Differential Geometry and Topology

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Editors' Preface

A large number of mathematical books begin as lecture notes; but, since mathematicians are busy, and since the labor required to bring lecture notes up to the level of perfection which authors and the public demand of formally published books is very considerable, it follows that an even larger number of lecture notes make the transition to book form only after great delay or not at all. The present lecture note series aims to fill the resulting gap. It will consist of reprinted lecture notes, edited at least to a satisfactory level of completeness and intelligibility, though not necessarily to the perfection which is expected of a book. In addition to lecture notes, the series will include volumes of collected reprints of journal articles as current developments indicate, and mixed volumes including both notes and reprints.

JACOB T. SCHWARTZ
MAURICE LÉVY

Differential Geometry and Topology
Author's Preface

The present set of notes was developed during a course given in the 1965–66 academic year. It is hoped that, in spite of the rather fragmentary character of the notes, they will be of use to graduate students and others wishing to survey the material with which they are concerned.

Our emphasis lies on the development and application of intersection theoretic methods for the calculation of various interesting topological invariants. Chapter 1 gives a summary of the usual basic generalities of differential topology. The fundamental lemma of Sard is proved and yields an elementary proof for the Brouwer fixed point theorem. Chapter 2 uses Sard's lemma, and the transversality arguments originally developed by Rene Thom, to derive the classical connections between geometric intersection theory and algebraic homology on a rigorous basis. In the following chapter we use these intersection theoretic results to calculate the cohomology ring of the Grassmann spaces; the facts derived in this way form the basis for our subsequent discussion of Whitney and Chern classes. In the second part of Chapter 3 we use intersection theoretic arguments, combined with arguments taken from Morse theory, to prove the Poincare duality theorem for differentiable manifolds. Chapter 4 summarizes various basic facts concerning fiber bundles, especially linear bundles. Chapter 5 gives an outline of the algebraic theory of spectral sequences. In Chapter 6 we combine the general principles discussed in the two preceding chapters with the intersection theoretic methods developed in Chapter 3 and discuss the characteristic classes of linear bundles. In the following chapter we develop various fundamental formulae of Riemannian geometry; then, combining these with the topological material developed in the preceding chapters, we derive the very interesting generalization of the Riemann–Roch theorem due to Chern.

The remaining chapters are intended to serve as a quick introduction to the generalized cohomology theory, and particularly to K-theory, which has played an important role in various recent topological advances. Chapter 8 begins with the simple proof of the fact that certain families of homotopy classes define generalized cohomology theories and then applies this general result to define *K*-theory as a particular generalized cohomology theory. Chapter 9 continues the discussion of *K*-theory, deriving various product relations belonging to this theory as well as the special properties of *K*-theory which follow from the Bott periodicity theorem. In the third section of Chapter 9, we show that the Chern classes define a homomorphism of the *K*-theory into ordinary cohomology theory. This fact is then used to derive a number of interesting properties of *K*-theory. The Atiyah–Hirzebruch spectral sequence for *K*-theory is derived in Section 4 of Chapter 9. Mapping this spectral sequence by the Chern character discussed in the preceding section of Chapter 9, we are able to develop the various interesting relations between the *K*-function and the ordinary cohomology theory. In Section 5 of Chapter 9 the *K*-theory operations originally introduced by Adams are discussed. Our concluding Chapter 10 gives an incomplete, but hopefully helpful, sketch of the proof of Adams' result on the number of linearly independent vector fields on *n*-spheres.

Contents

CHAPTER I

General Theory of Manifolds

We begin with the concept of a *differentiable manifold,* or more briefly *manifold.*

Definition (1): A manifold M is a separable metric space with a system of open subsets $\{U_\alpha\}$ satisfying the following properties: (i) $M \subseteq \bigcup_\alpha U_\alpha$, (ii) for each α, there is a map $h_\alpha : U_\alpha \to E^m$ (E^m is the m-dimensional Euclidean space) such that h_α is a homeomorphism of U_α with an open ball in E^m, (iii) for arbitrary α and β, the map $h_\alpha h_\beta^{-1}$ of $h_\beta(U_\alpha \cap U_\beta) \to E^m$ is smooth, i.e. C^∞. The sets U_α are called "coordinate patches on M" and the maps $h_\alpha h_\beta^{-1}$ are called "change of coordinate maps"; m is called the dimension of M.

One can apply the notions of ordinary local analysis to manifolds. For example, if $\phi : M \to M_1$ is a map from the manifold M into the manifold M_1, we say ϕ is C^∞ or *smooth* at the point $p \in M$ if there exist coordinate patches $U_p \subseteq M$, $V_{\phi(p)} \subseteq M_1$ such that $p \in U_p$, $\phi(p) \in V_{\phi(p)}$ and such that the composite map $g_{\phi(p)} \phi h_p^{-1} : h_p(U_p) \to E^{m_1}$ is C^∞ at $h_p(p) \in E^m$. Here, h_p is the map associated with U_p and $g_{\phi(p)}$ is the map associated with $V_{\phi(p)}$. We say that $\phi' : M \to M_1$ is a *diffeomorphism* between M and M' if ϕ is a homeomorphism which is C^∞ at each point $p \in M$ and which is such that ϕ^{-1} is C^∞ at each point $q \in M_1$. Let us note, incidentally, that if $\phi : M \to M_1$ is C^∞ at $p \in M$, then $g_\gamma \phi h_\alpha^{-1} : h_\alpha(U_\alpha) \to E^{m_1}$ is a C^∞ map at $h_\alpha(p)$ for all coordinate patches U_α, V_γ such that $p \in U_\alpha \subseteq M$, $\phi(p) \in V_\gamma \subseteq M_1$. In fact, $g_\gamma \phi h_\alpha^{-1}$, when restricted to an appropriate subdomain, is equal to

$$g_\gamma g_{\phi(p)}^{-1} g_{\phi(p)} \phi h_p^{-1} h_p h_\alpha^{-1} = (g_\gamma \circ g_{\phi(p)}^{-1}) \circ (g_{\phi(p)} \circ \phi \circ h_p^{-1}) \circ (h_p \circ h_\alpha^{-1}).$$

Condition (iii) of Definition (1) tells us that $g_\gamma g_{\phi(p)}^{-1}$ and $h_p h_\alpha^{-1}$ are C^∞, while $g_{\phi(p)} \phi h_p^{-1}$ is C^∞ by assumption. Hence the composite map is C^∞, as asserted. This shows that the notion of differentiability at a point is invariant, i.e. coordinate free.

Let $f: M \to E^1$ be a smooth function at each point $p \in M$. We say that f is *horizontal* at p if, in coordinates, all first partial derivatives of f vanish at p. It follows easily from the chain rule that the notion of being horizontal at a point is invariant under coordinate changes.

Definition (2): Let τ be a linear functional, acting on the set of real-valued smooth functions on M. If, whenever f is horizontal at p, we have $\tau f = 0$, we say τ is a *tangent vector* to M at p.

The following definition is easily seen to be equivalent to Definition (2).

Definition (2′): Let τ be a linear functional, acting on the set of real-valued smooth functions on M. If $\tau(fg) = f(p)\,\tau(g) + g(p)\,\tau(f)$, we say τ is a *tangent vector* to M at p.

If f is a C^∞ real function of a real variable, then integration by parts yields

$$f(1) - f(0) = \int_0^1 f'(s)\,ds = f'(0) + \cdots$$

$$+ \frac{1}{n!} f^{(n)}(0) + \frac{1}{n!} \int_0^1 (1-s)^n f^{(n+1)}(s)\,ds.$$

If now g is a C^∞ real function of m real variables and if $x \in E^m$, set $f(s) = g(sx)$, and then substitute in the above formula. We obtain

$$g(x) = g(0) + \sum (\partial_i g)(0)\,x^i + \frac{1}{2!}\sum (\partial_{i_1 i_2} g)(0)\,x^{i_1}x^{i_2} + \cdots$$

$$+ \frac{1}{n!}\sum (\partial_{i_1 \cdots i_n} g)(0)\,x^{i_1}\cdots x^{i_n}$$

$$+ \frac{1}{n!}\int_0^1 (1-s)^n \sum (\partial_{i_1 \cdots i_{n+1}} g)(sx)\,x^{i_1}\cdots x^{i_{n+1}}\,ds,$$

where

$$(\partial_{i_1 \cdots i_j} g)(0) = \left.\frac{\partial^j g}{(\partial x^{i_1})\cdots(\partial x^{i_j})}\right|_{x=0}, \qquad x = (x^1, \ldots, x^m).$$

Let $g: M \to E^1$ be smooth. We say that g is *n-horizontal* at $p \in M$ if, in coordinates, all partials of g up to and including order n vanish at p. In the above formula, it is clear that the integral term is n-horizontal at $x = (0, \ldots, 0)$. The term $g(0)$ is a constant and we call the sum of the remaining terms the principal part of $g(x)$. Hence, near $x = 0$, $g(x)$ can be written as the sum of a constant, a principal part, and an n-horizontal part.

Definition (3): Let τ be a linear functional, acting on the set of real-valued smooth functions on M. If, whenever f is n-horizontal at p, we have $\tau f = 0$, we say τ is a *jet of order n* (or more briefly, n-jet) to M at p.

Clearly, the notion of tangent vector and 1-jet coincide.

We shall now see how to express an n-jet at p in terms of coordinates at p. Se let $g: M^m \to E^1$ be smooth at $p \in M$ and suppose that in coordinates p corresponds to $(x^1, \ldots, x^m) = (0, \ldots, 0) = 0$. Suppose τ is an arbitrary n-jet at p. From above, we have $g(x) = C + p(x) + H_n(x)$ near $x = 0$. Here, C is a constant, $p(x)$ is the principal part of $g(x)$, and $H_n(x)$ is the n-horizontal part of $g(x)$.

By definition $\tau\,(H_n(x)) = 0$. Moreover, if C is a constant, C is k-horizontal at p for all $k \geqq 1$, in particular for $k = n$. So $\tau(C) = 0$. By the linearity of τ, we have

$$\tau(g) = \tau\,(p(x)) = \tau\,\Big(\sum (\partial_i g)\,(0)\,x^i + \cdots + \frac{1}{n!}\sum (\partial_{i_1} \cdots \partial_{i_n} g)\,(0)\,x^{i_1} \cdots x^{i_n}\Big)$$

$$= \sum (\partial_i g)\,(0)\,\tau(x^i) + \cdots + \frac{1}{n!}\sum (\partial_{i_1} \cdots \partial_{i_n} g)\,(0)\,\tau\,(x^{i_1} \cdots x^{i_n}).$$

Setting

$$\tau(x^i) = a_i(\tau), \ldots, \tau\,(x^{i_1} \cdots x^{i_n}) = a_{i_1 \ldots i_n}(\tau),$$

we obtain

$$\tau(g) = \sum (\partial_i g)\,(0)\,a_i(\tau) + \cdots + \frac{1}{n!}\sum (\partial_{i_1 \ldots i_n} g)\,(0)\,a_{i_1 \ldots i_n}(\tau).$$

The real numbers $a_i(\tau), \ldots, a_{i_1 \ldots i_n}(\tau)$ are called the coordinates of τ at p. If we put $n = 1$, we have

$$\tau(g) = \sum_{i=1}^{m} a_i(\tau)\,\frac{\partial g}{\partial x^i}$$

or

$$\tau = \sum a_i(\tau)\,\frac{\partial}{\partial x^i}.$$

We can define the sum of two n-jets τ, τ_1 and the product of an n-jet by a scalar in the obvious manner. So the set of n-jets at a point p forms a linear space, *the n-jet space at p*. We call the 1-jet space at p the *tangent space at p*.

If $f: N^n \to E^1$ is a smooth function on the manifold N, we write $f \in C^\infty(N)$. If now $\phi: M^n \to N^m$ is smooth, then $f \in C^\infty(N)$ implies $f \circ \phi \in C^\infty(M)$.

Lemma (1): If $\phi(p) = q$ and $f \in C^\infty(N)$ is k-horizontal at q, then $f \circ \phi$ is k-horizontal at p.

Proof: Locally, ϕ is represented in coordinates by $y_j = \phi_j(x_1, \dots, x_m)$ $(j = 1, 2, \dots, n)$. Here $(x_1, \dots, x_m)$ are coordinates at p and $(y_1, \dots, y_n)$ are coordinates at q. So

$$\frac{\partial f(\phi(x))}{\partial x_i} = \frac{\partial f(\phi_1(x), \dots, \phi_n(x))}{\partial x_i}$$

$$= \sum_{j=1}^{n} \frac{\partial f(\phi_1(x), \dots, \phi_n(x))}{\partial y_j} \cdot \frac{\partial \phi_j(x_1, \dots, x_m)}{\partial x_i}.$$

Therefore, since $\left.\dfrac{\partial f(\phi(x))}{\partial y_j}\right|_{x=0} = 0$ for $j = 1, 2, \dots, n$, $\left.\dfrac{\partial f(\phi(x))}{\partial x_i}\right|_{x=0} = 0$ for $i = 1, 2, \dots, m$. Similarly we can prove that all partials of $f \circ \phi$ up to order k vanish at $x = 0$.

Q.E.D.

If now τ is a k-jet on M at p, we define $\hat{\tau}(f) = \tau(f \circ \phi)$ for $f \in C^\infty(N)$. From Lemma (1), it follows that $\hat{\tau}$ is a k-jet on N at $q = \phi(p)$. We write $\hat{\tau} = \phi_*(\tau)$ and call $\hat{\tau}$ an "image k-jet". The following properties follow immediately from the definition.

(a) $\phi_*(\tau + \tau_1) = \phi_*(\tau) + \phi_*(\tau_1)$
(b) 1-jets $\subseteq$ 2-jets $\subseteq$ 3-jets $\subseteq \cdots$

Definition (4): If $\tau(p)$ is, for all $p \in M$, a tangent vector at p, we say $\tau(p)$ is a *tangent vector field.* Define the real-valued function g by $g(p) = \tau(p)(f)$, $f \in C^\infty(M)$. This gives rise to a mapping $T: f \to g$. If g is a smooth function for all $f \in C^\infty(M)$, we say T is a *smooth vector field.* Clearly $T(fg) = fTg + gTf$.

Similarly, we can define the notion of a k-jet field.

Lemma (2): If T_1 is a k_1-jet field and T_2 is a k_2-jet field, then $(T_1T_2)f = T_1(T_2f)$ is a $k_1 + k_2$-jet field.

Proof: If T_1 and T_2 are both smooth, the composite map T_1T_2 is also smooth. It remains to show that at each point $p \in M$, T_1T_2 gives rise to a $k_1 + k_2$-jet. But, as we have shown above, in coordinates at p, T_2 is a partial differential operator of order k_2 and T_1 is a partial differential operator of order k_1. Hence, T_1T_2 is a partial differential operator of order $k_1 + k_2$.

Q.E.D.

Let $\phi: M \to N$ be smooth, $p \in M$. Denote by $\tau_p(M)$ the tangent space at p. We have shown that there is a linear map $\phi_*: \tau_p(M) \to \tau_{\phi(p)}(N)$. We define the *rank* of ϕ at p to be the dimension of the image of $\tau_p(M)$ under ϕ_*. We denote the rank of ϕ at p by $\text{rank}_p\,\phi$. If x is a coordinate vector on M at p and y is coordinate vector on N at $\phi(p)$, then ϕ can be written locally as $\Phi(x) = y$. It is easily seen that the linear map ϕ_* is represented in matrix form by the Jacobian matrix $\left\|\frac{\partial \Phi_i}{\partial x_j}\Big|_{x=x(p)}\right\|$. Hence $\text{rank}_p\,\phi$ is equal to the rank of this Jacobian matrix.

Remark: If $p_n \to p$, then $\lim\limits_{n\to\infty} \text{rank}_{p_n}\phi \geqq \text{rank}_p\,\phi$. Indeed, choose a coordinate patch U around p. For n sufficiently large, $p_n \in U$. If now $\text{rank}_p\,\phi = r$, the Jacobian determinant evaluated at p has a nonvanishing minor of order r. But since the determinant is a continuous function of its entries, it follows that the chosen minor of order r will not vanish for p_n sufficiently near p.

Note however that the Jacobian determinant evaluated at p_n may contain nonvanishing minors of order exceeding r. When this does not occur, we say ϕ is locally of constant rank.

If $\phi: M^m \to N^n$, the maximum possible value of $\text{rank}_p\,\phi$ is min (m, n). If $\text{rank}_p\,\phi$ is maximal, we say p is a *regular* point of ϕ; otherwise, we say p is a *singular* point of ϕ.

Lemma (3): If at p, ϕ is locally of constant rank, then there are coordinates in M at p and N at $\phi(p)$ such that ϕ appears as $\Phi\,[\tilde{x}, \tilde{y}] = [\tilde{x}, 0]$.

Proof: Let $x = (x_1, x_2, \ldots, x_m)$ be a coordinate vector on M at p and suppose p corresponds to $x = 0$. By assumption, $\text{rank}_p\,\phi = r$ in a neighborhood of p. Hence, in a neighborhood of $x = 0$, the Jacobian matrix $\left\|\frac{\partial \Phi_i}{\partial x_j}\right\|$ has a nonvanishing $r \times r$ minor. Without loss of generality, assume

$$\det\left\|\frac{\partial \Phi_i}{\partial x_j}; \quad 1 \leqq i, j \leqq r\right\| \neq 0$$

near $x = 0$. Let

$$\begin{aligned} u_k &= \Phi_k(x_1, \ldots, x_m), & k &= 1, 2, \ldots, r, \\ u_k &= x_k, & k &= r+1, \ldots, m. \end{aligned}$$

Then $u = (u_1, \ldots, u_m)$ can be taken as a new coordinate vector on M at p. In fact, the determinant of the Jacobian of the transformation $(x_1, \ldots, x_m) \to (u_1, \ldots, u_m)$ at $x = 0$ is easily seen to be non-zero and we can apply the implicit function theorem. In this new coordinate system, ϕ is represented by a new function $\Phi'(u)$ and moreover,

$$\Phi'_k(u_1, \ldots, u_m) = u_k, \qquad k = 1, 2, \ldots, r.$$

The new Jacobian matrix is

$$\left\| \frac{\partial \Phi'_i}{\partial u_j}; \quad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq m \right\| = \left(\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline * & \begin{matrix} \dfrac{\partial \Phi'_{r+1}}{\partial u_{r+1}} & \cdots & \\ & \ddots & \\ & & \dfrac{\partial \Phi'_m}{\partial u_n} \end{matrix} \end{array} \right)$$

Since the rank of this matrix is exactly r in a whole neighborhood of $u = 0$, we must have $\partial \Phi'_\alpha / \partial u_\beta \equiv 0$ in this neighborhood for $\alpha, \beta > r$. Hence, for $\alpha > r$, $\Phi'_\alpha (u_1, \ldots, u_m)$ is actually a function of only the first r coordinates $u_1, \ldots, u_r$, i.e. $\Phi'_\alpha (u_1, \ldots, u_n) = \Phi''_\alpha (u_1, \ldots, u_r)$. If now y is the coordinate vector on N at $\phi(p)$, we set

$$z_k = y_k, \qquad k = 1, \ldots, r,$$

$$z_k = y_k - \Phi''_k (y_1, \ldots, y_r), \qquad k = r + 1, \ldots, n.$$

Since the Jacobian of this transformation is nonzero at p we see that the transformation $(y_1, \ldots, y_n) \to (z_1, \ldots, z_n)$ is a valid coordinate change at $\phi(p)$. Using the coordinate systems u on M at p and z on N at $\phi(p)$, we see that ϕ is represented by a new function $\tilde{\Phi}(u)$ and

$$\tilde{\Phi}_k (u_1, \ldots, u_m) = u_k, \qquad k = 1, \ldots, r,$$

$$\tilde{\Phi}_k (u_1, \ldots, u_m) = 0, \qquad k = r + 1, \ldots, n.$$

This is the desired result if we write $\tilde{x} = (u_1, \ldots, u_r)$, $\tilde{y} = (u_{r+1}, \ldots, u_m)$. Q.E.D.

Corollary (1): If $\operatorname{rank}_p \phi$ is maximal and $m \geqq n$, then for given coordinates x on N at $\phi(p)$, we can choose coordinates $[x, y]$ on M at p such that ϕ is represented by $\Phi [x, y] = x$.

Corollary (2): If $\operatorname{rank}_p \phi$ is maximal and $m \leqq n$, then for given coordinates $[x, y]$ on N at $\phi(p)$, we can choose coordinates x on M at p such that ϕ is given by $\Phi(x) = [x, 0]$.

Definition (5): Let M^m and V^v be manifolds such that $V \subseteq M$, $v \leqq m$. We say that V is a *submanifold* of M if in appropriate coordinates V appears locally as a plane. This means that there is a system of coordinate patches $\{U_\alpha\}$ on M such that $\{U_\alpha \cap V\}$ is a system of coordinate patches covering V

and if $x = (x_1, \ldots, x_m)$ is the coordinate vector on U_α, then $U_\alpha \cap V$ appears as $(x_{v+1}, \ldots, x_m) = (0, \ldots, 0)$.

Corollary (3): Let $m \geqq n$ and suppose $\phi : M^m \to N^n$. If W is a submanifold of N and if $\phi^{-1}(W)$ contains only regular points of ϕ, then $\phi^{-1}(W)$ is a submanifold of M and $\operatorname{codim}(\phi^{-1}(W)) = \operatorname{codim}(W)$.

Proof: Let $p \in \phi^{-1}(W)$ be arbitrary. Then $q = \phi(p) \in W$. Now, there is a coordinate neighborhood V on N at $\phi(p)$ with corresponding coordinate vector $x = (x_1, \ldots, x_n)$ such that $V \cap W$ is a coordinate neighborhood on W at $\phi(p)$ and where x restricted to W is of the form $(x_1, \ldots, x_w, 0, \ldots, 0)$. By Corollary (1), we can choose coordinates $[x, y]$ in M such that ϕ is given by $\Phi\,[x_1, \ldots, x_w, x_{w+1}, \ldots, x_n, y_{n+1}, \ldots, y_m] = (x_1, \ldots, x_n)$. Hence, if U is the coordinate patch on M at p corresponding to the coordinate vector $[x, y]$, $\phi^{-1}(W) \cap U$ appears as $[x_1, \ldots, x_w, 0, \ldots, 0, y_{n+1}, \ldots, y_m]$. This proves that $\phi^{-1}(W)$ is a submanifold of M. Finally, $\operatorname{codim}(\phi^{-1}(W)) = m - (w + (m - n)) = n - w = \operatorname{codim}(W)$.

Definition (6): A subset e of a manifold M is said to have *measure zero* if $h_\alpha(e \cap U_\alpha)$ has Lebesgue measure zero for every coordinate patch U_α on M.

Notice that a set of measure zero has empty interior. The following important lemmas will be very useful in the future discussion.

Lemma (4) (Sard): Let $\phi : M \to N$ be smooth and let $S \subseteq M$ be the set of all singular points of ϕ. Then $\phi(S)$ has measure zero.

Lemma (5): Every connected one-dimensional manifold is diffeomorphic to either an open interval or a circle.

Before proceding with the proofs of these lemmas, we shall show how the Brouwer Fixed-Point Theorem can be derived from them. The following proof of Brouwer's theorem is completely non-combinatorial and serves to illustrate the power of the differentiable approach to topological questions. Later, we shall apply these lemmas in a more serious way.

Recall that Brouwer's theorem states that every continuous self-map of the closed n-dimensional ball has a fixed point. Let $B_r^n = \{x = (x_1, \ldots, x_n) \mid \|x\| \leqq r\}$ be the closed n-ball of radius r. We claim that it suffices to prove Brouwer's theorem for differentiable maps. For suppose $f: B_1^n \to B_1^n$ is continuous. Let $\pi_\varepsilon : B_1^n \to B_{1-\varepsilon}^n$ be the retraction of B_1^n onto its subset $B_{1-\varepsilon}^n$. In formulas, we can write

$$\pi_\varepsilon(x) = \begin{cases} x & \text{if } \|x\| < 1 - \varepsilon \\ \dfrac{x}{\|x\|}(1 - \varepsilon) & \text{if } \|x\| \geqq 1 - \varepsilon. \end{cases}$$

It is clear that $d(f, \pi_\varepsilon f) \leqq \varepsilon$ where d is the supremum metric. Consider now the maps

$$\pi_{1/m}f : B_1^n \to B_{1-1/m}^n, \qquad m = 2, 3, \ldots$$

By the Weierstrass Approximation Theorem, we can find C^∞ maps ϕ_m such that $d(\pi_{1/m}f, \phi_m) \leqq 1/m$. In particular, this means $\phi_m(B_1^n) \subseteq B_1^n$. Assuming Brouwer's Theorem for smooth maps, there is an $x^m \in B_1^n$ such that $\phi_m(x^m) = x^m$, $m = 2, 3, \ldots$ Using the compactness of B_1^n and passing to a subsequence if necessary, we may assume $x^m \to x_\infty \in B_1^n$. It follows from the continuity of f and the triangle inequality that $f(x_\infty) = x_\infty$.

To prove Brouwer's theorem for smooth maps, it suffices to prove that there exists no smooth map $\phi : B_1^n \to \partial B_1^n$ such that $\phi | \partial B_1^n$ is the identity ($\partial B_1^n = \{x \mid \|x\| = 1\}$). It is a standard result that this last statement is equivalent to Brouwer's theorem and we do not stop to prove it [see Hurewicz and Wallman—*Dimension Theory*]. We should note, however, that the proof is geometric and completely elementary. What remains then is to prove the no-retraction theorem for smooth maps. Suppose that there were a C^∞ map $\phi : B^n \to \partial B^n$ whose restriction to ∂B^n is the identity map. To say that ϕ is smooth on B^n means that there is an extension of ϕ to an open neighborhood $N(B^n)$ of B^n which is C^∞. (Later, we shall use this definition of differentiability when we speak about manifolds with boundary.) Now $\phi : N(B^n) \to \partial B^n$ is a smooth map from an n-dimensional manifold onto an n-1-dimensional submanifold. If S is the set of singular points of ϕ, then by Sard's Lemma, $\phi(S)$ has measure zero. In particular, there is a $p \in \partial B^n$ such that $p \notin \phi(S)$. By Corollary (3), $\phi^{-1}(p)$ is a one dimensional submanifold of $N(B^n)$. Let K be the connected component of $\phi^{-1}(p)$ containing p. There are two possibilities (Lemma (5)):

(i) K is diffeomorphic to an open interval. Now K is a closed subset of $N(B^n)$ and hence $K \cap B^n$ is closed in B^n. Let K be parametrized by $\{\eta(s) : -\infty < s < +\infty\}$ and suppose $\eta(0) = p$. Now first of all, K must pierce ∂B^n at p i.e. it cannot be that $K \subseteq B^n$ or $K \subseteq N(B^n) - B^n$. For otherwise K would be tangent to ∂B^n at p which would contradict the regularity of ϕ at p. (Later, we shall meet this situation in a more general setting when we talk about transversality.) As $\eta(0) = p$, $\eta(-\varepsilon)$ must lie either inside or outside B^n for small ε. We suppose $\eta(-\varepsilon)$ is in the interior of B^n. Then $\eta(s) \in \operatorname{Int} B^n$ for all $s < 0$. Otherwise, $q = \eta(s_0) \in \partial B^n$ for some $s_0 < 0$. Now $q \neq p$ for otherwise K would not be diffeomorphic to an open interval. But, since $\phi | \partial B^n$ is the identity, $q = \phi(q) = \phi(\eta(s_0)) = p$. Hence s_0 cannot exist. Similarly, $\eta(s) \in N(B^n) - B^n$ for all $s > 0$. So consider the set of points in B^n of the form $\eta(s)$, $s < 0$. Let $\{s_n\}$ be a sequence of real numbers tending monotonely to $-\infty$ and consider the sequence of points $\{\eta(s_n)\}$. Passing to a subsequence if necessary, we may assume $\lim_{n\to\infty} \eta(s_n) = p_\infty \in B^n$ exists. Now $\phi(\eta(s_n)) = p$ for all n and by continuity of ϕ, $\phi(p_\infty) = p$. Thus $p_\infty \in \phi^{-1}(p) \cap \bar{K} \cap B^n = K \cap B^n$ since $K \cap B^n$ is closed in B^n. But this clearly contradicts the fact that K is diffeomorphic to an open interval. Hence case (i) is completed.

(ii) K is diffeomorphic to a circle—As in (i), it cannot be that K lies completely inside B^n or completely inside $N(B^n) - B^n$. But since K is essentially a circle, K must intersect ∂B^n in two points, i.e. $K \cap \partial B^n = \{p, q\}$, $q \neq$ p. But, as in (i), $q = \phi(q) = \phi(p) = p$, a contradiction. This proves case (ii) and completely proves the no-retraction theorem.

Proof of Lemma (4): Let $p \in M$ and set $q = \phi(p)$. Pick a coordinate patch V_q around q and then, by continuity of ϕ, find a coordinate patch U_p around p such that $\phi(U_p) \subseteq V_q$. We can then find a coordinate neighborhood U_p' around p such that $\bar{U}_p' \subseteq U_p$, $\bar{U}_p'$ is compact, $\bar{U}_p'$ is homeomorphic to an m-ball B^m and U_p' is homeomorphic to Int B^m. Now $\phi(U_p') \subseteq V_q$ and V_q is homeomorphic to the interior of some open ball B^n. So ϕ can be regarded locally as a map ϕ : Int $B^m \to$ Int B^n. Now M is a separable metric space and we can therefore cover M with a countable collection of coordinate patches of the form U_p'. If we can therefore prove Sard's lemma in the case ϕ : Int $B^m \to$ Int B^n, we may then use the complete additivity of Lebesgue measure to prove Sard's lemma in the general case. Denote Int B^k by W^k. We prove that if S is the set of singular points of the map ϕ: $W^m \to W^n$, then $\phi(S)$ has measure zero. We may regard ϕ: $W^m \to W^n$ as a restriction of the corresponding map $\phi : B^m \to B^n$, and we observe that all functions and their derivatives are uniformly continuous.

Case 1, $m < n$: In this case, we show $\phi(W^m)$ has measure zero. We write W^m as a union of small cubes, each of side ε (and therefore each of volume ε^m). We can choose the ε-cubes in such a way that their total number does not exceed c/ε^m where c is a constant independent of ε (c is essentially the volume of W^m). Denote the ε-cubes by $R_i^{(\varepsilon)}$. Now $\phi(W^m) \equiv \phi\left(\bigcup_i R_i^{(\varepsilon)}\right) = \bigcup_i \phi(R_i^{(\varepsilon)})$. Since ϕ is differentiable, diam $(\phi(R_i^{(\varepsilon)}) \leqq K\varepsilon$ where K is independent of ε. Hence, Vol $(\phi(R_i^{(\varepsilon)})) \leqq K'\varepsilon^n$ where K' is independent of ε. Finally, Vol $(\phi(W^m)) \leqq K'\varepsilon^n \dfrac{c}{\varepsilon^m} = K'c\,\varepsilon^{n-m}$. This converges to zero as $\varepsilon \to 0$ because $n > m$.

Case 2, $m \geqq n$: Let S_k be the set of points in W^m where the rank of ϕ exactly equals k. It is enough to prove that $\phi(S_k)$ has measure zero for each $k < n$. Let P be the mapping of E^n into itself which projects every n-vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ into $(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}, 0, \ldots, 0)$. Clearly $P\phi$ is still singular on S_k. If we suppose Sard's lemma holds for the case $n = k + 1$, then $P\phi(S_k)$ has measure zero (in this subspace). By a result in measure theory (essentially Fubini's theorem), we know that if a set $e \subseteq E^n$ is such that $P(e)$ has measure zero, then $P^{-1}(P(e))$ has measure zero. As $e \subseteq P^{-1}(P(e))$, this means that e itself has measure zero. In our case, we have $\phi(S_k) = e$ and hence $\phi(S_k)$ has measure zero. It remains therefore to prove the following statement: If $\phi : W^m \to W^{k+1}$, then $\phi(S_k)$ has measure zero. We proceed by induction on m. For $m = 1$, we must have $k = 0$ and $\phi : W^1 \to W^1$ i.e. ϕ is a smooth self-mapping of an open interval. S_0 is just the set of points where $d\phi(x)/dx = 0$. Cover S_0 with a set of intervals $R_i^{(\varepsilon)}$ of length ε in such a way that the total number of intervals does not exceed c/ε where c is independent of ε. For each i, we claim that $\phi(R_i^{(\varepsilon)})$ is a set of diameter not exceeding $K\varepsilon^2$ where K is independent of ε. In fact let $x_0 \in R_i^{(\varepsilon)} \cap S_0$. Then $d\phi/dx|_{x=x_0} = 0$. If x is any other point in $R_i^{(\varepsilon)}$, we apply Taylor's formula to obtain

$$|\phi(x) - \phi(x_0)| \leqq K\,|x - x_0|^2 \leqq K\varepsilon^2.$$

Hence, $\phi(S_0) \subseteq \phi\left(\bigcup_i R_i^{(\varepsilon)}\right) = \bigcup_i \phi(R_i^{(\varepsilon)})$ and hence

$$\mathrm{Vol}\,(\phi(S_0)) \leqq \frac{c}{\varepsilon} K\varepsilon^2 = cK \cdot \varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$\mathrm{Vol}\,(\phi(S_0)) = 0.$$

We now assume the theorem true for all integers less than m and try to deduce the theorem for the integer m. First of all, assume $k = 0$. Then $\phi : W^m \to W^1$ is a smooth real valued function. Let S_0^m be the set of all points in W^m where all the partials of ϕ up to and including order m vanish. By exactly the same argument as in the case $m = 1$, we can prove that $\phi(S_0^m)$ has measure zero. Now let $\hat{S}_0^j$ be the set of all points in W^m where all the partials of ϕ up to order j vanish, but where some $j + 1$-st partial does not vanish, i.e. if $p \in \hat{S}_0^j$, there is a $\psi = \partial^j\phi$ such that $\psi(p) = 0$ but $\partial\psi(p) \neq 0$. Now ψ is a map from $W^m \to W^1$ and is regular at p. Let U be a neighborhood of p in which $\partial\psi(x) \neq 0$. By Corrollary (3), the set $\psi^{-1}(0) \cap U$ is a submanifold of W^m of dimension $m - 1$. Since W^m is separable, this means that $\hat{S}_0^j$ is contained in a countable union of submanifolds of W^m of dimension $m - 1$.

By the induction assumption and the complete additivity of the measure, it follows that $\phi(\hat{S}_0^j)$ has measure zero. Now $S_0 = \hat{S}_0^1 \cup \hat{S}_0^2 \cup \cdots \cup \hat{S}_0^{m-1} \cup S_0^m$; hence $\phi(S_0)$ has measure zero.

Suppose now $\phi\colon W^m \to W^{k+1}$, $k + 1 \leqq m$, $k > 0$. We want to show $\phi(S_k)$ has measure zero. Let $p \in S_k$; in a neighborhood of p, there is a $k \times k$ minor of the Jacobian matrix $\|\partial\Phi_j/\partial x_i\|$ which is nonvanishing. By the implicit function theorem, we can find local coordinates at p and $\phi(p)$ such that the map ϕ is given by $\tilde{\Phi}(x, y) = [x, f(x, y)]$. Here $x = (x_1, x_2, \ldots, x_k)$, $y = (y_{k+1}, \ldots, y_m)$ and $f(x, y)$ is a real-valued function. If U is the coordinate neighborhood corresponding to the coordinate vector $[x, y]$, then clearly $S_k \cap U = \left\{[x, y] \,\middle|\, \dfrac{\partial f(x, y)}{\partial y_j} = 0\right\}$. Again using the separability of W^m, we see that S_k is contained in a countable union of sets of the type

$$\left\{[x, y] \,\middle|\, \frac{\partial f(x, y)}{\partial y_j} = 0\right\}.$$

Hence it suffices to prove that the measure of the image (under ϕ) of such a set has measure zero.

Now, $\phi(S_k) = \left\{[x, t] \mid t = f(x, y),\ \dfrac{\partial f(x, y)}{\partial y_j} = 0;\ j = k + 1, \ldots, m\right\}$.

Fix $x = \bar{x} = (\bar{x}_1, \ldots, \bar{x}_k)$ and consider $\left\{[\bar{x}, t] \mid t = f(\bar{x}, y),\ \dfrac{\partial f(\bar{x}, y)}{\partial y_j} = 0;\ j = k + 1, \ldots, m\right\}$. For fixed $\bar{x}$, f can be regarded as mapping the coordinates $y_{k+1}, \ldots, y_m$ into the real numbers $t = f(\bar{x}, y)$, i.e. $\tilde{f}\colon W^{m-k} \to W^1$. If we denote by $\tilde{S}_0$ the set of singular points of $\tilde{f}$, it is clear that $\tilde{S}_0$ coincides with the set of all y such that $\dfrac{\partial f(\bar{x}, y)}{\partial y_j} = 0$, $j = k + 1, \ldots, m$. But here we have just the situation discussed above (namely, when $k + 1 \leqq m$, $k = 0$) and it follows that $\phi(\tilde{S}_0)$ has measure zero. But $\phi(\tilde{S}_0)$ is essentially

$$\left\{[\bar{x}, t] \mid t = f(\bar{x}, y),\ \frac{\partial f(\bar{x}, y)}{\partial y_j} = 0,\ j = k + 1, \ldots, m\right\}$$

and hence this latter set has measure zero. Thus $\phi(S_k)$ is such that its intersection with each vertical line $\{[\bar{x}, t] \mid -t_1 < t < t_2\}$ has measure zero. Again appealing to Fubini's theorem, we deduce that $\phi(S_k)$ itself has measure zero. This completes the proof of Sard's lemma.

Q.E.D.

Let $\mathscr{D}$ be a smooth vector field on the smooth manifold M. If $\gamma(t)$ is a differentiable curve in M, then we can define $\frac{d}{dt}\gamma(t)\circ f = \frac{d}{dt}f(\gamma(t))$. This gives a tangent vector at each point $\gamma(t) \in M$. Given $p_0 \in M$, we want to find a differentiable curve $\gamma(t, p_0)$ in M passing through p_0 such that $\frac{d}{dt}\gamma(t)\circ f = \mathscr{D}\circ f$. This leads to the initial-value problem $\frac{d}{dt}\gamma(t, p_0) = \mathscr{D}(\gamma(t, p_0))$, $\gamma(0, p_0) = p_0$. Expressing the problem in local coordinates, we are able to apply the classical theory of ordinary differential equations. See S. Long's book for the relevant theory of differential equations.

We now proceed to the proof of Lemma (5) on one-dimensional connected manifolds. Suppose M is a connected one-dimensional manifold. Let U be a maximal coordinate patch on M. The existence of U will be established later. Now U is diffeomorphic to the open interval $(-1, 1)$; let $p(t)$, $-1 < t < 1$, be a $1-1$ parametrization of U. If U is closed in M, we are finished. In fact, U is open and closed in M and since M is connected, $U = M$. Thus, in this case, M is diffeomorphic to an open interval. We assume then that U is not closed. Then, there exists a sequence $t_n \to 1$ (for definiteness) such that $p(t_n) \to p_\infty \notin U$. We take a coordinate patch $\hat{V}$ around p_∞. Now $\hat{V}$ contains some point in U, say $p(A)$ (where $-1 < A < 1$). We choose a coordinate patch V around p_∞ such that V has compact closure $\bar{V}$ in $\hat{V}$. As $\bar{V}$ is diffeomorphic to a closed interval, we can moreover require that one of the endpoints of $\bar{V}$ be $p(A)$. It follows readily that all points $p(t)$ for $t > A$ lie in V and then plainly $p(t) \to p_\infty$ as $t \to 1$. We claim that $U \cap V$ is one of the two following sets:

(a) $\{p(t) \mid t > A\}$,
(b) $\{p(t) \mid t > A\} \cup \{p(t) \mid t < B$ for some B between -1 and $+1\}$.

To see this, we observe that one of the following statements is true:

(1) As $t \to -1$, $p(t)$ has no limit point,
(2) As $t \to -1$, $p(t) \to p_{-\infty}$ and $p_{-\infty} \neq p_\infty$,
(3) As $t \to -1$, $p(t) \to p_\infty$ i.e. $p_{-\infty} = p_\infty$.

If either (1) or (2) holds, we can choose V and $\varepsilon > 0$ so small that $V \cap \{p(t) \mid t < -1 + \varepsilon\} = \phi$. For otherwise, there would exist a sequence $p(t_n)$, $t_n \to -1$ such that $p(t_n) \to p_\infty$. But this would imply that $p_{-\infty}$ exists and that $p_{-\infty} = p$. Thus, if either (1) or (2) holds, then (a) is true. On the other hand, if (3) holds, then clearly (b) is true.

We consider first case (a): We shall prove that there exists on $U \cup V$ a vector field $\mathscr{D}$ such that (i) on U, and in the coordinates of U, $\mathscr{D}$ is positive and bounded, (ii) on V, and in the coordinates of V, $\mathscr{D}$ is positive and bounded. To see this, we note first of all that we may assume that the change of coordinates map from U to V has positive derivative. For otherwise, we reverse the coordinate on V (i.e. if s is the coordinate on V, replace s by $-s$). We now construct a vector field $\mathscr{D}_1$ on U which is nonnegative and bounded on U. To do this, construct a C^∞ real function f_1 on the interval $-1 < t < 1$ such that $f_1 = 1$ on $(-1, A)$ and such that f_1 tapers off smoothly to 0 before reaching $t = +1$. Let $\mathscr{D}_1$ be the vector field on U which, when expressed in coordinates, is of the form $f_1(t)\dfrac{d}{dt}$. Assume now $\hat{V}$ is diffeomorphic to the interval $-1 < s < 1$ while V is given by $-\frac{1}{2} < s < \frac{1}{2}$. Construct a C^∞ real function f_2 on the interval $-1 < s < 1$ such that $f_2 = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and such that f_2 tapers off smoothly to 0 before reaching $s = -1$ and $s = 1$. Let $\mathscr{D}_2$ be the vector field on V which, when expressed in coordinates, is of the form $f_2(s)\dfrac{d}{ds}$. If we put $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2$, then $\mathscr{D}$ clearly satisfies (i) and (ii) above. Take $p_0 \in U \cap V$ and consider the initial value problem

$$\frac{d\gamma(x, p_0)}{dx} = \mathscr{D}(\gamma(x, p_0)), \gamma(0, p_0) = p_0.$$

Let $\gamma(x, p_0)$ be the solution to this problem. It follows from the fact that $\mathscr{D}$ is positive and bounded that $x \to \gamma(x, p_0)$ covers all of V monotonely and all of U monotonely. But then γ gives a $1-1$ parametrization of $U \cup V$ which contradicts the maximality of U. Hence, we conclude that p_∞ does not exist in this case, i.e. $U = M$.

Case (b) may be treated in exactly the same way.

We consider case (c): We have $p(t) \to p_\infty$ as $t \to \pm 1$. If $K = \{p(t) \mid -1 < t < 1\} \cup \{p_\infty\}$, then K is both open and closed and therefore $K = M$. Now K is topologically a circle and it remains only to prove that M is also diffeomorphic to a circle. Note that, there is a continuous parametrization of M by θ $(0 \leqq 0 \leqq 2\pi)$ and any coordinate system can be oriented using θ. More precisely, we can cover M by coordinate patches such that each change of coordinates map has positive derivative. Hence, the notion of a positive vector field on M is well-defined. Since M is compact, we can construct (by using a partition of unity) a vector field $\mathscr{D}$ on M which is everywhere positive and bounded. Again, let $p_0 \in U \cap V$ and consider the initial value problem $d\gamma(x, p_0)/dx = \mathscr{D}(y(x, p_0), \gamma(0, p_0) = p_0$.

The range of the solution $\gamma(x, p_0)$ includes $U \cup V$. Now γ cannot be $1-1$ and so there is an $x_0 \neq 0$ such that $\gamma(x_0, p_0) = \gamma(0, p_0)$.

Then the set $\{\gamma(x, p_0) | 0 \leqq x \leqq x_0\}$ is a circle embedded in M and thus must be all of M.

It remains to prove the existence of a maximal coordinate patch U in M. We use Zorn's lemma: Take an ascending chain $\{U_n\}$ of coordinate patches (i.e. $U_j \subseteq U_k$ if $j \leqq k$). Note that since M has a countable base, it is unnecessary to consider uncountable chains. We can assume that all the change of coordinate maps have positive derivative. Let $L = \bigcup_{n=1}^{\infty} U_n$. If we can show that L is covered by a single coordinate patch, we will have proved that every chain has a least upper bound and we can then appeal to Zorn's lemma. We shall construct on L a positive and bounded vector field. On each U_j, we can construct a vector field $\mathscr{D}_j$ which is positive and bounded. We may even assume each $\mathscr{D}_j$ is bounded above by 1. Let $\mathscr{D} = \sum_{n=1}^{\infty} \frac{1}{n!} \mathscr{D}_n$. Then $\mathscr{D}$ is the desired vector field. Using the previous arguments, we find that L is indeed covered by a single coordinate patch. This completes the proof of the lemma.

Q.E.D.

Definition (7): Let M^m be a manifold. If U and V are two overlapping coordinate neighborhoods on M, we say U and V are *positively related* if the change of coordinates map from U to V has positive Jacobian determinant. M is said to be *orientable* if it can be covered by coordinate neighborhoods, any two of which are positively related. Otherwise, M is said to be *nonorientable*.

The following construction is useful: If N is a connected, nonorientable manifold, there exists a connected, orientable manifold $\hat{N}$ which is a twofold covering of N. To define $\hat{N}$, we take the collection S of all pairs (p, U) where U is a coordinate neighborhood around p. We take two copies of S, say S_+ and S_-. Then $S_+ = \{(p, U)_+ | p \in U\}$, $S_- = \{(p, U)_- | p \in U\}$. We make the following identification: Identify $(p, U)_+$ with $(q, V)_+$ iff $p = q$ and the change of coordinates map from U to V has positive Jacobian determinant at the point p, identify $(p, U)_+$ with $(q, V)_-$ iff $p = q$ and the change of coordinates map from U to V has negative Jacobian determinant at the point p, and identify $(p, U)_-$ with $(q, V)_-$ iff $p = q$ and the change of coordinates map from U to V has positive Jacobian determinant at p. Let $\hat{N}$ be the set obtained from $S_+ \cup S_-$ by virtue of this identification. To define a topology on $\hat{N}$, we define neighborhoods as follows: Denote by $[(p, U)_+]$ the equivalence class (i.e. the element in $\hat{N}$) containing $(p, U)_+$. For a neighborhood of $[(p, U)_+]$, take all points in $\hat{N}$ of the form $[(q, U)_+]$ where q ranges over U. Similarly, we define the neighborhood of $[(p, U)_-]$. It is easy to check that with this definition of neighborhoods $\hat{N}$ becomes a topological space. To make $\hat{N}$ a manifold, we define coordinates on the neighborhoods as follows: For coordinates around $[(p, U)_+]$, take the coordinates of U. For coordinates around $[(p, U)_-]$, take the reversed coordinates of U. That is, if $(x_1, \ldots, x_n)$ are the coordinates of U, take $(x_1, \ldots, x_{n-1}, -x_n)$ to be the coordinates around $[(p, U)_-]$. With this structure, $\hat{N}$ becomes a C^∞ manifold. It is easy to see that $\hat{N}$ is orientable and that the map $[(p, U)_\pm] \to p$ is a two to one differentiable covering map. It follows from the fact that $\hat{N}$ is nonorientable that N is connected.

