq j$, and $\rho(y_i, y_j) \leq \rho(x, y_j)$.

Proof. Define $S(m)$, $m \in Q^*$ satisfying the five conditions of C). Then define $\rho : R \times R \to [0,1]$ by

$$\rho(x,y) = \inf\{\, m \mid m \in Q^*,\ y \in S(x, S(m))\} \quad \text{if} \quad y \in S(X, S(m)) \quad \text{for some} \quad m ,$$

$$\rho(x,y) = 1 \quad \text{otherwise.}$$

It is obvious that $\rho(x,y) = \rho(y,x)$ and $\rho(x,x) = 0$. From ii) and iii) of C) it follows that $\rho(x,y) > 0$ if $x \neq y$.

Assume $\rho(x,y) < 1$ and $\rho(x,z) < 1$ for $x,y,z \in R$. Then for every $\varepsilon > 0$ there are $m,p \in Q^*$ such that $y \in S(x,S(m))$, $z \in S(y,S(p))$, $\rho(x,y) \leq m < \rho(x,y) + \varepsilon$, and $\rho(y,z) \leq p < \rho(y,z) + \varepsilon$.

By v) of C) there is $W \in S(m+p)$ such that $x,z \in W$. Hence

$$\rho(x,z) \leq m + p < \rho(x,y) + \rho(y,z) + 2\varepsilon \quad .$$

Thus it is proved that $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Therefore ρ is a metric function.

Note that

$$S(x,S(m/2)) \subset S_m(x) \subset S(x,S(m)) \quad \text{for each } m \in Q^* \; .$$

Thus by virtue of ii) the metric ρ is compatible with the topology of R .

Finally, let $x,y_1,\ldots,y_{n+2}$ be any $n+3$ points of R . If $\rho(x,y_k) = 1$ for some k , then $\rho(y_i,y_k) \leq \rho(x,y_k)$ for any i . So assume that $\rho(x,y_k) < 1$ for $k = 1,\ldots,n+2$. Let $\varepsilon > 0$ be given. Then choose $m(j) \in Q^*$, $j = 1,\ldots,n+2$ such that $\rho(x,y_j) \leq m(j) < \rho(x,y_j) + \varepsilon$, and $U_j \in S(m(j))$, $j = 1,\ldots,n+2$ such that $x,y_j \in U_j$.

Suppose $U_j \in S^{i(j)}(m(j))$, $j = 1,\ldots,n+2$. Then for some pair (j,k) of distinct indices we have $i(j) = i(k)$ because $1 \leq i(j) \leq n+1$ for $j = 1,\ldots,n+2$. Suppose $m(j) \leq m(k)$; then from i) and iv) of C) it follows that $U_j \subset U_k$, and thus y_j , $y_k \in U_k$. Hence

$$\rho(y_j,y_k) \leq m(k) < \rho(x,y_k) + \varepsilon \; .$$

The pair (y_j,y_k) may differ as ε differs. However, since there are only finitely many such pairs, we can fix a pair (y_j,y_k) satisfying

$$\rho(y_j,y_k) < \rho(x,y_k) + 1/n \quad \text{for} \quad n = 1,2,\ldots \; .$$

Thus $\rho(y_j,y_k) \leq \rho(x,y_k)$.

E) If one can introduce a metric ρ into R which satisfies the condition of D), then $\dim R \leq n$.

Proof. Actually we shall prove the following slightly stronger statement by use of induction on n : If one can introduce a metric ρ on R such that for a definite number $\delta > 0$ and for every $n+3$ points $x,y_1,\ldots,y_{n+2}$ of R with $\rho(x,y_j) < \delta$, $j = 1,\ldots,n+2$, there is a pair of distinct indices i,j for which $\rho(y_i,y_j) \leq \rho(x,y_j)$, then $\dim R \leq n$.

Let ρ be a metric satisfying the condition for $n = 0$. Then for any $x,y \in R$ and

$\varepsilon > 0$ with $\varepsilon < \delta$, $S_\varepsilon(x)$ and $S_\varepsilon(y)$ are disjoint or equal. More precisely $S_\varepsilon(x) = S_\varepsilon(y)$ if $y \in S_\varepsilon(x)$, and $S_\varepsilon(x) \cap S_\varepsilon(y) = \emptyset$ if $y \notin S_\varepsilon(x)$.

Thus $\mathcal{U}_k = \{ S_{\frac{1}{k}}(x) \mid x \in R \}$ for sufficiently large k's are disjoint open coverings of R which satisfy the conditions of Theorem V.1 for $n = 0$. Therefore $\dim R \leqq 0$.

Assume validity of the statement for $n = m - 1$. Suppose that ρ is a metric satisfying the condition for $n = m$ and $\delta > 0$. Then we claim that for each positive number $\varepsilon < \delta$ and for each closed set F of R $\dim B(S_\varepsilon(F)) \leqq m - 1$, where $S_\varepsilon(F) = \bigcup \{ S_\varepsilon(x) \mid x \in F \}$.

Assume the contrary; then by the induction hypothesis we can select $m + 2$ points $x, y_1, \dots, y_{m+1} \in B(S_\varepsilon(F))$ such that

$$\rho(x, y_j) < \varepsilon, \quad j = 1, \dots, m+1,$$
$$\rho(y_i, y_j) > \rho(x, y_j) \quad \text{for every } i, j \text{ with } i \neq j.$$

Now, choose a small neighbourhood $U(x)$ of x such that for every $x' \in U(x)$

$$\rho(x', y_j) < \varepsilon \quad \text{and} \quad \rho(y_i, y_j) > \rho(x', y_j) \quad \text{if} \quad i \neq j.$$

Then there is $y_{m+2} \in F$ satisfying $S_\varepsilon(y_{m+2}) \cap U(x) \neq \emptyset$ because $x \in B(S_\varepsilon(F))$. Select a point $x' \in S_\varepsilon(y_{m+2}) \cap U(x)$. Then

$$\rho(x', y_j) < \varepsilon < \delta, \quad j = 1, \dots, m+2,$$
$$\rho(y_i, y_j) > \rho(x', y_j) \quad \text{if} \quad i \neq j \text{ and } 1 \leqq i, j \leqq m+1,$$
$$\rho(y_i, y_{m+2}) \geqq \varepsilon > \rho(x', y_{m+2}), \quad i = 1, \dots, m+1,$$
$$\rho(y_{m+2}, y_j) \geqq \varepsilon > \rho(x', y_j), \quad j = 1, \dots, m+1.$$

This contradicts the property of ρ. Thus we have proved

$$\dim B(S_\varepsilon(F)) \leqq m - 1.$$

Now, let F and G be disjoint closed sets of R and m_0 a natural number satisfying $1/m_0 < \delta$. Then put

$$S = \bigcup_{m=m_0}^{\infty} \left(S_{\frac{1}{m}}(F) - \overline{S_{\frac{1}{m}}(G)}\right).$$

It is easy to see that S is an open set such that

$$F \subset S \subset R - G \quad \text{and} \quad B(S) \subset \left(\bigcup_{m=m_0}^{\infty} B(S_{\frac{1}{m}}(F)) \right) \cup \left(\bigcup_{m=m_0}^{\infty} B(S_{\frac{1}{m}}(G)) \right) .$$

It follows from the sum theorem and the claim proved above that $\dim B(S) \leqq m-1$. Therefore $\dim R \leqq m$ which completes the induction.

Combining D) and E) we obtain the following main theorem.

Theorem V. 4. *A metric space R has $\dim \leqq n$ $(n \geqq 0)$ if and only if one can introduce a topology-preserving metric ρ on R such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R, there is a pair of distinct indices i, j for which $\rho(y_i, y_j) \leqq \rho(x, y_j)$.*

Definition V. 4. *A metric ρ is called non-archimedean if it satisfies $\rho(x,z) \leqq \max(\rho(x,y), \rho(y,z))$ for every points x, y and z of the space.*

Corollary.[9] *A metric space R is 0-dimensional if and only if one can introduce a topology-preserving non-archimedean metric on R.*

Proof. This is a direct consequence of Theorem V.4 because the condition there for $n=0$ is precisely the one for a non-archimedean metric.

We can slightly weaken the condition of Theorem V.4 if R is separable.

Theorem V. 5.[10] *A separable metric space R has $\dim \leqq n$ if and only if one can introduce a topology-preserving totally bounded metric ρ into R such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R, there is a triplet of indices i, j, k satisfying $\rho(y_i, y_j) \leqq \rho(x, y_k)$, and $i \neq j$.*

Proof. The 'only if' part follows directly from Theorem V.4.

We can prove the 'if' part as follows if R is compact. Let $\varepsilon > 0$ and M a maximal subset of R such that $\rho(x,y) \geqq \varepsilon$ for every $x, y \in M$ with $x \neq y$. Then $\mathcal{U}_\varepsilon = \{ S_\varepsilon(x) \mid x \in M \}$ is obviously an open covering of R. Suppose $x \in S_\varepsilon(x_k)$, $k = 1, \ldots, n+2$; then $\rho(x, x_k) < \varepsilon$, $k = 1, \ldots, n+2$.

Hence for some distinct i and j we obtain $\rho(x_i, x_j) < \varepsilon$ which implies $x_i = x_j$. Thus $\operatorname{ord} \mathcal{U}_\varepsilon \leqq n+1$.

Since R is compact, we can construct a sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open coverings which satisfies the condition of Theorem V.1. Hence $\dim R \leqq n$.

---

[9] This theorem is due to J. de Groot [1].

[10] This theorem is due to J. de Groot [2].

Now, let R be a separable metric space with a totally bounded metric ρ satisfying the condition. Then the completion $\langle R^*, \rho^* \rangle$ of $\langle R, \rho \rangle$ is a compact metric space. We shall prove that ρ^* satisfies the same condition as ρ. Assume that $x^*, y^*, \ldots, y^*_{n+2}$ are $n+3$ points of R^* such that

$$\rho^*(y^*_i, y^*_j) > \rho^*(x^*, y^*_k) \quad \text{whenever} \quad i \neq j .$$

Since R is dense in R^*, we can choose points x and y_j of R which are so close to x^* and y^*_j, respectively that

$$\rho(y_i, y_j) > \rho(x, y_k) \quad \text{whenever} \quad i \neq j ,$$

which is a contradiction. Thus ρ^* satisfies the required condition, and hence, as proved in the above, $\dim R^* \leqq n$. Therefore it follows from the subspace theorem that $\dim R \leqq n$.

Corollary. [11] *A compact metric space R has $\dim \leqq n$ if and only if one can introduce a metric ρ into R such that for every $n+3$ points $x, y_1, \ldots, y_{n+2}$ of R, there is a triplet i, j, k of indices satisfying $\rho(y_i, y_j) \leqq \rho(x, y_k)$, and $i \neq j$.*

V. 4. Another metric that characterizes dimension

Every spherical neighbourhood of any point of E^n has a boundary of dimension $n-1$. J. H. Roberts' improvement of the corollary to Theorem IV.12 (p. 85) shows that we can characterize dimension of separable metric spaces in this direction. But we must face difficult circumstances to characterize analogously dimension of general metric spaces. Because the existence of a metric which satisfies the condition of the corollary to Theorem IV.12 may not be sufficient for a general metric space R to have dimension $\leqq n$ (even if it holds for every $r > 0$) since

[11] It is unknown yet if this theorem is true for every metric space. In the following is another characterization of dimension by a metric satisfying a stronger condition:

A metric space R has $\dim \leqq n$ if and only if one can introduce a topology-preserving metric ρ on R such that if $\rho(S_{\frac{\varepsilon}{2}}(x), y_i) < \varepsilon$, $i = 1, \ldots, n+2$ for $\varepsilon > 0$ and points $x, y_1, \ldots, y_{n+2}$ of R, then $\rho(y_i, y_j) < \varepsilon$ for some distinct indices i, j.

This theorem was proved by J. Nagata [2] and a simpler proof was given by S. Buzâsi [1].

ind $R \leqq n$ is not sufficient for that. However, we can develop this idea in general metric cases as seen in the following.

Theorem V. 6 [12]. *A space R has dimension $\leqq n$ if and only if we can introduce into R a topology-preserving metric ρ such that the spherical neighbourhoods $S_\varepsilon(p)$, $\varepsilon > 0$ of every point p of R have boundaries of dimension $\leqq n-1$ and such that $\{S_\varepsilon(p) \mid p \in R\}$ is closure-preserving for every $\varepsilon > 0$.*

Proof. To show the "if" part let us note that

$$B(S_\varepsilon(A)) = B(\cup\{S_\varepsilon(p) \mid p \in A\}) \subset \cup\{B(S_\varepsilon(p)) \mid p \in A\}$$

for every subset A of R because $\{S_\varepsilon(p) \mid p \in R\}$ is closure-preserving. Therefore

$$\{B(S_\varepsilon(A)) \cap B(S_\varepsilon(p)) \mid p \in A\}$$

is a closure-preserving closed covering of $B(S_\varepsilon(A))$. Since $\dim B(S_\varepsilon(p)) \leqq n-1$ for every $p \in A$, from the corollary to Theorem II.1 we can conclude that $\dim B(S_\varepsilon(A)) \leqq n-1$.

Now, let F and G be given disjoint closed sets of R; then by the above remark we can construct two sequences

$$U_1 \supset \bar{U}_2 \supset U_2 \supset \ldots \supset F ,$$
$$W_1 \supset \bar{W}_2 \supset W_2 \supset \ldots \supset G$$

of open sets U_i and W_i such that

$$F = \bigcap_{i=1}^{\infty} U_i , \quad G = \bigcap_{i=1}^{\infty} W_i ,$$

$$\dim (\bar{U}_i - U_i) \leqq n-1 , \quad \dim (\bar{W}_i - W_i) \leqq n-1 .$$

It is clear that $U = \bigcup_{i=1}^{\infty} (U_i - \bar{W}_i)$ is an open set and satisfies

$$F \subset U \subset R - G , \quad \bar{U} - U \subset \bigcup_{i=1}^{\infty} [(\bar{U}_i - U_i) \cup (\bar{W}_i - W_i)] .$$

---

[12] This theorem is due to J. Nagata [5]. S. Buzási's [1] idea is used here to simplify the proof. Also note that as indicated by the proof, one can introduce into every metrizable space R a topology-preserving metric ρ such that $\{S_\varepsilon(p) \mid p \in R\}$ is closure preserving for every $\varepsilon > 0$.

The latter relation implies $\dim(\bar{U}-U)\leqq n-1$ by virtue of the sum theorem. Thus we can conclude that $\dim R\leqq n$.

To show the 'only if' part let $\dim R\leqq n$; then we can choose a sequence $\{\mathcal{U}_i \mid i=1,2,\dots\}$ of open coverings such that

a) $\mathcal{U}_1 > \mathcal{U}_2^{***} > \mathcal{U}_2 > \mathcal{U}_3^{***} > \dots$,

b) $\operatorname{mesh} \mathcal{U}_m \to 0$ as $m\to\infty$,

c) $S^2(x,\mathcal{U}^*_{m+1})$ intersects at most $n+1$ members of $\mathcal{U}_m$ for every $x\in R$.

Note that for every $A\subset R$ and integers $1\leqq m_2<\dots<m_p$, the following holds:

$$(1)\qquad S^2(\dots S^2(A,\mathcal{U}_{m_2})\dots\mathcal{U}_{m_p})\subset S^3(A,\mathcal{U}_{m_2})\,.$$

The easy proof is left to the reader.

Now we define open coverings $\mathcal{S}(m_1,\dots,m_p)$ for integers $m_1,\dots,m_p$ satisfying $1\leqq m_1<m_2<\dots<m_p$ as follows.

$$S(U;m_1)=U \quad \text{for each} \quad U\in\mathcal{U}_{m_1}\,,$$
$$S(U;m_1,\dots,m_p)=S^2(\dots S^2(U;\mathcal{U}_{m_2}),\dots,\mathcal{U}_{m_p})\,,$$
$$\mathcal{S}(m_1,\dots,m_p)=\{S(U;m_1,\dots,m_p)\mid U\in\mathcal{U}_{m_1}\}\,.$$

It follows from (1) and a) that

$$S(U;m_1,\dots,m_p)\subset S^3(U,\mathcal{U}_{m_2})\subset S(U,\mathcal{U}_{m_1})\,.$$

Hence we obtain

$$(2)\qquad \mathcal{S}(m_1)=\mathcal{U}_{m_1}<\mathcal{S}(m_1,\dots,m_p)<\mathcal{U}^*_{m_1}\,.$$

Now we can prove that

$$\mathcal{S}(l_1,\dots,l_q)<\mathcal{S}(m_1,\dots,m_p)$$

holds whenever

$$1/2^{l_1}+\dots+1/2^{l_q}<1/2^{m_1}+\dots+1/2^{m_p}\,.$$

To do so assume that

(3) $\quad m_1 = l_1, \dots, m_{i-1} = l_{i-1}\,,\; m_i < l_i$ for some i with $1 \leqq i \leqq p, q$,

and put $C = S(U; l_1, \dots, l_{i-1})$ for any $U \in \mathfrak{U}_{m_1}$ ($C = U$ if $i = 1$). Then it follows from (1) and a) that

$$S(U; l_1, \dots, l_q) \subset S^3(C, \mathfrak{U}_{l_i}) \subset S^2(C, \mathfrak{U}_{m_i}) .$$

Since

$$S^2(C, \mathfrak{U}_{m_i}) \in S(m_1, \dots, m_i)$$

follows from (3) and $S(m_1, \dots, m_i) < S(m_1, \dots, m_p)$ is obvious, we have $S(l_1, \dots, l_q) < S(m_1, \dots, m_p)$.

Now, we define a mapping $\rho : R \times R \to [0,1]$ by

(4) $$\rho(x,y) = \inf \{ 1/2^{m_1} + \dots + 1/2^{m_p} \mid y \in S(x, S(m_1, \dots, m_p)) \} ,$$
$$\rho(x,y) = 1 \text{ if } y \notin S(x, S(m_1, \dots, m_p)) \text{ for every } S(m_1, \dots, m_p) .$$

First of all $\rho(x,y) = \rho(y,x)$ is obvious. It is almost obvious that $\rho(x,y) = 0$ if and only if $x = y$. To prove the triangle axiom for ρ, assume that

$$1 > \rho(x,y) = a \geqq b = \rho(z,y) ,$$

because the case of $\rho(x,y) = 1$ is trivial. Let $\varepsilon > 0$ be given; then by (4) we can choose $1 \leqq m_1 < \dots < m_p$ and $1 \leqq l_1 < \dots < l_q$ such that $2 \leqq p$, $2 \leqq q$, $l_1 < m_p$, and

(5) $$x \in S(y, S(m_1, \dots, m_p)) ,\quad z \in S(y, S(l_1, \dots, l_q)) ,$$
$$a < 1/2^{m_1} + \dots + 1/2^{m_p} < a + \varepsilon ,$$
$$b < 1/2^{l_1} + \dots + 1/2^{l_q} < b + \varepsilon ,$$
$$1/2^{l_1} + \dots + 1/2^{l_q} < 1/2^{m_1} + \dots + 1/2^{m_p} .$$

Then $m_i \leqq l_1 \leqq m_{i+1}$ holds for some i with $1 \leqq i < p$.

Case 1: $m_i < l_1 < m_{i+1}$.

Let $U \in \mathfrak{U}_{m_1}$, $V \in \mathfrak{U}_{l_1}$, and

(6) $$x, y \in S(U; m_1, \dots, m_p) = A ,\quad y, z \in S(V; l_1, \dots, l_q) = B$$

be given. Further, put $D = S(U;m_1,\ldots,m_i)$. Then we obtain

(7) $$A \subset S^3(D, U_{m_i+1}) \subset S(D, U^*_{l_1+1}) .$$

Since

$$B \subset S^3(V, U_{l_2}) \subset S(V, U^*_{l_1+1}) ,$$

there is $W \in U^*_{l_1+1}$ such that $y \in W$ and $W \cap V \neq \emptyset$. From (7) it follows that

$$y \in W \cap S(D, U^*_{l_1+1}) \neq \emptyset .$$

Thus

$$x, z \in A \cup V \subset S(S^2(D, U^*_{l_1+1}), U_{l_1}) \subset S^2(D, U_{l_1}) ,$$

which implies that

$$x, z \in S(U; m_1, \ldots, m_i, l_1, \ldots, l_q) .$$

Therefore we conclude that

$$\rho(x,z) \leq 1/2^{m_1} + \ldots + 1/2^{m_i} + 1/2^{l_1} + \ldots + 1/2^{l_q} < a + b + 2\varepsilon .$$

Case 2: $m_i = l_1$, and $m_1,\ldots,m_i$ are integers in succession, i.e. $m_i = m_{i-1} + 1 = m_{i-2} + 2 = \ldots = m_1 + (i-1)$.

Case 2.1: $m_1 = 1$.

Then

$$\rho(x,z) \leq 1 = 1/2^{m_1} + \ldots + 1/2^{m_i} + 1/2^{l_1} < a + b + 2\varepsilon .$$

Case 2.2: $m_1 > 1$.

Combining (2) and (5) we obtain

$$x \in S(y, U^*_{m_1}) \quad \text{and} \quad z \in S(y, U^*_{l_1}) .$$

Hence

$$x \in S(z, U^{**}_{m_1}) \subset S(z, U_{m_1-1}) = S(z, S(m_1-1)) ,$$

which implies

$$\rho(x,z) \leq 1/2^{m_1 - 1} = 1/2^{m_1} + \dots + 1/2^{m_i} + 1/2^{l_1} < a + b + 2\varepsilon .$$

Case 3: $m_i = l_1$; $m_s, m_{s+1}, \dots, m_i$ are integers in succession, and $m_{s-1} + 1 < m_s$, where $2 \leq s \leq i$.

We may suppose (6) holds. Further put $E = S(U; m_1, \dots, m_{s-1})$. Then it follows from (1) and a) that

$$A \subset S^3(E, U_{m_s}) \subset S(E, U_{m_s - 1}) \text{ and}$$
$$B \subset S(V, U_{l_1}) \in U^*_{l_1} = U^*_{m_i} < U_{m_s - 1} .$$

Since $y \in A \cap B \neq \emptyset$, these relations imply that

$$x, z \in A \cup B \subset S^2(E, U_{m_s - 1}) = S(U; m_1, \dots, m_{s-1}, m_s - 1) .$$

Hence

$$\rho(x,z) \leq 1/2^{m_1} + \dots + 1/2^{m_{s-1}} + 1/2^{m_s - 1} = 1/2^{m_1} + \dots + 1/2^{m_i} + 1/2^{l_1} < a + b + 2\varepsilon .$$

Thus in every case we obtain $\rho(x,z) < a + b + 2\varepsilon$, which proves the triangle axiom for ρ . Since

$$S(x, U_{m+1}) \subset S_{\frac{1}{2^m}}(x) \subset S(x, U_m)$$

is obvious, the metric ρ agrees with the topology of R .

Now for any sequence $m_1, m_2, \dots$ of countably many integers with $1 \leq m_1 < m_2 < \dots$ and for $U \in U_{m_1}$ we define an open set $S(U; m_1, m_2, \dots)$ by

$$S(U; m_1, m_2, \dots) = \bigcup_{k=1}^{\infty} S(U; m_1, \dots, m_k)$$

and also an open covering $S(m_1, m_2, \dots)$ by

$$S(m_1, m_2, \dots) = \{ S(U; m_1, m_2, \dots) \mid U \in U_{m_1} \} .$$

Let us assume in the remainder of the proof that

$$0 < \varepsilon = 1/2^{m_1} + 1/2^{m_2} + \dots, \text{ and } 1 \leq m_1 < m_2 < \dots .$$

Then we can assert that

(8) $S_\varepsilon(x) = S(x, S(m_1, m_2, \dots))$ for every $x \in R$.

For, if $y \notin S(x, S(m_1, m_2, \dots))$, then $y \notin S(x, S(m_1, \dots, m_k))$ for $k = 1, 2, \dots$. Thus $\rho(x,y) \geqq 1/2^{m_1} + 1/2^{m_2} + \dots$, which means $y \notin S_\varepsilon(x)$. Namely we obtain

$$S_\varepsilon(x) \subset S(x, S(m_1, m_2, \dots)) .$$

Conversely, if $y \in S(x, S(m_1, m_2, \dots))$, then there is $U \in \mathcal{U}_{m_1}$ such that $x, y \in S(U; m_1, m_2, \dots)$. In view of the definition of $S(U; m_1, m_2, \dots)$ we get $x, y \in S(U; m_1, \dots, m_k)$ for some $k \geqq 1$. Hence

$$\rho(x,y) \leqq 1/2^{m_1} + \dots + 1/2^{m_k} < \varepsilon$$

which means $y \in S_\varepsilon(x)$. Hence

$$S(x, S(m_1, m_2, \dots)) \subset S_\varepsilon(x) .$$

Thus we conclude that (8) is true.

To show $\dim B(S_\varepsilon(x)) \leqq n-1$, we note that (1) implies

$$S(U; m_1, \dots, m_k) \subset S^3(U, \mathcal{U}_{m_2}) \subset S^3(U, \mathcal{U}_{m_1+1}) .$$

Thus we obtain

(9) $S(U; m_1, m_2, \dots) \subset S^3(U, \mathcal{U}_{m_1+1})$.

Further note that by c) each $S(x, \mathcal{U}^*_{m_1+1})$ intersects at most $n+1$ members of $\mathcal{U}_{m_1}$. Hence each $x \in R$ is contained in $S^3(U, \mathcal{U}_{m_1+1})$ for at most $n+1$ distinct members U of $\mathcal{U}_{m_1}$. This note combined with (9) implies

(10) $\operatorname{ord} S(m_1, m_2, \dots) \leqq n+1$.

Now, let us turn to the proof of $\dim B(S_\varepsilon(x)) \leqq n-1$.

Let $y \in B(S_\varepsilon(x))$ be given. Then we can prove

(11) $\operatorname{ord}_y \mathcal{U}_{m_k} \leqq n$, $k = 1, 2, \dots$.

For, suppose

$$y \in U_i \in \mathcal{U}_{m_k}\,, \quad i=1,\dots,n+1\ .$$

Then, since $\bigcap_{i=1}^{n+1} U_i$ is a neighbourhood of y, by (8)

$$\Big(\bigcap_{i=1}^{n+1} U_i\Big) \cap S(x, \mathcal{S}(m_1,m_2,\dots)) \neq \emptyset\ .$$

Hence there exists $U' \in \mathcal{U}_{m_1}$ satisfying

$$x \in S(U';m_1,m_2,\dots) \subset S(x,\mathcal{S}(m_1,m_2,\dots)) = S_\varepsilon(x) \quad \text{and}$$

$$S(U';m_1,m_2,\dots) \cap \Big(\bigcap_{i=1}^{n+1} U_i\Big) \neq \emptyset\ .$$

Since

$$S(U';m_1,m_2,\dots) = \bigcup\{\, S(U;m_k,m_{k+1},\dots) \mid U \in \mathcal{U}_{m_k}\,,\ U \subset S(U';m_1,\dots,m_k)\,\}\,,$$

there is $U \in \mathcal{U}_{m_k}$ such that

(12) $\quad U \subset S(U';m_1,m_2,\dots) \subset S_\varepsilon(x)\,,$

(13) $\quad S(U,m_k,m_{k+1},\dots) \cap \Big(\bigcap_{i=1}^{n+1} U_i\Big) \neq \emptyset\ .$

Recall that

$$y \in U_i \cap B(S_\varepsilon(x)) \neq \emptyset\,, \quad i=1,\dots,n+1\ .$$

Thus it follows from (12) that U is distinct from each of U_i, $i=1,\dots,n+1$. Hence (13) implies that

$$\operatorname{ord} \mathcal{S}(m_k,m_{k+1},\dots) \geqq n+2\,,$$

which contradicts (10). Therefore (11) is proved.

This implies that the restrictions $\mathcal{U}'_{m_k}$ of $\mathcal{U}_{m_k}$ $(k=1,2,\dots)$ to $B(S_\varepsilon(x))$ are open coverings of the boundary satisfying

$$\mathcal{U}'_{m_1} > \mathcal{U}'_{m_2} > \dots\,, \ \text{mesh}\ \mathcal{U}'_{m_k} \to 0 \ \text{as}\ k \to \infty\,, \ \text{and}\ \operatorname{ord} \mathcal{U}'_{m_k} \leqq n\ .$$

Hence by Theorem V.1 we conclude that $\dim B(S_\varepsilon(x)) \leqq n-1$.

Finally we shall prove that $\{ S_\varepsilon(x) \mid x \in R \}$ is closure-preserving for every $\varepsilon > 0$. (Note that we still assume $\varepsilon = 1/2^{m_1} + 1/2^{m_2} + \dots$.) From (8) and (10) it follows that each $S_\varepsilon(x)$ is a finite sum of members of $S(m_1, m_2, \dots)$. Hence it suffices to prove that $S(m_1, m_2, \dots)$ is closure-preserving. To do so we note that because of the condition c) of $\{ \mathcal{U}_m \}$ each element U_0 of $\mathcal{U}_{m_1+1}$ intersects $S^3(U, \mathcal{U}_{m_1+1})$ for at most $n+1$ distinct members U of $\mathcal{U}_{m_1}$. Hence by (9) U_0 intersects at most $n+1$ distinct elements of $S(m_1, m_2, \dots)$. The covering $S(m_1, m_2, \dots)$ must therefore be locally finite and accordingly closure-preserving. This proves $\{ S_\varepsilon(x) \mid x \in R \}$ to be closure-preserving and the proof of the theorem is complete.

The following corollary is a direct consequence of the theorem.

Corollary. *A space* R *has* $\dim \leq n$ *if and only if we can introduce on* R *a topology-preserving metric such that* $\dim B(S_\varepsilon(F)) \leq n-1$ *for every* $\varepsilon > 0$ *and for every closed set* F *of* R.

CHAPTER VI

INFINITE-DIMENSIONAL SPACES

We shall first introduce notions which distinguish some types of infinite-dimensional spaces from general infinite-dimensional spaces. It will be a natural idea to extend definitions of Ind and ind by transfinite induction.

Definition VI. 1. *i) A space R has strong (weak) inductive dimension -1,* Ind $R = -1$ (ind $R = -1$) *if* $R = \emptyset$.

ii) Let α be a transfinite ordinal number. If for every disjoint closed sets F and G (for any neighbourhood $U(p)$ of any point p) of R there exists an open set V such that

$$F \subset V \subset R - G \ (p \in V \subset U(p)) \text{ and } \operatorname{Ind} B(V) < \alpha \ (\operatorname{ind} B(V) < \alpha),$$

then R has strong (weak) transfinite inductive dimension $\leq \alpha$,

$$\operatorname{Ind} R \leq \alpha \ (\operatorname{ind} R \leq \alpha).$$

In the above definition the word 'large (small)' may be used in place of 'strong (weak)'.

Definition VI. 2. *If for every countable number of pairs $\{F_i, G_i\}$, $i = 1, 2, \ldots$ of disjoint closed sets F_i and G_i of a space R, there exist open sets U_i, $i = 1, 2, \ldots$ such that*

$$F_i \subset U_i \subset R - G_i \text{ and } \bigcap_{i=1}^{k} B(U_i) = \emptyset \text{ for some number } k,$$

then we call R *a weakly infinite-dimensional space. Otherwise* R *is called strongly infinite-dimensional.*

The decomposition theorem states that a space R has dimension $\leq n$ if and only if R can be decomposed into $n+1$ 0-dimensional spaces. Therefore it is natural to define countable-dimensional spaces as follows:

Definition VI. 3. *A space* R *is called countable-dimensional if* $R = \bigcup_{i=1}^{\infty} R_i$ *for some subspaces* R_i *of dimension* ≤ 0 [1]. *By the decomposition theorem this is equivalent to the statement that* R *is the countable sum of finite-dimensional spaces.*

Definition VI. 4. *If a space* R *is the countable sum of finite-dimensional closed sets, then we call* R *a countable-dimensional space in the strong sense or strongly countable-dimensional.*

A countable dimensional space in the strong sense is countable-dimensional, but the converse is not true as shown by the counter example R^{ω}, the set of points in I^{ω} at most finitely many of whose coordinates are rational [2].

We shall discuss some relation between weak infinite-dimensionality and countable-dimensionality later in VI.1. A famous problem of P. Alexandroff in this aspect has recently been answered by R. Pol [1] in the negative [3]. Namely he gave an example of a compact metric space which is weakly infinite-dimensional but not countable-dimensional. Thus for compact metric spaces the latter is a stronger condition than the former.

---

[1] L. Tumarkin [5] proved that if R is a countable-dimensional separable space, then R can be decomposed into countably many 0-dimensional subspaces R_i, $i = 1,2,\ldots$ so that for every n, $\bigcup_{i=1}^{n} R_i$ is 0-dimensional. K. Nagami [6] generalized this result to general metric spaces. (See also A. Arhangelskii [1]).

[2] See Yu. Smirnov [6] or J. Nagata [3].

[3] In the construction of R. Pol the following two propositions play important roles: A. Lelek's [1] proposition that for every separable completely metrizable space X there is a metric compactification $\tilde{X}$ such that $\tilde{X} - X$ is strongly countable-dimensional, and L. Rubin - R. Schori - J. Walsh's [1] proposition that there is a separable completely metrizable space which is totally disconnected but not countable-dimensional.

As for relationships among compactification, weak-dimensionality, and countable-dimensionality, see (besides the above-mentioned paper of Lelek) E. Sklyarenko [1], A. W. Schurle [1], Z. Shmuely [1], and K. Nagami - J. H. Robers [1] in which it is proved that the set K^{ω} of points in I^{ω}, whose coordinates are all zero except for at most finitely many, has no countable-dimensional (metric) completion.

It is also known that a normal space X is weakly infinite-dimensional if and only if βX is weakly infinite-dimensional and that $\operatorname{Ind} X \leq \alpha$ if and only if $\operatorname{Ind} \beta X \leq \alpha$, where βX denotes the Stone-Čech compactification of X.

Another famous problem was Tumarkin's: Does every infinite-dimensional compact metric space contain closed sets of arbitrary high finite dimension? (Note that every $n(<\infty)$-dimensional space has an m-dimensional closed set for every $m \leq n$ and also that every countable-dimensional compact metric space has an m-dimensional closed set for every natural number m because of 3 C).) L. Tumarkin [4] himself proved that every infinite-dimensional compact metric space contains either compact sets of arbitrary high finite dimension or an *infinite-dimensional Cantor manifold*, i.e. an infinite-dimensional compact set F such that for every finite dimensional subset A of F the complement $F-A$ is connected. This problem has remained unsolved for a long time until D. Henderson [1] constructed an infinite-dimensional compact metric space which has no n-dimensional closed subset for every n with $1 \leq n < \infty$ in 1965. His complicated original construction was improved meanwhile by several authors, e.g. R. H. Bing [2] and A. V. Zarelua [3]. Further J. J. Walsh [1] constructed an infinite-dimensional compact metric space containing no n-dimensional $(1 \leq n < \infty)$ subsets.

Up to the present we do not know as much about infinite-dimensional spaces as about finite-dimensional spaces because of some difficult circumstances which are peculiar to infinite-dimensional cases. In this chapter we shall give a brief account of infinite-dimensional spaces and especially of countable-dimensional spaces, and strong transfinite inductive dimension [4].

VI. 1. Countable-dimensional spaces

The purpose of this section is to extend some results of the theory of finite-dimensional spaces to countable-dimensional spaces. As a matter of fact, the foundation of the theory of countable-dimensional spaces is established by III.4 A), III.7 A) and III.7 B).

To begin with, let us show that not every space is countable-dimensional [5].

---

[4] P. Fletcher - R. McCoy - R. Solver [1] defined another kind of infinite-dimensionality as follows and studied its properties: a space X is called *boundedly paracompact* if for each open cover U of X there is a natural number n (dependent on U) such that U has a locally finite open refinement of order $\leq n$. Another class of infinite-dimensional spaces was considered by D. Addis - J. Gresham [1].

[5] In contrast to this fact every metric space is homeomorphic to a subset of the topological product of countably many one-dimensional metric spaces. H. J. Kowalsky [1] proved that every metric space is homeomorphic to a subset of the topological product of countably many star spaces. A star space is a 1-dimensional metric space defined by Definition VI.6. As for finite-dimensional spaces J. Nagata [2] proved that every n-dimensional metric space can be imbedded in the product of $n+1$ 1-dimensional metric spaces.

On the other hand K. Borsuk [4] proved that S^2 cannot be imbedded as a subset of the product of two 1-dimensional spaces. Thus it is generally impossible to represent an n-dimensional space as a subset of the product of n 1-dimensional spaces.

A) Consider the n-dimensional cube

$$I_1^n = \{ x \mid |x_i| \leq 1 ,\ i = 1,\dots,n \}$$

where we denote by x_i the i-th coordinate of the point $x \in E^n$. Put

$$F_i = \{ x \mid x \in I_1^n ,\ x_i = 1 \} ,\ F_i' = \{ x \mid x \in I_1^n ,\ x_i = -1 \} ,$$

for $i = 1,\dots,n$. If V_i , $i = 1,\dots,n$ are open sets of I_1^n satisfying $F_i \subset V_i \subset I_1^n - F_i'$, then $\bigcap_{i=1}^n B(V_i) \neq \emptyset$.

Proof. Assume the contrary, i.e. $\bigcap_{i=1}^n B(V_i) = \emptyset$.
Then by use of the corollary to Theorem I.6, we can construct real-valued continuous functions f_i , $i = 1,\dots,n$ such that

$$f_i(F_i) = 1 ,\ f_i(F_i') = -1 ,\quad B(V_i) = \{ x \mid f_i(x) = 0 \} ,\quad |f_i| \leq 1 .$$

We define by $f(x) = (f_1(x),\dots,f_n(x))$ a continuous mapping of I_1^n into itself. It is clear that

$$f(F_i) \subset F_i ,\ f(F_i') \subset F_i' ,\ p_0 \notin f(I_1^n) ,$$

where p_0 denotes the origin of E^n . Letting $I_2^n = \{ x \mid |x_i| \leq 2 ,\ i = 1,\dots,n \}$ we consider a given point x of $I_2^n - (I_1^n \cup B(I_2^n))$. By a,b we denote the projections of x from p_0 to $B(I_1^n)$ and $B(I_2^n)$ respectively. Then we denote by $g(x)$ the point which divides the segment joining $f(a)$ and b , in the ratio $\rho(a,x)/\rho(x,b)$. Finally, we define a continuous mapping h of I_2^n into itself by

$$h(x) = \begin{cases} f(x) & \text{for } x \in I_1^n , \\ g(x) & \text{for } x \in I_2^n - (I_1^n \cup B(I_2^n)) , \\ x & \text{for } x \in B(I_2^n) . \end{cases}$$

Then h leaves every point of $B(I_2^n)$ fixed while $p_0 \notin h(I_2^n)$. But this contradicts IV.3 A). Thus our assertion is proved.

B) The Hilbert cube I^ω is neither countable-dimensional nor weakly infinite-dimensional.

Proof. To proof the first assertion let us assume the contrary. Let

$$I^\omega = \prod_{i=1}^{\infty} [-\tfrac{1}{i} , \tfrac{1}{i}] ,$$

$$F_i = \{ x \mid x \in I^\omega ,\ x_i = 1/i \},\ F_i' = \{ x \mid x \in I^\omega ,\ x_i = -1/i \},$$

where we denote by x_i the i-th coordinate of x. Since by assumption I^ω can be decomposed into countably many 0-dimensional subsets, III.4 A) implies the existence of a sequence V_i, $i = 1,2,\dots$ of open sets such that

$$F_i \subset V_i \subset I^\omega - F_i', \text{ and}$$
$$\operatorname{ord}_p \{ B(V_i) \mid i = 1,2,\dots \} < +\infty \quad \text{at every} \quad p \in I^\omega .$$

Define

$$I^n = \{ x \in I^\omega \mid |x_i| \leq 1/i \quad \text{for} \quad i = 1,\dots,n \ ;\ x_j = 0 \quad \text{for} \quad j \geq n+1 \}.$$

Then I^n is topologically equivalent to the n-cube. It is clear that $V_i \cap I^n$ is an open set of I^n such that

$$F_i \cap I^n \subset V_i \cap I^n \subset I^n - F_i' \cap I^n .$$

Hence by A) we obtain

$$\bigcap_{i=1}^{n} B(V_i) \supset \bigcap_{i=1}^{n} B_{I^n}(V_i \cap I^n) \neq \emptyset .$$

Since I^ω is compact, this implies

$$\bigcap_{i=1}^{\infty} B(V_i) \neq \emptyset ,$$

which is a contradiction. Thus I^ω is not countable-dimensional. Note that the second assertion is also implicitely proved in the above.

Theorem VI. 1 [6]. *A space* R *is countable-dimensional if and only if there exists a* σ*-locally finite open basis* V *such that* $\operatorname{ord}_p B(V) < +\infty$ *at every point* p *of* R.

[6] Theorems in Sections 1 and 2 are mainly due to J. Nagata [3]. Characterizations of strongly countable-dimensional spaces are discussed in J. Nagata [2] and A. Arhangelskii [1]. The latter proved that a metric space X is strongly countable-dimensional if and only if it has a base U such that $\operatorname{rank}_x U < +\infty$ at every point x of X.

Proof. Let R be countable-dimensional; then R is decomposable into countably many 0-dimensional spaces A_i, $i = 1,2,\dots$. By the corollary to Theorem I.1 there exists a sequence of locally finite open coverings $\mathcal{U}_i$ of R such that $\{ S(p,\mathcal{U}_i) \mid i = 1,2,\dots \}$ is a neighbourhood basis of each point p. Assume

$$\mathcal{U}_i = \{ U_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \},\ \tau_0 = 0 .$$

Then by virtue of the local finiteness of $\mathcal{U}_i$ there is a closed covering $\{ F_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \}$ such that $F_\alpha \subset U_\alpha$. Defining

$$\tau = \sup \{ \tau_i \mid i = 1,2,\dots \}$$

we get an open collection $\{ U_\alpha \mid \alpha < \tau \}$ and a closed collection $\{ F_\alpha \mid \alpha < \tau \}$ which satisfy the condition of III.4 A). Hence we can construct an open collection $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$ according to that proposition. Now, letting

$$\mathcal{V}_i = \{ V_\alpha \mid \tau_{i-1} \leqq \alpha < \tau_i \} ,$$

we know that $\mathcal{V}$ is the desired σ-locally finite open basis of R.

Conversely, if there exists an open basis $\mathcal{V}$ which satisfies the condition in the statement, then we let

$$A_n = \{ p \mid \operatorname{ord}_p B(\mathcal{V}) = n-1 \},\ n = 1,2,\dots .$$

Since $\mathcal{U} = \{ V \cap A_n \mid V \in \mathcal{V} \}$ is a σ-locally finite open basis of A_n such that

$$\operatorname{ord} B_{A_n}(\mathcal{U}) \leqq n-1$$

from Theorem II.9 we can deduce that $\dim A_n \leqq n-1$. Thus from the decomposition theorem it follows that $R = \bigcup_{n=1}^{\infty} A_n$ is countable-dimensional.

C) A space R is countable-dimensional if and only if for every open collection $\{ U_\alpha \mid \alpha < \tau \}$ and closed collection $\{ F_\alpha \mid \alpha < \tau \}$ such that $F_\alpha \subset U_\alpha$ and such that $\{ U_\beta \mid \beta < \alpha \}$ is locally finite for every $\alpha < \tau$, there exists an open collection $\{ V_\alpha \mid \alpha < \tau \}$ such that

$$F_\alpha \subset V_\alpha \subset U_\alpha ,$$
$$\operatorname{ord}_p \{ B(V_\alpha) \mid \alpha < \tau \} < +\infty \text{ at every point } p \in R .$$

Proof. The "only if" part is a direct consequence of III.4 A).

To prove the "if" part we consider a σ-locally finite open basis $\mathcal{U}$ of R which is decomposed into locally finite open coverings

$$\mathcal{U}_i = \{ U_\alpha \mid \tau_{i-1} \leq \alpha < \tau_i \}, \quad i = 1, 2, \ldots .$$

Then there exists a closed covering $\{ F_\alpha \mid \tau_{i-1} \leq \alpha < \tau_i \}$ of R such that $F_\alpha \subset U_\alpha$. If we put

$$\tau = \sup \{ \tau_i \mid i = 1, 2, \ldots \} ,$$

then

$$\mathcal{U} = \{ U_\alpha \mid \alpha < \tau \} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$$

and $\{ F_\alpha \mid \alpha < \tau \}$ satisfy the condition of the assertion. Hence we can construct the open collection $\mathcal{V} = \{ V_\alpha \mid \alpha < \tau \}$ in the assertion. Now, it is easy to see that $\mathcal{V}$ is an open basis satisfying the condition of Theorem VI.1, and hence R is countable-dimensional.

Theorem VI. 2. *A space R is countable-dimensional if and only if for all sequences $\{ U_i \mid i = 1, 2, \ldots \}$ of open sets and $\{ F_i \mid i = 1, 2, \ldots \}$ of closed sets satisfying $F_i \subset U_i$, $i = 1, 2, \ldots$ there exists a sequence $\mathcal{V} = \{ V_i \mid i = 1, 2, \ldots \}$ of open sets such that*

$$F_i \subset V_i \subset U_i , \quad i = 1, 2, \ldots \quad \textit{and}$$
$$\operatorname{ord}_p B(\mathcal{V}) < +\infty \quad \textit{at every point } p \textit{ of } R .$$

Proof. Since the "only if" part is an immediate consequence of C), we show only the "if" part. By use of Theorem I.4 we can find a σ-discrete open basis

$$\mathcal{U}' = \bigcup_{i=1}^{\infty} \mathcal{U}'_i$$

of the space R. Suppose $\mathcal{U}'_i = \{ U'_\gamma \mid \gamma \in \Gamma_i \}$ is a discrete open collection; then we decompose each U'_γ as

$$U'_\gamma = \bigcup_{j=1}^{\infty} F'_{\gamma j}$$

for closed sets $F'_{\gamma j}$. Furthermore, we let

$$U_i = \cup \{ U'_\gamma \mid \gamma \in \Gamma_i \} \text{ , and } F^j_i = \cup \{ F'_{\gamma j} \mid \gamma \in \Gamma_i \} .$$

Note that F^j_i is closed by virtue of the discreteness of U'_i . Then, since $F^j_i \subset U_i$, $i, j = 1, 2, \ldots$, by use of the condition of the theorem we construct an open collection $V = \{ V^j_i \mid i, j = 1, 2, \ldots \}$ such that

$$F^j_i \subset V^j_i \subset U_i \text{ , and}$$
$$\operatorname{ord}_p B(V) < +\infty \text{ at every } p \in R .$$

Defining

$$W^j_\gamma = V^j_i \cap U'_\gamma \text{ for } \gamma \in \Gamma_i ,$$

we get a locally finite open collection

$$W^j_i = \{ W^j_\gamma \mid \gamma \in \Gamma_i \} .$$

Here we note that

$$B(W^j_\gamma) \cap B(W^j_{\gamma'}) = \emptyset$$

if $\gamma, \gamma' \in \Gamma_i$, $\gamma \neq \gamma'$, because in this case it follows from the discreteness of U'_i that

$$\bar{W}^j_\gamma \cap \bar{W}^j_{\gamma'} \subset \bar{U}'_\gamma \cap \bar{U}'_{\gamma'} = \emptyset .$$

Hence

$$W = \cup \{ W^j_i \mid i, j = 1, 2, \ldots \}$$

is a σ-locally finite open basis of R such that $\operatorname{ord}_p B(W) < +\infty$ at every $p \in R$. Hence by Theorem VI.1, we conclude that R is countable-dimensional.

Corollary. *Every countable-dimensional compact space* R *is weakly infinite-dimensional* [7].

Theorem VI. 3. *A space* R *is countable-dimensional if and only if there exists a sequence* $\{\mathsf{F}_i \mid i=1,2,\dots\}$ *of locally finite closed coverings of* R *satisfying*

i) *for every neighbourhood* $U(p)$ *of every point* p *of* R *there exists some* i *with* $S(p,\mathsf{F}_i)\subset U(p)$,

ii) $\mathsf{F}_i=\{F(\alpha_1,\dots,\alpha_i)\mid \alpha_k\in\Omega,\ k=1,\dots,i\}$, *where* $F(\alpha_1,\dots,\alpha_i)$ *may be empty,*

iii) $F(\alpha_1,\dots,\alpha_{i-1})=\cup\{F(\alpha_1,\dots,\alpha_{i-1},\beta)\mid \beta\in\Omega\}$,

iv) $\sup\{\operatorname{ord}_p \mathsf{F}_i\mid i=1,2,\dots\}<+\infty$ *at every point* p *of* R .

Proof. Let R be a countable-dimensional space such that $R=\cup_{n=1}^{\infty}A_n$ for 0-dimensional spaces A_n . Then III.7 B) assures us that there exists a required sequence $\{\mathsf{F}_i\mid i=1,2,\dots\}$ of closed coverings.

Conversely, if there exists a sequence $\{\mathsf{F}_i\mid i=1,2,\dots\}$ satisfying i)-iv), then we define

$$A_n=\{p\mid \sup\{\operatorname{ord}_p \mathsf{F}_i\mid i=1,2,\dots\}=n+1\}\ ,\ n=0,1,\dots\ .$$

Since $\operatorname{ord}_p \mathsf{F}_i\leqq n+1$ at every point $p\in A_n$, by Theorem III.9 we obtain that $\dim A_n\leqq n$. Since the condition iv) of F_i implies that $R=\cup_{n=0}^{\infty}A_n$, R is countable-dimensional.

Theorem VI. 4. *A space* R *is countable-dimensional if and only if there exists a subset* P *of a generalized Baire's 0-dimensional space* $N(\Omega)$ *for suitable* Ω *and a closed continuous mapping* f *of* P *onto* R *such that for each point* q *of* R *the inverse image* $f^{-1}(q)$ *consists of finitely many points* [8] .

---

[7] This assertion is not true unless R is compact, because we can easily see that K^{ω} , the set of points in I^{ω} at most finitely many of whose coordinates are different from zero, is not weakly infinite-dimensional though it is countable-dimensional in the strong sense.

There is another definition of *weak infinite-dimensionality* which is obtained by replacing "$\cap_{i=1}^{k} B(U_i)=\emptyset$ for some k" in Definition VI.2 with "$\cap_{i=1}^{\infty} B(U_i)=\emptyset$". This definition is due to P. S. Alexandroff while the previous one is due to Yu. Smirnov. They are equivalent in compact spaces. The discrete sum $\cup_{n=1}^{\infty} I^n$ of n-dimensional cubes is strongly countable-dimensional and weakly infinite-dimensional in the sense of Alexandroff, but not in the sense of Smirnov. Every countably-dimensional space is weakly infinite-dimensional in the sense of Alexandroff. (In fact the same is true for every hereditarily normal space.)

[8] E. Sklyarenko [2] proved that a compact space R is countable-dimensional if and only if there exists a continuous mapping f of the Cantor's perfect set D^{∞} on R such that for each point q of R , $f^{-1}(q)$ consists of countably many points. J. Walker - B. Wenner [1] proved a similar theorem for strongly countable-dimensional spaces.