Definition (8): Let A be closed subset of a separable metric space M. M is a *relative manifold modulo* A if $M - A$ is a manifold. We say (M, A) or (M/A) is a relative manifold.

Definition (9): Let $M \subseteq N$ (N is a manifold), $p \in M$. Then p is a *j-corner* point if there is a coordinate patch U in N such that in these coordinates, $M \cap U$ appears as a j-corner in E^n ($n = \dim(N)$). A j-corner in E^n is $\{x = [x_1, x_2, \ldots, x_n] \mid \|x\| < 1,\ x_1 \geqq 0, \ldots, x_j \geqq 0\}$. Here, p corresponds to the origin $x = [0, 0, \ldots, 0]$.

Definition (10): A *manifold with corners* is a space such that every point is a j-corner point for some $j \geqq 0$. A *manifold with boundary* is a space such that every point is either a 0-corner point or a 1-corner point.

The notion of j-corner point is independent of the coordinate patch; in fact if p is a j-corner point, then the set of all vectors tangent "in both directions" to the manifold at p is an $m-j$-dimensional vector space and this number is invariant under any nonsingular linear transformation. If M^m is a manifold with boundary ∂M, then ∂M is a manifold of dim $m - 1$. ∂M inherits its differentiable structure from M.

Let M^m, A^a be manifolds, N^n a submanifold of M. Locally, if $[x, y]$ are coordinates on M, N appears as $x = 0$, i.e. the coordinates on N are $[0, y]$.

Definition (11): Let $\phi \cdot A \to M$ be C^∞; then ϕ is *transverse* to N at $q \in \phi^{-1}(N)$ if

$$T_p(N) + \phi_* (T_q(A)) = T_p(M).$$

Here $p = \phi(q)$, $T_p(N)$ and $T_p(M)$ are the respective tangent spaces, and ϕ_* is the map of tangent spaces induced by ϕ.

Note: (a) if $\dim(A) + \dim(N) < \dim(M)$, then $N \cap \phi(A) = \phi$;

(b) if $\dim(A) + \dim(N) = \dim(M)$, then the solutions of $\phi(q) \in N$ are isolated points;

(c) let P_x be the projection $[x, y] \to x$.
The definition of transversality is equivalent to: $P_x\phi$ is regular at q. (In this case, $P_x\phi$ is regular at q means $(P_x\phi)_*$ is an onto map).

(d) if ϕ is transverse to N at each point of $\phi^{-1}(N)$, then $\phi^{-1}(N)$ is a submanifold of A and codim $(\phi^{-1}(N))$ in A = codim (N) in M. This follows from the lemmas given previously.

Definition (12): Let N, $\bar{N}$ be submanifolds of the manifold M, $N \cap \bar{N} \neq \phi$. N and $\bar{N}$ are said to be *transverse* at their intersection if the injection i: $N \to M$ is transverse to $\bar{N}$ at each intersection point; i.e. $T_p(N) + T_p(\bar{N}) = T_p(M)$ $(p \in N \cap \bar{N})$.

As an example, let M be E^3, N the plane $x_3 = 0$, and $\bar{N}$ the line $x_1 = 0$, $x_2 = 0$. Then N, $\bar{N}$ are transverse at $(0, 0, 0)$.

Lemma (6): If N, $\bar{N}$ are transverse at their intersection, $p \in N \cap \bar{N}$, then it is possible to choose suitable coordinates near p such that both N, $\bar{N}$ appear as planes (in fact, as linear subspaces whose sum is the whole space).

Proof: Let $[x, y]$ be coordinates on M at p, such that N appears as $[0, y]$. Let P_x: $[x, y] \to x$. Introduce coordinates z on $\bar{N}$; $i(z) : \bar{N} \to M$. As $P_x i$ is regular at p, change coordinates $z \to [u, v]$ such that $P_x i\,[(u, v]) = u$ (see remark (c) above). Then $i\,([u, v]) = [u, \psi\,(u, v)]$. As $\bar{N}$ is regularly embedded, i has maximal rank. Hence

$$[u, \psi\,(u, v)] = [u, \eta\,(u, v), x\,(u, v)], \quad \det\left(\frac{\partial \eta}{\partial v}\right) \neq 0.$$

Change coordinates in $\bar{N}$: $[u, v] \to [u, \eta]$. Then $i\,([u, \eta]) = [u, \eta, x\,(u, \eta)]$. Change coordinates in M: $[u, \eta, x] \to [u, \eta, x - x\,(u, \eta)]$. Then $i\,([u, \eta]) = [u, \eta, 0]$ and the assertion is proved.

Q.E.D.

Corollary: $N \cap \bar{N}$ is a submanifold.

Proof: Clearly, $N \cap \bar{N}$ is locally a plane at $p \in N \cap \bar{N}$.

Lemma (7): Let $\phi : M \to A$ and let $N \subseteq M$, $B \subseteq A$ be submanifolds. Suppose ϕ is transverse to B everywhere and $\phi|N$ is transverse to B everywhere. Then N and $\phi^{-1}(B)$ are transverse in M.

Proof: Since transversality pertains to tangent spaces, we can restate the lemma in the following linearized form: if $\bar{N} \subseteq \bar{M}$, $\bar{B} \subseteq \bar{A}$ (considered as subspaces of linear spaces), $\bar{\phi} : \bar{M} \to \bar{A}$ is a linear map, $\bar{\phi}(\bar{M}) + \bar{B} = \bar{A}$ and $\bar{\phi}(\bar{N}) + \bar{B} = \bar{A}$, then $\bar{N} + \bar{\phi}^{-1}(\bar{B}) = \bar{M}$. To prove the last statement, take any $\bar{m} \in \bar{M}$. Then $\bar{\phi}(\bar{m}) = \bar{a} = \bar{\phi}(\bar{n}) + \bar{b} \Rightarrow \bar{m} - \bar{n} \in \bar{\phi}^{-1}(\bar{B})$.

Q.E.D.

If M is a manifold with boundary ∂M, we always assume M is contained in some larger manifold M_0. A map $\phi : M \to A$ (A is a manifold) is understood to be the restriction of the map $\phi : M_0 \to A$. Suppose now B is a submanifold of A. Suppose ϕ is transverse to B everywhere, $\phi | \partial M$ is transverse to B everywhere (in ∂M). By the first condition, we know $\phi^{-1}(B)$ is a submanifold of M_0. By the previous lemma, $\phi^{-1}(B)$ is transverse to ∂M wherever it intersects ∂M. It is easy to see that $M \cap \phi^{-1}(B)$ is a manifold with boundary $\partial\,(M \cap \phi^{-1}(B)) = \phi^{-1}(B) \cap \partial M$.

Example: Suppose the above hypotheses are satisfied. Moreover, assume M is *compact*, $B = \{b\}$ (i.e. only one point), $\dim(M) = 1 + \dim(A)$. Then $\phi^{-1}(b)$ is a submanifold of M of dimension 1. Therefore $\phi^{-1}(b)$ is a union of open arcs and circles. $C = \phi^{-1}(b) \cap M$ is then a collection of *closed* arcs and circles; ∂C is a set of points in $\partial M \cap \phi^{-1}(b)$. This set of points is finite because M is compact and the parity of this set of points is even (because the intersection of an open arc or a circle with ∂M consists of an even number of points).

This Lemma will play an important role in the next chapter, when we discuss the theory of degree.

We state this result for future reference as a lemma.

Lemma (8): Let M be a compact manifold with boundary ∂M, and A a manifold of the same dimension as ∂M. Let $\phi: M \to A$ be a smooth map. Suppose that $a \in A$ is chosen in such a way that 3 is transverse to a everywhere and $\phi | \partial M$ is transverse to a everywhere. Then $\phi^{-1}(a) \cap M$ decomposes into a number of smooth closed curves lying in $M - \partial M$ and a number of smooth closed arcs in M intersecting ∂M only at their end-points. These arcs intersect ∂M transversally at their end-points. All the points of $\phi^{-1}(a) \cap \partial M$ are end-points of such arcs.

CHAPTER II

Degree of a Map and Intersection Theory. Applications

Definition (1): Let M, A be manifolds, both of dimension m. Suppose M is *compact* and let $\phi : M \to A$ be C^∞. If ϕ is transverse to $\{p\}$, $p \in A$, then the parity of the number of points in $\phi^{-1}(p)$ is called *the degree of ϕ at p (mod 2)* and is denoted by $\deg_2(\phi)|_p$.

Let M and A be as before and let $\psi: M \to A$, $\bar{\psi}: M \to A$ be (smoothly) homotopic; i.e. there exists a $\Psi: M \times [0, 1] \to A$ such that $\Psi|M \times \{0\} = \psi$, $\Psi|M \times \{1\} = \bar{\psi}$. (We assume ψ, $\bar{\psi}$, Ψ are C^∞ and say that Ψ is a smooth homotopy.) Take $p \in A$ and suppose all maps involved are transverse to the submanifold $\{p\}$ of A. Let $M' = M \times [0, 1]$. Then M' is a manifold with boundary $\partial M' = (M \times \{0\}) \cup (M \times \{1\})$ and $\dim (M') = m + 1$. By Lemma 8 of the last chapter, $\partial (\Psi^{-1}(p) \cap M')$ contains an even number of points. This means that if the number of solutions of $\Psi(x) = p, x \in M \times \{0\}$, is n_0 and the number of solutions of $\Psi(x) = p$, $x \in M \times \{1\}$, is n_1, then $n_0 \equiv n_1 \pmod 2$. But $\Psi(x) = \psi(x)$ for $x \in M \times \{0\}$ and $\Psi(x) = \bar{\psi}(x)$ for $x \in M \times \{1\}$, and hence the parity number of points in $\psi^{-1}(p)$ = the parity number of points in $\bar{\psi}^{-1}(p)$. We shall show later that it is not necessary to assume that the homotopy Ψ is transverse to A at $\{p\}$. We will then have proved:

Theorem (1): $\deg_2(\phi)|_p$ is a (smooth) homotopy invariant.

Remark: It follows from an approximation theorem that we will discuss later that if two smooth maps are homotopic then they are actually smoothly homotopic.

Remark: Assume the manifolds M and A are compact and also that A is connected. Then the set $B \subseteq A$ of all points p such that a smooth map $\phi : M \to A$ is transverse to $\{p\}$ is open and everywhere dense (Sard's Lemma).

It will be shown below that $\deg_2 \phi|_p$ is the same for all points $p \in B$. It will then be called the *degree of ϕ (mod 2)*.

In fact, let x be a coordinate system with origin at the point $a \in A$, let V be a spherical neighborhood of $\mathscr{A}$ in this coordinate system, and further let b_0 and b_1 be two points of $V \cap B$. It is easy to construct a diffeomorphism ψ of A onto itself, under which all points of the set $A - V$ remain fixed and which maps b_0 to b_1. The map ψ is homotopic to the identity (see the "Homotopy Lemma" below). It is easy to see that the degree of $\psi\phi$ at b_1 is equal to the degree of ϕ at b_0; but since $\psi\phi$ and ϕ are homotopic, their degrees at b_1 coincide. Thus the degrees of ϕ at all points $b \in V \cap B$ are equal. Since A is connected and B is everywhere dense in A it follows that the degrees of ϕ at all points of B are the same.

Corollary: Let M be a compact manifold. Then, since the identity map of M onto itself has degree 1, and the constant map has degree zero, these two maps are not homotopic.

Remark: Assume the manifolds M and A in Definition (1) above are oriented, while A is also connected. $\phi^{-1}(p)$ consists of a finite number of points $m_1, m_2, \ldots, m_k$, at each of which the Jacobian determinant of ϕ is not zero and has a definite sign. Denote by $\varepsilon_i\,(= \pm 1)$ the sign of the Jacobian determinant of ϕ at m_i, $i = 1, 2, \ldots, k$. We may now state the following proposition.

Proposition: Let M^{m+1} be a compact oriented manifold with boundary ∂M, A^m a compact connected manifold which is oriented, ϕ a smooth map of $M \to A$ such that $\phi/\partial M$ is transverse to A at p. Then the map $\phi/\partial M$ has degree zero at p.

Proof: Using Lemma 8 of the preceding chapter, we have only to show that if $\sigma = \sigma(t)$, $0 \leqq t \leqq 1$, is a smooth closed arc in M intersecting ∂M only in its end-points, if $\phi(\sigma(t)) \equiv p$, and if σ intersects ∂M transversally at its end points, then the following statement holds:

A: Let N_q denote the tangent space to any manifold N at one of its points q. Suppose that the jacobian transformation ϕ_* maps a positively oriented basis for $(\partial M)_{\sigma(0)}$ to a positively oriented basis for A_p. Then ϕ_* maps a positively oriented basis for $(\partial M)_{\sigma(1)}$ to a negatively oriented basis for A_p.

To see this, note that ∂M is oriented by the following rule: call a basis β of $(\partial M)_q$ positively oriented if, when supplemented by a vector of M_q pointing into M, it gives a positively oriented basis of the (oriented) space M_q. Now let $\sigma'(t)$ denote the tangent vector to σ at $\sigma(t)$. Then, by hypothesis, ϕ_* in duces an isomorphism of the factor space $M_{\sigma(t)}/(\sigma'(t))$ onto A_p. Call a set $\alpha = \{v_1, \ldots, v_n\}$ of n vectors in $M_{\sigma(t)}$ *σ-positive* if $\{v_1, \ldots, v_n, \sigma'(t)\}$ is a positively oriented basis for $M_{\sigma(t)}$. Since $\sigma'(0)$ points into M, it follows from the above that if α is σ-positive set of vectors in $M_{\sigma(t)}$, $t = 0$, then $\phi_* \alpha$ is a positively oriented set of vectors in A_p. By continuity, the same holds for all $0 \leqq t \leqq 1$. Since $\sigma'(1)$ points out of M, a σ-positive set of vectors in $(\partial M)_{\sigma(1)}$ is negatively oriented in the space $(\partial M)_{\sigma(1)}$. Hence the above statement A follows. Q.E.D.

Definition (1A): Let M, A be oriented manifolds, both of dimension m. Suppose M is compact, and let $\phi: M \to A$ be C^∞. If ϕ is transverse to $\{p\}$, $p \in A$, then the number of points in $\phi^{-1}(p)$, counted with the sign convention of the above Remark, is the *degree of ϕ at p*, and is denoted by $\deg(\phi)|_p$.

Using the above proposition, and arguing exactly as in the proof of Theorem 1, we may obtain the following result.

Theorem (1A): $\deg(\phi)|_p$ is a (smooth) homotopy invariant of mappings between compact oriented manifolds.

In the next few pages, we shall show that the restrictions arising from the proof of Theorems 1 and 1A, namely that ϕ be smooth, that ϕ be transverse to p, as well as the condition of smoothness of homotopies, may be relaxed.

In the course of doing this, we will develop lemmas in terms of which a more general and powerful "intersection theory" can be built up.

Let X be a metric space, $Y \subseteq X$. Then Y is said to be of *first category* if Y is the countable union of nowhere dense subsets of X. Y is said to be a *prevalent* subset if $X - Y$ is of first category. If P is a property of points, P is said to be true at *most points* if the set of points on which P is true is prevalent.

Any manifold has a metric which makes it complete: For M compact, this is obvious. Otherwise take the one-point compactification $M \cup \{p_\infty\}$. If $\varrho(x, y)$ is a complete metric on $M \cup \{p_\infty\}$, introduce $\varrho_1(x, y) = \varrho(x, y) + |\varrho(x, p_\infty)^{-1} - \varrho(y, p_\infty)^{-1}|$ for $x, y \in M$. Then ϱ_1 is a complete metric on M.

As M is separable and locally compact, $M = \bigcup_{m=1}^{\infty} K_m$, with $K_1, K_2, \ldots,$ compact, $K_1 \subset \operatorname{int} K_2 \subset K_2 \subset \operatorname{int} K_3 \subset \ldots$ We consider the set of continuous maps of M into itself. Define

$$d(\phi, \psi) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\sup_{p \in K_i} \varrho(\phi(p), \psi(p))}{1 + \sup_{p \in K_i} \varrho(\phi(p), \psi(p))}.$$

Then the set of maps of M into M forms a complete metric space. Clearly, convergence in d is the same as uniform convergence on compact subsets.

If M is a manifold, let $J_K(M)$ be the collection of all K-jets at all points of M. $J_K(M)$ is called the K-jet space of M. We can make $J_K(M)$ a manifold. For $K = 1$ this manifold is called the tangent space of M and its dimension is $2m$. We introduce coordinates in $J_1(M)$ as follows: if (m^0, α^0) is a pair from $J_1(M)$ ($m^0 \in M$, α^0 is a tangent vector at m^0), there is a coordinate neighborhood U of m^0 with coordinates $(x_1, \ldots, x_m)$. Let $\pi : J_1(M) \to M$ be defined by $\pi(\tilde{m}, \alpha) = \tilde{m}$. Then $\pi^{-1}(U)$ is the set of all pairs $(\tilde{m}, \alpha)$ such that $\tilde{m} \in U$. If $(\tilde{m}, \alpha) \in \pi^{-1}(U)$, then $\tilde{m} = (x_1, \ldots, x_m)$ and

$$\alpha = a_1 \frac{\partial}{\partial x_1} + \cdots + a_m \frac{\partial}{\partial x_m}.$$

The map

$$(\tilde{m}, \alpha) \to (x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_m)$$

is a $1-1$ map between $\pi^{-1}(U)$ and an open subset of E^{2m}. There is a unique topology on $J_1(M)$ which makes this map a homeomorphism. We take $\{\pi^{-1}(U)\}$ as coordinate neighborhoods of $J_1(M)$ with homeomorphisms

$$(\tilde{m}, \alpha) \to (x_1, x_2, \ldots, x_m, a_1, a_2, \ldots, a_m)$$

as coordinate mappings.

It is left to the reader to verify that these coordinate neighborhoods are compatible with one another.

A similar construction, based on the fact that in local coordinates a K-jet has a unique expression as a partial differential operator of order K, can be used to define a manifold structure for $J_K(M)$. We leave details to the reader.

Let ϱ_K be a complete metric on $J_K(M)$. A map $\phi : M \to M$ induces a map $\phi_* : J_K(M) \to J_K(M)$ (Chapter I). Introduce $d_K(\phi_*, \psi_*)$ as above. Define $\hat{d}_K(\phi, \psi) = d_K(\phi_*, \psi_*)$. Then convergence in $\hat{d}_K$ is the same as uniform convergence of all derivatives up to order K on compact subsets. Finally, define

$$\hat{d}_\infty(\phi, \psi) = \sum_{K=1}^{\infty} \frac{1}{2^K} \frac{\hat{d}_K(\phi, \psi)}{1 + \hat{d}_K(\phi, \psi)}.$$

Then convergence in $\hat{d}_\infty$ is the same as uniform convergence of all derivatives on compact subsets. All the metrics defined above are complete.

Auxiliary Lemma (1): Let (M, A) be a relative manifold. Suppose $\phi : (M^m, A) \to N^m$, $p \in N^m - \phi(A)$. Assume M is compact and ϕ is transverse to $\{p\}$. If ψ is sufficiently close to ϕ in the C^1-topology then the number of points in $\psi^{-1}(p)$ = the number of points in $\phi^{-1}(p)$.

Proof: Since M is compact, $\phi^{-1}(p)$ has only a finite number of points, call them $\{q_1, \ldots, q_K\}$. Surround each q_j with a coordinate neighborhood U_j such that $U_j \cap U_i = \phi\ (i \neq j)$. We can do this because each $q_j \in M - A$ (which is a manifold). Then $\phi\left(M - \bigcup_{j=1}^{K} U_j\right) \not\ni p$. For ψ sufficiently close (even in the C^0-sense) $\psi\left(M - \bigcup_{j=1}^{K} U_j\right) \not\ni p$. We want to show that for ψ sufficiently close to ϕ in the C^1-sense, each U_j contains exactly one point of $\psi^{-1}(p)$. Now, as ϕ is transverse to $\{p\}$ at q_j,

$$T_p(\{p\}) + \phi_*\,(T_{q_j}(M)) = T_p(N).$$

As $T_p(\{p\})$ is 0-dimensional, $\phi_*\,(T_{q_j}(M))$ must be m-dimensional. Hence the Jacobian of ϕ must be nonzero. If ψ is close to ϕ in the C^1-sense, then the Jacobian of ψ at q_j must also be different from zero. By the implicit function theorem, ψ maps U_j diffeomorphically onto a neighborhood of p. In particular, there exists exactly one point $x_j \in U_j$ such that $\psi(x_j) = p$.

Q.E.D.

Corollary: Suppose the manifolds in the auxiliary lemma are oriented. We count a point in $\phi^{-1}(p)$ with a plus sign if the Jacobian of ϕ is positive at the point, and with a negative sign if the Jacobian is negative. Then with obvious modifications in the statement of the above lemma, the result is still true. The proof is along similar lines.

Lemma (2): Let N be a compact, connected manifold, $\phi: N \to N$ a self-diffeomorphism. If ψ is sufficiently close to ϕ in the C^1-sense, then ψ is also a self-diffeomorphism.

Proof: ϕ_* is nonsingular everywhere; therefore, for ψ sufficiently close to ϕ in the C^1 sense, ψ_* has the same property. So ψ is locally $1 - 1$ and takes open sets onto open sets by the implicit function theorem. As N is compact, ψ takes closed sets to closed sets. By the connectedness of N, $\psi(N) = N$. It remains to show ψ is globally $1 - 1$: Suppose it is not. Then there exists a sequence of maps $\{\psi_n\}$, $\psi_n \to \phi$ in the C^1-topology and sequences of points $\{q_n\}$, $\{q_n'\}$, such that $\psi_n(q_n) = \psi_n(q_n')$. Using the compactness of N and passing to subsequences if necessary, assume $q_n \to q_\infty$, $q_n' \to q_\infty'$. Clearly $\phi(q_\infty) = \phi(q_\infty')$. Hence $q_\infty = q_\infty'$. But then for n sufficiently large, q_n, q_n' lie in a single arbitrarily neighborhood. This contradicts the fact that ψ_n is locally $1 - 1$.

Q.E.D.

Let $\mathcal{M}$ be the space of all maps $\phi : N \to N$ with the C^∞-topology. The above lemma shows that the subset $D \subseteq \mathcal{M}$ of self-diffeomorphisms is open. Let $d_\infty(\phi, \psi)$ be the complete metric on the space of C^∞-maps of $N \to N$. Introduce

$$\hat{d}_\infty(\phi, \psi) = d_\infty(\phi, \psi) + \left| \left(\inf_{K \in \mathcal{M} - D} d_\infty(\phi, K) \right)^{-1} - \left(\inf_{K \in \mathcal{M} - D} d_\infty(\psi, K) \right)^{-1} \right|$$

for $\phi, \psi \in D$. $\hat{d}_\infty$ is a complete metric on D.

Jiggling Lemma (3): Let $\Psi : M^m \to N^n$ be C^∞ and let V be a submanifold of N. Assume M, N are compact. Then the set of self-diffeomorphisms η of N such that $\eta\Psi$ is transverse to V is a prevalent open set of D.

Corollary: Let $\phi, \psi : M^m \to N^m$ be smoothly homotopic and let $p \in N$. If ϕ and ψ are both transverse to $\{p\}$, then the parity of the number of points in $\phi^{-1}(p)$ is equal to the parity of the number of points in $\psi^{-1}(p)$.

Proof of Corollary: Let Φ be a homotopy between ϕ and ψ. By the Jiggling Lemma, there is a self-diffeomorphism η of N which is close to the identity such that $\eta\Phi$, $\eta\phi$, $\eta\psi$ are all transverse to N at $\{p\}$. Now, $\eta\phi$ and $\eta\psi$ are smoothly homotopic (via the homotopy $\eta\Phi$) and we are able to apply the remarks preceding Theorem 1. Thus, the parity of the number of points in $(\eta\phi)^{-1}(p)$ is equal to the parity of the number of points in $(\eta\psi)^{-1}(p)$. Since η is close to the identity, we can apply the auxiliary Lemma (1) to the maps ϕ, $\eta\phi$ as well as to the maps ψ, $\eta\psi$.

Q.E.D.

Proof of Jiggling Lemma: We shall show that the set of diffeomorphisms η of N such that $\eta\psi$ is transverse to V is a prevalent set. It clearly suffices to prove that the set of diffeomorphisms η such that $\eta\psi$ restricted to a small coordinate neighborhood U on M is transverse to V is a prevalent set. We call this set of diffeomorphisms $TR(U)$. For η close to a fixed η_1, $\eta\psi(U) \subseteq W$, a fixed coordinate neighborhood on N. We choose coordinates $[x, y]$ in W (x and y are themselves vectors) such that V appears locally as $y = 0$. Let P be the projection $[x, y] \to y$. Transversality of $\eta\psi$ restricted to U means that at each point in $(P\eta\psi)^{-1}(0)$, the Jacobian of $P\eta\psi$ is of maximal rank, i.e. there is an $r \times r$ nonvanishing minor, where r is as large as possible. It is clear that $TR(U)$ is open. We must show that $TR(U)$ is dense; for this it suffices to prove that arbitrarily near each fixed η_2, there is an η_3 such that $P\eta_3\psi$ has a Jacobian of maximum rank at each point of $(P\eta_3\psi)^{-1}(0)$. Let Σ be the set of points in U where $P\eta_2\psi$ does not have maximum rank. By

Sard's lemma, $P\eta_2\psi(\Sigma)$ has measure zero. If $\varepsilon > 0$ is an arbitrary number, there then exists a vector δ whose norm $\|\delta\| < \varepsilon$ and $\delta \notin P\eta_2\psi(\Sigma)$, i.e. the Jacobian of $P\eta_2\psi$ has maximal rank at every point of $(P\eta_2\psi)^{-1}(\delta)$. It is possible to define a diffeomorphism $\eta_0: N \rightarrow N$ which is such that $\eta_0([x, y]) = [x, y - \delta]$ holds for all $x, y \in W$ (we may assume that W is such that it is itself contained in a coordinate neighborhood W', thereby insuring that $[x, y - \delta] \in W'$) and such that η_0 tapers off smoothly to the identity away from W. If δ is small, η_0 is very near the identity, and $\eta_3 = \eta_0\eta_2$ is very near η_2. Now, $(P\eta_0\eta_2\psi)^{-1}(0) = P\eta_2(\psi)^{-1}(\delta)$ and the proof is complete.

Q.E.D.

Remark: If, in the above lemma, N is not a genuine manifold but rather a relative manifold modulo some closed subset A, and if $V \subseteq N - A$, then the set of diffeomorphisms $\eta: (N, A) \rightarrow (N, A)$ such that (i) $\eta\psi$ is transverse to V, (ii) η coincides with the identity in a neighborhood of A, is a prevalent open set.

The proof of this assertion closely resembles the proof of the Jiggling Lemma. Details are left to the reader.

Lemma (4): Any compact manifold N^n can be embedded in a sufficiently high dimensional Euclidean space E^K.

Proof: We can cover N with coordinate neighborhoods U_α such that for each α there exist coordinate neighborhoods V_α and W_α satisfying $U_\alpha \subseteq \bar{U}_\alpha \subseteq V_\alpha \subseteq \bar{V}_\alpha \subseteq W_\alpha$. By compactness, we may assume the number of U_α is finite, say $N \subseteq \bigcup_{i=1}^{l} U_{\alpha_i}$. We may also assume that none of the n-dimensional balls which are homeomorphic to the coordinate neighborhoods U_{α_j}, $j = 1, \ldots, l$, and which give the coordinates on these coordinate neighborhoods contain the origin. Let $f_\alpha: N^n \rightarrow E^n$ be a C^∞ function such that

$$f_\alpha(p) = \begin{cases} \text{coordinate vector of } p \text{ in } U_\alpha, \text{ if } p \in U_\alpha \\ 0, \text{ if } p \in N - V_\alpha \end{cases}$$

Let g_α be a C^∞ function such that

$$g_\alpha(p) = \begin{cases} \text{coordinate vector of } p \text{ in } V_\alpha, \text{ if } p \in V_\alpha \\ 0, \text{ if } p \in N - W_\alpha \end{cases}$$

Define $F: N^n \rightarrow E^{2nl}$ by $F(p) = [f_{\alpha_1}(p), \ldots, f_{\alpha_l}(p), g_{\alpha_1}(p), \ldots, g_{\alpha_l}(p)]$. The map F is $1-1$: For suppose $F(p) = F(q)$. If $p \in U_{\alpha_j}$, then $f_{\alpha_j}(p) \neq 0$ and therefore also $f_{\alpha_j}(q) \neq 0$. By the definition of the f_α, this means $p, q \in V_\alpha$. Now $g_{\alpha_i}(p) = g_{\alpha_i}(q)$, $i = 1, \ldots, l$, and by the definition of g_α, it follows that $g_{\alpha_j}(p)$ is the coordinate vector of p in V_{α_j} while $g_{\alpha_j}(q)$ is also the coordinate vector of q in V_{α_j}. But then we must have $p = q$ and F is $1-1$. On the other hand, F is C^∞ and since F is clearly of maximal rank also F^{-1} is C^∞. So F is a diffeomorphic embedding of N^n into E^K, where $K = 2nl$.

Corollary: Any compact manifold can be given a Riemannian metric. (This observation will be useful in our subsequent discussion of Riemannian geometry.)

Note: The previous lemma is a special case of a more general result called the Whitney Embedding Theorem: this asserts that if N^n is an arbitrary (not necessarily compact) manifold, then there is an embedding $\tau : N^n \to E^{2n+1}$ such that $\tau(N)$ is a closed subset of E^{2n+1}. The proof can be found (for example) in *Introduction to Differentiable Manifolds* by Auslander and MacKenzie.

Lemma (5): If N^n is a smooth manifold embedded in E^K, then there is a tubular neighborhood U of N^n in E^K and a map $\varrho : U \to N^n$ such that ϱ is a smooth retraction of U onto N^n.

Proof: Consider the K-manifold Σ^K consisting of pairs $\sigma = [a, b]$ such that $a \in N^n$, and b is a vector in E^K orthogonal to N at its foot a. It is clear the map f: $\Sigma^K \to E^K$ given by $f[a, b] = a + b$ is a smooth map whose Jacobian determinant is nonzero at every point σ in the set of points Σ_0 in which b has zero length. By the implicit function theorem, there is a neighborhood U of Σ_0 such that the map f: $U \to E^K$ is smooth, $1-1$, and maps onto an open set V in E^K; f^{-1} on V is also smooth by the implicit function theorem. We put $g[a, b] = a$ and $\varrho = g \circ f^{-1}$.

Q.E.D.

Homotopy Lemma (6): Let M^m and N^n be compact manifolds and let $\eta_1, \eta_2 : M \to N$ be smooth maps. Let d be the metric on N where N is considered as a topological subspace of some E^K (see Lemma 4). Then there exists an $\varepsilon > 0$ such that if $d(\eta_1(p), \eta_2(p)) < \varepsilon$ for all $p \in M$, then η_1 and η_2 are homotopic.

Proof: Considering N as an embedded submanifold of E^K, we take the neighborhood U guaranteed by Lemma 5. If $\varepsilon > 0$ is chosen small enough, $t\eta_1(p) + (1 - t)\,\eta_2(p) \in U$ for all $p \in M$, $t \in [0, 1]$. Set $\varrho_t(p) = \varrho\,(t\eta_1(p) + (1 - t)\,\eta_2)(p)$ where ϱ is the restraction of U on N. Clearly, ϱ_t is a homotopy between η_1 and η_2.

Q.E.D.

Approximation Lemma (7): Let M^m and N^n be compact manifolds and let $f: M^m \to N^n$ be a *continuous* map. Then (using the notation of Lemma 6) there exists, for any preassigned $\varepsilon > 0$, a smooth map $\phi_\varepsilon: M \to N$ such that $d(f(m), \phi_\varepsilon(m)) < \varepsilon$ for all $m \in H$.

Proof: Follows at once from Lemma 5 and the Weierstrass approximation theorem. Q. E. D.

Definition (2): Let M^m and N^n be compact manifolds and let $\phi: M \to N$ be a continuous map. Let $p \in N$ be arbitrary. By Lemma 7, we can find a sequence $\{\phi_j\}$ of smooth maps such that $d(\phi(m), \phi_j(m)) < 1/j$. By the Jiggling Lemma, we can assume ϕ_j is transverse to $\{p\}$ for all j. By the Homotopy Lemma, ϕ_k and ϕ_l are homotopic if $k, l \geqq A$, for some integer A. By the corollary to the Jiggling Lemma, the parity of the number of points in $\phi_k^{-1}(p)$ is equal to the parity of the number of points in $\phi_l^{-1}(p)$ (for $k, l \geqq A$). We define $\deg_2 \phi|_p$ to be this parity number.

Theorem (1′): $\deg_2 \phi|_p$ is a homotopy invariant.
The proof is obvious.

Theorem (1′A): $\deg \phi|_p$ is a homotopy invariant of maps between orientable manifolds.

Intersection Theory (mod 2)

We recall first a few definitions from singular homology theory. By a singular m-cube in a space X, we mean a map $\sigma: I^n \to X$ ($I^n = \{x = (x_1, x_2, \ldots, x_n) \in E^n \mid 0 \leqq x_i \leqq 1\}$.) If $n = 0$, then σ is interpreted as a single point in X. If $n > 0$, we define the i-th lower and upper faces $\lambda_i^0\sigma$ and $\lambda_i^1\sigma$ of σ to be the singular $(n-1)$-cubes given by

$$(\lambda_i^\varepsilon\sigma)(x_1, \ldots, x_{n-1}) = \sigma(x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_{n-1})$$

for every $i = 1, 2, \ldots, n$, where $\varepsilon = 0, 1$, and $(x_1, \ldots, x_{n-1}) \in I^{n-1}$. Then for $i < j$ we have $\lambda_i^\varepsilon\lambda_j^\eta = \lambda_{j-1}^\eta\lambda_i^\varepsilon$; ε, η are 0 or 1. Define $Q_n(X)$ to be the free abelian group generated by all singular n-cubes in X if $n \geqq 0$ and $Q_n(X) = 0$ if $n < 0$. Then the operation

$$\partial\sigma = \sum_{i=1}^{n} (-1)^i (\lambda_i^1\sigma - \lambda_i^0\sigma)$$

determines a homomorphism $\partial: Q_n(X) \to Q_{n-1}(X)$ for every n. It is straightforward to verify that $\partial\partial = 0$.

For each singular $(n-1)$-cube σ in X, $n > 0$, we define a singular n-cube $D\sigma$ in X by setting

$$(D\sigma)(x_1, \ldots, x_{n-1}, x_n) = \sigma(x_1, \ldots, x_{n-1}).$$

A singular n-cube V in X is said to be *degenerate* if $V = D\sigma$ for some σ. In other words, V is degenerate iff it does not depend on the last coordinate x_n of the point $(x_1, \ldots, x_n)$ in I^n. The degenerate singular n-cubes in X, $n > 0$, generate a subgroup $D_n(X)$ of $Q_n(X)$. Since $\lambda_i^\varepsilon D = D\lambda_i^\varepsilon$, $i < n$, $\lambda_n^\varepsilon D = 1$, it follows that

$$\partial D\sigma = \sum_{i=1}^{n} (-1)^i (\lambda_i^1 D\sigma - \lambda_i^0 D\sigma) = \sum_{i=1}^{n-1} (-1)^i (D\lambda_i^1\sigma - D\lambda_i^0\sigma),$$

and hence ∂ carries $D_n(X)$ into $D_{n-1}(X)$.

For each integer n, the quotient group $C_n(X) = Q_n(X)/D_n(X)$ is obviously a free abelian group and will be called the group of normalized cubical singular n-chains in X. Since ∂ carries $D_n(X)$ into $D_{n-1}(X)$, it induces a homomorphism

$$\partial\colon C_n(x) \to C_{n-1}(X),$$

for every n. Since $\partial\partial = 0$, we can define singular cycles and singular boundaries with coefficients in an arbitrary abelian groub G as in simplicial homology theory and then we can define the singular homology group $H_n(X, G) = \text{(cycles)/(boundaries)}$.

Let (M^m, A) be a relative manifold, $V^v \subseteq M$ a submanifold. Assume $V \cap A = \phi$. We say the chain $\Sigma\, k_\sigma \sigma$ is carried by A and write $\Sigma\, k_\sigma\sigma \subseteq A$ if $\sigma(I^n) \subseteq A$, all σ. A singular n-chain α called a *cycle modulo A* if its boundary $\partial\alpha \subseteq A$.

Suppose $\alpha = \Sigma k_\sigma \sigma$. Then arbitrarily near the identity, there exists a diffeomorphism η of M such that $\eta \equiv$ identity near A and such that $\eta\sigma$ is transverse to V for each σ, and such that $\eta\sigma$ restricted to any face of I^n is transverse to V. This follows immediately from the remark following the Jiggling Lemma.

Definition (3): Let $m - v = k$. (m, v as in the last paragraph.) Suppose α is a k-cycle of M mod A, $\alpha = \Sigma k_\sigma \sigma$. We wish to define $\alpha \cdot V$, *the intersection number of* α *and* V *(mod 2).*

Take any diffeomorphism η sufficiently close to the identity and with the properties described above. First of all there are no solutions of $\eta\sigma(x) \in V$ on the boundary of I^k. For suppose there is such an x on a face of I^k. Since, dim (face) $< k =$ codim (V), the existence of such an x contradicts remark (a) following the definition of transversality in Chapter I. From remark (b) it follows that the set of solutions of $\eta\sigma(x) \in V$ is an isolated set. As I^k is compact, there can be only finitely many such points. Call this finite number $\#(\sigma)$ and *define* $\sigma \cdot V = \textit{parity}\,(\#(\sigma)), \alpha \cdot V = \Sigma\, k_\sigma \cdot \text{parity}\,(\#\,\sigma)$. This defines the intersection number of α with V (mod 2). It is easy to see that $\alpha \cdot V$ as defined is independent of η. We argue as follows. If $\alpha = \Sigma\, k_\sigma \sigma$, then by definition $\eta\alpha = \Sigma\, k_\sigma \cdot \eta\sigma$. Let η' be another diffeomorphism having the same properties as η. By the Homotopy Lemma, η and η' are homotopic. We claim then that $\eta\alpha$ and $\eta'\alpha$ are homologous mod A, i.e. $\eta\alpha - \eta'\alpha = \partial\beta + \gamma$, where $\gamma \subseteq A$, and β a is a $k+1$-dim singular chain. To see this, note that there exists a homotopy η_t between η, η' i.e. $\eta_0 = \eta$, $\eta_1 = \eta'$. Define $\beta = \Sigma k_\sigma \cdot \hat{\sigma}$ where $\hat{\sigma}(t, x) = \eta_t \cdot \sigma(x)$. It follows from the preceding definition of the operator ∂ that $\eta\alpha - \eta'\alpha - \partial\beta \subseteq A$.

Lemma (8): If α is homologous to 0 mod A, then $\alpha \cdot V = 0$ i.e. if $\alpha = \Sigma\, k_\sigma \sigma = \partial\beta + \gamma$, $\gamma \subseteq A$, then $\Sigma\, k_\sigma \cdot \text{parity}\ \#(\sigma) = 0$.

Proof: By the preceding definition and the jiggling lemma, we may M slightly to make all the singular cubes of α, β, γ transverse to V. The singular cubes of γ lie on A and are therefore disjoint from V. $\beta = \Sigma\, l_\tau \tau$ implies that $\partial\beta = \Sigma\, l_\tau \cdot \partial\tau$. We shall show that for each cube τ, the number of points x in ∂I^{k+1} such that $\tau(x) \in V$ is even. Indeed, $\tau^{-1}(V)$ is a 1-dim manifold with boundary, and all the boundary points are in ∂I^{k+1} by the Lemma 8 of the preceding chapter. So $\tau^{-1}(V)$ is a set of arcs and circles and the set of end points $= \{x \in \partial I^{k+1} \,|\, \tau(x) \in V\}$ is even.

Q.E.D.

If the manifold M and its submanifold V are oriented, then we may state a more precise version of Lemma 8, i.e., one which refers to signed integers rather than merely to a parity. We proceed as follows. As above, we call the number of solutions of $\eta\sigma(x_0) \in V$ (in the transverse case) $\#(\sigma)$. Now, however, we count solutions with sign ± 1, in the following manner. Let $\{u_i\}$ be a positively oriented basis for the tangent space to I^k at x_0, and let $\{v_j\}$ be a positively oriented basis for the tangent space to V at $\eta\sigma(x_0)$. If $\{(\eta\sigma)_* u_i, v_j\}$ is a positively oriented basis for the tangent space to M, we count x_0 with the sign $+1$, otherwise -1. Adopting this sign convention, we put $\sigma \cdot V = \#(\sigma)$, $\alpha \cdot V = (\Sigma\, n_\sigma \sigma) \cdot V = \Sigma\, n_\sigma \cdot (\#(\sigma))$. Then we may state the following lemma.

Lemma 8A: If α is homologous to zero mod A, then $\alpha \cdot V = 0$, i.e., if $\alpha = \Sigma\, n_\sigma \sigma = \partial\beta + \gamma$, $\gamma \subseteqq A$, then $\Sigma\, n_\sigma(\#\sigma) = 0$.

The proof is very much like the proof of Lemma 8; however, it uses the Proposition following Theorem 1 of the present chapter in the place of Lemma 8 of the preceding chapter. We leave the details of the proof of Lemma 8*A* to the reader.

Corollary: $\alpha \cdot V$ depends only on the homology class $\{\alpha\}$ of α, *so we may define* $\{\alpha\} \cdot V = \alpha \cdot V$.

Corollary: $(\alpha_1 + \alpha_2) \cdot V = \alpha_1 \cdot V + \alpha_2 \cdot V$. i.e. $\{\alpha\} \cdot V$ is linear in $\{\alpha\}$.

Let $\sigma: I^k \to M$, $\sigma': I^l \to M'$. We define

$$\sigma \times \sigma': I^k \times I^l = I^{k+l} \to M \times M'$$

by

$$\sigma \times \sigma' (x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+l})$$
$$= [\sigma (x_l, \ldots, x_k), \sigma' (x_{k+1}, \ldots, x_{k+l})].$$

If $\alpha = \Sigma\, k_\sigma \cdot \sigma$, $\alpha' = \Sigma\, l_{\sigma'} \cdot \sigma'$, we define $\alpha \times \alpha' = \Sigma\, k_\sigma l_{\sigma'}.\ \sigma \times \sigma'$. It is left to the reader to verify that

$$\partial (\alpha \times \alpha') = (\partial\alpha) \times \alpha' + (-1)^k \alpha \times (\partial\alpha')$$

where ∂ is the boundary operator defined previously.

Now suppose (M, A) and (M', A') are relative manifolds. Let α be a cycle of M mod A, and α' a cycle of M' mod A'. Using the above formula, we can easily verify the following statements:

(1) $\alpha \times \alpha'$ is a cycle of $M \times M'$ mod $(M \times A') \cup (A \times M')$.

(2) If α is a boundary of M mod A or if α' is a boundary of M' mod A', then $\alpha \times \alpha'$ is a boundary of $M \times M'$ mod $(M \times A') \cup (M' \times A)$. (Note that $(\alpha + \beta) \times \alpha' = \alpha \times \alpha' + \beta \times \alpha'$.)

Combining (1) and (2) we see that the map $[\alpha, \alpha'] \to \alpha \times \alpha'$ induces a map of $H_k(M, A) \times H_l(M', A') \to H_{k+l}(M \times M', (M \times A') \cup (M' \times A))$. Assume now that both M, M' are contained in a genuine manifold M_0; moreover assume that M, M' and M_0 all have the same dimension m and that $A \cap M' = A' \cap M = \phi$.

Finally, define $N = \operatorname{diag} M_0 = \{[x, x] \mid x \in M_0\}$; N is an m-dimensional manifold. Assume that α is a cycle of M mod A, α' a cycle of M' mod A'. From the previous paragraph, $\alpha \times \alpha'$ is a cycle of $M \times M'$ mod $(M \times A') \cup (M' \times A)$. As $N \cap \{M \times A' \cup M' \times A\} = \phi$, we can define the intersection number $\{\alpha \times \alpha'\} \cdot N$ if we assume that $\dim \alpha + \dim \alpha' = k + l = m$. This follows from the theory developed above since codim (N) in M_0 is $2m - m = m$.

Definition (4): Define the intersection number $\{\alpha\} \cdot \{\alpha'\}$ of the two homology classes $\{\alpha\}$, $\{\alpha'\}$ (mod 2) by $\{\alpha\} \cdot \{\alpha'\} = \{\alpha \times \alpha'\} \cdot N$.

Lemma (9): The intersection number as thus defined is

(a) well defined for $\{\alpha\} \in H_k(M_1, A)$, $\{\alpha'\} \in H_l(M_2, A')$ if $k + l = m$.

(b) Bilinear in $\{\alpha\}$, $\{\alpha'\}$.

(c) Satisfies $\gamma \cdot \gamma' = \gamma' \cdot \gamma$ where $\gamma = \{\alpha\}$, $\gamma' = \{\alpha'\}$.

Proof: To check (a) we have to show that $\gamma \cdot \gamma'$ does not depend on the choice of representatives α, α'. But this is true because the homology class of $\alpha \times \alpha'$ depends only on the homology classes of α and α'.

(b) is obvious.

(c) Define a diffeomorphism ϱ of $M \times M$ by ϱ: $[m_1, m_2] \to [m_2, m_1]$, and a homeomorphism of $(I^{k+l}, \partial I^{k+l})$ onto itself by

$$\varrho_{k,l}: (x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+l}] \to [x_{k+1}, \ldots, x_{k+l}, x_1, \ldots, x_k].$$

Note that $\varrho_{k,l}$ is homotopic as a map of $(I^{k+l}, \partial I^{k+l})$ to the identity if kl is even, but to the map $[x_1, \ldots, x_{k+l}] \to [1 - x_1, x_2, \ldots, x_{k+l}]$ if kl is odd. If $\beta = \Sigma n_\sigma \sigma$ is a singular $k + l$ dimensional cycle, it follows that $\Sigma n_\sigma(\sigma \cdot \alpha_{k,l})$ belongs to the same homology class as $(-1)^{kl}\beta$. Plainly, if σ: $I^k \to M$ and τ: $I^l \to M$, we have $\varrho(\sigma \times \tau) = (\tau + \sigma)\alpha_{k,l}$. Thus, for k- and l-dimensional homology classes, we have $\delta_*(\gamma \times \gamma') = (-1)^{kl}\gamma' \times \gamma$. Since $N = \text{diag}(M \times M)$ is invariant under ϱ, we have $(\gamma \times \gamma') \cdot N = (-1)^k (\gamma' \times \gamma) \cdot N$, which by reduction mod 2 gives (c).