Proof. The "only if" part is an immediate consequence of Theorem VI.3. In fact, let R be a countable-dimensional space and $\{F_i \mid i = 1,2,\dots\}$ a sequence of closed coverings satisfying i)-iv) of Theorem VI.3. Then we define a subset P of $N(\Omega)$ by

$$P = \{(\alpha_1,\alpha_2,\dots) \mid \bigcap_{i=1}^{\infty} F(\alpha_1,\dots,\alpha_i) \neq \emptyset\}$$

and a mapping f of P onto R by

$$f(\alpha) = \bigcap_{i=1}^{\infty} F(\alpha_1,\dots,\alpha_i) \quad \text{for} \quad \alpha = (\alpha_1,\alpha_2,\dots) .$$

We can show, quite analogously to the proof of Theorem III.8, that f is a closed continuous mapping such that for every $q \in R$, $f^{-1}(q)$ consists of finitely many points.

Conversely, if there exist such $N(\Omega)$, P and f, then we let

$$A_n = \{q \mid f^{-1}(q) \text{ consists of } n+1 \text{ points}\} .$$

Since f restricted to $f^{-1}(A_n)$ is obviously a closed continuous mapping of $f^{-1}(A_n)$ onto A_n, by Theorem III.8 A_n is an at most n-dimensional subset of R. Thus from the fact that $R = \bigcup_{n=0}^{\infty} A_n$ we deduce that R is countable-dimensional.

Theorems VI.1, VI.2, VI.3 and Theorem VI.4 are extensions of Theorems II.9, II.8, III.9 and Theorem III.8 respectively to the countable-dimensional case.

VI. 2. Imbedding of countable-dimensional spaces

In this section we shall give a universal space for countable-dimensional spaces.

A) Let R be a countable-dimensional space for which $R = \bigcup_{n=1}^{\infty} A_n$, $\dim A_n = 0$. Let $\{U_m \mid m = 1,2,\dots\}$ be an open collection and $\{F_m \mid m = 1,2,\dots\}$ a closed collection such that $F_m \subset U_m$. Then there exists an open collection $V = \{V_{mr} \mid m = 1,2,\dots, |r| < \sqrt{2}/2m,\ r \text{ rational}\}$ satisfying

i) $F_m \subset V_{mr} \subset \bar{V}_{mr} \subset V_{mr'} \subset \bar{V}_{mr'} \subset U_m$ if $r < r'$.

ii) $\bar{V}_{mr} = \bigcap \{V_{mr'} \mid r' > r\}$, $V_{mr} = \bigcup \{\bar{V}_{mr'} \mid r' < r\}$,

iii) $\operatorname{ord}_p B(V) \leqq n-1$ at each point p of A_n.

Proof. First we consider m fixed and index all the rational numbers satisfying $|r| < \sqrt{2}/2m$ so that

$$\begin{array}{c} r_{m1} , \\ r_{m2} < r_{m1} < r_{m3} , \\ r_{m4} < r_{m2} < r_{m5} < r_{m1} < r_{m6} < r_{m3} < r_{m7} , \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

Note that these rational numbers are naturally ordered as

$$\frac{0}{1} ,$$

$$\frac{-k_2}{2} , \frac{-k_2+1}{2} , \ldots , \frac{k_2}{2} ,$$

$$\frac{-k_3}{3} , \frac{-k_3+1}{3} , \ldots , \frac{k_3}{3} ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

for $k_i = [\sqrt{2}/2m]i$; then to suit our requirement this order must only be modified slightly.

Now, we put

$$\begin{aligned} N_{m1} &= \{ r_{m1} \} , \\ N_{m2} &= \{ r_{m2}, r_{m3} \} , \\ N_{m3} &= \{ r_{m4}, r_{m5}, r_{m6}, r_{m7} \} , \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

We shall define open sets V_{mr} satisfying i) iii) and

iv) if r_{mk} succeeds r_{mi} in $\cup_{h=1}^{s-1} N_{mh}$, i.e. $r_{mi} < r_{mk}$ and there exists no number r in $\cup_{h=1}^{s-1} N_{mh}$ satisfying $r_{mi} < r < r_{mk}$, and if $r_{mj} \in N_{ms}$, $r_{mi} < r_{mj} < r_{mk}$, then

$V_{r_{mj}} \subset S_{1/s}(\bar{V}_{r_{mi}})$ for odd s,

$R - \bar{V}_{r_{mj}} \subset S_{1/s}(R - V_{r_{mk}})$ for even s,

where we denote, for brevity, $V_{mr_{mi}}$ by $V_{r_{mi}}$.

Then it will be easily seen that

$$\{ V_{r_{mi}} \mid r_{mi} \in \bigcup_{h=1}^{\infty} N_{mh} \} = \{ V_{mr} \mid |r| < \sqrt{2}/2m \}$$

satisfies also ii). In fact, assume that $\{V_{r_{mi}} \mid r_{mi} \in \bigcup_{h=1}^{\infty} N_{mh}\}$ satisfies iv). We consider a number $r_{mi} \in N_{m\,s-1}$ and a point p for which $p \notin \bar{V}_{r_{mi}}$. Then we select an odd t satisfying

$$t \geqq \max\left[s, \frac{1}{\rho(p, \bar{V}_{r_{mi}})} \right]$$

and the number $r_{mj} \in N_{mt}$ which succeeds r_{mi} in $\bigcup_{k=1}^{t} N_{mk}$. It follows from iv) that

$$V_{r_{mj}} \subset S_{1/t}(\bar{V}_{r_{mi}}) \not\ni p .$$

Hence

$$\bar{V}_{mr} = \bigcap \{V_{mr'} \mid r' > r\} .$$

Subsequently we consider a number $r_{mk} \in N_{m\,s-1}$ and a point p for which $p \in V_{r_{mk}}$. Then we choose an even t satisfying

$$t \geqq \max\left[s, \frac{1}{\rho(p, R - V_{r_{mk}})} \right]$$

and the number $r_{mj} \in N_{mt}$ which r_{mk} succeeds in $\bigcup_{k=1}^{t} N_{mk}$. It follows from iv) that

$$p \in R - S_{1/t}(R - V_{r_{mk}}) \subset \bar{V}_{r_{mj}} .$$

Hence

$$V_{mr} = \bigcup \{\bar{V}_{mr'} \mid r' < r\} ,$$

proving ii).

Now, we define V_{mr} for all m and r by induction in the following order

$$V_{r_{11}}, V_{r_{12}}, V_{r_{21}}, V_{r_{13}}, V_{r_{22}}, V_{r_{31}}, V_{r_{14}}, \ldots .$$

(The point is that the order of $\{r_{m1}, r_{m2}, \ldots\}$ for a fixed m is kept in the sequence $\{r_{11}, r_{12}, r_{21}, \ldots\}$. As long as this condition is satisfied we may choose any other ordering.)

First, by use of $\dim A_1 = 0$ we construct an open set $V_{r_{11}}$ such that

$$F_1 \subset V_{r_{11}} \subset \bar{V}_{r_{11}} \subset U_1 \quad \text{and} \quad B(V_{r_{11}}) \cap A_1 = \emptyset .$$

Since the construction procedure is quite analogous to that for V_0 in III.4 A) it will be omitted here.

Suppose that we have defined all $V_{r_{nh}}$ before $V_{r_{mj}}$ and that $r_{mj} \in N_{ms}$. Then to define $V_{r_{mj}}$ we choose numbers

$$r_{mi}, r_{mk} \quad \bigcup_{h=1}^{s-1} N_{mh}$$

such that $r_{mi} < r_{mj} < r_{mk}$ and such that r_{mk} succeeds r_{mi} in $\bigcup_{k=1}^{s-1} N_{mk}$. By use of $\dim A_n = 0$, we can construct $V_{r_{mj}}$ such that

$$\bar{V}_{r_{mi}} \subset V_{r_{mj}} \subset \bar{V}_{r_{mj}} \subset V_{r_{mk}} \cap S_{1/s}(\bar{V}_{r_{mi}}) \quad \text{if } s \text{ is odd,}$$
$$[R - S_{1/s}(R - V_{r_{mk}})] \cup \bar{V}_{r_{mi}} \subset V_{r_{mj}} \subset \bar{V}_{r_{mj}} \subset V_{r_{mk}} \quad \text{if } s \text{ is even}$$

and such that

$$\operatorname{ord}_p B(\mathcal{V}_{mj}) \leqq n - 1 , \quad \text{at each point } p \in A_n ,$$

where

$$\mathcal{V}_{mj} = \{ V_{r_{11}}, V_{r_{12}}, V_{r_{21}}, \ldots, V_{r_{mj}} \} .$$

The construction procedure for $V_{r_{mj}}$ is quite parallel to that for V_α in III.4 A) and hence its details are left to the reader. Thus the proof of the assertion is complete.

Recall that we defined $I^\omega = \prod_{i=1}^{\infty} [-\frac{1}{i}, \frac{1}{i}]$.

Theorem VI. 5. *Let R be a space with a σ-star-finite open base. Denote by R^ω the set of points in I^ω at most finitely many of whose coordinates are rational.*

Then R is countable-dimensional if and only if R is homeomorphic to a subset of $N(\Omega)\times R^{\omega}$ for suitable Ω [9] .

Proof. Let R be a space with a σ-star-finite open basis. Then, since by I.2 F) R is homeomorphic to a subset of $N(\Omega)\times I^{\omega}$, we obtain a sequence

$$N_1 > N_2 > \dots$$

of star-finite open coverings such that $\{ S(p,N_i) \mid i = 1,2,\dots \}$ is a neighbourhood basis of each point p of R . We put

$$S_i = \{ S^{\infty}(N,N_i) \mid N \in N_i \} ,$$

recalling that

$$S^{\infty}(N,N_i) = \bigcup_{n=1}^{\infty} S^{n}(N,N_i) .$$

Then, for brevity, we define

$$S_i = \{ S_{\alpha} \mid \alpha\in\Omega_i \} \text{ , where } S_{\alpha}\cap S_{\beta} = \emptyset \text{ for } \alpha\neq\beta .$$

Since N_i is star-finite, each set S_{α} is a sum of countably many elements of N_i . Hence we can decompose S_{α} as

$$S_{\alpha} = \bigcup \{ N_{\alpha j} \mid j = 1,2,\dots \} \text{ for } N_{\alpha j}\in N_i .$$

For each i we construct an open covering P_i of R such that

$$P_i = \{ P_{\alpha j} \mid \alpha\in\Omega_j , j = 1,2,\dots \} \text{ with } \bar{P}_{\alpha j}\subset N_{\alpha j} .$$

Defining

$$U_{ij} = \bigcup \{ N_{\alpha j} \mid \alpha\in\Omega_i \} \text{ and } F_{ij} = \bigcup \{ \bar{P}_{\alpha j} \mid \alpha\in\Omega_i \}$$

---

[9] Concerning the imbedding of countable-dimensional spaces in the strong sense Yu. Smirnov [6] and J. Nagata [3] proved that a space R with a σ-star-finite open basis is countable-dimensional in the strong sense if and only if it is homeomorphic to a subset of $N(\Omega)\times K^{\omega}$ for suitable Ω , where K^{ω} denotes the set of points in I^{ω} at most finitely many of whose coordinates are different from zero.
It is easy to see that in Theorem VI.5 Ω may be taken as the set whose cardinality is equal to the weight of R (namely the minimal cardinality of its base). The same applies in the strongly countable-dimensional case as well.

we obtain an open set U_{ij} and a closed set F_{ij} satisfying $F_{ij} \subset U_{ij}$. Then we put, for brevity,

$$\{U_{ij} \mid i,j = 1,2,\dots\} = \{U_m \mid m = 1,2,\dots\}$$
$$\{F_{ij} \mid i,j = 1,2,\dots\} = \{F_m \mid m = 1,2,\dots\} .$$

Now, suppose R is countable-dimensional. Then for these U_m and F_m we define V_{mr} by use of A) and construct a real-valued continuous function f_m over R by

$$f_m(p) = \begin{cases} \inf\{r \mid p \in V_{mr}\} & \text{if } p \in V_{mr} \text{ for some } r , \\ \sqrt{2}/2m & \text{if } p \notin V_{mr} \text{ for all } r . \end{cases}$$

Then by i) of A) it is clear that f_m is a continuous function such that

$$f_m(F_m) = -\sqrt{2}/2m , \; f_m(R - U_m) = \sqrt{2}/2m , \; |f_m| \leqq \sqrt{2}/2m .$$

Suppose

$$p \notin \cup \{B(V_{mr}) \mid |r| < \sqrt{2}/2m , r \text{ rational}\} ;$$

then we shall show that $f_m(p)$ is irrational. To this end we consider a given rational number r satisfying $|r| < \sqrt{2}/2m$. If $p \in V_{mr}$, then by ii) and i) of A) there exists r' for which $r' < r$, $p \in V_{mr'}$. Hence $f_m(p) \leqq r' < r$. If $p \notin \bar{V}_{mr}$, then by ii) of A) there exists r' for which $r' > r$, $p \notin V_{mr'}$. Hence $f_m(p) \geqq r' > r$. Thus $f_m(p) \neq r$ in either case. So from iii) of A) it follows that at most finitely many of $\{f_1(p), f_2(p), \dots\}$ are rational, and hence

$$f(p) = (f_1(p), f_2(p), \dots)$$

is a continuous mapping of R into R^ω .

Now, we put $\Omega = \cup_{i=1}^{\infty} \Omega_i$ and consider the generalized Baire's 0-dimensional space $N(\Omega)$. Then we define a continuous mapping c of R into $N(\Omega)$ by

$$c(p) = \alpha = (\alpha_1, \alpha_2, \dots) \text{ if } p \in S_{\alpha_i} , \; \alpha_i \in \Omega_i , \; i = 1,2,\dots .$$

Finally, we define a continuous mapping φ of R into $N(\Omega) \times R^\omega$ by

$$\varphi(p) = (c(p), f(p)), \; p \in R .$$

Note that φ is a one-to-one continuous mapping. To see that φ is homeomorphic, we suppose $U(p)$ is a given neighbourhood of a point p of R. Let us take i for which $S(p,N_i)\subset U(p)$. Since $\{F_{ij}\mid j=1,2,\dots\}$ is a covering of R, there exists F_{ij} satisfying $p\in F_{ij}$. We suppose $F_{ij}=F_m$.

Since $S_k=\{S_\alpha\mid \alpha\in\Omega_k\}$ is a covering of R, for each k, $1\leq k\leq i$, there exists $\alpha_k\in\Omega_k$ satisfying $p\in\bigcap_{k=1}^{i}S_{\alpha k}$. Then we define a neighbourhood $V(\varphi(p))$ of $\varphi(p)$ by

$$V(\varphi(p)) = N(\alpha_1,\dots,\alpha_i)\times N(f(p)),$$

where

$$N(\alpha_1,\dots,\alpha_i) = \{(\alpha_1',\alpha_2',\dots)\mid \alpha_k'=\alpha_k,\ k=1,\dots,i\}\subset N(\Omega),$$
$$N(f(p)) = \{(a_1,a_2,\dots)\mid a_m<0\}\subset R^\omega.$$

One can easily see that

$$\varphi^{-1}(V(\varphi(p)))\subset U(p),$$

which proves that φ is a homeomorphism. Thus the "only if" part of this theorem is established.

Conversely, it is clear that

$$N(\Omega)\times R^\omega = \bigcup_{n=1}^{\infty}[N(\Omega)\times R^n]$$

for

$$R^n = \{(a_1,a_2,\dots)\mid a_j,\ j>n\ \text{, are irrational; } |a_i|\leq 1/i \text{ for } i=1,2,\dots\}.$$

Since $\dim R^n=n$, the product theorem implies

$$\dim[N(\Omega)\times R^n]=n.$$

Hence $N(\Omega)\times R^\omega$ is countable-dimensional. This proves the "if" part of the theorem.

The following is an immediate consequence of this theorem.

Corollary. *Let R be a separable metric space. Then R is countable-dimensional if and only if R is homeomorphic to a subset of R^ω.*

VI. 3. Mappings and countable-dimensional spaces

Various good results were obtained on mappings and infinite-dimensional spaces. We have already seen such an example in Theorem VI.4 and shall show another example in the following. In particular we will discuss mappings which preserve countable-dimensionality.

A) Let A be a non-empty countable compact set in a space R. Then A contains an isolated point.

Proof. The easy proof is left to the reader.

B) Let $\mathcal{U}$ be an open covering of R. Then there is a σ-discrete open covering $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$.

Proof. Let $\mathcal{F}$ be a locally finite closed covering such that $\mathcal{F} < \mathcal{U}$. Apply the method used in the proof of the corollary of Theorem II.1 to get a σ-discrete closed covering $\mathcal{G}$ such that $\mathcal{G} < \mathcal{F}$. By swelling the members of $\mathcal{G}$ slightly we obtain a desired open covering $\mathcal{V}$.

C) Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a σ-locally finite covering of R such that each member of $\mathcal{U}$ is countable-dimensional. Then R is countable-dimensional.

Proof. First observe that the proposition is true if $\mathcal{U}$ is σ-discrete. Because in that case $\mathcal{U}_n$ is discrete, and thus each member of $\mathcal{U}_n$ is closed in the subspace $U_n = \cup\{U \mid U \in \mathcal{U}_n\}$, and hence U_n is countable-dimensional by the sum theorem.

Now, by use of B), we can find a σ-discrete open covering $\mathcal{V}_n = \bigcup_{m=1}^{\infty} \mathcal{V}_{nm}$ such that each member of $\mathcal{V}_n$ intersects at most finitely many members of $\mathcal{U}_n$. Then $\mathcal{V}_{nm} \cap \mathcal{U}_n$ is obviously σ-discrete. Hence $\bigcup_{n,m=1}^{\infty} \mathcal{V}_{nm} \wedge \mathcal{U}_n$ is a σ-discrete covering which refines $\mathcal{U}$. Thus by the previous observation we conclude that R is countable-dimensional.

D) Let f be a perfect mapping from R onto S. (Namely f is closed continuous, and $f^{-1}(y)$ is compact for each $y \in S$.) If $\mathcal{U}$ is a locally finite collection in R, then $f(\mathcal{U}) = \{f(U) \mid U \in \mathcal{U}\}$ is locally finite in S.

Proof. Let y be a given point of S. Then each $x \in f^{-1}(y)$ has an open neighbourhood $W(x)$ which intersects at most finitely many members of $\mathcal{U}$. Since $f^{-1}(y)$ is compact, it is covered by finitely many of $W(x)$'s, say $W(x_i)$, $i = 1, \ldots, k$. Then $V = S - f(R - \bigcup_{i=1}^{k} W(x_i))$ is an open neighbourhood of y because f is closed. It is obvious that V intersects at most finitely many members of $f(\mathcal{U})$. Thus the proposition is proved.

E) Let f be a perfect mapping from R onto S such that $f^{-1}(y)$ has an isolated point for each $y \in S$. If R is countable-dimensional, then so is S.

Proof. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a σ-discrete base, where each $\mathcal{U}_n = \{ U_\alpha \mid \alpha \in A_n \}$ is a discrete open collection. Pick an isolated point $x(y)$ from each $f^{-1}(y)$. For each $\alpha \in A_n$ we put

$$F_{n\alpha} = \{ x(y) \mid U_\alpha \cap f^{-1}(y) = \{ x(y) \} \}, \text{ and } G_{n\alpha} = f(F_{n\alpha}).$$

It is almost obvious that f maps $F_{n\alpha}$ topologically onto $G_{n\alpha}$. Since R is countable-dimensional, so is $F_{n\alpha}$ and $G_{n\alpha}$. Note that $F_{n\alpha} \subset \bar{U}_\alpha$, and $\{ \bar{U}_\alpha | \alpha \in A_n \}$ is discrete. Therefore $\{ F_{n\alpha} \mid \alpha \in A_n \}$ is discrete. Hence by D) $\{ G_{n\alpha} \mid \alpha \in A_n, n = 1,2,\dots \}$ is σ-locally finite. Hence S is countable-dimensional by C).

The following theorem is an infinite-dimensional counter-part of Theorem III.7.

Theorem VI.6. *Let f be a closed continuous mapping from a countable-dimensional space R onto S such that $B(f^{-1}(y))$ is countable for each $y \in S$. Then S is countable-dimensional* [10].

Proof. Put

$$S_1 = \{ y \in S \mid B(f^{-1}(y)) = \emptyset \} \text{ and } S_2 = \{ y \in S \mid B(f^{-1}(y)) \neq \emptyset \}.$$

To each $y \in S_1$ we assign a point $x(y)$ of $f^{-1}(y)$. Put $R_1 = \{ x(y) \mid y \in S_1 \}$; then it is obvious that f maps R_1 topologically onto S_1 because $f^{-1}(y)$ is open for each $y \in S_1$. Thus S_1 is countable-dimensional.

Theorem I.9 implies that $B(f^{-1}(y))$ is compact for each $y \in S$. Thus it follows from A) that $B(f^{-1}(y))$ has an isolated point for each $y \in S_2$. Put

$$R_2 = \bigcup \{ B(f^{-1}(y)) \mid y \in S_2 \}.$$

---

[10] Several people contributed to this theorem, especially E. G. Sklyarenko [2], A. V. Arhangelskii [2], K. Nagami [7] and A. Okuyama [2]; the last proved this theorem for spaces more general than metric spaces. E. G. Sklyarenko [3] and I. M. Leĭbo [1] proved similar types of theorems on weakly infinite-dimensional spaces.

The following theorem (a counter-part of Theorem III.6) is also due to Arhangelskii: Let f be a closed continuous mapping from R onto a countable-dimensional space S. If $\dim f^{-1}(q) < \infty$ for each $q \in S$, then R is countable-dimensional.

Then R_2 is countable-dimensional, and $f|R_2$ satisfies the condition of E). Thus $S_2 = f(R_2)$ is countable-dimensional. Since $S = S_1 \cup S_2$, S is also countable-dimensional.

Corollary. *Let f be a closed continuous mapping from a countable-dimensional space R onto S such that*

$$S' = \{ y \in S \mid |B(f^{-1}(y))| > \aleph_0 \}$$

is countable-dimensional. Then S is also countable-dimensional.

Proof. Let $R_0 = f^{-1}(S - S')$. Then $f | R_0$ satisfies the condition of the theorem. Hence $S - S'$ is countable-dimensional. and so is S .

Another aspect to the study of mappings (in relation to dimension) is the characterization of dimension in terms of mappings as shown in III.3. E. G. Sklyarenko characterized in [3] strongly infinite-dimensional spaces as follows:

A compact (T_2-) space X is strongly infinite-dimensional if and only if there is a continuous mapping f from X into the Hilbert cube I^{ω} satisfying the condition that for every finite dimensional face F of I^{ω} the restriction of f to $f^{-1}(F)$ is essential [11].

D. Henderson [3] defined a compact metric space J^{α} for every countable ordinal number α such that $J^{\alpha} = I^{\alpha}$ if $0 < \alpha < \infty$, and $\operatorname{Ind} J^{\alpha} = \alpha$, and also essential mappings onto J^{α}. Thus he proved that if there is an essential mapping from a normal space X onto J^{α}, then $\operatorname{Ind} X \geqq \alpha$ or $\operatorname{Ind} X$ does not exist.

VI. 4. Transfinite inductive dimension

In this section we shall study results obtained by Yu. Smirnov [7] and E. Sklyarenko [1] on the transfinite inductive dimension and related concepts [12].

A) If a space R has a strong transfinite inductive dimension, then it is weakly infinite-dimensional.

Proof. If $\operatorname{Ind} R = -1$, then this assertion is obviously true.

---

[11] See also B. Levšenko [1].

[12] "Weakly infinite-dimensional" is understood in the sense of Smirnov unless otherwise stated.

Given a transfinite ordinal number α, we assume the assertion for every R with $\operatorname{Ind} R<\alpha$. Then we suppose $\{F_i, G_i\}$, $i=1,2,\dots$ are pairs of disjoint closed sets of a space R with $\operatorname{Ind} R=\alpha$. There exists an open set U_1 such that

$$F_1\subset U_1\subset R-G_1 \quad\text{and}\quad \operatorname{Ind} B(U_1)<\alpha .$$

Thus by the induction hypothesis $B(U_1)$ is weakly infinite-dimensional, and hence there exist open sets V_2, V_3, ... of $B(U_1)$ which satisfy

$$F_i\cap B(U_1)\subset V_i\subset B(U_1)-G_i\cap B(U_1)\ ,\ i=2,3,\dots,$$

$$\bigcap_{i=2}^{k} B_{B(U_1)}(V_i)=\emptyset \quad\text{for some } k .$$

We can easily extend V_i to an open set U_i of R such that

$$F_i\subset U_i\subset R-G_i \quad\text{and}\quad B(U_i)\cap B(U_1)\subset B_{B(U_1)}(V_i) .$$

Hence $\bigcap_{i=1}^{k} B(U_i)=\emptyset$, which proves that R is weakly infinite-dimensional.

B) If a space R has a strong transfinite inductive dimension, then it is countable-dimensional.

Proof. If $\operatorname{Ind} R=-1$, then the assertion is clearly true. Assume B) for every space R of $\operatorname{Ind}<\alpha$. Given a space R of $\operatorname{Ind}=\alpha$, we can construct a σ-locally finite open basis $\mathcal{V}$ such that $\operatorname{Ind} B(V)<\alpha$ for every $V\in\mathcal{V}$. By the induction hypothesis each $B(V)$ is countable-dimensional. Therefore by use of the sum theorem we can easily verify that $\bigcup\{B(V)\mid V\in\mathcal{V}\}$ is countable-dimensional. Since it is obvious that

$$\dim(R-\bigcup\{B(V)\mid V\in\mathcal{V}\})\leq 0 ,$$

R itself is countable-dimensional.

The converse of B) is generally not true, but in the compact case equivalence holds between the two conditions as follows.

C) If R is compact, then it has a strong transfinite inductive dimension if and only if it is countable-dimensional.

Proof. Suppose that R is countable-dimensional. Then $R=\bigcup_{i=1}^{\infty}A_i$ for 0-dimensional subsets A_i. Assume that R has no $\operatorname{Ind} R$. Then there are disjoint closed sets F_1 and G_1 such that every open set U_1 satisfying $F_1 \subset U_1 \subset R - G_1$ has a boundary with no Ind. We can select an open set U_1 such that $B(U_1) \cap A_1 = \emptyset$. Put $B_1 = B(U_1)$ and note that $B_1 \neq \emptyset$. Since B_1 has no Ind, we can repeat the above argument, this time for the subspace B_1 to obtain an open set U_2 of B_1 such that

$$B_{B_1}(U_2) \cap A_2 = \emptyset ,$$

and $B_{B_1}(U_2)$ has no Ind. Put $B_2 = B_{B_1}(U_2)$ and note that $B_2 \neq \emptyset$. Repeating this process we get a decreasing sequence $B_1 \supset B_2 \supset \ldots$ of non-empty closed sets such that $B_i \cap A_i = \emptyset$. Since R is compact, $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$ in contradiction to $\bigcup_{i=1}^{\infty} A_i = R$. Thus R has Ind R.[13]

Theorem VI. 7. *A space R is weakly infinite-dimensional if and only if it can be decomposed as*

$$R = K \cup \left(\bigcup_{n=1}^{\infty} P_n \right)$$

for a weakly infinite-dimensional compact set K and finite-dimensional open sets P_n such that if $\{x_i \mid i = 1, 2, \ldots\}$, $x_i \in \bigcup_{n=1}^{\infty} P_n$ is a sequence of points having no accumulation point, then it is residual in some P_n, i.e. for some k and for all $i \geqq k$ $x_i \in P_n$.

Proof. First we suppose R is weakly infinite-dimensional. Define

$$P_n = \cup \{ U \mid U \text{ is an open set of } R \text{ with } \dim U \leqq n \}, \text{ and}$$

$$K = R - \bigcup_{n=1}^{\infty} P_n .$$

Then by the sum theorem P_n is an open set of dimension $\leq n$. (Note that every subset of R is paracompact.)

To prove that K is compact we assume the contrary. Then there exists a sequence $\{y_i \mid i = 1, 2, \ldots\}$ of points of K which has no accumulation point. Note that every open neighbourhood of each y_i has no finite dimension.

[13] As easily seen from this argument, a complete metric space R has a weak transfinite inductive dimension if and only if it is countable-dimensional. (Apply Baire's theorem.)

To prove simultaneously the property of P_n, $n = 1,2,\dots$ we assume that there exists a sequence $\{x_i \mid i = 1,2,\dots\}$ of points of $\bigcup_{n=1}^{\infty} P_n$ which has no accumulation point and is residual in no P_n. Then we can easily select a subsequence $\{x'_i\}$ of $\{x_i\}$ such that

$$x'_n \notin \bigcup_{i=1}^{n} P_i , \quad n = 1,2,\dots .$$

Note that by the definition of P_n every open neighbourhood of x'_n has dimension $> n$. Thus it follows from either of the preceding assumptions that there exists a sequence $\{z_i \mid i = 1,2,\dots\}$ of points, each z_i having an open neighbourhood $V(z_i)$ such that

(1) $\qquad \dim \overline{V(z_i)} > i$,

(2) $\qquad \{V(z_i) \mid i = 1,2,\dots\}$ is discrete.

Combining (1) with the corollary to Theorem II.8 for each $\overline{V(z_i)}$ we get closed sets F_i^j and G_i^j, $j = 1,\dots,i+1$ such that

(3) $$\begin{cases} F_i^j \cup G_i^j \subset \overline{V(z_i)} , \\ F_i^j \cap G_i^j = \emptyset , \end{cases}$$

and such that if W^j, $j = 1,\dots,i+1$ are open sets of R satisfying $F_i^j \subset W^j \subset R - G_i^j$, then

(4) $$\bigcap_{j=1}^{i+1} B(W^j) \neq \emptyset .$$

Further we define that

$$F_i^j = G_i^j = \emptyset \quad \text{for} \quad j > i+1 .$$

Letting

$$F^j = \bigcup_{i=1}^{\infty} F_i^j \quad \text{and} \quad G^j = \bigcup_{i=1}^{\infty} G_i^j ,$$

we get, by virtue of (2) and (3), disjoint closed sets F^j and G^j. Suppose W^j, $j=1,2,\ldots$ are given open sets such that $F^j \subset W^j \subset R - G^j$; then for any fixed number i

$$F_i^j \subset W^j \subset R - G_i^j, \quad j=1,\ldots,i+1 .$$

Therefore from (4) it follows that $\bigcap_{j=1}^{i+1} B(W^j) \neq \emptyset$ which contradicts the fact that R is weakly infinite-dimensional. Thus we have proved the compactness of K as well as the desired property of $\{P_n\}$. Since K is a closed subset of the weakly infinite-dimensional space R, it is easily seen to be also weakly infinite-dimensional.

Conversely, suppose R has a decomposition

$$R = K \cup \left(\bigcup_{n=1}^{\infty} P_n \right),$$

as described. Let $\{F_i, G_i\}$, $i=1,2,\ldots$ be given pairs of disjoint closed sets of R. Since K is weakly infinite-dimensional, there exist open sets U_i, $i=1,2,\ldots$ of K such that

$$F_i \subset U_i \subset \bar{U}_i \subset K - G_i, \quad \text{and} \quad \bigcap_{i=1}^{k} B_K(U_i) = \emptyset \quad \text{for some } k .$$

To each point p of $B_K(U_i)$ we assign $\varepsilon(p) > 0$ such that

$$S_{\varepsilon(p)}(p) \cap G_i = \emptyset \quad \text{and} \quad S_{\varepsilon(p)}(p) \cap B_K(U_j) = \emptyset$$

for every j with $1 \leqq j \leqq k$, $p \notin B_K(U_j)$. Then we put

$$V_i = K \cap (\cup \{ S_{\frac{1}{2}\varepsilon(p)}(p) \mid p \in B_K(U_i)\}) \quad \text{and} \quad W_i = V_i \cup U_i .$$

We can easily see that W_i is an open set of K which satisfies $\bar{U}_i \subset W_i \subset K - G_i$ and

$$(5) \qquad \bigcap_{i=1}^{k} (W_i - \bar{U}_i) = \emptyset .$$

Now, since $\dim P_n < +\infty$, using III.4 A) and the decomposition theorem we can construct open sets N_i, $i=1,2,\ldots$ such that

$$\bar{U}_i \subset N_i \subset \bar{N}_i \subset R - (K - W_i)\ ,\quad i = 1,\dots,k\ , \tag{6}$$
$$F_i \subset N_i \subset R - G_i\ ,\quad i = k+1,\ k+2,\dots,$$

$$\operatorname{ord}_p \{B(N_i) \mid i = 1,2,\dots\} \leq m_n < +\infty \tag{7}$$

at each point p of P_n. Assume that $\bigcap_{i=1}^{\ell} B(N_i) \neq \emptyset$ for every integer ℓ. Then we take $x_\ell \in \bigcap_{i=1}^{\ell} B(N_i)$. From (7) it follows that $x_\ell \notin P_n$ for every ℓ with $\ell > m_n$, and hence $\{x_\ell\}$ is residual in no P_n and has no accumulation point in $\bigcup_{n=1}^{\infty} P_n$. (Because every accumulation point of $\{x_\ell\}$ belongs to $\bigcap_{i=1}^{\infty} B(N_i)$, and hence by (7) it cannot be in any P_n.) Therefore $\{x_\ell\}$ must have an accumulation point x in K. Since $x_k, x_{k+1},\dots$ are contained in the closed set $\bigcap_{i=1}^{k} B(N_i)$, we obtain

$$x \in K \cap \left(\bigcap_{i=1}^{k} B(N_i)\right). \tag{8}$$

On the other hand, (6) implies

$$K \cap B(N_i) \subset W_i - \bar{U}_i\ ,$$

and hence (8) implies

$$\bigcap_{i=1}^{k} (W_i - \bar{U}_i) \neq \emptyset$$

which contradicts (5). Thus we can conclude that $\bigcap_{i=1}^{\ell} B(N_i) = \emptyset$ for some ℓ. This proves that R is weakly infinite-dimensional.

D) We define a mapping $\beta(\alpha)$ of the ordinal numbers ≥ -1 in itself as follows:

$\beta(-1) = \omega_0$, where ω_0 denotes the first countable ordinal number,

$\beta(\alpha) = \sup\{\beta(\alpha') \mid \alpha' < \alpha\} + 1$.

Then $\beta(\alpha)$ satisfies

i) if $\alpha' < \alpha$, then $\beta(\alpha') < \beta(\alpha)$,

ii) if $\alpha < \omega_1$, then $\beta(\alpha) < \omega_1$, where ω_1 denotes the first uncountable ordinal number

iii) $\alpha \leq \beta(\alpha)$ for every α.

E) Let a space R have a decomposition as in Theorem VI.6. If K has a strong transfinite inductive dimension $\operatorname{Ind} K$, then $\operatorname{Ind} R \leq \beta(\operatorname{Ind} K)$, where $\beta(\alpha)$ is the mapping defined in D).

Proof. We shall prove this assertion by induction on the dimension number of K. If $\operatorname{Ind} K = -1$, then $R = \bigcup_{n=1}^{\infty} P_n$. From the property of $\{P_n\}$ it easily follows that we can select a finite subcovering of $\{P_n \mid n = 1, 2, \ldots\}$. Since $\operatorname{Ind} P_n < +\infty$, by the sum theorem we conclude that $\operatorname{Ind} R < +\infty$, i.e. $\operatorname{Ind} R < \omega_0 = \beta(-1)$.

Assume that the assertion is true if $\operatorname{Ind} K < \alpha$. Let F and G be disjoint closed sets of a space R for which $\operatorname{Ind} K = \alpha$. Then there exists an open set U of R such that

$$F \subset U \subset R - G \quad \text{and} \quad \operatorname{Ind} (K \cap B(U)) = \alpha' < \alpha .$$

Then

$$B(U) = (K \cap B(U)) \cup \left(\bigcup_{n=1}^{\infty} (P_n \cap B(U)) \right)$$

is a decomposition of $B(U)$ satisfying the condition in Theorem VI.7. Hence by use of the induction hypothesis we obtain $\operatorname{Ind} B(U) \leq \beta(\alpha')$. Thus from i) in D) it follows that $\operatorname{Ind} B(U) < \beta(\alpha)$, which proves that $\operatorname{Ind} R \leq \beta(\alpha) = \beta(\operatorname{Ind} K)$.

F) If a compact space K has a strong transfinite inductive dimension, then $\operatorname{Ind} K < \omega_1$.

Proof. Assume the contrary and let α be the smallest ordinal, $\alpha \geq \omega_1$, for which there exists a compact space K of dimension α. Let $\{U_n \mid n = 1, 2, \ldots\}$ be an open basis of K. Suppose both V and W are finite sums of sets of $\{U_n\}$ such that $\bar{V} \subset W$. Then to each of such pair $\{V, W\}$ we assign an open set P which satisfies

$$\bar{V} \subset P \subset W \quad \text{and} \quad \operatorname{Ind} B(P) < \alpha .$$

Note that this implies

$$\operatorname{Ind} B(P) < \omega_1 . \tag{1}$$

We denote by $\mathcal{P}$ the countable collection of these open sets P. Since K is compact, we can easily verify that for any disjoint closed sets F and G of R there exists $P \in \mathcal{P}$ satisfying

$$F \subset P \subset R - G . \tag{2}$$

On the other hand, (1) implies that

$$\sup\{\operatorname{Ind} B(P) \mid P \in \mathcal{P}\} + 1 = \beta < \omega_1 \leqq \alpha .$$

Therefore from (2) we conclude that $\operatorname{Ind} K \leqq \beta$. Thus we achieve a contradiction, proving $\operatorname{Ind} K < \omega_1$.

Theorem VI. 8. *If a space R has a strong transfinite inductive dimension, then* $\operatorname{Ind} R < \omega_1$.

Proof. By A), R is weakly infinite-dimensional, and hence by Theorem VI.7 there exists a decomposition

$$R = K \cup \left(\bigcup_{n=1}^{\infty} P_n \right) .$$

Since R has $\operatorname{Ind} R$, K also has $\operatorname{Ind} K$. Since K is compact, by F) $\operatorname{Ind} K < \omega_1$. Therefore from E) and ii) in D) we obtain $\operatorname{Ind} R \leqq \beta(\operatorname{Ind} K) < \omega_1$.

Example IV. 1. (Yu. Smirnov [6]) Define a compact metric space Q^α for each $\alpha < \omega_1$ as follows: If α is finite, then Q^α = the α-dimensional (closed) cube. If $\alpha = \beta + 1$, then $Q^\alpha = Q^\beta \times [0,1]$. If α is a limit ordinal number, then Q^α = the Alexandroff one point compactification of the discrete sum $\bigcup_{\beta < \alpha} Q^\beta$. Then it can be proved that $\operatorname{Ind} Q^\alpha = \alpha$.

VI. 5. Sum theorem for transfinite inductive dimension

Generally speaking, properties of transfinite inductive dimension are considerably different from those of finite inductive dimension, and there are not so many beautiful theorems on the former as on the latter. The main purpose of this section is to discuss (finite) sum theorems for transfinite inductive dimension by use of G. H. Toulmin's [1] shuffling sum of ordinal numbers following the work by M. Landau [1] and A. R. Pears [1].

Definition VI. 5.[14] *Let α and β be ordinal numbers. Then we define the shuffling sums $\alpha \dotplus \beta$ and $\alpha \dotplus \beta$ as the infimum and supremum of the ordinal numbers of well-ordered sets obtained by shuffling the well-ordered sets $T_\alpha = \{x \mid 0 \leqq x < \alpha\}$ and $T_\beta = \{x \mid 0 \leqq x < \beta\}$ (without changing the order within T_α and T_β by the shuffling) of ordinal numbers. We make the convention:*

[14] G. H. Toulmin defined the shuffling sum to prove sum theorems for weak transfinite inductive dimension which were extended by Landau and Pears to strong transfinite inductive dimension.

$$-1 \dot{+} \alpha = \begin{cases} \alpha - 1 & \text{if } 0 \leqq \alpha < \omega_0 , \\ \alpha & \text{if } \alpha \geqq \omega_0 . \end{cases}$$

A) Let α and β be ordinal numbers such that $\alpha = \alpha' + p$, $\beta = \beta' + q$, where α', β' are limit ordinal numbers, and p, q are non-negative integers. Then

$$\alpha \dot{+} \beta = \begin{cases} \alpha & \text{if } \alpha' > \beta' , \\ \alpha + q = \beta + p & \text{if } \alpha' = \beta' , \\ \beta & \text{if } \alpha' < \beta' . \end{cases}$$

Proof. Assume $\alpha' > \beta'$; then $\beta < \alpha'$ and T_β is isomorphic with the subset $T'_\beta = \{ x \in T_\alpha \mid x < \beta \}$ of T_α. Shuffle T_β with T_α in such a way that the members of T'_β and those of T_β are placed alternatively. Then the shuffled set thus obtained has the ordinal number α, which obviously yields the minimum in the above definition. Hence $\alpha \dot{+} \beta = \alpha$.

Next, assume $\alpha' = \beta'$; then reasoning analogously as above we get a shuffled set whose ordinal number is $\alpha + q$. It is also easy to see that every shuffled set has an ordinal number not less than $\alpha + q$. Thus $\alpha \dot{+} \beta = \alpha + q = \beta + p$.

B) Put, for brevity, $\theta(\alpha, \beta) = \alpha \dot{+} (\beta + 1)$ for ordinal numbers α, β with $\alpha \geqq \beta$. Then

i) $\theta(\alpha, \beta) < \theta(\alpha, \alpha)$ if $\alpha > \beta$,

ii) $\theta(\gamma, \beta) < \theta(\alpha, \beta)$ if $\alpha > \gamma \geqq \beta$.

Proof. To prove i), assume $\alpha = \alpha' + p$, where α' is a limit ordinal number and p is a non-negative integer. If $\beta < \alpha'$, then by A)

$$\theta(\alpha, \beta) = \alpha < \alpha + p + 1 = \theta(\alpha, \alpha) .$$

If $\beta \geqq \alpha'$, then assume $\beta = \alpha' + q$, where q is a non-negative integer such that $q < p$. Then again by A)

$$\theta(\alpha, \beta) = \alpha + q + 1 < \alpha + p + 1 = \theta(\alpha, \alpha) .$$

To prove ii), we still assume $\alpha = \alpha' + p$. If $\gamma < \alpha'$, then by A) $\theta(\gamma, \beta) < \alpha' \leqq \alpha = \theta(\alpha, \beta)$.

If $\beta < \alpha' \leqq \gamma$, then by A) $\theta(\gamma, \beta) = \gamma < \alpha = \theta(\alpha, \beta)$.

If $\beta = \alpha'$, then put $\beta = \alpha' + q$, $\gamma = \alpha' + r$ for non-negative integers q, r, where $q \leqq r < p$. Now it follows from A) that

$$\theta(\gamma,\beta) = \alpha' + r + q + 1 < \alpha' + p + q + 1 = \theta(\alpha,\beta) .$$

C) If R has Ind R, and F is a closed subset of R, then Ind $F \leqq$ Ind R.

Proof. The easy proof (by use of induction on $\alpha =$ Ind R) is left to the reader.

D) Let $R = F \cup G$, where F and G are disjoint closed sets in R. If Ind F and Ind G exist, then Ind R = max (Ind F, Ind G).

Proof. Defining $\alpha = \max$ (Ind F, Ind G) we prove the proposition by induction on α.

D) is obviously true if $\alpha = -1$. So assume that it is true for all $\beta < \alpha$. Then suppose that Ind $G \leqq$ Ind $F = \alpha$. Let H and K be disjoint closed sets in R. Then there are open sets U and V such that

$$H \cap F \subset U \subset F - K , \quad \text{Ind } B_A(U) < \alpha ,$$
$$H \cap G \subset V \subset G - K , \quad \text{Ind } B_B(V) < \alpha .$$

Now $U \cap V = W$ is an open set satisfying $H \subset W \subset R - K$ and $B(W) = B_A(U) \cup B_B(V)$. By use of the induction hypothesis we get Ind $B(W) < \alpha$. Thus Ind $R \leqq \alpha$, and hence Ind $R = \alpha$ follows from C).

E) Let A be a subset of R such that Ind $A \leqq \alpha$. Suppose that F and G are disjoint closed sets in R. Then there is an open set U such that $F \subset U \subset R - G$ and Ind $B(U) \cap A < \alpha$.

Proof. Select open sets P and Q in R such that $F \subset P$, $G \subset Q$, $\bar{P} \cap \bar{Q} = \emptyset$. Since Ind $A \leqq \alpha$, there is an open set V of the subspace A such that

$$\bar{P} \cap A \subset V \subset A - \bar{Q} \quad \text{and} \quad \text{Ind } B_A(V) < \alpha .$$

Define $C = V \cup F$ and $D = (A - V) \cup G$. Then $\bar{C} \cap D = \emptyset$, $C \cap \bar{D} = \emptyset$, and hence there is an open set U of R such that $C \subset U$ and $\bar{U} \cap D = \emptyset$. Accordingly U satisfies $F \subset U$ and $U \cap G = \emptyset$. Now, since $B(U) \cap A \subset B_A(V)$, we obtain from C) Ind $B(U) \cap A < \alpha$.

Theorem VI. 9.[15] *Assume* $R = F \cup G$ *for closed subsets* F *and* G *of* R. *If* Ind F *and* Ind G *exist, then*

$$\text{Ind } R \leqq \max(\text{Ind } F, \text{ Ind } G) \dotplus (\text{Ind } F \cap G + 1) .$$

Proof. Define $\alpha = \max(\text{Ind } F, \text{ Ind } G)$. $\beta = \text{Ind } F \cap G$. Then by use of the symbol of B) it suffices to prove

$$(1)_{\alpha,\beta} \qquad \text{Ind } R \leqq \theta(\alpha,\beta) .$$

The proof will be carried out by induction on (α,β) as follows.

$(1)_{\alpha,-1}$ is true for every α by virtue of D).

Assume that $(1)_{\alpha,\beta}$ is true whenever $\beta < \beta_0$, or $\beta = \beta_0$ and $\beta \leqq \alpha < \alpha_0$. Then we shall prove that $(1)_{\alpha_0,\beta_0}$ is true.

Remember that

$$\alpha_0 = \max(\text{Ind } F, \text{ Ind } G) , \ \beta_0 = \text{Ind } F \cap G ,$$

where F and G are closed subsets of R such that $R = F \cup G$. Let H and K be given disjoint closed sets in R.

Case 1: $\alpha_0 = \beta_0$.

By use of E) we can select an open set U such that

$$H \subset U \subset R - K \quad \text{and} \quad \text{Ind } B(U) \cap F \cap G < \beta_0 = \alpha_0 .$$

Put $\gamma = \text{Ind } B(U) \cap F \cap G$; then $\gamma < \alpha_0$. Note that

$$\max(\text{Ind } B(U) \cap F, \text{ Ind } B(U) \cap G) \leqq \alpha_0 ,$$
$$B(U) = (B(U) \cap F) \cup (B(U) \cap G) .$$

Using the induction hypothesis we obtain $\text{Ind } B(U) \leqq \theta(\alpha_0,\gamma)$. It follows from B) i) that $\text{Ind } B(U) < \theta(\alpha_0,\alpha_0)$. Thus

$$\text{Ind } R \leqq \theta(\alpha_0,\alpha_0) = \theta(\alpha_0,\beta_0) .$$

[15] Actually this theorem and the following corollary hold for hereditarily normal spaces and totally normal spaces (to be defined later) respectively.

Case 2: $\alpha_0 > \beta_0$.
By E) there is an open set U such that $H \subset U \subset R - K$ and

(2) $\quad$ $\operatorname{Ind} B(U) \cap F < \alpha_0$.

Again by use of E) we can find an open set V such that

(3) $\quad$ $H \subset V \subset \bar{V} \subset U$,

(4) $\quad$ $\operatorname{Ind} B(V) \cap G < \alpha_0$.

Put $W = (U - G) \cup V$. Then W is an open set satisfying $H \subset W \subset R - K$. We claim that

(5) $\quad$ $B(W) \subset (B(U) \cap F) \cup (B(V) \cap G) \cup (F \cap G)$.

To prove it assume

$$x \notin (B(U) \cap F) \cup (B(V) \cap G) \cup (F \cap G) .$$

Then either $x \in F - G$ or $x \in G - F$.
Assume first $x \in F - G$. Since $x \notin B(U)$ follows from $x \notin B(U) \cap F$, either $x \in U$ or $x \in R - \bar{U}$ holds. If the former is the case, then $x \in U - G \subset W$. If the latter is the case, then $R - \bar{U}$ is a neighbourhood of x which is disjoint from W . Thus in either cases we obtain $x \notin B(W)$.
Next, assume that $x \in G - F$. Since $x \notin B(V)$ follows from $x \notin B(V) \cap G$, either $x \in V$ or $x \in R - \bar{V}$ holds. If the former is the case, then $x \in W$. Otherwise $(R - \bar{V}) - F$ is a neighbourhood of x which is disjoint from W . Thus we obtain $x \notin B(W)$ in either cases. Therefore our claim (5) is proved.
Put, for brevity, $B_1 = B(U) \cap F$ and $B_2 = B(V) \cap G$. Then observe that $B_1 \cap B_2 = \emptyset$ follows from (3). Hence by use of (2), (4) and D) we obtain $\operatorname{Ind} B_1 \cup B_2 < \alpha_0$. Thus

$$\max(\operatorname{Ind} B_1 \cup B_2 \ , \ \operatorname{Ind} F \cap G) = \delta < \alpha_0 \ ,$$
$$\operatorname{Ind}(B_1 \cup B_2) \cap (F \cap G) = \varepsilon \leqq \beta_0 \ .$$

Thus by the induction hypothesis we obtain

$$\operatorname{Ind}(B_1 \cup B_2) \cup (F \cap G) \leqq \theta(\delta, \varepsilon) \ .$$

It follows from C) and (5) that $\operatorname{Ind} B(W) \leqq \theta(\delta, \varepsilon)$. On the other hand B) ii) implies

$$\theta(\delta, \varepsilon) < \theta(\alpha_0 \ , \ \varepsilon) \leqq \theta(\alpha_0 \ , \ \beta_0) \ .$$

Therefore Ind $B(W) < \theta(\alpha_0, \beta_0)$, which proves

$(1)_{\alpha_0,\beta_0}$ Ind $R \leq \theta(\alpha_0,\beta_0)$.

and the induction is concluded.

Corollary. *Assume* $R = F \cup G$ *for closed sets* F *and* G . *If* Ind $F \cap G$ *is finite, and* Ind F *and* Ind G *exist, then* Ind $R = \max$ (Ind F , Ind G) .