Q.E.D.

Remark: If the manifolds above are oriented, we can, as in degree theory, develop a more specific integer-coefficient intersection theory. In this case, these two important formulas hold. We shall leave the proofs as an exercise, cf., however, Lemma 8*A*.

(1) $\alpha \cdot \beta = (-1)^{kl}\beta \cdot \alpha$, where $k = \dim \alpha$, $l = \dim \beta$,

(2) if ε and η are such that $\dim \varepsilon + \dim \eta = \dim M_0 + 1$ and if $\partial\varepsilon \cap \partial\eta = \phi$, then $\partial\varepsilon \cdot \eta = (-1)^l \varepsilon \cdot \partial\eta$ where $l = \dim \varepsilon$.

Lemma (10): Let α and α' be of dimensions k and l where $k + l = m$. If at each intersection point p of a cell σ of α with a cell σ' of α', the intersecting cells are "transverse" i.e. $\sigma_*(T_{p_1}(I^k)) + \sigma'_* T_{p_2}(I^l)) = T_p(M)$ (where $p_1 \in \sigma^{-1}(p)$, $p_2 \in \sigma'^{-1}(p)$ and where σ_*, σ'_* are the maps of tangent spaces induced by σ, σ'), then $\alpha \cdot \alpha' =$ parity $\#$ of intersections of all cells of α with all cells of α'.

The proof is left as an exercise to the reader.

Example (1): Let S^n be the n-sphere. We know $H_k(S^n, Z_2) = 0$ if $k \neq 0, n$, $H_k(S^n, Z_2) = Z_2$ if $k = 0, n$. Let γ^0, γ^n be the homology classes of dimensions 0 and n which generate their respective homology groups. Take representatives α^0, α^n of γ^0, γ^n (i.e. $\{\alpha^0\} = \gamma^0$, $\{\alpha^n\} = \gamma^n$) as follows: let α^n: $I^n \to S^n$ be the map that takes the interior of I^n onto the interior of I^n by the identity map, and let α^n take the entire boundary of I^n into a single boundary point p_∞. So $\alpha^n(I^n)$ is the n-cube with the boundary identified to one point p_∞, which is topologically S^n. Let α^0: $I^0 \to p_0$, $p_0 \in S^n$, $p_0 \neq p_\infty$. Then by the above lemma $\gamma^0 \cdot \gamma^n = 1$.

Theorem (2): Let M, M' be two manifolds and suppose γ_1, γ_2 are homology classes on M and γ_1', γ_2' are homology classes on M' such that $\dim(\gamma_1) + \dim(\gamma_2) = \dim M$, $\dim(\gamma_1') + \dim(\gamma_2') = \dim M'$. Then $(\gamma_1 \times \gamma_1') \cdot (\gamma_2 \times \gamma_2') = (\gamma_1 \cdot \gamma_2)(\gamma_1' \cdot \gamma_2')$. If the above conditions on dimensions are not satisfied, but the former intersection is defined, then $(\gamma_1 \times \gamma_1') \cdot (\gamma_2 \times \gamma_2') = 0$.

Proof: The proof follows immediately from the definition of intersection number and is left as an exercise to the reader.

We now quote a theorem of Künneth on homology groups of a topological product. For notations and proof, see *Homology Theory*—Hilton and Wylie, Chapter 5.

Theorem (Künneth): Let $|K|$, $|L|$ be polyhedra. Then

(i) $H_p(|K| \times |L|) \cong \sum_{m+n=p} H_m(|K|) \otimes H_n(|L|) \oplus \sum_{m+n=p-1} H_m(|K|) * H_n(|L|)$

where the group of coefficients is taken to be the group of integers. ("*" is the torsion product.)

(ii) If $\mathcal{J}$ is a field,

$$H_p(|K| \times |L|; \mathcal{J}) \cong \sum_{m+n=p} H_m(|K|) \otimes H_n(|L|) \otimes \mathcal{J}.$$

(iii) If q is prime, and Z_q is the group of integers mod q,

$$H_p(|K| \times |L|); Z_q) \cong \sum_{m+n=p} H_m(|K|; Z_q) \otimes H_n(|L|; Z_q).$$

Example (2): Consider $S^n \times S^n$. Künneth's theorem, the only non trivial homology classes of $S^n \times S^n$ are $\gamma^0 \times \gamma^0, \gamma^n \times \gamma^0, \gamma^0 \times \gamma^n$, $\gamma^n \times \gamma^n$. Using the previous theorem,

$$(\gamma^n \times \gamma^0) \cdot (\gamma^0 \times \gamma^n) = 1$$
$$(\gamma^0 \times \gamma^0) \cdot (\gamma^n \times \gamma^n) = 1$$

and all other intersections (which are defined) are 0.

Remark: If the intersection number of a homology class on $S^n \times S^n$ with all other classes (of appropriate complementary dimension) is 0, then the class is the zero class.

Proof: It suffices to consider n-cycles (because of the above formulas). The most general n-cycle on $S^n \times S^n$ is $C_1(\gamma^0 \times \gamma^n) + C_2(\gamma^n \times \gamma^0)$. By hypothesis, $C_2 = \{C_1(\gamma^0 \times \gamma^n) + C_2(\gamma^n \times \gamma^0)\} \cdot (\gamma^0 \times \gamma^n) = 0$. Similarly, $C_1 = 0$.

Suppose $\phi : S^n \to S^n$ is smooth. Let $\Gamma^\phi : S^n \to S^n \times S^n$ be defined by $\Gamma^\phi(x) = [x, \phi(x)]$ and let Λ_ϕ be the image of Γ^ϕ. Γ^ϕ induces a map $\Gamma^\phi_* : H_p(S^n) \to H_p(S^n \times S^n)$ for all p. Now $\Gamma^\phi_*(\gamma^n) = C_1(\gamma^n \to \gamma^0) + C_2(\gamma^0 \to \gamma^n)$ (because $\gamma^n \times \gamma^0$, $\gamma^0 \times \gamma^n$ form a basis for $H_n(S^n \times S^n)$). Take for representative of γ^n the singular n-cube σ described in example (1). A representative of $\Gamma^\phi_*(\gamma^n)$ is thus given by $x \to [\sigma(x), \phi(\sigma(x))]$. A representative of $\gamma^n \to \gamma^0$ is given by $\gamma \to [\sigma(x), x_0]$ where $x_0 = \gamma^0(I^0)$. A representative of $\gamma^0 \to \gamma^n$ is given by $x \to [x_0, \sigma(x)]$. By the last lemma, we find easily that $(\gamma^0 \times \gamma^n) \cdot \Gamma^\phi_*(\gamma^n) = 1$, $(\gamma^n \to \gamma^0) \cdot \Gamma_*(\gamma^n) =$ parity $\#$ of solution of $\phi(\sigma(x)) = x_0 =$ parity $\#$ of solutions of $\phi(p) = x_0$ which is just $\deg_2(\phi)|_{x_0}$. Therefore,

$$1 = (\gamma^0 \times \gamma^n) \cdot \Gamma^\phi_*(\gamma^n) = (\gamma^0 \times \gamma^n) \cdot \{C_1(\gamma^n \times \gamma^0) + C_2(\gamma^0 \times \gamma^n)\} = C_1$$

and

$$\begin{aligned} \deg_2(\phi)|_{x_0} &= (\gamma^n \times \gamma^0) \cdot \Gamma^\phi_*(\gamma^n) \\ &= (\gamma^n \times \gamma^0) \cdot \{C_1(\gamma^n \times \gamma^0) + C_2(\gamma^0 \times \gamma^n)\} = C_2) \end{aligned}$$

Thus,

$$\Gamma^\phi_*(\gamma^n) = (\gamma^n \times \gamma^0) + (\deg_2(\phi)|_{x_0})(\gamma^0 \times \gamma^n).$$

If ψ is another map $S^n \to S^n$, then $\Gamma^\psi_*(\gamma^n) = (\gamma^n \to \gamma^0) + (\deg_2(\psi)|_{x_0})(\gamma^0 \times \gamma^n)$. Therefore

$$\Gamma^\phi_*(\gamma^n) \cdot \Gamma^\psi_*(\gamma^n) = \deg_2(\phi)|_{x_0} + \deg_2(\psi)|_{x_0}.$$

Now $\Gamma^\phi_*(\gamma^n)$ is a chain carried by Λ_ϕ and $\Gamma^\psi_*(\gamma^n)$ is a chain carried by Λ_ψ. Therefore if $\deg_2(\phi)|_{x_0} \neq \deg_2(\psi)|_{x_0}$, then $\Gamma^\phi_*(\gamma^n) \cdot \Gamma^\psi_*(\gamma^n) = 1$ and consequently $\Lambda_\phi \cap \Lambda_\psi \neq \phi$. That is, there exists an $x_1 \in S^n$ such that $\phi(x_1) = \psi(x_1)$. This gives us a so-called coincidence theorem.

Corollary: If $\deg_2 \phi|_{x_0} = 0$, then ϕ has a fixed point.

Proof: Take $\psi =$ identity and apply the above result.

We now quote a fixed point theorem without proof. For the terminology and notation, see for example *Algebraic Topology* by Lefschetz.

Theorem (Lefschetz): Let X be a finitely triangulated space, $\phi : X \to X$. Consider the induced map

$$\phi_* : H_k(X) \to H_k(X)$$

Let $l(\phi) = \sum_{k=0}^{\dim X} (-1)^k \, tr\,(\phi_* | H_k(X))$. If $l(\phi) \neq 0$, then ϕ has a fixed point of homology with coefficients in some field.

Application: If $X = S^n$, $l(\phi) = 1 + (-1)^n \deg \phi$.

Let $\phi_1, \phi_2 : S^n \to S^n$, $\psi_1, \psi_2 : S^n \to S^n$. Define $\Phi : S^n \to S^n \times S^n$ by $\Phi(p) = [\phi_1(p), \phi_2(p)]$ and define Ψ analogously. Then

$$\Phi_*(\gamma^n) = \deg_2(\phi_1)(\gamma^n \times \gamma^0) + \deg_2(\phi_2)(\gamma^0 \times \gamma^n).$$

$$\psi_*(\gamma^n) = \deg_2(\psi_1)(\gamma^n \times \gamma^0) + \deg_2(\psi_2)(\gamma^0 \times \gamma^n).$$

$$\Phi_*(\gamma^n) \cdot \psi_*(\gamma^n) = \deg_2(\phi_1)\deg(\psi_2) + \deg_2(\psi_1)\deg_2(\phi_2).$$

If this number is $\neq 0$, there exists x_1, x_2 such that

$$[\phi_1(x_1), \phi_2(x_1)] = [\psi_1(x_2), \psi_2(x_2)].$$

We now define the cohomology ring of a space.

If X is a topological space, let $H(X)$ be the direct sum of the singular homology groups of X of all dimensions. If the coefficient group is a field, $H(X)$ can be considered as a vector space over this field. Define the (total) cohomology group $H^*(X)$ of X to be the dual space of $H(X)$.

Let $\phi : X \to Y$. Then ϕ induces a map $\phi_* : H(X) \to H(Y)$ and a dual map $\phi^* : H^*(Y) \to H^*(X)$.

Let diag: $X \to X \times X$ be the diagonal map defined by diag $(x) = (x, x)$.

Therefore diag: $H(X) \to H(X \times X) = H(X) \otimes H(X)$ and diag*: $H^*(X) \otimes H^*(X) \to H^*(X)$.

Let $\gamma_1^*, \gamma_2^* \in H^*(X)$; define the *cup product* of γ_1^* and γ_2^* (denoted by $\gamma_1^* \cup \gamma_2^*$) by

$$\gamma_1^* \cup \gamma_2^* = \text{diag}^* (\gamma_1^* \otimes \gamma_2^*).$$

This operation of product turns $H^*(X)$ into a ring, called the cohomology ring of X.

Let X be a topological space containing a closed subset A. A finite sequence $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 = A$ of closed subsets is called a *filtration* of X or a *normal series* of closed subsets of X. A filtration of X is said to give a *cell complex* if for each pair (X_j, X_{j-1}), $j = 1, \ldots, n$, there exists a set $C_1^{(j)}, C_2^{(j)}, \ldots, C_{m_j}^{(j)}$ of j-cubes and a corresponding set of maps $\sigma_1^{(j)}, \ldots, \sigma_{m_j}^{(j)}$ satisfying the following:

(i) For each $p = 1, 2, \ldots, m_j$, $\sigma_p^{(j)} : C_p^{(j)} \to X_j$

(ii) $\sigma_p^{(j)} : \partial C_p^{(j)} \to X_{j-1}$

(iii) $\sigma_p^{(j)}$ maps the interior of $C_p^{(j)}$, call it $C_p^{0(j)}$, homeomorphically onto its image.

(iv) $\bigcup_{p=1}^{m_j} \sigma_p^{(j)}(C_p^{0(j)}) = X_j - X_{j-1}$.

The $\sigma_p^{(j)}$ are called the *singular j-cells* of the cell complex and the subsets $\sigma_p^{(j)}(C_p^{0(j)})$ are called the *open j-cells* of the cell complex.

The reason for introducing the above notions is to aid us in computing the homology groups of various spaces. In order to carry this out, we need two basic lemmas (which we do not prove). For a proof, cf. Cooke and Finrey, *Homology of Cell Complexes*, Princeton University Press 1967.

Lemma (11): (a) Any singular j-cycle of (X, A) (X mod A) is homologous to a linear combination of the $\sigma_p^{(j)}$.

(b) The homology of (X, A) may be calculated from the cell complex as follows: We know $\partial C_p^{(j)}$ is a $(j-1)$-sphere. We take a basic $j-1$ dimensional integer homology class α on $\partial C_p^{(j)}$. As a map from $\partial C_p^{(j)}$ into X_{j-1}, $\sigma_p^{(j)}$ induces a homomorphism $\sigma_{p*}^{(j)}$ from $H(\partial C_p^{(j)})$ into $H(X_{j-1}, A)$. Therefore, $\sigma_{p*}^{(j)}(\alpha) \in H_{j-1}(X_{j-1}, A)$. From (a) any singular $j-1$ cycle of (X_{j-1}, A) can be expressed as a linear combination of certain of the $j-1$ cells of the cell complex. In particular,

$$\sigma_{p*}^{(j)}(\alpha) = \{\sum n_q \sigma_q^{(j-1)}\}$$

where, in general, $\{\beta\}$ is the homology class of β. We now introduce formal boundary operators $\tilde{\partial}$ by defining $\tilde{\partial}\sigma_p^{(j)} = \sum n_q \sigma_q^{(j-1)}$. It is easily proved that $\tilde{\partial}\tilde{\partial} = 0$. Hence, in the usual way, we can define formal cycles, formal boundaries, and show that the group of formal j-boundaries is contained in the group of formal j-cycles. We can then pass to the quotient group and obtain the formal j-th homology group. We assert that these formal homology groups are isomorphic with the actual singular homology groups. In addition, we assert that the integer n_q in the formula displayed above is the degree of the map

$$(\sigma_q^{(j-1)})^{-1}\,\sigma_p^{(j)}\colon\ \partial C_p^{(j)} \cap U \to C_q^{0(j-1)},$$

where U is the inverse image of $C_q^{0(j-1)}$ under this map.

These facts with be used as a basis for calculation in what follow.

Let (X, A), (Y, B) be two cell complexes,

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 = A$$

$$Y = Y_n \supseteq Y_{n-1} \supseteq \cdots \supseteq Y_0 = B.$$

Let $f: (X, A) \to (Y, B)$, i.e. $f(X) \subseteq Y, f(A) \subseteq B$. f is called a *cellular map* if:

(i) $f(X_j) \subseteq Y_j$,

(ii) f maps each open cell of $X_j - X_{j-1}$ homeomorphically onto an open cell of $Y_j - Y_{j-1}$ (recall that $X_j - X_{j-1}$ is the finite union of open j-cells).

An orientation of the open cells e of a cell complex is a choice for each e of a homeomorphism h from e to the open unit ball B_j. We say two homeomorphisms have the same orientation or opposite orientation depending on the following: let $h_1 : e \to B_j$, $h_2 : e \to B_j$, so that $h_1 h_2^{-1} : B_j \to B_j$. If $h_1 h_2^{-1}$ is orientation preserving, we say h_1, h_2 give the same orientation. If $h_1 h_2^{-1}$ is orientation reversing, we say h_1, h_2 give the opposite orientation to e.

Lemma (12): Let $f: (X, A) \to (Y, B)$ be a cellular map. Introduce the formal map $\tilde{f}$ on the open oriented cells e of (X, A) as follows: e has an orientation given by a homeomorphism $h: e \to B_j$. $f(e)$ has an orientation given by a homeomorphism $g: f(e) \to B_j$ (recall $f(e)$ is an open j-cell).

Therefore $gf: e \to B_j$. If now h and gf give the same orientation to e, we put $\tilde{f}(e) = +f(e)$. If h and gf give opposite orientations, we put $\tilde{f}(e) = -f(e)$. By linearity, we may introduce the formal map on formal linear combinations of open oriented cells. Then it is easily proved that $\tilde{f}$ commutes with $\tilde{\partial}$. Hence, as usual, $\tilde{f}$ induces a map $\tilde{f}_*$ of formal homology groups. We assert that this $\tilde{f}_*$ represents the map f_* of the singular homology groups.

We proceed now to the calculation of some homology groups.

(1) $X = S^n$, the n sphere: Note that we are cheating here because we have already used (in Lemma 11) the fact that $H_n(S^n)$ is isomorphic to the group of coefficients. However, we shall deduce that $H_k(S^n)$ is trivial for $0 < k < n$. For the cell decomposition $X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$, we choose $X_{n-1} = \cdots = X_0 =$ one point p. For the n-cell, we choose $\sigma^n: C^{(n)} \to S^n$ by mapping $C^{0(n)}$ identically and mapping $\partial C^{(n)}$ to the single point p. We define no k-cells for $0 < k < n$. We define one 0-*cell*, namely $\sigma^{(0)}: C^{(0)} \to p$. It is easy to verify that we have a cell complex. Moreover, all the operators $\tilde{\partial}$ are obviously 0. The calculation of the homology groups of S^n now results at once from the previous lemmas.

(2) $X = P_n^{\mathbb{C}}$, the complex projective n-space: For our filtration, we take $P_n^{\mathbb{C}} \supseteq P_{n-1}^{\mathbb{C}} \supseteq \cdots \supseteq P_0^{\mathbb{C}}$. Let $E_n^{\mathbb{C}}$ be the complex Euclidean space. We compactify $E_n^{\mathbb{C}}$ by introducing points ω^* at ∞ corresponding to all unit vectors ω as follows: We say $z(n) \to \omega^*$ if $z(n)/|z(n)| \to \omega$. It is easy to see that $E_n^{\mathbb{C}}$ so compactified is homeomorphic to the real $2n$-dimensional ball. For our $2n$-cell, we choose $\sigma^{(2n)}: B^{2n} \to P_n^{\mathbb{C}}$ to be the identification map. The map is

easily seen to be continuous. In the interior of B^{2n}(i.e. $E_n^{\mathbb{C}}$), $\sigma^{(2n)}$ is the identity and hence $\sigma^{(2n)}|E_n^{\mathbb{C}}$ is a homeomorphism onto $P_n^{\mathbb{C}} - P_{n-1}^{\mathbb{C}}$. Defining cells $\sigma^{(2n-2)}, \ldots, \sigma^{(0)}$ similarly and defining no odd dimensional cells, we obtain a cell decomposition of $P_n^{\mathbb{C}}$. Again, all boundaries $\tilde{\partial}$ are zero because there are no odd dimensional cells (check the definition of $\tilde{\partial}$). It follows readily that $H_j(P_n^{\mathbb{C}}) = Z, j = 0, 2, \ldots, 2n$, $H_j(P_n^{\mathbb{C}}) = 0, j = 1, 3, \ldots, 2n-1$.

(3) $X = P_n$, the real projective n-space: For a filtration, take $P_n \supseteq P_{n-1} \supseteq \cdots \supseteq P_0$. For each j, $0 \leqq j \leqq n$, we have one j-cell, namely $\sigma^{(j)} : C^{(j)} \to P_j$ where $C^{(j)}$ is the j-cube and $\sigma^{(j)}$ is the identification map. Thus the filtration gives a cell complex. We shall define a cellular map from (S^n, S^0) to (P_n, P_0) as follows: We begin by choosing a cell decomposition of S^n (different from the one defined in (1)). We write $S^n \supseteq S^{n-1} \supseteq \cdots \supseteq S^0$ where we regard S^{j-1} as the equator of S^j, $j = 1, 2, \ldots, n$. We define two j-cells $\sigma_1^{(j)}$, $\sigma_2^{(j)}$ for each dimension j. For $C_1^{(j)}$ we take the closed upper hemisphere U_j of S^j defined by $x_{j+1} \geqq 0$. For $C_2^{(j)}$ we take the closed lower hemisphere L_j of S^j defined by $x_{j+1} \leqq 0$. More precisely, since the standard j-cubes $C_1^{(j)}$, $C_2^{(j)}$ are homeomorphic to U_j, L_j, we let $\sigma_1^{(j)}$, $\sigma_2^{(j)}$ be the homeomorphisms taking $C_1^{(j)}$, $C_2^{(j)}$, onto U_j, L_j. Clearly $\sigma_1^{(j)}(\partial C_2^{(j)}) \subseteq S^{j-1}$ (because we regard S^{j-1} as the equator of S^j) and similarly $\sigma_2^{(j)}(\partial C_2^{(j)}) \subseteq S^{j-1}$. Moreover, it is clear that $S^j - S^{j-1} = \sigma_1^{(j)}(C_1^{0(j)}) \cup \sigma_2^{(j)}(C_2^{0(j)})$. So all the a cell complex conditions are satisfied. Now we define the cellular map f. Let $f: S^j \to P_j$ be the identification map, i.e. we regard P_j as S^j with x and $-x$ identified. From our construction, it is clear that f is indeed a cellular map. We now calculate the homology map $\tilde{f}$. The open j-cells of S^n are U_j^0 and L_j^0 (the open hemispheres). Moreover, the open j-cell of P_n can be regarded as U_j^0. Therefore $\tilde{f}(U_j^0) = U_j^0$. But f restricted to L_j^0 is the map taking x into $-x$. This map preserves (respectively, reserves) orientation if $j+1$ is even (respectively, odd), i.e. $\tilde{f}(L_j^0) = (-1)^{j+1} U_j^0$. Now, by Lemma (11), $\tilde{\partial}\sigma^{(j)} = k_j\sigma^{(j-1)}$ for some integer k_j. It follows easily by Lemma (12) that

$$k_j = \begin{cases} 2 & \text{if } j \text{ is even} \\ 0 & \text{if } j \text{ is odd} \end{cases}$$

since the fundamental cycle on S^j is $U_j + L_j$ and its image is either $2U_j$ or 0. Using this relation, the homology of P_n is readily calculated. We obtain the following result.

If n is even,

$$H_0(P_n, Z) \cong Z,\ H_1(P_n, Z) \cong H_3(P_n, Z) \cong \cdots \cong H_{n-1}(P_n, Z) \cong Z_2;$$

$$H_2(P_n, Z) \cong H_4(P_n, Z) \cong \cdots \cong H_n(P_n, Z) \cong \{0\}.$$

If n is odd,

$$H_0(P_n, Z) \cong H_n(P_n, Z) \cong Z;$$

$$H_1(P_n, Z) \cong H_3(P_n, Z) \cong \cdots \cong H_{n-2}(P_n, Z) \cong Z_2;$$

$$H_2(P_n, Z) \cong H_4(P_n, Z) \cong \cdots \cong H_{n-1}(P_n, Z) \cong \{0\}.$$

Cohomology Ring of P_n (mod 2)

We now calculate the cohomology ring (mod Z_2) of P_n, the real projective space. We saw above that $H_j(P_n, Z_2)$ is generated by one element, say α_j. We claim $\alpha_j \cdot \alpha_{n-j} = 1$. In fact α_j is represented by the map $\sigma^{(j)}$ which takes $C^{(j)}$ onto P_j. Similarly α_{n-j} is represented by the map $\sigma^{(n-j)}$ which takes $C^{(n-j)}$ onto P_{n-j}. As the maps $\sigma^{(j)}$ and $\sigma^{(n-j)}$ are nonsingular on the interiors of $C^{(j)}$ and $C^{(n-j)}$, it follows that $\alpha_j \cdot \alpha_{n-j}$ is equal to the number of intersections of P_j with P_{n-j} if we take P_j and P_{n-j} to be in general position in P_n. But $P_j \cap P_{n-j}$ is just one point and hence the assertion follows.

By Künneth's theorem, $H_j(P_n \times P_n, Z_2)$ has as generators $\alpha_0 \times \alpha_j, \ldots, \alpha_j \times \alpha_0$. By appealing to the product rule, we see that $(\alpha_0 \times \alpha_j) \cdot (\alpha_n \times \alpha_{n-j}) = 1$, $(\alpha_1 \times \alpha_{j-1}) \cdot (\alpha_{n-1} \times \alpha_{n-j+1}) = 1, \ldots, (\alpha_j \times \alpha_0) \cdot (\alpha_{n-j} \times \alpha_n) = 1$. All other intersections are 0. Consider the diagonal map of $P_n \to P_n \times P_n$. We calculate the image of α_j. We claim that $\text{diag}_*(\alpha_j) = (\alpha_0 \times \alpha_j) + (\alpha_1 \times \alpha_{j-1}) + \cdots + (\alpha_j \times \alpha_0)$. First of all, we know $\text{diag}_*(\alpha_j) = C_0(\alpha_0 \times \alpha_j) + \cdots + C_j(\alpha_j \times \alpha_0)$. If we can show that $\text{diag}_*(\alpha_j) \cdot (\alpha_l \times \alpha_m) = 1$ for all l such that $j + l + m = 2n$, we will have proved the assertion. Now, as before, $\text{diag}_*(\alpha_j) \cdot (\alpha_l \times \alpha_m)$ is just the number of points in $P_j \cap P_l \cap P_m$ when P_j, P_l, P_m are regarded as being in general position in $P_n \times P_n$. But $P_j \cap P_l \cap P_m = P_{j+l-n} = P_{j+l+m-2n} = P_0$ which is one point. Now introduce a dual basis $\beta_0, \beta_1, \ldots, \beta_n$ for the cohomology groups of P_n mod Z_2. Then β_j is the linear functional defined by $\beta_j(\alpha_i) = \delta_{ji}$, $i = 1, 2, \ldots, n$. We claim $\beta_j \cup \beta_k = \beta_{j+k}$. In fact,

$$\begin{aligned}
\beta_j \cup \beta_k(\alpha_{j+k}) &= \text{diag}^*(\beta_j \otimes \beta_k)(\alpha_{j+k}) \\
&= \beta_j \otimes \beta_k(\text{diag}_*(\alpha_{j+k})) \\
&= \beta_j \otimes \beta_k(\alpha_0 \times \alpha_{j+k} + \cdots + \alpha_j \times \alpha_k + \cdots + \alpha_{j+k} \times \alpha_0) \\
&= 1.
\end{aligned}$$

Similarly $\beta_j \cup \beta_k(\alpha_l) = 0$ if $l \neq j + k$. This proves the assertion. It follows that the cohomology ring of P_n (mod Z_2) is isomorphic to $Z_2[x]/(x^{n+1})$.

Exercise

Calculate the cohomology ring (mod Z) for $P_n^{\mathbb{C}}$. Using this and the Lefschetz fixed point theorem, prove that the spaces $P_{2k}^{\mathbb{C}}$ have the fixed point property. (A space X is said to have the fixed point property if every continuous self-mapping of X has a fixed point.)

CHAPTER III

Further Applications of Intersection Theory

This chapter is divided into two parts. The first part will be devoted to computing the homology groups and the cohomology ring of the Grassman manifold (see the definition below). The importance of the homology and cohomology structure of this manifold will become apparent later on when its central connection with the theory of fibre bundles is shown (see Chapters IV, VI). The method of computation itself is quite similar to the method used at the end of Chapter II in calculating the homology and cohomology structure of the projective spaces. The results of Chapter II are needed in the present chapter. On the other hand, in certain special cases (see below), the Grassman manifold turns out to be merely projective space. Hence, the results of this chapter can be regarded as a generalization of the results of Chapter II.

In the second part of this chapter, we shall prove the classical Poincare Duality Theorem by a differential method. Rather than using the cumbersome methods of combinational topology (see e.g. Alexandrov—*Combinatorial Topology*), we shall use as our main tool Morse Theory. This is in accord with the general philosophy developed in Chapters I and II, namely, to treat topological questions (as far as possible) by "differentiable" methods.

Part A: Homology of Grassman Manifolds

Let $G_{\varkappa,n}$ denote the set of $\varkappa$-dimensional linear subspaces ($\varkappa$-planes through the origin) of E^n. When we talk about real (resp. complex) $\varkappa$-planes in real (resp. complex) Euclidean n-space, we shall sometimes write $G^{\mathbb{R}}_{\varkappa,n}$ (resp. $G^{\mathbb{C}}_{\varkappa,n}$) for emphasis. It is easy to see that the orthogonal group $O(n)$ acts transitively on $G^{\mathbb{R}}_{\varkappa,n}$. Moreover, if π is a fixed $\varkappa$-plane and $\pi^{\perp}$ is its orthogonal complement, the subgroup of $O(n)$ mapping π onto itself splits up into the direct product $O(\varkappa) \times O(n-\varkappa)$, the first of which leaves $\pi^{\perp}$ pointwise fixed and the second of which leaves π pointwise fixed. Hence, we may identify $G^{R}_{\varkappa,n}$ with the (analytic) manifold $O(n)/O(\varkappa) \times O(n-\varkappa)$. In this way, we see that $G^{R}_{\varkappa,n}$ is a compact manifold of dimension $\varkappa(n-\varkappa)$. Similarly, we can identify $G^{C}_{\varkappa,n}$ with $U(n)/U(\varkappa) \times U(n-\varkappa)$ where $U(n)$ is the unitary group. Thus, $G^{C}_{\varkappa,n}$ is a compact manifold of complex dimension $\varkappa(n-\varkappa)$ and hence of real dimension $2\varkappa(n-\varkappa)$.

Observe that the correspondence between any $\varkappa$-plane and its orthogonal complement gives rise to a $1-1$ correspondence between $G_{\varkappa,n}$ and $G_{n-\varkappa,n}$. Note also that $G_{1,n} = P_{n-1}$, ordinary projective space.

In E^n, we pick an ascending sequence $E^0 \subseteq E^1 \subseteq \cdots \subseteq E^{n-1} \subseteq E^n$, where E^j is a subspace of E^n of dimension j. For a given plane $\pi \in G_{\varkappa,n}$ we set

$$m_j(\pi) = \inf_m \{m \mid \dim(\pi \cap E^m) \geqq j\}, \qquad j = 1, 2, \ldots, \varkappa.$$

From now on, we shall write πE^m instead of $\pi \cap E^m$. Observe that

$$m_1(\pi) < m_2(\pi) < \cdots < m_\varkappa(\pi).$$

We now set

$$C_{m_1,\ldots,m_\varkappa}(E^0, E^1, \ldots, E^n) = \{\pi \in G_{\varkappa,n} \mid m_j(\pi) = m_j, j = 1, 2, \ldots, \varkappa\}.$$

For brevity, we write $C_{m_1,\ldots,m_\varkappa}$ instead of $C_{m_1,\ldots,m_\varkappa}(E^0, E^1, \ldots, E^n)$. $C_{m_1,\ldots,m_\varkappa}$ is called the *Schubert Cell* corresponding to the multiindex $(m_1, m_2, \ldots, m_\varkappa)$.

Lemma (1): The Schubert cells are topological cells. Moreover, the collection of all Schubert cells gives a cell decomposition of $G_{\varkappa,n}$.

Proof: Let $\pi \in C_{m_1,\ldots,m_\varkappa}$. Then πE^{m_1} is one-dimensional and $\pi E^{m_1-1} = \{0\}$. As a basis for E^n, we can take the vectors

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0), \qquad i = 1, \ldots, n,$$

where the "1" in e_i appears in the i-th place. We can assume moreover that $E^\varkappa$ is generated by the vectors $e_1, e_2, \ldots, e_\varkappa$. Then πE^{m_1} is generated by a vector of the form $x = (x_1, \ldots, x_{m_1}, 0, \ldots, 0)$ where $x_{m_1} \neq 0$. In fact, if x_{m_1} were equal to zero, then we would have $\dim(\pi E^{m_1-1}) = 1$, which is impossible. Hence, we may assume $x_{m_1} = 1$. Now, πE^{m_2} is 2-dimensional and πE^{m_2-1} has dimension less than two. Then πE^{m_2} is generated by x and a vector of the form

$$\eta = (\eta_1, \ldots, \eta_{m_1}, \eta_{m_1+1}, \ldots, \eta_{m_2}, 0, \ldots, 0).$$

As the linear transformation $x \rightarrow x$, $\eta \rightarrow \eta - \eta_{m_1}x$ is invertible, it follows that the vectors x and $y = \eta - \eta_{m_1}x$ also generate πE^{m_2}. Now y is of the form

$$y = (y_1, \ldots, y_{m-1}, 0, y_{m_1+1}, \ldots, y_{m_2}, 0, \ldots, 0).$$

As above, $y_{m_2} \neq 0$ and hence we may suppose $y_{m_2} = 1$. We now continue the procedure, finding a basis for $\pi E^{m_2}, \ldots, \pi E^{m_\varkappa}$. We arrive at a $\varkappa \times n$ matrix

$$\begin{array}{lllllll}
x_1 \cdots x_{m_1-1} & 1 & 0 & \cdots & & & 0 \\
y_1 \cdots y_{m_1-1} & 0 & y_{m_1+1} \cdots y_{m_2-1} & 1 & 0 & \cdots & 0 \\
z_1 \cdots z_{m_1-1} & 0 & z_{m_1+1} \cdots z_{m_2-1} & 0 & z_{m_2+1} \cdots z_{m_3-1} & 1 & 0 \cdots 0 \\
\vdots & \vdots & \vdots & & \vdots & & \vdots
\end{array}$$

The first j rows represent j vectors generating πE^j $(j = 1, 2, \ldots, \varkappa)$. The components of this matrix are real or complex parameters (depending on whether we are considering $G^{\mathbb{R}}_{\varkappa,n}$ or $G^{\mathbb{C}}_{\varkappa,n}$) and can be chosen arbitrarily otherwise. Thus, we have defined coordinates on $C_{m_1,\ldots,m_\varkappa}$ and hence $C_{m_1,\ldots,m_\varkappa}$ becomes an open cell (although not necessarily an open subset of $G_{\varkappa,n}$) of dimension

$$(m_1 - 1) + (m_2 - 2) + \cdots + (m_\varkappa - \varkappa) = (m_1 + \cdots + m_\varkappa) - \frac{\varkappa(\varkappa + 1)}{2}.$$

There is a unique Schubert cell of dimension 0, namely $C_{1,2,\ldots,\varkappa}$ and a unique Schubert cell of dimension $\varkappa(n - \varkappa)$, namely $C_{n-\varkappa+1,\ldots,n}$.

We shall show that the Schubert cells give a cell decomposition of the space $G_{\varkappa,n}$. Denote by D^α the closed α-dimensional unit disc and by B^α the interior of D^α. For brevity, write $D_i = D^{m_i-1}$ and finally, write $D = D_1 \times D_2 \times \cdots \times D_\varkappa$, $B =$ interior of D. Clearly, D is homeomorphic to a closed disc of dimension $(m_1 + \cdots + m_\varkappa) - \varkappa(\varkappa + 1)/2$. We want to define a continuous map $\phi : D \rightarrow G_{\varkappa,n}$ such that

(1) $\phi(D) = C_{m_1,\ldots,m_\varkappa} \cup \{$Schubert cells of lower dimension$\}$

(2) ϕ maps B homeomorphically onto $C_{m_1,\ldots,m_\varkappa}$

(3) ϕ maps $\partial D = D - B$ into the union of Schubert cells of lower dimension.

Consider the set of all unit vectors $u \in E^{m_1}$ such that the m_1-th component $(u)_{m_1} \geqq 0$; this set is homeomorphic with $D_1 = D^{m_1-1}$. Let $R^{(1)}_u$ be a rotation of E^n, depending continuously on u, such that

$$\text{(a)} \quad R^{(1)}_u e_{m_1} = u, \qquad \text{(b)} \quad R^{(1)}_u \mid E^n \ominus E^{m_1} = \text{identity}.$$

Consider the set of all unit vectors $v \in E^{m_2} \ominus \{e_{m_1}\}$ such that $(v)_{m_2} \geqq 0$; this set is homeomorphic with $D_2 = D^{m_2-2}$. Let $R_v^{(2)}$ be a rotation of E^n, depending continuously on v, such that

$$\text{(a)}\quad R_v^{(2)} e_{m_2} = v, \qquad \text{(b)}\quad R_v^{(2)} \mid E^n \ominus [E^{m_2} \ominus \{e_{m_1}\}] = \text{identity}.$$

Consider the set of all unit vectors $w \in E^{m_3} \ominus \{e_{m_1}, e_{m_2}\}$ with $(w)_{m_3} \geqq 0$; this set is homeomorphic with $D_3 = D^{m_3-3}$. Let $R_w^{(3)}$ be a rotation of E^n, depending continuously on w, such that

$$\text{(a)}\quad R_w^{(3)} e_{m_3} = w, \qquad \text{(b)}\quad R_w^{(3)} \mid E^n \ominus [E^{m_3} \ominus \{e_{m_1}, e_{m_2}\}] = \text{identity}.$$

We continue this process $\varkappa$ times. We define ϕ as follows:

$$\phi\,((u, v, w, \ldots)) = \{u,\, R_u^{(1)} v,\, R_u^{(1)} R_v^{(2)} w, \ldots\} = \pi,$$

i.e. π is the plane spanned by the vectors u, $R_u^{(1)} v$, $R_u^{(1)} R_v^{(2)} w, \ldots$ The fact that π is actually a $\varkappa$-plane follows from the fact that the generating vectors of π defined above are mutually orthogonal. Let us show, for example, that $(u, R_u^{(1)} v) = 0$: $(u, R_u^{(1)} v) = (R_u^{(1)} e_{m_1}, R_u^{(1)} v) = (e_{m_1}, v) = 0$. Note that we use here the fact that $R_u^{(1)}$ is unitary.

Clearly, ϕ is continuous. We shall prove (1) above. By the construction, we have $\dim (\pi E^{m_i}) \geqq i$ for $i = 1, 2, \ldots, \varkappa$. By definition, we have then $m_i(\phi(d)) \leqq m_i$ for $d \in D$, $i = 1, 2, \ldots, \varkappa$. But this means

$$\phi(d) \in \bigcup_{(l_1, \ldots, l_\varkappa)} C_{l_1, \ldots, l_\varkappa}$$

where the summation extends over all multi-indices $(l_1, \ldots, l_\varkappa)$ satisfying $l_1 \leqq m_1, l_2 \leqq m_2, \ldots, l_\varkappa \leqq m_\varkappa$. This proves (1). To prove (2), we shall construct a continuous map

$$\psi : C_{m_1, \ldots, m_\varkappa} \to B$$

which will clearly be a 2-sided inverse of the map $\phi | B$. So let $\pi \in C_{m_1, \ldots, m_\varkappa}$. We know $\dim (\pi E^{m_1}) = 1$ and hence there is a *unique* unit vector $u \in \pi E^{m_1}$ with $(u)_{m_1} > 0$ such that u generates πE^{m_1}. As $\dim (\pi E^{m_2}) = 2$, πE^{m_2} is generated by u and a unit vector $v \in \{(R_u^{(1)})^{-1} \pi\} \cap \{E^{m_2} - \{e_{m_1}\}\}$. If we require $(v)_{m_2} > 0$, then v is uniquely determined by π. We continue the procedure in the obvious way and define

$$\psi(\pi) = (u, v, \ldots).$$

It is clear that ψ has the required properties.

Statement (3) above is easily verified also and the proof of Lemma 1 is completed.

To calculate the homology of $G_{\varkappa,n}$, we shall apply the previously developed theory. We must therefore look at the boundary relations.

In the complex case (i.e. $G^{\mathbb{C}}_{\varkappa,n}$), there are only even dimensional cells. Hence, all boundary relations are trivial.

In the real case, we let $(\partial D)_1$ be the part of the boundary ∂D which is mapped by ϕ into the union of Schubert cells, each of whose dimensions is exactly one less than the dimension of $C_{m_1,\dots,m_\varkappa}$. It is easy to see that such a cell has the form $C_{m_1,\dots,m_{s-1},m_s-1,m_{s+1},\dots,m_\varkappa}$ where $m_s - 1 > m_{s-1}$. Then, if $d \in (\partial D)_1$ and $d = (u, v, w, \dots)$, we assert that not more than one of the unit vectors $u, v, w, \dots$ can lie on the boundary of its corresponding hemisphere $D_1, D_2, D_3, \dots$ Suppose, for example, that u lies on the boundary of D_1 and v lies on the boundary of D_2. Then $\dim(\pi E^{m_1-1}) \geqq 1$, $\dim(\pi E^{m_2-1}) \geqq 2$ and hence $m_1(\pi) \leqq m_1 - 1$, $m_2(\pi) \leqq m_2 - 1$ contradicting our dimensional assumption. It follows readily that on $(\partial D)_1$, the map ϕ is exactly 2 to 1.

Let now $C = C_{m_1,\dots,m_\varkappa}$. Then $\tilde{\partial} C = \sum n_i C^i$ where the C^i are the cells of dimension one smaller than the dimension of C and the n_i are integers (or integers mod 2 if we are interested in homology mod 2) determined as follows (see the general discussion in Chapter 2): if C^i is not contained in $\phi(\partial D)$, then $n_i = 0$. If $C^i \subseteq \phi(\partial D)$, consider the map $\phi : \phi^{-1}(C^i) \to C^i$; then n_i is the degree of this map. Thus, n_i is the algebraic number of points in ∂D mapping into a single point of C^i. Since ϕ has been shown to be two to one on $(\partial D)_1$, it follows that, modulo 2, $\tilde{\partial} C = 0$.

We thus have the following

Theorem (1): $H_l\,(G^{\mathbb{C}}_{\varkappa,n}, Z)$ is isomorphic to the direct sum of N_l copies of Z where N_l is the number of Schubert cells of dimension l. $H_l(G^{\mathbb{R}}_{\varkappa,n}, Z_2)$ is isomorphic to the direct sum of N_l copies of Z_2 where N_l is as above.

Corollary: By duality, we see that $H^l(G^{\mathbb{C}}_{\varkappa,n}, Z) \cong H_l(G^{\mathbb{C}}_{\varkappa,n}, Z)$ and $H^l\,(G^{\mathbb{R}}_{\varkappa,n}, Z_2) \cong H_l(G^{\mathbb{R}}_{\varkappa,n}, Z_2)$. Hence, Theorem 1 gives us the cohomology groups of $G_{\varkappa,n}$.

We shall now carry out some intersection theoretic calculations on the manifold $G_{\varkappa,n}$. Denote by $E^{\beta}_{\perp}$ the orthogonal complement of E^{β} in E^n $(\beta \leqq n)$. Let $\pi \in G_{\varkappa,n}$ satisfy the following conditions:

$$\begin{array}{ll} \dim(\pi E^{m_1}) \geqq 1 & \dim(\pi E_{\perp}^{m_1-1}) \geqq \varkappa \\ \dim(\pi E^{m_2}) \geqq 2 & \dim(\pi E_{\perp}^{m_2-1}) \geqq \varkappa - 1 \\ \vdots & \vdots \\ \dim(\pi E^{m_\varkappa}) \geqq \varkappa & \dim(\pi E_{\perp}^{m_\varkappa-1}) \geqq 1 \end{array} \qquad (*)$$

The first condition tells us that π contains e_{m_1}, the unique vector (up to scalar multiples) in $E^{m_1} \cap E^{m_1-1}$. The second condition tells us that π contains e_{m_2}, the unique vector (up to scalar multiples) in $E^{m_2} \cap E^{m_2-1}$. In general, the j-th condition tells us that π contains e_{m_j}, the unique vector (up to scalar multiples) in $E^{m_j} \cap E^{m_j-1}$. Hence $\pi = \{e_{m_1}, e_{m_2}, \ldots, e_{m_\varkappa}\}$ and thus there is a unique plane π satisfying (*). It is therefore plain that (*) implies that fact $\pi \in C_{m_1,\ldots,m_\varkappa}$. Now, since

$$E^0 \subseteq E^1 \subseteq \cdots \subseteq E^n$$

we have also

$$E_\perp^0 \supseteq E_\perp^1 \supseteq \cdots \supseteq E_\perp^n.$$

Let $F^j = E_\perp^{n-j}$ $(j = 0, 1, \ldots, n)$. Then

$$F^0 \subseteq F^1 \subseteq \cdots \subseteq F^n.$$

We define

$$\hat{m}_j(\pi) = \inf_m \{m \mid \dim(\pi F^m) \geqq j\}$$

and we put

$$C^\perp_{m_1,\ldots,m_\varkappa} = \{\pi \in G_{\varkappa,n} \mid \hat{m}_j(\pi) = m_j\}.$$

Observe that $C^\perp_{m_1,\ldots,m_\varkappa} = C_{m_1,\ldots,m_\varkappa}(F^0, F^1, \ldots, F^n)$. Now, conditions (*) clearly imply $\pi \in C^\perp_{n-m_\varkappa+1,\ldots,n-m_1+1}$. We also have, by direct calculation,

$$\dim(C_{m_1,\ldots,m_\varkappa}) + \dim(C^\perp_{n-m_\varkappa+1,\ldots,n-m_1+1}) = \dim G_{\varkappa,n}.$$

It can moreover be shown that the cells $C_{m_1,\ldots,m_\varkappa}$ and $C^\perp_{n-m_\varkappa+1,\ldots,n-m_1+1}$ are in general position in $G_{\varkappa,n}$. We call each of these cells the *dual Shubert cell* of the other. It follows then that

$$C_{m_1,\ldots,m_\varkappa} \cdot C^\perp_{n-m_\varkappa+1,\ldots,n-m_1+1} = 1.$$

We shall now show that the set of $\pi \in G_{\varkappa,n}$ satisfying conditions

$$\begin{array}{ll}
\dim(\pi E^{m_1}) \geqq 1 & \dim(\pi E_\perp^{l_1-1}) \geqq \varkappa \\
\dim(\pi E^{m_2}) \geqq 2 & \dim(\pi E_\perp^{l_2-1}) \geqq \varkappa - 1 \\
\vdots & \vdots \\
\dim(\pi E^{m_\varkappa}) \geqq \varkappa & \dim(\pi E_\perp^{l_\varkappa-1}) \geqq 1
\end{array} \tag{**}$$

in addition to the condition $\sum_{i=1}^{\varkappa} l_i = \sum_{i=1}^{\varkappa} m_i$, is nonempty only if $l_j = m_j$ $(j = 1, \ldots, \varkappa)$.