Proof. If both Ind F and Ind G are finite, then the corollary is simply a special case of the sum theorem for finite dimension. If max (Ind F , Ind G) is infinite, then the corollary follows directly from the preceding theorem.[16]

[16] B. T. Levšenko [2] showed that the equality Ind $R = \max$ (Ind F , Ind G) does not generally hold even if R is compact: Assume $R = A \cup B$, Ind $A = \alpha = \alpha' + p$, Ind $B = \beta = \beta' + q$, where A and B are not necessarily closed in R , and α' , β' , p , q are as in A). Then B. T. Levšenko [3] proved that

$$\text{Ind } R \leq \max(\alpha,\beta) \quad \text{if} \quad \alpha' \neq \beta' ,$$
$$\text{Ind } R \leq \alpha' + p + q + 1 \quad \text{if} \quad \alpha' = \beta' .$$

M. Landau [1] proved

$$1 + \text{Ind } R \leq (1+\alpha) \dotplus (1+\beta) .$$

A. Pears [1] proved

$$\text{Ind } R \leq \max(\alpha,\beta) \dotplus (\min(\alpha,\beta) + 1) ,$$

which is a reformulation of Levšenko's result. (The above three theorems were actually proved for hereditarily normal spaces.) L. A. Luxemburg [1] gave an example of a compact metric space X such that Ind $X = \omega_0 + 2$, ind $X = \omega_0 + 1$, which sharply contrasts the coincidence of Ind and ind in the finite-dimensional case. The discrete sum of the n-dimensional cubes I^n , $n = 1,2,\ldots$ has ind but not Ind. (However, it is known that if a hereditarily normal compact space has ind, then it has Ind.)

On the other hand D. Henderson [2] defined a new transfinite dimension to prove a subspace theorem, a finite sum theorem, a product theorem etc., which have the same appearance as their counter-parts on finite dimension.

Also note that the locally countable sum of countable-dimensional closed sets is obviously countable-dimensional, and it is easy to prove that the countable sum of weakly infinite-dimensional (in the sense of Alexandroff) closed sets is weakly infinite-dimensional in the same sense. The locally countable sum theorem obviously holds for strong countable-dimensionality, too. See B. T. Levšenko [2] for further results on sum theorems of weakly infinite-dimensional spaces.

VI. 6. General imbedding theorem

The purpose of this section is to extend the imbedding theorems to general metric spaces [17]. Up to the present, one has not succeeded in finding an imbedding theorem for finite-dimensional general metric spaces which is very analogous to the one for the separable case. We can, however, imbed them into a kind of generalized Hilbert cube, i.e. a countable product of star spaces [18]. Although the contents of this section are only partially related with infinite-dimensional spaces, the method used here is analogous to that of Section 2 of this chapter, and this is the reason why this section is located here.

Definition VI. 6. *Let* $\{E_\alpha \mid \alpha \in A\}$ *be a collection of unit segments* $[0,1]$ *. By identifying all zeros in* $\bigcup\{E_\alpha \mid \alpha \in A\}$ *we get a star-shaped set* $S(A)$ *. We introduce a metric in* $S(A)$ *as follows,*

$$\rho(x,y) = \begin{cases} |x-y| & \text{if } x,y \text{ belong to the same segment } E_\alpha\,, \\ |x+y| & \text{if } x,y \text{ belong to distinct segments.} \end{cases}$$

Then we obtain a one-dimensional metric space $S(A)$ *called the star space with the index set* A *.*

Now we can assert the following.

Theorem VI. 10. [19] *A metric space* R *with weight* $|A|$ *has* $\dim \leqq n$ *if and only if it can be imbedded in the subset* $K_n(A)$ *of the countable product* $P(A) = \prod_{m=1}^{\infty} S_m(A)$ *of star spaces* $S_m(A)$, $m = 1, 2, \ldots$, *where we denote by* $K_n(A)$ *the set of points in* $P(A)$ *at most* n *of whose non-vanishing coordinates are rational.*

Proof. To show $\dim K_n(A) \leqq n$, we decompose $K_n(A)$ in the following way

$$K_n(A) = \bigcup_{m=0}^{n} K'_m\,,$$

where K'_m is the set of points in $P(A)$ exactly m of whose non-vanishing coordinates are rational. We consider a given class $\{a_j \mid j = 1, \ldots, m\}$ of m rational

[17] The content of this section is due to J. Nagata [6].
[18] We owe this terminology to H. J. Kowalsky [1].
[19] $|A|$ denotes the cardinal number of the set A . We mean by the *weight* of a topological space the cardinal number of an open basis which has the least cardinal number among the open bases of R .

numbers a_j such that $0 < a_j \leq 1$. Then the set of points in K'_m whose j-th coordinates are equal to a_j for every j is a 0-dimensional closed set of K'_m . This assertion is easily proved by the product theorem, because the set of irrational points and zero in a star space is a 0-dimensional space, and so is the set of a_j from the distinct branches of the star space $S_j(A)$. Hence it follows from the sum theorem that K'_m as the countable sum of those 0-dimensional closed sets is also 0-dimensional. By the decomposition theorem, this implies that $\dim K_n(A) \leq n$.

Conversely, we suppose R is a general metric space with weight $|A|$ and $\dim R \leq n$. By Bing's metrization theorem, there exists a σ-discrete open basis

$$\mathcal{W}_m = \{ W_{m\alpha} \mid \alpha \in A_m \} , \quad m = 1,2,\ldots .$$

We can assume without loss of generality that there exist open sets $V_{m\alpha}$, $\alpha \in A_m$, $m = 1,2,\ldots$ such that $\bar{V}_{m\alpha} = F_{m\alpha} \subset W_{m\alpha}$ and such that for every neighbourhood $U(p)$ of every point p of R , there exist m and $\alpha \in A_m$ for which $p \in F_{m\alpha} \subset W_{m\alpha} \subset U(p)$. Moreover, since $A_m \subset A$ can be assumed, we may further assume $A_m = A$ for every A by adding as many empty sets as desired to the original $\mathcal{W}_m$. Putting

$$W_m = \bigcup \{ W_{m\alpha} \mid \alpha \in A_m \} \quad \text{and} \quad F_m = \bigcup \{ F_{m\alpha} \mid \alpha \in A_m \} ,$$

we obtain open sets W_m and closed sets F_m satisfying $F_m \subset W_m$, $m = 1,2,\ldots$. Now by use of the decomposition theorem we decompose R in the following way:

$$R = \bigcup_{k=1}^{n+1} R_k$$

for 0-dimensional sets R_k , $k = 1,\ldots,n+1$. We define open sets U_{mr} , $m = 1,2,\ldots$; r = rational numbers with $0 < r < \sqrt{2}/2m$ such that

(1) $\quad F_m \subset U_{mr} \subset \bar{U}_{mr} \subset U_{mr'} \subset \bar{U}_{mr'} \subset W_m$ if $r > r'$,

(2) $\quad \bar{U}_{mr} = \bigcap \{ U_{mr'} \mid r' < r \}$, $\quad U_{mr} = \bigcup \{ \bar{U}_{mr'} \mid r' > r \}$,

(3) $\quad \operatorname{ord}_p \{ B(U_{mr}) \mid m = 1,2,\ldots;\ r = \text{a rational number with } 0 < r < \sqrt{2}/2m \} \leq k - 1$ for each point p of R_k .

The process to construct U_{mr} is parallel to that used in the argument of 2 A), so it will not be given here. Define

$$U^m_{r\alpha} = U_{mr} \cap W_{m\alpha} ;$$

then, since each $\{W_{m\alpha} \mid \alpha \in A_m\}$ is discrete, $U^m_{r\alpha}$ is an open set for all $\alpha \in A_m$, $m = 1, 2, \dots$, r = rational numbers with $0 < r < \sqrt{2}/2m$ such that

$$F_{m\alpha} \subset U^m_{r\alpha} \subset \bar{U}^m_{r\alpha} \subset U^m_{r'\alpha} \subset \bar{U}^m_{r'\alpha} \subset W_{m\alpha} \quad \text{if} \quad r > r' ,$$

$$B(U^m_{r\alpha}) = B(U_{mr}) \cap W_{m\alpha} .$$

Hence (3) implies that

$$(4) \qquad \operatorname{ord}_p \left\{ B(U^m_{r\alpha}) \,\middle|\, \begin{array}{l} \alpha \in A_m \text{ , } m = 1, 2, \dots \text{ , and } r \text{ is a} \\ \text{rational number with } 0 < r < \sqrt{2}/2m \end{array} \right\} \leq k - 1$$

for each point p of R_k. Now we construct star spaces $S_m(A_m)$ with the index set $A_m = A$ and denote by E_α, $\alpha \in A_m$ the unit segments of which $S_m(A_m)$ consists. Let f_m be the mapping of R into $S_m(A_m)$ defined as follows:

$$f_m(p) = \begin{cases} 0 & \text{if } p \notin W_m , \\ \sup \{ r \mid p \in U^m_{r\alpha} \} \in E_\alpha & \text{if } p \in W_{m\alpha} \text{ , and } p \in U^m_{r\alpha} \text{ for some } r , \\ 0 & \text{if } p \in W_{m\alpha} \text{ and } p \notin U^m_{r\alpha} \text{ for every } r . \end{cases}$$

Then it is easy to see that f_m is a continuous mapping with the properties $f_m(R - W_m) = 0$, and, if $x \in F_m$, then $f_m(x) = \sqrt{2}/2m \in E_\alpha$ for some $\alpha \in A_m$. By use of (2) we can also easily see that $f_m(p) = r \in E_\alpha$ if and only if $p \in B(U^m_{r\alpha})$. Hence $f_m(p)$ is non-vanishing and rational if and only if $p \in B(U^m_{r\alpha})$ for some $\alpha \in A_m$ and r.

Now we consider the product

$$P(A) = \prod_{m=1}^{\infty} S_m(A_m)$$

and its subset $K_n(A)$ mentioned in the theorem. Let us define a mapping f of R into $P(A)$ by

$$f(p) = \{ f_m(p) \mid m = 1, 2, \dots \} .$$

Then it follows from the properties of f_m and (4) that f is a homeomorphic mapping of R onto a subset of $K_n(A)$. Thus the proof of the theorem is complete.

As for the imbedding of countable-dimensional general metric spaces we can assert the following theorem.

Theorem VI. 11. *A metric space R with weight $|A|$ is countable-dimensional if and only if it can be imbedded in the subset $K_\infty(A)$ of a countable product $P(A) = \Pi_{m=1}^{\infty} S_m(A)$ of star spaces $S_m(A)$, $m = 1,2,\ldots$, where we denote by $K_\infty(A)$ the set of points in $P(A)$ at most finitely many of whose non-vanishing coordinates are rational.*

Proof. The proof is analogous to the proof for the finite-dimensional case and is therefore left to the reader.

In contrast to the separable case, it is impossible to find a universal space for n-dimensional general metric spaces among finite products of star-spaces, because such a product and accordingly every of its subsets can be decomposed into countably many closed subsets each of which has a σ-star-finite base while, as pointed out by Yu. Smirnov, not every finite-dimensional metric space has that property.
S. Lipscomb [1], however, has recently constructed such a universal space in a finite product of one-dimensional spaces as follows:

Let $N(A)$ denote the generalized Baire 0-dimensional metric space defined on the set A. Let $\alpha = (\alpha_1,\alpha_2,\ldots)$, $\beta = (\beta_1,\beta_2,\ldots) \in N(A)$. Then define a binary relation R in $N(A)$ by $\alpha R \beta$ if and only if

1) $\alpha = \beta$ or

2) $\alpha \neq \beta$ and there is a natural number j such that
for every $k < j$ $\alpha_k = \beta_k$, and
i) $\alpha_j = \beta_{j+s}$ for all $s \geqq 1$,
ii) $\beta_j = \alpha_{j+s}$ for all $s \geqq 1$.

Then the quotient space $J(A) = N(A)/R$ is a one-dimensional metric space, whose point is either a singleton or a doubleton class; a former type of point is called an irrational point and a latter type a rational point. Then he proved:
A metric space of weight $|A|$ has $\dim \leq n$ if and only if it is topologically imbedded in the set of points of $J(A)^{n+1}$ which have at most n rational coordinates.

An imbedding theorem for more general (non-metrizable) spaces is due to B. Pasynkov [4] and A. Zarelua [2], who constructed an n-dimensional compact space P with weight τ such that every completely regular space of $\dim \leqq n$ and weight $\leqq \tau$ is homeomorphic to a subset of P.[20]

[20] For other types of imbedding theorems see, e.g., B. R. Wenner [1], W. Kulpa [1], A. G. Nemec [1] and L. A. Luxemburg [2].

CHAPTER VII

DIMENSION OF NON-METRIZABLE SPACES

Throughout the preceding chapters we have considered only metric spaces except in Chapter I. Dimension theory for non-metrizable spaces has been greatly developed in these years, but still it is not as satisfactory as the theory for metric spaces. In the present chapter we shall discuss some of the principal results ever obtained in this field. Recall that, as noted before, all normal spaces and Tychonoff spaces are assumed to be T_1 (and are accordingly T_2) in this book. At some points of this chapter proofs may be somewhat sketchy to avoid bothering the reader with arguments similar to those used previously.

VII. 1. Sum theorem and subspace theorem for dim

A) A topological space X has $\dim \leq n$ if and only if for every finite open covering $\mathcal{U} = \{ U_i \mid i = 1, \dots, k \}$ there is an open covering $\mathcal{V} = \{ V_i \mid i = 1, \dots, k \}$ such that $V_i \subset U_i$, $i = 1, \dots, k$ (namely $\mathcal{U}$ shrinks to $\mathcal{V}$), and such that $\operatorname{ord} \mathcal{V} \leq n + 1$.

Proof. The easy proof is left to the reader.

We owe the following theorem to P. A. Ostrand [1].

Theorem VII. 1. *Let* $\{ F_\gamma \mid \gamma \in \Gamma \}$ *be a locally finite closed covering of a topological space* X *such that* $\dim F_\gamma \leq n$ *for all* $\gamma \in \Gamma$. *Then* $\dim X \leq n$.

Proof. Well-order the given locally finite closed covering and denote it by $\{F_\gamma \mid 0 \leqq \gamma < \tau\}$. We may assume that $F_0 = \emptyset$.

Now, let $\mathcal{U} = \{U_i \mid i = 1,\dots,k\}$ be a given open covering of X. We shall define open coverings $\mathcal{V}_\gamma = \{V(\gamma,i) \mid i = 1,\dots,k\}$, $0 \leqq \gamma < \tau$, such that

(1) if $\beta < \delta$, $\mathcal{V}_\beta$ shrinks to $\mathcal{V}_\delta$,

(2) if $\beta < \delta$, $V(\beta,i) - V(\delta,i) \subset \cup\{F_\gamma \mid \beta < \delta \leqq \delta\}$,

(3) if $x \in F_\gamma$, $\operatorname{ord}_x \mathcal{V}_\gamma \leqq n+1$.

We shall define $\mathcal{V}_\gamma$, $0 \leqq \gamma < \tau$, by use of induction on γ.

First we put $\mathcal{V}_0 = \mathcal{U}$.

Now assume that $\mathcal{V}_\gamma$ have been defined for all $\gamma < \delta$. Then we put

$$W_i = \cap\{V(\gamma,i) \mid \gamma < \delta\} \quad \text{and} \quad \mathcal{W} = \{W_i \mid i = 1,\dots,k\} .$$

Let $\beta < \delta$; then by the induction hypothesis (2) we have

$$\begin{aligned}(4) \qquad & V(\beta,i) - W_i = \cup\{V(\beta,i) - V(\gamma,i) \mid \beta < \gamma < \delta\} \subset \\ & \subset \cup\{\cup\{F_{\gamma'} \mid \beta < \gamma' \leqq \gamma\} \mid \beta < \gamma < \delta\} = \cup\{F_\gamma \mid \beta < \gamma < \delta\} .\end{aligned}$$

We can prove that each W_i is an open set. For this purpose assume $x \in W_i$, $x \in F_{\gamma_i}$, $i = 1,\dots,p$, and $\gamma_1 < \gamma_2 < \dots < \gamma_p < \delta$, while $x \notin F_\gamma$ if $\gamma \neq \gamma_1,\dots,\gamma_p$ and $\gamma < \delta$.

Then note that $x \in V(\gamma_p,i)$ follows from $V(\gamma_p,i) \supset W_i$. Put

$$F = \cup\{F_\gamma \mid x \notin F_\gamma\} ;$$

then $P = V(\gamma_p,i) \cap (X - F)$ is an open neighbourhood of x. Since it follows from (4) that

$$V(\gamma_p,i) - W_i \subset \cup\{F_\gamma \mid \gamma_p < \gamma < \delta\} \subset F ,$$

we obtain

$$P \subset V(\gamma_p,i) \cap [X - (V(\gamma_p,i) - W_i)] \subset W_i .$$

Thus W_i is an open set.

Next we shall prove that $\mathcal{W}$ is a covering of X. Let $x \in X$ be given. For each $\gamma < \delta$ we can find $i(\gamma)$ such that $x \in V(\gamma, i(\gamma))$. Assume that the set $\{\gamma \mid i(\gamma) = i_0\}$ is cofinal in the set $\{\gamma \mid \gamma < \delta\}$. (Namely for each element $\gamma < \delta$, there is γ' such that $i(\gamma') = i_0$ and $\gamma \leqq \gamma' < \delta$.) Then $x \in V(\gamma, i_0)$ for all $\gamma < \delta$, because of the induction hypothesis (1). Thus $x \in W_{i_0}$ follows, proving our assertion.

Using A) there is an open collection $\mathcal{W}' = \{W_i' \mid i = 1, \dots, k\}$ in X such that

$$W_i' \subset W_i, \quad i = 1, \dots, k, \quad \bigcup_{i=1}^{k} W_i' \supset F_\delta, \quad \text{and}$$

$$\operatorname{ord}_x \mathcal{W}' \leqq n+1 \quad \text{at each } x \in F_\delta,$$

because $\dim F_\delta \leqq n$. Put

$$V(\delta, i) = (W_i - F_\delta) \cup W_i' \quad \text{and} \quad \mathcal{V}_\delta = \{V(\delta, i) \mid i = 1, \dots, k\}.$$

Then $\mathcal{V}_\delta$ obviously satisfies (1), (2), (3). (Note that (2) follows from (4) and the definition of $V(\delta, i)$.)

Finally we put

$$V_i = \bigcap \{V(\gamma, i) \mid 0 \leqq \gamma < \tau\} \quad \text{and} \quad \mathcal{V} = \{V_i \mid i = 1, \dots, k\}.$$

Then a similar argument with the above leads us to the conclusion that $\mathcal{V}$ is an open covering of X to which $\mathcal{V}_0 = \mathcal{U}$ shrinks. It follows from (3) that $\operatorname{ord} \mathcal{V} \leqq n+1$. Hence $\dim X \leqq n$.

B) Let $\mathrm{F} = \{F_i \mid i = 1, \dots, k\}$ be a closed collection of a normal space X and $\{U_i \mid i = 1, \dots, k\}$ an open collection such that $F_i \subset U_i$, $i = 1, \dots, k$. Then there is an open collection $\mathcal{W} = \{W_i \mid i = 1, \dots, k\}$ such that $F_i \subset W_i \subset U_i$, $i = 1, \dots, k$, and $\operatorname{ord} \mathrm{F} = \operatorname{ord} \mathcal{W}$.

Proof. If $k = \operatorname{ord} \mathrm{F}$, then the proposition is obviously true.

Assume $k > \operatorname{ord} \mathrm{F} = n$. Then $\mathcal{V}' = \{V(x) \mid x \in X\}$ is an open covering of X, where

$$V(x) = X - \bigcup \{F_i \mid x \notin F_i\} \quad \text{for each } x \in X.$$

Since X is normal, there is an open covering $\mathrm{P} = \{P_j \mid j = 1, \dots, \ell\}$ of X such that $\mathrm{P}^\Delta < \mathcal{V}$. Put

$W_i = S(F_i, P) \cap U_i$ for $i = 1, \ldots, k$ and $\mathcal{W} = \{ W_i \mid i = 1, \ldots, k \}$.

Then it is obvious that $F_i \subset W_i \subset U_i$, $i = 1, \ldots, k$.

To prove $\operatorname{ord} \mathcal{W} = \operatorname{ord} \mathcal{F}$, let $x \in X$. Then $S(x, P) \subset V(y)$ for some $y \in X$. Note that because of its definition $V(y)$ intersects at most n members of $\mathcal{F}$. Hence $S(x, P)$, too, intersects at most n members of $\mathcal{F}$, which implies that $\operatorname{ord}_x \mathcal{W} \leqq n$. Therefore we obtain $\operatorname{ord} \mathcal{U} = \operatorname{ord} \mathcal{F}$.

C) A normal space X has $\dim \leqq n$ if and only if every finite open covering of X can shrink to a closed covering of $\operatorname{ord} \leqq n+1$.

Proof. The proposition follows directly from B).

Theorem VII. 2. *Let* $\{ G_i \mid i = 1, 2, \ldots \}$ *be a closed covering of a normal space* X *such that* $\dim G_i \leqq n$, $i = 1, 2, \ldots$. *Then* $\dim X \leqq n$.[1]

Proof. We may assume

(1) $\qquad G_1 \subset G_2 \subset \ldots ,$

because the sum theorem holds for every finite sum (Theorem VII.1.) Let $\mathcal{U} = \{ U_i \mid i = 1, \ldots, k \}$ be a given open covering of X . Since X is normal, we can construct an open covering $\mathcal{U}_1 = \{ U_i^1 \mid i = 1, \ldots, k \}$ of X such that $\bar{U}_i^1 \subset U_i$. By B) and C) there is an open covering $\mathcal{V}_1 = \{ V_i^1 \mid i = 1, \ldots, k \}$ of G_1 such that

$$V_i^1 \subset U_i^1 \quad \text{and} \quad \operatorname{ord} \bar{\mathcal{V}}_1 \leqq n+1 .$$

Now construct an open covering $\mathcal{U}_2 = \{ U_i^2 \mid i = 1, \ldots, k \}$ of X such that

$$\bar{U}_i^2 \subset V_i^1 \cup (U_i^1 - G_1) .$$

[1] This theorem was proved by E. Čech [2] and others. K. Morita [2] proved the following more general theorem: Let $\{ G_\gamma \mid \gamma < \tau \}$ be a closed covering of a normal space X and $\{ P_\gamma \mid \gamma < \tau \}$ an open covering such that $\dim G_\gamma \leqq n$, $G_\gamma \subset P_\gamma$ for every $\gamma < \tau$, and such that $\{ P_\gamma \mid \gamma < \delta \}$ is locally finite for each $\delta < \tau$. Then $\dim X \leqq n$. (Here τ denotes a not necessarily countable ordinal number.)

Again by use of B) and C) we obtain an open covering $\mathcal{V}_2 = \{V_i^1 \mid i=1,\dots,k\}$ of G_2 such that

$$V_i^2 \subset U_i^2 \quad \text{and} \quad \operatorname{ord} \bar{\mathcal{V}}_2 \leqq n+1 .$$

Continue the same process to define open coverings $\mathcal{U}_j = \{U_i^j \mid i=1,\dots,k\}$ and $\mathcal{V}_j = \{V_i^j \mid i=1,\dots,k\}$ of X and of G_j, respectively, satisfying

(2) $\quad \operatorname{ord} \bar{\mathcal{U}}_j \leqq n+1$

(3) $\quad \bar{U}_i^{j+1} \subset V_i^j \cup (U_i^j - G_j)$,

(4) $\quad V_i^j \subset U_i^j$,

and accordingly

(5) $\quad U_i^{j+1} \subset U_i^j$.

Put $F_i = \overline{(V_i^1 \cup V_i^2 \cup \dots)}$. Then it is obvious that $F_i \subset \bar{U}_i^1 \subset U_i$, because of (4) and (5). On the other hand, since $\bigcup_{i=1}^k V_i^j = G_j$, $\bigcup_{i=1}^k F_i = X$ follows. Namely $\mathcal{U}$ shrinks to the closed covering $\mathcal{F} = \{F_i \mid i=1,\dots,k\}$.

To prove $\operatorname{ord}_x \mathcal{F} \leqq n+1$ at each point $x \in X$, let $x \in G_j - G_{j-1}$ be given. (Assume $G_0 = \emptyset$.) Then if $x \notin \bar{V}_i^j$ for some i with $1 \leqq i \leqq k$, then $x \notin \bar{U}_i^{j+1}$ follows from (3). Thus

$$x \notin \overline{(U_i^{j+1} \cup U_i^{j+2} \cup \dots)} = \bar{U}_i^{j+1}$$

follows from (5). (4) implies

$$x \notin \overline{(V_i^{j+1} \cup V_i^{j+2} \cup \dots)} .$$

On the other hand

(7) $\quad x \notin G_{j-1} = \bar{G}_{j-1} \supset \bar{V}_i^1 \cup \dots \cup \bar{V}_i^{j-1}$

follows from (1). Combine (6), (7) and the original assumption $x \notin \bar{V}_i^j$ to get $x \notin F_i$. Now $\operatorname{ord}_x \mathcal{F} \leqq n+1$ follows from the above argument and (2). Thus by C) we conclude $\dim X \leqq n$.

Corollary. *Let* $\{F_\alpha \mid \alpha \in A\}$ *be a locally countable closed covering of a paracompact* T_2*-space* X . *If* $\dim F_\alpha \leq n$ *for all* $\alpha \in A$, *then* $\dim X \leq n$.

Proof. Combine Theorems VII.1, VII.2 and the following proposition D).

D) Let F be a closed subset of a topological space X . Then $\dim F \leq \dim X$, and $\operatorname{Ind} F \leq \operatorname{Ind} X$.

Proof. Obvious.

Definition VII. 1. *A normal space* X *is called totally normal if for every open set* U *of* X , *there is an open covering* $\mathcal{U}$ *of* U *such that*

(i) *each member of* $\mathcal{U}$ *is an* F_σ*-set (= a countable sum of closed sets) in* X ,

(ii) $\mathcal{U}$ *is locally finite in* U .

E) Every perfectly normal space and every hereditarily paracompact T_2-space are totally normal.

Proof. Omitted.

Remark. 1) Every F_σ-set in a normal space is normal.

2) A T_2-space X is normal if and only if every finite open covering of X shrinks to an open covering by cozero sets. (See I.1 A). The easy proofs of the remarks are left to the reader.)

F) Every totally normal space X is hereditarily normal. In fact every subset of a totally normal space is totally normal.

Proof. Let U be an open set of X . Then there is a covering $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ of U such that $\mathcal{U}$ is locally finite in U, and each U_α is a cozero set in X. Then by Remark 1) U_α is normal. Now suppose that $\{V_i \mid i = 1,\dots,k\}$ is a given open covering of U . Then by Remark 2) $\{V_i\}$ can shrink to a covering $\{V_i^\alpha \mid i = 1,\dots,k\}$ of U_α , where each V_i^α is a cozero set in U_α . Since U_α is a cozero set in U , so is V_i^α . Define

$$W_i = \bigcup \{V_i^\alpha \mid \alpha \in A\} .$$

Then W_i is a cozero set in U satisfying $W_i \subset V_i$. Hence $\{V_i\}$ shrinks to the covering $\{W_i \mid i = 1,\dots,k\}$ of U by cozero sets. Hence by Remark 2) U is normal. Thus X is hereditarily normal by I.1 B). Now the second statement of the proposition is almost obvious.[2]

Theorem VII. 3.[3] *Let* X *be a totally normal space and* Y *a subset of* X *. Then* $\dim Y \leqq \dim X$.

Proof. Let $\mathcal{U} = \{U_i \mid i = 1,\dots,k\}$ be an open covering of Y . Then $U_i = V_i \cap Y$ for some open sets V_i , $i = 1,\dots,k$, of X . Define $V = \cup\{V_i \mid i = 1,\dots,k\}$. Then, since X is totally normal, there is an open covering $\{W_\alpha \mid \alpha \in A\}$ of V such that $\{W_\alpha\}$ is locally finite in V and such that each W_α is F_σ in X , i.e. $W_\alpha = \cup_{i=1}^{\infty} F_{\alpha i}$ for closed sets $F_{\alpha i}$ in X .

Assume that $\dim X = n$; then by D) $\dim F_{\alpha i} \leqq n$ holds. Put $F_i = \cup\{F_{\alpha i} \mid \alpha \in A\}$; then $\{F_{\alpha i} \mid \alpha \in A\}$ is a locally finite closed covering of F_i , and hence by Theorem VII.1, we obtain $\dim F_i \leqq n$.

Since $\{F_i \mid i = 1,2,\dots\}$ is a closed covering of V , and V is normal, from Theorem VII.2 it follows that $\dim V \leqq n$. Thus $\{V_i \mid i = 1,\dots,k\}$ can shrink to an open covering $\{V_i' \mid i = 1,\dots,k\}$ of V with $\mathrm{ord} \leqq n+1$. Then $\mathcal{U}$ shrinks to $\mathcal{V}' = \{V_i' \cap Y \mid i = 1,\dots,k\}$, where $\mathcal{V}'$ is an open covering of Y such that $\mathrm{ord}\, \mathcal{V}' \leqq n+1$. This proves, because of A), that $\dim Y \leqq n$.

It is impossible to extend this theorem to hereditarily normal spaces. V. V. Filippov [3] and E. Pol - R. Pol [1] proved that there is a hereditarily normal, zero-dimensional space X which contains subspaces X_n , $n = 1,2,\dots$ with $\dim X_n = \mathrm{Ind}\, X_n = n$.[4]

---

[2] The space of all ordinal numbers $\leq \omega_1$ with the order topology is hereditarily normal but not totally normal. For an example of a totally normal space which is neither perfectly normal nor paracompact see R. H. Bing [1] (Example G).

[3] This theorem is due to C. H. Dowker [3].

[4] In fact V. V. Filippov proved this theorem assuming a set-theoretical hypothesis while E. Pol - R. Pol showed that the assumption could be dropped. Observe that this theorem implies that the subspace theorem does not hold for a compact T_2-space and its normal subspaces. Because, as easily seen, $\dim \beta X = \dim X$ and $\mathrm{Ind}\, \beta X = \mathrm{Ind}\, X$ hold for every normal space X . (See Section 6 of the present chapter.)

C. H. Dowker [3] defined the *local dimension* locdim X of a space X as the least number n such that every point of X has a closed neighbourhood $\bar{U}$ with $\dim \bar{U} \leqq n$ and studied relations between local dimension and the subset theorem.

K. Morita [2] proved that if X is a normal space and Y a subset of X with the star-finite property, then $\dim Y \leqq \dim X$.

VII. 2. Dimensions of non-metrizable spaces

Although we proved sum theorems in the previous section for considerably general spaces, it is not so easy to establish a satisfactory dimension theory for non-metrizable spaces. Perhaps a reason of the difficulty lies in the fact that different dimension functions do not easily coincide when the space is non-metrizable. In the present section we will study relations between different definitions of dimension on non-metrizable spaces.

Generally it is obvious that $\operatorname{ind} X \leqq \operatorname{Ind} X$ holds for every T_1-space X. Conditions under which ind and Ind coincide were studied by quite a few authors, among whom Y. Katsuta [1] proved that $\operatorname{ind} X = \operatorname{Ind} X$ holds whenever X is hereditarily paracompact T_2 and has the star-finite property.[5]

On the other hand V. V. Filippov [1] constructed a compact T_2-space X such that $\operatorname{ind} X = 2$, $\operatorname{Ind} X = 3$.

Concerning the relationships between $\dim X$ and $\operatorname{Ind} X$ the following assertion can be proved.[6]

A) For every normal space X : $\dim X \leqq \operatorname{Ind} X$.

Proof. Let us show by induction on the number n that $\operatorname{Ind} X \leqq n$ implies $\dim X \leqq n$.

---

[5] T. Mizokami [1] proved the same for every totally normal space X satisfying the following condition: For each open base $\mathcal{U}$ of X there is an ordered open covering $(\mathcal{V},<)$ of X such that (i) for each $V \in \mathcal{V}$, $\{V' \in \mathcal{V} \mid V' < V\}$ is locally finite in V, (ii) for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subset U$, $B(V) \subset B(U)$. This theorem unifies various other theorems including the above-mentioned result of Y. Katsuta and others due to K. Morita, R. Ford, B. Fitzpatrick and K. Nagami. See also L. F. McAuley [1], J. A. French [1] and V. V. Fedorčuk [2].

Perhaps another reason of the difficulty of general dimension theory is that there is no convenient theorem like Theorem II.2 to characterize dimension by a special base if the space is non-metrizable. J. Nagata [12] and T. Mizokami [2] proved $\operatorname{Ind} X \leqq n$ if and only if X has a σ-closure preserving base $\mathcal{U}$ such that $\operatorname{Ind} \overline{B(U)} \leqq n-1$ for all $U \in \mathcal{U}$, under certain conditions on X. The latter's condition is particularly satisfied if X is the image of a metric space under a closad continuous mapping. However, it is unknown yet if the same theorem holds whenever X is a regular space with a σ-closure-preserving base.

[6] This assertion is due to N. B. Vedenissoff [1].

Since the validity of the assertion is clear for $n=-1$, we shall assume it for $\operatorname{Ind} X \leq n-1$. Now, assume $\operatorname{Ind} X \leq n$, and suppose $\mathcal{U} = \{U_i \mid i=1,\dots,k\}$ is a finite open covering of X. Then by virtue of the normality of X and I.1 A), there exists an open covering $\mathcal{V} = \{V_i \mid i=1,\dots,k\}$ such that $\bar{V}_i \subset U_i$ and $\operatorname{Ind} B(V_i) \leq n-1$. Using the induction hypothesis we obtain $\dim B(V_i) \leq n-1$. Hence by Theorem VII.2 $B = \bigcup_{i=1}^{k} B(V_i)$ is a closed set with $\dim B \leq n-1$. Therefore there exists an open covering $\mathcal{W} = \{W_i \mid i=1,\dots,k\}$ of B such that $\bar{W}_i \subset U_i$ and $\operatorname{ord} \bar{\mathcal{W}} \leq n$.

Now we can easily construct a finite open covering $\mathcal{N}$ of X such that

$$\mathcal{N} < \wedge \{\{U_i, X-\bar{W}_i\} \mid i=1,\dots,k\},$$

and each member of $\mathcal{N}$ intersects at most n members of $\bar{\mathcal{W}}$. Since X is normal, by I.1 C) we can find a finite open covering $\mathcal{P}$ satisfying $\mathcal{P}^{\Delta} < \mathcal{N}$. Put

$$Q_i = S(\bar{W}_i, \mathcal{P}) \quad \text{and} \quad \mathcal{Q} = \{Q_i \mid i=1,\dots,k\}.$$

Then it is easy to see that $\mathcal{Q}$ is an open collection satisfying

$$\mathcal{Q} < \mathcal{U}, \ \operatorname{ord} \mathcal{Q} \leq n, \ \text{and} \ \bigcup\{Q \mid Q \in \mathcal{Q}\} \supset B.$$

On the other hand, we define

$$\mathcal{U}' = \wedge \{\{U_i, X-\bar{U}_i\} \mid i=1,\dots,k\};$$

then $\mathcal{U}'$ is an open collection which covers $X-B$ and satisfies $\operatorname{ord} \mathcal{U}' \leq 0$. This implies $\operatorname{ord} \mathcal{Q} \cup \mathcal{U}' \leq n+1$. Since $\mathcal{Q} \cup \mathcal{U}'$ is an open refinement of $\mathcal{U}$, we conclude that $\dim X \leq n$.

In a similar way we can easily show the following assertion which was first proved by P. S. Alexandroff.

B) For every compact T_2-space X. $\dim X \leq \operatorname{ind} X$.

On the other hand I. M. Leĭbo [1] proved that $\dim X = \operatorname{Ind} X$ holds whenever X is the image of a metric space by a closed continuous mapping and also that $\dim X = \operatorname{Ind} X = \operatorname{ind} X$ holds whenever X is the image of a separable metric space by a closed continuous mapping. B. Pasynkov [2], [5] also obtained important results in this aspect. In [2] he proved coincidence of the three dimension functions for every locally compact group. [7] In the following we shall discuss a remarkable result obtained in [5].

C) Let f be a continuous mapping from a normal space X onto a metric space Y and g a continuous mapping from Y onto a metric space Z such that $\varphi = g \cdot f$ is a closed mapping satisfying $\dim \varphi^{-1}(z) \leqq 0$ for all $z \in Z$. If $\dim Y \leqq n$, then $\operatorname{Ind} X \leqq n$.

Proof. The proof will be carried out by induction on n.

If $n = -1$, then the proposition is obviously true.

Assume that it is true if $\dim Y \leqq n-1$. First we remark that for each $z \in Z$ and each open neighbourhood U of $\varphi^{-1}(z)$ in X, the set $X - f(X-U)$ is a neighbourhood of $g^{-1}(z)$ in Y (i.e. the interior of the former set contains the latter), because the mapping φ is closed and

$$g^{-1}(Z - \varphi(X - U)) \subset X - f(X - U) .$$

Now, let F and G be given disjoint closed sets in X. Then for each $z \in Z$ there are open sets U_z and V_z in X such that

$$(1) \qquad \varphi^{-1}(z) \subset U_z \cup V_z \, , \; U_z \cap V_z = \emptyset \, , \; U_z \cap G = \emptyset \, , \; V_z \cap F = \emptyset \, .$$

because $\dim \varphi^{-1}(z) \leqq 0$. Put $W_z = U_z \cup V_z$. The remark above implies that $Q_z = Y - f(X - W_z)$ is a neighbourhood of $g^{-1}(z)$ in Y.

Hence there is a locally finite open covering $\mathcal{Q}$ of Y such that $\mathcal{Q} < \{ Q_z \mid z \in Z \}$ and

$$(2) \qquad \dim B(Q) \leqq n-1 \quad \text{for all} \quad Q \in \mathcal{Q} ,$$

because $\dim Y \leqq n$ for the metric space Y. Let $Q \in \mathcal{Q}$; then $Q \subset Q_z$ for some $z \in Z$. This implies that $f^{-1}(Q) \subset W_z$.

[7] In fact he proved more, namely $\dim X = \operatorname{Ind} X = \operatorname{ind} X = \operatorname{ind} G - \operatorname{ind} H$ if X is the factor space G/H of a locally compact group G by a closed subgroup H. K. Nagami [5] obtained a similar result. See also P. Alexandroff - V. Ponomarev [1] and V. V. Fedorčuk [3] for conditions implying the coincidence of different dimension functions.

Now define

$$(3) \qquad U'(Q) = U_z \cap f^{-1}(Q) \quad \text{and} \quad V'(Q) = V_z \cap f^{-1}(Q) .$$

Observe that

$$(4) \qquad f(B(f^{-1}(Q))) \subset B(Q) .$$

Note that $\{U'(Q) \mid Q \in \mathcal{Q}\}$ is a locally finite open collection covering F and that $U'(Q) \cap G = \emptyset$ for all $Q \in \mathcal{Q}$.

Put $P = \bigcup \{U'(Q) \mid Q \in \mathcal{Q}\}$.

Then P is an open set of X such that $F \subset P \subset X - G$ and

$$B(P) \subset \bigcup \{B(U'(Q)) \mid Q \in \mathcal{Q}\} \subset \bigcup \{B(f^{-1}(Q)) \mid Q \in \mathcal{Q}\} .$$

The last relation follows from the local finiteness of $\{U'(Q) \mid Q \in \mathcal{Q}\}$ and from the fact that $B(U'(Q)) \subset B(f^{-1}(Q))$, which easily follows from (1) and (3). Hence

$$(5) \qquad f(B(P)) \subset f(\bigcup \{B(f^{-1}(Q)) \mid Q \in \mathcal{Q}\}) \subset$$
$$\subset \bigcup \{f(B(f^{-1}(Q))) \mid Q \in \mathcal{Q}\} \subset \bigcup \{B(Q) \mid Q \in \mathcal{Q}\} = Q' .$$

The last part of the relation follows from (4). Since $\mathcal{Q}$ is locally finite, $\dim Q' \leqq n-1$ follows from (2) by use of the sum theorem. Thus $\dim f(B(P)) \leqq n-1$ is implied by (5). Hence by the induction hypothesis we obtain $\operatorname{Ind} B(P) \leqq n-1$. Therefore $\operatorname{Ind} X \leqq n$.

D) Let $\mathcal{U}$ be a locally finite open covering of a normal space X with $\dim X \leqq n$. Then $\mathcal{U}$ can be shrunk to an open covering $\mathcal{V}$ such that there is a locally finite open covering $\mathcal{V}'$ each of whose members intersects at most $n+1$ members of $\mathcal{V}$.

Proof. Use I.1 A) and the technique of the proof of Theorem II.6 to shrink $\mathcal{U}$ to $\bar{\mathcal{V}}$ where $\mathcal{V}$ is an open covering such that $\operatorname{ord} \bar{\mathcal{V}} \leqq n+1$. Suppose

$$\mathcal{U} = \{U_\alpha \mid \alpha \in A\} \quad \text{and} \quad \mathcal{V} = \{V_\alpha \mid \alpha \in A\} ,$$

where $\bar{V}_\alpha \subset U_\alpha$. Then for each $x \in X$ we define

$$V(x) = (\bigcap \{U_\alpha \mid x \in \bar{V}_\alpha\}) \cap (\bigcap \{X - \bar{V}_\alpha \mid x \notin \bar{V}_\alpha\}) .$$

Then $\mathcal{V}' = \{V(x) \mid x \in X\}$ satisfies the desired condition.

E) Let $\mathcal{U}$ be a locally finite open covering of a normal space X. Then there is a locally finite open covering $\mathcal{A}$ such that $\mathcal{A}^* < \mathcal{U}$

Proof. Assume $\mathcal{U} = \{ U_\alpha \mid \alpha \in A \}$. Then find an open covering $\{ V_\alpha \mid \alpha \in A \}$ satisfying $\bar{V}_\alpha \subset U_\alpha$. For each finite subset B of A we put

$$W(B) = (\cap \{ U_\alpha \mid \alpha \in B \}) \cap (\cap \{ X - \bar{V}_\alpha \mid \alpha \in A - B \}), \text{ and}$$
$$\mathcal{W} = \{ W(B) \mid B \text{ is a finite subset of } A \}.$$

Then $\mathcal{W}$ is a locally finite open covering such that $\mathcal{W}^\Delta < \mathcal{U}$. By repeating the same argument we construct a locally finite open covering $\mathcal{A}$ such that $\mathcal{A}^\Delta < \mathcal{W}$. Then $\mathcal{A}$ is the desired covering.

Theorem VII. 4. *Let* φ *be a closed continuous mapping from a normal space* X *onto a metric space* Z *such that* $\dim \varphi^{-1}(z) \leqq 0$ *for all* $z \in Z$. *Then* $\dim X =$ Ind X.

Proof. Since $\dim X \leqq \operatorname{Ind} X$ is proved in A), we shall prove $\operatorname{Ind} X \leqq \dim X$. Assume $\dim X \leqq n$. Let $\mathcal{W}_1, \mathcal{W}_2, \dots$ be a sequence of locally finite open coverings of Z such that

$$\mathcal{W}_1 > \mathcal{W}_2^* > \mathcal{W}_2 > \mathcal{W}_3^* > \dots \text{ and } \operatorname{mesh} \mathcal{W}_i \to 0 \text{ as } i \to \infty .$$

Put

$$\mathcal{U}_i = f^{-1}(\mathcal{W}_i) = \{ f^{-1}(W) \mid W \in \mathcal{W}_i \} ;$$

then each $\mathcal{U}_i$ is a locally finite open covering of X. Since $\dim X \leqq n$, by use of D) we can shrink $\mathcal{U}_1$ to an open covering $\mathcal{V}_1$ and construct a locally finite open covering $\mathcal{V}_1'$ each of whose members meets at most $n+1$ members of $\mathcal{V}_1$. By E) there is a locally finite open covering $\mathcal{A}$ of X such that

$$\mathcal{A}^* < \mathcal{V}_1 \wedge \mathcal{V}_1' \wedge \mathcal{U}_2 .$$

Shrink $\mathcal{A}$ to an open covering $\mathcal{V}_2$ and construct a locally finite open covering $\mathcal{V}_2'$ each of whose members meets at most $n+1$ members of $\mathcal{V}_2$. Then $\mathcal{V}_2^* < \mathcal{V}_1 \wedge \mathcal{U}_2$, and each member of $\mathcal{V}_2$ intersects at most $n+1$ members of $\mathcal{V}_1$. Repeating the same process we get a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of locally finite open coverings of X such that

$$V_1 > V_2^* > V_2 > V_3^* \ldots, \quad V_i < U_i \quad \text{for} \quad i = 1,2,\ldots ,$$

and each member of V_{i+1} meets at most $n+1$ members of V_i .

Next, let us define the equivalence relation $\sim$ among the points of X by

$$x \sim y \quad \text{if and only if} \quad y \in \bigcap_{i=1}^{\infty} S(x, V_i) .$$

Then each equivalence class is a subset of $\varphi^{-1}(z)$ for some $z \in Z$. Thus we can talk about the natural mapping f from X onto Y , the set of all equivalence classes of X and the natural mapping g from Y onto Z . (Namely f maps $x \in X$ to the class containing x .) Introduce a topology into Y by defining that $\{ S(y, f(V_i)) \mid i = 1,2,\ldots \}$ is a neighbourhood base at each point y of Y . Then it is easy to see that f and g are continuous mappings such that $\varphi = g \cdot f$ and that Y is a metric space with $\dim Y \leqq n$ (by virtue of Theorem V.3). Thus $\operatorname{Ind} X \leqq n$ follows by use of C). Hence $\dim X = \operatorname{Ind} X$.[8]

Negative results in this fields are due to A. L. Lunc [1] and O. V. Lokucievskiĭ [1], who constructed a compact T_2-space R such that $\dim R = 1$ and $\operatorname{ind} R = \operatorname{Ind} R = 2$.[9]

More generally, P. Vopenka [1] showed that for every m, n with $1 \leqq m \leqq n \leqq \infty$, there exist compact T_2-spaces R and S such that $\dim R = m$, $\operatorname{ind} R = n$, $\dim S = m$, and $\operatorname{Ind} S = n$.

On the other hand, C. H. Dowker [3] constructed a normal space R such that $\operatorname{ind} R = 0$ and $\dim R = \operatorname{Ind} R = 1$.

VII. 3. Sum theorem and subspace theorem for Ind

O. V. Lokucievskiĭ's compact T_2-space R which we quoted in the preceding section contains two closed subsets F_1 and F_2 such that

$$R = F_1 \cup F_2, \quad \operatorname{ind} F_1 = \operatorname{Ind} F_1 = 1 , \text{ and } \operatorname{ind} F_2 = \operatorname{Ind} F_2 = 1 .$$

---

[8] In this proof factorization of the mapping φ plays an important role. See S. Mardešić [1] and B. Pasynkov [8] for factorization theorems in dimension theory.

[9] V. Fedorčuk [1] gave a first countable compact T_2-space X with $\dim X < \operatorname{ind} X$.

Thus the finite sum theorem for inductive dimension does not hold even if the space is compact T_2. In the present section we shall prove a sum theorem and a subspace theorem for Ind of totally normal spaces, which are mainly due to C. H. Dowker [2]. In the rest of this section we shall be concerned only with large (strong) inductive dimension.

A) To prove the subspace theorem for every totally normal space, it suffices to prove the same for all open subsets.

Proof. Assume that $\operatorname{Ind} U \leqq \operatorname{Ind} X$ holds for every open set U of every totally normal space X. To prove the theorem in general, we use induction on $n = \operatorname{Ind} X$. If $n = -1$, then it is obviously true. Assume that the theorem has been proved for every $(n-1)$-dimensional space. Now, let $\operatorname{Ind} X = n$, and Y be an arbitrary subset of X. Suppose F and G are disjoint closed sets of Y. Then $U = X - \bar{F} \cap \bar{G}$ is an open set of X, and hence $\operatorname{Ind} U \leqq n$. Since $F \cap U$ and $G \cap U$ are disjoint closed sets of U, there is an open set V of U such that

$$F \cap U \subset V \subset U - G \cap U \quad \text{and} \quad \operatorname{Ind} B_U(V) \leqq n-1 \text{ .}^{10}$$

Put $W = V \cap Y$. Then W is an open set of Y such that

$$F \subset W \subset Y - G \quad \text{and} \quad B_Y(W) \subset B_U(V) \text{ .}$$

Hence by the induction hypothesis we get $\operatorname{Ind} B_Y(W) \leqq n-1$, which proves $\operatorname{Ind} Y \leqq n$.

B) Let X be a hereditarily normal space. If F is a closed subset of X such that $\operatorname{Ind} F \leqq n$ and $\operatorname{Ind} (X-F) \leqq n$, then $\operatorname{Ind} X \leqq n$.

Proof. For $n = -1$ the assertion is clearly true. Assume its validity for every F satisfying $\operatorname{Ind} F \leqq n-1$ and $\operatorname{Ind} (X-F) \leqq n-1$. Let G and H be disjoint closed sets of X. Since $\operatorname{Ind} F \leqq n$, there exists an open set U of F such that $F \cap G \subset U \subset \bar{U}^F \subset F - H$ [11] and

(1) $\qquad \operatorname{Ind} (\bar{U}^F - U) \leqq n-1$.

[10] $B_U(.)$ denotes the boundary in the subspace U.

[11] By $\bar{U}^F$ we denote the closure of U in the subspace F.

Then $G \cup U$ and $H \cup (F - \bar{U}^F)$ satisfy

$$\overline{G \cup U} \cap (H \cup (F - \bar{U}^F)) = \emptyset \quad \text{and} \quad (G \cup U) \cap \overline{H \cup (F - \bar{U}^F)} = \emptyset .$$

Hence, by virtue of the hereditary normality of X, and I.1 B), there exist open sets V and W such that

$$G \cup U \subset V , \; H \cup (F - \bar{U}^F) \subset W , \; V \cap W = \emptyset , \text{ and}$$
$$\bar{V} \cap \bar{W} \subset \overline{G \cup U} \cap \overline{H \cup (F - \bar{U}^F)} = \bar{U} \cap \overline{F - \bar{U}^F} \subset \bar{U}^F - U .$$

Thus $\bar{V} \cap (X - F)$ and $\bar{W} \cap (X - F)$ are disjoint closed sets of $X - F$. Since $\operatorname{Ind} (X - F) \leqq n$, there exists an open set P of $X - F$ such that $\bar{V} \cap (X - F) \subset P \subset (X - F) - \bar{W}$ and

(2) $\qquad \operatorname{Ind} (\bar{P}^{X - F} - P) \leqq n - 1 .$

Now, put $Q = V \cup P$; then Q is an open set of X which satisfies $G \subset Q \subset X - H$ and

(3) $\qquad B(Q) \subset (\bar{P}^{X - F} - P) \cup (\bar{U}^F - U) = S ;$

$\bar{U}^F - U$ is a closed subset of S such that

$$\bar{P}^{X - F} - P = S - (\bar{U}^F - U) .$$

Using the induction hypothesis we deduce from (1) and (2) that $\operatorname{Ind} S \leqq n - 1$. Therefore, by (3) and I D) we conclude that $\operatorname{Ind} B(Q) \leqq n - 1$. Hence $\operatorname{Ind} X \leqq n$.

C) Let X be a hereditarily normal space and

$$X = Y_0 \supset Y_1 \supset Y_2 \supset \dots$$

be a sequence of open sets such that $\bigcap_{i=1}^{\infty} Y_i = \emptyset$. If $\operatorname{Ind} (Y_{i-1} - Y_i) \leqq n$ for $i = 1, 2, \dots$, then $\operatorname{Ind} X \leqq n$.

Proof. This proposition can be proved by arguing similarly as in the proof of B). The proposition is obviously true if $n = -1$. Assume that the proposition is true for $(n - 1)$-dimensional sets. Let G and H be disjoint closed sets of X. Since $\operatorname{Ind} (Y_0 - Y_1) \leqq n$, there is an open set U_1 of $Y_0 - Y_1$ such that

$$G \cap (Y_0 - Y_1) \subset U_1 \subset (Y_0 - Y_1) - H \quad \text{and} \quad \operatorname{Ind} B_{Y_0 - Y_1}(U_1) \leqq n-1 .$$

Find open sets V_1 and W_1 in X such that

$$G \cup U_1 \subset V_1 , H \cup [(Y_0 - Y_1) - \bar{U}_1^{Y_0 - Y_1}] \subset W_1 ,$$
$$V_1 \cap W_1 = \emptyset , \text{ and } \bar{V}_1 \cap \bar{W}_1 \subset B_{Y_0 - Y_1}(U_1) .$$

Since $\operatorname{Ind} (Y_1 - Y_2) \leqq n$, there is an open set U_2 of $Y_1 - Y_2$ such that

$$\bar{V}_1 \cap (Y_1 - Y_2) \subset U_2 \subset (Y_1 - Y_2) - \bar{W}_1 \quad \text{and} \quad \operatorname{Ind} B_{Y_1 - Y_2}(U_2) \leqq n-1 .$$

Find open sets V_2 and W_2 in X such that

$$V_1 \cup U_2 \subset V_2 , W_1 \cup [(Y_1 - Y_2) - \bar{U}_2^{Y_1 - Y_2}] \subset W_2 ,$$
$$V_2 \cap W_2 = \emptyset , \text{ and } \bar{V}_2 \cap \bar{W}_2 \subset B_{Y_0 - Y_1}(U_1) \cup B_{Y_1 - Y_2}(U_2) .$$

Continuing the same process we get sequences $\{U_i\}$, $\{V_i\}$, $\{W_i\}$, where U_i is open in $Y_{i-1} - Y_i$, and V_i and W_i are open in X satisfying

$$\bar{V}_{i-1} \cap (Y_{i-1} - Y_i) \subset U_i \subset (Y_{i-1} - Y_i) - \bar{W}_{i-1} ,$$
$$\operatorname{Ind} B_{Y_{i-1} - Y_i}(U_i) \leqq n-1 ,$$
$$V_{i-1} \cup U_i \subset V_i , W_{i-1} \cup [(Y_{i-1} - Y_i) - \bar{U}_i^{Y_{i-1} - Y_i}] \subset W_i ,$$
$$V_i \cap W_i = \emptyset , \text{ and } \bar{V}_i \cap \bar{W}_i \subset \bigcup_{j=1}^{i} B_{Y_{j-1} - Y_j}(U_j) .$$

Then put $U = \bigcup_{i=1}^{\infty} U_i$. Now it is easy to verify that U is an open set such that

$$G \subset U \subset X - H \quad \text{and} \quad B(U) = \bigcup_{i=1}^{\infty} B_{Y_{i-1} - Y_i}(U_i) .$$

Thus by use of the induction hypothesis we obtain $\operatorname{Ind} B(U) \leqq n-1$. Hence $\operatorname{Ind} X \leqq n$.

D) Let U be an open set of a totally normal space X. Then U has coverings P and Q which are both σ-discrete in U and consist of closed sets of X such that P° shrinks to Q, where $P^{\circ} = \{P^{\circ} \mid P \in P\}$. ($P^{\circ}$ denotes the interior of P.)

Proof. By the definition of totally normal space U has an open covering $U = \{U_\alpha \mid \alpha \in A\}$ which is locally finite in U, and each U_α is an F_σ-set of X. Thus each U_α can be expressed as

$$U_\alpha = \bigcup_{i=1}^{\infty} W_{\alpha i}, \quad \bar{W}_{\alpha i} \subset W_{\alpha i+1},$$

where $W_{\alpha i}$ are open sets of X. We may assume that the members of U are well-ordered, i.e. $U = \{U_\alpha \mid 0 \leqq \alpha < \tau\}$.

Now define

$$P_{\alpha i} = \bar{W}_{\alpha i} - \bigcup_{\beta < \alpha} W_{\beta i+1}, \quad Q_{\alpha i} = \bar{W}_{\alpha i-1} - \bigcup_{\beta < \alpha} W_{\beta i+2},$$

$$P_i = \{P_{\alpha i} \mid 0 \leqq \alpha < \tau\}, \quad Q_i = \{Q_{\alpha i} \mid 0 \leqq \alpha < \tau\}.$$

Then each P_i is discrete in U and consists of closed sets of X, and the same is true for Q_i, too, while $Q_{\alpha i} \subset P^{\circ}_{\alpha i}$. Thus $P = \bigcup_{i=1}^{\infty} P_i$ and $Q = \bigcup_{i=1}^{\infty} Q_i$ are coverings as desired.

E) Assume that $\{F_\alpha \mid \alpha \in A\}$ is a discrete covering of X. (Thus each F_α is closed and open in X.) If $\operatorname{Ind} F_\alpha \leqq n$ for all $\alpha \in A$, then $\operatorname{Ind} \bigcup \{F_\alpha \mid \alpha \in A\} \leqq n$.

Proof. Obvious. (Use induction on n.)

F)$_n$ Let X be a totally normal space and $\{F_i \mid i = 1, 2, \dots\}$ a closed covering of X such that $\operatorname{Ind} F_i \leqq n$, $i = 1, 2, \dots$. Then $\operatorname{Ind} X \leqq n$.

G)$_n$ Let U be an open subset of a totally normal space X with $\operatorname{Ind} X \leqq n$. Then $\operatorname{Ind} U \leqq n$.

Proof of F)$_n$ *and* G)$_n$. We will prove the two propositions simultaneously by induction on n.

i) F)$_{-1}$ and G)$_{-1}$ are obviously true.

ii) F)$_{n-1}$ implies G)$_n$.

First observe that F)$_{n-1}$ implies the following assertion

(1) Let $\{F_i \mid i = 1, 2, \dots\}$ be a closed covering of a totally normal space X and $\{D_i \mid i = 1, 2, \dots\}$ a closed covering of X such that $F_i \subset D^{\circ}_i$, $i = 1, 2, \dots$. If $\operatorname{Ind} D_i \leqq n$, $i = 1, 2, \dots$, then $\operatorname{Ind} X \leqq n$.

Proof of the assertion (1) is as follows. Let G and H be disjoint closed sets of X. For each i we can find open sets V_{ik}, W_{ik}, $k=1,2,\dots$ such that

$$G\cap F_i\subset V_{ik}\subset\bar V_{i\,k+1}\subset D_i^\bullet - H\ ,$$
$$H\cap F_i\subset W_{ik}\subset\bar W_{i\,k+1}\subset D_i^\bullet - G\ ,$$
$$\text{Ind } B(V_{ik})\leqq n-1 \quad\text{and}\quad \text{Ind } B(W_{ik})\leqq n-1\ ,$$

because $\text{Ind } D_i\leqq n$. Put

$$V_k = X - \bar V_{1k}\cup\dots\cup\bar V_{kk} \quad\text{and}\quad W_k = X-\bar W_{1k}\cup\dots\cup\bar W_{kk}\ .$$

Then V_k and W_k are open sets such that

$$V_k\supset\bar V_{k+1}\supset H\ ,\ G\cap(\bigcap_{k=1}^{\infty}\bar V_k)=\emptyset\ ,$$
$$W_k\supset\bar W_{k+1}\supset G\ ,\ H\cap(\bigcap_{k=1}^{\infty}W_k)=\emptyset\ ,$$
$$B(V_k)\subset\bigcup_{i=1}^{k}B(V_{ik})\ ,\ B(W_k)\subset\bigcup_{i=1}^{k}B(W_{ik})\ .$$

Hence by $\text{F})_{n-1}$ we obtain $\text{Ind } B(V_k)\leqq n-1$ and $\text{Ind } B(W_k)\leqq n-1$. Now define

$$P=(W_1-\bar V_1)\cup(W_2-\bar V_2)\cup\dots\ .$$

Then P is an open set such that

$$G\subset P\subset X-H \quad\text{and}\quad B(P)\subset\bigcup_{k=1}^{\infty}(B(V_k)\cup B(W_k))\ .$$

(A somewhat similar technique was used to prove II.1 E).) Hence by $\text{F})_{n-1}$ and 1 D) we get $\text{Ind } B(P)\leqq n-1$. Thus $\text{Ind } X\leqq n$, i.e. (1) is proved.

Now, to prove ii) we assume that U is an open set of X with $\text{Ind } X\leqq n$. Let $\mathcal{P}=\bigcup_{i=1}^{\infty}\mathcal{P}_i$ and $\mathcal{Q}=\bigcup_{i=1}^{\infty}\mathcal{Q}_i$ be the coverings of U obtained in D), where $\mathcal{P}_i=\{P_\alpha\mid\alpha\in A_i\}$ and $\mathcal{Q}_i=\{Q_\alpha\mid\alpha\in A_i\}$ are discrete in U and satisfy $Q_\alpha\subset P_\alpha^\bullet$. By 1 D) $\text{Ind } P_\alpha\leqq n$, because P_α is closed in X. Thus by E) $\text{Ind } P_i\leqq n$, where $P_i=\cup\{P_\alpha\mid\alpha\in A_i\}$. If we write $Q_i=\cup\{Q_\alpha\mid\alpha\in A_i\}$, then P_i and Q_i are closed sets in U such that $Q_i\subset P_i^\bullet$. Thus from (1) it follows that $\text{Ind } U\leqq n$, which proves ii).

iii) $\text{G})_n$ implies $\text{F})_n$.

Put

$$Y_0 = X\ ,\ Y_i = X - (F_1 \cup \ldots \cup F_i)\ ,\ i = 1, 2, \ldots\ .$$

Then Y_i are open sets with

$$Y_0 \supset Y_1 \supset Y_2 \supset \ldots \quad \text{and} \quad \bigcap_{i=1}^{\infty} Y_i = \emptyset\ .$$

Now $Y_{i-1} - Y_i = F_i - (F_1 \cup \ldots \cup F_{i-1})$ is an open subset of F_i. Since $\operatorname{Ind} F_i \leqq n$, it follows from G)$_n$ that $\operatorname{Ind} (Y_{i-1} - Y_i) \leqq n$. Thus by C) we get $\operatorname{Ind} X \leqq n$, which proves F)$_n$.

iv) Now combine i), ii), iii) to establish F)$_n$ and G)$_n$ for all integers $n \geqq -1$.

From A), F)$_n$ and G)$_n$ we obtain the following theorems.

Theorem VII. 5. *Let* $\{ F_i \mid i = 1, 2, \ldots \}$ *be a closed covering of a totally normal space* X *such that* $\operatorname{Ind} F_i \leqq n$, $i = 1, 2, \ldots$. *Then* $\operatorname{Ind} X \leqq n$.

Theorem VII. 6. *Let* Y *be a subset of a totally normal space* X . *Then* $\operatorname{Ind} Y \leqq \operatorname{Ind} X$.

The following statement is another sum theorem whose proof is similar to that of C).

Theorem VII. 7. *Let* $\{ F_\alpha \mid \alpha \in A \}$ *be a locally finite closed covering of a totally normal space* X *such that* $\operatorname{Ind} F_\alpha \leqq n$ *for all* $\alpha \in A$. *Then* $\operatorname{Ind} X \leqq n$.

Proof. Well-order the given covering as $\{ F_\alpha \mid 0 \leqq \alpha \leqq \tau \}$. We shall prove the theorem by induction on n .

It is obviously true if $n = -1$.

Assume its validity in the $(n-1)$-dimensional case. Let G and H be given disjoint closed sets of X . Put

$$F(\alpha) = \cup \{ F_\beta \mid 0 \leqq \beta \leqq \alpha \}\ ,\ 0 \leqq \alpha \leqq \tau\ .$$

Then $F(\alpha)$ is a closed set. We shall construct sets U_α for $0 \leqq \alpha \leqq \tau$ by use of induction on α such that

$(1)_\alpha$ U_α is an open set in $F(\alpha)$,

$(2)_\alpha$ $G \cap F(\alpha) \subset U_\alpha$, $H \cap F(\alpha) \subset F(\alpha) - \bar{U}_\alpha^{F(\alpha)}$,

$(3)_\alpha$ $\operatorname{Ind} B_{F(\alpha)}(U_\alpha) \leqq n-1$,

$(4)_\alpha$ for every $\beta < \alpha$, $U_\beta = U_\alpha \cap F(\beta)$, $B_{F(\beta)}(U_\beta) = B_{F(\alpha)}(U_\alpha) \cap F(\beta)$.

Since $\operatorname{Ind} F_0 \leqq n$, we can choose U_0 satisfying these conditions for $\alpha = 0$. Assume that U_β has been defined for all $\beta < \alpha$. Define $F = \cup \{ F(\beta) \mid \beta < \alpha \}$; then F is a closed set. Put

(5) $C = G \cup (\bigcup_{\beta < \alpha} U_\beta)$,

(6) $B = H \cup (\cup \{ F(\beta) - \bar{U}_\beta^{F(\beta)} \mid \beta < \alpha \})$.

Then $(\bar{C} \cap B) \cup (C \cap \bar{B}) = \emptyset$, because $\{ F_\alpha \}$ is a locally finite collection of closed sets, and $(1)_\beta$, $(2)_\beta$ and $(4)_\beta$ hold for all $\beta < \alpha$. Since X is hereditarily normal, there are open sets P and Q of X such that

(7) $C \subset P$, $B \subset Q$,

(8) $P \cap Q = \emptyset$, $\bar{P} \cap \bar{Q} \subset \bar{C} \cap \bar{B}$.

Since F is closed, we have $\bar{C} \cap \bar{B} \subset F$, which implies $\bar{P} \cap \bar{Q} \subset F$.
On the other hand it follows from Theorem VII.6 that $\operatorname{Ind} (F_\alpha - F) \leqq n$, because $\operatorname{Ind} F_\alpha \leqq n$. Thus we can find out an open set V of $F_\alpha - F$ such that

(9) $\bar{P} \cap (F_\alpha - F) \subset V$,

(10) $\bar{V}^{F_\alpha - F} \subset (F_\alpha - F) - \bar{Q}$,

(11) $\operatorname{Ind} B_{F_\alpha - F}(V) \leqq n-1$.

Then put

(12) $U_\alpha = (\bigcup_{\beta < \alpha} U_\beta) \cup V$.

Now, $(1)_\alpha$ follows from $(1)_\beta$ $(4)_\beta$ for $\beta < \alpha$, (5), (7), (9) and (12). $(2)_\alpha$ follows from $(2)_\beta$ for $\beta < \alpha$, (5), (6), (7), (9), (10) and (12). $(4)_\alpha$ follows from (12), $(4)_\beta$ for $\beta < \alpha$, (6), (7) and (10).
To prove $(3)_\alpha$, we express the boundary of U_α in $F(\alpha)$ as

$$B_{F(\alpha)}(U_\alpha) = B_1 \cup B_2 ,$$

where

$$B_1 = B_{F(\alpha)}(U_\alpha) \cap (F(\alpha) - \tilde{F}) \quad \text{and} \quad B_2 = B_{F(\alpha)}(U_\alpha) \cap F .$$

Note that by $(4)_\beta$ for $\beta < \alpha$,

$$B_2 = \cup \{ B_{F(\beta)}(U_\beta) \mid \beta < \alpha \} = \cup \{ B_{F(\beta)}(U_\beta) \cap F_\beta \mid \beta < \alpha \} .$$

Thus by use of the induction hypothesis and $(3)_\beta$ for $\beta < \alpha$, we obtain

(13) $\operatorname{Ind} B_2 \leqq n - 1$.

Further note that $B_1 = B_{F_\alpha - F}(V)$. Thus from (11) it follows that $\operatorname{Ind} B_1 \leqq n - 1$, which combined with (13) implies that

$$\operatorname{Ind} B_{F(\alpha)}(U_\alpha) \leqq n - 1 ,$$

i.e. $(3)_\alpha$ is proved. Thus the induction process is complete, and so we can eventually construct U_τ . Since U_τ satisfies $(1)_\tau - (3)_\tau$ and since $F(\tau) = X$, U_τ is an open set of X such that

$$G \subset U_\tau \subset X - H \quad \text{and} \quad \operatorname{Ind} B(U_\tau) \leqq n - 1 .$$

Thus we conclude that $\operatorname{Ind} X \leqq n$.

Corollary. *Let* $\{ F_\alpha \mid \alpha \in A \}$ *be a locally countable closed covering of a hereditarily paracompact* T_2*-space* X . *If* $\operatorname{Ind} F_\alpha \leqq n$ *for all* $\alpha \in A$, *then* $\operatorname{Ind} X \leqq n$.

Proof. Combine Theorems VII.5 and VII.7.[12]

---

[12] We have proved sum theorems for dim and Ind under considerably general conditions. On the other hand another dimension, ind, behaves very badly in this respect. E. van Douwen [1] and T. Przymusiński [1] proved that the finite sum theorem does not hold for ind in the class of complete metric spaces. In fact the former showed that adding a single point could raise ind from 0 to 1 in the class of metric spaces.
As for Katětov-Smirnov's dimension (Definition I.4'), E. Pol [1] proved that there is a Tychonoff space X such that Katětov-Smirnov $\dim X > 0$, and $X = X_1 \cup X_2$ for functionally closed sets X_1 and X_2 with Katětov-Smirnov $\dim X_i = 0$, $i = 1,2$; this fact rather sharply contrasts Theorem VII.1. However, this dimension is useful to some extent when one tries to extend dimension theory beyond normal spaces. See K. Morita [8] for dimension theory in general spaces.

VII. 4. Characterisation of dim by partitions

In the present section we will discuss a generalization of Theorem II.8 to normal spaces (due to E. Hemmingsen and K. Morita).

A) Let $U = \{U_a \mid a = 1,\dots,k\}$ be a finite open collection in a topological space X and $F = \{F_a \mid a = 1,\dots,k\}$ a closed collection such that $F_a \subset U_a$. We define binary coverings V_a, $a = 1,\dots,k$ by $V_a = \{U_a, X - F_a\}$. Then $V = \wedge\{V_a \mid a = 1,\dots,k\}$ is a finite open covering of R satisfying $S(F_a, V) \subset U_a$.

Proof. It is clear that V is a finite open covering. Let

$$V = [\cap\{U_a \mid a \in \gamma\}] \cap [\cap\{X - F_a \mid a \notin \gamma\}]$$

be a given member of V, where γ denotes a subset of $\{1,\dots,k\}$. If $V \cap F_a \neq \emptyset$, then $a \in \gamma$, which implies $V \subset U_a$. Hence we obtain $S(F_a, V) \subset U_a$.

B) Let $U = \{U_a \mid a = 1,\dots,k\}$ be a finite open collection in a normal space X and $F = \{F_a \mid a = 1,\dots,k\}$ a closed collection such that $F_a \subset U_a$. If F is a closed subset of X of dimension $\leqq n$, then there exist open sets V_a and W_a such that

$$F_a \subset V_a \subset \bar{V}_a \subset W_a \subset U_a \quad \text{and} \quad \operatorname{ord}\{F \cap (\bar{W}_a - V_a) \mid a = 1,\dots,k\} \leqq n\,.$$[13]

Proof. Denoting by Γ the family of all finite subsets of $\{1,\dots,k\}$, we let $L = \{L_\gamma \mid \gamma \in \Gamma\}$, where

$$(1) \qquad L_\gamma = [\cap\{U_a \mid a \in \gamma\}] \cap [\cap\{X - F_a \mid a \notin \gamma\}]\,.$$

Since $\dim F \leqq n$, we obtain open coverings

$$M = \{M_\gamma \mid \gamma \in \Gamma\} \quad \text{and} \quad N = \{N_\gamma \mid \gamma \in \Gamma\}$$

such that

$$(2) \qquad \bar{N}_\gamma \subset M_\gamma \subset L_\gamma \quad \text{and} \quad \operatorname{ord}\{F \cap M_\gamma \mid \gamma \in \Gamma\} \leqq n+1\,.$$

[13] We can prove this theorem for every locally finite open covering U in an analogous way, by applying Theorem II.6 to the normal space X. (Theorem II.6 is valid for every normal space R.)

Define $\delta(\gamma) = \{ a \mid F_a \cap M_\gamma \neq \emptyset \}$; then $\delta(\gamma) \subset \gamma$. In order to show this, suppose a is an element of $\delta(\gamma)$. Then $F_a \cap M_\gamma \neq \emptyset$, which implies by (2) $F_a \cap L_\gamma \neq \emptyset$. Hence from (1) it follows that $a \in \gamma$, which proves $\delta(\gamma) \subset \gamma$.

By virtue of the normality of X we can define open sets $P_{\gamma a}$, $Q_{\gamma a}$ for each $\gamma \in \Gamma$ and $a \in \delta(\gamma)$ such that

(3) $$\bar{N}_\gamma \subset P_{\gamma a} \subset \bar{P}_{\gamma a} \subset Q_{\gamma a} \subset \bar{Q}_{\gamma a} \subset P_{\gamma a'} \subset M_\gamma \quad \text{if } a, a' \in \delta(\gamma) \text{ and } a < a' .$$

Now, for a fixed a with $1 \leqq a \leqq k$ we put

(4) $$V_a = \cup \{ P_{\gamma a} \mid \gamma \in \Gamma \ , \ a \in \delta(\gamma) \} ,$$

(5) $$W_a = \cup \{ Q_{\gamma a} \mid \gamma \in \Gamma \ , \ a \in \delta(\gamma) \} .$$

It follows from (3) and (4) that $F_a \subset S(F_a, N) \subset V_a$, because $F_a \cap N_\gamma \neq \emptyset$ implies $F_a \cap M_\gamma \neq \emptyset$, and hence $N_\gamma \subset P_{\gamma a}$ and $a \in \delta(\gamma)$. From (3) and (5) we obtain $W_a \subset S(F_a, M)$, because $a \in \delta(\gamma)$ implies $F_a \cap M_\gamma \neq \emptyset$, and hence $Q_{\gamma a} \subset M_\gamma \subset S(F_a, M)$. On the other hand, from (2) and A), it follows that

$$S(F_a, M) \subset S(F_a, L) \subset U_a .$$

Therefore $W_a \subset U_a$. Moreoever, from (3), (4) and (5), we obtain

$$\bar{V}_a = \cup \{ \bar{P}_{\gamma a} \mid \gamma \in \Gamma \ , \ a \in \delta(\gamma) \} \subset W_a .$$

Thus it remains only to prove

$$\operatorname{ord} \{ F \cap (\bar{W}_a - V_a) \mid a = 1, \dots, k \} \leqq n .$$

We suppose a_i , $i = 1, \dots, n+1$ are given distinct numbers between 1 and k . From (5) we obtain

$$\bar{W}_{a_i} = \cup \{ \bar{Q}_{\gamma a_i} \mid \gamma \in \Gamma \ , \ a_i \in \delta(\gamma) \} ,$$

and hence

$$\begin{aligned} \bar{W}_{a_i} - V_{a_i} &= \cup \{ \bar{Q}_{\gamma a_i} \mid \gamma \in \Gamma \ , \ a_i \in \delta(\gamma) \} - \cup \{ P_{\gamma a_i} \mid \gamma \in \Gamma , \ a_i \in \delta(\gamma) \} \\ &\subset \cup \{ \bar{Q}_{\gamma a_i} - P_{\gamma a_i} \mid \gamma \in \Gamma , a_i \in \delta(\gamma) \} . \end{aligned}$$

Therefore to prove

$$F \cap [\bigcap_{i=1}^{n+1} (\bar{W}_{a_i} - V_{a_i})] = \emptyset$$

it suffices to show

(6) $$F \cap [\bigcap_{i=1}^{n+1} (\bar{Q}_{\gamma_i a_i} - P_{\gamma_i a_i})] = \emptyset$$

for every choice of $n+1$ indices $\gamma_1, \dots, \gamma_{n+1}$. If $\gamma_i = \gamma_j$ for some distinct i, j, in view of (3), we may assume

$$\bar{Q}_{\gamma_i a_i} \subset P_{\gamma_j a_j} .$$

Hence

$$F \cap (\bar{Q}_{\gamma_i a_i} - P_{\gamma_i a_i}) \cap (\bar{Q}_{\gamma_j a_j} - P_{\gamma_j a_j}) = \emptyset$$

which implies (6). If $\gamma_1, \dots, \gamma_{n+1}$ are all different, we assume the contrary of (6). Then we select a point

(7) $$p \in F \cap [\bigcap_{i=1}^{n+1} (\bar{Q}_{\gamma_i a_i} - P_{\gamma_i a_i})] ,$$

which implies

$$p \notin P_{\gamma_i a_i} \supset N_{\gamma_i} , \quad i = 1, \dots, n+1 .$$

Since $\mathcal{N} = \{ N_\gamma \mid \gamma \in \Gamma \}$ covers X, there exists N_γ which contains p, and in consequence $\gamma \neq \gamma_i$, $i = 1, \dots, n+1$. Thus by (2) we obtain

(8) $$p \in F \cap M_\gamma .$$

On the other hand, from (7) combined with (3), we obtain

(9) $$p \in F \cap [\bigcap_{i=1}^{n+1} M_{\gamma_i}] .$$

But (9) combined with (8) contradicts $\operatorname{ord}\{F\cap M_\gamma \mid \gamma\in\Gamma\}\leqq n+1$. Thus in either case we conclude the validity of (6). Therefore

$$\operatorname{ord}\{F\cap(\bar{W}_a - V_a) \mid a=1,\dots,k\}\leqq n ,$$

which completes the proof of this assertion.

Theorem VII. 8. *A normal space X has dimension $\leqq n$ if and only if for every finite open collection $\{U_i \mid i=1,\dots,k\}$ and closed collection $\{F_i \mid i=1,\dots,k\}$ satisfying $F_i\subset U_i$, $i=1,\dots,k$, there exists an open collection $\{V_i \mid i=1,\dots,k\}$ such that $F_i\subset V_i\subset U_i$, $i=1,\dots,k$, and $\operatorname{ord}\{B(V_i) \mid i=1,\dots,k\}\leqq n$.*

Proof. The "only if" part is a direct consequence of B). As for the proof of the "if" part, we refer to the proof of Theorem II.8. Although in that theorem the metrizability was assumed, only the normality was used in the proof.

The following corollary can be deduced from Theorem VII.8 in the same way as in the case of metric spaces.

Corollary. *A normal space X has dimension $\leqq n$ if and only if for every open collection $\{U_i \mid i=1,\dots,n+1\}$ and closed collection $\{F_i \mid i=1,\dots,n+1\}$ satisfying $F_i\subset U_i$, there exists an open collection $\{V_i \mid i=1,\dots,n+1\}$ such that $F_i\subset V_i\subset U_i$, $i=1,\dots,n+1$, $\bigcap_{i=1}^{n+1} B(V_i)=\emptyset$.*[14]

VII. 5. Dimension and mappings

Many theorems on dimension and mappings can be extended to considerably general non-metrizable spaces. In this section we shall prove for non-metrizable spaces only a few of the relevant theorems in Chapter III and leave the reexamination of the remaining ones to the reader.

We take Definition III.1 as the definition of the unstable value of a mapping defined on non-metrizable spaces. Then we obtain the following theorem which runs exactly as Theorem III.1.

---

[14] Generally, let F and G be disjoint sets and C a closed set such that $X-C=U\cup V$ for disjoint open sets U and V satisfying $U\supset F$, $V\supset G$. Then C is called a *partition* between F and G . If W is an open set such that $F\subset W\subset\bar{W}\subset X-G$, then $B(W)$ is a partition between F and G . Thus Theorem VII.8 and its corollary can be stated in terms of partitions in place of $B(V_i)$.

Theorem VII. 9. *A normal space* X *has* $\dim \leqq n$ *if and only if all values of every continuous mapping of* X *into* I^{n+1} *are unstable.*

Proof. As for the "if" part, we can prove its validity using the corollary to Theorem VII.8 by the same argument as in the proof of III.1 B). In that proof we in fact did not use the metrizability of R but only its normality.

As for the "only if" part, we define closed sets F_i and G_i as in the proof of III.1 A). Then we define by use of 4 B) two open sets V_i and W_i, $i = 1, \dots, n+1$ satisfying

$$F_i \subset V_i \subset \bar{V}_i \subset W_i \subset X - G_i \quad \text{and} \quad \bigcap_{i=1}^{n+1} (\bar{W}_i - V_i) = \emptyset .$$

We can construct a continuous function φ_i over X such that $|\varphi_i| \leqq \varepsilon$, $\{x \mid \varphi_i(x) = 0\} \subset \bar{W}_i - V_i$, $\{x \mid \varphi_i(x) = \varepsilon\} = F_i$, and $\{x \mid \varphi_i(x) = -\varepsilon\} = G_i$. The remainder of the proof runs parallel to the metric case.

From Theorem VII.9 we can easily deduce the following theorem which corresponds to Theorem III.2 in the metric case.

Theorem VII. 10. *A normal space* X *has* $\dim \leqq n$ *if and only if for every closed set* C *of* X *and every continuous mapping* f *of* C *into* S^n *there exists a continuous extension of* f *over* X.

Proof. All we have to do is to check in the non-metrizable case the proof of Theorem III.2; this will be left to the reader.

Theorem VII. 11. *A normal space* X *has* $\dim \leqq n$ *if and only if every mapping of* X *into* I^{n+1} *is inessential, where we take the same Definition III.5 for non-metrizable spaces.*

Proof. In fact the proof of Theorem III.5 only uses the normality of R, so we can apply it to the non-metrizable case.

A) Let $\{U_i \mid i = 1, \dots, k\}$ be an open covering of a normal space X. If there exists an open covering $V < \{U_i\}$ with $\operatorname{ord} V \leqq n+1$, then there exists a closed covering $\{F_i \mid i = 1, \dots, k\}$ such that $F_i \subset U_i$ and $\operatorname{ord} \{F_i \mid i = 1, \dots, k\} \leqq n+1$.

Proof. Without loss of generality, we may suppose

$$V = \{V_i \mid i = 1, \dots, k\}, \quad V_i \subset U_i, \quad i = 1, \dots, k .$$

Since X is normal, we can find an open covering $\{W_i \mid i = 1, \dots, k\}$ for which $\bar{W}_i \subset V_i$, $i = 1, \dots, k$. Thus $\{F_i \mid i = 1, \dots, k\}$, $F_i = \bar{W}_i$ is the desired closed covering.

B) Let $\{F_\alpha \mid \alpha<\tau\}$ be a locally finite closed covering of a normal space X and $U = \{U_i \mid i=1,\dots,k\}$ a finite open covering of X. If

i) there exists an open covering U_α of F_α such that $\operatorname{ord} U_\alpha \leqq n+1$ and $U_\alpha < U$,

ii) $\dim F_\alpha \cap F_\beta \leqq n-1$ for $\alpha \neq \beta$,

then there exists an open covering W of X satisfying $\operatorname{ord} W \leqq n+1$, $W<U$.

Proof. We shall define by induction closed sets $G_{\alpha i}$ for $\alpha<\tau$, $i=1,\dots,k$ such that

$$G_{\alpha i} \subset F_\alpha \cap U_i , \quad \bigcup_{i=1}^{k} \bigcup_{\beta \leqq \alpha} G_{\beta i} = \bigcup_{\beta \leqq \alpha} F_\beta , \text{ and}$$

$$\operatorname{ord} \{ \bigcup_{\beta \leqq \alpha} G_{\beta i} \mid i=1,\dots,k \} \leqq n+1 .$$

For $\alpha=0$, we can define G_{0i}, $i=1,\dots,k$ by use of A) combined with condition i).

Let us assume that $G_{\beta i}$, $i=1,\dots,k$ have been constructed for every $\beta<\alpha$. Let

$$G_i = \bigcup_{\beta<\alpha} G_{\beta i} ;$$

then G_i is closed because by virtue of the assumption $\{F_\beta \mid \beta<\alpha\}$ and accordingly $\{G_{\beta i} \mid \beta<\alpha\}$ are locally finite. Hence by the induction hypothesis $\{G_i \mid i=1,\dots,k\}$ is a closed collection of order $\leqq n+1$. By 1 B) there are open sets M_i, $i=1,\dots,k$, for which

$$G_i \subset M_i \subset U_i \text{ and } \operatorname{ord} \{M_i \mid i=1,\dots,k\} \leqq n+1 .$$

It follows from condition i) combined with A) and 1 B) that there exist open sets N_i and closed sets H_i, $i=1,\dots,k$, for which

$$H_i \subset N_i \subset U_i , \quad \operatorname{ord} \{N_i \mid i=1,\dots,k\} \leqq n+1 , \text{ and}$$

$$\bigcup_{i=1}^{k} H_i = F_\alpha . \tag{1}$$

It follows from condition ii) combined with Theorem VII.1 that

$$\dim (F_\alpha \cap (\bigcup_{\beta<\alpha} F_\beta)) \leqq n-1 .$$

Hence we can construct, by use of A) and I B), a finite open collection $\mathcal{W} = \{ W_j \mid j = 1,\dots,l \}$ such that

(2) $$F_\alpha \cap (\bigcup_{\beta < \alpha} F_\beta) \subset \bigcup_{j=1}^{l} W_j ,$$

(3) $$\bar{\mathcal{W}} < \mathcal{U} \wedge [\bigwedge_{i=1}^{k} (\{ M_i , X - G_i \} \wedge \{ N_i , X - H_i \})] ,$$

(4) $$\operatorname{ord} \bar{\mathcal{W}} \leq n .$$

We define closed sets J_i , J and F by

(5) $$J_i = H_i \cap (X - \bigcup_{j=1}^{l} W_j) ,$$

(6) $$J = \bigcup_{i=1}^{k} J_i ,$$

(7) $$F = \bigcup_{\beta < \alpha} F_\beta = \bigcup_{i=1}^{k} G_i .$$

Now, we shall construct closed sets K_j , L_j , $j = 1,\dots,l$, such that

(8) $$\bar{W}_j = K_j \cup L_j , \quad K_j \subset X - \bigcup_{h=1}^{j-1} (K_h \cap L_h) \cup J , \quad L_j \subset X - F .$$

Since by (2), (5), (6) and (7) $F \cap J = \emptyset$, we can find closed sets K_1 and L_1 for which

$$\bar{W}_1 = K_1 \cup L_1 , \quad K_1 \subset X - J , \quad L_1 \subset X - F .$$

Suppose we have defined $K_1,\dots,K_{j-1}$; $L_1,\dots,L_{j-1}$. Then $\bigcup_{h=1}^{j-1} (K_h \cap L_h) \cup J$ and F are disjoint closed sets. Therefore, we can construct closed sets K_j and L_j satisfying (8).

We note that the second formula of (8) implies that

(9) $$(K_h \cap L_h) \cap (K_j \cap L_j) = \emptyset \quad \text{if} \quad h \neq j .$$

We shall prove

(10) $$\operatorname{ord} \{ K_j , L_j \mid j = 1,\dots,l \} \leq n + 1 .$$

Let

$$K = \Big(\bigcap_{p=1}^{r} K_{i_p} \Big) \cap \Big(\bigcap_{q=1}^{s} L_{j_q} \Big) \neq \emptyset .$$

If $\{ i_p \mid p=1,\ldots,r \} \cap \{ j_q \mid q=1,\ldots,s \}$ contains two distinct numbers i_p and $i_{p'}$, then

$$K \subset (K_{i_p} \cap L_{i_p}) \cap (K_{i_{p'}} \cap L_{i_{p'}})$$

which contradicts (9). Hence $\{ i_p \mid p=1,\ldots,r \} \cap \{ j_q \mid q=1,\ldots,s \}$ contains at most one number. Thus it follows from (4) and

$$K \subset \Big(\bigcap_{p=1}^{r} \bar{W}_{i_p} \Big) \cap \Big(\bigcap_{q=1}^{s} \bar{W}_{j_q} \Big) ,$$

which is implied by (8), that $r+s-1 \leqq n$. So we have shown (10).

Now, for $i=1,\ldots,k$, let

$$K'_i = \bigcup \{ K_j \mid K_j \cap G_h = \emptyset \text{ for } h=1,\ldots,i-1 \, ; \, K_j \cap G_i \neq \emptyset \} ,$$

$$L'_i = \bigcup \{ L_j \mid L_j \cap J_h = \emptyset \text{ for } h=1,\ldots,i-1 \, ; \, L_j \cap J_i \neq \emptyset \} .$$

Then it follows from (3), (5) and from $G_i \subset M_i$, $H_i \subset N_i$ that

(11) $$K'_i \cup G_i \subset M_i , \quad L'_i \cup J_i \subset N_i .$$

Furthermore, let us suppose that

$$K_{j_p} \cap F = \emptyset \quad \text{for } p=1,\ldots,r ,$$

$$K_j \cap F \neq \emptyset \quad \text{for } j \neq j_p ,$$

$$L_{j'_q} \cap J = \emptyset \quad \text{for } q=1,\ldots,s ,$$

$$L_{j'} \cap J \neq \emptyset \quad \text{for } j' \neq j'_q .$$

Then we have from (8)

$$(12)\qquad \begin{cases} F\cap J=\emptyset\ , \\ K_{j_p}\cap(F\cup J)=\emptyset\ ,\ L_{j'_q}\cap(F\cup J)=\emptyset\ . \end{cases}$$

We note that by (7) $j\neq j_p$ implies

$$K_j\cap\left(\bigcup_{i=1}^{k} G_i\right)\neq\emptyset\ ,$$

i.e.

$$(13)\qquad K_j\subset K'_i \quad\text{for some}\quad i$$

and that by (6) $j'\neq j'_q$ implies

$$L_{j'}\cap\left(\bigcup_{i=1}^{k} J_i\right)\neq\emptyset\ ,$$

i.e.

$$(14)\qquad L_{j'}\subset L'_{i'}\quad\text{for some}\quad i'\ .$$

We put

$$\mathcal{F}=\{(K'_i\cap F_\alpha)\cup G_i\ ,\ (L'_i\cap F_\alpha)\cup J_i\ ,\ K_{j_p}\cap F_\alpha\ ,\ L_{j'_q}\cap F_\alpha\mid$$
$$i=1,\dots,k\ ;\ p=1,\dots,r\ ;\ q=1,\dots,s\}\ .$$

We shall prove that $\mathcal{F}$ is a closed covering of $\bigcup_{\beta\leqq\alpha}F_\beta$ with $\operatorname{ord}\mathcal{F}\leqq n+1$. From (5) and (1) it follows that

$$\bigcup_{i=1}^{k} J_i=\left(\bigcup_{i=1}^{k} H_i\right)\cap\left(X-\bigcup_{j=1}^{l} W_j\right)=F_\alpha\cap\left(X-\bigcup_{j=1}^{l} W_j\right)\ .$$

From (13), (14) and (8) we deduce that

$$\left(\bigcup_{i=1}^{k} K'_i\right)\cup\left(\bigcup_{p=1}^{r} K_{j_p}\right)\cup\left(\bigcup_{i=1}^{k} L'_i\right)\cup\left(\bigcup_{q=1}^{s} L_{j'_q}\right)\supset\bigcup_{j=1}^{l}(K_j\cup L_j)=\bigcup_{j=1}^{l}\bar{W}_j\ .$$

Thus a given point x of F_α satisfies at least one of the following conditions,

a) $x \in J_i \subset (L'_i \cap F_\alpha) \cup J_i$ for some i,

b) $x \in K'_i$ for some i,

which implies $x \in (K'_i \cap F_\alpha) \cup G_i$ and

c) $x \in K_{j_p}$ for some p,

which implies $x \in K_{j_p} \cap F_\alpha$ and

d) $x \in L'_i$ for some i,

which implies $x \in (L'_i \cap F_\alpha) \cup J_i$ and

e) $x \in L_{j'_q}$ for some q,

which implies $x \in L_{j'_q} \cap F_\alpha$.

If x is a given point of $\bigcup_{\beta<\alpha} F_\beta$, then by (7) $x \in G_i \subset (K'_i \cap F_\alpha) \cup G_i$ for some i. Hence $\mathcal{F}$ is a closed covering of $\bigcup_{\beta \leqq \alpha} F_\beta$.

To verify ord $\mathcal{F} \leqq n+1$ it suffices to prove it on F_α, because the same obviously holds outside of F_α by virtue of the induction hypothesis. Let

$$L = [\bigcap_{m=1}^{m_0} ((K'_{i_m} \cap F_\alpha) \cup G_{i_m})] \cap [\bigcap_{n=1}^{n_0} ((L'_{i'_n} \cap F_\alpha) \cup J_{i'_n})] \cap$$
$$\cap [\bigcap_{t=1}^{t_0} (K_{j_{p(t)}} \cap F_\alpha)] \cap [\bigcap_{u=1}^{u_0} (L_{j'_{q(u)}} \cap F_\alpha)] \cap F_\alpha \neq \emptyset .$$

Unless $t_0 = u_0 = 0$, $m_0 n_0 = 0$, from (6), (7) and (12) we obtain

$$L \subset (\bigcap_{m=1}^{m_0} K'_{i_m}) \cap (\bigcap_{n=1}^{n_0} L'_{i'_n}) \cap (\bigcap_{t=1}^{t_0} K_{j_{p(t)}}) \cap (\bigcap_{u=1}^{u_0} L_{j'_{q(u)}}) \cap F_\alpha ,$$

where generally the intersection of A_m, m ranging through an empty set of indices, means X. Hence, in view of the definitions of K'_i, K_{j_p}, L'_i, $L_{j'_q}$, we obtain

$$L \subset [\bigcap_{m=1}^{m_0} \bigcap_{n=1}^{n_0} (K_{i''_m} \cap L_{i'''_n})] \cap [\bigcap_{t=1}^{t_0} \bigcap_{u=1}^{u_0} (K_{j_{p(t)}} \cap L_{j'_{q(u)}})] \cap F_\alpha$$

for some i''_m and i'''_n for which

$$K_{i''_m} \subset K'_{i_m}, \quad L_{i'''_n} \subset L'_{i_n}, \quad i''_m \neq j_{p(t)}; \quad i'''_n \neq j'_{q(u)} .$$

Thus from (10) it follows that

$$m_0 + n_0 + t_0 + u_0 \leqq n + 1 .$$

If $t_0 = u_0 = 0$, $m_0 n_0 = 0$, then assume, for example, $m_0 \geqq 1$, $n_0 = 0$. From (11) we obtain

$$L = \bigcap_{m=1}^{m_0} [(K'_{i_m} \cap F_\alpha) \cup G_{i_m}] \cap F_\alpha \subset \bigcap_{m=1}^{m_0} M_{i_m} ,$$

which implies $m_0 \leqq n+1$, because $\operatorname{ord}\{M_i \mid i = 1,\dots,k\} \leqq n+1$.

Similarly we can prove $n_0 \leqq n+1$ in case $t_0 = u_0 = 0$, $m_0 = 0$, $n_0 \geqq 1$. Thus we can conclude $\operatorname{ord} \mathrm{F} \leqq n+1$.

Now, we define $G_{\alpha i}$, $i = 1,\dots,k$, by

$$\begin{aligned} G_{\alpha i} = {} & (K'_i \cap F_\alpha) \cup (L'_i \cap F_\alpha) \cup J_i \cup \\ & \cup [\cup\{K_{j_p} \mid K_{j_p} \subset U_i ;\ K_{j_p} \not\subset U_{i'}, \text{ for } i' < i\} \cap F_\alpha] \cup \\ & \cup [\cup\{L_{j'_q} \mid L_{j'_q} \subset U_i ;\ L_{j'_q} \not\subset U_{i'}, \text{ for } i' < i\} \cap F_\alpha] . \end{aligned}$$

Then it follows from the above property of F, (1), (11), (7), the induction hypothesis and $G_i = \bigcup_{\beta<\alpha} G_{\beta i}$, that $G_{\alpha i}$, $i = 1,\dots,k$, are the closed sets which were wanted at the beginning of the proof. We shall leave the details of the proof to the reader.

To complete our proof, let $G_i = \bigcup_{\alpha<\tau} G_{\alpha i}$. Since $\{F_\alpha \mid \alpha < \tau\}$ is locally finite, and $G_{\alpha i} \subset F_\alpha$, it follows that $\{G_{\alpha i} \mid \alpha < \tau\}$ is locally finite, and hence G_i is closed. Thus $G = \{ G_i \mid i = 1, \dots, k \}$ is a closed covering of $\bigcup_{\alpha < \tau} F_\alpha = X$ such that $G < U$ and $\operatorname{ord} G \leqq n + 1$. Hence by use of 1 B) we obtain the desired open covering W.

Theorem VII. 12. *Let f be a closed continuous mapping of a normal space X onto a paracompact space Y such that $\dim f^{-1}(q) \leqq k$ for each point q of Y. Then* $\dim X \leqq \operatorname{Ind} Y + k$.[15]

Proof. Since the theorem is clearly true if $\operatorname{Ind} Y = -1$, we shall assume it for Y with $\operatorname{Ind} Y < n$, and proceed to prove it for Y with $\operatorname{Ind} Y = n$.

Let $U = \{ U_i \mid i = 1, \dots, s \}$ be a given open covering of X. Since $\dim f^{-1}(q) \leqq k$ for each point $q \in Y$, by A) and 1. B) there exist open sets V_i, $i = 1, \dots, s$ such that

$$f^{-1}(q) \subset \bigcup_{i=1}^{s} V_i, \quad V_i \subset U_i, \text{ and}$$

$$\operatorname{ord} \{ V_i \mid i = 1, \dots s \} \leqq k + 1 . \tag{1}$$

Let

$$V'(q) = \bigcup_{i=1}^{s} V_i ; \tag{2}$$

then $\{ V'(q) \mid q \in X \}$ is an open covering of X which satisfies $V'(q) \supset f^{-1}(q)$. We take for each q an open set $V(q)$ for which

$$V'(q) \supset \overline{V(q)} \supset V(q) \supset f^{-1}(q) . \tag{3}$$

It follows from $q \in Y - f(X - V(q))$ and the closedness of f, that $V = \{ Y - f(X - V(q)) \mid q \in Y \}$ is an open covering of Y. Since Y is a paracompact space with $\operatorname{Ind} Y \leqq n$, there exists a locally finite open covering $W = \{ W_\alpha \mid \alpha < \tau \}$ such that $W < V$ and $\operatorname{Ind} B(W_\alpha) \leqq n - 1$.

[15] Due to K. Morita [6].

Since $ff^{-1}(B(W_\alpha)) = B(W_\alpha)$, f is a closed continuous mapping of $f^{-1}(B(W_\alpha))$ onto $B(W_\alpha)$. Hence it follows from the induction hypothesis that

(4) $\dim f^{-1}(B(W_\alpha)) \leqq n + k - 1$.

Since by the continuity of f , $B(f^{-1}(W_\alpha)) \subset f^{-1}(B(W_\alpha))$, we obtain from (4)

(5) $\dim B(f^{-1}(W_\alpha)) \leqq n + k - 1$.

Now, let

(6) $K_\alpha = \overline{f^{-1}(W_\alpha)}$;

then

(7) $\{ K_\alpha \mid \alpha < \tau \}$ is a locally finite open covering of X .

We put

(8) $F_\alpha = K_\alpha - \bigcup_{\beta < \alpha} f^{-1}(W_\beta)$.

For a given α there is q for which $\bar{W}_\alpha \subset Y - f(X - V(q))$. Then one can easily see that

$$F_\alpha \subset K_\alpha \subset \overline{V(q)} \subset V'(q) ,$$

and hence by (1) and (2) there exists an open covering $\mathcal{U}_\alpha$ of F_α such that

(9) $\begin{cases} \mathcal{U}_\alpha < \mathcal{U} \\ \operatorname{ord} \mathcal{U}_\alpha \leqq k + 1 \leqq n + k + 1 . \end{cases}$

On the other hand, from (6) and (8) we can easily see that

$$F_\alpha \cap F_\beta \subset B(f^{-1}(W_\beta)) \quad \text{if} \quad \beta < \alpha .$$

Therefore, by virtue of (5),

(10) $\dim (F_\alpha \cap F_\beta) \leqq n + k - 1$.

Thus $\{F_\alpha \mid \alpha < \tau\}$ satisfies (7), (9) and (10), and hence by B) there exists an open covering P with $P < U$ and $\operatorname{ord} P \leq n + k + 1$. Therefore $\dim X \leq n + k$.

Theorem VII. 13. *Let f be a closed continuous mapping of a totally normal space X onto a normal space Y such that for each point q of Y, $B(f^{-1}(q))$ contains at most $m + 1$ points $(m \geq 0)$. Then* $\operatorname{Ind} Y \leq \operatorname{Ind} X + m$.

Proof. The proof of this theorem is similar to that of Theorem III.7. All we have to do is to modify it as follows. Throughout the proof of Theorem III.7 we change 'dim' into 'Ind'. In iii) of that proof we consider the open subset $J = B(V) - H$ of $B(V)$ instead of H_k, $k = 1, 2, \ldots$. Define

$$J' = f^{-1}(J) \cap (X - \bar{U}) = \cup \{ f^{-1}(q)_2 \mid q \in J \} .$$

As is easily seen, f restricted to J' is a closed continuous mapping of J' onto J such that each point of J has an inverse image of at most m points. Hence by the induction hypothesis and Theorem VII.6 $\operatorname{Ind} J \leq n + m - 1$. Thus by 3 B) combined with $\operatorname{Ind} H \leq n + m - 1$ we conclude that $\operatorname{Ind} B(V) = \operatorname{Ind} (H \cup J) \leq n + m - 1$. (Note that Y is actually a hereditarily normal space.) This proves $\operatorname{Ind} Y \leq n + m$.

E. Sklyarenko [4] proved $\dim X \leq \dim Y + n$ if there is a closed continuous mapping f from a paracompact T_2-space X onto a paracompact T_2-space Y such that $\dim f^{-1}(q) \leq n$ for every $q \in Y$.

K. Morita [6] proved $\dim Y \leq \operatorname{Ind} X + m$ if there is a closed continuous mapping f from a normal space X onto a normal space Y such that each $f^{-1}(q)$ contains at most $m + 1$ points.

A. Zarelua [4] proved $\dim Y \leq \dim X + m$ under the same assumption.

A. V. Arhangelskii [3] obtained a different type of theorem as follows: Let f be a closed continuous mapping from a normal space X onto a normal space Y. Put $T = \{ y \in Y \mid f^{-1}(y)$ is not a singleton$\}$, $\operatorname{rd} T = \sup \{ \dim G \mid G \subset T$, and G is closed in $Y\}$. Then $\dim Y \leq \dim X + \operatorname{rd} T + 1$.

VII. 6. Product theorem

A) Let X be a compact T_2-space and Y a metric space. Suppose $X \neq \emptyset$ or $Y \neq \emptyset$. Then $\dim X \times Y \leq \dim X + \dim Y$.