Suppose, for example, $l_1 > m_1$. Then $E^{m_1} \cap E_\perp^{l_1-1} = \{0\}$ and the first line of (**) would be violated. Therefore, $l_1 \leqq m_\perp^1$. Similarly, $l_2 \leqq m_2, \ldots, l_\varkappa \leqq m_\varkappa$. As $\sum_{i=1}^{\varkappa} l_i = \sum_{i=1}^{\varkappa} m_i$, we must have $l_j = m_j$ $(j = 1, \ldots, \varkappa)$.

Thus, if $(l_1, l_2, \ldots, l_\varkappa)$ is a $\varkappa$-tuple of strictly increasing integers such that $l_{j_0} \neq m_{j_0}$ for some $j_0 \in \{1, 2, \ldots, \varkappa\}$, while $\sum_{i=1}^{\varkappa} l_i = \sum_{i=1}^{\varkappa} m_i$, then

$$C_{m_1, \ldots, m_\varkappa} \cdot C^{\perp}_{n-l_\varkappa+1, \ldots, n-l_1+1} = 0.$$

We have already observed that the Schubert cells depend upon the sequence $E^0 \subseteq E^1 \subseteq \cdots \subseteq E^n$. Suppose then we have two such sequences,

$$E^0 \subseteq E^1 \subseteq \cdots \subseteq E^n \quad \text{and} \quad \tilde{E}^0 \subseteq \tilde{E}^1 \subseteq \cdots \subseteq \tilde{E}^n$$

where $E^0 = \tilde{E}^0$, $E^n = \tilde{E}^n$ and where E^i and $\tilde{E}^i$ are Euclidean i-dimensional spaces embedded in Euclidean n-space. Let

$$C_{m_1, \ldots, m_\varkappa} = C_{m_1, \ldots, m_\varkappa}(E^0, \ldots, E^n),$$

$$\tilde{C}_{m_1, \ldots, m_\varkappa} = C_{m_1, \ldots, m_\varkappa}(\tilde{E}^0, \ldots, \tilde{E}^n).$$

We have the following

Lemma (2): $C_{m_1, \ldots, m_\varkappa}$ and $\tilde{C}_{m_1, \ldots, m_\varkappa}$ define the same homology class in $H(G_{\varkappa,n})$.

Proof: The sequence $E^0 \subseteq \cdots \subseteq E^n$ gives rise to a basis $e_1, e_2, \ldots, e_n$ where $E^i = \{e_1, \ldots, e_i\}$. The sequence $\tilde{E}^0 \subseteq \tilde{E}^1 \subseteq \cdots \subseteq \tilde{E}^n$ gives risc to a similar basis $\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n$. Let $T\colon E^n \to \tilde{E}^n = E^n$ be the linear transformation defined by

$$Te_i = \tilde{e}_i, \qquad i = 1, \ldots, n.$$

The map T induces naturally a map

$$T' : G_{\varkappa,n} \to G_{\varkappa,n}$$

which in turn induces a map

$$T_* : H(G_{\varkappa,n}) \to H(G_{\varkappa,n}).$$

Clearly, $T'(C_{m_1, \ldots, m_\varkappa}) = \tilde{C}_{m_1, \ldots, m_\varkappa}$. If we are dealing with $G^{\mathbb{C}}_{\varkappa,n}$, then $T \in GL(n, \mathbb{C})$; since $GL(n, \mathbb{C})$ is arcwise connected, T can be joined to I (the identity matrix) by a path in $GL(n, \mathbb{C})$. This induces a homotopy between T' and I' (the identity map of $G_{\varkappa,n}$). Hence, T_* is the identity map of $H(G^{\mathbb{C}}_{\varkappa,n})$ and the assertion is proved. If we are dealing with $G^{\mathbb{R}}_{\varkappa,n}$, then $T \in GL(n, \mathbb{R})$ and then either $\det T > 0$ or $\det T < 0$. If $\det T > 0$, then T can be joined to I by a path in $GL(n, \mathbb{R})$ since T lies in the connected component of $GL(n, \mathbb{R})$ which contains I. In this case, T_* is the identity map, as above. If $\det T < 0$, then T can be joined to the matrix

$$\hat{I} = \begin{pmatrix} -1 & & & & \\ & 1 & & 0 & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

by a path in $GL(n, \mathbb{R})$ since T lies in the connected component of $GL(n, \mathbb{R})$ which contains $\hat{I}$. But the map

$$\hat{I}' : G_{\varkappa,n} \to G_{\varkappa,n}$$

induced by $\hat{I}$ is the identity map on $G_{\varkappa,n}$ because we have assumed that $G_{\varkappa,n}$ consists of unoriented $\varkappa$-planes. Hence, as above, T_* is the identity map. This proves the lemma.

Observing that $C^{\perp}_{m_1,\dots,m_\varkappa} = C_{m_1,\dots,m_\varkappa}(F^0, F^1, \dots, F^n)$, we have

Theorem (2): Let $m_1 + m_2 + \cdots + m_\varkappa = l_1 + l_2 + \cdots + l_\varkappa$. Then

$$C_{m_1,\dots,m_\varkappa} \cdot C_{n-l_\varkappa+1,\dots,n-l_1+1} = \delta^{l_1 l_2 \dots l_\varkappa}_{m_1 m_2 \dots m_\varkappa}$$

where

$$\delta^{l_1 \dots l_\varkappa}_{m_1 \dots m_\varkappa} = 1 \quad \text{if} \quad l_i = m_i, \qquad (i = 1, \dots, \varkappa),$$
$$= 0 \quad \text{otherwise}.$$

That is, the intersection of each Schubert cell with its dual cell is 1, while all other intersections are 0.

We now consider the cohomology ring of $G_{\varkappa,n}$. Let $P^{(1)}_{n-1} \times \cdots \times P^{(\varkappa)}_{n-1}$ be the product of $\varkappa$ copies of projective space P_{n-1}. From a result of Chapter 2,

$$H^*(P^{C}_{n-1}, \mathbb{Z}) \cong \mathbb{Z}(z)/(z^n), \qquad H^*(P^{R}_{n-1}, \mathbb{Z}_2) \cong \mathbb{Z}_2(z)/(z^n).$$

From Künneth's Theorem,

$$H^*(P^{\mathbb{C}}_{n-1} \times \cdots \times P^{\mathbb{C}}_{n-1}) \cong \mathbb{Z}(z_1, \dots, z_\varkappa)/(z_1^n, \dots, z_\varkappa^n)$$
$$H^*(P^{\mathbb{R}}_{n-1} \times \cdots \times P^{\mathbb{R}}_{n-1}) \cong \mathbb{Z}_2(z_1, \dots, z_\varkappa)/(z_1^n, \dots, z_\varkappa^n).$$

Again from Künneth's theorem, the homology classes of $P^{(1)}_{n-1} \times \cdots \times P^{(\varkappa)}_{n-1}$ can be represented by products

$$P^{(1)}_{l_1} \times P^{(2)}_{l_2} \times \cdots \times P^{(\varkappa)}_{l_\varkappa}, \quad l_i \leqq n-1; \quad (i = 1, 2, \dots, \varkappa).$$

We define a "natural" map

$$N: P^{(1)}_{n-1} \times \cdots \times P^{(\varkappa)}_{n-1} \to G_{\varkappa,\varkappa n}$$

as follows. An element of $P_{n-1}^{(i)}$ is the equivalence class of an n-vector $(x_0, x_1, \ldots, x_{n-1})$. This equivalence class gives a one-dimensional subspace of a Euclidean space E_i^n $(i = 1, 2, \ldots, \varkappa)$.

Let $E^{\varkappa n} = E_1^n \oplus \cdots \oplus E_\varkappa^n$. If $v_i \in P_{n-1}^{(i)}$, define

$$N((v_1, v_2, \ldots, v_\varkappa)) = \{v_1, v_2, \ldots, v_\varkappa\} = \pi$$

i.e. π is the plane in $E^{\varkappa n}$ generated by the vectors $v_1, v_2, \ldots, v_\varkappa$.

Let $\{P_{\alpha_1}^{(1)} \times \cdots \times P_{\alpha_\varkappa}^{(\varkappa)}\}$ denote the homology class of $P_{\alpha_1}^{(1)} \times \cdots \times P_{\alpha_\varkappa}^{(\varkappa)}$ in $H(P_{n-1}^{(1)} \times \cdots \times P_{n-1}^{(n)})$; similarly, let $\{C_{\beta_1, \ldots, \beta_\varkappa}\}$ denote the homology class of $C_{\beta_1, \ldots, \beta_\varkappa}$ in $H(G_{\varkappa,n})$.

Suppose that we introduce a lexicographic ordering of the homology classes of a Grassmann manifold by writing $\{C_{m_1, \ldots, m_k}\} > \{C_{n_1, \ldots, n_k}\}$ if $m_1 > n_1$, or if $m_1 = n_1$ while $m_2 > n_2$, or if $m_1 = n_1$ and $m_2 = n_2$ while $m_3 > n_3$, etc. Then, if $l_1 \le l_2 \le \cdots \le l_x \le n - 1$, we will show that (†) $N_*(\{P_{l_1}^{(1)} \times \cdots \times P_{l_x}^{(x)}\}) = \{C_{l_1+1, l_2+2, \ldots, l_x+x}\}$ + lower classes of like dimension, where N_* denotes the mapping of homology induced by the continuous mapping N. We prove this as follows. Let p, m, k be positive integers such that $p \ge m + k$; let $C_{m_1, \ldots, m_k}$ be a Schubert cell of the Grassmann manifold $G_{k,p}$; and suppose that $m_k < m + k$. Consider the mapping M: $G_{k,p} \times P^m \to G_{k,p+m+1}$ which maps $\pi \times (x) \in G_{k,p} \times P^m$ into the hyperplane π_x in E^{p+m+1} generated by the vectors $[y, x]$, where $y \in \pi$. We shall show, using intersection theory, that
(††) $M_*(\{C_{m_1, \ldots, m_k}\} \times \{P^m\}) = a\{C_{m_1, \ldots, m_k, m+k}\}$ + lower classes of like dimension, and that the coefficient a is equal to 1. Indeed, regard $E^1 \subseteq E^2 \subseteq E^3 \subseteq \ldots$ as in the preceding paragraphs, let $n_1 < n_2 < \cdots n_{k+1}$, and suppose that π_x lies in the dual cell of $C_{n_1, \ldots, n_{k+1}}$, so that $\dim(\pi_x E_{n_j-1}^{\perp})$ $\ge k + 2 - j$. Since $\pi \subseteq \pi_x$ is a hyperplane of codimension 1, it follows that $\dim(\pi E_{n_j-1}^{\perp}) \ge k + 1 - j$, so that $\dim(\pi E_{n_j-1}^{\perp} E_{m_j}) \ge 1$ and $n_j \le m_j$, $j = 1, \ldots, k$. Thus, if the image $M(C_{m1, \ldots, m_k} \times P^m)$ has a non-zero intersection with the dual cell of $C_{n_1, \ldots, n_{k+1}}$, we have $\{C_{n_1, \ldots, n_{k+1}}\} < \{C_{m_1, \ldots, m_k. m+k}\}$ in the lexicographic ordering. Using Theorem 2, we may at once conclude that (††) is true; however, it remains for us to evaluate the coefficient a. To this end, we proceed as follows. Let Tx: $E^{m+1} \to E^p$ be the linear transformation defined by

$$Tx = (x_1, \ldots, x_{n_1-1}, 0, x_{n_1}, \ldots, x_{n_2-2}, 0, \ldots, x_{m-1}, x_m, 0, \ldots, 0).$$

Let M^t: $G_{k,p} \times P^m \to G_{k,p+m+1}$ map $\pi \times (x)$ into the plane $\pi_x^{(t)}$ generated by the vectors $[y + tTx, x]$, where $y \in \pi$. Then $M^0 = M$, while M^t clearly depends continuously upn t. To prove that $a = 1$ in (††) we have therefore only to show that the intersection number of $M_*^{(1)}(\{C_{m_1 \cdots m_k}\} \times \{P^m\})$ with the dual class of $C_{m_1, \cdots, m_k, m+k}$ is 1. But if $\pi_x^{(1)}$ belongs to the corresponding dual cell, we have $\dim (\pi_x^{(1)} E_{m_j - 1}^{\perp}) \geq k + 2 - j$ as above, and it follows as above that $\dim (\pi_x^{(1)} E_{m_j} E_{m_j - 1}^{\perp}) \geq 1$. Thus the basis vector e_{m_j} belongs to $\pi_x^{(1)}$ for $1 \leq j \leq k$. We may therefore write $[e_{m_j}, 0] = [y + zTx, zx]$ for some $y \in \pi$ and scalar z, from which it follows that $z = 0$, $e_{m_j} \in \pi$, $1 \leq j \leq k$, so that π is uniquely determined. Moreover, since $\dim (\pi_x^{(1)} E_{m+k}^{\perp}) = 1$, $Tx = 0$; thus the $m + 1$-dimensional vector x is a multiple of the basis vector e_{m+1}, i.e., x is determined up to a scalar factor. The intersection with which we are concerned thus consists of precisely one point; since it is readily seen to be transverse, the intersection number a of (††) is equal to 1, and (††) is fully proved.

It is clear that (†) follows from (††), by induction on x; thus (†) is also established.

We assert further that

$$N_* (\{P_{l_1}^{(1)} \times \cdots \times P_{l_\varkappa}^{(\varkappa)}\}) = \{C_{l'_1 + 1, l'_2 + 2, \ldots, l'_\varkappa + \varkappa}\}$$
$$+ \text{ lower classes of like dimension,}$$

where $l_1, l_2, \ldots, l_\varkappa$ is an arbitrary sequence of integers such that $l_i \leqq n - 1$ $(i = 1, \ldots, \varkappa)$ and where the sequence $l'_1, l'_2, \ldots, l'_n$ is the sequence $l_1, l_2, \ldots, l_n$ rearranged in ascending order. To see this, let

$$\psi : P_{n-1}^{(1)} \times \cdots \times P_{n-1}^{(\varkappa)} \rightarrow P_{n-1}^{(1)} \times \cdots \times P_{n-1}^{(\varkappa)}$$

be defined by $\psi((v_1, \ldots, v_\varkappa)) = (v_{i_1}, \ldots, v_{i_\varkappa})$. Then, ψ permutes the vectors $v_1, \ldots, v_\varkappa$ among themselves, and

$$\psi (P_{l_1}^{(1)} \times \cdots \times P_{l_\varkappa}^{(\varkappa)}) = P_{l_{i_1}}^{(1)} \times \cdots \times P_{l_{i_\varkappa}}^{(\varkappa)}.$$

By choosing ψ appropriately, we can insure $l_{i_1} \leqq l_{i_2} \leqq \cdots \leqq l_{i_\varkappa}$.

Observe that $N \circ \psi = T' \circ N$ where $T' : G_{\varkappa, n} \rightarrow G_{\varkappa, n}$ is induced by an appropriate (nonsingular) linear transformation of $E^{\varkappa n}$. From the proof of Lemma 2, it follows that

$$N_* \circ \psi_* = N_*, \quad \psi^* \circ N^* = N^*$$

where N^*, ψ^* are the induced mappings in chomology. The desired result follows directly.

Let now $\gamma_{m_1, \ldots, m_\varkappa}$ be the cohomology class dual to $\{C_{m_1, \ldots, m_\varkappa}\}$. By the above assertion, it follows that $N^* \gamma_{m_1, \ldots, m_\varkappa}(\{P_{l_1}^{(1)} \times \cdots \times P_{l_\varkappa}^{(\varkappa)}\}) = 1$ if $(l_1, l_2, \ldots, l_\varkappa)$ is a permutation of $(m_1 - 1, m_2 - 2, \ldots, m_\varkappa - \varkappa)$ and equals zero if $(\mathrm{l}_1, \ldots, \mathrm{l}_x) < (\mathrm{m}_1 - 1, \mathrm{m}_2 - 2, \ldots, \mathrm{m}_x - x)$ in the lexicographic ordering of x-tuples introduced above.

Let $\delta_l^{(i)}$ be the cohomology class in $H^l(P_{n-1}^{(i)})$ dual to $\{P_l^{(i)}\}$ $(i = 1, \ldots, \varkappa)$.

Any cohomology class of $H^*(P_{n-1}^{(1)} \times \cdots \times P_{n-1}^{(x)})$ can be written uniquely as a combination of the basis vectors $\delta_{l_1}^{(1)} \otimes \cdots \otimes \delta_{l_x}^{(x)}$. Suppose that we order these basis vectors by the same lexicographic ordering of $(l_1, \ldots, l_x)$. We know from Chapter 2 that $\delta_l^{(i)} = (\delta_1^{(i)})^l = \delta_1^{(i)} \cup \delta_1^{(i)} \cup \cdots \cup \delta_1^{(i)}$ (l times). It follows from the above that

$$N^*\gamma_{m_1, \ldots, m_\varkappa} = \mathrm{Symm}\,(\delta_{m_1-1}^{(1)} \otimes \delta_{m_2-2}^{(2)} \otimes \cdots \otimes \delta_{m_\varkappa-\varkappa}^{(\varkappa)})$$
$$+ \text{ a combination of higher basis vectors}$$

where "Symm" is the symmetrization operator. Writing $\delta_{m_j-j}^{(j)} = (\delta_1^{(j)})^{m_j-j}$, we get

$$N^*\gamma_{m_1, \ldots, m_\varkappa} = \mathrm{Symm}\,((\delta_1^{(1)})^{m_1-1} \otimes \cdots \otimes (\delta_1^{(\varkappa)})^{m_\varkappa-\varkappa})$$
$$+ \text{ a combination of higher basis vectors.}$$

From this, the following theorem is evident.

Theorem (3): N^* maps $H^*(G_{\varkappa.\varkappa n}^{\mathcal{C}})$ onto the subring of $Z(z_1, \ldots, z_\varkappa)/(z_1^n, \ldots, z_\varkappa^n)$ consisting of symmetric polynomials. Also, N^* maps $H^*(G_{\varkappa,\varkappa n}^{R})$ onto the subring of $Z_2(z_1, \ldots, z_\varkappa)/(z_1^n, \ldots, z_\varkappa^n)$ consisting of symmetric polynomials.

Consider now an increasing sequence of Euclidean spaces $E^\varkappa \subseteq E^{\varkappa+1} \subseteq \cdots \subseteq E^N \subseteq \cdots$. Then,

$$E^n = \{(x_1, \ldots, x_n)\,(x_n \in R)\}, \qquad n \geqq \varkappa.$$

The inclusion $E^n \subseteq E^{n+1}$ gives rise to an inclusion map

$$i_n : G_{\varkappa,n} \to G_{\varkappa,n+1}$$

which is clearly a cellular map. Since there are no nontrivial boundary relations among the Schubert cells, it follows that

$$(i_n)_* : H(G_{\varkappa,n}) \to H(G_{\varkappa,n+1})$$

is an injective homomorphism.

Now let E^∞ denote the set of all sequences of real numbers, all but a finite number of which are 0. Then E^∞ can be viewed as the direct limit space of the sequence $E^\varkappa \subseteq E^{\varkappa+1} \subseteq \cdots$. Then define $G_{\varkappa,\infty}$ to be the direct limit space of the sequence $G_{\varkappa,\varkappa} \subseteq G_{\varkappa,\varkappa+1} \subseteq \cdots$. Clearly, $G_{\varkappa,\infty}$ can be regarded as the set of $\varkappa$-planes in E^∞. Using the fact that the maps $(i_n)_*$ are injective, it can be shown that

$$H(G_{\varkappa,\infty}) = \operatorname*{dir\,lim}_{n\to\infty} H(G_{\varkappa,n}).$$

Using this, the following theorem can be demonstrated.

Theorem (4): $H^*(G_{\varkappa,\infty})$ is isomorphic to the ring of all symmetric polynomials in $\varkappa$ variables $z_1, \dots, z_\varkappa$.

From a classical theorem in algebra, any symmetric polynomial in $\varkappa$ variables can be written as a polynomial in the elementary symmetricpolynomials $\sigma_1, \dots, \sigma_\varkappa$. Denote by $c_1, \dots, c_\varkappa$ the elements of $H^*(G_{\varkappa,\infty})$ which correspond to $\sigma_1, \dots, \sigma_\varkappa$ under the isomorphism given by Theorem 4. The elements $c_1, \dots, c_\varkappa$ are called the *Whitney Classes* (in the real case) and the *Chern Classes* (in the complex case). Thus, Theorem 4 asserts that any cohomology class in $H^*(G_{\varkappa,\infty})$ can be written as a sum of cup products of Whitney (Chern) Classes.

Note that in the real case, $c_j \in H^j(G^{\mathbb{R}}_{\varkappa,\infty}, \mathbb{Z}_2)$, while in the complex case, $c_j \in H^{2j}(G^{\mathbb{C}}_{\varkappa,\infty}, \mathbb{Z})$.

Part B: The Poincaré Duality Theorem

We recall some of the facts from the theory of character groups. (For a full exposition of results, see Loomis—*An Introduction to Abstract Harmonic Analysis* or Pontryagin—*Topological Groups.*) Let G be a locally compact abelian group with a countable basis. Then we can define the character group $\hat{G}$ of G to be the group of continuous homomorphisms of G into R mod 1, the additive group of reals modulo 1. It is known that (i) if $G \cong \mathbb{R}^n$, then $\hat{G} \cong \mathbb{R}^n$, (ii) if G is finite and has the discrete topology, then $G \cong \hat{G}$, (ii) if $G \cong \mathbb{Z}$, then $\hat{G} \cong \mathbb{R}$ mod 1. There is a pairing of the groups G, $\hat{G}$ to the group $\mathbb{R}$ mod 1, i.e. for $g \in G$, $\hat{g} \in \hat{G}$, there is a natural "multiplication" $g\hat{g} = r$, $r \in \mathbb{R}$ mod 1. In particular, if $g, \hat{g} \in \mathbb{R}$, then $g\hat{g}$ is just the ordinary multiplication reduced modulo 1. It is known that G is compact $\Leftrightarrow$ $\hat{G}$ is discrete and that $\hat{\hat{G}} \cong G$ for any locally compact G. We say that two groups G and G' are dually paired to $\mathbb{R}$ mod 1 if there is a multiplication $gg' \in R$ mod 1 such that if ϕ_g is the element of $\hat{G}'$ defined by $\phi_g(g') = gg'$ and if $\psi_{g'}$ is the element of $\hat{G}$ defined by $\psi_{g'}(g) = gg'$, then the maps $g \to \phi_g$ and $g' \to \psi_{g'}$ are isomorphisms of G onto $\hat{G}'$ and G' onto $\hat{G}'$.

We are now in a position to state the *Poincaré Duality Theorem: Let M^m be a compact orientable manifold without boundary and let G be a locally compactabelian group with a countable basis. Then $H_k(M; G)$ and $H_{m-k}(M; \hat{G})$*

are dually paired to R mod 1 *if the multiplication between* $H_k(M; G)$ *and* $H_{m-k}(M; \hat{G})$ *is defined as the intersection number of homology classes.*

Corollary (1): Let M^m be a compact orientable manifold without boundary. If α is a singular (k)-cycle on M with real (respectively integer) coefficients, and if $\alpha \cdot \beta = 0$ for all singular $(m-k)$-cycles β on M with real (respectively R mod 1) coefficients, then α is a boundary.

If G is a field, the singular homology groups of H are vector spaces. We define the p-h Betti number b_p of M to be the dimension of $H_p(M; G)$. As a consequence of the Poincaré Duality Theorem, we have the following

Corollary (2): $b_k = b_{m-k}$.

If M is not orientable, the intersection theory mod Z cannot be applied. However, we still have *Poincaré Duality Theorem Mod* 2: *If* M^m *is a compact manifold without boundary, then* $H_k(M; Z_2)$ *and* $H_{m-k}(M; Z_2)$ *are dually paired to* R mod 1 *if the multiplication between* $H_k(M; Z_2)$ *and* $H_{m-k}(M; Z_2)$ *is defined as the intersection number*mod 2 *of homology classes.*

Our proof of the Poincaré Duality Theorem will make use of Morse Theory. We shall state the results we need without proof. The proofs can be found in Milnor—*Morse Theory.*

Suppose then that f is a smooth real-valued function on a manifold M^m. A point $p \in M$ is called a *critical* point of f if, in local coordinates at p, $\dfrac{\partial f}{\partial x_1}(p) = \cdots = \dfrac{\partial f}{\partial x_m}(p) = 0$. A critical point p is called *nondegenerate* if the matrix $\left\| \dfrac{\partial^2 f}{\partial x_i \partial x_j}(p) \right\|$ is nonsingular. A lemma of Morse states that if p is a nondegenerate critical point of f, then there exist local coordinates at p such that $f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2$ in the entire coordinate patch. The integer k is called the *index* of the critical point p. It is an immediate consequence of Morse's lemma that nondegenerate critical points are isolated. Hence, if M is a compact manifold, the number of nondegenerate critical points is finite. The following lemma, which is closely related to the jiggling lemma, and the details of whose proof we leave to the reader, plays a crucial role in what follows.

Lemma (1): Let f be a smooth real-valued function on the compact manifold M. Arbitrarily near f (in the sense of the d_k-metric, for any k), there exists a smooth real-valued function g on M such that g has onlynondegenerate critical points. If, moreover, $p_1, p_2, \ldots, p_k$ are the critical points of g, then g can be chosen so that $g(p_1), g(p_2), \ldots, g(p_k)$ are distinct real numbers.

The following theorem is one of the central results of Morse Theory. First, we give some definitions. Let f be a smooth real-valued function on a compact manifold M. The real number c is called a critical *level* of f if there exists a critical point p of f such that $f(p) = c$. Denote by $(r_1 \leqq f \leqq r_2)$ the set of all points $q \in M$ such that $r_1 \leqq f(q) \leqq r_2$, where r_1 and r_2 are any real numbers. Similarly, we denote by $(f \leqq r)$ (respectively $(f \geqq r)$) the set of all points $q \in M$ such that $f(q) \leqq r$ (respectively $f(q) \geqq r$). We then have

Morse's Theorem: *Let a and b be real numbers, $a < b$. If there are no critical levels of f between a and b, then $(f \leqq a)$ is a deformation retract of $(f \leqq b)$. In particular, the relative singular homology group $H((f \leqq b)/(f \leqq a)) = 0$. On the other hand, suppose there is exactly one critical level of f between a and b corresponding to exactly one nondegenerate critical point of f of index k. Then $H_l((f \leqq b)/(f \leqq a)) = 0$ for $k \neq l$ and $H_k((f \leqq b)/(f \leqq a); G) \cong G$.*

For a proof of this result, cf. Milnor, *op. cit.*

We now proceed to the proof of the Poincaré Duality Theorem. By Lemma (1), we can find a smooth real-valued function f on M, all of whose critical points are nondegenerate. Moreover, if $p_1, p_2, \ldots, p_{n-1}$ are these critical points, we may suppose that $f(p_1), f(p_2), \ldots, f(p_{n-1})$ are all distinct. Without loss of generality, we may assume $f(p_1) < f(p_2) < \cdots < f(p_{n-1})$. It is clear that $f(p_1)$ is the minimum value, and $f(p_{n-1})$ the maximum value, of the function f. We now choose any real numbers $c_1, c_2, \ldots, c_n$ such that $c_1 < f(p_1) < c_2 < f(p_2) < \cdots < c_{n-1} < f(p_{n-1}) < c_n$. It suffices to prove (by induction on i) that $H_k((f \leqq c_i); G)$ and $H_{m-k}((f \leqq c_i)/(f = c_i); \hat{G})$ are dually paired to R mod 1 by the intersection number. Indeed, setting $i = n$, we have $(f \leqq c_n) = M$, $(f = c_n) = \phi$ and the assertion follows directly. For $i = 1$, the proposition is obvious because $(f \leqq c_1) = \phi$. Let $c = f(p_1)$ denote the minimum value of f. There is a coordinate patch at p_1 such that f is given by $f = c + x_1^2 + \cdots + x_m^2$. Since p_1 is the only point in M at which f takes on its minimum, it follows easily from the compactness of M that there exists a number $\varepsilon > 0$ such that $(f \leqq c + \varepsilon)$ is homeomorphic to $B^m_\varepsilon = \{(x_1, \ldots, x_m) | x_1^2 + \cdots + x_m^2 \leqq \varepsilon\}$. Now $H_k(B^m_\varepsilon, G)$ and $H_{m-k}(B^m_\varepsilon/\partial B^m_\varepsilon, \hat{G})$ are dually paired to R mod 1 by the intersection number. In fact, if $k \neq 0$, $H_k(B^m_\varepsilon, G) = 0 = H_{m-k}(B^m_\varepsilon/\partial B^m_\varepsilon, \hat{G})$. Moreover, $H_0(B^m_\varepsilon, G) \cong G$ and $H_m(B^m_\varepsilon/\partial B^m_\varepsilon, \hat{G}) \cong \hat{G}$. It follows from the first part of Morse's theorem that $H_k((f \leqq c_2), G) \cong H_k(B^m_\varepsilon, G)$ and $H_{m-k}((f \leqq c_2)/(f = c_2), \hat{G}) \cong H_{m-k}(B^m_\varepsilon/\partial B^m_\varepsilon, \hat{G})$. The proposition is thus verified for $i = 2$. We now proceed inductively, assuming the result for $i \leqq j$ and proving it for $i = j + 1$. Let the index of the critical point p_j be λ. Set $T = (f \leqq c_{j+1})$, $U = (f = c_{j+1})$, $T_- = (f \leqq c_{j+1} - \varepsilon)$, $T_0 = (c_j + \varepsilon \leqq f \leqq c_{j+1})$, $L = (f \leqq c_j)$. Here, ε is chosen so small that $c_{j+1} - \varepsilon > f(p_j) > c_j + \varepsilon$. We consider two exact homology sequences; the sequence of the pair (T_-, L) and the sequence of the triple (T, T_0, U). We have

$$\begin{aligned} \cdots \rightarrow H_{k+1}(T_-/L, G) \rightarrow H_k(L, G) \rightarrow H_k(T_-, G) \rightarrow \\ \rightarrow H_k(T_-/L, G) \rightarrow H_{k-1}(L, G) \rightarrow \cdots \end{aligned} \tag{1}$$

$$\begin{aligned} \cdots \leftarrow H_{m-k-1}(T_0/U, \hat{G}) \leftarrow H_{m-k}(T/T_0, \hat{G}) \leftarrow H_{m-k}(T/U, \hat{G}) \leftarrow \\ \leftarrow H_{m-k}(T_0/U, \hat{G}) \leftarrow H_{m+k-1}(T/T_0, \hat{G}) \leftarrow \cdots \end{aligned} \tag{2}$$

We want to show that $H_k(T, G)$ and $H_{m-k}(T/U, \hat{G})$ arc dually paired to R mod 1 by the intersection number. By virtue of Morse's theorem, it is just as well to consider $H_k(T_-, G)$ as $H_k(T, G)$. In general, $\alpha \cdot \beta$ is defined whenever $\partial\alpha \cap \beta = \phi, \alpha \cap \partial\beta = \phi$. Hence, intersection numbers are defined between elements of $H_r(L, G)$ and $H_{m-r}(T/T_0, \hat{G})$, $H_r(T_-, G)$ and $H_{m-r}(T/U, \hat{G})$, and $H_r(T_-/L, G)$ and $H_{m-r}(T_0/U, \hat{G})$. Indeed, $L \cap T_0 = T_- \cap U = \phi$. We shall show that $H_k(L, G)$ and $H_{m-k}(T/T_0, G)$ are dually paired to R mod 1 by the intersection number. By excision, $H_{m-k}(T/T_0, \hat{G}) \cong H_{m-k}((f \leqq c_j + \varepsilon)/(f = c_j + \varepsilon), \hat{G})$ and by the first part of Morse's theorem, $H_k(L, G) \cong H_k((f \leqq c_j + \varepsilon), G)$. The assertion now follows by induction and another application of Morse's theorem. The same result is true, of course, for the groups $H_{k-1}(L, G)$ and $H_{m-k+1}(T/T_0, \hat{G})$. We claim moreover that $H_k(T_-/L)$. $G)$ and $H_{m-k}(T_0/U, \hat{G})$ are dually paired to R mod 1 by the intersection number. Indeed, it follows directly the second part of Morse's theorem that $H_k(T_-/L, G) \cong \begin{cases} 0 & \text{if } k \neq \lambda \\ G & \text{if } k = \lambda. \end{cases}$ By excision,

$$\begin{aligned} H_{m-k}(T_0/U, \hat{G}) &\cong H_{m-k}((f \geqq c_j + \varepsilon)/(f \geqq c_{j+1}), \hat{G}) \\ &\cong H_{m-k}((-f \leqq -c_j - \varepsilon)/(-f \leqq -c_{j+1})\, \hat{G}). \end{aligned}$$

As p_j is a critical point of f of index λ, it is clearly a critical point of $-f$ of of index $m - \lambda$. It follows again from the second part of Morse's theorem that $H_{m-k}(T_0/U, \hat{G}) \cong \begin{cases} 0 & \text{if } k \neq \lambda \\ \hat{G} & \text{if } k = \lambda \end{cases}$: This proves the claim. The same result is true, of course, for the groups $H_{k+1}(T_-/L, G)$ and $H_{m-k-1}(T_0/U, \hat{G})$. We claim, now, that the sequences (1) and (2) are dual cxact sequences in the following sense. Consider abstractly two exact sequences of groups

$$\cdots \longrightarrow A_{j+1} \xrightarrow{\phi_{j+1}} A_j \xrightarrow{\phi_j} A_{j-1} \dashrightarrow \cdots \tag{1'}$$

$$\cdots \leftarrow - B_{j+1} \xleftarrow{\psi_j} B_j \xleftarrow{\psi_{j-1}} B_{j-1} \leftarrow \cdots \tag{2'}$$

Suppose we have defined a product $a \cdot b$ for $a \in A_j$, $b \in B_j$. We say that the sequences (1′) and (2′) are dual exact sequences if $\phi_{j+1} a' \cdot b = a' \cdot \psi_j b$ holds for $a' \in A_{j+1}$, $b \in B_j$. Let us assume for the moment that (1) and (2) above are dual exact sequences. Our theorem will then be deduced from the

Algebraic Lemma: Suppose (1′) and (2′) are dual exact sequences. Suppose further that A_{3j} and B_{3j} are dually paired to R mod 1 and also that A_{3j+1} and B_{3j+1} are dually paired to R mod 1. Then, A_{3j+2} and B_{3j+2} are dually paired to R mod 1.

To prove this, let $C_i = \hat{B}_i$, the character group of B_i. The exact sequence (2′) gives rise to a corresponding exact sequence $\longrightarrow C_{j+1} \xrightarrow{\hat{\psi}_j} C_j \xrightarrow{\hat{\psi}_{j-1}} C_{j-1}$. We consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A_{j+1} \arrow[r, "\phi_{j+1}"] \arrow[d, "f_{j+1}"'] & A_j \arrow[r, "\phi_j"] \arrow[d, "f_j"'] & A_{j-1} \arrow[r] \arrow[d, "f_{j-1}"'] & \cdots \\
{} \arrow[r] & C_{j+1} \arrow[r, "\psi_j"] & C_j \arrow[r, "\psi_{j-1}"] & C_{j-1} \arrow[r] & \cdots
\end{tikzcd}$$

The maps $f_i : A_i \to C_i$ are given by $a_i \to g_{a_i}$, where $g_{a_i}(b_i) = a_i \cdot b_i$. The relation $\phi_{j+1} a' \cdot b = a' \cdot \psi_j b$ insures that the diagram is commutative. Moreover it follows from the hypothesis that f_{3j}, g_{3j+1} are isomorphisms. By the "Five Lemma" (see Eilenberg-Steenrod), f_{3j+2} is also an isomorphism. The assertion of the lemma now follows easily.

Thus, if we can show that (1) and (2) are dual exact sequences, it will follow that $H_k(T_-; G)$ and $H_{m-k}(T/U; \hat{G})$ are dually paired to R mod 1 by the intersection number. The fact that (1) and (2) are dual exact sequences follows easily from properties of the intersection number which we have developed. We leave the details as an easy exercise.

CHAPTER IV

Introduction to the Theory of Fiber Bundles

In this chapter, we begin our study of fiber bundles and their applications to differential topology. Throughout these notes, emphasis is placed on the theory of linear bundles and their associated principal bundles (see definitions below and in Chapter VI). A most important particular case which we shall consider is the tangent bundle of a manifold (see below).

Geometrically, a fiber bundle may be considered as a generalization of a product space. In fact, as we shall see below, a fiber bundle "appears locally as a product space". However, is allowed a "twisting in the large".

The formal definition of a fiber bundle is actually rather involved. Moreover, many of the theorems in the subject are of a technical nature. Since there is a quite detailed book on the subject available, namely, Steenrod—*Topology of Fiber Bundles*, we shall be rather lax in supplying full arguments. Frequently, we state without proof theorems which can be found in Stenrod's book. As another reference for the subject (especially for the theory of linear bundles), we suggest the newer book Hirzebruch—*Neue Topologische Methoden in der Algebraischen Geometrie*.

To begin with, recall that if G is a topological group and Y is a space, then G is a *group of homeomorphisms* of Y if there is a continuous map $\eta : G \times Y \to Y$ (we write $\eta(g, y) = g \cdot y$) such that

(i) $ey = y$ for all $y \in Y$ (where e is the unit of G),
(ii) $g_1(g_2y) = (g_1g_2)y$ for all $g_1, g_2 \in G$ and $y \in Y$.

G is called *effective* if $gy = y$ for all $y \in Y$ implies $g = e$.

Definition (1): A fiber bundle is a collection (X, B, F, G, π) as follows:

(1) A space X called the *bundle space* (or *total space*).
(2) A space B called the *base space*.
(3) A space F called the *fiber*.

(4) An effective group G of homeomorphisms of F called the group of the bundle.

(5) A map $\pi : X \xrightarrow{\text{onto}} B$ called the projection such that $\pi^{-1}(b)$ is homeomorphic to the space F for all $b \in B$.

(6) A family $\{\mathscr{U}_\alpha\}$ of open sets covering B; the $\mathscr{U}_\alpha$'s are called coordinate neighborhoods. For each $\mathscr{U}_\alpha$, there exists a homeomorphism im_α such that $im_\alpha : \pi^{-1}(\mathscr{U}_\alpha) \xrightarrow{\text{onto}} \mathscr{U}_\alpha \times F$ is given by

$$\text{Im}_\alpha (x) = [b, f] = [\pi(x), f_\alpha(x)], \quad (b \in \mathscr{U}_\alpha, f \in F).$$

(7) If $b \in \mathscr{U}_\alpha \cap \mathscr{U}_\beta$ then $\text{im}_\beta \circ \text{im}_\alpha^{-1} [b, f] = [b, U_b(\beta, \alpha) f]$. We require that the maps $U_b(\beta, \alpha) : F \to F$ depend continuously on b and belong to the group of homeomorphisms G. The $U_b(\beta, \alpha)$ are called the *coordinate transformations* of the bundle.

Examples: (For more details concerning these examples, see Steenrod, loc. cit.)

(i) If X is a Lie group and F a closed subgroup, then X is a bundle over the base $B = X/F$ with fiber F and group F acting on itself by left translations.

(ii) The Möbius strip is a bundle over the circle as base and the line segment as fiber. The group G is the cyclic group of order 2.

(iii) As a very simple example, we have the product space $B \times F$, which is a bundle over B with fiber F. In this case, the group G is the trivial group.

(iv) Let M^m be a manifold, $\tau(M)$ the collection of all tangent spaces at all points of M, i.e. the collection of all pairs $[p, v]$ where $p \in M$ and v is a tangent vector to M at p. Let $\pi : \tau(M) \to M$ be defined by $\pi[p, v] = p$. Then $\tau(M)$ becomes a fiber bundle over M called the *tangent bundle.* In fact, for coordinate neighborhoods $\mathscr{U}_\alpha$ we take the coordinate neighborhoods of M. Write $\mathscr{W}(m)$ for the Lie group of all non-singular $m \times m$ matrices. The maps $U_b(\beta, \alpha)$ of the last definition are just the Jacobian matrices of transformations of coordinate maps and therefore belong to $\mathscr{W}(m)$; therefore $\tau(M)$ is a *$\mathscr{W}(m)$-bundle* over M.

Remark: The coordinate change functions $U_b(\beta, \alpha)$ of a bundle satisfy the following properties

(i) $U_b(\alpha, \alpha) =$ identity homeomorphism $e \in G$

(ii) $U_b(\gamma, \beta) \circ U_b(\beta, \alpha) = U_b(\gamma, \alpha)$

Clearly, (i) and (ii) imply that $U_b(\alpha, \beta) \circ U_b(\beta, \alpha) = e$. A set of functions $U_b(\beta, \alpha)$ defined on the set $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, which satisfy conditions (i) and (ii) and which are continuous functions of b (i.e. the map from $\mathscr{U}_\alpha \cap \mathscr{U}_\beta \to G$ given by $b \to U_b(\beta, \alpha)$ is continuous) is said to define a *G-bundle structure* on B.

Now let B be a space and let $\{\mathscr{U}_\alpha\}$ be an open covering of B. Suppose that a topological G acts on a space F as a group of homeomorphisms. Finally, suppose functions $U_b(\beta, \alpha)$ are given on $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$ which define a G-bundle structure on B. Then we have the

Existence Theorem (See Steenrod, loc. cit., p. 14)

There exists a G-bundle X over B with fiber F and coordinate transformations $U_b(\beta, \alpha)$. X can be constructed as follows. Let $\tilde{X}$ be the collection of all triples $\{[b, f, \alpha] \mid b \in \mathscr{U}_\alpha, f \in F\}$. $\tilde{X}$ is given the relative topology of the product space (where the index set is assumed to have the discrete topology). X is obtained from $\tilde{X}$ via the equivalence relation $[b, f, \alpha] \sim [b, U_b(\beta, \alpha) f, \beta]$, i.e. X is defined as the resulting set of equivalence classes. The theorem also asserts that X is unique up to equivalence (see directly below).

Equivalence of G-bundles

Let X, X' be two G-bundles with the same base B and fiber F, and with projections π, π' respectively. Then the two bundle structures are *equivalent* if there exists a homeomorphism $\phi : X \xrightarrow{\text{onto}} X'$ satisfying the following:

(i) $\pi'\phi = \pi$. Thus, $\phi : \pi^{-1}(b) \to \pi'^{-1}(b)$ is a homeomorphism,

(ii) if $\mathscr{U}_\alpha, \mathscr{V}_{\beta'}$ are coordinate neighborhoods of the G-bundles X, X' respectively and if $\mathrm{im}_{\beta'} \circ \phi \circ \mathrm{im}_\alpha^{-1}[b, f] = [b, \Phi_b(\beta', \alpha) f]$, then we require that Φ_b depend continuously on b and belong to G.

Notation: We use the symbol "$\cong$" for equivalence.

Remark: The following relation follows immediately from the definitions:

$$U_b(\gamma', \beta')\, \Phi_b(\beta', \alpha)\, U_b(\alpha, \delta) = \Phi_b(\gamma', \delta).$$

Another way of putting this is as follows:

$$U_b(\gamma', \beta') = \Phi_b(\gamma', \delta)\, U_b(\delta, \alpha)\, \Phi_b^{-1}(\beta', \alpha).$$

The last relation could be taken as the definition of equivalence of G-bundle structures over B.

Definition (2): Let ξ be a bundle structure over B with fiber F and group G, and let $\psi : B' \to B$ be a continuous map. The *induced* bundle structure $\psi^\dagger \xi$

having base space B', fiber F, and group G is defined as follows. The coordinate neighborhoods are the inverse images of those of ξ: $\mathscr{U}'_\alpha = \psi^{-1}(\mathscr{U}_\alpha)$. The coordinate transformations are given by $U_{\psi(b')}(\beta, \alpha)$ where $\psi(b') \in \mathscr{U}_\alpha \cap \mathscr{U}_\beta$.

Notice that if ξ_1, ξ_2 are equivalent bundle structures over B, then $\psi^\dagger\xi_1$ and $\psi^\dagger\xi_2$ are equivalent over B'. Moreover, $(\psi_1\psi_2)^\dagger = (\psi_2)^\dagger(\psi_1)^\dagger$, $(\text{ident})^\dagger = \text{ident}$.

Definition (3): Let ξ be a G-bundle structure on B, G' a subgroup of G. We say ξ is *reducible* to G' if there exists an equivalent bundle structure ξ' such that all coordinate changes $U_b(\beta, \alpha)$ are in G'. If $G' = \{e\}$, we call ξ a *trivial bundle*. Then ξ is equivalent to the product space bundle.

If $G = \mathscr{W}(n)$, the general linear group, and if ξ is reducible to $\mathscr{W}^+(n)$, i.e. the subgroup of $\mathscr{W}(n)$ of all $n \times n$ matrices with positive determinant, then we say that ξ is an *orientable bundle.*

Lemma (1): A manifold M^m is orientable iff the tangent bundle $\tau(M)$ is an orientable bundle (see last definition).

Proof: Recall that a manifold M is orientable if it can be covered by coordinate neighborhoods such that the Jacobian matrices of coordinate maps have positive determinants. From this and from the definition of tangent bundle the sufficiency follows.

Necessity: Let $\tau_1, \ldots, \tau_m$ be a frame of vectors at some point $p \in M$, i.e. a set of linearly independent vectors at p. Let x be a coordinate system in some neighborhood of p. Let the components of τ_1 in this coordinate system be given by $\tau_i^{(j)}(x)$ $(j = 1, 2, \ldots, m)$. We say that the frame of vectors τ_i is positively oriented if $\|\tau_i^j\|$ has positive determinant. This notion is well defined since $\tau(M)$ is oriented. Now, given a coordinate system x_α on M, let $\tau_1, \ldots, \tau_m$ be the directional derivatives in the coordinate directions. Call this system positive if $\tau_1, \ldots, \tau_m$ forms a positive frame, negative otherwise. Cover M with coordinates as follows: take x_α if x_α is positive, take x'_α = the reverse of x_α otherwise. (Recall that the reverse of $[x_1, \ldots, x_n]$ is $[x_1, \ldots, x_{n-1}, -x_n]$.) This orients M.