Proof. Assume $X \neq \emptyset$ and $Y \neq \emptyset$. Let $\dim X = n$, $\dim Y = m$. Then it is easy to see that the projection $\pi: X \times Y \to Y$ is a closed continuous mapping because X is compact. $(\pi(x,y) = y.)$ It is also easy to see that $X \times Y$ is normal. (See e.g.

J. Nagata [8].) Since $\dim \pi^{-1}(y) = \dim X = n$ and $\dim Y = \operatorname{Ind} Y = m$, by Theorem VII.12 we obtain $\dim X \times Y \leqq n+m$.

We owe the following theorem to K. Morita [7].

Theorem VII. 14. *Let X be a compact T_2-space and $X \times Y$ a normal space. Suppose $X \neq \emptyset$ or $Y \neq \emptyset$. Then* $\dim X \times Y \leqq \dim X + \dim Y$.

Proof. Suppose $X \neq \emptyset$, $Y \neq \emptyset$, $\dim X = n$ and $\dim Y = m$. Let $\mathcal{A}$ be a given finite open covering of $X \times Y$. Then by I.1 C) there is a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of finite open coverings of $X \times Y$ such that

$$\mathcal{A} > \mathcal{W}_1^* > \mathcal{W}_1 > \mathcal{W}_2^* > \dots .$$

Define for $y_1, y_2 \in Y$

$$y_1 \underset{i}{\sim} y_2 \quad \text{if and only if} \quad (x,y_1) \in S((x,y_2),\mathcal{W}_i) \quad \text{for every } x \in X .$$

(The binary relation $\underset{i}{\sim}$ is no equivalence relation.) Define

$$V_i(y) = \{ y' \in Y \mid y \underset{i}{\sim} y' \} \quad \text{for } y \in Y, \text{ and } \mathcal{V}_i = \{ V_i(y) \mid y \in Y \} .$$

Then we can prove that $\mathcal{V}_i$ is an open covering of Y.

To prove that $V_i(y)$ is open, let $y' \in V_i(y)$ be given. Then for each $x \in X$ there is $W_x \in \mathcal{W}_i$ such that (x,y), $(x,y') \in W_x$. Choose open neighbourhoods $U(x)$, $V_x(y)$, $V_x(y')$ of x, y, y', respectively such that

$$U(x) \times V_x(y) \subset W_x \quad \text{and} \quad U(x) \times V_x(y') \subset W_x .$$

Since X is compact, $X = \bigcup_{j=1}^{k} U(x_j)$ for some finitely many x_j's. Then

$$V(y') = \bigcap_{j-1}^{k} V_{x_j}(y') \subset V_i(y) .$$

Because if $y'' \in V(y')$, then for any $x \in X$ there is j for which $x \in U(x_j)$. Thus

$$(x,y) \in U(x_j) \times V_{x_j}(y) \subset W_{x_j} ,$$
$$(x,y'') \in U(x_j) \times V_{x_j}(y') \subset W_{x_j} ,$$

which implies $y \underset{i}{\sim} y''$. Hence $V(y') \subset V_i(y)$ which proves that $V_i(y)$ is open.

It is also easy to see that $\mathcal{V}^*_{i+1} < \mathcal{V}_i$ follows from $\mathcal{W}^*_{i+1} < \mathcal{W}_i$, because $S(\mathcal{V}_{i+1}(y), \mathcal{V}_{i+1}) \subset \mathcal{V}_i(y)$. Thus by Theorem I.1' each $\mathcal{V}_i$ has a locally finite open refinement $\mathcal{V}'_i$. Since $\dim Y \leqq m$, we can construct, as we did in the proof of Theorem VII.4, a sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of locally finite open coverings of Y such that

$$\mathcal{P}_1 > \mathcal{P}^*_2 > \mathcal{P}_2 > \mathcal{P}^*_3 > \ldots, \quad \mathcal{P}_i < \mathcal{V}'_i < \mathcal{V}_i,$$

and each member of $\mathcal{P}_{i+1}$ meets at most $(m+1)$ members of $\mathcal{P}_i$. We continue to use the method of the above mentioned proof to define a metric space Z with $\dim Z \leqq m$ and a continuous mapping f from Y onto Z such that $\{ S(z, f(\mathcal{P}_i)) \mid i = 1, 2, \ldots \}$ is a neighbourhood base at each $z \in Z$. Note that $f^{-1}(z)$ for each $z \in Z$ is an equivalence class in Y with respect to $\sim$ defined by

$$y \sim y' \text{ if and only if } y' \in \bigcap_{i=1}^{\infty} S(y, \mathcal{P}_i).$$

Further observe that $y \sim y'$ implies

$$y \in \bigcap_{i=1}^{\infty} S(y', \mathcal{P}_i) \subset \bigcap_{i=1}^{\infty} S(y', \mathcal{V}_i)$$

and accordingly

(1) $$y \underset{i}{\sim} y', \quad i = 1, 2, \ldots .$$

Now we define a continuous mapping φ from $X \times Y$ onto $X \times Z$ by $\varphi((x,y)) = (i(x), f(y))$, where i is the identity mapping of X.

Let

$$\mathcal{A} = \{ A_l \mid l = 1, \ldots, k \},$$
$$\mathcal{B} = \{ (X \times Z - \varphi(X \times Y - A_l))^{\circ} \mid l = 1, \ldots, k \}.$$

Then $\mathcal{B}$ is an open covering of $X \times Z$. To prove this let $q = (x, z) \in X \times Z$ be given. Then $\varphi^{-1}(q) \cap W \neq \emptyset$ for some $W \in \mathcal{W}_2$. Select $p = (x, y) \in \varphi^{-1}(q) \cap W$. Then x has an open neighbourhood U in X such that

(2) $$U \times \{ y \} \subset W .$$

Since $\mathcal{W}_2^* < \mathcal{W}_1 < \mathcal{W}_1^* < \mathcal{A}$, there is ℓ for which

(3) $\qquad S^4(W,\mathcal{W}_2) \subset A_\ell$.

Now, put $V = S(z,f(\mathcal{P}_2))$. Then $U \times V$ is a neighbourhood of q in $X \times Z$. We claim that $U \times V \subset X \times Z - \varphi(X \times Y - A_\ell)$.

To prove it, let $q' = (x',z') \in U \times V$ be given. Then $x' \in U$, and there is $V(y_0) \in \mathcal{V}_2$ such that $z, z' \in f(V(y_0))$, because $\mathcal{P}_2 < \mathcal{V}_2$. Hence we can choose y_1 and y_2 such that

(4) $\qquad y_1 \in f^{-1}(z) \cap V(y_0)$, $y_2 \in f^{-1}(z') \cap V(y_0)$.

Note that

(5) $\qquad y \in f^{-1}(z)$

follows from $p \in \varphi^{-1}(q)$. Define $p' = (x',y') \in \varphi^{-1}(q')$; then

(6) $\qquad y' \in f^{-1}(z')$.

Thus it follows from (4), (5), (6) that

$$y \underset{2}{\sim} y_1 \underset{2}{\sim} y_0 \underset{2}{\sim} y_2 \underset{2}{\sim} y' .$$

Thus there are $W_i \in \mathcal{W}_2$, $i = 1,\dots,4$ such that every successive two points of the sequence (x',y), (x',y_1), (x',y_0), (x',y_2), (x',y') are contained in some W_i . On the other hand, since $x' \in U$, $(x',y) \in W$ follows from (2). Thus $(x',y') \in S^4(W,\mathcal{W}_2)$, which combined with (3) implies $(x',y') \in A_\ell$. Hence we have proved $\varphi^{-1}(q') \subset A_\ell$. Thus $q' \in X \times Z$ but $q' \notin \varphi(X \times Y - A_\ell)$.

Recall that q' is a given point of the neighbourhood $U \times V$ of q , and hence

$$U \times V \cap \varphi(X \times Y - A_\ell) = \emptyset ,$$

proving

$$q \in (X \times Y - \varphi(X \times Y - A_\ell))^{\circ} .$$

Therefore $\mathcal{B}$ covers $X \times Z$.

Now, since X is compact T_2 and Z metric, by A) we have

$$\dim X \times Z \leq \dim X + \dim Z \leq n + m \ .$$

Hence there is an open covering $\mathcal{D}$ of $X \times Z$ such that $\mathcal{D} < \mathcal{B}$ and ord $\mathcal{D} \leq n+m+1$. Then $\mathcal{D}' = \{\varphi^{-1}(D) \mid D \in \mathcal{D}\}$ is an open covering of $X \times Y$ such that $\mathcal{D}' < \mathcal{A}$ and ord $\mathcal{D}' \leq n+m+1$. This proves that $\dim X \times Y \leq n+m$.

Corollary. *Suppose that X is a paracompact T_2-space, and $X = \bigcup_{i=1}^{\infty} F_i$ for locally compact closed sets F_i, $i = 1, 2, \ldots$. Further suppose that $X \times Y$ is normal, and $X \neq \emptyset$ or $Y \neq \emptyset$. Then* $\dim X \times Y \leq \dim X + \dim Y$.[16]

Proof. Assume $\dim X = n$, $\dim Y = m$. Each F_i is a locally finite sum of compact sets. Thus by Theorem VII.14 and the sum theorem (for locally finite sums) we obtain $\dim F_i \times Y \leq n+m$. By use of the countable sum theorem we conclude that $\dim X \times Y \leq n+m$.

Let us review here some simple facts about the Stone-Čech compactification.

B) Let X be a Tychonoff space; then there is a unique compact T_2-space βX (called Stone-Čech compactification of X) satisfying

(i) X is a dense subset of βX,

(ii) $\{\bar{Z} \mid Z \in \mathcal{Z}\}$ is a closed base for βX, where $\mathcal{Z}$ denotes the collection of all zero sets of X,

(iii) if $Z_i \in \mathcal{Z}$, $i = 1, \ldots, k$, then $\bigcap_{i=1}^{k} \bar{Z}_i = \overline{\bigcap_{i=1}^{k} Z_i}$.

(In (ii) and (iii) the closure is taken in βX.) Every real-valued bounded continuous function on X can be continuously extended to βX, and thus for every cozero set D of X there is a cozero set D' of βX such that $D' \cap X = D$.[17]

C) Let X be a normal space; then $\dim X = \dim \beta X$, $\operatorname{Ind} X = \operatorname{Ind} \beta X$.

Proof. Let $\dim X = n$, $\dim \beta X = m$. Suppose $\mathcal{U}$ is a given finite open covering of X. Then there is a finite covering $\mathcal{V}$ of X by cozero sets such that $\mathcal{V} < \mathcal{U}$. (Use I.1 A) and Urysohn's lemma.) Now, by B), $\beta\mathcal{V} = \{\beta X - \overline{(X - V)} \mid V \in \mathcal{V}\}$ is an open

---

[16] K. Morita [7] proved this theorem for the Katětov-Smirnov dimension without assuming the normality of $X \times Y$. K. Morita [3] proved $\dim X \times Y \leq \dim X + \dim Y$ in case that X and Y are T_2 and $X \times Y$ has the star-finite property.

[17] See e.g. J. Nagata [8] for a proof.

covering of βX. Hence there is an open covering $\mathcal{V}'$ of βX such that $\mathcal{V}' < \beta\mathcal{V}$, ord $\mathcal{V}' \leqq m+1$. The restriction of $\mathcal{V}'$ to X is an open refinement of $\mathcal{U}$ with ord $\leqq m+1$. Thus $n \leqq m$.

Let $\mathcal{W}$ be a finite open covering of βX. Then by B) there is a finite covering $\mathcal{W}'$ of βX such that $\mathcal{W}' < \mathcal{W}$, and each member of $\mathcal{W}'$ is of the form $\beta X - \overline{(X-Z)}$ for a zero set Z of X. Then select a finite covering $\mathcal{A}$ of X by cozero sets of X such that $\mathcal{A} < \mathcal{W}'$, ord $\mathcal{A} \leqq n+1$. Now it is easy to check $\beta\mathcal{A} < \mathcal{W}$ and ord $\beta\mathcal{A} \leqq n+1$, where

$$\beta\mathcal{A} = \{\beta X - \overline{(X-A)} \mid A \in \mathcal{A}\}.$$

Thus $m \leqq n$, i.e. $n = m$. The proof of the second equality is omitted.

D) Let $\mathcal{U}$ be a σ-locally finite open covering of a countably paracompact space X. Then there is a locally finite open covering $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$.

Proof. Let $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$, where each $\mathcal{U}_i$ is locally finite. Put $U_i = \bigcup\{U \mid U \in \mathcal{U}_i\}$. Then, since X is countably paracompact, there is a locally finite open covering $\{V_i \mid i = 1,2,\dots\}$ such that $V_i \subset U_i$. Put $\mathcal{V}_i = \{V_i \cap U \mid U \in \mathcal{U}_i\}$; then $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a locally finite open refinement of $\mathcal{U}$.

E) Let Y be a metric space and $X \times Y$ a countably paracompact normal space. Suppose that $\mathcal{B} = \{B_h \mid h = 1,\dots,k\}$ is an open covering of $X \times Y$. Then there is a cozero set D of $\beta X \times Y$ and an open covering $\mathcal{D} = \{D_h \mid h = 1,\dots,k\}$ of D such that $X \times Y \subset D$ and $D_h \cap (X \times Y) \subset B_h$, $h = 1,\dots,k$.

Proof. Let $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$ be a σ-locally finite base for Y, where $\mathcal{V}_i = \{V_\alpha \mid \alpha \in A_i\}$, $i = 1,2,\dots$ are locally finite open coverings of Y. We may assume $\mathcal{V}_i \subset \mathcal{V}_{i+1}$, and thus $A_i \subset A_{i+1}$. Define

$$A_i' = \{(\alpha_1,\dots,\alpha_i) \mid \alpha_1 \in A_1,\dots,\alpha_i \in A_i,\ \bigcap_{\ell=1}^{i} V_{\alpha_\ell} \neq \emptyset\}$$

and

$$V(\alpha_1,\dots,\alpha_i) = \bigcap_{\ell=1}^{i} V_{\alpha_\ell}, \quad (\alpha_1,\dots,\alpha_i) \in A_i'.$$

Then $V(\alpha_1,\dots,\alpha_i)$ is a non-empty open set of Y. Further we define open sets of X by

$$U(\alpha_1,\dots,\alpha_i\ ;\ h) = \bigcup\left\{ U \;\middle|\; \begin{array}{l} U \text{ is an open set of } X \text{ such that} \\ U\times V\ (\alpha_1,\dots,\alpha_i)\subset B_h \end{array} \right\},$$
$(\alpha_1,\dots,\alpha_i)\in A_i'\ ,\ 1\leqq h\leqq k$.

Then we obtain

(1) $$U(\alpha_1,\dots,\alpha_i\ ;\ h)\times V(\alpha_1,\dots,\alpha_i)\subset B_h\ .$$

Then we put

$$U(\alpha_1,\dots,\alpha_i) = \bigcup_{h=1}^{k} U(\alpha_1,\dots,\alpha_i\ ;\ h)\ .$$

Now it is obvious that

$$\mathcal{W} = \{\, U(\alpha_1,\dots,\alpha_i)\times V(\alpha_1,\dots,\alpha_i) \mid (\alpha_1,\dots,\alpha_i)\in A_i'\ ;\ i=1,2,\dots \}$$

is an open covering of $X\times Y$. For each fixed i, $\{ V(\alpha_1,\dots,\alpha_i) \mid (\alpha_1,\dots,\alpha_i)\in A_i' \}$ is a locally finite open covering of Y. Hence $\mathcal{W}$ is a σ-locally finite open covering of $X\times Y$. Since $X\times Y$ is countably paracompact and normal, by use of D) we can find a locally finite open covering $\mathcal{N}=\{N_\gamma \mid \gamma\in\Gamma\}$ of $X\times Y$ such that $\mathcal{N}<\mathcal{W}$. Shrink $\mathcal{N}$ to an open covering $\mathcal{L}=\{L_\gamma \mid \gamma\in\Gamma\}$ satisfying $\bar{L}_\gamma\subset N_\gamma$. Then put

(2) $$L(\alpha_1,\dots,\alpha_i\ ;\ \gamma) = \bigcup\left\{ U \;\middle|\; \begin{array}{l} U \text{ is an open set of } X \text{ such that} \\ U\times V(\alpha_1,\dots,\alpha_i)\subset L_\gamma \end{array} \right\}$$
$(\alpha_1,\dots,\alpha_i)\in A_i'\ ,\ \gamma\in\Gamma$.

Each $L(\alpha_1,\dots,\alpha_i\ ;\gamma)$ is an open set of X satisfying

(3) $$L(\alpha_1,\dots,\alpha\ \ ;\gamma)\times V(\alpha_1,\dots,\alpha_i)\subset L_\gamma\subset\bar{L}_\gamma\subset N_\gamma\ .$$

For $(\alpha_1,\dots,\alpha_i\ .\ \beta_1,\dots,\beta_j)\in A_{i+j}'$, we define

(4) $$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j) = \bigcup\{\overline{L(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j\ ;\gamma)} \mid \gamma \text{ satisfies } N_\gamma\subset U(\alpha_1,\dots,\alpha_i)\times V(\alpha_1,\dots,\alpha_i)\}\ .$$

Then $H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)$ is a closed set of X because of (3) and the local finiteness of $\mathcal{L}$. Furthermore, observe that

(5) $$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\times\overline{V(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)} \subset U(\alpha_1,\dots,\alpha_i)\times V(\alpha_1,\dots,\alpha_i)$$

follows from (3). Hence

$$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset U(\alpha_1,\dots,\alpha_i)$$

holds in X, because $V(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j)\neq\emptyset$.

Since X is a normal space, for each $(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j)\in A'_{i+j}$ we can select a cozero set $M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)$ of X such that

$$H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset U(\alpha_1,\dots,\alpha_i)\ .$$

Since

$$U(\alpha_1,\dots,\alpha_i)=\bigcup_{h=1}^{k} U(\alpha_1,\dots,\alpha_i\ ;\ h)\ ,$$

there are cozero sets $M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j\ ;\ h)$, $h=1,\dots,k$, of X such that

(6) $$\bigcup_{h=1}^{k} M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j\ ;\ h)=M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\ ,$$
$$M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j\ ;\ h)\subset U(\alpha_1,\dots,\alpha_i\ ;\ h)\ .$$

Now, let $p=(x,y)$ be a given point of $X\times Y$. Then $p\in L_\gamma$ for some $\gamma\in\Gamma$, and $N_\gamma\subset U(\alpha_1,\dots,\alpha_i)\times V(\alpha_1,\dots,\alpha_i)$ for some i and $(\alpha_1,\dots,\alpha_i)\in A'_i$, because $N<\mathcal{W}$. Since $\mathcal{V}$ is a base for Y and $A_i\subset A_{i+1}$, there are $\beta_1\in A_{i+1},\dots,\beta_j\in A_{i+j}$ such that

$$p=(x,y)\in U\times V(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j)\subset L_\gamma$$

for some open neighbourhood U of x . Hence by (2)

$$x\in L(\alpha_1,\dots,\alpha_i\ ,\ \beta_1,\dots,\beta_j\ ;\ \gamma)\ .$$

By (4)

$$x\in H(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\subset M(\alpha_1,\dots,\alpha_i\ ;\ \beta_1,\dots,\beta_j)\ .$$

and hence

$$p \in M(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) .$$

This proves that $\{ M(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \in A'_{i+j} ; i,j = 1,2,\dots\}$ covers $X \times Y$.

Therefore $\{ M(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \in A'_{i+j} ; i,j = 1,2,\dots ; h = 1,\dots,k \}$ covers $X \times Y$.

By B) there is a cozero set $D(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h)$ of βX whose intersection with X is $M(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h)$. Then $\{ D(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \in$ $\in A'_{i+j} ; i,j = 1,2,\dots ; h = 1,\dots,k \}$ is a collection of cozero sets in $\beta X \times Y$ which covers $X \times Y$.

Define for $h = 1,\dots,k$

$$D_h = \cup \{ D(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \in A'_{i+j} ; i,j = 1,2,\dots \} .$$

Then $D_h \cap (X \times Y) \subset B_h$ can be proved as follows. $D(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times$ $\times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \cap (X \times Y) = M(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times$ $\times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j)$. On the other hand (6) and (1) imply

$$M(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \subset U(\alpha_1,\dots,\alpha_i ; h) \times V(\alpha_1,\dots,\alpha_i) \subset B_h .$$

Thus it is proved that $D_h \cap (X \times Y) \subset B_h$.

Finally we shall prove that D_h is a cozero set of $\beta X \times Y$. For fixed i, j $\{ V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \in A'_{i+j} \}$ is locally finite in Y. Thus $\{ D(\alpha_1,\dots,\alpha_i ; \beta_1,\dots,\beta_j ; h) \times V(\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \mid (\alpha_1,\dots,\alpha_i , \beta_1,\dots,\beta_j) \in A'_{i+j} ; i,j = 1,2,\dots\}$ is a collection of cozero sets of $\beta X \times Y$, which is σ-locally finite in $\beta X \times Y$. Thus D_h is a cozero set of $\beta X \times Y$. Put $D = \cup_{h=1}^{k} D_h$ and $\mathcal{D} = \{ D_1,\dots,D_k \}$. We obtained the desired result.

We owe the following theorem to Y. Kodana [3].

Theorem VII. 15. *Let Y be a metric space and $X\times Y$ a countably paracompact normal space. Suppose $X\neq\emptyset$ or $Y\neq\emptyset$. Then* $\dim X\times Y\leqq\dim X+\dim Y$.[18]

Proof. Assume $\dim X=n$, $\dim Y=m$. Then by C) $\dim \beta X=n$. Thus by Theorem VII.14 $\dim \beta X\times Y\leqq n+m$. (Note that the product of a compact T_2-space and a paracompact T_2-space is normal. See J. Nagata [8] for a proof.) Let $\mathcal{B}$ be a given finite open covering of $X\times Y$. Then we consider a cozero set D of $\beta X\times Y$ and its open covering $\mathcal{D}$ mentioned in E). Since D is an F_σ-set in the normal space $\beta X\times Y$, it is normal as a subspace, and thus by the countable sum theorem $\dim D\leqq \dim \beta X\times Y\leqq n+m$. Hence there is a finite open covering $\mathcal{D}'$ of D such that $\mathcal{D}'<\mathcal{D}$, $\operatorname{ord}\mathcal{D}'\leqq n+m+1$. Then the restriction of $\mathcal{D}'$ to $X\times Y$ is an open refinement of $\mathcal{B}$ with $\operatorname{ord}\leqq n+m+1$. Hence $\dim X\times Y\leqq n+m$.

V. Filippov [3] generalized Theorem VII.15 as follows: If $X\times Y$ is countably paracompact and normal, and if there is a perfect mapping from Y onto a metric space, then $\dim X\times Y\leqq\dim X+\dim Y$ ($X\neq\emptyset$ or $Y\neq\emptyset$).

This theorem implies practically Theorem VII.14.

B. Pasynkov [6] further extended this theorem as follows: $\dim X\times Y\leqq\dim X+\dim Y$ whenever $X\times Y$ is a rectangular product and X, Y Tychonoff ($X\neq\emptyset$ or $Y\neq\emptyset$), where dim is the Katětov-Smirnov dimension, and where it is defined that a product $X\times Y$ is called *rectangular* if every finite covering of $X\times Y$ by cozero sets has a σ-locally finite open refinement by sets of the form $U\times V$ for cozero sets U and V of X and Y, respectively.[19]

T. Hoshina - K. Morita [1] studied general conditions under which $X\times Y$ is rectangular and extended Pasynkov's theorem further to non-Tychonoff spaces.

As for large induction dimension, $\operatorname{Ind} X\times Y\leqq\operatorname{Ind} X+\operatorname{Ind} Y$ (where $X\neq\emptyset$ or $Y\neq\emptyset$) was proved under various conditions, for examples in case that $X\times Y$ is hereditarily paracompact T_2 and has the star-finite property (Y. Katsuta [1]), each of X and Y is the inverse image of a metric space by a perfect mapping and

[18] M. E. Rudin - M. Starbird [1] proved that if the product $X\times Y$ of a countably paracompact space X and a metric space Y is normal, then $X\times Y$ is countably paracompact. Thus Theorem VII.15 can be improved as follows:
Let Y be a metric space, X countably paracompact and $X\times Y$ normal. Suppose $X\neq\emptyset$ or $Y\neq\emptyset$. Then $\dim X\times Y\leqq\dim X+\dim Y$.

[19] The product $X\times Y$ of Tychonoff spaces is rectangular, e.g., in the following cases: (1) Y is locally compact and paracompact, (2) the projection $\pi: X\times Y\to X$ is a closed mapping, (3) there is a perfect mapping from Y onto a metric space, and $X\times Y$ is countably paracompact and normal, (4) (implied by (3)) there is a perfect mapping from each of X and Y onto a metric space.

$X \times Y$ totally normal (J. Nagata [12]), $X \times Y$ is normal, X locally compact and paracompact, and the finite sum theorem holds for Ind in X and Y (V. Filippov [3]), and more generally in case that $X \times Y$ is a rectangular product and normal, and the finite sum theorem holds for Ind in X and Y (B. Pasynkov [6]).[20]

On the other hand the equality $\dim X \times Y = \dim X + \dim Y$ holds under rather restrictive conditions. K. Morita [7] proved it in case that $\dim X = 1$, and Y is a paracompact T_2-space which is a countable sum of locally compact closed subsets, where dim is Katětov-Smirnov dimension.

It is obvious that $\dim X \times Y \leq \dim X + \dim Y$ does not hold in general. Because $\dim Z = 0$ implies that Z is a normal space, and hence if X, Y are 0-dimensional spaces such that $X \times Y$ is not normal, then the product theorem does not hold. For example, let $X = Y =$ the Sorgenfrey line (= the real numbers with the open base $\{ [x, x+\varepsilon) \mid -\infty < x < +\infty,\ \varepsilon > 0 \}$.)

What can be said if $X \times Y$ is normal? This had been a long standing question until it was negatively answered by M. Wage [1] under the assumption of the continuum hypothesis and by T. Przymusiński [2] without the hypothesis.[21] The former gave a space X such that $\dim X = 0$, $\dim X \times X \neq 0$, and $X \times X$ is locally compact and perfectly normal. The latter gave, for every natural number n, a separable first-countable space X such that X^n is Lindelöf and $\dim X^n = 0$, X^{n+1} is normal and $\dim X^{n+1} > 0$ (X^n denotes the n-ple product of X).

Since $\dim X = 0$ if and only if $\operatorname{Ind} X = 0$, the above examples show that the product theorem for Ind does not hold either even if $X \times Y$ is normal.[22]

V. Filippov [2] constructed compact T_2-spaces X and Y such that $\operatorname{ind} X = \operatorname{Ind} X = 1$, $\operatorname{ind} Y = \operatorname{Ind} Y = 2$, $\operatorname{ind} X \times Y \geq 4$. Thus the product theorem for the inductive dimension does not hold even for compact T_2-spaces.

---

[20] Also note that $\operatorname{Ind} X \times Y \leq \operatorname{Ind} X + \operatorname{Ind} Y$ holds for regular spaces X and Y ($X \neq \emptyset$ or $Y \neq \emptyset$) if the characterization theorem by means of a σ-closure-preserving base (footnote 5) holds for X and Y. Because it is known that if a regular space has a σ-closure-preserving base, then it is hereditarily paracompact and thus the product theorem can be proved in a similar way as done for metric spaces by use of the finite sum theorem and induction. Thus the product theorem is true especially if X and Y are the images of metric spaces by closed continuous mappings.

[21] The problem still remains open if $X \times Y$ is paracompact T_2.

[22] S. Mrowka [1] and P. Nyikos [1] studied relations between 0-dimensional spaces and N-compact spaces (= closed sets in products of countable discrete spaces).

VII. 7. Characterization of dim by $\Delta_k(X)$

Pontrjagin-Schnirelmann's theorem (Theorem IV.6) was generalized to non-metrizable spaces by J. Bruijning [1] by use of totally bounded pseudo-metrics. However, in the present section we are going to define a new function $\Delta_k(X)$, which has a remote resemblance to Pontrjagin-Schnirelmann's function $N(\varepsilon, R, \rho)$ in its basic idea, and we shall see that $\Delta_k(X)$ is a more appropriate function to characterize dim of non-metrizable spaces. The main results and method of this section are due to J. Bruijning - J. Nagata [1]. In the following we denote by C^k_m the set of all m-element subsets of $\{1, 2, \dots, k\}$ and by $\binom{k}{m}$ its cardinality, i.e. $\binom{k}{m} = k!/m!(k-m)!$.

A) A normal space X has $\dim \leqq n$ if and only if every open covering $\{U_1, \dots, U_{n+2}\}$ can be shrunk to an open covering $\{V_1, \dots, V_{n+2}\}$ such that $\bigcap_{i=1}^{n+2} V_i = \emptyset$.

Proof. The 'only if' part follows directly from I A).

To prove the 'if' part, assume the condition and let $\mathcal{B} = \{B_1, \dots, B_k\}$ be an open covering of X. Assume $k \geqq n+2$ and put $C^k_{n+2} = \{A_1, \dots, A_p\}$. Suppose $A_1 = \{i_1, \dots, i_{n+2}\}$.

Then, let $\{F_1, \dots, F_k\}$ be a closed covering of X such that $F_i \subset B_i$, $i = 1, \dots, k$. Put

$$K_1 = F_{i_1} \cup \dots \cup F_{i_{n+2}} \quad \text{and} \quad U_j = (X - K_1) \cup B_{i_j} \quad \text{for} \quad j = 1, \dots, n+2 .$$

Then $\{U_1, \dots, U_{n+2}\}$ is an open covering of X, and hence it can be shrunk to an open covering $\{V_1, \dots, V_{n+2}\}$ such that $\bigcap_{j=1}^{n+2} V_j = \emptyset$. Now put

$$U^1_{i_j} = V_j \cap B_{i_j}, \quad j = 1, \dots, n+2 ,$$
$$U^1_i = B_i \quad \text{for} \quad i \quad \text{satisfying} \quad 1 \leqq i \leqq k, \quad i \neq i_1, \dots, i_{n+2} .$$

Then $\mathcal{U}^1 = \{U^1_1, \dots, U^1_k\}$ is an open covering of X because $V_j \cap K_1 \subset B_{i_j} \cap K_1$. Observe that $\mathcal{U}^1$ shrinks $\mathcal{B}$ and satisfies $\bigcap\{U^1_i \mid i \in A_1\} = \emptyset$.

By use of the same argument we can shrink $\mathcal{U}^1$ to an open covering $\mathcal{U}^2 = \{U^2_1, \dots, U^2_k\}$ such that $\bigcap\{U^2_i \mid i \in A_2\} = \emptyset$, and accordingly $\bigcap\{U^2_i \mid i \in A_1\} = \emptyset$.

Repeating the same argument we obtain an open covering $\mathcal{U}^p = \{U^p_1, \dots, U^p_k\}$ which shrinks $\mathcal{B}$ and satisfies $\bigcap\{U^p_i \mid i \in A_l\} = \emptyset$ for $l = 1, \dots, p$. Thus $\operatorname{ord} \mathcal{U}^p \leqq n+1$, proving $\dim X \leqq n$.

Definition VII. 2. *Let* X *be a topological space and* k *a natural number. Then we define* $\Delta_k(X)$ *as the least natural number* m *such that for every open covering* $\mathcal{U}$ *of* X *with* $|\mathcal{U}| \leqq k$ *, there is an open covering* $\mathcal{V}$ *of* X *such that* $|\mathcal{V}| \leqq m$ *,* $\mathcal{V} \overset{\Delta}{<} \mathcal{U}$ *.*

B) Let X be a normal space with $\dim X \leqq n$ $(n \geqq 0)$, and k a natural number. Then

$$\Delta_k(X) \leqq 2^k - 1 \quad \text{if} \quad k \leqq n+1 ,$$
$$\Delta_k(X) \leqq \binom{k}{1} + \dots + \binom{k}{n+1} \quad \text{if} \quad k \geqq n+1 .$$

Proof. We shall prove the inequality in case of $k \geqq n+1$. Let $\mathcal{U}$ be an open covering of X with $|\mathcal{U}| \leqq k$. Then we may put $\mathcal{U} = \{ U_1, \dots, U_k \}$ by counting a same element repeatedly if necessary. Now, shrink $\mathcal{U}$ to an open covering $\mathcal{V} = \{ V_1, \dots, V_k \}$ with $\operatorname{ord} \mathcal{V} \leqq n+1$. Then select an open covering $\mathcal{W} = \{ W_1, \dots, W_k \}$ such that $\bar{W}_i \subset V_i$. For each $A \in \bigcup_{m=1}^{n+1} C_m^k$, we put

$$P(A) = (\cap \{ V_i \mid i \in A \}) \cap (\cap \{ X - \bar{W}_i \mid i \notin A \, , \; 1 \leqq i \leqq k \}) .$$

Then it is obvious that

$$\mathcal{P} = \{ P(A) \mid A \in \bigcup_{m=1}^{n+1} C_m^k \}$$

is an open covering of X such that

$$\mathcal{P} \overset{\Delta}{<} \mathcal{V} < \mathcal{U} \, , \quad \text{and} \quad |\mathcal{P}| \leqq \binom{k}{1} + \dots + \binom{k}{n+1} .$$

Thus

$$\Delta_k(X) \leqq \binom{k}{1} + \dots + \binom{k}{n+1} .$$

In case of $k \leqq n+1$, we can use the above argument to get

$$\Delta_k(X) \leqq \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k} = 2^k - 1 .$$

C) Let X be an infinite normal space with $\dim X \geqq n \geqq 0$. Let k be a natural number. Then there are k closed subsets of X with $\dim \geqq n$ which are disjoint with each other.

Proof. If $n=0$, then, since X is infinite (contains infinitely many points), just choose k distinct points.

If $n>0$, then it suffices to prove that X contains two disjoint closed sets with $\dim \geqq n$. Since $\dim X \geqq n$, there is an open covering $\mathcal{U}=\{U_1,\dots,U_j\}$ of X which cannot be shrunk to an open covering of $\operatorname{ord} \leqq n$. Let $\mathcal{F}=\{F_1,\dots,F_j\}$ be a closed covering of X such that $F_i \subset U_i$. Then $\dim F_i \geqq n$ for some i because of the sum theorem. Let V be an open set such that $F_i \subset V \subset \bar{V} \subset U_i$.

Then $\dim (X-V) \geqq n$: Assume the contrary, $\dim (X-V)<n$. Since $\{U_\ell - \bar{V},\ U_i \mid \ell \neq i\}$ covers $X-V$, there is a collection $\mathcal{W}=\{W_1,\dots,W_j\}$ of open sets of X such that $W_i \subset U_i$, $W_\ell \subset U_\ell - \bar{V}$ for $\ell \neq i$, $\operatorname{ord} \mathcal{W} \leqq n$, and $\bigcup_{k=1}^{j} W_k \supset X-V$. Now, put $P_i = W_i \cup V$ and $P_\ell = W_\ell$ for $\ell \neq i$. Then $\mathcal{P}=\{P_1,\dots,P_j\}$ is an open covering of X such that $\operatorname{ord} \mathcal{P} \leqq n$, and $P_k \subset W_k$ for $k=1,\dots,j$, which is a contradiction. Thus we obtain two disjoint closed sets F_i and $X-V$ each of which has $\dim \geqq n$.

D) Let X be an infinite normal space with $\dim X \geqq n$. Let k be a natural number. Then

$$\Delta_k(X) \geqq 2^k - 1 \quad \text{if} \quad k \leqq n+1 \text{, and}$$

$$\Delta_k(X) \geqq \binom{k}{1} + \dots + \binom{k}{n+1} \quad \text{if} \quad k \geqq n+1 .$$

Proof. Assume $k \geqq n+1$. By C) there is a collection $\{C(\alpha) \mid \alpha \in C^k_{n+1}\}$ of disjoint closed sets of X with $\dim C(\alpha) \geqq n$. By A) there is an open covering $\mathcal{U}(\alpha)=\{U^\alpha_i \mid i \in \alpha\}$ of $C(\alpha)$ which cannot be shrunk to an open covering of $C(\alpha)$ with an empty intersection. Define open sets of X by

$$(1) \qquad U_i = (X - \cup \{C(\alpha) \mid \alpha \in C^k_{n+1}\}) \cup (\cup \{U^\alpha_i \mid i \in \alpha\}),\quad i=1,\dots,k .$$

Then $\mathcal{U}=\{U_i \mid i=1,\dots,k\}$ is obviously an open covering of X. Let $\mathcal{V}$ be an open covering of X satisfying $\mathcal{V}^\Delta < \mathcal{U}$. Then we claim that

$$(2) \qquad |\mathcal{V}| \geqq \binom{k}{1} + \dots + \binom{k}{n+1} .$$

To prove it, we shall show that for every $\beta \in C^k_1 \cup C^k_2 \cup \dots \cup C^k_{n+1}$, there is a member $V(\beta)$ of $\mathcal{V}$ such that

$$(3) \qquad V(\beta) \subset U_i \quad \text{if and only if} \quad i \in \beta .$$

If this is shown, we know that, for distinct β's, the $V(\beta)$'s are distinct sets, which implies that $\mathcal{V}$ contains at least $\binom{k}{1} + \dots + \binom{k}{n+1}$ distinct members.

Now, select a subset γ of $\{1,\dots,k\}$ such that $\beta \cap \gamma = \emptyset$, and $\alpha = \beta \cup \gamma \in C^k_{n+1}$. Put

(4) $$K = C(\alpha) - \cup \{ U^\alpha_i \mid i \in \gamma \}.$$

Then observe that $\{ U^\alpha_j \mid j \in \beta \}$ is a collection of open sets of $C(\alpha)$ which covers K. Further we note that

(5) $\{ U^\alpha_j \mid j \in \beta \}$ is a covering of K which cannot be shrunk to a closed covering of K with an empty intersection.

We assume the contrary. Then there is a closed covering $\{ K_j \mid j \in \beta \}$ of K such that

$$K_j \subset U^\alpha_j \quad \text{and} \quad \bigcap_{j \in \beta} K_j = \emptyset .$$

Use 1 B) to get open sets H_j, $j \in \beta$, of $C(\alpha)$ such that

$$K_j \subset H_j \subset U^\alpha_j \quad \text{and} \quad \bigcap_{j \in \beta} H_j = \emptyset .$$

Then $\{ H_j \mid j \in \beta \} \cup \{ U^\alpha_i \mid i \in \gamma \}$ is an open covering of $C(\alpha)$ with an empty intersection, and this covering shrinks $U(\alpha)$, which contradicts the definition of $U(\alpha)$.

Now, we define closed sets G_j, $j \in \beta$, by

(6) $$G_j = \{ x \in K \mid S(x,V) \subset U_j \}, \ j \in \beta .$$

Then $G_j \subset U^\alpha_j$ follows from (1) and (4). It follows from (4) and $V^\Delta < U$ that $\cup \{ G_j \mid j \in \beta \} = K$. Hence by (5) we obtain $\cap \{ G_j \mid j \in \beta \} \neq \emptyset$.

Select a point $x \in \cap \{ G_j \mid j \in \beta \}$. Let $V(\beta)$ be an arbitrary element of V such that $x \in V(\beta)$. Then $V(\beta) \subset U_j$ for each $j \in \beta$, because $x \in G_j$. (Recally (6).) On the other hand $x \notin U_i$ for each $i \notin \beta$ follows from (1) and (4). Hence $x \in V(\beta) - U_i$, proving $V(\beta) \not\subset U_i$ for each $i \notin \beta$.

Thus (3) and accordingly our claim (2) is proved. This implies that

$$\Delta_k(X) \geq \binom{k}{1} + \dots + \binom{k}{n+1} .$$

In case of $k \leqq n+1$, $\dim X \geqq k-1$ holds. Thus by use of the above results we obtain

$$\Delta_k(X) \geqq \binom{k}{1} + \ldots + \binom{k}{k} = 2^k - 1 \ .$$

Theorem VII. 16. *Let X be an infinite normal space with* $\dim X = n$ $(0 \leqq n \leqq \infty)$. *Let k be a natural number. Then*

$$\Delta_k(X) = 2^k - 1 \quad \text{if} \quad k \leqq n+1 \ ,$$
$$\Delta_k(X) = \binom{k}{1} + \ldots + \binom{k}{n+1} \quad \text{if} \quad k \geqq n+1 \ .$$[23]

Proof. Combine B) and D).

Corollary. *Let X be an infinite normal space. Then*

$$\dim X = \lim_{k \to \infty} \frac{\log \Delta_k(X)}{\log k} - 1 \ .$$

Proof. This follows directly from Theorem VII.16.

VII. 8. Characterizations in terms of $C(X)$

In IV.8 we characterized dim of a compact metric space R by the ring $C(R)$ of all real-valued continuous functions on R . In the present section we shall discuss two characterizations of $\dim X$ of a normal space X in terms of $C(X)$. These characterizations are based on quite different ideas as compared to those of IV.8 and are due to M. Canfell [1] and J. Hejcman [2], respectively.

Definition VII. 3. *Let us denote by C the ring $C(X)$ of continuous real-valued functions of a topological space X. A subset $fC = \{ f\varphi \mid \varphi \in C \}$ of C is called a principal ideal, where f is a fixed element of C . A finite set $\{ f_i C \mid i = 1, \ldots, n \}$ of principal ideals is called uniquely generated if the following holds:*

[23] Theorem VII.16 and its corollary can be extended to a general topological space if we adopt the Katětov-Smirnov dimension. (See J. Bruijning - J. Nagata [1].)

Whenever $f_iC = g_iC$, $i = 1,\dots,n$, *there are* $\theta_i \in C$, $i = 1,\dots,n$ *such that* $f_i = g_i\theta_i$, *and* $\theta_1 C + \dots + \theta_n C = \{\theta_1\varphi_1 + \dots + \theta_n\varphi_n \mid \varphi_1,\dots,\varphi_n \in C\} = C$.

Further we define $\mathrm{p}\text{-}\dim C$ *as the least integer* n *such that every set of* $(n+1)$ *principal ideals is uniquely generated, and* $\mathrm{p}\text{-}\dim C = \infty$ *if there is no such integer.*

Theorem VII. 17. *Let* X *be a non-empty normal space. Then* $\dim X = \mathrm{p}\text{-}\dim C(X)$.

Proof. Let $\dim X \leq n < \infty$. Then suppose $\{f_iC \mid i = 1,\dots,n+1\}$ is a given set of principal ideals. Assume $f_iC(X) = g_iC(X)$ for $i = 1,\dots,n+1$. Then

(1) $$f_i = g_i\varphi_i\ ,\quad i = 1,\dots,n+1\ ,$$
$$g_i = f_i\psi_i\ ,\quad i = 1,\dots,n+1\ ,$$

for some φ_i , $\psi_i \in C(X)$. Hence $g_i = g_i\varphi_i\psi_i$, i.e.

(2) $$g_i(1 - \varphi_i\psi_i) = 0\ .$$

Let

(3) $$Z_i = Z(\varphi_i)\ ,\quad i = 1,\dots,n+1\ ,$$
$$Z_i' = Z(\varphi_i + 1 - \varphi_i\psi_i)\ ,\quad i = 1,\dots,n+1\ ,$$

where we denote by $Z(\varphi)$ the zero set $\{x \in X \mid \varphi(x) = 0\}$ for $\varphi \in C(X)$. Then Z_i and Z_i' are disjoint zero sets, and hence there is $u_i \in C(X)$ such that

(4) $$u_i(Z_i) = 1\ ,\quad u_i(Z_i') = 0\ ,\quad 0 \leqq u_i \leqq 1\ .$$

Define a continuous mapping u from X to I^{n+1} by $u(x) = (u_1(x),\dots,u_{n+1}(x))$, $x \in X$. Then $u(Z_i) \subset S^n$, $u(Z_i') \subset S^n$, where we denote by S^n the boundary of I^{n+1} in E^{n+1} . Since $\dim X \leq n$, by Theorem VII.11, the mapping u is inessential. Thus there is a continuous mapping v from X into S^n such that $v(x) = u(x)$ for $x \in u^{-1}(S^n)$.

Let us express v as $v = (v_1,\dots,v_{n+1})$, where $v_i \in C(X)$, $i = 1,\dots,n+1$. Then put $\theta_i = \varphi_i + v_i(1 - \varphi_i\psi_i)$, $i = 1,\dots,n+1$. Let us prove

(5) $$Z(\theta_1) \cap \dots \cap Z(\theta_{n+1}) = \emptyset\ .$$

To do so, let x be an arbitrary point of X . Since $v(x) \in S^n$, either $v_i(x) = 0$ or $v_i(x) = 1$ holds for some i . Suppose $v_i(x) = 0$; then $u_i(x) \neq 1$, because otherwise we reach the contradiction $v_i(x) = u_i(x) = 1$.

Hence by (4) $x \notin Z_i$. By (3) this implies $\varphi_i(x) \neq 0$. Hence $\theta_i(x) = \varphi_i(x) \neq 0$, i.e. $x \notin Z(\theta_i)$.

Suppose $v_i(x) = 1$; then $u_i(x) \neq 0$, and thus $x \notin Z'_i$ follows from (4). Hence by (3)

$$\theta_i(x) = \varphi_i(x) + 1 - \varphi_i(x)\psi_i(x) \neq 0 \ .$$

Therefore $x \notin Z(\theta_i)$. Thus (5) is proved. This implies that

$$\frac{\theta_i}{\theta_1^2 + \ldots + \theta_{n+1}^2} \in C(X) \ , \ i = 1, \ldots, n+1 \ .$$

Hence

$$\theta_1 C(X) + \ldots + \theta_{n+1} C(X) = C(X) \ ,$$

and it follows from (1) and (2) that

$$g_i \theta_i = g_i \varphi_i + g_i v_i (1 - \varphi_i \psi_i) = g_i \varphi_i = f_i \ .$$

Namely $\{ f_1 C(X), \ldots, f_{n+1} C(X) \}$ is uniquely generated, and hence $\text{p-dim}\, C(X) \leqq n$.

Conversely, assume that $\text{p-dim}\, C(X) \leqq n < \infty$. Let Z_i , Z'_i , $i = 1, \ldots, n+1$, be given pairs of disjoint closed sets of X . Find, by use of Urysohn's lemma, $p_i \in C(X)$ such that $p_i(Z_i) = 1$ and $p_i(Z'_i) = -1$. Put

$$F_i = \{ x \in X \mid p_i(x) \geqq 1/2 \}, \ F'_i = \{ x \in X \mid p_i(x) \leqq -1/2 \} \ ,$$
$$G_i = \{ x \in X \mid p_i(x) \leqq 1/2 \}, \ G'_i = \{ x \in X \mid p_i(x) \geqq -1/2 \} \ .$$

Now find $q_i \in C(X)$ such that $q_i(F_i) = 1$, $q_i(F'_i) = -1$, $-1 \leqq q_i \leqq 1$. Furthermore, select s_i , $t_i \in C(X)$ such that

(1) $$s_i(Z_i) = 1 \ , \ s_i(G_i) = 0 \ , \ 0 \leqq s_i \leqq 1 \ ,$$
$$t_i(Z'_i) = -1 \ , \ t_i(G'_i) = 0 \ , \ -1 \leqq t_i \leqq 0 \ .$$

Put $f_i = s_i + t_i$. Then we claim that

(2) $$f_i = q_i \mid f_i \mid .$$

If $x \in F_i$, then $q_i(x) = 1$, and $f_i(x) = s_i(x) + t_i(x) = s_i(x) \geqq 0$.
If $x \in F'_i$, then $q_i(x) = -1$, and $f_i(x) = s_i(x) + t_i(x) = t_i(x) \leqq 0$.
If $x \in X - (F_i \cup F'_i)$, then $x \in G_i \cap G'_i$, and hence $f_i(x) = s_i(x) + t_i(x) = 0$.

Thus the equality (2) holds in every case. Observe that (2) implies

(3) $$|f_i| = q_i f_i \ .$$

Therefore

$$f_i C(X) = |f_i| C(X) \ , \quad i = 1,\dots,n+1 \ .$$

Since p-dim $C(X) \leq n$, there are $\theta_1,\dots,\theta_{n+1} \in C(X)$ such that $f_i = \theta_i |f_i|$, $i = 1,\dots,n+1$, and $\theta_1 C(X) + \dots + \theta_{n+1} C(X) = C(X)$. The last equality implies that

$$Z(\theta_1) \cap \dots \cap Z(\theta_{n+1}) = \emptyset \ .$$

Put $U_i = \{ x \in X | \ \theta_i(x) > 0 \}$. Then, since for each $x \in Z_i$ $f_i(x) = s_i(x) + t_i(x) = 1$ follows from (1), we obtain $Z_i \subset U_i$. On the other hand, for each $x \in Z_i'$, $f_i(x) = -1$ follows from (1), and hence $Z_i' \cap U_i = \emptyset$.
It is obvious that

$$\bigcap_{i=1}^{n+1} B(U_i) \subset \bigcap_{i=1}^{n+1} Z(\theta_i) = \emptyset \ .$$

Thus by Theorem VII.8, dim $X \leq n$. Hence the theorem is proved. [24]

We now turn to another method of characterization. It is interesting that dim can be characterized in terms of the concept "partition of unity" which is important in topology and analysis, though the characterization is not purely of algebraic nature.

Definition VII. 4. *A subset D of $C(X)$ is called a partition of unity if every element of D is non-negative and $\sum_{f \in D} f = 1$ (meaning $\sum_{f \in D} f(x) = 1$ for all $x \in X$) . If D contains only finitely many elements, the partition is called finite. A partition of unity D is said to be subordinated to a covering $\mathfrak{U}$ of X if $\{ P(f) \mid f \in D \} < \mathfrak{U}$, where $P(f) = \{x \in X | \ f(x) \neq 0 \}$.*

To every partition of unity D and covering $\mathfrak{U}$ the following real numbers can be associated:

$$\lambda D = \inf \{ \sup \{ f(x) \mid f \in D \} \mid x \in X \} \quad \text{and}$$

[24] M. Canfell [1] proved the theorem in a more general form.

$$c(U) = \sup \left\{ \lambda D \;\middle|\; \begin{array}{l} D \text{ is a finite partition of} \\ \text{unity subordinated to } U \end{array} \right\} .$$

Theorem VII. 18. *Let X be a non-empty normal space and Ω the collection of all finite open coverings of X. Then* $\inf\{c(U) \mid U \in \Omega\} = 1/(\dim X + 1)$.

Proof. Assume $\dim X \leqq n < \infty$. Then for each $U \in \Omega$ there is $V = \{V_i \mid i = 1,\dots,k\}$ such that $V < U$, $\operatorname{ord} V \leqq n+1$. Let $W = \{W_i \mid i = 1,\dots,k\} \in \Omega$ satisfy $\overline{W}_i \subset V_i$. Then select $f_i \in C(X)$, $i = 1,\dots,k$, such that $f_i(\overline{W}_i) = 1$, $f_i(X - V_i) = 0$, $0 \leqq f_i \leqq 1$.