Q.E.D.

Cross-Section of a Bundle

Definition (4): Let X be a bundle over the base B, with projection $\pi: X \to B$. Then a cross-section of X is a continuous map $k: B \to X$ such that $\pi k = \text{identity}$.

Example: A continuous vector field on a manifold is a cross-section of the tangent bundle of the manifold.

Note: Not all bundles have cross-sections; for example let $\hat{\tau}(M)$ be the set of all non-zero vectors in $\tau(M)$. Then $\hat{\tau}(M)$ is a bundle over M and $\hat{\tau}(M)$ may not have a cross-section. Indeed, the following result holds.

Theorem: There is a continuous nonvanishing vector field on M iff the Euler characteristic of $M = 0$.

See Steenrod, op. cit., for a proof.

Linear Bundles

Definition (4): A fiber bundle is said to be a *linear bundle* or a *vector space bundle* if the fiber is a vector space V^k of some dimension k and the group is $\mathscr{W}(k)$.

Definition (5): Let $(X, B, V^k, \mathscr{W}(k), \pi)$ be a linear bundle. Let $X' \subseteq X$ be such that $\pi(X') = B$. Then $(X', B, V^j, \mathscr{W}(j), \pi|X')$ is said to be a *linear sub-bundle* of $(X, B, V^k, \mathscr{W}(k), \pi)$ if

(1) $\pi^{-1}(b) \cap X' = V'(b)$ is a linear subspace of V^k, always of the same dimension j, i.e. $j = \dim(V'(b))$ does not depend on b.

(2) $V'(b)$ depends continuously on b, i.e. $V'(b)$ has a basis which depends continuously on b.

Let $b_0 \in B$. We pick a basis $\{v_1(b_0), \dots, v_j(b_0)\}$ for $V'(b_0)$ and then we find vectors $v_{j+1}, \dots, v_k$ in $V(b_0) = \pi^{-1}(b_0)$ such that the system $\{v_1(b_0), \dots, v_j(b_0), v_{j+1}, \dots, v^k\}$ forms a basis for all of $V(b_0)$. By property (2) of Definition 5, we find a basis $\{v_1(b), \dots, v_j(b)\}$ for $V'(b)$ such that $v_i(b)$ is close to $v_i(b_0)$ whenever b is close to b_0 $(1 \leq i \leq j)$. Then, if b is sufficiently near b_0, the system $\{v_1(b), \dots, v_j(b), v_{j+1}, \dots, v_k\}$ forms a basis for $V'(b)$ (as one clearly sees). Thus, there is a unique nonsingular linear transformation $L(b)$, which is continuous in b, and which maps

$$\{v_1(b_0), \dots, v_j(b_0), v_{j+1}, \dots, v_k\} \to \{v_1(b), \dots, v_j(b), \dots, v_j(b), v_{j+1}, \dots, v_k\}.$$

Now, X "appears locally" as $\mathscr{U} \times V(b_0)$ for some coordinate neighborhood $\mathscr{U}$ and some $b_0 \in B$ (see Definition 1). The above argument shows that X' "appears locally" as $\mathscr{U} \times V'(b_0)$. The reader should now be able to satisfy himself that a linear sub-bundle is actually a bundle.

Definition (6): Let X be a linear bundle and X' a linear sub-bundle. (By abuse of language, we call X and X' bundles, although they are actually only the bundle spaces.) We define an equivalence relation in X as follows: $x_1 \sim x_2$ if

(i) $\pi(x_1) = \pi(x_2)$, and
(ii) $x_2 - x_1 \in X'$.

Denote by X/X' the resulting set of equivalence classes. X/X' then forms the bundle space of a linear bundle called the *factor bundle.*

To verify that X/X' can be turned into the bundle space of a linear bundle, we observe that locally, we can write $X = \mathscr{U} \times V(b_0)$, $X' = \mathscr{U} \times V'(b_0)$; whence $X/X' = \mathscr{U} \times V(b_0)/V'(b_0)$. To define a natural linear bundle structure on X/X', use this "coordinatization".

Lemma (2): Suppose

(1) $X \xrightarrow{\pi} B$ is a linear bundle,
(2) X' is a linear sub-bundle,
(3) X is orientable.

Then X' is orientable iff X/X' is orientable. Moreover, any orientation of either of these bundles induces naturally an orientation of the other.

Proof: If X is a linear bundle with fiber V^k, then a k-frame in X is a set $x_1, x_2, \ldots, x_k$ of k linearly independent elements of X such that $\pi(x_i) = b$. (This makes sense since $\pi^{-1}(b) = V^k$.) Notice that a bundle is orientable if the set of frames can be divided "continuously" into right handed frames and left handed frames, i.e. if $x_1, \ldots, x_k$ is a right handed frame, then any frame sufficiently close to it is a right handed frame. Now, suppose X' is orientable. Call a frame $x_{j+1}, \ldots, x_k$ of X/X' right handed if, when supplemented with a right handed frame of X', we get a right handed frame for X. This orients X/X'. Conversely, suppose X/X' is orientable. Call a frame $x_1, \ldots, x_j$ in X' right handed if, when supplemented with a right handed frame for X/X', we get a right handed frame for X. This orients X', and proves our lemma. Q.E.D.

Let M be a manifold, $M' \subseteq H$ a submanifold. Let $\tau(M)$ be the tangent bundle of M, $\tau_0(M)$ the restriction of $\tau(M)$ to M', i.e. the set of all vectors tangent to M at points of M'. Then

$$\tau_0(M) \supseteq \tau(M').$$

Suppose M has a Riemannian metric, and let $\nu(M')$ be the set of vectors in $\tau_0(M)$ which are normal to M'. We call $\nu(M')$ the *normal bundle* of M'. It is easy to see that $\tau_0(M') \cong \tau(M) \oplus \nu(M')$. (A precise definition of $\oplus$

is given in a later section.) If M is orientable, then $\tau(M)$ is orientable and hence $\tau_0(M)$ is orientable (because it is a restriction of the orientable bundle $\tau(M)$). Therefore by the last lemma, $\tau(M')$ is orientable iff $\nu(M')$ is orientable, which implies that M' is orientable if $\nu(M')$ is orientable by Lemma 1.

Special Case: Assume M is an orientable manifold with boundary, $M' = \partial M$. Then $\nu(M')$ is trivial. To see this, let $n(b)$ be the unit normal vector at a point b in the boundary M' of M pointing inward; $n(b)$ depends continuously on b. Given any normal vector v at b, assign coordinates $[b, v \cdot n(b)]$ to v. (The dot means inner product.) Then

$$\nu(M') \cong M' \times R$$

where R is the real line. But this means that $\nu(M')$ is the trivial bundle, hence orientable. Therefore, if M is an orientable manifold, then ∂M is also orientable.

Remark: As we have observed above, we have the relation $\nu(M') \oplus \tau(M') \cong \tau_0(M)$. This means that we have a natural equivalence of the bundles $\nu(M')$ and $\tau_0(M)/\tau(M')$. Now, $\tau_0(M)/\tau(M')$ is clearly independent of the choice of a Riemannian metric for the manifold. Hence, $\nu(M')$, which seems to depend on the choice of the metric, is actually determined (up to equivalence) by the internal structure of M.

Now suppose M and M' are manifolds, both contained in another manifold M_0. Assume that M and M' intersect transversely, i.e. at each point $p \in M \cap M' = MM'$, we have

$$\tau(M)|_p + \tau(M')|_p = \tau(M_0)|_p.$$

We claim that if M_0, M and M' are all orientable and oriented in some way, then MM' is also orientable (and inherits a natural orientation). To see this, note that

$$\{\tau(M)|_{MM'} + \tau(M')|_{MM'}\}/\tau(MM') \cong \tau(M_0)|_{MM'}.$$

(By $\tau(M)|_{MM'}$, we mean the set of all tangent vectors to M at points of MM'. Similarly for $\tau(M')|_{MM'}$ and $\tau(M_0)|_{MM'}$.) Now, by assumption, $\tau(M)|_{MM'}$, $\tau(M')|_{MM'}$ and $\tau(M_0)|_{MM'}$ are all oriented. Therefore, $\tau(MM')$ is oriented; whence MM' is oriented, as asserted.

To complete the circle of ideas centering about orientation of submanifolds, etc., we state the following lemma and corollary. The proof of the lemma is left as an exercise (see Steenrod, loc. cit., pp. 47–49).

Lemma (3): Suppose that $\phi : M \to M'$ is a smooth map between two manifolds M and M'. Let N' be a submanifold of M' and suppose ϕ is everywhere transverse to N'. Then, as we know, $N = \phi^{-1}(N')$ is a submanifold of M. We assert that

$$\nu(N) \cong \phi^{\dagger}(\nu(N'))$$

where $\phi^{\dagger}$ is the induced map on bundles.

Corollary: If M, M' and N' are orientable, then N is also orientable.

Proof: Since M' and N' are orientable, $\nu(N')$ is orientable (see discussion after Lemma 2 above). This means that $\nu(N')$ is reducible to $\mathscr{W}^+(n')$ where n' is the dimension of N'. Now, $\phi^{\dagger}$ does not "enlarge" the group $\mathscr{W}^+(n)$, i.e. $\phi^{\dagger}(\nu(N'))$ is an orientable bundle. By Lemma 3, $\nu(N)$ is also orientable. But then, since M is orientable, it follows that N is orientable.

Q.E.D.

CHAPTER V

Spectral Sequences

If X is a topological space and A a closed subset of X, we call (X, A) a pair. A *normal series* for (X, A) is a sequence

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 = A$$

where all the X_j are closed subsets of X.

Let $H(X, A) = \sum_{d=1}^{\infty} \oplus H_d(X, A)$ where $H_d(X, A)$ is the dth relative homology group of (X, A). The inclusion map $(X_j, A) \to (X_n, A)$ induces a homomorphism of $H(X_j, A) \to H(X_n, A)$ and we denote by $\operatorname{Im} H(X_j, A)$ the image of the group $H(X_j, A)$ in the group $H(X_n, A)$. (In general, if (R, U) and (S, V) are pairs such that $R \subseteq S$, $U \subseteq V$, we denote by "$\operatorname{Im} H(R, U)$ in $H(S, V)$" the image of the group $H(R, U)$ in the group $H(S, V)$ under the homomorphism which is induced by the inclusion map $(R, U) \to (S, V)$.) We consider the sequence

$$\begin{aligned} H(X, A) = H(X_n, A) &\supseteq \operatorname{Im} H(X_{n-1}, A) \\ &\supseteq \operatorname{Im} H(X_{n-2}, A) \supseteq \cdots \supseteq \operatorname{Im} H(A, A) = 0. \end{aligned}$$

Since all the groups involved are abelian, the above sequence of groups is actually a normal series for $H(X, A)$ (in the group-theoretic sense). Denote by F_j the j-th factor of this normal series, i.e.

$$F_j = \frac{\operatorname{Im} H(X_j, A)}{\operatorname{Im} H(X_{j-1}, A)}.$$

By convention, set $X_{-k} = A$ $(k = 0, 1, 2, \ldots)$ and $X_l = X$ $(l = n, n+1, \ldots)$. We wish to find relations between the groups $H(X_j, X_{j-1})$ and the groups of the normal series for (X, A). For this purpose, we define the *Leray intermediate groups* E_r^j: we set

$$E_r^j = \frac{\operatorname{Im} H(X_j, X_{j-r}) \text{ in } H(X_{j+r-1}, X_{j-r})}{\operatorname{Im} H(X_{j-1}, X_{j-r}) \text{ in } H(X_{j+r-1}, X_{j-r})}.$$

It is easy to verify that $E_1^j = H(X_j, X_{j-1})$ and $E_\infty^j = F_j$. The main result to be proved in this section is the following.

Theorem (1): *We can define boundary operators* $\partial_r^j : E_r \to E_r^{j-r}$ *such that* $\partial^r \partial^r = 0$. *If* $Z(E_r^j) = \{z \in E_r^j | \partial^r z = 0\}$ *and* $B(E_r^j) = \{\partial^r \omega | \omega \in E_r^{j+r}\}$, *then, as usual,* $B(E_r^j) \subseteq Z(E_r^j)$. *Set* $H(E_r^j) = Z(E_r^j)/B(E_r^j)$. *Then* $H(E_r^j) \cong E_{r+1}^j$.

Proof: We define ∂^r as follows: Consider the sequence

$$H(X_j, X_{j-r}) \xrightarrow{\partial} H(X_{j-r}) \longrightarrow H(X_{j-r}, X_{j-2r}) \longrightarrow H(X_{j-1}, X_{j-2r}).$$

Here ∂ is a boundary operator and the other maps are induced by the respective inclusions. The composite map from $H(X_j, X_{j-r})$ to $H(X_{j-1}, X_{j-2r})$ will be denoted by ∂_0^r. Note that ∂_0^r can be regarded as a map from $H(X_j, X_{j-r})$ to "Im $H(X_{j-r}, X_{j-2r})$ in $H(X_{j-1}, X_{j-2r})$" and a fortiori as a map from $H(X_j, X_{j-r})$ to E_r^{j-r} where

$$E_r^{j-r} = \frac{\operatorname{Im} H(X_{j-r}, X_{j-2r}) \text{ in } H(X_{j-1}, X_{j-2r})}{\operatorname{Im} H(X_{j-r-1}, X_{j-2r}) \text{ in } H(X_{j-1}, X_{j-2r})}.$$

If now $\{\gamma\} \in E_r^j$ and if $\gamma \in$ "Im $H(X_j, X_{j-r})$ in $H(X_{j+r-1}, X_{j-r})$" is a representative of $\{\gamma\}$, we set $\partial^r\{\gamma\} = \{\partial_0^r \alpha\}$ where $\alpha \in H(X_j, X_{j-r})$ is an element whose image in $H(X_{j+r-1}, X_{j-r})$ is γ and where $\{\partial_0^r \alpha\}$ is the equivalence class (i.e. the element of E_r^{j-r}) containing $\partial_0^r \alpha$. To see that ∂^r is a well-defined homomorphism from E_r^j to E_r^{j-r}, the following assertions must be verified:

(a) Let $\alpha \in H(X_j, X_{j-r})$. If the image of α in $H(X_{j+r-1}, X_{j-r})$ is 0, then $\partial_0^r \alpha = 0$.

(b) If $\alpha \in$ "Im $H(X_{j-1}, X_{j-r})$ in $H(X_j, X_{j-r})$", then $\partial_0^r \alpha = 0$.

We prove (a): Consider the sequence

$$H(X_{j+r-1}, X_j) \xrightarrow{\partial} H(X_j, X_{j-r}) \longrightarrow H(X_{j+r-1}, X_{j-r}).$$

This is the exact sequence of the triple $(X_{j+r-1}, X_j, X_{j-r})$. Since α is mapped to 0, we must have $\alpha = \partial\beta$ (by exactness) where $\beta \in H(X_{j+r-1}, X_j)$. We now consider the sequence

$$H(X_{j+r-1}, X_j) \xrightarrow{\partial} H(X_j) \longrightarrow H(X_j, X_{j-r}) \xrightarrow{\partial} H(X_{j-r})$$
$$\longrightarrow H(X_{j-r}, X_{j-2r}) \longrightarrow H(X_{j-1}, X_{j-2r}).$$

Now ∂ in the second display above is just the composite map of ∂ and the "inclusion" $H(X_j) \to H(X_j, X_{j-r})$. To show that the image of $\partial\alpha$ in

$H(X_{j-1}, X_{j-2r})$ is 0, it suffices to show that the image of $\partial\beta$ in $H(X_{j-1}, X_{j-2r})$ is 0. Indeed, the image of $\partial\beta$ in $H(X_{j-r})$ is already 0 because the section $H(X_j) \to H(X_j, X_{j-r}) \to H(X_{j-r})$ of the above sequence is just part of the exact sequence of the pair (X_j, X_{j-r}). This proves (a).

We next prove (b). Consider the following diagram.

$$\begin{tikzcd}
H(X_j, X_{j-r}) \arrow[r, "\partial"] & H(X_{j-r}) \arrow[rr] \arrow[dr, "i_2"] & & H(X_{j-r}, X_{j-2r}) \arrow[r] & H(X_{j-1}, X_{j-2r}) \\
H(X_{j-1}, X_{j-r}) \arrow[u, "i_1"] \arrow[ur, "\partial'"] & & H(X_{j-1}) \arrow[urr, "i_3"] & &
\end{tikzcd}$$

Let $\alpha \in H(X_j, X_{j-r})$ be such that $i_1(\beta) = \alpha$ where $\beta \in H(X_{j-1}, X_{j-r})$. Now $\partial^r_0\alpha = \partial^r_0 i_1(\beta) = i_3 i_2 \partial'(\beta)$. But $i_2\partial'(\beta) = 0$ because the section

$$H(X_{j-1}, X_{j-r}) \xrightarrow{\partial'} H(X_{j-r}) \xrightarrow{i_2} H(X_{j-1}),$$

is a portion of the exact sequence of the pair (X_{j-1}, X_{j-r}). Hence $\partial^r_0\alpha = 0$, as asserted.

We now prove that $\partial^r\partial^r = 0$. We show more, namely that $\partial^r_0\partial^r_0 = 0$. In fact, consider the following diagram.

$$\begin{tikzcd}
H(X_j, X_{j-r}) \arrow[r, "\partial_1"] & H(X_{j-r}) \arrow[r, "i_1"] & H(X_{j-r}, X_{j-2r}) \arrow[r] \arrow[dr, "\partial^1"] & & H(X_{j-1}, X_{j-2r}) \arrow[dl, "\partial_2"'] \\
& H(X_{j-2}, X_{j-3r}) & H(X_{j-2r}, X_{j-3r}) \arrow[l, "i_3"'] & H(X_{j-2r}) \arrow[l, "i_2"'] &
\end{tikzcd}$$

Now $\partial^r_0\partial^r_0\alpha = i_3 i_2 \partial^1 i_1 \partial_1(\alpha)$. Put $\beta = \partial_1(\alpha)$. Then as in our previous arguments, $\partial^1 i_1(\beta) = 0$. Hence, $\partial^r_0\partial^r_0\alpha = 0$.

We now prove $H(E^j_r) \cong E^j_{r+1}$. For each $x_0 \in H(X_j, X_{j-r})$ such that $\{x_0\} \in Z(E^j_r)$ (i.e. $\{\partial^r_0 x_0\} = 0$), we will define an element $\{\omega_0\} \in E^j_{r+1}$ such that the map $\{x_0\} \to \{\omega_0\}$ determines a homomorphism of $Z(E^j_r)$ into E^j_{r+1}. We then prove that this map is onto and that the kernel of the map is precisely $B(E^j_r)$. The proof will be split into three lemmas.

Lemma (1): Let $\partial^r\{x_0\} = 0$. Then there exist $z_0 \in H(X_{j-1}, X_{j-r})$ and $\omega_0 \in H(X_j, X_{j-r-1})$ such that $x_0 = \operatorname{im} z_0 + \operatorname{im} \omega_0$. (Recall that by $\operatorname{im} z_0$ we mean the image of z_0 in $H(X_j, X_{j-r})$. Similarly for $\operatorname{im} \omega_0$.)

Proof: Consider first the diagram

$$\begin{tikzcd}
H(X_j, X_{j-r}) \arrow[r, "\partial"] & H(X_{j-r}, X_{j-2r}) \arrow[r] & H(X_{j-1}, X_{j-2r}) \\
& H(X_{j-r-1}, X_{j-2r}) \arrow[u, "i"'] &
\end{tikzcd}$$

Since $\partial^r\{x_0\} = 0$, we see by looking at the definition of E_r^{j-r} that there exists $y_0 \in H(X_{j-r-1}, X_{j-2r})$ such that $\partial x_0 - iy_0$ has zero image in $H(X_{j-1}, X_{j-2r})$. Consider next the diagram

$$\begin{tikzcd}
H(X_j, X_{j-r}) \arrow[r, "\partial"] & H(X_{j-r}, X_{j-r-1}) \arrow[r] & H(X_{j-1}, X_{j-r-1}) \\
& H(X_{j-r-1}, X_{j-r-1}) \arrow[u] &
\end{tikzcd}$$

Since $H(X_{j-r-1}, X_{j-r-1}) = 0$, it follows easily from the above that ∂x_0 in $H(X_{j-r}, X_{j-r-1})$ has zero image in $H(X_{j-1}, X_{j-r-1})$. Using the exact sequence of the triple $(X_{j-1}, X_{j-r}, X_{j-r-1})$, we see that there exists $z_0 \in H(X_{j-1}, X_{j-r})$ such that $\partial x_0 = \partial z_0$ in $H(X_{j-r}, X_{j-r-1})$. Consider the image of z_0 under the map $H(X_{j-1}, X_{j-r}) \to H(X_j, X_{j-r})$; call this element $\operatorname{im} z_0$. Then $\partial(x_0 - \operatorname{im} z_0) = 0$ in $H(X_{j-r}, X_{j-r-1})$. Using the exact sequence of the triple $(X_j, X_{j-r}, X_{j-r-1})$, we see that there exists $\omega_0 \in H(X_j, X_{j-r-1})$ such that $x_0 - \operatorname{im} z_0 = \operatorname{im} \omega_0$, i.e. $x_0 = \operatorname{im} z_0 + \operatorname{im} \omega_0$.

Q.E.D.

We define the map from $Z(E_r^j)$ to E_{r+1}^j as follows: Let $\{x_0\} \in Z(E_r^j)$ and let x_0 be an element in $H(X_j, X_{j-r})$ representing $\{x_0\}$. By Lemma 1, $x_0 = \operatorname{im} z_0 + \operatorname{im} \omega_0$. Now $\omega_0 \in H(X_j, X_{j-r-1})$ and

$$E_{r+1}^j = \frac{\operatorname{Im} H(X_j, X_{j-r-1}) \text{ in } H(X_{j+r}, X_{j-r-1})}{\operatorname{Im} H(X_{j-1}, X_{j-r-1}) \text{ in } H(X_{j+r}, X_{j-r-1})}.$$

Let $\{\omega_0\}$ be the element in E_{r+1}^j determined by ω_0. We define the map by sending $\{x_0\} \to \{\omega_0\}$. We show this map is well-defined.

Lemma (2): If $\{x_0\} = 0$, then $\{\omega_0\} = 0$.

Proof: We have to prove two statements:

(a) If the image of x_0 in $H(X_{j+r-1}, X_{j-r})$ is 0, then $\{\omega_0\} = 0$.
(b) If $x_0 \in$ "$\operatorname{Im} H(X_{j-1}, X_{j-r})$ in $H(X_j, X_{j-r})$" then $\{\omega_0\} = 0$.

We prove (a): From Lemma 1, write $x_0 = \operatorname{im} z_0 + \operatorname{im} \omega_0$. It follows directly from the hypothesis that the image of x_0 in $H(X_{j+r}, X_{j-1})$ is 0.

Moreover, the image of z_0 in $H(X_{j+r}, X_{j-1})$ is 0. This follows immediately from the commutativity of the diagram

$$\begin{tikzcd} H(X_{j-1}, X_{j-r}) \arrow[rr] \arrow[dr] & & H(X_{j+r}, X_{j-1}) \\ & H(X_{j-1}, X_{j-1}) \arrow[ur] & \end{tikzcd}$$

Hence, the image of ω_0 in $H(X_{j+r}, X_{j-1})$ is 0. Let $\bar{\omega}_0$ be the image of ω_0 under the map $H(X_j, X_{j-r-1}) \to H(X_{j+r}, X_{j-r-1})$. Then the image of $\bar{\omega}_0$ in $H(X_{j+r}, X_{j-1})$ is 0. Using the exact sequence of the triple $(X_{j+r}, X_{j-1}, X_{j-r-1})$, we find that $\bar{\omega}_0 \in$ "Im $H(X_{j-1}, X_{j-r-1})$ in $H(X_{j+r}, X_{j-r-1})$". It follows immediately from the definition of E^j_{r+1} that $\{\omega_0\} = 0$. This proves (a).

To prove (b), we note that, by hypothesis, there exists z_0 in $H(X_{j-1}, X_{j-r})$ such that $x_0 = \text{im } z_0'$. Thus $\text{im } \omega_0 = \text{im } (z_0' - z_0)$; and therefore the image of ω_0 in the group $H(X_j, X_{j-1})$ is zero. By the exactness of the homology sequence for the triple $(X_j, X_{j-1}, X_{j-r-1})$ it follows that there exists $u_0 \in H(X_{j-1}, X_{j-r-1})$ such that $\omega_0 = \text{im } u_0$, proving that $\{\omega_0\} = 0$.

Q.E.D.

Lemmas 1 and 2 show that the map $\{x_0\} \to \{\omega_0\}$ is a homomorphism of $Z(E^j_r)$ into E^j_{r+1}. This map is onto because given $\{\omega_0\} \in E^j_{r+1}$, we may just choose $x_0 \in H(X_j, X_{j-r})$ such that $x_0 = \text{im } \omega_0$. To complete the proof of the theorem, we show that the kernel of the map $\{x_0\} \to \{\omega_0\}$ is $B(E^j_r)$.

Lemma (3): If $\{\omega_0\} = 0$, then $\{x_0\} \in B(E^j_r)$.

Proof: We have $x_0 = \text{im } z_0 + \text{im } \omega_0$. Now, $\{\omega_0\} = 0$ iff there exists $u_0 \in H(X_{j-1}, X_{j-r-1})$ such that the image of ω_0 in $H(X_{j+r}, X_{j-r-1})$ coincides with the image of u_0 in $H(X_{j+r}, X_{j-r-1})$. Let us denote by $\tilde{u}_0$, $\tilde{z}_0$ the images of u_0, z_0 in $H(X_j, X_{j-r})$. Also, denote by $\bar{x}_0, \bar{u}_0, \bar{z}_0$ the images of x_0, u_0, z_0 in $H(X_{j+r}, X_{j-r})$. It follows that $\bar{x}_0 = \bar{z}_0 + \bar{u}_0$. If we put $r_0 = x_0 - \tilde{u}_0 - \tilde{z}_0$, then r_0 has image zero in $H(X_{j+r}, X_{j-r})$. Using the exact sequence of the triple (X_{j+r}, X_j, X_{j-r}), we see that $r_0 = \partial s_0$, $s_0 \in H(X_{j+r}, X_j)$. Therefore $x_0 = \partial s_0 + \tilde{z}_0 + \tilde{u}_0$. If $\{\tilde{z}_0\}$ and $\{\tilde{u}_0\}$ are the elements in E^j_r determined by $\tilde{z}_0$ and $\tilde{u}_0$, it is easy to see that $\{\tilde{z}_0\} = \{\tilde{u}_0\} = 0$. Noting that

$$E^{j+r}_r = \frac{\text{Im } H(X_{j+r}, X_j) \text{ in } H(X_{j+2r-1}, X_j)}{\text{Im } H(X_{j+r-1}, X_j) \text{ in } H(X_{j+2r-1}, X_j)}$$

and hence that s_0 determines an element $\{s_0\} \in E^{j+r}_r$, it follows from the definition of ∂^r that $\{x_0\} = \partial^r\{s_0\}$. Thus $\{\omega_0\} = 0$ iff $\{x_0\} \in \partial^r E^{j+r}_r$.

Q.E.D.

This completes the proof of the theorem. Notice that in the course of the proof, we have not made use of all of the axioms for a homology theory (see Eilenberg-Steenrod). In particular, we have not used the dimension axiom. The fact that the Leray relation $H(E_r^j) \cong E_{r+1}^j$ holds independent of this axiom lends importance to the so-called generalized homology theories where the dimension axiom is not postulated. Notice also that the proof of the Leray relation is completely algebraic; there are no really vital references to topological spaces or continuous maps. Indeed the theory of spectral sequences has a wide range of applicability and is not restricted solely to topological questions. Before discussing an application of the theory of spectral sequences to fiber bundles, we make a few further remarks. If we set

$$E_{r,d}^j = \frac{\operatorname{Im} H_d(X_j, X_{j-r}) \text{ in } H_d(X_{j+r-1}, X_{j-r})}{\operatorname{Im} H_d(X_{j-1}, X_{j-r}) \text{ in } H_d(X_{j+r-1}, X_{j-r})},$$

then ∂^r is a homomorphism taking $E_{r,d}^j$ into $E_{r,d-1}^{j-r}$. As before

$$Z(E_{r,d}^j)/B(E_{r,d}^j) \cong E_{r+1,d}^j.$$

A sequence E_r^j of groups, with boundary operators ∂_r^j as in Theorem 1, and such that $H(E_r^j) \cong E_{r+1}^j$, is called a *spectral sequence.* If there exists r_0 such that $r \geqq r_0$ implies $\partial_r^j = 0$, so that $E_r^j \cong E_{r+1}^j$ for $r \geqq r_0$, we say that the spectral sequence is *convergent.* The groups E_r^j for $r \geqq r_0$ are then written as E_∞^j, and called the *limit groups* of the spectral sequence. If G is a group with a normal sequence whose factor groups are isomorphic to the groups E_∞^j, we say that the spectral sequence converges to normal factors for G, and we write $E_1^j \Rightarrow G$, $E_2^j \Rightarrow G$, $E_3^j \Rightarrow G$, etc.

We mention finally the spectral sequence in cohomology. Here we take a different normal series from the one we took in homology. The series is given as

$$\begin{aligned} H^*(X, A) &= \ker(H^*(X, A) \to H^*(A, A)) \supseteq \ker(H^*(X, A) \to H^*(X_1, A)) \\ &\supseteq \ker(H^*(X, A) \to H^*(X_2, A)) \supseteq \cdots \\ &\supseteq \ker(H^*(X, A) \to H^*(X, A)) = 0. \end{aligned}$$

We set

$$E_r^{*j} = \frac{\ker(H^*(X_{j+r-1}, X_{j-r}) \to H^*(X_{j-1}, X_{j-r}))}{\ker(H^*(X_{j+r-1}, X_{j-r}) \to H^*(X_j, X_{j-r}))}.$$

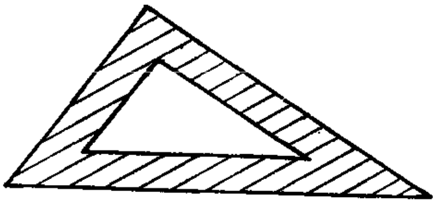
As before, we can prove that $E_1^{*j} = H^*(X_j, X_{j-1})$ and that E_∞^{*j} is the j-th factor of the above normal series. The theory then proceeds in much the same was as for homology.

One of the main applications of spectral sequences is to help find relationships between the homology and cohomology groups of the bundle space, base space and fiber of a fiber bundle. We shall give a brief glimpse of the method.

Let then X be a bundle over a (finitely) triangulable base B, with fiber F and projection π. We shall need the following proposition about bundles, the proof of which can be found in Steenrod—*The Topology of Fiber Bundles*, p. 53.

Proposition: Any bundle over a cell is a trivial bundle.

Let σ be a single simplex in B, $\partial\sigma$ its boundary. Let $(\partial\sigma)_\tau$ be the "thickened" boundary and let $\hat{\sigma} = \sigma - (\partial\sigma)_\tau$. Thus, in the figure to the right, the whole (closed) triangle represents σ, the shaded part (including the boundaries) represents $(\partial\sigma)_\tau$ and the inner (closed) triangle represents $\hat{\sigma}$.



By the above proposition, $\pi^{-1}(\sigma) = \sigma \times F, \pi^{-1}(\partial\sigma) = \partial\sigma \times F, \pi^{-1}((\partial\sigma)_\tau) = (\partial\sigma)_\tau \times F$. Since $\partial\sigma$ is clearly a deformation retract of $(\partial\sigma)_\tau$, $\partial\sigma \times F = \pi^{-1}(\partial\sigma)$ is also a deformation retract of $(\partial\sigma)_\tau \times F = \pi^{-1}((\partial\sigma)_\tau)$. Let X_j denote the part of X lying over the j-skeleton of B and let $(X_k)_\tau$ denote the part of X lying over the union of the $k-1$ skeleton of B and the thickened boundaries of dimension k. Then, X_{j-1} is a deformation retract of $(X_{j-1})_\tau$ and therefore $H(X_j, X_{j-1}) \cong H(X_j, (X_{j-1})_\tau)$. By excision, $H(X_j, X_{j-1}) \cong H\left(\pi^{-1}\left(\bigcup_\alpha \hat{\sigma}_\alpha^j\right), \pi^{-1}\left(\bigcup_\alpha \partial\hat{\sigma}_\alpha^j\right)\right)$ where we sum over all j-simplices in B. But observe that the simplices $\hat{\sigma}_\alpha^j$ are mutually disjoint. Hence, we may write

$$H\left(\pi^{-1}\left(\bigcup_\alpha \hat{\sigma}_\alpha^j\right), \pi^{-1}\left(\bigcup_\alpha \partial\hat{\sigma}_\alpha^j\right)\right) = \sum_\alpha \oplus H(\pi^{-1}(\hat{\sigma}_\alpha^j), \pi^{-1}(\partial\hat{\sigma}_\alpha^j))$$
$$= \sum_\alpha \oplus H(\hat{\sigma}_\alpha^j \times F, \partial\hat{\sigma}_\alpha^j \times F).$$

An elementary calculation, using Kunneth's theorem, shows that $\sum_\alpha \oplus H(\hat{\sigma}_\alpha^j \times F, \partial\hat{\sigma}_\alpha^j \times F) \cong \sum_\alpha \oplus H(F)$. Recalling the summation range of α, it follows that

$$H(X_j, X_{j-1}) \cong C_j(B, H(F))$$

where $C_j(B, H(F))$ is the group of j-chains of B with coefficients in $H(F)$.

Now, $X = X_n \supseteq X_{n-1} \supseteq \cdots$ is a filtration of X, corresponding to the filtration $B = B_n \supseteq B_{n-1} \supseteq \cdots$ of the complex B. By definition, $E_1^j = H(X_j, X_{j-1})$. Hence, the above result can be restated as

$$E_1^j \cong C_j(B, H(F)).$$

Finally, we assert without proof that

$$E_2^j \cong H_j(B, H(F)).$$

This formula can be considered to be the fundamental relationship which we have been seeking. The proof of the formula follows without difficulty from the Leray relation $H(E_r^j) \cong E_{r+1}^j$. We leave the details to the reader.

The preceding theory now implies that there exists, a convergent spectral sequence through which $H(B, H(F)) \Rightarrow H(X)$. This fundamental result in the homology theory of fiber bundles, due to Levay, has played a very significant role in the advance of topology during the last three decades.

CHAPTER VI

Introduction to the Theory of Characteristic Classes

In this chapter, we study of the some the homotopy and cohomology theory of bundles. Proofs of many standard theorems are omitted; they can all be found in Steenrod—*Topology of Fiber Bundles.*

Our main aim in this chapter is to show that, at least in certain respects, the study of an arbitrary principal G-bundle (see definition below) can be reduced to the study of a particular principal G-bundle, the so-called universal G-bundle. In particular, if G is one of the classical Lie groups U_n or O_n, the base space of the corresponding universal bundle turns out to be the space $G_{n,\infty}$. We can then use our knowledge of the cohomology structure of $G_{n,\infty}$ to define and study the Chern and Whitney classes of an arbitrary principal U_n or O_n bundle. Our study of these "characteristic classes" will be continued in subsequent chapters. Finally, we mention briefly (and do not use again) the Pontryagin classes.

We begin by recalling some facts about homotopy groups. For details, see Hu—*Homotopy Theory,* Chapter IV.

Let X be a topological space, A a subspace and $x_0 \in A$ a fixed point. Denote by I^{n-1} the face of the cube I^n defined by $t_n = 0$, and by J^{n-1} the set $\overline{\partial I^n - I^{n-1}}$. Consider the set $F^n(X, A, x_0)$ of all maps f of the cube I^n into X which carry I^{n-1} into A and J^{n-1} into the point x_0. An "addition" is defined in $F^n(X, A, x_0)$ for $n > 1$ by setting

$$(f+g)(t) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leqq t_1 \leqq \frac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \leqq t_1 \leqq 1 \end{cases}$$

for $f, g \in F^n(X, A, x_0)$. We say two maps $f_0, f_1 \in F^n(X, A, x_0)$ are equivalent if there is a deformation f_t connecting f_0 and f_1 such that

$$f_t(I^{n-1}) \subseteq A, \quad f_t(J^{n-1}) = x_0, \quad 0 \leqq t \leqq 1.$$

We denote the set of equivalence classes thus obtained by $\pi_n(X, A, x_0)$. The addition in $F^n(X, A, x_0)$ induces an addition in $\pi_n(X, A, x_0)$ which turns $\pi_n(X, A, x_0)$ into a group. We call $\pi_n(X, A, x_0)$ the n-th relative homotopy group of the pair (X, A) with respect to the basepoint x_0; it is known to be abelian for $n \geqq 3$. If, in particular, $A = x_0$, we put $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$ and call this group the n-th absolute homotopy group of X with respect to x_0; $\pi_n(X, x_0)$ is abelian for $n \geqq 2$. For $n = 1$, we get the fundamental group of X at x_0. If the spaces X and A are arcwise connected, then the n-th homotopy groups are isomorphic for any two basepoints x_0, x_1. In this case, we will write $\pi_n(X, A)$ instead of $\pi_n(X, A, x_0)$.

If $f: (X, A, x_0) \to (Y, B, y_0)$ is such that $f(A) \subseteq B$ and $f(x_0) = y_0$, then it can be shown that f induces a homomorphism

$$f_*: \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0).$$

Also, if $f \in F^n(X, A, x_0)$, then the restriction of f to the face I^{n-1} will define an element in $\pi_{n-1}(A, x_0)$ and in this way we get a homomorphism

$$\partial: \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

called the boundary homomorphism. We now have the

Lemma (1): The following sequence of homotopy groups of the pair (X, A) is exact:

$$\cdots \longrightarrow \pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \xrightarrow{j_*} \pi_{n-1}(X, A) \longrightarrow \cdots$$

Here, i_* and j_* are the maps induced by the inclusion map i, j.

Remark: There is a useful extension of the notion of homotopy sequence to that of a triple (X, A, B) where $X \supseteq A \supseteq B$ and where the basepoint $x_0 \in B$. It is the sequence

$$\cdots \longrightarrow \pi_n(A, B) \xrightarrow{i_*} \pi_n(X, B) \xrightarrow{j_*} \pi_n(X, A) \xrightarrow{\tilde{\partial}} \pi_{n-1}(A, B) \longrightarrow$$
$$\cdots \longrightarrow \pi_2(X, A),$$

where i, j are inclusion maps and $\tilde{\partial}$ is the composition

$$\pi_n(X, A) \xrightarrow{\partial} \pi_{n-1}(A) \xrightarrow{k_*} \pi_{n-1}(A, B)$$

where k is the inclusion map. It follows without much difficulty from Lemma (1) that this sequence is exact.

We now turn to the homotopy properties of fiber bundles. Here, our constant reference is Steenrod, loc. cit. The key property which fiber bundles possess and from which many theorems about them follow is

Theorem (1) (Covering Homotopy Theorem): (cf. Steenrod, p. 50, and Hu, p. 65).

If X is a bundle over B with projection p and if $f: K \to X$ is a map of a "nice" space (e.g. a finite complex) into X such that there exists a homotopy $g_t: K \to B$ with $g_0 = p \circ f$, then there exists a homotopy $f_t: K \to X$ with $f_0 = f$, $p \circ f_t = g_t$.

From this theorem and Lemma (1), we can construct the

Exact Homotopy Sequence of a Bundle: (Steenrod, p. 90).

In the notation of the theorem above, let $x_0 \in X$, $b_0 = p(x_0)$, $F_0 = p^{-1}(b_0)$, so that F_0 is homeomorphic to the fiber F. It follows from Theorem 1 that $p_*: \pi_n(X, F_0, x_0) \to \pi_n(B, b_0, b_0)$ is an isomorphism for $n \geqq 2$. Using this fact, we define a homomorphism

$$\Delta : \pi_n(B, b_0) \to \pi_{n-1}(F_0, x_0)$$

by $\Delta = \partial \circ p_*^{-1}$ where $\partial: \pi_n(X, F_0) \to \pi_{n-1}(F_0)$ is the boundary homomorphism. Then the exact sequence of the pair (X, F_0) (see Lemma 1) is transformed into an exact sequence

$$\cdots \xrightarrow{\Delta} \pi_n(F_0) \xrightarrow{i_*} \pi_n(X) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\Delta} \pi_{n-1}(F_0) \xrightarrow{i_*} \cdots$$
$$\cdots \xrightarrow{p_*} \pi_2(B) \xrightarrow{\Delta} \pi_1(F_0) \xrightarrow{i_*} \pi_1(X) \xrightarrow{p_*} \pi_1(B).$$

Definition (1): A bundle ξ is called a *principal* G-bundle if (a) the fiber F = the group G and (b) G operates on itself by left translation.

Given a bundle ξ, with base B, group G and fiber F, then we can construct the *associated principal bundle* $\tilde{\xi}$ in the following way. Let $\tilde{\xi}$ have the same base B, the same coordinate neighborhoods U_α, the same $U_b(\alpha, \beta)$ and the same group G as ξ, but replace the fiber F of ξ by the group G and allow G to operate on itself by left translation.

Remark: Let us observe that, up to equivalence, there is a $1-1$ correspondence between G-bundle structures over a space B and principal G-bundles over B. This follows from the fact that given a G-bundle structure over B, there is a unique way of constructing a principal G-bundle over B (see the Existence Theorem in Chapter IV).

Theorem (2) (Bundle Structure Theorem): (Steenrod, p. 30).

Let G be a topological group, H and K closed subgroups such that $K \subsetneqq H$, and let H admit a local cross-section. If p is the map induced by inclusion of cosets, then G/K is a bundle over G/H with projection p, fiber H/K and group H/K_0 acting on H/K by left translation (where K_0 is the largest subgroup of K invariant in H). Furthermore, any two local cross-sections lead to equivalent bundles. Finally, the left translations of G/K by elements of G are bundle maps of this bundle onto itself.

As a special case note that if we take $K = \{e\}$ then G is a principal bundle over G/H with fiber and group H.

Let us use the above ideas to prove the following lemma, which will be useful in what follows.

Lemma (2): If O_m is the group of orthogonal transformations of E^m, then O_N/O_n is "aspherical" for all $k < n$, i.e. $\pi_k(O_N/O_n) = 0$ for $k < n$.

(Notice that O_N/O_n is arc-wise connected because O_N has two components and O_n contains matrices with both positive and negative determinant.)

Proof: O_m is a Lie group, and therefore O_n has a local cross-section in O_{n+1}. We have $O_n \subset O_{n+1} \subset O_{n+2} \subset \cdots \subset O_N$. By the Bundle Structure Theorem, O_N/O_n is a bundle over O_N/O_{n+1} with fiber $O_{n+1}/O_n = S^n$. Now, $\pi_k(S^n) = 0$ if $k < n$.

By writing down the homotopy exact sequence of this bundle, we see that every third term in the sequence is zero. Therefore,

$$\pi_k(O_N/O_n) \cong \pi_k(O_N/O_{n+1}) \quad \text{if} \quad k < n.$$

By repeating this argument, we get

$$\pi_k(O_N/O_n) \cong \pi_k(O_N/O_{n+1}) \cong \cdots \cong \pi_k(O_N/O_N) = 0.$$

Q.E.D.

Lemma (2′): If U_n is the group of all unitary transformations of $\mathbb{C}^n$, then

$$\pi_k(U_N/U_n) = 0 \quad \text{if} \quad k < 2n + 1.$$

The proof is analogous to that of Lemma 2.

Definition (2): A k-frame in E^N is an ordered set of k independent vectors in E^N. Call the set of all orthonormal $(N-n)$-frames in E^N the *Stiefel Manifold* $V_{N-n,N}$. Clearly, O_N acts transitively on $V_{N-n,N}$. The subgroup leaving fixed a certain frame f_0^{N-n} is just the orthogonal group $O_{N-(N-n)} = O_n$ operating in the space orthogonal to all the vectors of f_0^{N-n}. Thus, $V_{N-n,N} = O_N/O_n$.

If we work in $\mathbb{C}^N$, we can define $V^{\mathbb{C}}_{N-n,N}$ simularly and show that $V^{\mathbb{C}}_{N-n,N} = U_N/U_n$.

We have $O_n \subseteq O_n \times O_m$. By Theorem 2, $V_{m,n+m} = O_{n+m}/O_n$ is a bundle over $O_{n+m}/O_n \times O_m = G_{m,m+n}$, with fiber $O_m \times O_n/O_n \cong O_m$ and group $O_m \times O_n/O_n \cong O_m$. I.e. the Stiefel manifold $V_{m,m+n}$ is a principal bundle over the *Grassmann manifold* $G_{m,m+n}$ with group O_m and fiber O_m.

Definition (3): Let ξ be a principal bundle over a space B with group G. Then ξ is called n-universal for the group G if for any n-complex K, subcomplex L of K, principal bundle ξ' over K with group G, and bundle map f of $\xi'|L \to \xi$, there exists an extension of f to a bundle map of $\xi' \to \xi$. (Notice that if η is a bundle with bundle space X, base space A, and projection map $p: X \to A$, and if $\hat{A} \subset A$, then we can construct, in the obvious way, a bundle $\hat{\eta}$ with bundle space $p^{-1}(\hat{A})$, base space $\hat{A}$ and projection $p|p^{-1}(\hat{A})$.)

Characterization of an n-Universal Bundle

Theorem (3): (Steenrod, p. 102).

A principal bundle ξ, with bundle space X, is n-universal iff $\pi_i(X) = 0$ for $0 \leqq i < n$. (Note: $\pi_0(X) = 0$ means X is arcwise connected.)

Corollary: The real (resp. complex) Grassmannian $G_{m,n}$, with the bundle structure defined over it as described above, is an $(n-m)$-universal O_m-bundle (resp. $[2(n-m)+1]$-universal U_m-bundle).

Proof: We showed above that $V_{m,n}$ is a principal O_m bundle over $G_{m,n}$, and $\pi_k(V_{m,n}) = 0$ if $0 \leqq k < n-m$. Similarly for the complex case.

Definition (4): We have a sequence of natural inclusions $E^m \subseteq E^{m+1} \subseteq \cdots$ which induces a sequence of inclusions $V_{m,n} \subseteq V_{m,n+1} \subseteq \cdots$. Let $V_{m,\infty}$ be the direct limit space; $V_{m,\infty}$ is then the set of orthogonal m-frames in E^∞.

Definition (5): Consider $V_{m,\infty}$ is a principal O_m-bundle over $G_{m,\infty}$ (U_m-bundle in the complex case). The bundle constructed in this way is r-universal for O_m-bundles (resp. U_m-bundles) for all integers $r > 0$. We are going to call the space $G_{m,\infty}$ a *universal classifying space* for O_m-bundles (resp. U_m-bundles). We will denote this principal bundle and its bundle structure by α_m.

Theorem (4) (Classification Theorem): (Steenrod, p. 101).

Let ξ be a principal G-bundle with the base space B and bundle space X. Assume $\pi_k(X) = 0$ $(0 \leqq k \leqq n)$. Let K be a complex of dimension n. Then the operation of assigning to each map $f: K \to B$ its induced bundle over K gives a $1-1$ correspondence between G-bundle structures on K and homotopy classes of maps of $K \to B$.

Corollary: Let K be a complex of finite dimension. Then O_m-bundle structures (resp. U_m-bundle structures) on K are in $1-1$ correspondence with the homotopy classes of maps of $K \to G^{\mathbb{R}}_{m,\infty}$ (resp. $G^{\mathbb{C}}_{m,\infty}$).

Definition (6): Recall that if $\phi : X \to Y$, then ϕ induces a map ϕ^* of the cohomology ring $H^*(Y) \to H^*(X)$, and induces a map $\phi^\dagger$ of G-bundle structures on Y into G-bundle structures on X. Let now Γ be a map assigning to each G-bundle structure ξ on an arbitrary space Y an element of $H^*(Y; C)$ such that $\Gamma(\phi^\dagger\xi) = \phi^*\Gamma(\xi)$ for all spaces X, Y and all maps $\phi : X \to Y$. Then Γ is called a *C-cohomology invariant of G-bundle structures.* Here C is a commutative ring taken as a coefficient ring for the cohomology rings of spaces.

Example of a Cohomology Invariant

Let β be an O_m-bundle over a triangulable space B with fiber F, bundle space X. Assume $\pi_i(F) = 0$ for $i \leqq n-1$ and let $n > 1$. Let B^k be the k-skeleton of B, σ^{k+1} a $(k+1)$-dimensional simplex of B, and $\partial\sigma^{k+1}$ its boundary (which is homeomorphic to S^k, the k-dimensional sphere). Let ϕ be a cross-section defined on B^k; we want to investigate the possibility of extending ϕ to B^{k+1}. The restriction of β to the call σ^{k+1} is a trivial bundle since every bundle over a cell is trivial. Therefore $\phi|\partial\sigma^{k+1}$ is a map of $\partial\sigma^{k+1} \to \partial\sigma^{k+1} \times F$ defined by $\phi(c) = (c, f)$, and hence defines a map (call it ϕ again) of $\partial\sigma^{k+1} \to F$. Since $\partial\sigma^{k+1}$ is homeomorphic to S^k and F is simply connected, ϕ defines an element α_0 in $\pi_k(F)$. It is easy to see that ϕ can be extended to σ^{k+1} iff $\alpha_\sigma = 0$. Therefore, if $k \leqq n-1$, ϕ can always be extended to all of B^{k+1}. Assume now $k = n$; then the operation of assigning to each σ^{k+1} the element α_σ in $\pi_k(F)$ as above defines a $(k+1)$-cochain α on B. It can be shown that α is a cocycle and does not depend on the successive extensions of ϕ on lower dimensions. Therefore we could start by ϕ defined arbitrarily on B^0. It is evident therefore that to each β, we can associate a uniquely defined element in $H^{n+1}(B, \pi_n(F))$; denote this element by $\text{obst}_{n+1}(\beta, F)$. The map "obst" of O_m-bundle structures on B into $H^{n+1}(B, \pi_n(F))$ may be shown to be a $\pi_n(F)$-cohomology invariant of O_m-bundle structures.

Theorem (5): The set S of Z-cohomology invariants of U_m-bundle structures is in $1-1$ correspondence with the set of cohomology classes of $G^{\mathbb{C}}_{m,\infty}$ with integer coefficients.

Proof: Recall that $V^{\mathbb{C}}_{m,\infty}$ is a principal U_m-bundle over $G^{C}_{m,\infty}$. Let α_m be the corresponding bundle structure. Let Γ be an element of S. Then $\Gamma(\alpha_m)$ is an element γ of cohomology ring $H^*(G_{m,\infty}, Z)$. Let β_1 be a U_m-bundle structure over a base B. Then, by the classification theorem, there exists a map ϕ (unique up to homotopy) of $B \to G^{\mathbb{C}}_{m,\infty}$ such that $\beta_1 = \phi^{\dagger}(\alpha_m)$. But then

$$\Gamma(\beta_1) = \Gamma(\phi^{\dagger}\alpha_m) = \phi^{*}\Gamma(\alpha_m) = \phi^{*}(\gamma).$$

If Γ' is another cohomology invariant such that $\Gamma'(\alpha_m) = \gamma$, then $\Gamma'(\beta_1) = \phi^{*}(\gamma)$, and therefore $\Gamma(\beta_1) = \Gamma'(\beta_1)$ for all spaces B and U_m-bundle structures β_1, i.e. $\Gamma = \Gamma'$. Conversely, let γ be any element in $H^{*}(G^{\mathbb{C}}_{m,\infty}, Z)$; we want to define an element Γ_{γ} in S which corresponds to γ. Let B be any space, ξ a U_m-bundle structure on B. There is a map ϕ (unique up to homotopy) from B into $G^{\mathbb{C}}_{m,\infty}$ such that $\xi = \phi^{\dagger}(\alpha_m)$. Define $\Gamma_{\gamma}(\xi) = \phi^{*}(\gamma) \in H^{*}(B, Z)$. It is easy to check that Γ_{γ} is a well defined Z-cohomology invariant of U_m-bundle structures. We have only to show that if $\Gamma_{\gamma} = \Gamma_{\gamma'}$, then $\gamma = \gamma'$. It is clearly enough to check that $\Gamma_{\gamma}(\alpha_m) = \gamma$. But this is clear since if i is the identity map of $G^{\mathbb{C}}_{m,\infty} \to G^{\mathbb{C}}_{m,\infty}$, then $i^{\dagger}$ = identity, i^{*} = identity.

This completes the proof of the theorem.

Recalling that the Chern classes $c_j \in H^{2j}(G_{m,\infty}, Z)$ form a set of ring generators of $H^{*}(G_{m,\infty}, Z)$, we have the following

Corollary: Every Z-cohomology invariant of U_m-bundle structures can be identified in a unique way with a linear combination of cup products of Chern classes.

Similarly, we can prove the following theorem and corollary.

Theorem (6): The set S' of Z_2-cohomology invariants of O_m-bundle structures is in $1-1$ correspondence with the set of cohomology classes of $G^{R}_{m,\infty}$ with Z_2 coefficients.

Corollary: Every Z_2-cohomology invariant of O_m-bundle structures can be identified in a unique way with a linear combination of cup products of Whitney classes.

Definition (7): Let $\xi = (X, B, U_m, p)$ be a principal U_m-bundle with base B, bundle space X and projection p. Since $G^{\mathbb{C}}_{m,\infty}$ is a universal classifying space, there exists a map $\psi : B \to G^{\mathbb{C}}_{m,\infty}$, which is unique up to homotopy, such that $\xi = \psi^{\dagger}(\alpha_m)$ where $\alpha_m = (V_{m,\infty}, G_{m,\infty}, U_m, p_m)$. The map ψ induces a map ψ^{*}, on the cohomology level, from $H^{*}(G^{\mathbb{C}}_{m,\infty})$ into $H^{*}(X)$. Notice that if ψ is homotopic to $\tilde{\psi}$, then $\psi^{*} = \tilde{\psi}^{*}$. Thus, ψ^{*} is determined by ξ itself and does not depend on the mapping ψ. We can then define the *Chern class* $c_j(\xi)$ of the principal bundle ξ by $c_j(\xi) = \psi^{*}(c_j)$, where c_j is the j-th Chern class of α_m. Notice that $c_j(\xi)$ belongs to $H^{2j}(X)$. Define the Chern classes of an arbitrary U_m-bundle to be the Chern classes of the associated principal bundle.

Definition (8): $c(\xi) = 1 + c_1(\xi) + \cdots + c_m(\xi)$ is called the *total Chern class* of ξ.

One can define the *Whitney classes* of O_m-bundles and total Whitney class of O_m-bundles analogously by using the real Grassmann manifold $G^R_{m,\infty}$ and the cohomology ring of $G^R_{m,\infty}$ with coefficients in Z_2.

Conjugate Bundle

Let ξ be a U_m-bundle structure on B with coordinate neighborhoods N_i, and coordinate transformations $U_b(i,j)$. Now $U_b(i,j)$, which is an element in U_m, can be considered as an $m \times m$ unitary matrix. Let $\bar{U}_b(i,j)$ be the complex conjugate of $U_b(i,j)$, again an element in U_m. Denote by $\bar{\xi}$ the bundle structure over B with coordinate neighborhoods N_i and coordinate transformations $\bar{U}_b(i,j)$. $\bar{\xi}$ is called the *conjugate* of ξ.

We want to find the relation between the Chern classes of ξ and those of $\bar{\xi}$.

The bundle structure ξ is induced by a map $\phi : B \to G_{m,\infty}$. There is a map

$$C : G_{m,\infty} \to G_{m,\infty}$$

defined by mapping an element $\pi \in G_{m,\infty}$ (π is an m-plane through the origin in E^∞) into its complex conjugate $\bar{\pi} \in G_{m,\infty}$. It is easy to see then that the map

$$C \circ \phi : B \to G_{m,\infty}$$

induces the bundle structure $\bar{\xi}$. Let

$$C' : P_n^{(1)} \times P_n^{(2)} \times \cdots \times P_n^{(m)} \to P_n^{(1)} \times P_n^{(2)} \times \cdots \times P_n^{(m)}$$

be defined by $C'((z^{(1)}, z^{(2)}, \ldots, z^{(m)})) = (\bar{z}^{(1)}, \bar{z}^{(2)}, \ldots, \bar{z}^{(m)})$. Recall that in Chapter IIIA, we introduced a map $N : P_n^{(1)} \times \cdots \times P_n^{(m)} \to G_{m,m(n+1)}$. Letting $\tilde{N}$ be the composition of N and the inclusion map $G_{m,m(n+1)} \to G_{m,\infty}$, we see that

$$\begin{tikzcd} P_n^{(1)} \times \cdots \times P_n^{(m)} \arrow[r, "\tilde{N}"] \arrow[d, "C'"] & G_{m,\infty} \arrow[d, "C"] \\ P_n^{(1)} \times \cdots \times P_n^{(m)} \arrow[r, "\tilde{N}"] & G_{m,\infty} \end{tikzcd}$$

is a commutative diagram.

Recall that a cell decomposition of P_n is given by

$$P_n \supseteq P_{n-1} \supseteq \cdots \supseteq P_1 \supseteq P_0.$$

Clearly, the map C' takes each cell of P_n homeomorphically onto itself, but with a possible change of orientation. By looking at the coordinates on

P_j, it is not hard to see that in fact C' maps P_j onto itself homeomorphically with degree $(-1)^j$. Therefore, the induced map of cohomology classes takes the cohomology class t^j into $(-1)^j t^j$, where t is a ring generator of $H^*(P_n, Z)$. By using the commutativity of the above diagram, one can see that C^* has the same effect on $H^*(G_{m,\infty})$ as $(C')^*$ has on $H^*(P_n^{(1)} \times \cdots \times P_n^{(m)})$. If ψ is the map of Definition 7, we have $c_j(\xi) = \psi^*(c_j)$. Moreover,

$$c_j(\bar{\xi}) = (C \circ \psi)^* (c_j) = \psi^* C^* (c_j) = \psi^* ((-1)^j c_j)$$

We thus arrive at the formula

$$c_j(\bar{\xi}) = (-1)^j c_j(\xi). \tag{*}$$

Pontryagin Classes

Let γ be an O_m-bundle structure on B, with coordinate neighborhoods N_i and coordinate transformations $U_b(i,j)$. As before, $U_b(i,j)$ can be considered as an $m \times m$ real orthogonal matrix. If we "enlarge" the group of γ by considering O_m as a subgroup of U_m, one can consider $U_b(i,j)$ as an element of U_m and get a U_m-bundle structure γ^c on B. Now, it is clear that $\gamma^c = \bar{\gamma}^c$. Therefore, by (*) above,

$$c_j(\gamma^c) = c_j(\bar{\lambda}^c) = (-1)^j c_j(\gamma^c).$$

Thus, we have $2c_j(\gamma^c) = 0$ if j is odd. Define the *Pontryagin classes* $p_j(\gamma)$ of the O_m-bundle structure γ by

$$p_j(\gamma) = c_{2j}(\gamma^c).$$

It is clear that $p_j(\gamma)$ is an element of $H^{4j}(B)$.

CHAPTER VII

Riemannian Geometry. An Application of Characteristic Classes

First, we shall discuss some of the aspects of Riemannian Geometry; later we shall consider some applications of the previous theory to certain geometric problems. As a reference, we suggest Milnor—*Morse Theory.*

To begin with, we recall that a vector field V on a manifold M is a linear map from $C^\infty(M)$ into itself (here $C^\infty(M)$ denotes the algebra of C^∞ real-valued functions defined on M) which satisfies $V(\phi\psi) = (V\phi)\psi + \phi(V\psi)$ for $\phi, \psi \in C^\infty(M)$. If U and V are vector fields, it is easy to verify that the Poisson Bracket $[U, V] = UV - VU$ is also a vector field. Let $D^1(M)$ (or more briefly, D^1) denote the set of all vector fields on M.

Definition (1): A *Riemannian metric* on M is a bilinear map $g : D^1 \times D^1 \to C^\infty(M)$ such that

(i) g is local, i.e. $g(\phi U, \psi V) = \phi\psi g(U, V)$ for $U, V \in D^1$ and $\phi, \psi \in C^\infty(M)$;
(ii) g is symmetric, i.e. $g(U, V) = g(V, U)$ for $U, V \in D^1$;
(iii) g is positive definite, i.e. $g(U, U) \geqq 0$ and $g(U, U) = 0 \Leftrightarrow U = 0$.

Definition (2): An *affine connection* on M is a bilinear map $\vdash : D^1 \times D^1 \to D^1$ (we write $\vdash(U, V) = V \vdash U$) such that

(i) $\phi V \vdash U = \phi(V \vdash U)$,
(ii) $V \vdash \phi U = (V\phi)U + \phi(V \vdash U)$.

Definition (3): An affine connection is said to be *symmetric* if $(V \vdash U) - (U \vdash V) = [V, U]$ for $U, V \in D^1$.

Definition (4): An affine connection is said to be *appropriate to a given metric* if

$$W(U \circ V) = (W \vdash U) \circ V + U \circ (W \vdash V) \quad \text{for} \quad U, V, W \in D^1$$

(Here, we write $U \circ V$ for $g(U, V)$.)

Let us examine the above definitions in local coordinates. We set $g_{ij} = g(\partial_i, \partial_j) = \partial_i \circ \partial_j$. $\left(\text{Here, we have written } \partial_i = \dfrac{\partial}{\partial x_i}\right)$. Thus, if $U = u^i\partial_i, V = v^j\partial_j$ (we employ the summation convention), we have $g(U, V) = g_{ij}u^iv^j$. If $\partial_j \vdash \partial_i = \Gamma^k_{ji}\partial_k$, $V \vdash U = v^i\partial_j \vdash u^i\partial_i = v^j(\partial_j \vdash u^i\partial_j) = v^j(u^i\partial_j \vdash \partial_i + (\partial_j u^i)\,\partial_i) = v^i\,(u^i\Gamma^k_{ji}\partial_k + (\partial_j u^i)\,\partial_i) = (v^ju^i\Gamma^k_{ji} + v^j\partial_j u^k)\,\partial_k$. It is an easy matter to check that the connection is symmetric iff $\Gamma^k_{ij} = \Gamma^k_{ji}$. Concerning Definition 4, we have the following

Theorem (1): Let M be a Riemannian manifold, i.e. a manifold with a given Riemannian metric g. Then there exists exactly one symmetric connection on M which is appropriate to g.

Proof: We shall first prove the uniqueness. Let $\vdash$ be a connection on M with the required properties. Locally, $\vdash$ is completely determined by the Γ^k_{ij}. We shall obtain an explicit formula for the Γ^k_{ij} in terms of the metric. It is clear that this establishes uniqueness. We begin by writing out the equation in Definition 4 in local coordinates. We have

$$W(U \circ V) = w^k\,(\partial_k\,(g_{ij}\,u^iv^j)) \quad \text{where} \quad W = w^k\partial_k.$$

Now,

$$\begin{aligned}(W \vdash U) \circ V + U \circ (W \vdash V) &= (w^ku^i\Gamma^l_{ki} + (\partial_k u^l)\,w^k)\,\partial_l \circ v^j\partial_j \\ &\quad + u^i\partial_i \circ (w^kv^j\Gamma^l_{kj} + (\partial_k v^l)\,w^k)\,\partial_l \\ &= g_{lj}\,\{u^iv^jw^k\Gamma^l_{ik} + v^jw^k\partial_k u^l\} \\ &\quad + g_{il}\,\{u^iv^jw^k\Gamma^l_{kj} + u^iw^k\partial_k v^l\}.\end{aligned}$$

We put $\Gamma_{\varrho\sigma\tau} = g_{\varrho l}\Gamma^l_{\sigma\tau}$. Then $\Gamma^\varrho_{l\sigma} = g^{\varrho l}\Gamma_{l\sigma\tau}$ where the matrix $\|g^{ij}\|$ is the inverse of the (positive definite, and hence invertible) matrix $\|g_{ij}\|$. As

$$w^k\,(\partial_k\,(g_{ij}u^iv^j)) = w^k\,\{(\partial_k g_{ij})\,u^iv^j + g_{ij}u^i\,(\partial_k v^j) + g_{ij}v^j\,(\partial_k u^i)\},$$

we get

$$\begin{aligned}&\{(\partial_k g_{ij})\,u^iv^jw^k\} + \{g_{ij}u^iw^k\,(\partial_k v^j) + g_{ij}v^jw^k\,(\partial_k u^i)\} \\ &\qquad = \{g_{lj}\Gamma^l_{ki}u^iv^jw^k + g_{il}\Gamma^l_{kj}u^iv^jw^k\} + \{g_{ij}u^iw^k\,(\partial_k v^j) + g_{ij}v^jw^k\,(\partial_k u^i)\}.\end{aligned}$$

Hence, $\partial_k g_{ij} = \Gamma_{jki} + \Gamma_{ikj}$. We write $\partial_\alpha g_{\beta\gamma} = \{\alpha\beta\gamma\}$, $\Gamma_{\alpha\beta\gamma} = [\alpha\beta\gamma]$. The previous equation gives us $\{kij\} = [jki] + [ikj]$. Taking cyclic permutations of the indices yields $\{ijk\} = [kij] + [jik]$, $\{jki\} = [ijk] + [kji]$. Since the connection is symmetric, $[\alpha\beta\gamma] = [\alpha\gamma\beta]$. Thus, the three equations give us $[ijk] = \frac{1}{2}(\{kij\} + \{jki\} - \{ijk\})$, i.e. $\Gamma_{ijk} = \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})$.

Therefore $\Gamma^i_{jk} = \frac{1}{2}(\partial_k g_{\alpha j} + \partial_j g_{k\alpha} - \partial_\alpha g_{jk})\, g^{\alpha i}$ and we are finished. To prove existence, we just define the Γ^i_{jk} by the above formula. It is easy to verify that the resulting connection satisfies all of the required properties.

Definition (5): A covariant n-tensor field on a manifold M is an n-linear map $T\colon \overbrace{D^1 \times \cdots \times D^1}^{n} \to C^\infty(M)$ which is local, i.e. $T(\phi_1 U_1, \ldots, \phi_n U_n) = \phi_1 \cdots \phi_n T(U_1, \ldots, U_n)$.

A Riemannian metric is a covariant 2-tensor field. We shall now construct two other important tensor fields.

1) The covariant derivative of a covariant n-tensor field

T: Define
$$\begin{aligned} DT(U_1, \ldots, U_n, W) &= W(T(U_1, \ldots, U_n)) \\ &\quad - T(W \vdash U_1, U_2, \ldots, U_n) - T(U_1, W \vdash U_2, U_3, \ldots, U_n) \\ &\quad - \cdots - T(U_1, \ldots, U_{n-1}, W \vdash U_n). \end{aligned}$$

It is obvious that DT is $(n+1)$-linear and a short calculation shows that DT is local. In local coordinates, if $t_{i_1 \ldots i_n}$ are the coordinates of T, it turns out that the coordinates of DT are $t_{i_1 \ldots i_n; j} = \partial_j (t_{i_1 \ldots i_n}) - t_{\alpha i_2 \ldots i_n} \Gamma^\alpha_{i_1 j} - t_{i_1 \alpha i_3 \ldots i_n} \Gamma^\alpha_{i_2 j} - \cdots - t_{i_1 \ldots i_{n-1} \alpha} \Gamma^\alpha_{i_n j}$.

2) The Riemann Curvature Tensor:
Define

$$[R(U, V)\, W] \circ Z = \{U \vdash (V \vdash W) - V \vdash (U \vdash W) - [U, V] \vdash W\} \circ Z.$$

Again, it is obvious that R is 4-linear and it is not difficult to verify that R is local. If we put $R(\partial_j, \partial_k) \cdot \partial_i = R^m_{ijk} \partial_m$, then it can easily be checked that
$$R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_k \Gamma^l_{ij} + \Gamma^m_{ik} \Gamma^l_{mj} - \Gamma^m_{ij} \Gamma^l_{mk}.$$

Now, $R_{hijk} = g_{hm} R^m_{ijk} = [R(\partial_j, \partial_k) \cdot \partial_i] \circ \partial_h$ and we get by direct calculation $R_{hijk} = \frac{1}{2}(g_{hk,ij} + g_{ij,hk} - g_{hj,ik} - g_{ik,hj})$ + terms in the first derivatives of the metric tensor. (Note $g_{\alpha\beta,\gamma\delta}$ means $\partial^2 g_{\alpha\beta}(x)/\partial x^\gamma\, \partial x^\delta$.)

Let now γ be a (smooth) curve in the manifold M^n and let $V(t)$ be a vector field along γ, i.e. if $\gamma(t) = (p^1(t), \ldots, p^n(t))$ $(0 \leqq t \leqq 1)$, then $V(t)$ is a tangent vector at $\gamma(t)$. We say that $V(t)$ is parallel along $\gamma(t)$ if $\frac{d\gamma}{dt} \vdash V(t) = 0$. If now γ is parametrized by arc length s, we say that γ is a geodesic if $\frac{d\gamma}{ds} \vdash \frac{d\gamma}{ds} = 0$. If $\gamma(s)$ is geodesic, then we have in local coordinates

$$\frac{d^2 p^i(s)}{ds^2} + \Gamma^i_{jk}(p(s)) \frac{dp^j(s)}{ds} \cdot \frac{dp^k(s)}{ds} = 0.$$

Definition (6): A local coordinate system $(x^1, \dots, x^n)$ on a Riemannian manifold M^n is said to be a geodesic coordinate system if we have

$$\Gamma^i_{jk}(x)\, x^j x^k = 0, \qquad i = 1, 2, \dots, n.$$

The statement that a coordinate system is geodesic is equivalent to the assertion that geodesics through the origin appear in these coordinates as straight radii parametrized linearly. It is therefore clear that, if $\exp_p(v)$ denotes the point $\sigma(t)$ along the geodesic $\sigma(t)$ originating at p and with initial tangent vector v, then the mapping $v \to \exp_p(v)$ establishes geodesic coordinates at p. Thus geodesic coordinates exist in the neighborhood of any given point. Observe that in geodesic coordinates, $\Gamma_{ijk}(x)\, x^j x^k = 0$. Hence, since $\Gamma_{ijk} = \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk})$, we must have

$$2\,(\partial_k g_{ji})\, x^j x^k - (\partial_i g_{jk})\, x^j x^k = 0.$$

Let us assume the $g_{\alpha\beta}(x)$ are real analytic. By direct calculation,

$$(x^k\, \partial_k)\,(g_{ji} x^j) = x^j x^k\,(\partial_k g_{ji}) + g_{ji} x^j,$$

$$\partial_i\,(g_{jk} x^j x^k) = (\partial_i g_{jk})\, x^j x^k + 2 g_{ij} x^j.$$

Combining results, we find that in geodesic coordinates

$$2\,(x^k \partial_k)\,(g_{ij} x^j) - \partial_i\,(g_{jk} x^j x^k) = 0. \tag{*}$$

We now set $v_i(x) = g_{ij}(x)\, x^j$ and $v_i^{(\delta)}(x) = \partial_{j_1 \dots j_\delta}(v_i(0))\, x^{j_1} \cdots x^{j_\delta}$ where δ is any non-negative integer. Formula (*) now reads

$$2\,(x^k \partial_k)\, v_i = \partial_i\,(v_j x^j). \tag{**}$$

Taking the homogeneous terms of order δ from each side of this equation gives

$$2\,(x^k \partial_k)\, v_i^{(\delta)} = \partial_i\,(v_j^{(\delta)} x^j).$$

Now, $v_i^{(\delta)}(tx) = t^\delta v_i^{(\delta)}(x)$ and differentiation with respect to t yields

$$(x^k \partial_k)\, v_i^{(\delta)}(x) = \delta v_i^{(\delta)}(x) \qquad (\text{placing } t = 1).$$

Formula (**) is now transformed to

$$2\delta v_i^{(\delta)} = \partial_i(v_j^{(\delta)} x^j). \tag{***}$$

We now put $f^{(\delta)} = v_j^{(\delta)} x^j$, so that (***) yields

$$v_i^{(\delta)} = \frac{1}{2\delta}\, \partial_i f^{(\delta)}, \qquad \delta > 0.$$

We thus obtain, using all the previous formulas,

$$f^{(\delta)} = \frac{1}{2\delta}\,(x^j \partial_j f^{(\delta)}) = \frac{\delta + 1}{2\delta}\, f^{(\delta)}.$$

Hence, $f^{(\delta)} = 0$ unless $\delta = 1$. Since, without loss of generality, we may assume that $g_{ij}(0) = \delta_{ij}$, it follows from $f^{(\delta)} = 0\,(\delta > 1)$ that in geodesic coordinates,

$$g_{ij}(x)\,x^j = x^i.$$

Expanding the left hand side in a Taylor series, and comparing coefficients with the right hand side, we find easily

$$g_{ij,k_1\ldots k_m}(0)\,x^j x^{k_1}\cdots x^{k_m} = 0$$

and, in particular, since $g_{ij,k}(0) + g_{ik,j}(0) = g_{jk,i}(0)$, we have

$$g_{ij,k}(0) = 0.$$

Using the previous formula for R_{hijk}, we see that in geodesic coordinates

$$R_{hijk}(0) = \tfrac{1}{2}\,(g_{hk,ij}(0) + g_{ij,hk}(0) - g_{hj,ik}(0) - g_{ik,hj}(0)).$$

Additional calculation also gives us the following result.

Theorem (2): At the center of geodesic coordinates, the m-h order partial derivatives of the metric tensor are expressible algebraically in terms of the covariant derivatives of order $\leqq m-2$ of the Riemann tensor.

Proof: From the expression

$$R_{ijkl} = \tfrac{1}{2}\,(g_{ij,kl} - g_{ik,jl} + g_{kl,ij} - g_{jl,ik}) + T^{(1)},$$

where here and below we write $T^{(n)}$ for any algebraic expression in the metric tensor g and its m-th derivatives, $m \leqq n$, we find on interchanging i, k and adding that

$$(1) \qquad 2\,(R_{ijkl} + R_{kjil}) + T^{(1)} = g_{kl,ij} + g_{il,kj} - 2g_{jl,ik}.$$

Differentiating the equation $g_{ri}(x)\,x^r = x^i$ twice we find that $-g_{ri,jk}x^r = g_{ij,k} + g_{ik,j}$. Repeated differentiation now gives (E_n) $-\,g_{ri,i_1,\ldots i_n}x^r = g_{ii_1,i_2\ldots i_n} + \cdots + g_{ii_n,i_1\ldots i_{n-1}}$. Using equation (E_2) to replace the first two terms on the right of (1), we find that

$$(2) \qquad g_{rl,ijk}x^r - 2\,(R_{ijkl} + R_{kjil}) + T^{(1)} = 3g_{jl,ik}.$$

Setting $x = 0$ in (2), we obtain an expression for the second derivative of the metric tensor at the origin in terms of the Riemann tensor. We now prove Theorem 2 by induction on the integer m. We have just given the proof for $m = 2$; let $m > 2$, and suppose Theorem 2 valid for $m_1 < m$. Then, taking the m-2'nd derivative of equation (2), and noting that the m-2'nd derivative of the Riemann tensor can be expressed in terms of its m-2'nd covariant derivatives and derivatives of the metric tensor of order at most $m - 1$, we see that to prove Theorem 2 it is sufficient to prove that, at the center of geodesic coordinates, the derivative $h = g_{ij,i_1 \ldots i_n}$ can be written in terms of the set of expressions

$$3g_{ij_0,j_1 \ldots j_n} + g_{ij_1,j_2 \ldots j_n j_0} + \cdots + g_{ij_{n-2},j_{n-1}j_n j_0 \ldots j_{n-3}}.$$

Let σ denote the cyclic permutation $j_0 \cdots j_n \to j_1 \cdots j_n j_0$ of indices, so that $\sigma^{n+1} = 1$. Then we must prove that h can be expressed in terms of $3h + \sigma h + \cdots + \sigma^{n-2}h$, given the fact that $(1 + \sigma + \cdots + \sigma^n)\, h = 0$ which follows from (E_n) on setting $x = 0$. To prove this note that the polynomials $P(y) = 3 + y + \cdots + y^{n-2}$ and $Q(y) = 1 + y + \cdots + y^n$ have no roots in common and hence are relatively prime. Indeed, a root y_0 of $P(y) = 0$ satisfies $P(y_0) - y_0 P(y_0) = 0$, i.e. $3 - 2y_0 - y_0^{n-1} = 0$, and hence lies on the unit circle if and only if $2y_0 = 2$ and $y_0^{n-1} = 1$, so that $y_0 = 1$ and $Q(y_0) \neq 0$. We may therefore find rational polynomials $A(y)$ and $B(y)$ such that $A(y)\,P(y) + B(y)\,Q(y) = 1$, so that $h = A(\sigma)\,P(\sigma)\,h$ and the proof of Theorem 2 is complete. Q.E.D.

Corollary: The covariant derivatives of the Riemann tensor give a complete set of invariants for the Riemann metric in geodesic coordinates.

Proof: By Theorem 2, the covariant derivatives of the Riemann tensor completely determine the values of g_{ij} and all its derivatives at the center of geodesic coordinates. But g_{ij} is analytic and hence g_{ij} is completely determined in the whole coordinate neighborhood by the values of all its derivatives at the center.

We shall now consider another important instance of covariant n-tensor fields, the alternating n-tensor fields (or n-forms). By definition, ω is an alternating n-tensor field if

$$\omega\,(V_{\sigma(1)}, \ldots, V_{\sigma(n)}) = \operatorname{sgn} \sigma \cdot \omega\,(V_1, \ldots, V_n)$$

where $V_i \in D^1$, σ is a permutation of $1, 2, \ldots, n$ and $\operatorname{sgn} \sigma$ is the sign of σ. If ω_i is an alternating k_i-tensor field ($i = 1, 2$), then the outer product of ω_1 and ω_2 is defined by

$$\omega_1 \cdot \omega_2\,(V_1, \ldots, V_{k_1}, \ldots, V_{k_1+k_2}) = \omega_1\,(V_1, \ldots, V_{k_1}) \cdot \omega_2\,(V_{k_1+1}, \ldots, V_{k_1+k_2})$$

and the exterior product of ω_1 and ω_2 is defined by

$$\omega_1 \wedge \omega_2 = A\,(\omega_1 \cdot \omega_2)$$

where A is the alternating function, defined as follows:

$$A\omega(V_1, \ldots, V_n) = (n!)^{-1}\Sigma_\sigma \operatorname{sgn}\sigma \cdot \omega(V_{\sigma(1)}, \ldots, V_{\sigma(n)}).$$

One shows easily that exterior multiplication is associative and that

$$\omega_1 \wedge \omega_2 = (-1)^{k_1 k_2}\omega_2 \wedge \omega_1.$$

If $D\omega$ is the covariant derivative of ω, we put

$$d\omega = A(D\omega)$$

and call d exterior differentiation. We then have the relations

$$d^2 = 0, \quad d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2.$$

Let us fix our attention on E^N, a Euclidean space of arbitrary dimension. Consider the collection of all pairs (S, T) where S is a compact subset of E^N and T is a closed subset of S. We define a collection of spaces as follows (see J. Schwartz, Amer. J. of Math., vol. 77, No. 1, Jan. 1965, pp. 29–44 for the following definitions and proofs of the next few theorems):

(i) F^k is the space of all (smooth) k-forms defined on E^N.
(ii) $V^k(S)$ is the subspace of F^k consisting of the forms which vanish near S.
(iii) $Z^k(S, T) = \{\omega \in V^k(T) | d\omega \in V^{k+1}(S)\}$.
(iv) $B^k(S, T) = \{d\omega + \theta | \omega \in V^{k-1}(T), \theta \in V^k(S)\}$.
(v) $D^k(S, T) = Z^k(S, T)/B^k(S, T)$.

We then have the following theorem, which we state without proof.

Theorem (3): *De Rham's Theorem* for every triangulable pair (S, T), $D^k(S, T)$ is naturally isomorphic to the k-h real relative (singular) cohomology group of (S, T).

If we let $S = M^m$, a manifold and $T = \phi$, we then obtain the classical theorem of De Rham. We call $Z^k(M)$ the space of closed k-forms and $B^k(M)$ the space of exact k-forms.

If $\phi: M^m \to N^n$ is a smooth map and if ω is a k-form on N, then we can define a k-form $\phi'\omega$ on M by setting

$$\phi'\omega(V_1, \ldots, V_k) = \omega(\phi_1 V_1, \ldots, \phi_1 V_k)$$

where $V_i \in D^1(M)$ $(i = 1, \ldots, k)$ and ϕ_1 is the map on vector fields induced by ϕ. It is easy to see that $d\phi'\omega = \phi' d\omega$. Hence ϕ' induces a map, call it ϕ' again, from $D^k(N) \to D^k(M)$. It turns out that the following diagram is commutative.

$$\begin{tikzcd}
H^k(N, \mathbb{R}) \arrow[r, "\phi^*"] & H^k(M, \mathbb{R}) \\
D^k(N) \arrow[u, "\nu_N"'] \arrow[r, "\phi'"] & D^k(M) \arrow[u, "\nu_M"']
\end{tikzcd}$$

Here, ν_N and ν_M are the natural isomorphisms given by De Rham's theorem and ϕ^* is the map on cohomology induced by the (continuous) map ϕ.

Let M^n be a Riemannian manifold and let $V_1, \dots, V_n$ be smooth vector fields defined on some coordinate neighborhood $\mathcal{N}$ such that on $\mathcal{N}$, $V_i \circ V_j = g(V_i, V_j) = \delta_{ij}$. Let ω_i be the 1-form given by

$$\omega_i(V_j) = \delta_{ij}.$$

We define the 1-form ω_{ij} by

$$\omega_{ij}(V) = \omega_i(V \vdash V_j)$$

where V is a vector field on $\mathcal{N}$. If we differentiate $V_i \circ V_j = \delta_{ij}$ covariantly with respect to V, we obtain

$$(V \vdash V_j) \circ V_i + V_j \circ (V \vdash V_i) = 0$$

and, since $\omega_i(V) = V \circ V_i$, we get

$$\omega_{ij} = -\omega_{ji}.$$

If U is any vector field on $\mathcal{N}$, then $U = \omega_j(U)\, V_j$ (sum over repeated indices). By the definition of an affine connection,

$$\begin{aligned}\omega_i(V \vdash \omega_j(U)\, V_j) &= \omega_j(U)\, \omega_{ij}(V) + \omega_i(V_j) \cdot V(\omega_j(U)) \\ &= \omega_j(U)\, \omega_{ij}(V) + V(\omega_i(U)).\end{aligned}$$

Thus,

$$D\omega_i(U, V) = V(\omega_i(U)) - \omega_i(V \vdash U) = -\omega_j(U)\, \omega_{ij}(V)$$

from above. Applying the alternation operator A to both sides of this equation gives

$$d\omega_i = -\omega_j \wedge \omega_{ij}. \tag{C-1}$$

Let now $\Omega_{ij}(U, V) = \omega_i(R(U, V)\, V_j)$. We now prove

$$\omega_i(R(U, V)\, Z) = \omega_k(Z)\, \{\omega_{il} \wedge \omega_{lk}(U, V) - d\omega_{ik}(U, V)\}.$$

By the product rule for covariant differentiation we have $U\omega(V) = (D\omega)(V, U) + \omega(U \vdash V)$ for any 1-form ω, and thus, antisymmetrizing, we have $U\omega(V) - V\omega(U) - \omega\,[(U, V]) = -(d\omega)(U, V)$. By definition of the Riemann form R, we have

$$\omega_i(R(U, V)\, Z) = \omega_i(U \vdash (V \vdash Z) - V \vdash (U \vdash Z) - [U, V] \vdash Z).$$

Hencc, using the fundamental equation $\omega_i(V \vdash U) = \omega_{ij}(V)\,\omega_j(U) + V\omega_i(U)$, we have

$$\begin{aligned}
\omega_i(R(U, V)Z) &= \omega_{ij}(U)\,\omega_j(V \vdash Z) + U\omega_i(V \vdash Z)\\
&\quad - \text{same terms with } U, V \text{ interchanged}\\
&\quad - \omega_{ij}([U, V])\,\omega_j(Z) - [U, V]\,\omega_i(Z)\\
&= \omega_{ij}(U)\,\omega_{jk}(V)\,\omega_k(Z) + \omega_{ij}(U)\,V\omega_j(Z)\\
&\quad + U(\omega_{ij}(V)\,\omega_j(Z)) + UV\omega_i(Z)\\
&\quad - \text{same terms with } U, V \text{ interchanged}\\
&\quad - \omega_{ij}([U, V])\,\omega_j(Z) - [U, V]\,\omega_i(Z)\\
&= \{(\omega_{ik} \wedge \omega_{kj})(U, V) + U\omega_{ij}(V) - V\omega_{ij}(U)\\
&\quad - \omega_{ij}([U, V])\}\,\omega_j(Z)\\
&= \{\omega_{ik} \wedge \omega_{kj}(U, V) - d\omega_{ij}(U, V)\}\,\omega_j(Z),
\end{aligned}$$

proving the desired fromula.

(Summing over repeated indices here and below.) In particular, if we put $Z = V_j$, we get

$$d\omega_{ij} = \omega_{il} \wedge \omega_{lj} - \Omega_{ij}. \tag{C-2}$$

Now, taking the exterior dcrivative of each side of (C-2) and using the fact that $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, we get

$$d\Omega_{ij} = d\omega_{il} \wedge \omega_{lj} - d\omega_{jl} \wedge \omega_{li}. \tag{C-3}$$

The formulas (C-1), (C-2), (C-3) are called the *Cartan structure equations.*

We shall now construct an important class of forms on a Riemannian manifold M^n, the so-called curvature forms. We definc

$$\Omega_{(k)} = \Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_{k-1} i_k} \wedge \Omega_{i_k i_1}.$$

If n is even and M is orientable, we also define

$$\Omega = \varepsilon^{i_1 \cdots i_n} \Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_{n-1} i_n}$$

where $\varepsilon^{i_1 \cdots i_n} = \operatorname{sgn}\begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$ if $i_1, \dots, i_n$ are distinct and $\varepsilon^{i_1 \cdots i_n} = 0$ if $i_1, \dots, i_n$ arc not all distinct.

Theorem (4): The curvature forms $\Omega_{(k)}$ are invariantly defined on M, i.e. if $\hat{V}_1, \ldots, \hat{V}_n$ is another orthonormal system of vector fields on $\mathcal{N}$, and if $\hat{\omega}_i$, $\hat{\Omega}_{ij}$ and $\hat{\Omega}_{(k)}$ are obtained using the new system, then $\hat{\Omega}_{(k)} = \Omega_{(k)}$. Moreover, the forms $\Omega_{(k)}$ are closed i.e., $d\Omega_{(k)} = 0$. For k odd, $\Omega_{(k)} = 0$.

Proof: There exist orthogonal matrices $\|a_{ij}\|$ such that $\hat{V}_i = a_{ki} V_k$ (sum over k as usual). If then $\hat{\omega}_i$ is defined by $\hat{\omega}_i(\hat{V}_j) = \delta_{ij}$, we easily get $\hat{\omega}_i = a_{ki}\omega_k$. Using the definitions of $\hat{\Omega}_{ij}$ and Ω_{ij}, we get easily

$$\hat{\Omega}_{ij}(U, V) = a_{ki} a_{lj} \Omega_{kl}(U, V).$$

If we use this formula on each factor of $\hat{\Omega}_{j_1 j_2} \wedge \hat{\Omega}_{j_2 j_3} \wedge \cdots \wedge \hat{\Omega}_{j_k j_l}$ and use the fact that $\|a_{ij}\|$ is orthogonal, the first assertion follows directly.

To prove that $d\Omega_{(k)} = 0$ we make use of the formula

$$d\Omega_{ij} = -\Omega_{il} \wedge \omega_{lj} + \omega_{il} \wedge \Omega_{lj} \tag{*}$$

which is readily deduced from the Cartan structure equations. Now,

$$d\Omega_{(k)} = d\Omega_{i_1 i_2} \wedge (\Omega_{i_2 i_3} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1})$$
$$+ \Omega_{i_1 i_2} \wedge d(\Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_k i_1}).$$

But,

$$\begin{aligned}
&\Omega_{i_1 i_2} \wedge d(\Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_k i_1}) \\
&= \Omega_{i_1 i_2} \wedge \{d\Omega_{i_2 i_3} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1} + \Omega_{i_3 i_2} \wedge d(\Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1})\} \\
&= \Omega_{i_1 i_2} \wedge d\Omega_{i_2 i_3} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1} \\
&\quad + \Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge d(\Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1}) \\
&= d\Omega_{i_2 i_3} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1} \wedge \Omega_{i_1 i_2} \\
&\quad + \Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge d(\Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1})
\end{aligned}$$

since $\Omega_{i_1 i_2}$, being a 2-form, commutes with any form. Hence, we have

$$d\Omega_{(k)} = 2d\Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_k i_1}$$
$$+ \Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge d(\Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_k i_1}).$$

By induction, we finally get

$$d\Omega_{(k)} = k \cdot d\Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_k i_1}.$$

To see that $d\Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_k i_1} = 0$, just apply formula (*) to $d\Omega_{i_1 i_2}$. The result follows by direct calculation.

To see that $\Omega_{(k)}$ is already 0 for k odd, observe first that $\Omega_{ij} = -\Omega_{ji}$. This follows from (C-2). Now,

$$\begin{aligned}\Omega_{(k)} &= \Omega_{i_1 i_2} \wedge \Omega_{i_2 i_3} \wedge \cdots \wedge \Omega_{i_k i_1} \\ &= (-1)^k \Omega_{i_2 i_1} \wedge \Omega_{i_3 i_2} \wedge \cdots \wedge \Omega_{i_k i_{k-1}} \wedge \Omega_{i_1 i_k} \\ &= (-1)^k \Omega_{i_1 i_k} \wedge \Omega_{i_k i_{k-1}} \wedge \cdots \wedge \Omega_{i_3 i_2} \wedge \Omega_{i_2 i_1} \\ &= -\Omega_{(k)}\end{aligned}$$

since k is odd. This completes the proof.

Theorem (5): The curvature form Ω is invariantly defined on an even dimensional oriented M^n, i.e. if $\hat{V}_1, \ldots, \hat{V}_n$ is another orthonormal system of vector fields on $\mathcal{N}$, giving the same orientation as $V_1, \ldots, V_n$, then $\hat{\Omega} = \Omega$. Moreover, Ω is a closed form.

Proof: As in Theorem 3, $\hat{\Omega}_{ij} = a_{ki}a_{lj}\Omega_{kl}$. But now we also have $|a_{ij}| = \det(\|a_{ij}\|) = 1$. By direct calculation, we get

$$\begin{aligned}\hat{\Omega} &= \varepsilon^{j_1 \cdots j_n} \hat{\Omega}_{j_1 j_2} \wedge \hat{\Omega}_{j_3 j_4} \wedge \cdots \wedge \hat{\Omega}_{j_{n-1} n} \\ &= \varepsilon^{j_1 \cdots j_n} a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n} \Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_{n-1} i_n} \\ &= \varepsilon^{i_1 \cdots i_n} \Omega_{i_1 i_2} \wedge \Omega_{i_3 i_4} \wedge \cdots \wedge \Omega_{i_{n-1} i_n},\end{aligned}$$

because of the definition of determinant. Since $d\Omega$ is of order $n + 1$ and since M is n-dimensional, it follows that $d\Omega = 0$.

Definition (7): By Theorems 4 and 5, the curvature forms $\Omega_{(k)}$, Ω define elements in the De Rham cohomology ring; we call these elements the *curvature cohomology classes*. By virtue of De Rham's theorem, these elements may be identified with elements of the ordinary cohomology ring (with real coefficients).

Suppose now that M_0^ν is a submanifold of the Riemannian manifold M^n. In a natural way, M_0 inherits a Riemannian metric g_0 from the metric g on M. By Theorem 1, g_0 determines a symmetric connection, call it $\vdash_0$, which is appropriate to g_0.

Lemma (1): $V \vdash U_0 = P_0(V \vdash U)$, where $U, V \in D^1(M_0)$, P_0 is the orthogonal projection of a tangent space to M onto the corresponding tangent space to M_0.

Proof: Put $V \vdash_1 U = P_0(V \vdash U)$. It is easy to check that $\vdash_1$ is an affine connection which is symmetric and appropriate to g_0. Appealing to the uniqueness assertion of Theorem 1, the lemma is proved.

Let now $V_1, \ldots, V_n$ be a system of orthonormal vector fields on a coordinate neighborhood $\mathcal{N}$ of M^n such that $V_1, \ldots, V_\nu$ form a system of tangent vector fields to M_0^ν while $V_{\nu+1}, \ldots, V_n$ form a system of normal vector fields to M_0^ν. Let $\omega_i^{(0)}$, $\omega_{ij}^{(0)}$, $\Omega_{ij}^{(0)}$ denote the forms on M_0 defined using the vector fields $V_1, \ldots, V_\nu$. Thus, the forms are defined only for the values $1 \leqq i$, $j \leqq \nu$.

If $1 \leqq i \leqq \nu$ and $V \in D^1(M_0)$, then $\omega_i(V) = \omega_i^{(0)}(V)$, i.e. $\omega_i^{(0)}$ is the restriction of ω_i $(1 \leqq i \leqq \nu)$.