Put

$$g_i = f_i \Big/ \sum_{j=1}^{k} f_j , \quad i = 1,\dots,k .$$

Then $D = \{g_1,\dots,g_k\}$ is a partition of unity subordinated to U . Let x be an arbitrary point of X ; then $g_i(x) \neq 0$ for at most $n+1$ distinct elements of D , because $\operatorname{ord} V \leqq n+1$, and $g_i = f_i = 0$ holds outside of V_i . Since $\sum_{i=1}^{k} g_i(x) = 1$ there exists i with $g_i(x) \geqq 1/(n+1)$. Thus $\lambda D \geqq 1/(n+1)$, which implies $c(U) \geqq 1/(n+1)$, and accordingly

$$\inf\{c(U) \mid U \in \Omega\} \geqq 1/(n+1) .$$

Thus

$$\inf\{c(U) \mid U \in \Omega\} \geqq 1/(\dim X + 1) .$$

Conversely, assume $\dim X \geqq n$. Then there is $U \in \Omega$ such that every open refinement V of U satisfies $\operatorname{ord} V \geqq n+1$. Assume that $\lambda D > 1/(n+1)$ holds for some finite partition of unity $D = \{f_i \mid i = 1,\dots,k\}$ subordinated to U . Then $\lambda D > 1/(n+1) + \varepsilon$ for some $\varepsilon > 0$, and hence $\sup\{f_i(x) \mid i = 1,\dots,k\} > 1/(n+1)$ for every $x \in X$, i.e. $f_i(x) > 1/(n+1)$ holds for every $x \in X$ and some i .

Now, put $V_i = \{x \mid f_i(x) > 1/(n+1)\}$; then $V = \{V_1,\dots,V_k\} \in \Omega$. Since $V < U$ follows from the fact that D is subordinated to U , we have $\operatorname{ord} V \geqq n+1$.

Assume that $\operatorname{ord}_x V \geqq n+1$ holds at $x \in X$. Thus there are $n+1$ distinct elements, say $f_1,\dots,f_{n+1}$ of D such that $f_i(x) > 1/(n+1)$ for $i = 1,\dots,n+1$. Hence

$$\sum_{i=1}^{k} f_i(x) \geqq \sum_{i=1}^{n+1} f_i(x) > 1 ,$$

which is a contradiction.

Therefore $\lambda D \leq 1/(n+1)$ holds for all finite partitions of unity subordinated to U. Hence $c(U) \leq 1/(n+1)$, which implies

$$\inf \{ c(U) \mid U \in \Omega \} \leq 1/(\dim X + 1) \ ,$$

proving the theorem. [25]

[25] Let Ω' be the set of all locally finite open coverings of X. Define $c'(U) = \sup \{ D \mid D$ is a (not necessarily finite) partition of unity subordinated to $U \}$. Then $\inf \{ c'(U) \mid U \in \Omega' \} = \inf \{ c'(U) \mid U \in \Omega \} = \inf \{ c(U) \mid U \in \Omega \} = 1/(\dim X + 1)$ can be easily proved for a normal space X in a similar way. J. Hejcman [2] proved the theorem in a more general form.

CHAPTER VIII

DIMENSION AND COHOMOLOGY

The purpose of this chapter is to investigate dimension theory from the point of view of algebraic topology. This point of view is originally due to P. Alexandroff [2] and has been greatly developed by his school [1]. Here we shall not try to cover the extensive developments in this field which could easily be the subject of another book. We shall only aim at a characterization of dimension in terms of cohomology adopting the method due to C. H. Dowker [1] which goes back to E. Čech [3]. Recent investigations on the subject show that for non-compact spaces cohomology is more practical than homology, and that is the reason why we deal mainly with cohomology in this chapter. The final section will contain a brief account of homology in dimension theory. Throughout the chapter every space considered will be a paracompact T_2-space unless the contrary is explicitly stated, though some discussions are valid for more general spaces. [2]

VIII. 1. Homology group and cohomology group of a complex

Definition VIII. 1. *Let E be a set of elements which are called (abstract) vertices and K a collection of finite subsets of E such that every subset of a*

[1] See P. Alexandroff [6], [8].
[2] K. Morita [8] gave a cohomological characterization of dimension of general topological spaces.

set belonging to K *also belongs to* K *; then we call* K *an (abstract) complex. We call a set of* $n+1$ *vertices* $a_0, \ldots, a_n$ *belonging to* K *an (abstract) n-simplex or n-dimensional simplex of* K *and denote it by* $|a_0, \ldots, a_n|$[3] *. An m-simplex* $|a_{i_0}, \ldots, a_{i_m}|$ *whose vertices are chosen from* $\{a_0, \ldots, a_n\}$ *is called an m-face of* $|a_0, \ldots, a_n|$ *. If* K *contains an n-simplex, but no n'-simplex for* $n' > n$ *, then* K *is called an n-complex or n-dimensional complex. A complex* K *is called a finite complex if it contains only finitely many simplices and is called a star-finite complex if every simplex of* K *is a face of at most finitely many simplices of* K *.*

Example VIII. 1. Let K_0 be a geometrical complex [4] in a Euclidean space and E the set of the vertices of K_0 .

We say that $n+1$ vertices of E form an n-simplex if and only if they are the vertices of an n-dimensional (geometrical) simplex belonging to K_0 . Then we get an abstract complex K . Generally every finite abstract n-complex K is *isomorphic* to a geometrical complex K_0 in E^{2n+1} , i.e. the vertices of K are mapped onto the vertices of K_0 by a one-to-one mapping which gives rise to a one-to-one correspondence between the simplices of K and those of K_0 . K_0 is called an n-*polyhedron* if it is regarded as a point set in the Euclidean space. In the case where K is infinite we can realize K as a metric space in the following way. Let V be the set of all vertices of K . Put $K_0 = \{x \mid x$ is a real-valued function on V such that $0 \leqq x \leqq 1$, $\sum_{p \in V} x(p) = 1$, $\{p \in V \mid x(p) > 0\}$ is a simplex of $K\}$. The distance between two points x and y of K_0 is defined by $\sum_{p \in V} |x(p) - y(p)|$. We shall call K_0 the *realization* of K . For each $x \in K_0$, $\{x(p) \mid p \in V\}$ is called the *barycentric coordinate* of x . Let $s = |a_0, \ldots, a_n| \in K$. Then the subset $K_0(s) = \{x \in K_0 \mid x(p) = 0$ if $p \neq a_0, \ldots, a_n\}$ of K_0 is homeomorphic to the geometrical simplex $[a_0, \ldots, a_n]$, and $K_0 = \cup\{K_0(s) \mid s \in K\}$. Where there is no fear of confusion the same notation is used for an abstract complex and for its realization.

[3] On the other hand, we shall denote by $[a_0, \ldots, a_n]$ the geometrical simplex with the vertices $a_0, \ldots, a_n$, where $a_0, \ldots, a_n$ are points of E^m in a general position and $[a_0, \ldots, a_n] = \{x \in E^m \mid x = \sum_{i=0}^n \lambda_i a_i$, $\sum_{i=0}^n \lambda_i = 1$, $0 \leqq \lambda_i \leqq 1$ for $i = 0, \ldots, n\}$.

[4] A finite set K_0 of geometrical simplices in Euclidean space is called a *geometrical complex* if (i) every face of a simplex belonging to K_0 also belongs to K_0 . (ii) the intersection of two simplices belonging to K_0 is a face of each of those simplices.

Example VIII. 2. Let U be a covering of a topological space R. We regard the members of U as the vertices and define that $U_0, \ldots, U_n \in U$ will make up an n-simplex if and only if $U_0 \cap \ldots \cap U_n \neq \emptyset$. Then we obtain an abstract complex which is called the *nerve* of U and denoted by $N(U)$ through this chapter. As a matter of fact essentially we used this concept in the proof of the imbedding theorem. The nerve of a finite or star-finite covering is a finite or star-finite complex respectively.

Definition VIII. 2. *Let $|a_0, \ldots, a_n|$ be an n-simplex belonging to a complex K. We consider an oriented n-simplex with the vertices $a_0, \ldots, a_n$ in this order and denote it by $(a_0, \ldots, a_n)$. For an oriented n-simplex x^n we denote by $|x^n|$ the non-oriented simplex consisting of the vertices of x^n and call it the absolute simplex of x^n. We define $(a_{i_0}, \ldots, a_{i_n}) = (a_0, \ldots, a_n)$ or $(a_{i_0}, \ldots, a_{i_n}) = -(a_0, \ldots, a_n)$ according to whether the permutation $\begin{pmatrix} 0 \ldots n \\ i_0 \ldots i_n \end{pmatrix}$ is even or odd. For $n = 0$ we consider two oriented 0-simplices a_0 and $-a_0$.*

Definition VIII. 3. *Let us denote by G and K a commutative group and an abstract complex, respectively. We shall write the operation of G additively. Let us denote by $\{|x_\alpha^n| \mid \alpha \in A\}$ the totality of the abstract simplices of K. Then an (n, G)-chain of K is a linear form*

$$\varphi^n = \sum_{\alpha \in A} g^\alpha x_\alpha^n$$

of the n-simplices x_α^n, $\alpha \in A$ of K with the coefficients $g^\alpha \in G$. Let $\varphi^n = \sum_\alpha g^\alpha x_\alpha^n$ *and* $\psi^n = \sum_\alpha h^\alpha x_\alpha^n$ *be two (n, G)-chains of K; then we define the sum of φ^n and ψ^n by*

$$\varphi^n + \psi^n = \sum_\alpha (g^\alpha + h^\alpha) x_\alpha^n .$$

We denote by 0 the (n, G)-chain $\sum_\alpha g^\alpha x_\alpha$ with $g^\alpha = 0$ for every α. Then the (n, G)-chains of K form a commutative group denoted by $C_n(K, G)$.

Definition VIII. 4. *Let x_α^n and x_β^{n-1} be oriented simplices of a complex K. Then we define the incidence number η_β^α between them as follows:*

$$\eta_\beta^\alpha = 0 \quad \text{if} \quad x_\beta^{n-1} \text{ is no face of } x_\alpha^n ,$$
$$\eta_\beta^\alpha = (-1)^i \varepsilon \quad \text{if} \quad x_\alpha^n = (a_0, \ldots, a_n) , \quad x_\beta^{n-1} = \varepsilon(a_0, \ldots, \hat{a}_i, \ldots, a_n) ,$$

where $\varepsilon = +1$ *or* -1, *and* $(a_0, \dots, \hat{a}_i, \dots, a_n) = (a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$. *To define the incidence number for* $n = 0$, *we consider one -1-simplex* x^{-1} *containing no vertex and belonging to* K. *We define the incidence number between* a *and* x^{-1} *as* 1 *and the incidence number between* $-a$ *and* x^{-1} *as -1. Let* $\varphi^n = \sum_{\alpha \in A} g^\alpha x^n_\alpha$ *be an* (n,G)*-chain of* K. *We express by*

$$\{|x^{n+1}_\gamma| \mid \gamma \in C\},\ \{|x^n_\alpha| \mid \alpha \in A\},\ \{|x^{n-1}_\beta| \mid \beta \in B\},$$

the totalities of the absolute $(n+1)$, n, $(n-1)$*-simplices of* K, *respectively. Then the boundary* $\partial\varphi^n$ *of* φ^n *is the* $(n-1,G)$*-chain defined by*

$$\partial\varphi^n = \sum_{\beta \in B} \Big(\sum_{\alpha \in A} g^\alpha \eta^\alpha_\beta \Big) x^{n+1}_\beta ,$$

where we assume that K *is star-finite.*[5] *The coboundary* $\delta\varphi^n$ *of* φ^n *is the* $(n+1,G)$*-chain defined by*

$$\delta\varphi^n = \sum_{\gamma \in C} \Big(\sum_{\alpha \in A} g^\alpha \eta^\gamma_\alpha \Big) x^{n+1}_\gamma ,$$

where K *is not necessarily star-finite.* ($\delta\varphi^n = 0$ *if* K *contains no* $(n+1)$*-simplex.*)

A) $\partial\partial\varphi^n = 0$, $\delta\delta\varphi^n = 0$.

Proof. Assume $\varphi^n = \sum_\alpha g^\alpha x^n_\alpha$. Then

$$\partial\varphi^n = \sum_{\alpha,\beta} g^\alpha \eta^\alpha_\beta x^{n-1}_\beta , \quad \text{and} \quad \partial\partial\varphi^n = \sum_{\alpha,\beta,\gamma} g^\alpha \eta^\alpha_\beta \eta'^\beta_\gamma x^{n-2}_\gamma ,$$

where η'^β_γ denotes the incidence number between x^{n-1}_β and x^{n-2}_γ. Suppose

$$x^n_\alpha = (a_0, \dots, a_n) ,$$
$$x^{n-1}_\beta = \varepsilon(a_0, \dots, \hat{a}_i, \dots, a_n) ,\ x^{n-1} = \varepsilon'(a_0, \dots, \hat{a}_k, \dots, a_n) ,$$
$$x^{n-2}_\gamma = \varepsilon''(a_0, \dots, \hat{a}_i, \dots, \hat{a}_k, \dots, a_n) .$$

[5] In case all but a finite number of the g^α's vanish, K need not be star-finite.

Then

$$g^{\alpha}\eta^{\alpha}_{\beta}\eta'^{\beta}_{\gamma} = \varepsilon(-1)^{i}\varepsilon\varepsilon''(-1)^{k-1}g^{\alpha} = \varepsilon''(-1)^{i+k-1}g^{\alpha}$$

and

$$g^{\alpha}\eta^{\alpha}_{\beta'}\eta'^{\beta'}_{\gamma} = \varepsilon'(-1)^{k}\varepsilon'\varepsilon''(-1)^{i}g^{\alpha} = \varepsilon''(-1)^{i+k}g^{\alpha}$$

cancel each other. Therefore $\sum_{\beta} g^{\alpha}\eta^{\alpha}_{\beta}\eta'^{\beta}_{\gamma} = 0$ for every α, γ, which implies $\partial\partial\varphi^{n} = 0$. $\delta\delta\varphi^{n} = 0$ can be proved in a similar way.

Definition VIII. 5. *An (n,G)-chain is called an (n,G)-cycle if its boundary is equal to zero. An (n,G)-chain is called an (n,G)-cocycle if its coboundary is equal to zero. We denote by $Z_n(K,G)$ and $Z^n(K,G)$ the set of the (n,G)-cycles and of the (n,G)-cocycles of K, respectively. It is easily seen that they are both subgroups of $C_n(K,G)$.*

Furthermore, by A), the (n,G)-chains which are the boundaries of $(n+1,G)$-chains form a subgroup of $Z_n(K,G)$ which is denoted by $B_n(K,G)$. The chains in $B_n(K,G)$ are called bounding cycles. The (n,G)-chains which are the coboundaries of $(n-1,G)$-chains form a subgroup of $Z^n(K,G)$ which is denoted by $B^n(K,G)$. The chains in $B^n(K,G)$ are called cobounding cocycles.

Now we call the difference group

$$H_n(K,G) = Z_n(K,G) - B_n(K,G)$$

the n-dimensional homology group of K with the coefficient group G and the difference group

$$H^n(K,G) = Z^n(K,G) - B^n(K,G)$$

the n-dimensional cohomology group of K with the coefficient group G. The elements of $H_n(K,G)$ and $H^n(K,G)$ are called n-dimensional homology classes and n-dimensional cohomology classes, respectively. Two cycles (cocycles) belonging to the same homology (cohomology) classes are called homologous (cohomologous). We denote by $\varphi^n \sim \psi^n$ in K the fact that two cocycles φ^n and ψ^n of K are cohomologous.

Definition VIII. 6. *Let K be a complex and L a subcollection of K. If every face of a simplex of L belongs to L, then we call L a subcomplex of K. Suppose $\varphi^n = \sum_{\alpha \in A} g^\alpha x^n_\alpha$ is an (n,G)-chain of K. Then we denote by $\{ | x^n_\beta | \,|\, \beta \in B \}$ the totality of the absolute n-simplices of L. Now, to φ^n there corresponds an (n,G)-chain: $\psi^n = h_L \varphi^n$ of L such that*

$$\psi^n = h_L \varphi^n = \sum_{\beta \in B} g^\beta x^n_\beta .$$

We call ψ^n the restriction of φ^n to L and φ^n an extension of ψ^n over K. Conversely to an (n,G)-chain $\psi^n = \sum_{\beta \in B} g^\beta x^n_\beta$ of L there corresponds an (n,G)-chain $\varphi^n = h_K \psi^n$ of K such that

$$\varphi^n = h_K \psi^n = \sum_{\alpha \in A} g^\alpha x^n_\alpha ,$$

where $g^\alpha = 0$ for $\alpha \notin B$.

B) h_L induces a homomorphism also denoted by h_L of $H^n(K,G)$ into $H^n(L,G)$. h_K induces a homomorphism also denoted by h_K of $H_n(L,G)$ into $H_n(K,G)$.

Proof. It is clear that h_L is a homomorphism of $C_n(K,G)$ into $C_n(L,G)$ and satisfies $h_L \delta \varphi^n = \delta h_L \varphi^n$. Hence h_L maps every cocycle of K to a cocycle of L and every cobounding cocycle of K to a cobounding cocycle of L. Therefore h_L induces a homomorphism of $H^n(K,G)$ into $H^n(L,G)$. The proof for h_K is similar.

Definition VIII. 7. *The homomorphisms h_L and h_K in B) are called the natural homomorphisms. Suppose h_L maps $\tilde{e} \in H^n(K,G)$ to $e \in H^n(L,G)$; then $\tilde{e}$ is called an extension of e, and e is said to be extendible over K.*

C) Let $\tilde{e} \in H^n(K,G)$ be an extension of $e \in H^n(L,G)$. If ψ^n is a cocycle of L which represents e, then there exists a cocycle φ^n of K which represents $\tilde{e}$ and is an extension of ψ^n, i.e. $h_L \varphi^n = \psi^n$.

Proof. If $\tilde{e} = 0$, then $e = 0$. Hence ψ^n is a cobounding cocycle of L, i.e. $\psi^n = \delta \psi^{n-1}$ for some $(n-1,G)$-chain ψ^{n-1} of L. Then $\delta h_K \psi^{n-1}$ is a cobounding cocycle of K satisfying

$$h_L \delta h_K \psi^{n-1} = \delta h_L h_K \psi^{n-1} = \delta \psi^{n-1} = \psi^n .$$

Thus $\delta h_K \psi^{n-1} = \varphi^n$ is the desired extension of ψ^n. If $\tilde{e}$ is an arbitrary element, we take a cocycle $\varphi_1^n \in \tilde{e}$. Then $h_L \varphi_1^n \in e$, and hence $\psi^n - h_L \varphi_1^n$ is a cobounding cocycle of L. Hence, as above, there exists a cobounding cocycle φ_2^n of K which is an extension of $\psi^n - h_L \varphi_1^n$ over K. Put $\varphi^n = \varphi_1^n + \varphi_2^n$; then since $\varphi^n \in \tilde{e}$, and $h_L \varphi^n = h_L \varphi_1^n + h_L \varphi_2^n = \psi^n$, φ^n is the desired extension.

Definition VIII. 8. *Let K and L be two complexes and f a mapping which maps each vertex of K to a vertex of L. f is called a simplicial mapping if it satisfies the following condition: If vertices $a_0, \dots, a_n$ form a simplex of K, then $f(a_0), \dots, f(a_n)$ form a simplex of L, where some of $f(a_0), \dots, f(a_n)$ may coincide. If for an n-simplex $x^n = (a_0, \dots, a_n)$, $(f(a_0), \dots, f(a_n))$ is also an n-simplex, then we denote it by $f(x^n)$.*

Let f be a simplicial mapping of K into L. For an (n,G)-chain $\psi^n = \sum_\beta h^\beta y_\beta^n$ of L, we define an (n,G)-chain φ^n of K by

$$\varphi^n = f^* \psi^n = \sum_\alpha \Big(\sum_\beta h^\beta f_\beta^\alpha\Big) x_\alpha^n ,$$

where

$$f_\beta^\alpha = \begin{cases} 1 & \text{if } f(x_\alpha^n) = y_\beta^n \\ -1 & \text{if } f(x_\alpha^n) = -y_\beta^n , \\ 0 & \text{otherwise.} \end{cases}$$

Suppose for every y_β^n of L, there exist only finitely many simplices x_α^n of K such that $f(x_\alpha^n) = \pm y_\beta^n$. Then for an (n,G)-chain $\varphi^n = \sum_\alpha g^\alpha x_\alpha^n$ of K we define an (n,G)-chain ψ^n of L by

$$\psi^n = f\varphi^n = \sum_\beta \Big(\sum_\alpha g^\alpha f_\beta^\alpha\Big) y_\beta^n .$$

D) $\delta f^* \psi^n = f^* \delta \psi^n$, $\partial f \varphi^n = f \partial \varphi^n$.

Proof. We shall prove the first formula only; the second one can be proved in a similar way. Let $\psi^n = \sum_\beta h^\beta y_\beta^n$. Then

$$\delta f^* \psi^n = \delta \sum_\alpha \Big(\sum_\beta h^\beta f_\beta^\alpha\Big) x_\alpha^n = \sum_{\alpha,\gamma} \Big(\sum_\beta h^\beta f_\beta^\alpha\Big) \eta_\alpha^\gamma x_\gamma^{n+1} =$$

$$= \sum_\gamma \Big(\sum_{\alpha,\beta} h^\beta f_\beta^\alpha \eta_\alpha^\gamma\Big) x_\gamma^{n+1} = \sum_\gamma \Big(\sum_{\gamma',\beta} h^\beta f_{\gamma'}^\gamma \eta_\beta^{\gamma'}\Big) x_\gamma^{n+1} = f^* \delta \psi^n ,$$

because $\sum_{\alpha,\beta} h^{\beta} f^{\alpha}_{\beta} \eta^{\gamma}_{\alpha} = \sum_{\gamma',\beta} h^{\beta} f^{\gamma}_{\gamma'} \eta^{\gamma'}_{\beta}$ is easy to check.

Since f^* is easily seen to be a homomorphism of $C_n(L,G)$ into $C_n(K,G)$, we can prove, by use of D) and an argument like the one in B), that

E) f^* induces a homomorphism of $H^n(L,G)$ into $H^n(K,G)$. f induces a homomorphism of $H_n(K,G)$ into $H_n(L,G)$.[6]

From now on we shall be chiefly concerned with cohomology with coefficient group I , the additive group of integers, though some of the following results may be true even for general coefficient groups. Accordingly, we mean by an n-chain, n-cocycle, $H^n(K)$ etc. an (n,I)-chain, (n,I)-cocycle, $H^n(K,I)$ etc. respectively.

Definition VIII. 9. *Let K be a complex with the not necessarily countable set of vertices $a_1, a_2, \ldots$. In addition we consider the vertices $b_1, b_2, \ldots$ corresponding to $a_1, a_2, \ldots$. We define $(b_{i_0}, b_{i_1}, \ldots, b_{i_\lambda}, a_{i_\lambda}, \ldots, a_{i_n})$ $(0 \leq \lambda \leq n)$ to be a simplex if $(a_{i_0}, a_{i_1}, \ldots, a_{i_\lambda}, \ldots, a_{i_n})$ is an n-simplex of K. We denote by PK the complex which consists of those simpleces and their faces. For an n-simplex $x^n = (a_{i_0}, \ldots, a_{i_n})$ of K we define an $(n+1)$-chain Px^n of PK by*

$$Px^n = \sum_{\lambda=0}^{n} (-1)^{\lambda} (b_{i_0}, b_{i_1}, \ldots, b_{i_\lambda}, a_{i_\lambda}, \ldots, a_{i_n}) ,$$

and more generally

$$P\varphi^n = \sum_{\alpha} g^{\alpha} P x^n_{\alpha}$$

for a chain $\varphi^n = \sum_{\alpha} g^{\alpha} x^n_{\alpha}$ of K .

F) Let $x^n = (a_0, \ldots, a_n)$ be an n-simplex of a complex K and $y^n = (b_0, \ldots, b_n)$ the n-simplex of PK which consists of the vertices $b_0, \ldots, b_n$ corresponding to $a_0, \ldots, a_n$, respectively. Then $\partial Px^n = x^n - y^n - P\partial x^n$.

[6] As a matter of fact B) is a special case of E) in which f is the identity mapping of L into K .

Proof. We put, for brevity.

$$\begin{aligned}
z_j^{n+1} &= (b_0,\dots,b_j,a_j,\dots,a_n)\ ,\\
z_j^n[b_i] &= (b_0,\dots,\hat{b}_i,\dots,b_j,a_j,\dots,a_n)\ ,\\
z_j^n[a_k] &= (b_0,\dots,b_j,a_j,\dots,\hat{a}_k,\dots,a_n)\ ,\\
x_i^{n-1} &= (a_0,\dots,\hat{a}_i,\dots,a_n)\ .
\end{aligned}$$

Then

$$\begin{aligned}
Px^n &= \sum_{j=0}^{n} (-1)^j z_j^{n+1}\\
\partial z_j^{n+1} &= \sum_{i=0}^{j} (-1)^i z_j^n[b_i] - \sum_{i=j}^{n} (-1)^i z_j^n[a_i]\ ,\\
\partial x^n &= \sum_{i=0}^{n} (-1)^i x_i^{n-1}\ ,\\
Px_i^{n-1} &= \sum_{j=0}^{i-1} (-1)^j z_j^n[a_i] - \sum_{j=i+1}^{n} (-1)^j z_j^n[b_i]\ .
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\partial Px^n + P\partial x^n &= \sum_{j=0}^{n} (-1)^j\Big(\sum_{i=0}^{j} (-1)^i z_j^n[b_i] - \sum_{i=j}^{n} (-1)^i z_j^n[a_i]\Big) +\\
&\quad \sum_{i=0}^{n} (-1)^i\Big(\sum_{j=0}^{i-1} (-1)^j z_j^n[a_i] - \sum_{j=i+1}^{n} (-1)^j z_j^n[b_i]\Big) =\\
&= \sum_{i=0}^{n} (-1)^{2i} z_i^n[b_i] - \sum_{i=0}^{n} (-1)^{2i} z_i^n[a_i] =\\
&= z_0^n[b_0] + \Big(\sum_{i=1}^{n} z_i^n[b_i] - \sum_{i=0}^{n-1} z_i^n[a_i]\Big) - z_n^n[a_n] =\\
&= x^n - y^n\ ,
\end{aligned}$$

because

$$z_{i+1}^n[b_{i+1}] = z_i^n[a_i].$$

This implies the desired relation.

G) Let φ^n be an n-cocycle of PK. We denote by g_0^α the coefficient of x_α^n in the linear form φ^n and by g_1^α that of y_α^n, where x_α^n and y_α^n denote simplices which correspond to each other as mentioned in F). Then the n-chains $\varphi_0^n = \sum_\alpha g_0^\alpha x_\alpha^n$ $(= h_K \varphi^n)$ and $\varphi_1^n = \sum_\alpha g_1^\alpha x_\alpha^n$ of K are cohomologous in K.

Proof. Generally, for an n-finite chain $\psi_1^n = \sum_\mu a^\mu z_\mu^n$ and an n-chain $\psi_2^n = \sum_\mu b^\mu z_\mu^n$ of PK we define

$$I(\psi_1^n, \psi_2^n) = \sum_\mu a^\mu b^\mu ,$$

where we mean by a finite chain a chain only finitely many of whose coefficients do not vanish. Then, for an $(n+1)$-finite chain $\psi_1^{n+1} = \sum_\lambda a^\lambda z_\lambda^{n+1}$ and an n-chain $\psi_2^n = \sum_\mu b^\mu z_\mu^n$ we obtain

$$I(\partial\psi_1^{n+1}, \psi_2^n) = \sum_{\lambda,\mu} a^\lambda \eta_\mu^\lambda b^\mu = I(\psi_1^{n+1}, \delta\psi_2^n) .$$

Hence, we obtain

$$I(\partial\psi^{n+1}, \varphi^n) = I(\psi^{n+1}, \delta\varphi^n) = 0$$

for the cocycle φ^n and an arbitrary $(n+1)$-finite chain ψ^{n+1} because $\delta\varphi^n = 0$. Thus, it follows from F) that

$$0 = I(\partial P x_\alpha^n, \varphi^n) = I(x_\alpha^n, \varphi^n) - I(y_\alpha^n, \varphi^n) - \sum_\beta \eta_\beta^\alpha I(P x_\beta^{n-1}, \varphi^n) .$$

Since

$$I(x_\alpha^n, \varphi^n) = g_0^\alpha \quad\text{and}\quad I(y_\alpha^n, \varphi^n) = g_1^\alpha ,$$

putting

$$I(P x_\beta^{n-1}, \varphi^n) = g^\beta ,$$

we obtain

$$g_0^\alpha - g_1^\alpha - \sum_\beta \eta_\beta^\alpha g^\beta = 0 .$$

We consider an $(n-1)$-chain $\varphi^{n-1} = \sum_\beta g^\beta x_\beta^{n-1}$ of K. Then it follows from the above formula that in K

$$\delta\varphi^{n-1} = \sum_{\alpha,\beta} g^\beta \eta_\beta^\alpha x_\alpha^n = \sum_\alpha (g_0^\alpha - g_1^\alpha) x_\alpha^n = \sum_\alpha g_0^\alpha x_\beta^n - \sum_\alpha g_1^\alpha x_\alpha^n = \varphi_0^n - \varphi_1^n$$

proving that φ_0^n and φ_1^n are cohomologous in K.

Example VIII. 3. The oriented n-simplex $(-1)^k (a_0, \dots, \hat{a}_k, \dots, a_{n+1})$ is called a (positively) oriented face of the oriented $(n+1)$-simplex $(a_0, \dots, a_{n+1})$, i.e. the oriented faces are oriented so that the incidence numbers are equal to 1.

Let T^{n+1} be the complex consisting of an $(n+1)$-simplex $x_0^{n+1} = (a_0, a_1, \dots, a_{n+1})$ and its faces. Let $\varphi^n = \sum_{k=0}^{n+1} g^k x_k^n$ be an n-cocycle where $x_k^n = (-1)^k (a_0, \dots, \hat{a}_k, \dots, a_{n+1})$, $k = 0, \dots, n+1$, are the oriented faces of x_0^{n+1}. Then

$$\delta\varphi^n = \sum_{k=0}^{n+1} g^k x_0^{n+1} = 0 \text{ , i.e. } \sum_{k=0}^{n+1} g^k = 0 .$$

Let $x_{k\ell}^{n-1} = (a_0, \dots, \hat{a}_k, \dots, \hat{a}_\ell, \dots, a_{n+1})$, $\psi^{n-1} = \sum h^{k\ell} x_{k\ell}^{n-1}$, where $h^{k\,k+1} = g^0 + g^1 + \dots + g^k$ and $h^{k\ell} = 0$, otherwise. Then we easily see that $\delta\psi^{n-1} = \varphi^n$.

Therefore we obtain $H^n(T^{n+1}) = 0$. As a matter of fact, we can show that $H^m(T^{n+1}) = 0$ for every m.

On the other hand, we easily see that if K is a union of disjoint complexes K_α, then $H^n(K)$ $(n > 0)$ is the direct sum of $H^n(K_\alpha)$. Hence, if K consists of disjoint simplices and their faces, then $H^n(K) = 0$ for $n > 0$.

Example VIII. 4. Let K be an abstract complex. Then we define the *barycentric subdivision* K' of K as follows. We associate a vertex $a(x^n)$ with each absolute simplex $|x^n|$ of K. Geometrically, $a(x^n)$ is the barycenter of the simplex. Then K' consists of all the simplices

$$|a(x_{\alpha_0}^{n_0}), a(x_{\alpha_1}^{n_1}), \dots, a(x_{\alpha_i}^{n_i})|$$

where $n_0 > n_1 > \dots > n_i$, and $x_{\alpha_h}^{n_h}$ is a face of $x_{\alpha_{h-1}}^{n_{h-1}}$. If the vertices $a(x_{\alpha_h}^{n_h})$ are all barycenters of $|x^n|$ or of its faces, then this simplex is called a *subsimplex* of $|x^n|$. If $x^n = (a_0, \dots, a_n)$ is an oriented simplex, then we can decide the orientations of the n-subsimplices of x^n by deciding just one of them as

$$(a((a_0, \dots, a_n)); a((a_1, \dots, a_n)), \dots, a((a_n)))$$

which is called the *oriented n-subsimplex* of x^n . (For example, in case of $n=2$

$$|a((a_0,a_1,a_2)),a((a_1,a_2)),a((a_1))|$$

is oriented as

$$-(a((a_0,a_2,a_1)),a((a_2,a_1)),a((a_1))$$

because $(a_0,a_2,a_1)=-(a_0,a_1,a_2)$.) Suppose the n-simplices of K and K' are oriented as above. To each n-chain $\psi^n=\sum_\beta g^\beta y^n_\beta$ of K' we assign an n-chain $\varphi^n=\pi\psi^n$ of K by

$$\varphi^n=\pi\psi^n=\sum_{\alpha,\beta} g^\beta\xi^\alpha_\beta x^n_\alpha ,$$

where

$$\xi^\alpha_\beta=\begin{cases} 1 & \text{if } y^n_\beta \text{ is an oriented } n\text{-subsimplex of } x^n_\alpha , \\ 0 & \text{otherwise.}\end{cases}$$

Then we can show that π satisfies $\pi\delta\varphi^n=\delta\pi\varphi^n$ and, moreover, that π induces an isomorphism of $H^n(K')$ onto $H^n(K)$.[7]

Repeating this process of barycentric subdivision we obtain complexes K' , K'' , K''',... whose cohomology groups are all isomorphic to that of K . We call them *successive barycentric subdivisions.*

Let L be a subcomplex of K . If, in constructing a barycentric subdivision of K , we choose as the additional vertices only the barycenters of the simplices which do not belong to L , then we obtain the *barycentric subdivision* K' of K *mod* L . For such a barycentric subdivision we can also show that the above mapping induces an isomorphism of $H^n(K')$ onto $H^n(K)$.

Example VIII. 5. Let S^n be the complex consisting of all the faces of $\dim\leq n$ of an $(n+1)$-simplex. We call S^n the *elementary n-sphere.* If we suppose the original simplex is oriented and its faces are its oriented faces, then we can speak of the *oriented elementary n-sphere* S^n .

---

[7] A detailed proof can be found in W. Hurewicz and H. Wallman [1].

Let us denote by S^n the oriented elementary n-sphere. Then every n-chain $\varphi^n = \sum g^k x_k^n$ is clearly a cocycle. φ^n is cobounding in S^n if and only if $h_{T^{n+1}}\varphi^n$ is cobounding in T^{n+1}, and, as seen in Example VIII.3, $h_{T^{n+1}}\varphi^n$ is cobounding if and only if $\sum g^k = 0$. Thus each cohomology class in $H^n(S^n)$ is characterized by the value of $\sum g^k$, and hence $H^n(S^n)$ is isomorphic to the coefficient group I. In view of Example VIII.4 this assertion, including the fact that φ^n is cobounding if and only if $\sum g^k = 0$, is also true for any of the successive barycentric subdivisions of S^n.

VIII. 2. Cohomology group of a topological space

Definition VIII. 10. *Let $\mathfrak{U}_\alpha$ and $\mathfrak{U}_\beta$ be open coverings of a topological space R such that $\mathfrak{U}_\beta < \mathfrak{U}_\alpha$. For every member U of $\mathfrak{U}_\beta$ we choose a member U' of $\mathfrak{U}_\alpha$ such that $U' \supset U$. Then we obtain a simplicial mapping of $N(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$ which is called a projection of $N(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$ and denoted by π_α^β. Then, by 1 E), $\pi_\alpha^{*\beta}$ induces a homomorphism of $H^n(N(\mathfrak{U}_\alpha))$ into $H^n(N(\mathfrak{U}_\beta))$.*

A) Let us consider two projections f_1 and f_2 of $N(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$, where $\mathfrak{U}_\beta < \mathfrak{U}_\alpha$. Then they induce the same homomorphism of $H^n(N(\mathfrak{U}_\alpha))$ into $H^n(N(\mathfrak{U}_\beta))$.

Proof. We denote by $a_1, a_2, \dots$ the not necessarily countably many vertices of $N(\mathfrak{U}_\beta)$ and by $b_1, b_2, \dots$ vertices which correspond to $a_1, a_2, \dots$ in a one-to-one manner. We construct the complex $PN(\mathfrak{U}_\beta)$ by use of Definition VIII.9. Let us define a mapping f by $f(a_i) = f_1(a_i)$ and $f(b_i) = f_2(a_i)$.

Then f is obviously a simplicial mapping of $PN(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$. We denote by $N'(\mathfrak{U}_\beta)$ the subcomplex of $PN(\mathfrak{U}_\beta)$ consisting of the vertices $b_1, b_2, \dots$. It is clear that the restrictions of f to $N(\mathfrak{U}_\beta)$ and $N'(\mathfrak{U}_\beta)$ turn out to be f_1 and f_2, respectively.

Now, let φ^n be an n-cocycle of $N(\mathfrak{U}_\alpha)$. It is easy to see that

$$f_1^*\varphi^n \sim f_2^*\varphi^n \quad \text{in } N(\mathfrak{U}_\beta).$$

For $f^*\varphi^n$ is an n-cocycle of $PN(\mathfrak{U}_\beta)$, and $f^*\varphi^n$ restricted to $N(\mathfrak{U}_\beta)$ and $N'(\mathfrak{U}_\beta)$ turns out to be $f_1^*\varphi^n$ and $f_2^*\varphi^n$, respectively.[8] Hence, by 1 G), the desired result follows. Thus A) is proved.

[8] Occasionally, we have regarded f_2 as a mapping of $N'(\mathfrak{U}_\beta)$ into $N(\mathfrak{U}_\alpha)$, since $N(\mathfrak{U}_\beta)$ and $N'(\mathfrak{U}_\beta)$ are isomorphic. But this should cause no confusion.

Definition VIII. 11. *Let* A *be a directed set, i.e. a set between some of whose elements the order is defined such that* $\alpha < \beta$ *and* $\beta < \gamma$ *implies* $\alpha < \gamma$ *and such that for every two elements* α, β *there is a third element* γ *with* $\gamma > \alpha$, $\gamma > \beta$. *Then we consider a collection* $\{ G_\alpha \mid \alpha \in A \}$ *of commutative groups* G_α . *Suppose for each* $\alpha, \beta \in A$ *with* $\alpha < \beta$ *a homomorphism* $\overset{*}{\pi}{}^{\beta}_{\alpha}$ *of* G_α *into* G_β *is defined such that*

$$\overset{*}{\pi}{}^{\gamma}_{\beta}\overset{*}{\pi}{}^{\beta}_{\alpha} = \overset{*}{\pi}{}^{\gamma}_{\alpha} \quad \text{for} \quad \alpha < \beta < \gamma .$$

Then we call $\{ G_\alpha, \overset{*}{\pi}{}^{\beta}_{\alpha} \mid \alpha, \beta \in A ,\ \alpha < \beta \}$ *a directed spectrum. We call* $g_\alpha \in G_\alpha$ *and* $g_\beta \in G_\beta$ *equivalent in this directed spectrum if there exists* $\gamma \in A$ *such that*

$$\gamma > \alpha ,\ \gamma > \beta ,\ \overset{*}{\pi}{}^{\gamma}_{\alpha} g_\alpha = \overset{*}{\pi}{}^{\gamma}_{\beta} g_\beta \quad \text{in} \quad G_\gamma .$$

Classifying all the elements of $\cup \{ G_\alpha \mid \alpha \in A \}$ by this equivalence relation we obtain a collection G of equivalence classes. We define the sum of two elements g and g' of G as follows. Let g_α and g'_β be representatives of the classes g and g' respectively. Taking $\gamma \in A$ with $\gamma > \alpha$, $\gamma > \beta$, we define the sum $g + g'$ as the class represented by the element

$$\overset{*}{\pi}{}^{\gamma}_{\alpha} g_\alpha + \overset{*}{\pi}{}^{\gamma}_{\beta} g_\beta .$$

It is clear that such a class is uniquely determined by g and g' . In this way G becomes a commutative group called the *limit group* of the directed spectrum $\{ G_\alpha, \overset{*}{\pi}{}^{\beta}_{\alpha} \}$.

Definition VIII. 12. *Let* $\{ \mathfrak{U}_\alpha \mid \alpha \in A \}$ *be the collection of the locally finite open coverings of a space* R . *We define that* $\alpha < \beta$ *if and only if* $\mathfrak{U}_\alpha > \mathfrak{U}_\beta$. *Then* A *is a directed set. We denote by* $\overset{*}{\pi}{}^{\beta}_{\alpha}$ *the homomorphism of* $H^n(N(\mathfrak{U}_\alpha))$ *into* $H^n(N(\mathfrak{U}_\beta))$ *dealt with in A). Then* $\overset{*}{\pi}{}^{\gamma}_{\beta}\overset{*}{\pi}{}^{\beta}_{\alpha} = \overset{*}{\pi}{}^{\gamma}_{\alpha}$ *holds obviously for every* α, β, γ *with* $\alpha < \beta < \gamma$. Hence

$$\{ H^n(N(\mathfrak{U}_\alpha)),\ \overset{*}{\pi}{}^{\beta}_{\alpha} \mid \alpha, \beta \in A ,\ \alpha < \beta \}$$

is a directed spectrum. Thus we can define by use of Definition VIII. 11 its limit group $H^n(R)$, *called the n-dimensional cohomology group of* R .[9]

[9] The definition can also be used to define the cohomology group $H^n(R,G)$ more generally for every topological space R and every commutative coefficient group G.

Let R and S be two spaces and f a continuous mapping of R into S. Let $\{ V_\alpha \mid \alpha \in A \}$ be the collection of the locally finite open coverings of S. Then $U_\alpha = f^{-1}(V_\alpha) = \{ f^{-1}(V) \mid V \in V_\alpha \}$, $\alpha \in A$ are locally finite open coverings of R. Mapping $f^{-1}(V) \in U_\alpha$ to $V \in V_\alpha$ we obtain a simplicial mapping of $N(U_\alpha)$ into $N(V_\alpha)$ which we shall also denote by f. Now we can prove the following:

B) Let $e^n = \{e^n_\alpha\}$ be an element of $H^n(S)$. Then the element of $H^n(R)$ containing $f^*e^n_\alpha$ is uniquely determined by e^n, where $e^n_\alpha \in H^n(N(U_\alpha))$ is a representative of e^n. We shall denote this element of $H^n(R)$ by f^*e^n.

Proof. Let $V_\gamma < V_\alpha$; then we put

$$\tilde{\pi}^\gamma_\alpha = f^{-1}\pi^\gamma_\alpha f$$

for the mapping f of the vertices of $N(U_\gamma)$ onto the vertices of $N(V_\gamma)$, the projection π^γ_α of $N(V_\gamma)$ into $N(V_\alpha)$ and the mapping f^{-1} of the vertices of $N(V_\alpha)$ onto the vertices of $N(U_\alpha)$. Then $\tilde{\pi}^\gamma_\alpha$ is easily seen to be a projection of $N(U_\gamma)$ into $N(U_\alpha)$. We now assert that

$$(1) \qquad \overset{*}{\tilde{\pi}}{}^\gamma_\alpha f^* \varphi^n_\alpha = f^* \overset{*}{\pi}{}^\gamma_\alpha \varphi^n_\alpha \quad \text{in } N(U_\gamma)$$

for every n-cocycle φ^n_α of $N(V_\alpha)$. This obviously follows from the definition of $\tilde{\pi}^\gamma_\alpha$, i.e. $f\tilde{\pi}^\gamma_\alpha = \pi^\gamma_\alpha f$, and this implies the desired relation.

Let e^n_α, $e^n_\beta \in e^n$; then for some γ with $U_\gamma < U_\alpha \wedge U_\beta$ we obtain

$$\overset{*}{\pi}{}^\gamma_\alpha e^n_\alpha = \overset{*}{\pi}{}^\gamma_\beta e^n_\beta .$$

Therefore

$$f^* \overset{*}{\pi}{}^\gamma_\alpha e^n = f^* \overset{*}{\pi}{}^\gamma_\beta e^n_\beta .$$

By use of the assertion (1) we conclude

$$\overset{*}{\tilde{\pi}}{}^\gamma_\alpha f^* e^n_\alpha = \overset{*}{\tilde{\pi}}{}^\gamma_\beta f^* e^n_\beta ,$$

i.e. $f^*e^n_\alpha$ and $f^*e^n_\beta$ represent the same element of $H^n(R)$.

The easy proof of the following assertion is left to the reader.

C) The mapping f^* in B) is a homomorphism of $H^n(S)$ into $H^n(R)$.

Definition VIII. 13. *Let C be a closed subset of a space R. The homomorphism h^* of $H^n(R)$ into $H^n(C)$ induced by the identity mapping (imbedding) h of C into R is called the natural homomorphism. If an element e^n of $H^n(C)$ is the image of an element $\tilde{e}^n$ of $H^n(R)$ by the natural homomorphism, then $\tilde{e}^n$ is called the extension of e^n and e^n is said to be extendible over R.*

Let $e^n \in H^n(R)$, and let $e^n_\alpha \in H^n(N(V_\alpha))$ be a representative of e^n, where V_α is a locally finite open covering of R. Then $U_\alpha = \{ U \cap C \mid U \in V_\alpha \}$ is a locally finite open covering of C such that $U_\alpha = h^{-1}(V_\alpha)$. Put, for brevity, $K = N(V_\alpha)$, $L = N(U_\alpha)$; then L is a subcomplex of K. Now h^*e^n is the element of $H^n(C)$ represented by $h_L e^n_\alpha$ for the natural homomorphism h_L of $H^n(K)$ into $H^n(L)$ since $h_L e^n_\alpha = h^* e^n_\alpha$.

Example VIII. 6. Let R be a polyhedron and K the complex associated with R. Let $K_1, K_2, \dots$ be the successive barycentric subdivisions of K. For a vertex p of K_i we denote by $S(p, K_i)$ the open star around p, i.e. the set of the points of (the realized) K_i whose barycentric coordinate for p is positive. Then $U_i = \{ S(p, K_i) \mid p:$ vertices of $K_i \}$ is a finite open covering. We note that $N(U_i)$ is isomorphic to K_i. Since $\{ U_i \mid i = 1, 2, \dots \}$ is a *cofinal subset* [10] of the set of all locally finite open coverings of R, we can easily see that $H^n(R)$ is isomorphic to the limit group of $\{ H^n(N(U_i)), \pi^{*i}_j \mid i < j ; i, j = 1, 2, \dots \}$. It is also easy to see that the homomorphism π^{*i}_j is an isomorphism of $H^n(N(U_j)) = H^n(K_j)$ onto $H^n(N(U_i)) = H^n(K_i)$. Hence $H^n(R)$ is isomorphic to $H^n(K_i)$ for each i and accordingly to $H^n(K)$. Thus we can say that the cohomology group of a polyhedron is isomorphic to the cohomology group of the associated complex.

VIII. 3. Dimension and cohomology

Definition VIII. 14. *Let $\mathfrak{U}$ be an open covering of a space R and N the realization of the nerve $N(\mathfrak{U})$. Let φ be a continuous mapping of R into N. We denote by $x(U_\alpha)$ the vertex of N corresponding to $U_\alpha \in \mathfrak{U}$ and by S_α the open star around $x(U_\alpha)$. If $\varphi^{-1}(S_\alpha) \subset U_\alpha$ for every $U_\alpha \in \mathfrak{U}$, then we call φ a canonical mapping relative to $\mathfrak{U}$.*

---

[10] Let B be a subset of a directed set A. If for every element a of A there is an element b of B satisfying $b > a$, then we call B a *cofinal subset* of A.

A) Let φ be a continuous mapping of a space R into the realized nerve N of a point-finite open covering U of R. Then φ is canonical relative to U if and only if for every point p of R, $\varphi(p) \in \overline{S(p)}$, where $S(p)$ denotes the geometrical simplex spanned by the vertices $\{x(U_\alpha) \mid p \in U_\alpha \in U\}$.

Proof. Let φ be a canonical mapping and p a point of R. Let $\varphi(p) \in S$ for a simplex S of N and denote by $x(U_{\alpha_i})$, $i = 1, \dots, k$ its vertices. Then

$$\varphi(p) \in S_{\alpha_i}, \quad i = 1, \dots, k,$$

which implies

$$p \in \varphi^{-1}(S_{\alpha_i}) \subset U_{\alpha_i}, \quad i = 1, \dots, k$$

by the definition of canonical mapping. Therefore

$$S \subset \overline{S(p)}, \text{ i.e. } \varphi(p) \in \overline{S(p)}.$$

Conversely, if φ satisfies $\varphi(p) \in \overline{S(p)}$ for every point p of R, then we assume that p is a given point of $\varphi^{-1}(S_\alpha)$, where $U_\alpha \in U$. Then

$$\varphi(p) \in S_\alpha \cap \overline{S(p)}.$$

Hence $x(U_\alpha)$ is a vertex of $S(p)$, and hence $p \in U_\alpha$, proving that φ is canonical.

B) Let U be a locally finite open covering of a space R; then there exists a canonical mapping φ of R into the realized nerve N of U.

Proof. Let $U = \{U_\alpha \mid \alpha \in A\}$ and denote by $x(U_\alpha)$ the vertex of N corresponding to U_α. For each choice of $\alpha_1, \dots, \alpha_k \in A$ satisfying $U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \neq \emptyset$ we define a closed set $F(\alpha_1, \dots, \alpha_k)$ of R by

$$F(\alpha_1, \dots, \alpha_k) = R - \bigcup \{U_\alpha \mid \alpha \neq \alpha_1, \dots, \alpha_k\}.$$

Let $p \in F(\alpha_1, \dots, \alpha_k)$; then we define as follows a point

$$\varphi(p) = \{x_1(\alpha_1, \dots, \alpha_k, p), \dots, x_k(\alpha_1, \dots, \alpha_k, p)\}$$

belonging to the simplex $[x(U_{\alpha_1}),\ldots,x(U_{\alpha_k})]$ by induction on the number k, where we denote by $x_1(\alpha_1,\ldots,\alpha_k,p),\ldots,x_k(\alpha_1,\ldots,\alpha_k,p)$ the barycentric coordinates of the point $\varphi(p)$ associated with the vertices $x(U_{\alpha_1}),\ldots,x(U_{\alpha_k})$, respectively.

For $k=1$ we define

$$\varphi(p)=\{x_1(\alpha_1,p)\}$$
$$x_1(\alpha_1,p)=1 \quad \text{for} \quad p\in F(\alpha_1) .$$

Namely

$$\varphi(p)=x(\alpha_1) \quad \text{for} \quad p\in F(\alpha_1) .$$

Assume that we have defined $\varphi(p)$ for $k=\ell$. Let $p\in F(\alpha_1,\ldots,\alpha_l,\alpha_{l+1})$. Then we put

$$\begin{aligned} y_1(\alpha_1,\ldots,\alpha_l,\alpha_{l+1};p) &= x_1(\alpha_1,\hat\alpha_2,\ldots,\alpha_{l+1},p) \quad \text{for } p\in F(\alpha_1,\hat\alpha_2,\ldots,\alpha_{l+1}) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= x_1(\alpha_1,\ldots,\alpha_l,\hat\alpha_{l+1},p) \quad \text{for} \quad p\in F(\alpha_1,\ldots,a_l,\hat a_{l+1}) \\ &= 0 \quad \text{for} \quad p\in F(\hat\alpha_1,\alpha_2,\ldots,\alpha_{l+1}) . \end{aligned}$$

Since $y_1(\alpha_1,\ldots,\alpha_l,\alpha_{l+1},p)$ is a continuous function defined on a closed subset of $F(\alpha_1,\ldots,\alpha_{l+1})$, by Tietze's extension theorem we can continuously extend it over $F(\alpha_1,\ldots,\alpha_{l+1})$. Note that we assume R to be paracompact T_2 and accordingly normal.