If $1 \leqq i, j \leqq \nu$ and $V \in D^1(M_0)$, then

$$\begin{aligned} \omega_{ij}(V) &= \omega_i(V \vdash V_j) = \omega_i(P_0(V \vdash V_j)) = \omega_i(V \vdash_0 V_j) \\ &= \omega_i^{(0)}(V \vdash_0 V_j) = \omega_{ij}^{(0)}(V), \end{aligned}$$

i.e. $\omega_{ij}^{(0)}$ is the restriction of ω_{ij} $(1 \leqq i, j \leqq \nu)$.

On M^n, we have (C-2): $\Omega_{ij} = \sum_{l=1}^{n} \omega_{il} \wedge \omega_{lj} - d\omega_{ij}$ $(1 \leqq i,\ j \leqq n)$. Similarly, on M_0^ν, we have $\Omega_{ij}^{(0)} = \sum_{l=1}^{\nu} \omega_{il}^{(0)} \wedge \omega_{lj}^{(0)} - d\omega_{ij}^{(0)}$ $(1 \leqq i, j \leqq \nu)$. By the above remarks, it follows that actually

$$\Omega_{ij}^{(0)} = \sum_{l=1}^{\nu} \omega_{il} \wedge \omega_{lj} - d\omega_{ij}, \qquad 1 \leqq i, j \leqq \nu.$$

Subtracting, we get

$$\Omega_{ij}^{(0)} = \Omega_{ij} - \sum_{l=\nu+1}^{n} \omega_{il} \wedge \omega_{lj}, \qquad 1 \leqq i, j \leqq \nu. \tag{**}$$

In the sequel we make use of the following deep theorem of Nash, which we state without proof. (For a proof, cf. J. Schwartz, NYU Lecture Notes on Nonlinear Functional Analysis, 1963/64.)

Theorem (6) (Nash): Any Riemannian manifold can be isometrically embedded in E^N for some sufficiently large integer N.

Using this theorem, we take an arbitrary Riemannian manifold M_0^ν and embed it isometrically in E^N. In the above discussion, we replace M^n by E^N. Since in E^N, $\Omega_{ij} = 0$, we get from (**) above

Theorem (7) (Generalized Theorema Egregium):

$$\Omega_{ij}^{(0)} = -\sum_{l=\nu+1}^{N} \omega_{il} \wedge \omega_{lj}, \qquad 1 \leqq i, j \leqq \nu.$$

We change our notation and let M^n be an arbitrary Riemannian manifold, sometrically embedded in E^N. We now make a fundamental definition which will enable us to relate the curvature cohomology classes defined above with the characteristic classes of Chapter VI. (Cf. Pontryagin—"Some Topological Invariants of Closed Riemannian Manifolds", AMS Translations, Series 1, Volume 7.)

Definition (8): Let 0 be the origin in E^N. We define a map

$$T: M^n \to G_{n,N}$$

where, as in Chapter III, $G_{n,N}$ is the set of n-dimensional planes in N-space, as follows. For each $x \in M^n$, let T_x be the tangent space to M^n at x. Then, since $M^n \subseteq E^N$, T_x may be viewed as an n-plane in E^N. Let $T(x)$ be the n-dimensional subspace passing through 0 which is parallel to T_x. The map $x \to T(x)$ so defined is called a *tangential representation* of M^n.

Let $W_{n,N}$ be the subspace of $G_{n,N} \times G_{1,N}$ consisting of all pairs (π, e) such that e is a vector lying in π; $W_{n,N}$ is given the relative topology of the product space. A projection $p_2: W_{n,N} \to G_{n,N}$ is given by $p_2(\pi, e) = \pi$. Then $W_{n,N}$ is the bundle space of a vector bundle ξ over $G_{n,N}$ (see Chapter IV, Definition 4). We can suppose the structural group of this bundle to be O_n, acting as usual on an n-dimensional vector space. If now $\tau(M)$ is the tangent bundle of M (recall that $\tau(M)$ consists of pairs (x, e_x) with $x \in M$ and e_x a vector lying in T_x) and if $p_1: \tau(M) \to M$ is the projection $(x, e_x) \to x$, then there is a bundle map

$$\tilde{T}: \tau(M) \to W_{n,N}$$

that is,

$$\begin{tikzcd}
\tau(M) \arrow[r, "\tilde{T}"] \arrow[d, "p_1"] & W_{n,N} \arrow[d, "p_2"] \\
M \arrow[r, "T"] & G_{n,N}
\end{tikzcd}$$

is a commutative diagram. In fact, $\tilde{T}$ is given by

$$\tilde{T}((x, e_x)) = (T_x, e(x))$$

where $e(x)$ is the vector passing through 0 which is parallel to e_x. It is clear that the associated principal bundle $\tilde{\xi}$ of ξ (see Chapter VI, discussion after Definition 1) is the bundle with bundle space the Stiefel manifold $V_{n,N}$ (see Chapter VI, discussion after Definition 2). Also $T^+\tilde{\xi}$ is just the associated principal bundle of $\tau(M)$. By the Classification Theorem (see Chapter VI, Theorem 4), the homotopy class of T is uniquely determined independently of the particular imbedding of M used, for N sufficiently large. In particular, T^*, the induced map on cohomology, is an invariant of the associated principal O_n-bundle of $\tau(M)$. Thus, T^* does not depend on the embedding of M^n in E^n, nor does it depend on N (as long as N is sufficiently large).

We now construct a collection of forms on the manifold $G_{n,N}$, in a manner analogous to the construction of the curvature forms on M^n. It will turn out that the curvature forms on M^n can be obtained from this new collection of forms by an application of T' (where T' is the map on forms induced by T).

Let π be the generic element of $G_{n,N}$. We choose an orthonormal basis in π, say $e_1, e_2, \ldots, e_n$ (where $e_i = e_i(\pi)$) and supplement this basis by an orthonormal basis $e_{n+1}(\pi), \ldots, e_N(\pi)$ for $\pi^\perp$. We assume the vectors $e_i(\pi)$ vary smoothly on $G_{n,N}$ in a neighborhood wherein we calculate.

Recall that if f is a real valued smooth function defined on $G_{n,N}$ (or for that matter, on any manifold), and if V is a vector field on $G_{n,N}$, then $df(V) = Vf$. If now g is a smooth map from $G_{n,N}$ into E^N, then g can be regarded as a collection of N functions $g_1, \ldots, g_N$ with $g_i : G_{n,N} \to R$ ($i = 1, 2, \ldots, N$). We then define dg by

$$dg(V) = (Vg_1, Vg_2, \ldots, Vg_n).$$

Thus, dg is a mapping

$$dg : D^1(G_{n,N}) \to \underbrace{C^\infty(G_{n,N}) \times \cdots \times C^\infty(G_{n,N})}_{N \text{ factors}}$$

(Recall the notations $D^1(M)$, $C^\infty(M)$ introduced at the beginning of this chapter.)

The vectors $e_i(\pi)$ introduced above can be regarded as maps

$$e_i : G_{n,N} \to E^N.$$

We now define 1-forms $\tilde{\omega}_{\alpha\beta}$ on $G_{n,N}$ ($\alpha, \beta = 1, 2, \ldots, N$) by

$$\tilde{\omega}_{\alpha\beta} = de_\alpha \circ e_\beta$$

where $\circ$ is the Riemannian metric on E^N, i.e. the usual dot product. We then define 2-forms $\tilde{\Omega}_{ij}$ on $G_{n,N}$ ($i, j = 1, \ldots, n$) by

$$\tilde{\Omega}_{ij} = -\sum_{r=n+1}^{N} \tilde{\omega}_{ir} \wedge \tilde{\omega}_{rj}.$$

As before, we introduce forms $\tilde{\Omega}_{(k)}$, $\tilde{\Omega}$ as follows:

(a) $$\tilde{\Omega}_{(k)} = \tilde{\Omega}_{i_1 i_2} \wedge \tilde{\Omega}_{i_2 i_3} \wedge \cdots \wedge \tilde{\Omega}_{i_{k-1} i_k} \wedge \tilde{\Omega}_{i_k i_1}.$$

and

(b) $$\tilde{\Omega} = \varepsilon^{i_1 \cdots i_n} \tilde{\Omega}_{i_1 i_2} \wedge \tilde{\Omega}_{i_3 i_4} \wedge \cdots \wedge \tilde{\Omega}_{i_{n-1} i_n},$$

if n is even.

Remark: In fact, the form $\tilde{\Omega}$ is defined only up to sign on the (nonorientable) manifold $G_{n,N}$. To remove the ambiguity, we may consider $\hat{\Omega}$ as defined on the two-sheeted covering space $\tilde{G}_{n,N}$ (see Chapter I, discussion after Definition 7) of $G_{n,N}$. On $\tilde{G}_{n,N}$, SO_n acts transitively and in fact, $\tilde{G}_{n,N} = SO_N/SO_n \times SO_{N-n}$. We shall not dwell on this technical point and leave to the reader the task of making rigorous any imprecise statement in the discussion below. Observe that the forms $\tilde{\Omega}_{(k)}$ are already invariantly defined on $G_{n,N}$ since no question of orientation enters in this case.

Observe again that O_n acts transitively on $G_{n,N}$. Thus, if $g \in O_n$, there is induced a smooth map

$$g: G_{n,N} \to G_{n,N}.$$

If ω is an r-form on $G_{n,N}$, we say that ω is invariant under O_n when $g'\omega = \omega$ for all $g \in O_n$, where g' is the induced map on forms.

We assert that the forms $\tilde{\Omega}_{(k)}$, $\tilde{\Omega}$ constructed above are independent of the bases e_i used to define them. As the proof of this assertion very much resembles the proof of Theorems 4 and 5, we leave its details to the reader. It follows very readily from the definition of the forms $\tilde{\omega}_{ij}$ and from the fact that the vectors e_α form an orthonormal basis that $d\tilde{\omega}_{ij} = \sum_{\alpha=1}^{N} \omega_{i\alpha} \wedge \omega_{\alpha j}$. Therefore $d\tilde{\omega}_{ij} = \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj} - \tilde{\Omega}_{ij}$, where k is summed from 1 to n. Using this formula to replace $(C-2)$, we may deduce that the forms $\tilde{\Omega}_{(k)}$ and $\tilde{\Omega}$ are closed, in much the same way as the corresponding facts were proved for $\Omega_{(k)}$ and Ω; all additional details are left to the reader.

Finally, we can easily convince ourselves that

$$T'\left(\tilde{\Omega}_{(k)}\right) = \Omega_{(k)}, \quad T'\left(\tilde{\Omega}\right) = \Omega.$$

Hence, we also have

$$T^*\left(\{\tilde{\Omega}_{(k)}\}\right) = \{\Omega_{(k)}\}, \quad T^*\left(\{\tilde{\Omega}\}\right) = \{\Omega\},$$

where T^* is the induced map on cohomology classes and where { } denotes cohomology class. We now prove

Theorem (8): The curvature cohomology classes of M^n can be expressed as real linear combinations of cup products of the Whitney classes of the associated principal O_n-bundle of $\tau(M)$.

Proof: Let $T_N: M^n \to G_{n,N}$ be a tangential representation of M^n. Let $\tilde{\Omega}^{(N)}_{(k)}$, $\tilde{\Omega}^{(N)}$ be the forms constructed above (where we called them $\tilde{\Omega}_{(k)}$, $\tilde{\Omega}$). Thus,

$T^*\left(\{\tilde{\Omega}^{(N)}_{(k)}\}\right) = \{\Omega_{(k)}\}$, $T^*\left(\{\tilde{\Omega}^{(N)}\}\right) = \{\Omega\}$. If $i_N : G_{n,N} \to G_{n,\infty}$ is the natural injection, we obtain a map

$$i_N \circ T : M^n \to G_{n,\infty}.$$

Since $i_N \circ T \cong i_P \circ T$ for any N, P which are sufficiently large, the choice of N is immaterial. Moreover, we have

$$(i_N \circ T)^\dagger (\alpha_n) = \widetilde{\tau(M)},$$

where α_n is the bundle introduced in Chapter VI, Definition 5 and $\widetilde{\tau(M)}$ is the associated principal bundle of $\tau(M)$. As the cohomology map i_N^* is onto, there exist elements $\gamma_{(k)}, \gamma \in H^*(G_{n,N})$ such that

$$i_N^*(\gamma_{(k)}) = \{\tilde{\Omega}^{(N)}_{(k)}\}, \; i_N^*(\gamma) = \{\tilde{\Omega}^{(N)}\};$$

whence

$$(i_N \circ T)^* (\gamma_{(k)}) = \{\Omega_{(k)}\}, \; (i_N \circ T)^* (\gamma) = \{\Omega\}.$$

But the elements $\gamma_{(k)}, \gamma$ can be written as linear combinations of cup products of Whitney classes and the theorem follows if we just use the definition of the Whitney classes of the bundle $\widetilde{\tau(M)}$.

Let M^n be an orientable even-dimensional manifold. We have expressed the curvature forms of M^n, originally given in terms of the Riemann tensor on M^n, in topologically invariant form. Let us look, in particular, at the form Ω. Clearly, the integral $\int_{M^n} \Omega$ must also be a topological invariant of M^n. To find this numerical invariant, one may simply try to compute the integral as it stands. However, by using the relation between Ω and $\tilde{\Omega}^{(N)}$ developed above, we may reduce the problem to a computation of the integral $\int_{T_N(M^n)} \tilde{\Omega}^{(N)}$. Indeed, by the usual change of variable formula, we have

$$\int_{T_N(M^n)} \tilde{\Omega}^{(N)} = \int_{M^n} T_N'\left(\tilde{\Omega}^{(N)}\right) = \int_{M^n} \Omega.$$

Thus, all of the computational burden has been thrown on one particular form $\tilde{\Omega}^{(N)}$ on one particular space $G_{n,N}$. This, of course, reflects the "universal" character of the space $G_{n,N}$. The actual computation of the integral $\int_{T_N(M^n)} \tilde{\Omega}^{(N)}$ can be carried out without too much difficulty (see Pontryagin, loc. cit.). Here, we omit the calculations and content ourselves with stating the final result, as follows.

Theorem (9) (Generalized Gauss-Bonnet Theorem:) If $\chi(M^n)$ is the Euler characteristic of M^n and σ_n is the volume of the n-dimensional unit ball, then

$$\int_{M^n} \Omega = \tfrac{1}{2}\sigma_n \cdot \chi(M^n).$$

CHAPTER VIII

Generalized Cohomology Theories

In the final three chapters of these notes, we continue to develop the cohomology theory of bundles. We shall introduce, for any finite complex X (or, more generally, for any finite cellular pair (X, Y)), a sequence of groups defined in terms of the bundles on X. These groups, together with certain homomorphisms between them, form a system which is very much like a cohomology theory, satisfying all of the Eilenberg-Steenrod axioms except for the dimension axiom. This "bundle cohomology theory" has found many important applications in topology; one of these is discussed in Chapter 10.

This chapter is divided into two parts. In part A, a general notion of "generalized" or "nonstandard" cohomology theory is introduced (having nothing to do with bundles). In part B, we introduce our "bundle cohomology theory", the so-called K-Theory. It is shown that K-Theory is a nonstandard cohomology theory in the sense of part A. A deeper and more detailed study of K-Theory will be carried out in Chapter 9.

Part A: A Nonstandard Cohomology Theory

All the topological spaces we will speak of here will be assumed to be based, i.e. with each space X, some point $x_0 \in X$ will be singled out. Moreover, we shall only consider maps $f: X \to Y$ which are basepoint preserving.

Let T be an arcwise connected space; call T a *test space*. For any space X, we have T^X, the set of all continuous (basepoint preserving) maps of X into T. We let (T^X) denote the set of all homotopy classes of maps of X into T. If T^X is given the compact-open topology, then the elements of (T^X) are just the arc-connected components of T^X. It is easy to check that (T^X) is a functor of two variables; covariant in T, contravariant in X.

If A is a closed subset of X such that the basepoint $x_0 \in A$, we denote by X/A the set obtained by identifying all points of A. X/A becomes a topological space if we define a subset W of X/A to be open iff $p^{-1}(W)$ is open in X (where $p: X \to X/A$ is the natural projection).

We define the *one point union* $X \vee Y$ to be the space obtained from the disjoint union $X \cup Y$ by identifying the basepoints x_0, y_0. We define the *smash product* $X \wedge Y$ by

$$X \wedge Y = X \times Y/X \vee X.$$

Let the basepoint of S^n be v, the north pole. We put

$$S^n(X) = S^n \wedge X.$$

Then $S^n(X)$ is a covariant functor of X. Indeed, any map $f: Y \to X$ gives rise to a map

$$f': S^n \times Y \to S^n \times X$$

defined as $f'(p, y) = (p, f(y))$, $p \in S^n$, $y \in Y$. Since $f'(S^n \times y_0) \subseteq S^n \times x_0$, $f'(v \times Y) \subseteq v \times X$, it is clear that f' induces a map

$$f': S^n(Y) \to S^n(X).$$

One sees easily that the correspondence $X \to S^n(X), f \to \tilde{f}$ is functorial.

We shall now define a non-standard cohomology theory on the class of all pairs (X, Y) with X a cell complex, Y a closed subcomplex. We shall moreover require that the basepoint x_0 be a vertex in Y (provided $Y \neq \phi$). The theory is non-standard in the following senses:

(i) in lowest dimension, the cohomology "groups" are not usually actual groups,

(ii) the dimension axiom is not usually satisfied.

For a pair (X, Y), we set

$$T^{-n}(X, Y) = (T^{S^n(X/Y)})$$

and call $T^{-n}(X, Y)$ the n-th T-cohomology "group" of (X, Y). Although $T^{-n}(X, Y)$ is not actually a group in general, it nevertheless has a "distinguished" element (which is a substitute for a unit). Indeed, if U and V are spaces, with U arcwise connected, we may consider the element of (U^V) which is the class of all mappings homotopic to a constant as the distinguished element.

Having defined $T^{-n}(X, Y)$, we must define for any $f: (X, Y) \to (X', Y')$ a corresponding "homomorphism"

$$f^*: T^{-n}(X', Y') \to T^{-n}(X, Y).$$

This is easily done: f induces a map $\tilde{f}\colon X/Y \to X'/Y'$ which in turn induces a map $\tilde{f}\colon S^n(X/Y) \to S^n(X'/Y')$. Since (T^X) is contravariant in X, $\tilde{f}$ induces the desired map.

We must now define a coboundary operator

$$\delta\colon T^{-(n+1)}(Y) \to T^{-n}(X, Y).$$

Before doing this, we shall list three important properties of the operations $\vee$ and $\wedge$. Their proofs are left as exercises.

(i) $$(X \vee Y) \vee Z = X \vee (Y \vee Z)$$

(ii) $$(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$$

(iii) $$S^1 \wedge S^1 = S^2; \text{ in general, } \overbrace{S^1 \wedge \cdots \wedge S^1}^{n} = S^n.$$

Equality above must be interpreted up to homeomorphism.

We shall now state a topological result which is essential in what follows.

Homotopy Extension Theorem: Let (X, Y) be a pair, consisting of a cell complex X and a subcomplex Y. Let $f\colon X \to T$ be a map and suppose $h_t\colon Y \to T$ is a partial homotopy of f (i.e. $f|Y = h_0$). Then there exists a homotopy $H_t\colon X \to T$ such that $H_0 = f$, $H_t|Y = h_t$ $(0 \leqq t \leqq 1)$.

The proof when (X, Y) is a triangulable pair can be found in Hu—*Homotopy Theory*, p. 14.

Corollary (1): Let A be a subcomplex of the cell complex B. Let t, s denote generic points of $I = [0, 1]$ and let b denote the generic point of B. Let C denote the following subset of $I \times I \times B$:

$$C = \{(0, s, b) \mid s \in I, b \in B\} \cup \{(t, s, b) \mid t \in I, s \in I, b \in A\}.$$

Let $f\colon C \to T$ be a given map. Then f can be extended to all of $I \times I \times B$.

Proof: Just apply the above theorem (abbrev. HET) to the pair $(I \times B, I \times A)$.

Corollary 1 can be reformulated as follows. Let z be a complex variable ranging over the unit square in the complex plane. Suppose $f(z, b)$ is defined for (a) all z on the bottom of the unit square and all $b \in B$, (b) all z in the unit square and all $b \in A$. Then f can be extended to all z in the unit square and all $b \in B$.

Corollary (2): Suppose $f(z, b)$ is defined for

(a) all z on the left face, bottom face and right face of the unit square and all $b \in B$,

(b) as above. Then f can be extended to all z in the unit square and all $b \in B$.

Proof: Topologically, the bottom face of the unit square is the same as the union of the left, bottom and right faces. Hence, Corollary 2 follows from the reformulation of Corollary 1.

We proceed now to define δ. Let $\alpha \in T^{-(n+1)}(Y) = (T^{S^{n+1}(Y)})$ and let $\phi : S^{n+1}(Y) \to T$ be a representative of α. From (ii) and (iii) above, $S^{n+1}(Y) = S^1(S^n(Y))$. From the definition of smash product, we can easily convince ourselves that ϕ can be regarded as a map

$$\phi : I \times S^n(Y) \to T$$

satisfying $\phi : 0 \times S^n(Y) \to t_0$, $\phi : 1 \times S^n(Y) \to t_0$, $\phi : I \times p_0 \to t_0$, where t_0 is the basepoint of T and p_0 is the basepoint of $S^n(Y)$. We define a family of maps

$$\phi_t : S^n(Y) \to T$$

by setting $\phi_t(p) = \phi(t, p)$, $p \in S^n(Y)$. Thus, ϕ_0 is constant on $S^n(Y)$ and may therefore be extended as a constant to all of $S^n(X)$. (Since $X \supseteq Y$, also $S^n(X) \supseteq S^n(Y)$. Note also that $S^n(X)$ is a cell complex and $S^n(Y)$ is a subcomplex.) Thus, if $f_0 : S^n(X) \to T$ is defined by $f_0(p) = t_0$ for all $p \in S^n(X)$, it follows that ϕ_t is a partial homotopy of f_0. By HET, there is a homotopy

$$f_t : S^n(X) \to T$$

extending ϕ_t. Now $f_1 : S^n(X) \to T$ and since f_1 extends ϕ_1, we see that $f_1 : S^n(Y) \to t_0$. Hence f_1 can be regarded as a map

$$f_1 : S^n(X)/S^n(Y) \to T.$$

An easy check shows that $S^n(X)/S^n(Y) = S^n(X/Y)$. Thus f_1 is a map

$$f_1 : S^n(X/Y) \to T.$$

Let $\{f_1\}$ be the class of f_1 in $(T^{S^n(X/Y)})$ and put

$$\delta\alpha = \delta\{\phi\} = \{f_1\}.$$

We have to show that $\{f_1\}$ is well-defined by α. So let $\phi^{(0)}$ and $\phi^{(1)}$ be two representatives of α. Then $\phi^{(0)}$ and $\phi^{(1)}$ give rise to families of maps $\phi_t^{(0)}$, $\phi_t^{(1)}$ satisfying conditions mentioned above. Moreover, there is a homotopy $\phi_t^{(s)}$ between $\phi_t^{(0)}$ and $\phi_t^{(1)}$ and we have extensions $f_t^{(0)}$, $f_t^{(1)}$ of $\phi_t^{(0)}$, $\phi_t^{(1)}$ to all of $S^n(X)$ which are identically constant for $t = 0$. Define $f_0^{(s)}$ by

$$f_0^{(s)}(p) = t_0, \quad \text{all} \quad p \in S^n(X),$$

and also put

$$f_t^{(s)}(p) = \phi_t^{(s)}(p), \qquad \text{all} \qquad p \in S^n(Y).$$

We thus have a map f satisfying the conditions of Corollary 2 of the HET and therefore f can be extended to be a map from all of $I \times I \times S^n(X)$ into T. This map in turn gives a map $f_t^{(s)}(p)$ defined for all t, $s \in I$, $p \in S^n(X)$. Clearly, $f_1^{(s)}$ is a homotopy connecting $f_1^{(0)}$ and $f_1^{(1)}$ and since $f_1^{(s)}$ is constant on $S^n(Y)$, it follows that as elements of $T^{-n}(X, Y)$, $\{f_1^{(0)}\} = \{f_1^{(1)}\}$, i.e. δ is well-defined.

We shall now verify the first six Eilenberg–Steenrod axioms for a cohomology theory. The first three Eilenberg–Steenrod axioms assert the existence and basic properties of the cohomology maps induced by continuous maps and of the coboundary map. They have essentially been proved already; all additional details are left as an exercise.

To prove the exactness axiom, let us consider the sequence

$$\cdots \longrightarrow T^{-n}(Y) \xrightarrow{\delta} T^{-n+1}(X, Y) \xrightarrow{j^*} T^{-n+1}(X) \xrightarrow{i^*} T^{-n+1}(Y) \longrightarrow \cdots$$

where i^* and j^* are induced by the inclusion maps

$$i: Y \to X, \quad j: X \to (X, Y)$$

and δ is defined above.

There are six assertions to check. We shall only do two of them; the others are of the same order of difficulty and are left to the reader.

(1) $\ker i^* \subseteq \operatorname{image} j^*$: Let $\{\phi\} \in \ker i^*$. Then $\phi: S^{n-1}(X) \to T$ and $\phi | S^{n-1}(Y): S^{n-1}(Y) \to T$ is homotopic to the constant map. Call this homotopy ϕ_t. By HET, there is a homotopy $\Phi_t: S^{n-1}(X) \to T$ which extends ϕ_t. Hence $\phi \simeq \Phi_1$ and $\Phi_1 | S^{n-1}(Y) = \phi_1 =$ the constant map, i.e. $\{\phi\} = j^*\{\Phi_1\}$.

(2) $\ker j^* \subseteq \operatorname{image} \delta$: Let $\{\psi\} \in \ker j^*$. Then $\psi: S^{n-1}(X) \to T$ is the constant map on $S^{n-1}(Y)$ and ψ is homotopic to the constant map. Let ψ_t be the homotopy and for convenience, let $\psi_1 = \psi$, $\psi_0 =$ constant. But then $\psi_t | S^{n-1}(Y)$ is the constant map for $t = 0$ and $t = 1$ and hence induces a map of $S^1(S^{n-1}(Y)) \to T$; call this map Ψ. Then, by definition $\{\psi\} = \{\psi_1\} = \delta\{\Psi\}$, i.e. $\{\psi\} \in \operatorname{image} \delta$.

We now prove the Homotopy Axiom.

Let $f_0, f_1: (X, Y) \to (X', Y')$ be homotopic with homotopy f_t. Then $f_t: (X, Y) \to (X', Y')$ induces $f_t': X/Y \to X'/Y'$ which in turn induces $\bar{f}_t: S^n \wedge (X/Y) \to S^n \wedge (X'/Y')$. Now the map

$$f_i^*: (T^{S^n(X'/Y')}) \to (T^{S^n(X/Y)})$$

is given by $\{h\} \to \{h\bar{f}_i\}$, where $h \in T^{S^n(X'/Y')}$ and $i = 0, 1$. But $h\bar{f}_0$ is homotopic to $h\bar{f}_1$ via the homotopy $h\bar{f}_t$ and hence $\{h\bar{f}_0\} = \{h\bar{f}_1\}$, i.e. $f_0^* = f_1^*$.

We now prove the Excision Axiom. (In fact, in this theory, we have a strong form of excision.) Indeed, let U be any open set in Y and consider the inclusion map

$$f:(X - U, Y - U) \to (X, Y).$$

The fact that

$$f^*: T^{-n}(X, Y) \to T^{-n}(X - U, Y - U)$$

is an isomorphism follows directly from the fact that X/Y is homeomorphic to $X - U/Y - U$.

We note that the dimension axiom is false in general. Indeed, let $\{p, p_0\}$ be the space consisting of two points, with basepoint p_0. Then, since

$$S^n \wedge \{p, p_0\} = (S^n \times p) \cup (S^n \times p_0)/(\nu \times p) \cup (\nu \times p_0) \cup (S^n \times p_0),$$

it follows that $T^{-n}(\{p, p_0\}, p_0) = \pi_n(T)$.

We shall now give conditions on T in order that the T-cohomology theory be a genuine generalised cohomology theory.

Definition (1): A space T is said to be a *group-like* space if there exists a continuous map $\eta : T \times T \to T$ (we shall write $t_1 \times t_2 = \eta(t_1, t_2)$) having the following properties.

(i) There exists an element $e \in T$ such that $e \times e = e$ and such that
  a) the map $t \to t \times e$ is homotopic to the identity map $t \to t$.
  b) the map $t \to e \times t$ is homotopic to the identity map $t \to t$.

(ii) For each $t \in T$, there exists $I(t) \in T$ depending continuously on T such that
  c) the map $t \to I(t)$ is homotopic to the constant map $t \to e$.
  d) the map $t \to I(t) \times t$ is homotopic to the constant map $t \to e$.

(iii) The two maps $(t_1, t_2, t_3) \to (t_1 \times t_2) \times t_3$ and $(t_1, t_2, t_3) \to t_1 \times (t_2 \times t_3)$ are homotopic to each other.

We say that e is a homotopy unit of T, $I(t)$ is a homotopy in verse of t, and η is homotopy-associative. If, in addition to (i), (ii) and (iii) above, we also have

(iv) The map $\eta' : T \times T \to T$ defined by $\eta(t_1, t_2) = t_2 \times t_1$ is homotopic to the map η, we say that T is an *abelian* group-like space.

Note: We take e to be the basepoint of T. Then all the homotopies in the definition must be restricted to be basepoint preserving.

If T is a group-like space, then (T^X) can be given a group structure as follows.

If $\{f\}, \{g\} \in (T^X)$, define $\{f\} \cdot \{g\} = \{f \cdot g\}$ where $f \cdot g$ is given by

$$f \cdot g(x) = f(x) \times g(x).$$

Note that the idempotence of e insures that $f \cdot g$ is basepoint preserving. We define the unit element to be the homotopy class of the map

$$E: X \to T$$

given by $E(x) = e$ for all $x \in X$. If $\{f\} \in (T^X)$, define $\tilde{f}: X \to T$ by $\tilde{f}(x) = I(f(x))$ Then one can see easily that $\{f\} \cdot \{\tilde{f}\} = \{\tilde{f}\} \cdot \{f\} = \{E\}$. The associativity of the multiplication of homotopy classes follows from Definition 1, (iii). Finally, if T is an abelian group-like space, (T^X) is easily seen to be an abelian group.

According to a construction due to Eilenberg and MacLane (see Hu—*Homotopy Theory*, pp. 168–9, 198–204), there exists, for an arbitrary abelian group G, an abelian group-like space T_G such that

$$\pi_1(T_G) \approx G, \quad \pi_n(T_G) \approx 0 \qquad \text{for} \qquad n \neq 1.$$

Consider then the T_G-cohomology theory. If we set $H^n(X, Y) = T_G^{-n+1}(X,Y)$. then, as is easily seen, the dimension axiom is satisfied; in this special case, our general construction leads to normal cohomology theory with coefficients G.

Appendix to Part A

The purpose of this appendix is to point out that, without restricting T in any way, a group structure can be given to $(T^{S^n(X/Y)})$ for all $n \geqq 1$. It even turns out that for $n \geqq 2$, the group is abelian.

We shall give a rough indication of how this group structure is defined. The reader may consult Bourgin—*Modern Algebraic Topology*, pp. 414–421, for the details. The existence of a group structure in $(T^{S^n}) = \pi_n(T)$ can be deduced from the existence of a map $S^n \xrightarrow{\mu} S_1^n \vee S_2^n$ (which pinches the equator to a point) with certain properties. The fact that $\pi_n(T)$ is abelian for $n \geqq 2$ follows from the fact that the two halves S_1^n, S_2^n may be interchanged in a suitable manner. But the map μ naturally induces a map $S^n \wedge W \xrightarrow{\mu'} (S_1^n \wedge W) \vee (S_2^n \wedge W)$ and it turns out that the properties of μ needed to give a group structure (abelian if $n \geqq 2$) to (T^{S^n}) are also possessed by μ'.

The reader can now try to decide for himself whether the induced maps in T-cohomology and the coboundary map, both defined in Part A, are group homomorphisms.

Finally, one may ask whether the "natural" group structure outlined here agrees with the group structure defined in Part A in the case where T is an abelian grouplike space. The answer is that they do agree (cf. Bourgin, loc. cit.).

In Part B, we shall introduce a generalized cohomology theory whose test space T can be shown to be an abelian grouplike space. By virtue of the preceding remark, the group structure of the generalized cohomology groups to be introduced can be thought of as being derived from the grouplike structure of the test space. Thus, the remarks made in this appendix are not essential to the reading of the remainder of these notes.

Part B: *K*-Theory

We begin by describing certain algebraic operations on bundles. We derive from these operations a (commutative) ring $K(X)$. We then show how $K(X)$ relates to the theory developed in Part A.

Whitney Sum of Bundles

1. For $\alpha = 1, 2$, let γ_α be a $U(m_\alpha)$-bundle structure on a space B_α. Let the coordinate neighborhoods be denoted by $\{U_i^{(1)}\}$, $\{U_k^{(2)}\}$ respectively. Also, let $\{U_{b_1}^{(1)}(i, j)\}$, $\{U_{b_2}^{(2)}(k, l)\}$ denote the respective coordinate transformations. Now, the collection $\{U_k^{(1)} \times U_k^{(2)}\}$ covers $B_1 \times B_2$. For $(b_1, b_2) \in (U_i^{(1)} \times \times U_k^{(2)}) \cap (U_j^{(1)} \times U_i^{(2)})$, we define

$$U_{(b_1, b_2)}(i, k; j, l) = U_{b_1}(i, j) \oplus U_{b_2}(k, l).$$

Here, $\oplus$ denotes the direct sum of matrices. The group $U(m_1) \oplus U(m_2)$ is regarded as embedded in $U(m_1 + m_2)$ via the map which takes $(u, v) \in U(m_1) \oplus U(m_2)$ into the $(m_1 + m_2) \times (m_1 + m_2)$ matrix

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}.$$

Using the functions $U_{(b_1, b_2)}(i, k; j, l)$ as coordinate transformations, we clearly get a $U(m_1 + m_2)$-bundle structure on $B_1 \times B_2$; we call this bundle structure the *exterior sum* of γ_1 and γ_2 and denote it by $\gamma_1 \textcircled{\#} \gamma_2$.

If $B_1 = B_2 = B$, then we have the diagonal map $\delta\colon B \to B \times B$ given by $x \to (x, x)$. Using the notation introduced in Chapter IV, we have the induced bundle structure $\delta^\dagger(\gamma_1 \textcircled{\#} \gamma_2)$ on the space B. We call this bundle structure the *Whitney sum* of γ_1 and γ_2 and we write

$$\gamma_1 \oplus \gamma_2 = \delta^\dagger(\gamma_1 \textcircled{\#} \gamma_2).$$

2. Another way of looking at the Whitney sum of bundles (over the same base) is the following:

Let $\xi_\alpha = (E_\alpha, B, F_\alpha, U(m_\alpha), p_\alpha)$ ($\alpha = 1, 2$) be a complex vector bundle (i.e. a bundle whose fiber F_α is an m_α-dimensional complex vector space) with base B, total space E_α, projection p_α. Let E be the subset $\{(e_1, e_2) \in E_1 \times \times E_2 \mid p_1(e_1) = p_2(e_2)\}$ of the space $E_1 \times E_2$. Define a projection map

$$p : E \to B$$

by $p(e_1, e_2) = p_1(e_1) = p_2(e_2)$. The fiber is homeomorphic to $F_1 \times F_2$. Since $U(m_1 + m_2)$ acts on $F_1 \times F_2$, we can construct a bundle $\xi = (E, B, F_1 \times F_2, U(m_1 + m_2), p)$ and we call ξ the Whitney sum of ξ_1 and ξ_2.

3. Still another way of viewing the Whitney sum is the following: Let γ_1, γ_2 be bundle structures on B_1, B_2 with groups $U(m_1)$, $U(m_2)$ respectively. By the Universal Classification Theorem, there exist maps

$$\phi_i : B_i \to G_{m_i,\infty} \qquad (i = 1, 2)$$

which induce the bundles γ_i. On the other hand there exists a map N

$$N : G_{m_1,\infty} \times G_{m_2,\infty} \to G_{m_1+m_2,\infty},$$

defined by $N(\pi_1, \pi_2) = \pi_1 \oplus \pi_2$ where $\pi_i \in G_{m_i,\infty}$ ($i = 1, 2$). Therefore we have the following diagram:

$$B_1 \times B_2 \xrightarrow{\phi_1 \times \phi_2} G_{m_1,\infty} \times G_{m_2,\infty} \xrightarrow{N} G_{m_1+m_2,\infty}.$$

The composite map $N \circ (\phi_1 \times \phi_2)$ induces a bundle structure on $B_1 \times B_2$. One can convince oneself that this is the exterior sum $\gamma_1 \text{ⓘ} \gamma_2$ of γ_1 and γ_2. Again, if $B_1 = B_2 = B$, then we have the composite map

$$N \circ (\phi_1 \times \phi_2) \circ \delta : B \to G_{m_1+m_2,\infty}$$

and this map induces the Whitney sum of γ_1 and γ_2 on B.

Tensor Product (or Kronecker Product) of Bundles

We given only one version of this construction, corresponding to 3. above. Let γ_i, B_i, ϕ_i be as in 3. The natural bilinear map

$$h : U(m_1) \times U(m_2) \to U(m_1 m_2)$$

given by $(u, v) \to u \otimes v$ induces a map

$$h' : G_{m_1,\infty} \times G_{m_2,\infty} \to G_{m_1 m_2,\infty}$$

and hence we get the diagram

$$B_1 \times B_2 \xrightarrow{\phi_1 \times \phi_2} G_{m_1,\infty} \times G_{m_2,\infty} \xrightarrow{h'} G_{m_1m_2,\infty}.$$

The composite map $h' \circ (\phi_1 \times \phi_2)$ induces a bundle structure on $B_1 \times B_2$. We call this bundle structure the *exterior tensor product* of γ_1 and γ_2 and denote it by $\gamma_1 \circledast \gamma_2$. As before, if $B_1 = B_2 = B$, we can apply the diagonal map δ and get

$$\delta^\dagger(\gamma_1 \mathbin{\textcircled{\#}} \gamma_2) = \gamma_1 \otimes \gamma_2.$$

Then $\gamma_1 \otimes \gamma_2$ is called the *tensor product* of γ_1 and γ_2.

Remark: The following statements are easily proved:

(a) $\gamma_1 \oplus \gamma_2 \cong \gamma_2 \oplus \gamma_1$,

(b) $\gamma_1 \oplus (\gamma_2 \oplus \gamma_3) \cong (\gamma_1 \oplus \gamma_2) \oplus \gamma_3$,

(c) $\gamma_1 \otimes \gamma_2 \cong \gamma_2 \otimes \gamma_1$,

(d) $\gamma_1 \otimes (\gamma_2 \otimes \gamma_3) \cong (\gamma_1 \otimes \gamma_2) \otimes \gamma_3$,

(e) $\gamma_1 \otimes (\gamma_2 \oplus \gamma_3) \cong (\gamma_1 \otimes \gamma_2) \oplus (\gamma_1 \otimes \gamma_3)$,

(f) if $f: \hat{B} \to B$, then $f^\dagger(\gamma_1 \oplus \gamma_2) \cong f^\dagger(\gamma_1) \oplus f^\dagger(\gamma_2)$.

(g) if $f: \hat{B} \to B$, then $f^\dagger(\gamma_1 \otimes \gamma_2) \cong f^\dagger(\gamma_1) \otimes f^\dagger(\gamma_2)$.

Lemma (1): Given a principal $U(n)$-bundle ξ over a base X, there exists a principal $U(n')$-bundle $\xi^\perp$ over X (where n' is some suitable integer) such that $\xi \oplus \xi^\perp$ is trivial.

Proof: The lemma is proved first for the case $\xi = \alpha_{n,N} = (V_{n,N}, G_{n,N}, U(n), U(n), p_{n,N})$. Recall that the Stiefel manifold $V_{n,N}$ consists of all orthonormal n-frames at the origin in N-space. Equivalently, we may view V as the set of all pairs (π, v_n), π $\in G_{n,N}$ and where v_n is an orthonormal basis for π. The projection $p_{n,N}$ is therefore given by

$$p_{n,N}((\pi, v_n)) = \pi.$$

We define a set $E_{n,N}$ as follows: $E_{n,N} = \{(\pi, v^\perp_{N-n}) \mid \pi \in G_{n,N},\ v^\perp_{N-n}$ is an orthonormal $(N-n)$-frame forming an orthonormal basis of $\pi^\perp\}$. We define a projection map

$$p'_{n,N}: E_{n,N} \to G_{n,N}$$

by setting

$$p'_{n,N}((\pi, v^\perp_{N-n})) = \pi.$$

We then get a principal bundle $\alpha^\perp_{n,N} = (E_{n,N}, G_{n,N}, U(N-n), U(N-n), p'_{n,N})$. It is clear that $\alpha_{n,N} \oplus \alpha^\perp_{n,N}$ is equivalent to the product bundle $G_{n,N} \times U(N)$, i.e. $\alpha_{n,N} \otimes \alpha^\perp_{n,N}$ is trivial.

To prove the lemma for a general ξ, we choose a "classifying" map for ξ,

$$\phi : X \to G_{n,N},$$

where N is a sufficiently large integer (depending on the dimension of X). Thus,

$$\phi^{\dagger}(\alpha_{n,N}) = \xi.$$

We define $\xi^{\perp}$ by setting

$$\xi^{\perp} = \phi^{\dagger}(\alpha^{\perp}_{n,N}).$$

Then, $\xi \otimes \xi^{\perp} = \phi^{\dagger}(\alpha_{n,N}) \otimes \phi^{\dagger}(\alpha^{\perp}_{n,N}) \cong \phi^{\dagger}(\alpha_{n,N} \otimes \alpha^{\perp}_{n,N})$ by the preceding remark. The lemma now follows easily.

Definition (1): Let ξ_1 and ξ_2 be two principal unitary bundles on a base X. Then ξ_1 and ξ_2 are s-equivalent if there exist trivial bundles τ_1 and τ_2 such that

$$\xi_1 \oplus \tau_1 \cong \xi_2 \oplus \tau_2.$$

It is easy to check that s-equivalence is actually an equivalence relation. We denote it by $\overset{s}{\sim}$ and we denote the s-equivalence class containing the bundle α by $\{\alpha\}$. For the set of all s-equivalence classes, we write $K_0(X)$. Notice that (ordinary) equivalence of bundles implies s-equivalence but s-equivalence does not imply equivalence. E.g., the tangent bundle of S^{2n} is s-equivalent to a trivial bundle but not equivalent to a trivial bundle.

Lemma (2): The set $K_0(X)$ of all s-equivalence classes of principal unitary bundles on X forms an abelian group under the operation induced by the Whitney sum of bundles.

Proof: Let $\{\xi_1\}$ and $\{\xi_2\}$ be any two elements in $K_0(X)$; ξ_1 and ξ_2 are then principal unitary bundles representing the classes $\{\xi_1\}$ and $\{\xi_2\}$. We define

$$\{\xi_1\} \oplus \{\xi_2\} = \{\xi_1 \oplus \xi_2\}.$$

It is easy to check that this operation is well defined. Moreover it is associative and commutative by the remark preceding Lemma 1. The identity of the group is the class $\{\tau\}$ of all trivial bundles τ; this is a well defined object since all trivial bundles are s-equivalent. Moreover,

$$\{\xi\} \oplus \{\tau\} = \{\xi\} = \{\tau\} \oplus \{\xi\}.$$

Let $\{\xi\} \in K_0(X)$, ξ a representation of $\{\xi\}$. Let $\xi^{\perp}$ be the bundle constructed in the previous lemma, i.e. $\xi \oplus \xi^{\perp} \cong \tau$ where τ is a trivial bundle. Then

$$\{\xi\} \oplus \{\xi^{\perp}\} = \{\xi \oplus \xi^{\perp}\} = \{\tau\}.$$

Thus $\{\xi^{\perp}\}$ is the inverse element of $\{\xi\}$, and this completes the proof that $K_0(X)$ is an abelian group.

Note: We will see later that $K_0(X)$ can be made into a ring.

Atiyah-Hirzebruch-Grothendieck Functor

Let X be a finite dimensional cell complex, α a principal unitary bundle on X. Denote by (α) the bundle structure equivalence class containing α. We denote by $K_F(X)$ the free abelian group generated by all equivalence classes (α) of principal unitary bundles on X. Let further $K_F^+(X)$ be the subgroup of $K_F(X)$ generated by all elements of the form $(\alpha_1 \oplus \alpha_2) - (\alpha_1) - (\alpha_2)$. We set

$$K(X) = K_F(X)/K_F^+(X).$$

$K(X)$ is called the *Grothendieck Group* of X, or the A-H-G *Functor*. Indeed, $K(X)$ is a contravariant functor from the category of cell complexes and continuous maps to the category of abelian groups and homomorphisms. To see this, note that a map $f\colon X \to Y$ induces a map $f^{\dagger}$ from bundles over Y to bundles over X. As $f^{\dagger}$ carries equivalent bundles into equivalent bundles, we have an induced map, call it $f^{\dagger}$ again, from $K_F(Y)$ into $K_F(X)$. Now, by the remark preceding Lemma 1, we have

$$f^{\dagger}((\alpha_1 \oplus \alpha_2)) = f^{\dagger}((\alpha_1)) \oplus f^{\dagger}((\alpha_2)).$$

Hence, $f^{\dagger}$ also maps $K_F^+(Y)$ into $K_F^+(X)$ and thus induces a map, again called $f^{\dagger}$, from $K(Y)$ into $K(X)$. It is evident that the correspondence $X \to K(X)$, $f \to f^{\dagger}$ is functiorial.

Again using the above mentioned remark, we see that the tensor product of bundles induces naturally a (commutative) ring structure on $K(X)$. This ring also possesses a multiplicative unit, namely the class of the trivial bundle of dimension one, i.e. the trivial $U(1)$-bundle.

Notation: For a principal unitary bundle α over X, we write $[\alpha]$ for the class of α in $K(X)$. Recall that (α) denotes the equivalence class of α and $\{\alpha\}$ the s-equivalence class of α. We have already observed that $(\alpha) = (\beta) \Rightarrow \{\alpha\} = \{\beta\}$ but not conversely. We also clearly have $(\alpha) = (\beta) \Rightarrow [\alpha] = [\beta]$.

We define a ring homomorphism

$$\dim : K(X) \to Z$$

by setting

$$\dim\,(\Sigma n_i[\alpha_i]) = \Sigma\, n_i \cdot \dim \alpha_i.$$

Here, $\Sigma n_i[\alpha_i]$ is an arbitrary element of $K(X)$ so that the n_i are integers and all but a finite number of them are 0. Note carefully that if α_i is a $U(m_i)$-bundle, then by dim α_i is meant the integer m_i and not the dimension of the manifold $U(m_i)$. The integer m_i is also referred to sometimes as the *rank* of $U(m_i)$ and of the bundle α_i. The map dim is called the *dimension homomorphism*. We define

$$\tilde{K}(X) = \ker(\dim).$$

Thus, $\tilde{K}(X)$ is an ideal in $K(X)$. It is clear that $\tilde{K}(X)$, like $K(X)$, is a contravariant functor from the category of finite cell complexes and continuous maps into the category of abelian groups and homomorphisms.