We define the continuous functions

$$y_2(\alpha_1,\ldots,\alpha_{l+1},p),\ldots,y_l(\alpha_1,\ldots,\alpha_{l+1},p)$$

over $F(\alpha_1,\ldots,\alpha_{i+1})$ in a similar way. Moreoever, we define a continuous function $y_{l+1}(\alpha_1,\ldots,\alpha_{l+1},p)$ such that

$$\begin{aligned} y_{l+1}(\alpha_1,\ldots,\alpha_{l+1},p) &= x_{l+1}(\hat\alpha_1,\alpha_2,\ldots,\alpha_{l+1},p) \quad \text{for} \quad p\in F(\hat\alpha_1,\alpha_2,\ldots,\alpha_{l+1}) , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= x_{l+1}(\alpha_1,\ldots,\hat\alpha_l,\alpha_{l+1},p) \quad \text{for} \quad p\in F(\alpha_1,\ldots,\hat\alpha_l,\alpha_{l+1}) , \\ &= 0 \qquad\qquad \text{for} \quad p\in F(\alpha_1,\ldots,\alpha_l,\hat\alpha_{l+1}) , \\ &= 1 \quad \text{if} \quad y_1(\alpha_1,\ldots,\alpha_{l+1},p)=\ldots=y_l(\alpha_1,\ldots,\alpha_{l+1},p)=0 . \end{aligned}$$

Putting for $i = 1,\dots,l+1$

$$x_i(\alpha_1,\dots,\alpha_{l+1},p) = \frac{y_i(\alpha_1,\dots,\alpha_{l+1},p)}{y_1(\alpha_1,\dots,\alpha_{l+1},p)+\dots+y_{l+1}(\alpha_1,\dots,\alpha_{l+1},p)}$$

we define a mapping

$$\varphi(p) = \{ x_1(\alpha_1,\dots,\alpha_{l+1},p),\dots,x_{l+1}(\alpha_1,\dots,\alpha_{l+1},p) \}$$

of $F(\alpha_1,\dots,\alpha_{l+1})$ into $[x(U_{\alpha_1}),\dots,x(U_{\alpha_{l+1}})]$ which coincides with the previously defined mapping on

$$F(\alpha_1,\dots,\alpha_l) \cup \dots \cup F(\alpha_2,\dots,\alpha_{l+1}) .$$

Thus we have a mapping $\varphi(p)$ of R into N which is continuous on each $F(\alpha_1,\dots,\alpha_k)$. Since U is locally finite, for every point p of R there is a neighbourhood $U(p)$ which meets only finitely many members $U_{\alpha_1},\dots,U_{\alpha_k}$ of U. Then since $\varphi(p)$ is continuous on $F(\alpha_1,\dots,\alpha_k)$, $\varphi(p)$ is continuous at p i.e. it is a continuous mapping.

Let $S(p) = [x(U_{\alpha_1}),\dots,x(U_{\alpha_k})]$ be the simplex determined by p as in A). Then it is clear that $p \in F(\alpha_1,\dots,\alpha_k)$, and hence $\varphi(p) \in \overline{S(p)}$. Therefore, by A), φ is canonical.

C) Let U be a locally finite open covering of a space R and N its realized nerve. Let φ be a canonical mapping of R into N and f a mapping of N into a space S which is continuous on each finite subcomplex of N. Then $f\varphi$ is a continuous mapping of R into S.

Proof. Let p be a given point of R. Take a neighbourhood $U(p)$ of p which meets only finitely many members $U_1,\dots,U_k$ of U. Then $\varphi(U(p))$ is contained in the finite subcomplex of N spanned by $x(U_1),\dots,x(U_k)$ and continuously mapped into S by f. Thus $f\varphi$ is continuous at p.

Let K, L be two complexes and K_0, L_0 be the realizations of K, L, respectively. We consider a simplicial mapping f of K into L. Let $[a_0,\dots,a_n]$ be a simplex of K_0 and p a point in it with barycentric coordinates $\{x_0,\dots,x_n\}$. Then we define a point $f(p)$ in the simplex $[f(a_0),\dots,f(a_n)]$ as the point having the barycentric coordinates $\{x_0,\dots,x_n\}$. Then f is a mapping of K_0 into L_0 which is continuous on the closure of each simplex of K_0. We shall make no distinction between the mapping of K into L and the induced mapping of K_0 into L_0.

Definition VIII. 15. *Let either of* K *and* S^n *be an oriented elementary* n*-sphere or one of its successive barycentric subdivisions. We fix a positively oriented simplex* y_0^n *of* S^n *. Let* f *be a simplicial mapping of* K *into* S^n *. Then we define* $d_f(K)$ *by*

$$d_f(K) = \sum_{\alpha} f_0^{\alpha} ,$$

where $\sum_{\alpha}$ *denotes the sum over all positively oriented simplices* x_{α}^n *of* K *, and*

$$f_0^{\alpha} = \begin{cases} 1 & \text{if } f(x_{\alpha}^n) = y_0^n , \\ -1 & \text{if } f(x_{\alpha}^n) = -y_0^n , \\ 0 & \text{otherwise} . \end{cases}$$

D_n) Let either of K and S^n be an oriented n-sphere or one of its successive barycentric subdivisions. Suppose f is a simplicial mapping of K into S^n. If $d_f(K) = 0$, then f is homotopic to 0.

E_n) Let L be a subcomplex of an $(n+1)$-complex K and S^n an oriented elementary n-sphere. Suppose f is a simplicial mapping of L into S^n. For a fixed positively oriented simplex y_0^n of S^n and n-simplices x_{α}^n of L we define f_0^{α} as in Definition VIII. 15. If the n-cocycle

$$\varphi^n = f^* y_0^n = \sum_{\alpha \in A} f_0^{\alpha} x_{\alpha}^n , \text{ [11]}$$

of L is the restriction of a cocycle of K, then f can be extended to a mapping F of K into S^n which is continuous on the closure of every simplex of K.

Proof. We shall prove D_n) and E_n) simultaneously by induction on the number n.

Proof of D_1). Let us consider the 1-cocycle

$$\varphi^1 = f^* y_0^1 = \sum_{\alpha} f_0^{\alpha} x_{\alpha}^1$$

[11] We denote by y_0^n the n-chain $1 y_0^n$.

of K for a fixed 1-simplex $y_0^1 = (q_0, q_1)$ of S^1. Then since

$$d_f(K) = \sum_\alpha f_0^\alpha = 0 ,$$

by Example VIII.5 φ^1 is cobounding, i.e. there is a 0-chain ψ^0 of K such that $\delta\psi^0 = \varphi^1$.

Let $\psi^0 = \sum_{i=1}^{k+1} g^i x_i^0$. Then the 0-simplex x_i^0 is also a vertex of K. We suppose (x_i^0, x_{i+1}^0), $i = 1,\dots,k$ $(x_{k+1}^0 = x_1^0)$ are the 1-simplices of S^1. Then $g^{i+1} - g^i$ is the coefficient of (x_i^0, x_{i+1}^0) in $\delta\psi^0$. Hence

$$g^{i+1} - g^i = \begin{cases} 1 & \text{if } f((x_i^0, x_{i+1}^0)) = (q_0, q_1) , \\ -1 & \text{if } f((x_i^0, x_{i+1}^0)) = -(q_0, q_1) , \\ 0 & \text{otherwise.} \end{cases}$$

We regard S^1 as the space R^1 of reals mod 1 identifying the segment $y_0^1 = (q_0, q_1)$ in S^1 with the segment $(0, \frac{1}{2})$ in R^1. For a real number x we denote by (x) the congruence class mod 1 of x. We define a real valued function $g(x)$ over K by

$$(g(x)) = f(x) ,$$
$$\tfrac{1}{2}(g^i + g^{i+1}) - \tfrac{1}{2} \leq g(x) < \tfrac{1}{2}(g^i + g^{i+1}) + \tfrac{1}{2} \quad \text{for } x \in \overline{[x_i^0, x_{i+1}^0]} .$$

It is easily seen that $g(x)$ is continuous. Now $f(x,t) = (tg(x))$ gives a homotopy between f and a constant mapping. Therefore f is homotopic to 0.

Proof that D$_n$) implies E$_n$). Assume that

$$\varphi^n = \sum_{\alpha \in A} f_0^\alpha x_\alpha^n$$

is the restriction to L of the cocycle

$$\psi^n = \sum_{\beta \in B} g^\beta x_\beta^n$$

of K. Then

$$g^{\beta} = f_0^{\beta} \text{ if } \beta \in A ,$$

$$\sum_{\beta \in B} \eta_{\beta}^{\gamma} g^{\beta} = 0 \text{ for every } (n+1)\text{-simplex } x_{\gamma}^{n+1} \text{ of } K .$$

Let us denote by K_n and L_n the complexes consisting of the simplices with dim $\leq n$ of K and L respectively. By use of the successive barycentric subdivisions of K_n mod L_n, we obtain a complex K_n' such that each simplex x_{β}^n of $K_n - L_n$ contains an oriented n-subsimplex $x_{\beta}'^n$ belonging to K_n' such that the *star* [12] of $x_{\beta}'^n$ in K_n' has no vertex in common with $\dot{x}_{\beta}^n$. For x_{β}^n we denote by X_{β} the subcomplex of K_n' consisting of the n-subsimplices of x_{β}^n and their faces. We apply the successive barycentric subdivisions to X_{β} mod $X_{\beta} - \{x_{\beta}'^n\}$ to obtain a complex X_{β}' such that there are g^{β} subsimplices

$$x_{\beta k}^n , \ k = 1, \dots, g^{\beta} \text{ of } x_{\beta}'^n$$

belonging to X_{β}' such that the stars of $x_{\beta k}^n$ in X_{β}' have no common vertex with each other, where we are assuming $g^{\beta} \geqq 0$ without loss of generality. Put

$$K_n'' = \cup \{X_{\beta}' \mid \beta \in B - A\} .$$

Then note that X_{β}' is a complex obtained by triangulating x_{β}^n, and thus $K_n'' \cup L_n$ is a complex obtained from K_n by bestowing a finer triangulation. Further note that $x_{\beta k}^n$, $k = 1, \dots, g^{\beta}$ are oriented n-subsimplices of x_{β}^n belonging to K_n'' such that the stars of $x_{\beta k}^n$ have no vertices in common either with $\dot{x}_{\beta}^n$ or with each other.

Suppose

$$x_{\beta k}^n = (a_1^{\beta k} \dots a_{n+1}^{\beta k}) , \ k = 1, \dots, g^{\beta} ,$$

and denote by $b_0, b_1, \dots, b_{n+1}$ the vertices of S^n assuming that $y_0^n = (b_1 \dots b_{n+1})$. Then we extend f to a simplicial mapping g of $K_n'' \cup L$ into S^n by

$$g(a_l^{\beta k}) = b_l , \ l = 1, \dots, n+1 ,$$

$$g(a) = b_0 \text{ for the other vertices } a \text{ of } K_n'' .$$

[12] Let x^n be a simplex belonging to a complex K. Then we mean by the star of x^n in K the collection of the simplices of K which have a common vertex with x^n.

We note that $x^n_{\beta k}$, $k=1,\dots,g^\beta$ are the only simplices of K''_n which are mapped by g to y^n_0. Thus each X'_β contains exactly g^β n-simplices which are mapped by g to y^n_0.

Let x^{n+1}_γ be an arbitrary $(n+1)$-simplex of K and K_γ the complex consisting of the simplices which belong to $K''_n \cup L_n$ and are on the boundary of x^{n+1}_γ . Then it is easily seen that

$$d_g(K_\gamma) = \sum_{\beta\in B} \eta^\gamma_\beta g^\beta = 0 \ .$$

Thus it follows from D$_n$) that g is homotopic to 0 on the boundary of x^{n+1}_γ . Hence, by Borsuk's theorem, we can extend g over $\bar{x}^{n+1}_\gamma$. In consequence, f can be extended over K such that the extension is continuous on the closure of every simplex of K .

Proof that E$_n$) implies D$_{n+1}$). Let us consider a fixed simplex y^{n+1}_0 of S^{n+1} and denote by L the complex consisting of the $(n+1)$-simplices of K which are mapped onto y^{n+1}_0 by f , and their faces. We may assume that the $(n+1)$-simplices of L have no vertex in common. Because, if they have, then we can subdivide K and S^{n+1} such that there appears a subsimplex y^{n+1}_1 of y^{n+1}_0 which has no vertex in common with y^{n+1}_0 . If we replace y^{n+1}_0 by y^{n+1}_1 then it satisfies the desired condition keeping the value of $d_f(K)$ at 0 .

Assume

$$\varphi^{n+1} = f^* y^{n+1}_0 \ .$$

Then, since

$$\varphi^{n+1} = \sum f^\alpha_0 x^{n+1}_\alpha \quad \text{and} \quad d_f(K) = \sum f^\alpha_0 = 0 \ ,$$

by Example VIII.5 φ^{n+1} is cobounding, i.e.,

$$(1) \qquad \varphi^{n+1} = \delta\psi^n$$

for some n-chain ψ^n of K . We denote by L_n the complex consisting of the simplices of L of $\dim \leqq n$. Then we regard the restriction f_0 of f to L_n as a mapping of L_n into the elementary n-sphere S^n , the boundary of y^{n+1}_0 . Moreover, we denote by f_1 the restriction of f to L and regard f_1 as a mapping of L into the complex T^{n+1} consisting of y^{n+1}_0 and its faces.

Writing

$$\psi_1^n = f_1^* y_0^n$$

for the n-chain y_0^n of T^{n+1} we obtain an n-chain ψ_1^n of L. Since

$$\delta y_0^n = y_0^{n+1}$$

in T^{n+1}, we obtain, using 1 D),

(2) $$\delta\psi_1^n = \delta f_1^* y_0^n = f_1^* \delta y_0^n = f_1^* y_0^{n+1} = \tilde{\varphi}^{n+1} ,$$

where $\tilde{\varphi}^{n+1}$ denotes the restriction of φ^{n+1} to L. On the other hand, by $\tilde{\psi}^n$ we denote the restriction of ψ^n to L. Then it follows from (1) that

$$\tilde{\varphi}^{n+1} = \delta\tilde{\psi}^n \quad \text{in} \quad L ,$$

and hence by (2)

$$\delta(\psi_1^n - \tilde{\psi}^n) = 0 .$$

Therefore, by Example VIII.3, we obtain $\psi_1^n - \tilde{\psi}^n \sim 0$ in L, and accordingly

(3) $$h_{L_n} \psi_1^n \sim h_{L_n} \tilde{\psi}^n \quad \text{in} \quad L_n .$$

We consider the complex $K' = (K - L) \cup L_n$. Then, since the coefficient f_0^α of φ^{n+1} vanishes for $x_\alpha^{n+1} \notin L$, we obtain

$$\delta\psi^n = h_{K'} \varphi^{n+1} = 0 \quad \text{in} \quad K' ,$$

regarding ψ^n as a chain of K'. Thus ψ^n, or more precisely $h_{K'} \psi^n$ is a cocycle of K'. Hence the element of $H^n(L_n)$ represented by $h_{L_n} \tilde{\psi}^n$ is extendible over K'. Therefore, by 1 C) and (3),

$$h_{L_n} \psi_1^n = f_0^* y_0^n$$

is the restriction of a cocycle of K'. Thus it follows from E_n) that f_0 can be extended to a mapping F of K' into the boundary of y_0^{n+1} which is continuous on the closure of every simplex of K' and accordingly on K'.

Defining

$$g(x)=\begin{cases} f(x) & \text{for } x\in L\,, \\ F(x) & \text{for } x\notin L\,, \end{cases}$$

we obtain a continuous mapping g of K into $\overline{y_0^{n+1}}$. We may assume that y_0^{n+1} is a hemisphere of S^{n+1}. Therefore $f(x)$ and $g(x)$ are never antipodal, and hence f and g are homotopic. Since $g(K)\subset y_0^{n+1}$, g is homotopic to zero. Thus f is also homotopic to zero.

Theorem VIII. 1 (Hopf's Extension Theorem). *Let R be a (paracompact T_2)-space of* $\dim R\leqq n+1$ *and C a closed set of R. Suppose f is a continuous mapping of C into S^n. Then f can be extended to a continuous mapping F of R into S^n if and only if $f^*e\in H^n(C)$ is extendible over R for every element $e\in H^n(S^n)$.*

Proof. Necessity. Let us denote by h the identity mapping of C into R and by h^* the associated natural homomorphism of $H^n(R)$ into $H^n(C)$. Since $f=Fh$, we obtain $f^*e=h^*F^*e$ for every element e of $H^n(S^n)$. Hence f^*e is the image of the element F^*e of $H^n(R)$ by h^*, i.e. it is extendible over R.

Sufficiency. We regard S^n as an oriented elementary n-sphere as well as a topological space. We denote by $S(a)$ for a vertex a of S^n the union of the simplices of S^n having a as a vertex. Put $V_0=\{S(a)\mid a$ is a vertex of $S^n\}$. Then $N(V_0)$ is a complex isomorphic to S^n itself. We denote by y_0^n an oriented n-simplex of S^n or of $N(V_0)$. Let $e_0^n=\{y_0^n\}$ be the element of $H^n(N(V_0))$ represented by y_0^n. Put $U_0'=f^{-1}(V_0)$; then $f^*e_0^n=d_0^n$ is an element of $H^n(N(U_0'))$. By the assumption the element of $H^n(C)$ represented by d_0^n is extendible over R. Hence there exist locally finite open coverings U_β', U_γ' of C and $d_\gamma^n\in H^n(N(U_\gamma'))$ satisfying

(i) d_γ^n is extendible over $N(U_\gamma)$ for an extended locally finite open covering U_γ of U_γ' over R,

(ii) $U_\beta'<U_\gamma'\wedge U_0'$,

(iii) $\overset{*}{\pi}{}_\gamma^\beta d_\gamma^n=\overset{*}{\pi}{}_0^\beta d_0^n$,

where we denote by π_γ^β, π_0^β the projections of $N(U_\beta')$ into $N(U_\gamma')$, of $N(U_\beta')$ into $N(U_0')$, respectively.

Denote by U_β an extended locally finite open covering of U_β' over R such that $U_\beta<U_\gamma$. Then since R is a paracompact space of $\dim R\leqq n+1$, there is a locally

finite open covering $\mathfrak{U}_\alpha$ such that

$$\text{ord } \mathfrak{U}_\alpha \leqq n+2 \ , \ \mathfrak{U}_\alpha < \mathfrak{U}_\beta \ .$$[13]

Now

$$\overset{*}{\pi}{}^{\alpha}_{\beta}\overset{*}{\pi}{}^{\beta}_{\gamma}d^n_\gamma = \overset{*}{\pi}{}^{\alpha}_{\beta}\overset{*}{\pi}{}^{\beta}_{0}d^n_0 \ ;$$

hence

$$\overset{*}{\pi}{}^{\alpha}_{\gamma}d^n_\gamma = \overset{*}{\pi}{}^{\alpha}_{0}d^n_0 \ .$$

Since d^n_γ is extendible,

$$d^n_\gamma = h^*\tilde{d}^n_\gamma$$

for some $\tilde{d}^n_\gamma \in H^n(N(\mathfrak{U}_\gamma))$ and the natural mapping h^* of $H^n(N(\mathfrak{U}_\gamma))$ into $H^n(N(\mathfrak{U}'_\gamma))$. Hence

$$\overset{*}{\pi}{}^{\alpha}_{\gamma}h^*\tilde{d}^n_\gamma = \overset{*}{\pi}{}^{\alpha}_{0}d^n_0 \ ,$$

which implies, as seen in the proof of 2 B),

$$h^*\overset{*}{\tilde{\pi}}{}^{\alpha}_{\gamma}\tilde{d}^n_\gamma = \overset{*}{\pi}{}^{\alpha}_{0}d^n_0 \ ,$$

where $\tilde{\pi}^\alpha_\gamma$ denotes the projection of $N(\mathfrak{U}_\alpha)$ into $N(\mathfrak{U}_\gamma)$. Therefore, by 1 C), $(f\pi^\alpha_0)^*y^n_0$ is the restriction of an n-cocycle of $N(\mathfrak{U}_\alpha)$. Since $N(\mathfrak{U}_\alpha)$ is an $(n+1)$-complex, by use of E_n) we can extend $f\pi^\alpha_0$ to a mapping π of $N(\mathfrak{U}_\alpha)$ into S^n which is continuous on the closure of each simplex of $N(\mathfrak{U}_\alpha)$.

By virtue of B) we can construct a canonical mapping φ of R into $N(\mathfrak{U}_\alpha)$. It follows from C) that $\pi\varphi$ is a continuous mapping of R into S^n. Now we denote by f_0 the restriction of $\pi\varphi$ to C. Since φ is canonical, for every $x \in C$, $\varphi(x)$ belongs to the closed simplex of $N(\mathfrak{U}'_\alpha)$ determined by x as seen in A). Therefore, for $x \in C$

$$f_0(x) = \pi\varphi(x) = f\pi^\alpha_0\varphi(x)$$

[13] Here we use Theorem II.6 for a paracompact space. Though Theorem II.6 is stated for metric spaces, it has been proved essentially for every paracompact space R.

is contained in the same closed simplex of S^n which contains $f(x)$. Hence f is homotopic to f_0 in C. Since f_0 is extendible over R, by Borsuk's theorem [14] we can extend f over R.

Theorem VIII. 2. *Let R be a space of finite dimension. Then R has* $\dim \leqq n$ *if and only if for every integer $m \geqq n$ and every closed set C of R, the natural homomorphism of $H^m(R)$ into $H^m(C)$ is an onto mapping.*

Proof. Necessity. Let $e \in H^m(C)$ and U'_α be a locally finite open covering of C such that e has a representative e_α in $H^m(N(U'_\alpha))$. We extend the covering U'_α to a locally finite open covering U_α of R. Choose a locally finite open covering U_β of R such that $U_\beta < U_\alpha$, and $\operatorname{ord} U_\beta \leqq n+1$.

By U'_β we denote the restriction of U_β to C. Then $N(U_\beta)$ is a complex of $\dim \leqq n$ and $N(U'_\beta)$ is a subcomplex of $N(U_\beta)$. e has a representative e_β in $H^m(N(U'_\beta))$. Since every m-chain of $N(U'_\beta)$ is an m-chain of $N(U_\beta)$ and also an m-cocycle, e_β has an extension $\tilde{e}_\beta$ in $H^m(N(U_\beta))$. Thus the element $\tilde{e} \in H^m(R)$ represented by $\tilde{e}_\beta$ is an extension of e, i.e. $\tilde{e}$ is mapped by the natural homomorphism to e.

Sufficiency. Let $\dim R = m+1 > n$. Then $m \geqq n$, and by Theorem VII.10 there is a closed set C of R and a continuous mapping f of C into S^m which cannot be extended over R. Thus by Hopf's extension theorem $f^*e \in H^m(C)$ for some $e \in H^m(S^m)$ is not extendible over R, i.e. the natural homomorphism of $H^m(R)$ into $H^m(C)$ is not onto.

VIII. 4. Dimension and homology

For dimension theory in compact spaces the concept of homology is also useful. It seems to be more convenient to intuitive understanding than the concept of cohomology. In this final section we shall give a brief description, without proof, of the relation between homology and dimension. Throughout the section all spaces are assumed to be compact T_2-spaces unless the contrary is explicitly stated.

---

[14] We have proved Borsuk's theorem only in case R is metric. But it is easily seen that the same is true for every paracompact T_2-space R. The only problem is to show that $[0,1] \times R$ is normal if R is paracompact T_2. But this proposition was proved by C. H. Dowker [4] for every countably paracompact normal space R.

Definition VIII. 16. *Let* A *be a directed set. Suppose* $\{G_\alpha \mid \alpha \in A\}$ *is a collection of commutative groups. If for every* $\alpha, \beta \in A$ *with* $\alpha > \beta$ *a homomorphic mapping* π_β^α *of* G_α *into* G_β *is given such that* $\pi_\gamma^\alpha = \pi_\beta^\alpha \pi_\gamma^\beta$ *, then we call the collection* $\{G_\alpha ; \pi_\beta^\alpha \mid \alpha, \beta \in A, \alpha > \beta\}$ *an inverse spectrum.*

Let $\{G_\alpha ; \pi_\beta^\alpha\}$ be an inverse spectrum. We consider an element $\{g_\alpha \mid \alpha \in A\}$ of the cartesian product $\Pi_\alpha G_\alpha$ such that $\pi_\beta^\alpha g_\alpha = g_\beta$ whenever $\alpha > \beta$. We define the sum $g + g'$ of two such elements $g = \{g_\alpha\}$ and $g' = \{g'_\alpha\}$ by

$$g + g' = \{g_\alpha + g'_\alpha \mid \alpha \in A\}.$$

Then the collection of all such elements turns out to be a commutative group G which is called the *limit group* of the inverse spectrum $\{G_\alpha ; \pi_\beta^\alpha\}$.

We consider the collection $\{U_\alpha \mid \alpha \in A\}$ of all finite open coverings of a space R. If $U_\alpha < U_\beta$, then, by definition, $\alpha > \beta$; in this way A becomes a directed set. Denote by π_β^α a projection of $N(U_\alpha)$ into $N(U_\beta)$. Then, by I E), π_β^α induces a homomorphism of $H_n(N(U_\alpha), G)$ into $H_n(N(U_\beta), G)$. Moreover, we can prove that all projections of $N(U_\alpha)$ into $N(U_\beta)$ induce the same homomorphism $\hat{\pi}_\beta^\alpha$ of $H_n(N(U_\alpha), G)$ into $H_n(N(U_\beta), G)$.

Definition VIII. 17. *The limit group of the inverse spectrum*

$$\{H_n(N(U_\alpha), G) ; \hat{\pi}_\beta^\alpha \mid \alpha, \beta \in A, \alpha > \beta\}$$

is denoted by $H_n(R,G)$ *and is called the n-dimensional (Čech) homology group of* R.

In this definition we often adopt the additive group R_1 of the real numbers mod 1 as the coefficient group G and denote the homology group by $H_n(R)$.

As easily seen, we obtain the same limit group if we replace A by a cofinal subset A' of A in Definition VIII.12 or Definition VIII.16. Hence for compact spaces we may take the collection $\{U_\alpha \mid \alpha \in A\}$ of all finite open coverings, instead of locally finite open coverings, to define the cohomology group of R by means of Definition VIII.12 [15]. For homology and cohomology groups founded on the finite coverings there exists a duality which enables us to translate theorems on

[15] For non-compact spaces homology and cohomology groups based on the finite coverings are not practical. As shown by C. H. Dowker [1] even the 1-dimensional cohomology group of E^1 has a complicated structure.

cohomology to those on homology. As a matter of fact $H_n(R)$ is the character group of $H^n(R)$ [16].

Let C be a closed set of R. Suppose $e = \{e_\alpha\}$ is an element of $H_n(C)$ represented by $e_\alpha \in H_n(N(U'_\alpha))$. Let U_α be an extension of the finite open covering U'_α over R. Then, since $N(U'_\alpha)$ is a subsimplex of $N(U_\alpha)$, we can consider the image $he_\alpha \in H_n(N(U_\alpha))$ of e_α under the natural homomorphism. We denote by $\tilde{e}$ the element of $H_n(R)$ represented by he_α. Then, mapping e to $\tilde{e}$, we obtain the *natural homomorphism* of $H_n(C)$ into $H_n(R)$. As for the relation between homology and dimension of compact spaces it is known that

Theorem VIII. 3. *A finite dimensional space R has $\dim \leq n$ if and only if for every integer $m \geq n$ and for every closed set C of R, the natural homomorphism of $H_m(C)$ into $H_m(R)$ is an isomorphism.*

To characterize homologically the dimension of compact metric spaces the original method due to P. S. Alexandroff [2] is also useful.

Definition VIII. 18. *Let $a_0,\dots,a_n$ be points of a metric space R. If the diameter of the set $\{a_0,\dots,a_n\}$ is less than $\varepsilon > 0$, then we call $\{a_0,\dots,a_n\}$ an n-dimensional ε-simplex in R. We denote by K the abstract complex of the ε-simplices in R.*

Let G be a commutative group. We mean by an ε-chain (or more precisely (n,G)-ε-chain) a finite (n,G)-chain of K and by an ε-cycle a finite (n,G)-cycle of K.

Let φ^n be an (n,G)-ε-cycle. If $\partial\psi^{n+1} = \varphi^n$ for some $(n+1, G)$-δ-chain ψ^{n+1}, then we write $\varphi^n \underset{\delta}{\sim} 0$.

Let $\{\varepsilon_i \mid i = 1,2,\dots\}$ be a sequence of positive numbers such that $\lim \varepsilon_i = 0$, and φ^n_i be an ε_i-cycle. Then we call the sequence $\Phi^n = (\varphi^n_1, \varphi^n_2, \dots)$ of those ε_i-cycles a true cycle or more precisely (n,G)-true cycle.

If $\varphi^n_i \underset{\delta_i}{\sim} 0$, $i = 1,2,\dots$ and $\lim \delta_i = 0$, then we call Φ^n homologous to zero in R and denote it by $\Phi^n \sim 0$.

A true cycle Φ^n of R may also be a true cycle for some closed set C of R. We call such a closed set C a carrier of Φ^n. Φ^n is called essential if it has a carrier in which $\Phi^n \sim 0$ does not hold.

[16] See W. Hurewicz and H. Wallman [1].

Now the following theorem holds.

Theorem VIII. 4. *A finite dimensional compact metric space* R *has* $\dim \leq n$ *if and only if every* (n, R_1)*-true cycle* Φ^n *homologous to* 0 *in* R *is not essential* [17].

Referring to compact subsets of a Euclidean space, one can characterize their dimension in connection with their complementary spaces as demonstrated in the following obstruction theorem due to P. Alexandroff.

Theorem VIII. 5. *Let* R *be an r-dimensional compact subset of* E^n. *Then there is* $\delta > 0$ *such that for every* $\varepsilon > 0$ *one can find an* $(n-(r+1), 1)$*-cycle* φ^{n-r-1} *lying in* $E^n - R$ *having diameter* $< \varepsilon$ *but being no boundary of a chain in* $E^n - R$ *of diameter* $< \delta$. *On the other hand, for* $k > r+1$ *every* $(n-k, 1)$*-cycle of diameter* $< \varepsilon$ *lying in* $E^n - R$ *is the boundary of a chain in* $E^n - R$ *of diameter* $< \varepsilon$ [18].

The homological method provides also a useful tool for finding conditions in order that $\dim (R \times S) = \dim R + \dim S$. Here we quote a famous problem posed by P. Alexandroff: To determine a finite dimensional compact metric space R such that

$$\dim (R \times S) = \dim R + \dim S$$

for every compact metric space S.

This problem was solved by V. Boltyanski [2] [3] by means of homological methods. Y. Kodama [1] [4] solved this problem for more general spaces. [19]

---

[17] In this theorem we may replace true cycle with *convergent true cycle.* If a true cycle $\Phi^n = (\varphi_1^n, \varphi_2^n, \dots)$ satisfies $\varphi_i^n - \varphi_{i+1}^n \underset{\eta_i}{\sim} 0$ for a sequence $\{\eta_i\}$ of positive numbers such that $\lim \eta_i = 0$, then it is called a convergent true cycle. This theorem can be deduced from Theorem VIII.3 by use of the relation between Čech's homology theory and Vietori's homology theory concerning convergent true cycles; see S. Lefschetz [1].

[18] In this theorem the concepts of chain and cycle are based on geometrical simplices, so a diameter of a chain means the diameter of the point set which is the union of all simplices occurring in the chain with non-null coefficients. This theorem was further developed by K. Sitnikov [2].

[19] This problem is related with another problem, to look for conditions in order that the homology or cohomology dimension coincide with the covering dimension. See E. Dyer [1], Y. Kodama [2].

M. F. Bokštein [1] established an interesting theory to show that dim of the topological product of compact metric spaces R and S is determined by the homological dimensions of R and S.

Several dimension functions, *homology dimensions* and *cohomology dimensions* can be defined in connection with the homological and cohomological concepts.

Definition VIII. 19. *Let* G *be a commutative group and* R *a space. We denote by* $D_C(R,G)$ *the greatest integer* n *such that the natural homomorphism of* $H^{n-1}(R,G)$ *into* $H^{n-1}(C,G)$ *is not onto for some closed set* C *of* R [20]. *Similarly,* $D_h(R,G)$ *will be the greatest integer* n *such that the natural homomorphism of* $H_{n-1}(C,G)$ *into* $H_{n-1}(R,G)$ *is not isomorphic for some closed set* C *of* R. *By* $D_t(R,G)$ *we denote the greatest integer* n *such that there exists an essential* $(n-1,G)$*-true cycle in* R *which is homologous to* 0. *In each of these three definitions, we define* $D_C(R,G)=0$, $D_h(R,G)=0$ *or* $D_t(R,G)=0$ *respectively if there is no* n *satisfying the condition. If for every* n *the condition is satisfied, then the respective dimension is* ∞, *and it is* -1 *if* $R=\emptyset$.

The theorems VIII.2.3.4. imply that for a finite dimensional compact metric space R

$$D_C(R,I)=D_h(R,R_1)=D_t(R,R_1)=\dim R\ . \quad [21]$$

Generally, we can show that

$$\dim R \geq D_t(R,G) \geq D_h(R,G)$$

for every G and every compact metric space.

While we have so many different definitions of dimension, [22] it is an interesting problem to characterize a certain dimension as a function (defined on a class of spaces) satisfying certain axioms. Let us denote by d a function on a class $\mathcal{C}$ of spaces taking values in the set $\{-1,0,1,2,\dots\}$. P. Alexandroff considered the following axioms to characterize dim on the class of all compact metric spaces.

A$_1$) $d(T^n)=n$ for the n-dimensional simplex T^n $(T^{-1}=\emptyset)$.

A$_2$) If R and S are homeomorphic, then $d(R)=d(S)$.

A$_3$) If $R=R_1\cup R_2$ for closed sets R_1, R_2 of R, then $d(R)=\max(d(R_1),\ d(R_2))$.

---

[20] The dimension $D_C(R,G)$ and a similar dimension was investigated by H. Cohen [1], A. Okuyama [1] and others.

[21] In case $\dim R=\infty$ it is unknown whether this relation remains still valid.

[22] There are more attempts to find out good new dimension functions, which were not discussed in this book. See, e.g. O. V. Lokucievskiĭ [2], J. M. Aarts [3], J. M. Aarts - T. Nishiura [1], [2], V. I. Ponomarev [2], V. A. Valiev [1].

A_4) If $R \subset E^n$, then there exists $\varepsilon > 0$ such that for every continuous mapping f of R onto $S \subset E^n$ satisfying $\rho(p, f(p)) < \varepsilon$ the inequality $d(R) \leqq d(S)$ is implied.

Obviously dim satisfies these axioms. On the other hand L. S. Pontrjagin [1] proved that every $D_t(R, I_m)$ $(m \geqq 2)$ also satisfies the same conditions though it is different from dim, where I_m is the additive group of integers mod m. He also proved that $D_t(R, I_m)$ for different prime numbers m are all different from each other. [23]

Further reasonable axioms for d are

A_5) If $d(R) = n$, then there is a finite open covering V of R such that for every V-mapping f from R into S, $d(f(R)) \geqq n$ holds.

A_6) If R contains more than one point, and $\dim R = n$, then there is a closed set F of R such that $d(F) < d(R)$, and $R - F$ is disconnected.

P. S. Alexandroff [2] proved that for the class C of all finite dimensional compact metric spaces, dim is the only function satisfying A_1, A_2, A_3, A_5 and A_6. But the same conditions are not sufficient to characterize dim for metric spaces.

E. Ščepin [1] replaced condition A_3 with

A_3') If $R = \bigcup_i R_i$ for at most countably many closed sets R_i, then $d(R) = \sup_i d(R_i)$,

and, by use of A_1, A_2, A_3', A_5, A_6, he succeeded to characterize dim on the class of all finite-dimensional metric spaces (and also on the class of all finite-dimensional separable metric spaces). O. V. Lokucievskiĭ [3] characterized dim on the class of finite-dimensional compact T_2-spaces by modifying A_6.

K. Menger [2] posed the problem to characterize dim for separable metric spaces by A_1, A_2, A_3', M_7 and M_8, where

M_7) If $S \subset R$, then $d(S) \leqq d(R)$.

M_8) For every R, there is a compact metric space $S \supset R$ such that $d(R) = d(S)$.

According to V. I. Kuzminov [1], I. Švedov solved Menger's problem in the negative when d is a function on the class C of all subsets of Euclidean spaces. If C is the class of all subsets of E^n and if $n \leqq 2$, then dim is the only function on C satisfying Menger's five conditions. But the problem is not yet answered if $n \geqq 3$.

J. M. Aarts [2], S. Sakai [1], T. Nishiura [1] considered further axioms to characterize dim for metric spaces and separable metric spaces.

---

[23] Relations between $D_t(R, G)$ for various G were studied by M. Bokštein [1].

BIBLIOGRAPHY *)

Aarts, J. M.
[1] Completeness degree, Fundam. Math. 63 (1968) 27-41.
[2] A characterization of strong inductive dimension, Fundam. Math. 70 (1971) 147-155.
[3] A new dimension function, Proc. Am. Math. Soc. 50 (1975) 419-425.

Aarts, J. M. and Nishiura, T.
[1] Kernels in dimension theory, Trans. Am. Math. Soc. 178 (1973) 227-240.
[2] Covering dimension modulo a class of spaces, Fundam. Math. 78 (1973) 75-97.

Addis, D. F. and Gresham, J. H.
[1] A class of infinite-dimensional spaces, I: Dimension theory and Alexandroff's problem, Fundam. Math. 101 (1978) 195-205.

Alexandroff, P. S.
[1] Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension, Ann. Math., II. Ser. 30 (1928) 101-187.
[2] Dimensionstheorie, Math. Ann. 106 (1932) 161-238.
[3] Über die Dimension der bikompakten Räume, Dokl. Akad. Nauk SSSR 26 (1940) 619-622.
[4] On the dimension of normal spaces, Proc. R. Soc. Lond., Ser. A 189 (1947) 11-39.
[5] Kombinatornaya topologiya (Combinatorial topology), Moskwa-Leningrad 1947.
[6] The present status of the theory of dimension (R), Usp. Mat. Nauk 6 (1951) 43-68 (Transl., II. Ser., Am. Math. Soc. 1 (1955) 1-26.)
[7] Die Kontinua (V^p) - eine Verschärfung der Cantorschen Mannigfaltigkeiten, Monatsh. Math. 61 (1957) 67-76.
[8] On some results in the theory of topological spaces obtained during the last twenty five years (R), Usp. Mat. Nauk 15 (1960) 25-95.

*) Abbreviations according to Zentralblatt für Mathematik Vol. 480.
(R) or (RE) indicates that the paper is written in Russian or in Russian with an English summary respectively, but some papers in Russian with a too short English summary are indicated merely with (R).

Alexandroff, P. S. and Hopf, H.
[1] Topologie I, Berlin 1935.

Alexandroff, P. S. and Kolmogoroff, A.
[1] Endliche Überdeckung topologischer Räume, Fundam. Math. 26 (1936) 267-271.

Alexandroff, P. S. and Ponomarev, V. I.
[1] On certain class of n-regular spaces (R), Sib. Math. J. 1 (1960) 1-13.

Arhangelskii, A. V.
[1] The rank of systems of sets and the dimension of spaces (R), Dokl. Akad. Nauk SSSR 143 (1962) 735-758.
[2] On closed mappings, bicompact sets and a problem of P. S. Alexandroff, Mat. Sb., Nov. Ser. 96 (1966) 13-34.
[3] On closed mappings increasing dimension, Czech. J. Math. 18 (1968) 389-391.

Arhangelskii, A. V. and Filippov, V. V.
[1] Spaces with bases of finite rank, Mat. Sb., Nov. Ser. 16 (1972) 147-158.

Begle, E. G.
[1] A note on S-spaces, Bull. Am. Mat. Soc. 55 (1949) 577-579.

Bing, R. H.
[1] Metrization of topological spaces, Can. J. Math. 3 (1951) 175-186.
[2] A hereditarily infinite-dimensional space, Proc. 2nd Prague Topol. Symp. (1966) 56-62.

Bokštein, M. F.
[1] On the dimension of a topological product (R), Dokl. Akad. Nauk SSSR 63 (1948) 221-223.

Boltyanskii, V. G.
[1] Mappings of compacta into the Euclidean spaces (R), Izv. Akad. Nauk SSSR, Ser. Mat. 23 (1959) 871-892.
[2] On the dimension fullvaluedness of compacta (R), Dokl. Akad. Nauk SSSR 67 (1949) 773-776 (Transl. Am. Math. Soc. 48 (1951) 7-11).
[3] On the theorem of addition of dimensions (R), Usp. Mat. Nauk 6 (1951) 99-128.

Borsuk, K.
[1] Sur les prolongements des transformations continues, Fundam. Math. 28 (1937) 99-110.
[2] Ensembles dont les dimensions modulaires de Alexandroff coincident avec la dimension de Menger-Urysohn, Fundam. Math. 27 (1936) 77-93.
[3] Concerning the dimension of A.N.R.-sets, Bull. Acad. Pol. Sci., Sér. Sci. Math. 9 (1961) 685-687.
[4] Remarks on the cartesian product of two 1-dimensional spaces, Bull. Acad. Pol. Sci., Sér. Sci. Math. 23 (1975) 971-973.

Bruijning, J.
[1] A characterization of dimension of topological spaces by totally bounded pseudometrics, Pac. J. Math. 84 (1979) 283-289.

Bruijning, J. and Nagata, J.
[1] A characterization of covering dimension by use of $\Delta_k(X)$, Pac. J. Math. 80 (1979) 1-8.

Bushaw, D.
[1] Elements of general topology, New York 1963.

Buzási, S.
[1] Nagata's metric for uniformities, Publ. Math. 26 (1979) 91-94.

Canfell, M. J.
[1] Uniqueness of generators of principal ideals in rings of continuous functions, Proc. Am. Math. Soc. 26 (1970) 571-573.

Čech, E.
[1] Sur la dimension des espaces parfaitement normaux, Bull. Acad. Bohême 33 (1932) 38-45.
[2] Contribution à la théorie de la dimension (in Czech), Cas. Mat. 62 (1933) 277-290.
[3] Théorie générale de l'homologie dans un espace quelconque, Fundam. Math. 19 (1932) 149-183.

Cohen, H.
[1] A cohomological definition of dimension for locally compact spaces, Duke Math. J. 21 (1954) 209-222.

Dijkstra, J. J.
[1] A space with maximal discrepancy between its Katětov and covering dimensions, Topol. Appl. 12 (1981) 45-48.

Douwen, E. van
[1] The small inductive dimension can be raised by the adjunction of a single point, Indagationes Math. 35 (1973) 434-442.

Dowker, C. H.
[1] Mapping theorems for non-compact spaces, Am. J. Math. 69 (1947) 200-242.
[2] Inductive dimension of completely normal spaces, Q. J. Math. Oxf., II. Ser. 4 (1953) 267-281.
[3] Local dimension of normal spaces, Q. J. Math. Oxf., II. Ser. 6 (1955) 101-120.
[4] On countably paracompact spaces, Can. J. Math. 3 (1951) 219-224.

Dowker, C. H. and Hurewicz, W.
[1] Dimension of metric spaces, Fundam. Math. 43 (1956) 83-87.

Dugundji, J.
[1] Topology, Boston, 1966.

Dyer, E.
[1] On the dimension of products, Fundam. Math. 47 (1959) 141-160.

Egorov, V. I.
[1] On the metric dimension of point sets (R), Mat. Sb., Nov. Ser. 48 (1959) 227-250.

Eilenberg, S. and Otto, E.
[1] Quelques propriétés caractéristiques de la dimension, Fundam. Math. 31 (1938) 149-153.

Engelking, R.
[1] General Topology, Warszawa, 1977.
[2] Dimension Theory, Amsterdam-Warszawa, 1978.

Erdös, P.
[1] The dimension of the rational points in Hilbert space, Ann. Math., II. Ser. 41 (1940) 734-736.

Fedorčuk, V. V.
[1] Bicompacta with non-coinciding dimensionalities, Sov. Math., Dokl. 5 (1968) 1148-1150.
[2] On the dimension of κ-metrizable bicompacta, Sov. Math., Dokl. 18 (1977) 605-609.
[3] On the dimension of hereditarily normal spaces, Proc. Lond. Math. Soc., III. Ser. 36 (1978) 163-175.

Filippov, V. V.
[1] A bicompactum with non-coinciding inductive dimensionalities, Sov. Math., Dokl. 10 (1969) 208-211.
[2] The inductive dimension of a product of bicompacta, Sov. Math., Dokl. 13 (1972) 250-254.
[3] The dimension of normal spaces, Sov. Math., Dokl. 14 (1973) 547-550.

Fitzpatrick, B. and Ford, R.
[1] On the equivalence of small and large inductive dimensions in certain metric spaces, Duke Math. J. 34 (1967) 33-39.

Fletcher, P. and McCoy, R. A. and Solver, R.
[1] On boundedly metacompact and boundedly paracompact spaces, Proc. Am. Math. Soc. 25 (1970) 335-342.

Franz, W.
[1] Topologie I, Berlin, 1960.

French, J. A.
[1] Some completely normal spaces in which small and large inductive dimensions coincide, Houston J. Math. 2 (1976) 181-193.

Freudenthal, H.
[1] Entwicklungen von Räumen und ihren Gruppen, Compos. Math. 4 (1937) 145-234.

Gelfand, I. and Šilov, G.
[1] Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Rings, Mat. Sb., Nov. Ser. 51 (1941) 25-39.

Groot, J. de
[1] Non-archimedean metrics in topology, Proc. Am. Math. Soc. 7 (1956) 948-953.
[2] On a metric that characterizes dimension, Can. J. Math. 9 (1957) 511-514.

Groot, J. de and Nishiura, T.
[1] Inductive compactness as a generalization of semi-compactness, Fundam. Math. 58 (1966) 201-218.

Groot, J. de and Vries, H. de
[1] A note on non-archimedean metrizations, Indagationes Math. 17 (1955) 222-224.

Gruenhage, G. and Nyikos, P.
[1] Spaces with bases of countable rank, Gen. Topol. Appl. 8 (1978) 233-257.

Hawkes, J.
[1] Hausdorff measure, entropy and the independence of small sets, Proc. Lond. Math. Soc., III. Ser. 28 (1974) 700-724.

Hejcman, J.
[1] Uniform dimension and rings of bounded uniformly continuous functions, Colloq. Math. Soc. Janos Bolyai 8 (1972) 379-386.
[2] Dimension and partition of unity, Seminar Uniform Spaces, 1973-1974 directed by Z. Frolík, MO ČSAV Praha, 195-200.

Hemmingsen, E.
[1] Some theorems in dimension theory for normal Hausdorff spaces, Duke Math. J. 13 (1946) 495-504.

Henderson, D.
[1] An infinite-dimensional compactum with no positive-dimensional compact subsets - a simpler construction, Am. J. Math. 89 (1967) 105-121.
[2] D-dimension I, II, Pac. J. Math. 26 (1968) 91-107, 109-113.
[3] A lower bound for transfinite dimension, Fundam. Math. 63 (1968) 167-173.

Herrlich, D.
[1] Some topological theorems which fail to be true, Categorical Topology (Proc. Confer. Mannheim '75) (1976) 461-468.

Hodel, R. E.
[1] Open functions and dimension, Duke Math. J. 30 (1963) 461-468.

Hoshina, T. and Morita, K.
[1] On rectangular products of topological spaces (to appear).

Hurewicz, W.
[1] Normalbereich und Dimensionstheorie, Math. Ann. 96 (1927) 736-764.
[2] Ein Theorem der Dimensionstheorie, Ann. Math., II. Ser. 31 (1930) 176-180.
[3] Über dimensionserhöhende stetige Abbildungen, J. Reine Angew. Math. 169 (1933) 71-78.
[4] Über Abbildungen von endlichdimensionalen Räumen auf Teilmengen cartesischer Räume, Sitzb. Preuss. Akad. Wiss. Phys. Math. Kl. (1933) 754-768.
[5] Sur la dimension des produits cartesiens, Ann. Math., II. Ser. 36 (1935) 194-197.
[6] Über Abbildung der topologischen Räume auf die n-dimensionale Sphäre, Fundam. Math. 24 (1935) 144-150.

Hurewicz, W. and Wallman, H.
[1] Dimension Theory, Princeton, 1941.

Isbell, J. R.
[1] Uniform Spaces, Providence, 1964.

Janos, L.
[1] A metric characterization of zero-dimensional spaces, Proc. Am. Math. Soc. 31 (1972) 268-270.
[2] Dimension theory via reduced bisector chains, Can. Math. Bull. 21 (1978) 305-311.

Janos, L. and Martin, H. W.
[1] Metric characterizations of dimension for separable metric spaces, Proc. Am. Math. Soc. 70 (1978) 209-212.

Katětov, M.
[1] On rings of continuous functions and the dimension of compact spaces (RE). Čas. Pěst. Mat. 75 (1950) 1-16.
[2] On the dimension of non-separable spaces I (RE), Czech. Math. J. 2 (1952) 333-368.

[3] On the dimension of non-separable spaces II (RE), Czech. Math. J. 6 (1956) 485-516.

[4] On the relations between the metric and topological dimensions (RE), Czech. Math. J. 8 (1958) 163-166.

Katsuta, Y.

[1] A note on the inductive dimension of product spaces. Proc. Japan. Acad. 42 (1966) 1011-1015.

Keesling, J.

[1] Mappings and dimension in metric spaces, Gen. Topol. Appl. 2 (1972) 181-192.

Kelley, J. L.

[1] General Topology, New York, 1955.

Kimura, N.

[1] On the covering dimension of product spaces, Proc. Japan. Acad. 40 (1964) 267-271.

[2] On the covering dimension of certain product spaces, Proc. Japan. Acad. 41 (1965) 114-119.

Kljušin, V.

[1] Perfect mappings of paracompact spaces (R), Dokl. Akad. Nauk SSSR 159 (1964) 734-737.

Kodama, Y.

[1] On a problem of Alexandroff concerning the dimension of product spaces II, J. Math. Soc. Japan 11 (1959) 94-111.

[2] Test spaces for homological dimension, Duke Math. J. 29 (1962) 41-50.

[3] On subset theorems and the dimension of products, Am. J. Math. 91 (1969) 486-497.

[4] A remark on the dimensional fullvaluedness, Proc. Japan Acad. 53 (1977) 113-115.

Kowalsky, H. J.

[1] Einbettung metrischer Räume, Arch. Math. 8 (1957) 336-339.

[2] Topologische Räume, Basel-Stuttgart, 1961.

Kulpa, W.

[1] On a uniform universal space, Fundam. Math. 69 (1970) 243-251.

Kunugi, K.

[1] Sur la théorie du nombre de dimensions, Thése, Paris 1930.

Kuratowski, C.

[1] Sur l'application des espaces fonctionnels à la théorie de la dimension, Fundam. Math. 18 (1932) 285-292.

[2] Topologie I, Warszawa, 1934.

[3] Sur les théorèmes de "prolongement" dans la théorie de la dimension, Fundam. Math. 28 (1937) 336-342.

[4] Topologie II, Warszawa, 1950.

Kuzminov, V.

[1] Homological dimension theory, Russ. Math. Surv. 23 (1968) No. 5, 1-45.

[2] Test spaces for cohomological dimension of paracompact spaces, Sov. Math. Dokl. 9 (1968) 874-877.

Landau, M.

[1] Strong transfinite ordinal dimension, Proc. Am. Math. Soc. 21 (1969) 591-596.

Lefschetz, S.
[1] Algebraic Topology, Princeton, 1942.

Leĭbo, I. M.
[1] On the equality of dimensions for closed images of metric spaces, Sov. Math. Dokl. 15 (1974) 835-839.
[2] On closed mappings of infinite-dimensional spaces, Sov. Math. Dokl. 12 (1971) 1111-1113.

Lelek, A.
[1] On the dimension of remainders in compactifications, Sov. Math., Dokl. 6 (1965) 136-140.

Levšenko, B. T.
[1] Strongly infinite-dimensional spaces (R), Vestn. Mosk. Univ., Ser. Mat. No. 5 (1957) 219-228.
[2] On infinite-dimensional spaces, Sov. Math., Dokl. 2 (1961) 915-918.
[3] Spaces of transfinite dimension (R), Mat. Sb., Nov. Ser. 67 (1965) 255-266.

Lipscomb, S.
[1] On imbedding finite-dimensional metric spaces, Trans. Am. Math. Soc. 211 (1975) 143-160.

Lokucievskiĭ, O. V.
[1] On the dimension of bicompacta (R), Dokl. Akad. Nauk SSSR 67 (1949) 217-219.
[2] On the theory of dimension, Sov. Math., Dokl. 9 (1968) 359-361.
[3] An axiomatic definition of the dimension of bicompacta, Sov. Math., Dokl. 14 (1973) 1477-1480.

Lunc, A. L.
[1] A bicompactum whose inductive dimension is greater than its dimension defined by means of coverings (R), Dokl. Akad. Nauk SSSR 66 (1949) 801-803.

Luxemburg, L. A.
[1] Compacta with non-identical transfinite dimension, Sov. Math., Dokl. 14 (1973) 1593-1597.
[2] On universal metric locally finite-dimensional space, Gen. Topol. Appl. 10 (1979) 283-290.

Mañé, R.
[1] Expansive homeomorphisms and topological dimension, Trans. Am. Math. Soc. 252 (1979) 313-319.

Mardešić, S.
[1] On covering dimension and inverse limit of compact spaces, Ill. J. Math. 4 (1960) 278-291.

Martin, H. W.
[1] Strongly rigid metrics and zero-dimensionality, Proc. Am. Math. Soc. 67 (1977) 157-161.

McAuley, L. F.
[1] Conditions for the equality of the inductive dimension, Port. Math. J. 24 (1965) 21-30.

Menger, K.
[1] Über umfassendste n-dimensionale Mengen, Proc. Ned. Akad. Wet. 29 (1926) 1125-1128.
[2] Zur Begründung einer axiomatischen Theorie der Dimension, Monatsh. Math. Phys. 36 (1929) 193-218.

Mizokami, T.
[1] The equality of large and small inductive dimensions, J. Lond. Math. Soc. 20 (1979) 541-543.
[2] On Nagata's problem for paracompact σ-metric spaces, Topol. Appl. 11 (1980) 211-221.

Morita, K.
[1] On the dimension of normal spaces I, Japan. J. Math. 20 (1950) 5-36.
[2] On the dimension of normal spaces II, J. Math. Soc. Japan 2 (1950) 16-33.
[3] On the dimension of product spaces, Am. J. Math. 75 (1953) 205-223.
[4] Normal families and dimension theory for metric spaces, Math. Ann. 128 (1954) 350-362.
[5] A condition for metrizability of topological spaces and for n-dimensionality, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A (1955) 33-36.
[6] On closed mappings and dimension, Proc. Japan. Acad. 32 (1956) 161-165.
[7] On the dimension of the product of topological spaces, Tsukuba J. Math. 1 (1977) 1-6.
[8] Dimension of general topological spaces, Surveys in General Topology (1980) 297-336.

Morita, K. and Hanai, S.
[1] Closed mappings and metric spaces, Proc. Japan. Acad. 32 (1956) 10-14.

Mrowka, S.
[1] Recent results of E-compact spaces, TOPO'72 (Proc. 2nd Pittsburgh Internat. Conf. 1972) 298-301.

Nagami, K.
[1] Some theorems in dimension theory for non-separable spaces, J. Math. Soc. Japan 7 (1957) 80-92.
[2] Mappings of finite order and dimension theory, Japan. J. Math. 30 (1960) 25-54.
[3] A note on Hausdorff space with the star-finite property I, II, III, Proc. Japan. Acad. 37 (1961) 131-134, 189-192, 356-367.
[4] Mappings defined on 0-dimensional spaces and dimension, J. Math. Soc. Japan 14 (1962) 101-118.
[5] Dimension-theoretical structure of locally compact groups, J. Math. Soc. Japan 14 (1962) 379-396.
[6] Monotone sequence of 0-dimensional subsets of metric spaces, Proc. Japan. Acad. 41 (1965) 771-772.
[7] Closed images of countable-dimensional spaces, J. Math. Soc. Japan 19 (1967) 457-459.
[8] Dimension Theory, New York-London, 1970.
[9] On the dimension of product spaces, Proc. Japan. Acad. 36 (1960) 560-564.

Nagami, K. and Roberts, J. H.
[1] A study of metric-dependent dimension functions, Trans. Am. Math. Soc. 129 (1967) 414-435.

Nagata, J.
[1] On a necessary and sufficient condition of metrizability, J. Inst. Polytechn. Osaka City Univ. 1 (1950) 93-100.
[2] Note on dimension theory for metric spaces, Fundam. Math. 45 (1958) 143-181.
[3] On the countable sum of zero-dimensional metric spaces, Fundam. Math. 48 (1960) 1-14.
[4] On a universal n-dimensional set for metric spaces, J. Reine Angew. Math. 204 (1960) 132-138.
[5] Two theorems for the n-dimensionality of metric spaces, Compos. Math. 15 (1963) 227-237.

[6] A remark on general imbedding theorem in dimension theory, Proc. Japan. Acad. 39 (1963) 197-199.
[7] On a special metric and dimension, Fundam. Math. 55 (1964) 181-194.
[8] Modern General Topology, Amsterdam, 1974.
[9] A survey of dimension theory, Proc. 2nd Prague Topol. Symp. (1966) 259-270.
[10] A survey of dimension theory II, Gen. Topol. Appl. 1 (1971) 65-77.
[11] A survey of dimension theory III, Proc. Int. Topol. Conf. Moscow, 1979.
[12] On Hyman's M-space, Proc. Topol. Conf. VPI and State Univ. 1973 (1974) 198-207.

Nagata, J. and Kodama, Y.
[1] Introductory Topology, to appear from this Publisher.

Nemec, A. G.
[1] Metric dimension, Sov. Math., Dokl. 16 (1975) 1451-1455.

Nichols, J. C. and Smith, J.C.
[1] Examples concerning sum properties for metric-dependent dimension functions, Pac. J. Math. 38 (1971) 151-154.

Nishiura, T.
[1] Inductive invariants and dimension theory, Fundam. Math. 59 (1966) 243-262.

Nöbeling, G.
[1] Über eine n-dimensionale Universalmenge im R_{2n+1}, Math. Ann. 104 (1930) 71-80.

Nyikos, P.
[1] Prabir Roy's space Δ is not N-compact, Gen. Topol. Appl. 3 (1973) 197-210.

Okuyama, A.
[1] On cohomological dimension for paracompact spaces I, Proc. Japan. Acad. 38 (1962) 489-494.
[2] σ-spaces and closed mappings II, Proc. Japan. Acad. 44 (1968) 478-481.

Ostrand, P. A.
[1] Dimension of metric spaces and Hilbert's problem 13, Bull. Am. Math. Soc. 71 (1965) 619-622.
[2] A conjecture of J. Nagata on dimension and metrization, Bull. Am. Math. Soc. 71 (1965) 623-625.
[3] Covering dimension in general spaces, Gen. Topol. Appl. 1 (1971) 209-221.

Pasynkov, B.
[1] On polyhedral spectra and dimension of compact spaces, especially of compact groups (R), Dokl. Akad. Nauk SSSR 121 (1958) 45-48.
[2] On the coincidence of different definitions of the dimension for the locally bicompact groups (R), Dokl. Akad. Nauk SSSR 132 (1960) 1035-1037.
[3] ω-mappings and inverse spectra (R), Dokl. Akad. Nauk SSSR 150 (1963) 488-491.
[4] On universal bicompacta of given weight and dimension, Sov. Math., Dokl. 5 (1964) 245-246.
[5] On a class of mappings and on the dimension of normal spaces (R), Sib. Mat. Zh. 5 (1964) 356-376.
[6] The dimension of rectangular products, Sov. Math., Dokl. 16 (1975) 344-347.
[7] Almost metrizable topological groups, Sov. Math., Dokl. 6 (1965) 404-407.
[8] On the dimension of normal spaces, Sov. Math., Dokl. 12 (1971) 1784-1787.

Pears, A. R.
[1] A note on transfinite dimension, Fundam. Math. 71 (1971) 215-221.

Peck, N. T.
[1] Extreme points and dimension theory, Pac. J. Math. 25 (1968) 341-351.

Pol, E.
[1] Some examples in the dimension theory of Tychonoff spaces, Fundam. Math. 102 (1979) 29-43.
[2] On the dimension of the product of metrizable spaces, Bull. Acad. Pol. Sci., Sér. Sci. Math. 26 (1978) 525-534.

Pol, E. and Pol, R.
[1] A hereditarily normal strongly zero-dimensional space containing subspaces of arbitrary large dimension, Fundam. Math. 102 (1979) 29-43.

Pol, R.
[1] A weakly infinite-dimensional compactum which is not countable-dimensional, to appear.

Ponomarev, V. I.
[1] Normal spaces as images of zero-dimensional spaces (R), Dokl. Akad. Nauk SSSR 132 (1960) 1269-1270.
[2] Projective spectra and continuous mappings of paracompacta, Transl., II. Ser., Am. Math. Soc. 39 (1964) 133-164.

Pontrjagin, L.
[1] Sur une hypothèse fondamentale de la théorie de la dimension, C. R. Acad. Sci., Paris, Sér. A 190 (1930) 1105-1107.

Pontrjagin, L. and Schnirelmann, L.
[1] Sur une propriété métrique de la dimension, Ann. Math., II. Ser. 33 (1932) 152-162.

Pontrjagin, L. and Tolstowa, G.
[1] Beweis des Mengerschen Einbettungssatzes, Math. Ann. 105 (1931) 634-746.

Prymusiński, T. C.
[1] A note on dimension theory of metric spaces, Fundam. Math. 85 (1974) 277-284.
[2] On the dimensions of product spaces and an example of M. Wage, Proc. Am. Math. Soc. 76 (1979) 315-321.

Pust, H.
[1] Normalität und Dimension für Nearness-Räume, Dissertation Freie Univ. Berlin (1977).

Roberts, J. H.
[1] A theorem on dimension, Duke Math. J. 8 (1941) 565-574.
[2] Dimension function d_2 and covering dimension, Duke Math. J. 37 (1970) 467-472.

Roy, P.
[1] Failure of equivalence of dimension concepts for metric spaces, Bull. Am. Math. Soc. 68 (1962) 609-613.

Royden, H. L.
[1] Real Analysis, London, 1968.

Rozanskaya, Yu. A.
[1] Open mappings and dimension (R), Usp. Mat. Nauk 4 (1949) 178-179.

Rubin, L. R. and Schori, R. M. and Walsh, J. J.
[1] New dimension-theory techniques for constructing infinite-dimensional examples, Gen. Topol. Appl. 10 (1979) 93-102.

Rudin, M. E. and Starbird, M.

[1] Product with a metric factor, Gen. Topol. Appl. 5 (1975) 235-240.

Sakai, S.

[1] An axiomatic characterization of the large inductive dimension, Proc. Japan. Acad. 44 (1968) 782-785.

Saks, S.

[1] Theory of the Integral, Warszawa, 1934.

Ščepin, E. V.

[1] Axiomatics of the dimension of metric spaces, Sov. Math., Dokl. 13 (1972) 1177-1179.

[2] A finite-dimensional compact absolute neighborhood retract is metrizable, Sov. Math., Dokl. 18 (1977) 402-406.

Schurle, A. W.

[1] Compactification of strongly countable-dimensional spaces, Trans. Am. Math. Soc. 136 (1969) 25-32.

Šersnëv, M.

[1] Characterization of dimension of metric spaces by continuous mappings into Euclidean spaces (R), Usp. Mat. Nauk 12 (1957) 245-247.

Shmuely, Z.

[1] On strongly countable-dimensional sets, Duke Math. J. 38 (1971) 169-173.

Sitnikov, K.

[1] An example of two-dimensional set in the three-dimensional Euclidean space, which allows a deformation as small as desired in one-dimensional polyhedron, and some new character of the dimension of the sets in Euclidean spaces (R), Dokl. Akad. Nauk SSSR 88 (1953) 21-24.

[2] The definition of the dimension of a closed set by a metric property of the complimentary space (R), Dokl. Akad. Nauk SSSR 66 (1949) 1059-1062.

Sklyarenko, E.

[1] On dimensional properties of infinite-dimensional spaces, Transl., II. Ser., Am. Math. Soc. 21 (1962) 35-50.

[2] Some remarks on infinite-dimensional spaces (R), Dokl. Akad. Nauk SSSR 126 (1959) 1203-1206.

[3] Two theorems on infinite-dimensional spaces, Sov. Math., Dokl. 3 (1962) 547-550.

[4] A theorem on dimension lowering mappings, Bull. Acad. Polon. Sci., Sér. Sci. Math. 10 (1962) 429-432.

Smirnov, Yu. M.

[1] A necessary and sufficient condition for metrizability of topological spaces (R), Dokl. Akad. Nauk SSSR 77 (1951) 197-200.

[2] On the metrizability of bicompacta decomposable into a sum of sets with countable basis (R), Fundam. Math. 43 (1956) 387-393.

[3] On the dimension of proximity spaces (R), Mat. Sb., Nov. Ser. 38 (1956) 283-302.

[4] On strongly paracompact spaces (R), Izv. Akad. Nauk SSSR, Ser. Mat. 20 (1956) 253-274.

[5] On the metric dimension in the sense of P. S. Alexandroff (R), Izv. Akad. Nauk SSSR, Ser. Mat. 20 (1956) 679-684.

[6] On universal spaces for certain class of infinite-dimensional spaces, Transl., II. Ser., Am. Math. Soc. 21 (1962) 21-34.

[7] Some remarks on transfinite dimension (R), Dokl. Akad. Nauk SSSR 141 (1961) 814-817.

[8] On the dimension of proximity spaces, Transl., II. Ser., Am. Math. Soc. 21 (1962) 1-20.
[9] Über die Dimension der Adjunkten bei Kompaktifizierungen, Monatsber. Deut. Akad. Wiss. Berlin 7 (1965) 230-232.

Sorgenfrey, R. H.
[1] On the topological product of paracompact spaces, Bull. Am. Math. Soc. 53 (1947) 631-632.

Sperner, E.
[1] Neuer Beweis für die Invarianz der Dimensionszahl und der Gebiete, Abh. Math. Sem. Hamb. 6 (1928) 265-272.

Štan'ko, M. A.
[1] Solution of Menger's problem in the case of compacta, Sov. Math., Dokl. 12 (1971) 1846-1848.

Stone, A. H.
[1] Paracompactness and product spaces, Bull. Am. Math. Soc. 54 (1948) 977-982.
[2] Metrizability of decomposition spaces, Proc. Am. Math. Soc. 7 (1956) 690-700.
[3] Metrizability of unions of spaces, Proc. Am. Math. Soc. 10 (1959) 361-366.

Suzuki, J.
[1] Note on a theorem for dimension, Proc. Japan. Acad. 35 (1959) 201-202.

Szpilrajn, E. (Marczewski)
[1] La dimension et la mesure, Fundam. Math. 28 (1937) 81-87.

Toulmin, G. H.
[1] Shuffling ordinals and transfinite dimension, Proc. Lond. Math. Soc. 4 (1954) 177-195.

Tukey, J. W.
[1] Convergence and Uniformity in Topology, Princeton, 1940.

Tumarkin, L.
[1] Beitrag zur allgemeinen Dimensionstheorie, Mat. Sb., Nov. Ser. 33 (1926) 57-86.
[2] Über die Dimension nicht abgeschlossener Mengen, Math. Ann. 98 (1928) 637-656.
[3] Sur la structure dimensionelle des ensembles fermés, C. R. Acad. Sci., Paris, Sér. A 186 (1928) 420-422.
[4] On Cantorian manifolds of infinite number of dimension (R), Dokl. Akad. Nauk SSSR 115 (1957) 244-246.
[5] Sur la décomposition des espaces en une somme dénombrable d'ensembles de dimension nulle (R), Mosc. Univ. Math. Bull. 1 (1960) 25-32.

Urysohn, P.
[1] Works on topology and other fields of mathematics. Moscow-Leningrad, 1951.

Valiev, V. A.
[1] On the spectral dimension of bicompact extensions and products of topological spaces, Sov. Math., Dokl. 12 (1971) 1358-1362.

Vedenissoff, N.
[1] Généralisation de quelques théorèmes sur la dimension, Compos. Math. 7 (1939) 194-200.

Vopenka, P.
[1] On the dimension of compact spaces (R), Czech. J. Math. 8 (1958) 319-327.
[2] Remarks on the dimension of metric spaces (R), Czech. J. Math. 9 (1959) 519-522.

Wage, M.
[1] The dimension of product spaces, Proc. Natl. Acad. Sci. 75 (1978) 4671-4672.

Walker, J. W. and Wenner, B. R.
[1] Characterizations of certain classes of infinite-dimensional metric spaces, Topol. Appl. 12 (1981) 101-104.

Wallace, A. D.
[1] Dimensional types, Bull. Am. Math. Soc. 51 (1945) 679-681.

Walsh, J. J.
[1] An infinite-dimensional compactum containing no n-dimensional ($n \geq 1$) subsets, Topology 18 (1979) 91-95.

Wenner, B. R.
[1] A universal separable metric locally finite-dimensional space, Fundam. Math. 80 (1973) 283-286.

Zarelua, A.
[1] On a theorem of Hurewicz, Transl., II. Ser., Am. Math. Soc. 55 (1966) 141-152.
[2] A universal bicompactum of given weight and dimension, Sov. Math., Dokl. 5 (1964) 214-218.
[3] Construction of strongly infinite-dimensional compacta by means of rings of continuous functions, Sov. Math., Dokl. 15 (1974) 106-110.
[4] On finite-to-one mappings, Sov. Math., Dokl. 8 (1967) 172-175.

Zervos, S. P.
[1] Some relations between old and modern considerations on topological dimension: Its fundamental role in the foundation of ancient Greek geometry, Proc. internat. Sympos. topology, and its applications, Topologija i ee Primenenija, Budva 1972 (Beograd 1973) 252-259.

ADDITIONAL BIBLIOGRAPHY
(not referred to in the text).

Adnadzevic, D.
[1] Some properties of A-spaces, Sov. Math., Dokl. 14 (1973) 492-496 / Zbl. 287.54032.

Aleksandrov, P.S.
[1] Über den axiomatischen Aufbau der Dimensionstheorie, Proc. internat. Sympos. Topology Appl., Budva 1972, (1973) 16-17 (R) / Zbl. 282.54020.
[2] Introduction to homological dimension theory and general combinatorial topology. (Vvedenie v homologiceskuju teoriju razmernosti i obscuju kombinatornuju topologiju.), Moskva (1975) (R) / Zbl. 441.55002.

Assouad, P.
[1] Étude d'une dimension métrique liée à la possibilité de plongements dans R^n, C. R. Acad. Sci., Paris, Sér. A 288 (1979) 731-734 / Zbl. 409.54020.

Bakovic, V.
[1] A classification of infinite dimensional spaces, Mat. Vesnik 15(30) (1978) 11-19 (R) / Zbl. 407.54030.

Baladze, V.H.
[1] On the dimensions modulo a classes of spaces, Soobsc. Akad. Nauk Gruzin. SSR 89 (1978) 33-36 (RE) / Zbl. 403.54022.

Bel'nov, V.K.
[1] On zero-dimensional topological groups, Sov. Math., Dokl. 17 (1976) 164-168 / Zbl. 332.22001.
[2] Dimension of topologically homogeneous spaces and free homogeneous spaces, Sov. Math., Dokl. 19 (1978) 86-89 / Zbl. 398.54025.

Bennett, R.
[1] On the fundamental dimension of compact spaces, Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 21 (1973) 65-68 / Zbl. 246.54044.

Berg, G.; Julian, W.; Mines, R.; Richman, F.
[1] The constructive equivalence of covering and inductive dimensions, Gen. Topol. Appl. 7 (1977) 99-108 / Zbl. 359.54025.

Bogatyi, S.
[1] On metric retracts, Sov. Math., Dokl. 13 (1972) 674-677 / Zbl. 262.54007.

[2] Properties of the topological and uniform dimension in terms of rings of continuous functions, Sib. Math. J. 14 (1973) 199-205 / Zbl. 269.54026.

Bogatyi. S. A. and Madirimov, M.
[1] On the dimension theory of metrizable spaces with periodic homeomorphisms, Math. USSR, Sb. 27 (1975) 67-76 (1977) / Zbl. 357.54027.

Bothe, H. G.
[1] Relative embeddability of compacta in euclidean spaces, Topics in Topol., Colloq. Keszthely 1972, Colloq. Math. Soc. Janos Bolyai 8 (1974) 127-133 / Zbl. 289.54023.

Brechner, B. L.
[1] On the non-monotony of dimension, Pac. J. Math. 41 (1972) 587-600 / Zbl. 217, 483.

Broughan, K. A.
[1] Uniformization of topological spaces with small inductive dimension zero, Math. Ann. 205 (1973) 241-242 / Zbl. 247.54030.
[2] Metrization of spaces having Čech dimension zero, Bull. Aust. math. Soc. 9 (1973) 161-168 / Zbl. 279.54016.
[3] Four metric conditions characterizing Čech dimension zero, Proc. Am. Math. Soc. 64 (1977) 176-178 / Zbl. 335.54027.

Buzasi, S.
[1] Dimension and metrization of uniform spaces, Russ. Math. Surv. 35, No. 3 (1980) 176-178 / Zbl. 459.54018.

Canfell, M. J.
[1] Dimension theory in zero-set spaces, Pac. J. Math. 47 (1973) 393-400 / Zbl. 236.54027.

Castellet, M.
[1] A product theorem in dimension, Math. Z. 128 (1972) 195-197 / Zbl. 232.55010.

Charalambous. M. G.
[1] A new covering dimension function for uniform spaces, J. Lond. Math. Soc., II. Ser. 11 (1975) 137-143 / Zbl. 306.54048.
[2] The dimension of inverse limits, Proc. Am. Math. Soc. 58 (1976) 289-295 / Zbl. 348.54029.
[3] A note on the dimension of products, Gen. Topol. Relat. mod. Anal. Algebra, Proc. 4th Prague Topol. Symp. 1976, Part B (1977) 70-71 / Zbl. 371.54055.
[4] Inductive dimension and inverse sequences of compact spaces, Proc. Am. Math. Soc. 81 (1981) 482-484 / Zbl. 451.54029.
[5] A note on analytic dimension, J. Lond. Math. Soc., II. Ser. 17 (1978) 545-546 / Zbl. 397.54052.
[6] Spaces with increment of dimension n, Fundam. Math. 93 (1976) 97-107 / Zbl. 341.54045.

Chigogidze, A. Ch.
[1] On some question in dimension theory, Topology, Vol. I, 4th Colloq. Budapest 1978, Colloq. Math. Soc. Janos Bolyai 23 (1980) 273-286 / Zbl. 458.54026.
[2] On the dimension of increments of Tychonoff spaces, Fundam. Math. 111 (1981) 25-36 / Zbl. 378.54021.

Coban, M. M.
[1] The topological structure of subsets of topological groups and their quotient spaces, Sov. Math., Dokl. 17 (1976) 669-673 / Zbl. 346.22007.

Deak, E.
[1] On "nice" covers, Topology, Vol. I, 4th Colloq. Budapest 1978, Colloq. Math. Soc. Janos Bolyai 23 (1980) 343-350 / Zbl. 439.54035.

Deak, J.
[1] Subbase and dimension. I., Stud. Sci. Math. Hung. 11 (1976) 389-397 / Zbl. 449.54039.
[2] A new characterization of the class of subspaces of a Euclidean space, Stud. Sci. Math. Hung. 11 (1976) 253-258 / Zbl. 416.54035.

Duda, R.
[1] The origins of the concept of dimension, Colloq. Math. 42 (1979) 95-110 / Zbl. 459.54027.

Dvalisvili, B.
[1] On dimension and some other problems of bitopological spaces, Tr. Tbilis. Mat. Inst. Razmadze 56 (1977) 15-51 (RE) / Zbl. 396.54029.

Engelking, R.
[1] Some new proofs in dimension theory, Symp. math. 16, Topol. insiem. gen., Gruppi topol. Gruppi di Lie, Convegni 1973/1974, (1975) 83-91 / Zbl. 315. 54041.
[2] Transfinite dimension, Surveys in general topology, (1980) 131-161 / Zbl. 447.54046.

Erdös, P.
[1] Some remarks on subgroups of real numbers, Colloq. Math. 42 (1979) 119-120 / Zbl. 433.04002.

Fedorčuk, V. V.
[1] Infinite-dimensional compact Hausdorff spaces, Math. USSR, Izv. 13 (1979) 445-460 / Zbl. 423.54024.
[2] Perfectly normal compact space without intermediate dimensions, Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 23 (1975) 975-979 / Zbl. 313. 54043.
[3] Compatibility of some theorems of general topology with the axioms of set theory, Sov. Math., Dokl. 16 (1975) 160-162 / Zbl. 331.54025.
[4] A compact Hausdorff space all of whose infinite closed subsets are n-dimensional, Math. USSR, Sb. 25 (1975) 37-57 / Zbl. 327.54031.
[5] On dimensional components of compact spaces, Sov. Math., Dokl. 15 (1974) 505-509 / Zbl. 312.54037.
[6] Compact spaces without intermediate dimensions, Sov. Math., Dokl. 14 (1973) 1808-1811 / Zbl. 288.54039.

Filippov, V. V.
[1] Dimensionality of spaces with the action of a bicompact group, Math. Notes 25 (1979) 171-174 / Zbl. 436.54030.
[2] On the dimension of closed mappings, Sov. Math., Dokl. 13 (1972) 895-900 / Zbl. 273.54027.
[3] A survey of dimension theory, Am. Math. Soc., Translat., II. Ser. 109 (1977) 47-50 / Zbl. 368.54010.

Firby, P. A.
[1] Independence structures and topological dimension, J. Lond. Math. Soc., II. Ser. 22 (1980) 11-13 / Zbl. 443.54027.

Fora, A. A.
[1] On the covering dimension of subspaces of product of Sorgenfrey lines, Proc. Am. Math. Soc. 72 (1978) 601-606 / Zbl. 404.54029.
[2] On some test spaces in dimension theory, Fundam. Math. 111 (1981) 19-23 / Zbl. 378.54022.

French, J. A.
[1] A condition equivalent to covering dimension for normal spaces, Compos. Math. 28 (1974) 223-227 / Zbl. 288.54036.

Gehring, F. W. and Vaeisaelae, J.
[1] Hausdorff dimension and quasiconformal mappings, J. Lond. Math. Soc., II. Ser. 6 (1973) 504-512 / Zbl. 258.30020.

Goto T.
[1] On the realization of the metricdependent dimension function d, Proc. Am. Math. Soc. 58 (1976) 265-271 / Zbl. 341.54046.

Gresham, J. H.
[1] A class of infinite-dimensional spaces. II: An extension theorem and the theory of retracts, Fundam. Math. 107 (1980) 237-245 / Zbl. 437.54023.

Gruenhage, G.
[1] Large basis dimension and metrizability, Proc. Am. Math. Soc. 54 (1976) 397-400 / Zbl. 317.54037.

Gruenhage, G. and Zenor, P.
[1] Metrization of spaces with countable large basis dimension, Pac. J. Math. 59 (1975) 455-460 / Zbl. 316.54029.

Hadziivanov, N.
[1] On infinite-dimensional Cantor manifolds, Topics in Topol., Colloq. Keszthely 1972, Colloq. Math. Soc. Janos Bolyai 8 (1974) 355-363 / Zbl. 292.54048.

Hall, J. jun.
[1] Covering dimension in finite-dimensional metric spaces, Proc. Am. Math. Soc. 41 (1973) 274-277 / Zbl. 278.54034.

Hansen, A. B. and Sternfeld, Y.
[1] On the characterization of the dimension of a compact metric space K by the representing matrices of $C(K)$, Isr. J. Math. 22 (1975) 148-167 / Zbl. 318.46036.

Hejcman, J.
[1] Remarks on dimensions of mappings, Gen. Topol. Relat. mod. Anal. Algebra III, Proc. 3rd Prague Topol. Symp. 1971, (1972) 181-182 / Zbl. 308.54029.

Ishii, T.
[1] Some results on ω-compact spaces, Usp. Mat. Nauk 35, No. 3(213) (1980) 61-66 (R) / Zbl. 441.54011.

Ivanov, A. V.
[1] On the dimension of incompletely normal spaces, Mosc. Univ. Math. Bull. 31, No. 3/4 (1976) 64-69 / Zbl. 351.54028.
[2] On hereditary separability and dimension of products of compact Hausdorff spaces, Sov. Math., Dokl. 19 (1978) 460-464 / Zbl. 403.54002.
[3] On the embedding of countable products of separable metric spaces in weakly infinite-dimensional compacta, Russ. Math. Surv. 33, No. 6 (1978) 231-232 (R) / Zbl. 422.54029.

Janos, L.
[1] Dimension theory via bisector chains, Can. Math. Bull. 20 (1977) 313-317 / Zbl. 378.54020.
[2] Topological dimension as a first order property, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 64 (1978) 572-577 / Zbl. 429. 54018.
[3] Rigid subsets in Euclidean and Hilbert spaces, J. Aust. Math. Soc. 20 (1975) 66-72 / Zbl. 305.54033.

Jobe, J.
[1] Dendrites, dimension, and the inverse arc function, Pacific J. Math. 45 (1973) 245-256 / Zbl. 231.54038.

Johnson, D. M.
[1] The problem of the invariance of dimension in the growth of modern topology, I. Arch. Hist. Exact. Sci. 20 (1979) 97-188 / Zbl. 424.54020.

Joiner, Ch.
[1] Free topological groups and dimension, Trans. Am. Math. Soc. 220 (1976) 401-418 / Zbl. 331.54026.

Jung, C. F. K.
[1] Mappings and inductive invariants, Colloq. Math. 28 (1973) 29-31 / Zbl. 265.54040.
[2] On the deficiency of product spaces, Colloq. Math. 27 (1973) 245-246 / Zbl. 267.54034.

Kandil, A.
[1] On the dimension of θ-proximity spaces, Russ. Math. Surv. 33, No. 6 (1978) 233-234 / Zbl. 425.54008.

Keesling, J.
[1] The Stone-Čech compactification and shape dimension, Topology proceedings, Vol. 2, No. 1, 2, Proc. Topol. Conf. LA State Univ. 1977, (1978) 483-508 / Zbl. 413.54041.

Klebanov, B. S.
[1] A sum theorem for inductive dimensionalities, Mosc. Univ. Math. Bull. 34, No. 1 (1979) 54-60 / Zbl. 428.54026.

Kodama, Y.
[1] On θ-spaces and fundamental dimension in the sense of Borsuk, Fundam. Math. 89 (1975) 13-22 / Zbl. 314.55011.

Kodama, Y. and Watanabe, T.
[1] A note on Borsuk's n-movability, Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 22 (1974) 289-294 / Zbl. 274.54020.

Kozlovskii, I. M.
[1] Two theorems on metric spaces, Sov. Math., Dokl. 13 (1972) 743-747 / Zbl. 268. 54030.

Kulpa, W.
[1] A note on the dimension Dind, Colloq. Math. 24 (1972) 181-183 / Zbl. 242. 54036.
[2] Homeomorphisms of inverse limits of metric spaces, Fundam. Math. 86 (1975) 233-235 / Zbl. 296.54012.

Lau, A. Y. W.
[1] A note on monotone maps and hyperspaces, Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 24 (1976) 121-123 / Zbl. 319.54011.
[2] Acyclicity and dimension of hyperspace of subcontinua, Bull. Acad. Polon. Sci., Ser. Sci. math. astron. phys. 22 (1974) 1139-1141 / Zbl. 296.54033.

Leibo, I. M.
[1] On closed images of metric spaces, Sov. Math., Dokl. 16 (1975) 1292-1295 / Zbl. 329.54034.

Lelek, A.
[1] Remarks on strong homotopic stability, Bull. Acad. Pol. Sci., Sér. Sci. Math. 27 (1979) 365-368 / Zbl. 423.54027.

Lelek, A. and Tymchatyn, E. D.
[1] Pseudo-confluent mappings and a classification of continua, Can. J. Math. 27 (1975) 1336-1348 / Zbl. 323.54008.

Lifanov, I. K.
[1] On the dimension of normal spaces, Sov. Math. Dokl. 14 (1973) 383-387 / Zbl. 285.54029.

Lifanov, I. K. and Pasynkov, B. A.
[1] Two classes of spaces and two types of dimension, Mosc. Univ. Math. Bull. 25 No. 3-4 (1970) 26-29 / Zbl. 246.54045.

Linicuk, R. S.
[1] Dimension of hyperspaces, Ukr. Math. J. 30 (1978) 292-294 / Zbl. 409.54013.
[2] Dyadic subspaces of certain hyperspaces, Ukr. Math. J. 30 (1978) 177-178 / Zbl. 409.54012.

Lisica, Ju.
[1] The extension of mappings and a factorization theorem, Sov. Math., Dokl. 13 (1972) 1660-1662 / Zbl. 268.54015.
[2] Extension of continuous maps and the factorization theorem, Sib. Math. J. 14 (1973) 90-96 / Zbl. 266.54003.
[3] Obstructions to extension of mappings, Sib. Math. J. 20 (1979) 240-245 / Zbl. 435.54011.

Lokucievskii, O. V. and Scepin, E. V.
[1] On P. S. Alexandroff's axiomatic definition of the dimension, Topics in Topol., Colloq. Keszthely 1972, Colloq. Math. Soc. Janos Bolyai 8 (1974) 477-480 / Zbl. 292.54043.

Luxemburg, L. A.
[1] On transfinite inductive dimensions, Sov. Math., Dokl. 14 (1973) 388-393 / Zbl. 283.54019.
[2] On compact metric spaces with noncoinciding transfinite dimensions, Pac. J. Math. 93 (1981) 339-386 / Zbl. 397.54048.
[3] On transfinite dimension of metric spaces, Russ. Math. Surv. 34, No. 1 (1979) 229-230 / Zbl. 425.54017.

Madirimov, M.
[1] On the extension of equivariant mappings, Math. USSR, Sb. 27 (1975) 77-84 / Zbl. 351.54011.

Mancuso, V. J.
[1] Expansion and dimension, J. Aust. Math. Soc., Ser. A 22 (1976) 313-320 / Zbl. 343.54029.

Mogyorosy, J.
[1] A note on covering dimension of paracompact spaces, Ann. Mat. Pura Appl., IV. Ser. 106 (1975) 305-313 / Zbl. 318.54034.

Morita, K.
[1] On the dimension of the product of Tychonoff spaces, Gen. Topol. Appl. 3 (1973) 125-133 / Zbl. 258.54034.
[2] The Hopf extension theorem for topological spaces, Houston J. Math. 1 (1975) 121-129 / Zbl. 313.55005.
[3] Cech cohomology and covering dimension for topological spaces. Fundam. Math. 87 (1975) 31-52 / Zbl. 336.55003.

Nagami, K.
[1] Dimension of free L-spaces, Fundam. Math. 108 (1980) 211-224 / Zbl. 456.54027.
[2] Dimension of non-normal spaces, Fundam. Math. 109 (1980) 113-121 / Zbl. 456.54026.

Nemec, A. G.
[1] On the embedding of a zero-dimensional perfect mapping into a projection, Vestnik Mosk. Univ., Ser. I 28, No. 4 (1973) 56-58 (RE) / Zbl. 267.54010.
[2] On compactifications that do not raise weight or dimension, Sov. Math., Dokl. 15 (1974) 189-192 / Zbl. 321.54013.
[3] The metric dimension of complete spaces, Mosc. Univ. Math. Bull. 33, No. 2 (1978) 58-61 / Zbl. 421.54024.

Nichols, J. C.
[1] The realization of dimension function d_2 , Fundam. Math. 77 (1973) 211-217 / Zbl. 265.54038.

Nishiura, T.
[1] An axiomatic characterization of covering dimension in metrizable spaces, General Topology Appl., 2nd Pittsburgh internat. Conf. 1972, Lect. Notes Math. 378 (1974) 341-353 / Zbl. 286.54025.
[2] A subset theorem in dimension theory, Fundam. Math. 95 (1977) 105-109 / Zbl. 354.54021.
[3] Inductive invariants of closed extensions of mappings, Colloq. Math. 25 (1972) 73-78 / Zbl. 239.54009.

Nishiura, T. and Tymchatyn, E. D.
[1] Hereditarily locally connected spaces, Houston J. Math. 2 (1976) 581-599 / Zbl. 341.54043.

Nowak, S.
[1] Some remarks concerning the fundamental dimension of the Cartesian product of two compacta, Fundam. Math. 103 (1979) 31-41 / Zbl. 439.55006.
[2] Some properties of fundamental dimension, Fundam. Math. 85 (1974) 211-227 / Zbl. 301.54041

Nyikos, P.
[1] A survey of zero-dimensional spaces, Topol., Proc. Memphis State Univ. Conf. 1975 (1976) 87-114 / Zbl. 355.54028.
[2] The Sorgenfrey plane in dimension theory, Fundam. Math. 79 (1973) 131-139 / Zbl. 264.54028.
[3] Inverse preservation of small inductive dimension, Topol. Proc. Vol. 1, Conf. Auburn Univ. 1976 (1976) 63-66 / Zbl. 397.54053.

[4] Some surprising base properties in topology, Stud. Topol., Proc. Conf. Charlotte, N. C., 1974 (1975) 427-450 / Zbl. 337.54014.

Oka, S.
[1] A note on the covering dimension of Lasnev spaces, Proc. Jap. Acad., Ser. A 54 (1978) 73-75 / Zbl. 411.54044.
[2] Dimension of finite unions of metric spaces, Math. Jap. 24 (1979) 341-362 / Zbl. 429.54017.

Onyszkiewicz, J.
[1] Dimensions of PC space of models, Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 22 (1974) 773-777 / Zbl. 291.02041.

Ormocadze, R. N.
[1] On an inductive dimension and estimation of metric dimension, Soobsc. Acad. Nauk Gruzin. SSR 90 (1978) 17-20 (RE) / Zbl. 391.54023.

Ostaszewski, A. J.
[1] A perfectly normal countably compact scattered space which is not strongly zero-dimensional, J. Lond. Math. Soc., II. Ser. 14 (1976) 167-177 / Zbl. 348.54015.

Parfenov, P. G.
[1] Equality of dimensions of completely normal compact spaces, Sib. Math. J. 19 (1979) 461-467 / Zbl. 424.54022.
[2] On the dimension theory of bicompacta, Sov. Math., Dokl. 14 (1973) 398-400 / Zbl. 301.54038.
[3] On some classes of perfectly normal bicompacta, Mosc. Univ. Math. Bull. 26, No. 5/6 (1971) 54-56 / Zbl. 252.54009.
[4] On a general approach to the proof of some theorems in dimension theory, Sov. Math., Dokl. 15 (1974) 1526-1530 / Zbl. 315.54043.

Pasynkov, B. A.
[1] On the dimension of products of normal spaces, Sov. Math., Dokl. 14 (1973) 530-533 / Zbl. 287.54036.
[2] On the dimension and geometry of mappings, Sov. Math., Dokl. 16 (1975) 384-388 / Zbl. 355.54027.
[3] On the dimension of spaces with a compact group of transformations, Russ. Math. Surv. 31, No. 5 (1976) 128-137 / Zbl. 355.54026.
[4] On spaces with a compact group of transformations, Sov. Math., Dokl. 17 (1976) 1522-1526 / Zbl. 352.54019.
[5] A factorization theorem for nonclosed sets, Sov. Math., Dokl. 13 (1972) 292-295 / Zbl. 247.54037.

Pasynkov, B. A. and Nemec, A. G.
[1] On two general approaches to factorization theorems in dimension theory, Sov. Math., Dokl. 18 (1977) 502-506 / Zbl. 375.54025.

Pears, A. R.
[1] Dimension theory of general spaces, Cambridge etc. Cambridge University Press. VII, 428 p. (1975) / Zbl. 312.54001.

Pears, A. and Mack, J.
[1] Closed covers, dimension, and quasi-order spaces, Proc. Lond. Math. Soc., III. Ser. 29 (1974) 289-316 / Zbl. 319.54026.

Pol, E.
[1] A remark about the Juhasz-Kunen-Rudin construction of a hereditarily separable non-Lindelöf space, Bull. Acad. Polon. Sci., Sér. Sci. math. astron. phys. 24 (1976) 749-751 / Zbl. 339.54036.
[2] Strongly metrizable spaces of large dimension each separable subspace of which is zero-dimensional, Colloq. Math. 39 (1978) 25-27 / Zbl. 392.54016.

Pol, R. and Pol, E.
[1] A hereditarily normal strongly zero-dimensional space with a subspace of positive dimension and an N-compact space of positive dimension, Fundam. Math. 97 (1977) 43-50 / Zbl. 374.54034.

Prat, U.
[1] On a weak sum theorem in dimension theory, Isr. J. Math. 19 (1974) 277-283 / Zbl. 305.54038.

Przymusinski, T. C.
[1] Product spaces, Surveys in general topology, (1980) 399-429 / Zbl. 451.54007.

Rancin, D. V.
[1] On sums of n-dimensional topologies, Comment. Math. Univ. Carol. 19 (1978) 595-618 (RE) / Zbl. 403.54024.

Reichaw, M.
[1] On a theorem of W. Hurewicz on mappings which lower dimension, Colloq. Math. 26 (1972) 323-329 / Zbl. 225.54035.

Reichel, H.-C.
[1] Zur Entwicklung des Dimensionsbegriffes, Math.-Phys. Semesterber., Neue Folge 27 (1980) 16-54 / Zbl. 459.54028.
[2] A note on zero-dimensional spaces with the star-finite property, Proc. Am. Math. Soc. 42 (1974) 307-311 / Zbl. 276.54015.

Reilly, I. L.
[1] Zero-dimensional bitopological spaces, Nederl. Akad. Wet., Proc., Ser. A 76 (1973) 127-131 / Zbl. 258.54031.

Rhee, C. J.
[1] On dimension of the hyperspace of a pseudo-circle, Bull. Soc. R. Sci. Liège 43 (1974) 5-8 / Zbl. 281.54004.

Richman, F.; Berg, G.; Cheng, H.; Mines R.
[1] Constructive dimension theory, Compos. Math. 33 (1976) 161-177 / Zbl. 337. 55008.

Rogers, J. T. jun.
[1] Dimension and the Whitney subcontinua of $C(X)$, Gen. Topol. Appl. 6 (1976) 91-100 / Zbl. 318.54005.

Rubin, L. R.
[1] Noncompact hereditarily strongly infinite dimensional spaces, Proc. Am. Math. Soc. 79 (1980) 153-154 / Zbl. 445.54021.
[2] Hereditarily strongly infinite dimensional spaces, Mich. Math. J. 27 (1980) 65-73 / Zbl. 406.54017.

Savinov, N. V.
[1] An example of a perfectly normal bicompactum with no intermediate dimensions, Mosc. Univ. Math. Bull. 31, No. 3/4 (1976) 40-43 / Zbl. 356.54038.
[2] Two examples in the dimension theory of bicompacts, Soc. Math., Dokl. 18 (1977) 299-303 / Zbl. 372.54033.

Ščepin, E. V.
[1] Axiomatics of the dimension of metric spaces, Math. USSR, Sb. 21 (1973) 137-143 / Zbl. 288.54038.

Schori, R. M. and Walsh, J. J.
[1] Examples of hereditarily strongly infinite-dimensional compacta, Topology proceedings, Vol. 3, No. 2, Proc. Topol. Conf. Univ. Okla. 1978 (1979) 495-506 / Zbl. 442.54024.

Shmuely, Z.
[1] Some embeddings of infinite-dimensional spaces, Isr. J. Math. 12 (1972) 5-10 / Zbl. 237.54028.

Singal, M. K. amd Arya, S. P.
[1] Sum theorems for the covering dimension of totally normal spaces, Publ. Math. 22 (1975) 195-198 / Zbl. 315.54042.

Štan'ko, M. A.
[1] Approximation of imbeddings of compacta in codimensions greater than two, Sov. Math., Dokl. 12 (1971) 906-909 / Zbl. 237.57005.

Sternfeld, J.
[1] Dimension theory and superpositions of continuous functions, Isr. J. Math. 20 (1975) 300-320 / Zbl. 312.54038.
[2] Uniformly separating families of functions, Isr. J. Math. 29 (1978) 61-91 / Zbl. 384.54007.

Su Li Pi
[1] Wallman-type compactifications on 0-dimensional spaces, Proc. Am. Math. Soc. 43 (1974) 455-460 / Zbl. 253.54016.

Telgarsky, R. and Yajima, Y.
[1] On order locally finite and closure-preserving covers, Fundam. Math. 109 (1980) 211-216 / Zbl. 367.54008.

Terada, T.
[1] On the dimension of the remainder of Stone-Čech compactifications, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A 12 (1974) 218-221 / Zbl. 306.54049.

Terasawa, J.
[1] Spaces NUR need not be strongly 0-dimensional, Bull. Acad. Polon. Sci., Ser. Sci. math. astron. phys. 25 (1977) 279-281 / Zbl. 353.54024.
[2] Spaces NUR and their dimensions, Topology Appl. 11 (1980) 93-102 / Zbl. 424.54024.
[3] On the zero-dimensionality of some non-normal product spaces, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A 11 (1972) 167-174 / Zbl. 253.54032.

Todorov, V. T.
[1] Solution of Alexandroff's problem on Cantor manifolds, C. R. Acad. Bulg. Sci. 33 (1980) 445-447 / Zbl. 439.54034.

Tomaszewski, B.
[1] On weakly n-dimensional spaces, Fundam. Math. 103 (1979) 1-8./ Zbl. 408.54026.

Tumarkin, L. A.
[1] On a lemma and sets of the first and second category in the generalized sense, Mosc. Univ. Math. Bull. 26, No. 5/6 (1971) 84-86 / Zbl. 252.54021.

Ul'janov, V. M.
[1] First countable compactifications that do not raise weight or dimension, Sov. Math., Dokl. 15 (1974) 1218-1222 / Zbl. 303.54006.

Ury, L.
[1] On relative universal embedding spaces, Acta Math. Acad. Sci. Hung. 28 (1976) 181-188 / Zbl. 338.54023.

Vaeisaelae, J.
[1] Dimension and measure, Proc. Am. Math. Soc. 76 (1979) 167-168 / Zbl. 394.57019.

Venema, G. A.
[1] Neighbourhoods of compacta in Euclidean space, Fundam. Math. 108 (1980) 71-78 / Zbl. 372.57006.

Walsh, J. J.
[1] Infinite-dimensional compacta containing no n-dimensional ($n \geq 1$) subsets, Bull. Am. Math. Soc. 84 (1978) 137-138 / Zbl. 378.54023.
[2] A class of spaces with infinite cohomological dimension, Mich. Math. J. 27 (1980) 215-222 / Zbl. 406.54016.

Weller, G. P.
[1] The intersection multiplicity of n-dimensional paracompact spaces, Proc. Am. Math. Soc. 50 (1975) 402-404 / Zbl. 313.54020.
[2] The intersection multiplicity of compact n-dimensional metric spaces, Proc. Am. Math. Soc. 36 (1972) 293-294 / Zbl. 257.54029.

Wenner, B. R.
[1] A new class of paracompact spaces, J. Lond. Math. Soc., II. Ser. 5 (1972) 356-361 / Zbl. 242.54015.
[2] Finite-dimensional properties of infinite-dimensional spaces, Pac. J. Math. 42 (1972) 267-276 / Zbl. 215, 239.

Williams, S.
[1] Transfinite cardinal dimension and separability, Port. Math. 32 (1973) 139-145 / Zbl. 261.54027.

Yajima, Y.
[1] On order star-finite and closure-preserving covers, Proc. Jap. Acad., Ser. A 55 (1979) 19-22 / Zbl. 424.54008.

Zambahidze, L. G.
[1] On dimension type functions, Tr. Tbilis. Mat. Inst. Razmadze 56 (1977) 52-98 (RE) / Zbl. 396.54033.
[2] On the relations between the dimensions of free bases of free topological groups, Soobshch. Akad. Nauk Gruz. SSR 97 (1980) 569-572 (RE) / Zbl. 452.54031.
[3] On the interrelations among the peripheral properties of Tikhonov spaces, their powers, exponentials, free groups, and semigroups, Russ. Math. Surv. 34, No. 6 (1979) 256-260 / Zbl. 445.54020.
[4] On some relations between the dimensions of normal spaces and their subspaces of special type, Soobshch. Akad. Nauk Gruz. SSR 89 (1978) 41-44 (RE) / Zbl. 411.54040.

Zambahidze, L. G. and Tovodros, S. F.
[1] On the dimension *dm* and its applications in the classification of locally connected continua, Soobsc. Akad. Nauk Gruz. SSR 84 (1976) 49-52 (RE) / Zbl. 342.54030.

Zemke, C. F.
[1] Dimension and decompositions, Fundam. Math. 95 (1977) 157-165 / Zbl. 391.54024.

Zolotarev, V. P.
[1] On finite-to-one quotient mappings raising the dimension, Sov. Math., Dokl. 14 (1973) 202-206 / Zbl. 285.54028.
[2] On the dimensions of subspaces, Mosc. Univ. Math. Bull. 30, No. 5/6 (1975) 7-8 / Zbl. 325.54016.

LIST OF THEOREMS

LIST OF DEFINITIONS

AUTHOR INDEX

SUBJECT INDEX