We now try to establish a relationship between $\tilde{K}(X)$ and $K_0(X)$. We define a homomorphism of groups

$$h' : K_F(X) \to K_0(X)$$

by setting

$$h'((\alpha)) = \{\alpha\}$$

and then extending by linearity. Observe that if m is a positive integer, then

$$h'(m(\alpha)) = \{\alpha\} \oplus \{\alpha\} \oplus \cdots \oplus \{\alpha\} \quad (m \text{ factors})$$

while if m is a negative integer,

$$h'(m(\alpha)) = \{\alpha^\perp\} \oplus \{\alpha^\perp\} \oplus \cdots \oplus \{\alpha^\perp\} \quad (m \text{ factors}).$$

Notice also that

$$\begin{aligned} h'((\alpha_1 \oplus \alpha_2) - (\alpha_1) - (\alpha_2)) &= \{\alpha_1 \oplus \alpha_2\} + \{\alpha_1^\perp\} \oplus \{\alpha_2^\perp\} \\ &= \{\alpha_1 \oplus \alpha_1^\perp \oplus \alpha_2 \oplus \alpha_2^\perp\} \\ &= \{\tau_1 \oplus \tau_2\} \\ &= \{\tau\} \end{aligned}$$

and $\{\tau\}$ is the identity element of $K_0(X)$. Hence, the subgroup $K_F^+(X)$ of $K_F(X)$ lies in the Kernel of h' and thus h' gives rise to a homomorphism

$$h : K(X) \to K_0(X)$$

which is such that

$$h([\alpha]) = \{\alpha\}, \quad h(-[\alpha]) = \{\alpha^\perp\}.$$

Now, the restriction of h to $\tilde{K}(X)$ is a homomorphism

$$\tilde{h} : \tilde{K}(X) \to K_0(X).$$

We then have the

Lemma (3): The map $\hbar$ is a (group) isomorphism.

Proof: We begin by defining a map

$$i: Z \to K(X)$$

as follows. Let τ_n be the trivial bundle of dimension n. If $m \geqq 0$, set $i(m) = [\tau_m]$; if $m \leqq 0$, set $i(m) = -[\tau_{-m}]$. The lemma will be proved if we can show that the sequence

$$0 \longrightarrow Z \longrightarrow K(X) \xrightarrow{h} K_0(X) \longrightarrow 0$$

is exact. In fact, we shall show that the sequence splits. To do this, we define a homomorphism

$$j: K_0(X) \to K(X)$$

as follows. Let $\{\beta\} \in K_0(X)$ and let β be a representative of $\{\beta\}$. Let $n = \dim \beta$. We set $j(\{\beta\}) = [(\beta) - (\tau_n)]$. If β and β_1 are s-equivalent, there exist trivial bundles $\bar{\tau}$ and $\bar{\tau}_1$ such that $\beta + \bar{\tau}$ and $\beta_1 + \bar{\tau}_1$ are equivalent. Then, if $n_1 m_1 k_1 l$ are the dimensions of $\beta_1 \beta_1$, $\bar{\tau}_1 \bar{\tau}_1$ respectively, we have $\beta \oplus \bar{\tau} \oplus \tau_m \sim \beta_1 \oplus \bar{\tau}_1 \oplus \tau_m$, while $\bar{\tau}_1 \oplus \tau_m$ is clearly the same as the trivial bundle of dimension $n + m + k - m = n + k$, i.e., $\bar{\tau} \oplus \tau_n$. Thus $\gamma = \beta \oplus \bar{\tau} \oplus \tau_m \sim \beta_1 \oplus \bar{\tau} \oplus \tau_n$, and then clearly $[(\beta) - (\tau_n)] = [(\gamma) - (\bar{\tau} \oplus \tau_m \oplus \tau_n)] \sim [(\beta_1) - (\tau_m)]$. This shows that j is well-defined. Furthermore, the relations

$$\dim \circ\, i = \text{identity}, \quad h \circ j = \text{identity}$$

$$\dim \circ\, j = 0, \qquad h \circ i = 0$$

are obvious. Finally, if $[\alpha]$ is an arbitrary generator of $K(X)$ and $\dim \alpha = n$, we can write

$$[\alpha] = [(\alpha) - (\tau_n)] + [\tau_n] = j(\{\alpha\}) + i(n),$$

i.e. $K(X) = \operatorname{im}(i) \otimes \operatorname{im}(j)$. This proves the lemma.

Since $\tilde{K}(X)$ has a ring structure, $K_0(X)$ can be given a ring structure by using the (group) isomorphism $\hbar$.

We come now to our main result.

Theorem (1): There exists a space BU such that there is a $1-1$ correspondence $K_0(X) \leftrightarrow (BU^X)$.

Corollary (1): There is a $1-1$ correspondence $\tilde{K}(X) \leftrightarrow (BU^X)$.

Proof: This follows immediately from Lemma 3.

Corollary (2): The is a 1 − 1 correspondence $K(X) \leftrightarrow ((BU \times Z)^X)$ where Z is given the discrete topology.

Proof: $K(X) \cong \tilde{K}(X) \times Z$ and $((BU \times Z)^X) \leftrightarrow (BU^X) \times (Z^X) \leftrightarrow (BU^X) \times Z$. Then, Corollary 2 follows from Corollary 1.

Proof of Theorem 1: We define a sequence of spaces $E^{\infty}_{-n} (n = 1, 2, \ldots)$ as follows. Let E^{∞}_{-n} consist of all sequences of complex numbers $(\ldots, x_{-1}, x_0, x_1, \ldots)$, infinite in both directions, such that

(a) All but a finite number of the x_i are 0.
(b) All the x_i with $i \leqq -n$ are 0.

We shall regard the infinite Grassmannian $G_{n,\infty}$ as the set of n-planes in the space E^{∞}_{-n} (cf. Chapter III). We then have a sequence of natural inclusions

$$G_{1,\infty} \subseteq G_{2,\infty} \subseteq G_{3,\infty} \subseteq \cdots$$

defined as follows. Let $\pi_n \in G_{n,\infty}$. We have clearly

$$E^{\infty}_{-(n+1)} = E^{\infty}_{-n} \oplus \text{a one-dimensional space.}$$

Let $e_{-(n+1)}$ be a generator of this one-dimensional space and let

$$i_{n,n+1} : G_{n,\infty} \to G_{n+1,\infty}$$

be defined by

$$i_{n,n+1}(\pi_n) = \{\pi_n, e_{-(n+1)}\},$$

i.e. the plane generated by π_n and $e_{-(n+1)}$. We denote the composite inclusion of $G_{n,\infty}$ into $G_{n+k,\infty}$ by $i_{n,n+k}$. We define BU as the union of all the spaces $G_{n,\infty} (n = 1, 2, \ldots)$ taken in the direct limit topology; we write

$$BU = \lim_{n \to \infty} G_{n,\infty}.$$

Recall that each $G_{n,\infty}$ it itself a direct limit space. Thus,

$$G_{n,\infty} = \lim_{m \to \infty} G_{n,m}.$$

Hence, by extracting a "diagonal sequence", we may write

$$BU = \lim_{n \to \infty} G_{n,2n},$$

i.e. as a direct limit of compact spaces.

It follows immediately from this that any compact subset of BU is contained in some $G_{n,2n}$ (and hence in $G_{n,\infty}$). Note also that BU is itself *not* compact.

We now define a map $K_0(X) \to (BU^X)$ as follows: Let $\{\alpha\} \in K_0(X)$, α a representative of $\{\alpha\}$, and let $\dim \alpha = k$. (There is then a classifying map

$$\phi_k : X \to G_{k,\infty}$$

which induces α. If $i_k : G_{k,\infty} \to BU$ is the inclusion map, then we define

$$\Phi : X \to BU$$

by $\Phi = i_k \circ \phi_k$. If $\beta \overset{s}{\sim} \alpha$ and $\dim \beta = l$, choose a classifying map

$$\psi_l : X \to G_{l,\infty}$$

inducing β. We then have a map

$$\Psi : X \to BU$$

given by $\Psi = i_l \circ \psi_l$. We claim that Φ and Ψ are homotopic. Indeed, since $\alpha \overset{s}{\sim} \beta$, there exist trivial bundles τ_1, τ_2 such that

$$\alpha \oplus \tau_1 \cong \beta \oplus \tau_2 .$$

Let $\dim (\alpha \otimes \tau_1) = \dim (\beta \otimes \tau_2) = r$. Then $r \geqq k, l$ and we have maps

$$\phi_r, \psi_r : X \to G_{r,\infty}$$

given by $\phi_r = i_{k,r} \circ \phi_k$, $\phi_r = i_{l,r} \circ \psi_l$. It is easy to see that

$$i^{\dagger}_{k,r}(\alpha_r) \cong \alpha_k \oplus \tau_1, \quad i^{\dagger}_{l,r}(\alpha_r) \cong \alpha_l \oplus \tau_2$$

where, in general, α_n is the standard principal $U(n)$-bundle over $G_{n,\infty}$ (see Chapter VI, Definition 5). But then

$$\phi^{\dagger}_r(\alpha_r) \cong \alpha \oplus \tau_1, \quad \psi^{\dagger}_r(\alpha_r) \cong \beta \oplus \tau_2$$

and hence

$$\phi^{\dagger}_r(\alpha_r) \cong \psi^{\dagger}_r(\alpha_r).$$

By Chapter VI, Theorem 4, we obtain

$$\phi_r \simeq \psi_r$$

and since $\Phi = i_r \circ \phi_r$, $\Psi = i_r \circ \psi_r$, it follows that

$$\Phi \simeq \Psi,$$

as claimed. Thus, the correspondence

$$\{\alpha\} \to [\Phi],$$

where $[\Phi]$ is the homotopy class of Φ, is well-defined. The fact that this correspondence is $1 - 1$ follows immediately from the converse of the theorem just referred to. We prove finally that the correspondence is onto. Pick any $\Phi : X \rightarrow BU$. As X is a finite cell complex, X is compact and so also is $\Phi(X)$. From a remark made above, this means that $\Phi(X) \subseteq G_{n,\infty}$ for sufficiently large n. Thus, Φ can be factored in the form

$$\Phi = i_n \circ \phi_n$$

where ϕ_n is a map from X into $G_{n,\infty}$. We set

$$\alpha = \phi_n^{\dagger}(\alpha_n).$$

The reader can check that α is uniquely determined by Φ up to s-equivalence. Even more, α is uniquely determined by $[\Phi]$ up to s-equivalence. This can be proved by using the compactness of $X \times I$. Thus, the correspondence $\{\alpha\} \rightarrow [\Phi]$ is indeed invertible and we are finished.

Remark (1): It can be shown that BU is an abelian grouplike space. If (BU^X) is endowed with its natural group structure (see Part A and the appendix to Part A), it turns out that $\tilde{K}(X)$ and (BU^X) are group isomorphic.

Remark (2): Let $U = \lim_{n \rightarrow \infty} U(n)$ be the direct limit of the unitary groups. U is called the infinite unitary group. Then BU can be regarded as the universal classifying space for U; hence the notation BU.

Definition (2): Let (X, Y) be a cellular pair, i.e. Y is a subcomplex of the finite cell complex X. If $Y = \phi$ and if X has no a priori basepoint, we replace X by the disjoint union $X \cup \{p\}$ of X with a point p which becomes the basepoint. For each $n \geqq 0$, we set

$$K^{-n}(X, Y) = \tilde{K}(S^n(X/Y)).$$

By the results of Part A, the groups $K^{-n}(X, Y)$ form a generalized cohomology theory, the so-called K-theory. In the next chapter, we examine the particular features of this cohomology theory in more detail.

CHAPTER IX

Continuation of K-Theory

In this chapter, we shall examine in more detail the structure of the groups $K^{-n}(X, Y)$ defined at the end of the last chapter. As a reference for the material in this chapter, we suggest Atiyah and Hirzebruch—*Vector Bundles and Homogeneous Spaces,* Proceedings of Symposia in Pure Mathematics, Vol. 3, Differential Geometry, pp. 7–38, American Mathematical Society, 1961.

1. Product Theory

First of all, let us consider an arbitrary generalized cohomology theory, based on some grouplike space T. In Chapter 8, we used the notation $T^{-n}(X, Y)$ for $(T^{S^n(X/Y)})$. We look at the exact sequence of the pair $(X \times Y, X \vee Y)$. Using the facts that $T^{-n}(X, Y) = T^{-n}(X/Y, *)$ (where $*$ means the basepoint) and $X \times Y/X \vee Y = X \wedge Y$, we arrive at $(E) \cdots \xrightarrow{\delta} T^{-n}(Y \wedge Y) \xrightarrow{\pi^*} T^{-n}(X \times Y) \xrightarrow{i^*} T^{-n}(X \vee Y) \xrightarrow{\delta} \cdots$ Here, $i: X \vee Y \to X \times Y$ and $\pi: X \times Y \to X \wedge Y$ are the natural maps and i^*, π^* are the induced maps.

Lemma (1): The exact sequence (E) breaks up into short split exact sequences

$$0 \longrightarrow T^{-n}(X \wedge Y) \xrightarrow{\pi^*} T^{-n}(X \times Y) \xrightarrow{i^*} T^{-n}(X \vee Y) \longrightarrow 0.$$

Proof: First, we make some preliminary remarks. If (S, s_0) is any based space, let ΩS be the space of loops on S which begin and end at s_0. Inductively, we set $\Omega^n S = \Omega(\Omega^{n-1} S)$. We observe two properties of the operation Ω^n: (a) $\Omega^n S$ is grouplike for $n > 0$ regardless of whether or not S is grouplike, (*b*) there is a natural $1-1$ correspondence between $(S^{S^n W})$ and $((\Omega^n S)^W)$ (W any space) which can be described as follows. Suppose $n = 1$, the general case following inductively. A map $f: S^1 \wedge W \to S$ can be viewed simply as a map $f: I \times W \to S$ such that $f(0 \times W) = s_0, f(1 \times W) = s_0$, $f(I \times w_0) = s_0$. A map $f': W \to \Omega S$ is defined by letting $f'(w)$ be the loop, which evaluated at time t, is just $f(t, w)$; $f'(w)(t) = f(t, w)$. The restrictions on f guarantee that f' is a basepoint preserving map from W to ΩS. The correspondence $[f] \to [f']$ (where $[\ ]$ means homotopy class) is the desired correspondence.

Returning to the proof of the lemma, we begin by showing that the map $((\Omega^n T)^{X \times Y}) \xrightarrow{i^*} ((\Omega^n T)^{X \vee Y})$ is onto. So let $[f] \in ((\Omega^n T)^{X \vee Y})$, $f: X \vee Y \to \Omega^n T$ a representative of $[f]$. Let $f_1 = f|X \times y_0$, $f_2 = f|x_0 \times Y$. Define a map $F: X \times Y \to \Omega^n T$ by setting $F(x, y) = f_1(x, y_0) \cdot f_2(y, x_0)$, the product arising from the grouplike structure of $\Omega^n T$. Clearly, F extends f up to homotopy, i.e. $i^*([F]) = [f]$. It is even easier to see that the map $((\Omega^n T)^{X \wedge Y}) \xrightarrow{\pi^*} ((\Omega^n T)^{X \times Y})$ is $1 - 1$. Hence, the short sequence described in the lemma is indeed exact. To see that the sequence splits, we observe first that $T^{-n}(X \vee Y) \cong T^{-n}(X) \oplus T^{-n}(Y)$. (This assertion is clear if we appeal to the very definition of T^{-n}.) Now, the natural projections $\pi_1 : X \times Y \to X$, $\pi_2 : X \times X \to Y$ induce maps $\pi_1^* : T^{-n}(X) \to T^{-n}(X \times Y)$, $\pi_2^* : T^{-n}(Y) \to T^{-n}(X \times Y)$ and hence $\pi_1^* \oplus \pi_2^* : T^{-n}(X) \oplus T^{-n}(Y) \to T^{-n}(X \times Y)$ is defined. Identifying $T^{-n}(X \vee Y)$ with $T^{-n}(X) \oplus T^{-n}(Y)$, we see then that $\pi_1^* \oplus \pi_2^*$ provides the desired splitting.

Let us now return to K-Theory. The (external) tensor product of bundles induces naturally a multiplication

$$K(X) \oplus K(Y) \xrightarrow{\mu} K(X \times Y).$$

Recall that $\tilde{K}(X) = Ker\ (\dim)$, where $\dim: K(X) \to Z$ is the homomorphism defined in Chapter 8. dim may be thought of as the homomorphism induced by the inclusion map $x_0 \to X$ (where x_0 is the basepoint of X). Now, if $a \in \tilde{K}(X)$, $b \in \tilde{K}(Y)$, then, as can easily be seen,

$$\mu(a, b) \in Ker\ [\tilde{K}(X \times Y) \to \tilde{K}(X \vee Y)].$$

By Lemma 1, it follows that $\mu(a, b) \in \tilde{K}(X \wedge Y)$. Thus, μ induces a multiplication, call it $\tilde{\mu}$

$$\tilde{K}(X) \oplus \tilde{K}(Y) \xrightarrow{\tilde{\mu}} \tilde{K}(X \wedge Y).$$

For $m, n \geqq 0$, it is now possible to define a multiplication

$$K^{-m}(X, X_0) \oplus K^{-n}(Y, Y_0) \to K^{-(m+n)}(X \times Y, (X_0 \times Y) \cup (X \times Y_0)).$$

This is done as follows:

$$K^{-m}(X, X_0) = \tilde{K}(S^m(X/X_0)),$$

$$K^{-n}(Y, Y_0) = \tilde{K}(S^n(Y/Y_0)),$$

$$K^{-(m+n)}(X \times Y, (X_0 \times Y) \cup (X \times Y_0)) = \tilde{K}(S^{m+n}(X \times Y/X_0 \times Y \cup X \times Y_0)).$$

But
$$X \times Y/(X_0 \times Y) \cup (X \times Y_0) = X/X_0 \wedge Y/Y_0$$
and
$$S_m (X/X_0) \wedge S^n (Y/Y_0) = S^{m+n} (X/X_0 \wedge Y/Y_0).$$

Thus, we have only to a multiplication

$$\tilde{K}(S^m(X/X_0)) \oplus \tilde{K}(S^n(Y/Y_0)) \to \tilde{K}(S^m(X/X_0) \wedge S^n(Y/Y_0)).$$

And such a multiplication, namely $\tilde{\mu}$, has just been defined. For simplicity, we shall denote this multiplication by juxtaposition. Using the natural homeomorphism between $X \times Y$ and $Y \times X$, we get a product

$$K^{-n}(Y, Y_0) \oplus K^{-m}(X, X_0) \to K^{-(m+n)}(X \times Y, (X_0 \times Y) \cup (X \times Y_0)).$$

Lemma (2): If $a \in K^{-m}(X, X_0)$, $b \in K^{-n}(Y, Y_0)$, then $ab = (-1)^{mn}ba$.

Proof: We have the homeomorphisms $S^m \wedge S^n \leftrightarrow S^{m+n}$, $S^n \wedge S^m \leftrightarrow S^{m+n}$ and the composition

$$S^{m+n} \to S^n \wedge S^m \to S^m \wedge S^n \to S^{m+n}$$

is a map of degree $(-1)^{mn}$. The lemma follows easily from this (see Atiyah-Hirzebruch, loc. cit.).

Let now (X, X_0) be a cellular pair with basepoint $x_0 \in X_0$. We have diagonal maps

$$X \xrightarrow{d} X \times X, \quad X_0 \xrightarrow{d} (x_0 \times X) \cup (X \times X_0).$$

Passing to the quotient, we obtain a map

$$X/X_0 \xrightarrow{d} (X/x_0) \wedge (X/X_0).$$

By considering the composition

$$K^{-m}(X) \oplus K^{-n}(X, X_0) \longrightarrow K^{-(m+n)}(X \wedge (X/X_0)) \xrightarrow{d^*} K^{-(n+m)}(X/X_0)$$

we see that $\sum_{n \geq 0} K^{-n}(X, X_0)$ is a $\sum_{m \geq 0} K^{-m}(X)$-module. Summing up, we have the following theorem.

Theorem (1): $\sum_{m \geq 0} K^{-m}(X)$ is a graded anticommutative ring. There is a product mapping of degree 0

$$\Big(\sum_{m \geq 0} K^{-m}(X)\Big) \oplus \Big(\sum_{n \geq 0} K^{-n}(X, X_0)\Big) \to \sum_{p \geq 0} K^{-p}(X, X_0)$$

turning $\sum_{n \geq 0} K^{-n}(X, X_0)$ into a graded $\sum_{m \geq 0} K^{-m}(X)$-module.

Addendum: The multiplications defined above have evident functorial properties. As an example, let $f: (X, X_0) \to (X', X_0')$, $g: (Y, Y_0) \to (Y', Y_0')$ be mappings of pairs. Then the diagram

$$\begin{tikzcd}
K^{-m}(X', X_0') \oplus K^{-n}(Y', Y_0') \arrow[r] \arrow[d, "f^* \oplus g^*"] & K^{-(m+n)}(X' \times Y', (X_0' \times Y') \cup (X' \times Y_0')) \arrow[d, "(f \times g)^*"] \\
K^{-m}(X, X_0) \oplus K^{-n}(Y, Y_0) \arrow[r] & K^{-(m+n)}(X \times Y, (X_0 \times Y) \cup (X \times Y_0))
\end{tikzcd}$$

is easily seen to be commutative. Notice also that a map $f: X \to Y$ induces a homomorphism

$$f^*: \sum_{m \geqq 0} K^{-m}(Y) \to \sum_{m \geqq 0} K^{-m}(X)$$

which is a ring homomorphism.

Remark: Observe that in the above discussion we have used the particular properties of K-Theory only once, namely in deducing the existence of a map $\mu: K(X) \oplus K(Y) \to K(X \times Y)$. The rest of the development merely made use of the fact that K-Theory is a generalized cohomology theory. Thus, a product theory as above can be established for any T-cohomology theory which admits a map

$$\mu: T^0(X) \oplus T^0(Y) \to T^0(X \times Y).$$

2. The Bott Theorem and its Consequences

The special properties of a generalized cohomology theory depend entirely on the structure of the test space T of the cohomology theory. The homotopy structure of the space BU, which is the test space in K-Theory, has been completely determined by Bott. We present here Bott's results (in a formulation due to Bott and Atiyah) without proof. These results are then used to deduce some of the main results in K-Theory. For a proof of Bott's Theorem, see Milnor-*Morse Theory* and J. Schwartz—*Nonlinear Functional Analysis*, Gordon and Breach 1968, as well as Bott's original papers.

Theorem (2) (Bott-Atiyah-Hirzebruch): $\tilde{K}(S^1) = 0$, $\tilde{K}(S^2) = Z$. The generator of $\tilde{K}(S^2)$ can be described as follows: Let S^2 be represented as $P_1^{\mathbb{C}}$ and let η be the canonical line bundle (the Hopf bundle) over $P_1^{\mathbb{C}}$, i.e. over each point $[z_1, z_2] \in P_1^{\mathbb{C}}$ ($[z_1, z_2]$ means projective class of the pair (z_1, z_2)), the fiber is the complex line consisting of all pairs $(\lambda z_1, \lambda z_2)$, $\lambda \in C$. Let 1 denote the trivial 1-dimensional bundle over S^2 and set $\mu = \eta - 1$. Then

$g = [\mu] \in \tilde{K}(S^2)$ (because $\dim g = 0$) and g is a free generator of $\tilde{K}(S^2)$. Finally, let $a \in K^{-m}(X, X_0)$. Then the map

$$\beta : K^{-m}(X, X_0) \to K^{-(m+2)}(X, X_0),$$

given by $\beta(a) = ag$, is an isomorphism, the so-called Bott isomorphism.

Proof: The assertion concerning $\tilde{K}(S^2)$ made in Theorem 2 is elementary and may be proved as follows. Since S^2 may be covered by two coordinate discs P^+ and P^- intersecting in a circle C, and since any $U(n)$-bundle structure over P^+ or P^- is trivial, a $U(n)$-bundle structure over S^2 is determined by a coordinate change mapping h on C, which is a mapping $h: C \to U(n)$. Two such mappings h and h' determine equivalent bundle structures if and only if there exists maps $g_+ : P^+ \to U(n)$ and $g_- : P^+ \to U(n)$ such that $g_+ h g^{-1} = h'$. (Cf. the Remark preceding Definition 2 of Chapter 4.) It is trivial to verify that this is the case if and only if h and h' are homotopic. In this way, we established a correspondence between $U(n)$-bundle structures on S^2, and homotopy classes in $\pi_1(U(n))$. Since, as is developed in more detail below, any map $h: C \to U(n)$ can be deformed into a map $h: C \to U(1) \subseteq U(n)$, it follows that any $U(n)$-bundle structure over S^2 is equivalent to the direct sum of a $U(1)$-bundle structure and a trivial bundle. Thus the structure of $\tilde{K}(S^2)$ is determined by the $U(1)$ bundles which it contains. If a $U(1)$-bundle μ corresponds in the above sense to a given map $h: C \to U(1)$, then $\mu \otimes \mu$ corresponds to $h^2 : C \to U(1)$. Since it is easy to see that the Hopf bundle η corresponds to a generator h of the group $\pi_1(U(1))$, it follows that every $U(n)$-bundle structure on S^2 is s-equivalent either to a Kronecker power of η or to the inverse of one of these bundles. From this, verification of the asserted structure of $\tilde{K}(S^2)$ is easy.

We sketch the proof of the remaining assertion of Theorem 2. Let X be any finite-dimensional complex. Then, according to Theorem 1 of the preceding section, and its proof, the elements of $\tilde{K}(X)$ are in 1–1 correspondence with the homotopy classes of maps $\phi: X \to G_{n,2n}$ for any sufficiently large n. Examination of the details of the proof of the Bott periodicity theorem (Cf. J. Schwartz, Nonlinear Functional Analysis, Lemma 7.22, Corollaries 7.23, 7.24, Lemma 7.27, and Corollary 7.28) shows that there exists an imbedding of $G_{n,2n}$ into the loop space $\Omega(U(2n))$ which induces an isomorphism of $(G^X_{n,2n}) \to (\Omega(U(2n))^X)$ for low-dimensional spaces X. This imbedding may be described as follows. If p is an n-plane in E^{2n}, let E_p be the orthogonal projection onto p. Then $c_p(t) = \exp(2\pi i (I - E_p)\, t)$, $0 \leqq t \leqq \frac{1}{2}$; $c_p(t) = \exp(2\pi i t)\, I$, $\frac{1}{2} \leqq t \leqq 1$, defines a closed loop in $U(2n)$, whose end-points are both the identity map; $p \to c_p(t)$ is the desired imbedding.

For n large, $V_{2n,4n} = U(4n)/U(2n)$ is simply connected in all low dimensions, and is a bundle with fiber $U(2n)$ over $G_{2n,4n} = U(4n)/U(2n) \times U(2n)$. Thus, any map $\phi\colon Y \to U(2n)$ is homotopic in $V_{2n,4n}$ to a constant, and the resultant homotopy ϕ_S, projected by the natural map $V_{2n,4n} \to G_{2n,4n}$, defines an element of $((\Omega(G_{2n,4n}))^Y)$. The map $(U(2n)^Y) \to ((\Omega(G_{2n,4n}))^Y)$ defined in this way is an isomorphism of homotopy classes for low dimensional Y. (See below for a more familiar algebraic representation of this isomorphism.) Composing our two isomorphisms, and using the trivial identity $(\Omega(Z_1)^{Z_2}) \cong (Z_1^{S^1 \wedge Z_2})$, we obtain an explicit isomorphism $(G^X_{n,2n}) \to (U(2n)^{S^1 \wedge X}) \to (G_{2n,4n}^{S^2 \wedge X})$ for any X of dimension low compared with n. This composite isomorphism is the Bott isomorphism β of Theorem 2.

Finally, Theorem 2 identifies the isomorphism β as the product operation $a \to ag$. We omit the proof of this identification, noting however that it is used only occasionally in what follows, and that it may be derived from the explicit form of β as given above. The reader interested in additional details is referred to the cited papers of Bott and Atiyah-Hirzebruch Q.E.D.

Corollary (1): $K^{-s}((p_1, p_2), p_2) \cong Z$ if s is even, $\cong 0$ if s is odd.

Remark: Corollary 1 shows that the homotopy groups $\pi_s(BU)$ are periodic of order 2. The usual Bott Periodicity Theorem asserts that the homotopy groups $\pi_s(U)$ of the infinite unitary group are periodic of order 2, with $\pi_s(U) \cong Z$ if s is odd, $\cong 0$ if s is even. Actually, this assertion is equivalent to Corollary 1 as we now show.

We shall prove 3 statements, which together will readily establish our claim.

(1) For given s and for k sufficiently large,

$$\pi_{s-1}(U(k)) \cong \pi_{s-1}(U(k+1)) \cong \cdots \cong \pi_{s-1}(U).$$

(2) For given s and for k sufficiently large,

$$\pi_s(G_{k,2k}) \cong \pi_{s-1}(U(k)).$$

(3) For given s and for k sufficiently large,

$$\pi_s(G_{k,2k}) \cong \pi_s(G_{k+1,2k+1}) \cong \cdots \cong \pi_s(BU).$$

Proof of (1): $U(k)$ is a bundle over $U(k+1)/U(k) = S^{2k+1}$ with fiber $U(k)$. By the exact homotopy sequence of a bundle (see Chapter 6), we obtain

$$\cdots \longrightarrow \pi_s(S^{2k+1}) \xrightarrow{\partial} \pi_{s-1}(U(k)) \longrightarrow \pi_{s-1}(U(k+1)) \longrightarrow \pi_{s-1}(S^{2k+1}) \longrightarrow \cdots$$

But for $2k + 1 > s$, $\pi_s(S^{2k+1}) = 0$ and we find

$$0 \to \pi_{s-1}(U(k)) \to \pi_{s-1}(U(k+1)) \to 0$$

to be exact. That is, $\pi_{s-1}(U(k)) \cong \pi_{s-1}(U(k+1))$; moreover, the isomorphism is induced by the inclusion map $U(k) \to U(k+1)$. Using the same kind of compactness argument that has been given before, we conclude easily that the inclusion $U(k) \to U$ also induces an isomorphism $\pi_{s-1}(U(k)) \cong \pi_{s-1}(U)$. Thus, (1) is proved.

Proof of (2): We know that $V_{k,2k}$ is a bundle over $G_{k,2k}$ with fiber $U(k)$. Again by the exact bundle sequence,

$$\cdots \longrightarrow \pi_s(V_{k,2k}) \longrightarrow \pi_s(G_{k,2k}) \xrightarrow{\partial} \pi_{s-1}(U(k)) \longrightarrow \pi_{s-1}(V_{k,2k}) \longrightarrow \cdots$$

By Lemma 2′ of Chapter 6, $\pi_s(V_{k,2k}) = 0$ for $2k + 1 > s$. The conclusion (2) follows.

Proof of (3): The inclusion maps $U(k) \to U(k+1)$, $U(2k) \to U(2k+1)$ give rise to maps

$$\phi : V_{k,2k} = U(2k)/U(k) \to U(2k+1)/U(k+1) = V_{k+1,2k+1}$$

$$\psi : G_{k,2k} = U(2k)/U(k) \times U(k) \to U(2k+1)/U(k+1) \times U(k) = G_{k+1,2k+1}.$$

Clearly, ϕ is a bundle map which covers ψ. Therefore we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_s(V_{k,2k}) \arrow[r] \arrow[d] & \pi_s(G_{k,2k}) \arrow[r, "\partial"] \arrow[d, "\psi_*"] & \pi_{s-1}(U(k)) \arrow[r] \arrow[d, "i_*"] & \pi_{s-1}(V_{k,2k}) \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & \pi_s(V_{k+1,2k+1}) \arrow[r] & \pi_s(G_{k+1,2k+1}) \arrow[r, "\partial"] & \pi_{s-1}(U(k+1)) \arrow[r] & \pi_{s-1}(V_{k+1,2k+1}) \arrow[r] & {}
\end{tikzcd}$$

It follows from (1) and (2) that for $2k + 1 > s$, ∂ and i_* are isomorphisms. But then ψ_* is also an isomorphism. The fact that the inclusion $G_{k,2k} \to BU$ also induces an isomorphism $\pi_s(G_{k,2k}) \to \pi_s(BU)$ follows again by a conpactness argument.

Combining (1), (2) and (3), we see that $\pi_s(BU) \cong \pi_{s-1}(U)$. This establishes our original claim.

We now proceed to obtain some more information about the groups $K^{-n}(X, X_0)$ from the Bott Theorem.

Lemma (3): Let $f\colon (X, X_0) \to (X', X_0')$ be a mapping of pairs. Then the diagram

$$\begin{tikzcd}
K^{-n}(X', X_0') \arrow[r, "\beta"] \arrow[d, "f^*"] & K^{-n-2}(X', X_0') \arrow[d, "f^*"] \\
K^{-n}(X, X_0) \arrow[r, "\beta"] & K^{-n-2}(X, X_0)
\end{tikzcd}$$

is commutative.

Proof: This follows immediately from the diagram in the addendum to Theorem 1 and from the definition of β.

Lemma (4): If (X, X_0) is a pair, then the diagram

$$\begin{tikzcd}
K^{-n-1}(X_0) \arrow[r, "\delta"] \arrow[d, "\beta"] & K^{-n}(X, X_0) \arrow[d, "\beta"] \\
K^{-n-3}(X_0) \arrow[r, "\delta"] & K^{-n-2}(X, X_0)
\end{tikzcd}$$

is commutative.

Proof: This follows from Lemma 3 by the following rather standard argument. Let $X \cdot X_0$ be the space $(X \cup X_0 \times [0, 1])/X_0 \times \{1\}$, wherein we identify $X_0 \times \{0\}$ with X_0. Write CX_0 for the subspace $(X_0 \times [0, 1])/X_0 \times \{1\}$ of $X \cdot X_0$. Then CX_0 is easily seen to be deformable to a point. The natural identification map $X \cdot X_0 \to X/X_0 = (X \cdot X_0)/CX_0$ is therefore easily seen to induce an isomorphism $j\colon K^{-n}(X/X_0) \to K^{-n}(X \cdot X_0)$. On the other hand, $(X \cdot X_0)/X \times \{0\} = S^1 \wedge X_0$, so that the identification map $X \cdot X_0 \to S^1 \wedge X_0$ induces a homomorphism $k\colon K^{-n-1}(X_0) = K^{-n}(S^1 \wedge X_0) \to K^{-n}(X \cdot X_0)$. Comparison with the definition of the coboundary map δ (cf. Chapter 8, part A, paragraph following Corollary 2) will show that $\delta = j^{-1}k$. Thus $\beta\delta = \beta j^{-1}k = j^{-1}k\beta$ by Lemma 4. Q.E.D.

Theorem (3): Let (X, X_0) be a pair and let $X_0 \xrightarrow{i} X$ and $(X, \phi) \xrightarrow{j} (X, X_0)$ denote the natural inclusions. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K^{-2}(X) \arrow[r, "i^*"] & K^{-2}(X_0) \arrow[r, "\delta"] & K^{-1}(X, X_0) \arrow[r, "j^*"] & K^{-1}(X) \arrow[dr, "i^*"] & \\
& & K^0(X_0) \arrow[u, "\beta"] \arrow[ur, "\delta \circ \beta"'] & & & K^{-1}(X_0) \arrow[dl, "\delta"] \\
& & & K^0(X) \arrow[ul, "i^*"] & K^0(X, X_0) \arrow[l, "j^*"'] &
\end{tikzcd}$$

where the top row is the exact sequence of the pair (X, X_0). Then the hexagonal part of the diagram is exact.

Proof: Exactness at $K^{-1}(X)$, $K^{-1}(X_0)$, $K^0(X, X_0)$ and $K^0(X)$ being obvious, we have only to check exactness at $K^0(X_0)$ and $K^{-1}(X, X_0)$. Exactness at $K^{-1}(X, X_0)$ is also obvious, because β is an isomorphism. We check exactness at $K^0(X_0)$:

(a) By Lemma 4, $\delta\beta \circ i^* = \beta\delta \circ i^*$. But since $\delta \circ i^* = 0$, it follows that $\beta\delta \circ i^* = 0$, i.e. $\operatorname{Im} i^* \subseteq \operatorname{Ker} \delta\beta$.

(b) Suppose $\delta\beta(a) = 0$, $a \in K^0(X_0)$. Then $\beta(a) \in \operatorname{Ker} \delta$, and therefore $\beta(a) = i^*(a')$, $a' \in K^{-2}(X)$. Since β is onto, $a' = \beta(a'')$, $a'' \in K^0(X)$. Thus, $\beta(a) = i^*\beta(a'') = \beta i^*(a'')$ (by Lemma 3). Since β is $1-1$, $a = i^*(a'')$, i.e. $\operatorname{Ker} \delta\beta \subseteq \operatorname{Im} i^*$.

Definition (1): If $n \geqq 0$ is any integer and (X, X_0) a pair, we put

$$K^n(X, X_0) = K^0(X, X_0) \quad \text{if } n \text{ is even,}$$

$$K^n(X, X_0) = K^{-1}(X, X_0) \quad \text{if } n \text{ is odd,}$$

$$K^*(X, X_0) = K^0(X, X_0) \oplus K^1(X, X_0).$$

The following theorem is an immediate consequence of the above definitions and results.

Theorem (4): $K^*(X)$ is a Z_2-graded anticommutative ring, i.e. $K^0(X) \cdot K^0(X) \subseteq K^0(X)$, $K^0(X) \cdot K^1(X) \subseteq K^1(X)$, $K^1(X) \cdot K^1(X) \subseteq K^0(X)$. Moreover $K^*(X, X_0)$ is a Z_2-graded $K^*(X)$-module. Finally, we have an exact hexagon

$$\begin{tikzcd}
 & K^1(X, X_0) \arrow[r] & K^1(X) \arrow[dr] & \\
K^0(X_0) \arrow[ur, "\delta"] & & & K^1(X_0) \arrow[dl, "\delta"] \\
 & K^0(X) \arrow[ul] & K^0(X, X_0) \arrow[l] &
\end{tikzcd}$$

3. The Chern Character

This section will be devoted to constructing a certain ring homomorphism between $K(X)$ and the ordinary (e.g. singular) cohomology ring of X with rational coefficients. This homomorphism, the so-called Chern character, plays a very important role in the study and calculation of the ring $K(X)$. We shall, for example, perform such a calculation in the next chapter, in the case that X is projective space.

The Chern character is defined using the Chern classes. Thus, to begin with, we establish a fundamental theorem on Chern classes.

Theorem (5) (The Whitney Product Theorem): Let ξ_1, ξ_2 be complex bundles on complexes X_1, X_2. The following relation between Chern classes holds:

$$c(\xi_1 \textcircled{\#} \xi_2) = c(\xi_1) \times c(\xi_2); \quad \text{(see Chapter 8).}$$

If $X_1 = X_2 = X$, we may apply the diagonal map to this relation and obtain

$$c(\xi_1 \oplus \xi_2) = c(\xi_1) \cup c(\xi_2).$$

Proof: Let us denote by c_j, $c_k^{(1)}$, $c_l^{(2)}$ the Chern classes of $G_{n_1+n_2,\infty}$, $G_{n_1,\infty}$, $G_{n_2,\infty}$ respectively. We have a mapping

$$\tilde{N}: G_{n_1,\infty} \times G_{n_2,\infty} \to G_{n_1+n_2,\infty}$$

defined as follows: $G_{n_i,\infty}$ is regarded as the set of n_i-planes in E_i^∞, an infinite dimensional Euclidean space and $G_{n_1+n_2,\infty}$ is regarded as the set of (n_1+n_2)-planes in $E^\infty = E_1^\infty \oplus E_2^\infty$. $\tilde{N}$ is given by $\tilde{N}(\pi_1, \pi_2) = \{\pi_1, \pi_2\}$, i.e. the plane generated by π_1 and π_2. We know that there exist classifying maps of ξ_1, ξ_2, i.e. maps

$$\phi_1 : X_1 \to G_{n_1,\infty}, \quad \phi_2 : X_2 \to G_{n_2,\infty}$$

such that $\phi_i^\dagger(\alpha_{n_i}) = \xi_i$ (α_{ni}) is the universal bundle over $G_{n_i,\infty}$. Evidently, the bundle $\xi_1 \textcircled{\#} \xi_2$ is induced by

$$\tilde{N} \circ (\phi_1 \times \phi_2) : X_1 \times X_2 \to G_{n_1+n_2,\infty}.$$

Since the cohomology rings $H^*(G_{n_i,\infty}; Z)$ are torsion-free (see Chapter 3), it follows from Künneth's Theorem that

$$\tilde{N}^* c_j = Q_j(c_k^{(1)}, c_l^{(2)})$$

where Q_j is a homogeneous polynomial of degree j. Now,

$$c_j(\xi_1 \textcircled{\#} \xi_2) = [\tilde{N} \circ (\phi_1 \times \phi_2)]^* c_j = (\phi_1 \times \phi_2)^* \circ \tilde{N}^*(c_j)$$

$$= (\phi_1 \times \phi_2)^* Q_j(c_k^{(1)}, c_l^{(2)}).$$

Therefore,

$$c_j(\xi_1 \textcircled{\#} \xi_2) = Q_j(c_k^{(1)}(\xi_1), c_l^{(2)}(\xi_2)).$$

It remains to determine the polynomial Q_j. To do this, let $X_i = P_n^{(1)} \times \cdots \times P_n^{(n_i)}$ and let $X_{12} = X_1 \times X_2$. Here, n is chosen so large that $n(n_1 + n_2)$

$\geq j$. In Chapter 3 (cf. the proof of Theorem 3 of that chapter), we have defined maps

$$N_l : P_n^{(1)} \times \cdots \times P_n^{(n_l)} \to G_{n_l,\infty}$$

$$N : X_{12} = P_n^{(1)} \times \cdots \times P_n^{(n_1+n_2)} \to G_{n_1+n_2,\infty}.$$

Recall that N_l is defined as follows: An element $v_j \in P_n^{(j)}$ determines a one-dimensional subspace of a Euclidean space E_j^n. Putting $E^{n_l \cdot n} = E_1^n \oplus \cdots \oplus E_{n_l}^n$, we define $N_l((v_1, v_2, \ldots, v_{n_l}))$ as the plane π in $E^{nn_l} \subseteq E^\infty$ spanned by the one-dimensional spaces $v_1, \ldots, v_{n_l}$. Clearly,

$$N = \tilde{N} \circ (N_1 \times N_2).$$

Therefore, using Theorem 3 of Chapter 3, we find that if σ_j (resp. $\sigma_k^{(1)}$, resp. $\sigma_l^{(2)}$) is the j-th (resp. k-th, resp. l-th) elementary symmetric polynomial on $n_1 + n_2$ (resp. n_1, resp. n_2) generators, then

$$\sigma_j (t_1, \ldots, t_{n_1+n_2}) = Q_j (\sigma_k^{(1)} (t_1, \ldots, t_{n_1}), \sigma_l^{(2)} (t_{n_1+1}, \ldots, t_{n_1+n_2})).$$

But we also have

$$\sum_{j=1}^{n_1+n_2} z^j \sigma_j (t_1, \ldots, t_{n_1+n_2}) = \prod_{j=1}^{n_1+n_2} (z - t_j) = \prod_{j=1}^{n_1} (z - t_j) \cdot \prod_{j=n_1+n1}^{n_1+n_2} (z - t_j)$$

$$= \left(\sum_{k=1}^{n_1} z^k \sigma_k^{(1)} (t_1, \ldots, t_{n_1}) \right) \cdot \left(\sum_{l=1}^{n_2} z^l \sigma_l^{(2)} (t_{n_1+1}, \ldots, t_{n_1+n_2}) \right).$$

Thus,

$$Q_j (\sigma_k^{(1)}, \sigma_l^{(2)}) = \sigma_j = \sigma_j^{(1)} + \sigma_{j-1}^{(1)} \cdot \sigma_1^{(2)} + \cdots + \sigma_1^{(1)} \cdot \sigma_{j-1}^{(2)} + \sigma_j^{(2)}.$$

Since $c(\xi) = 1 + c_1(\xi) + c_2(\xi) + \cdots$, Theorem 5 follows. Q.E.D.

Our next aim is to show the existence of a very important homomorphism $K(X) \to H^*(X, \mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers, called the *Chern character* of a virtual bundle. Before defining this homomorphism, however, we prepare some technical lemmas concerning unitary bundles, which we will need in order to establish that the map to be defined is in fact a homomorphism. The object of these lemmas is to show that any algebraic relation between Chern classes that holds for all Whitney sums of U_1-bundles holds for all U_n-bundles generally (cf. Lemma 5 below). To establish this basic fact, we show that, given any U_n-bundle ξ over a base X, there exists a space Y and a mapping $\varrho : Y \to X$ such that

a) $\varrho^\dagger(\xi)$ is reducible to the torus subgroup T_n of U_n, i.e., is a direct sum of U_1-bundles. (This is the point of Theorems 6 and 7 below, and of their corollaries.)

b) $\varrho^*: H^*(X) \to H^*(Y)$ is a 1–1 mapping (this is the point of Theorems 8 and 9 below).

The space Y is constructed from the space E of the principal bundle ξ by an elementary general procedure, which it is the aim of the next few paragraphs to describe. Let $\alpha = (E, X, G, G, p)$ be a principal G-bundle, G a topological group. By definition, G acts on itself by *left* translations. We observe that there is a natural action of G on the total space E. In fact, E may be regarded (see Chapter 4) as the set of triples (x, g, U) with $x \in X$, $g \in G$, and U an open neighborhood of x, divided out by the equivalence relation $(x, g, U) \sim (x, g', V) \Leftrightarrow g' = g_{uv}(x) \cdot g$ (where g_{uv} is the coordinate transformation). Then G acts on E on the *right* by

$$[(x, g, U)] \cdot g_0 = [(x, g \cdot g_0, U)]$$

where [ ] means equivalence class. One checks easily that this is a well-defined action.

Suppose now Y is a space on which G acts on the left. We can define an action of G on the Cartesian product $E \times Y$ on the right,

$$\phi: (E \times Y) \times G \to E \times Y$$

by setting

$$\phi((e, y), g) = (eg, g^{-1} y).$$

Therefore we may also form identification space, denoted by $(E \times Y)/G$ and endowed with the identification topology. We denote the equivalence class of (e, y) in $(E \times Y)/G$ by $\{(e, y)\}$. If we define the map

$$p': (E \times Y)/G \to X$$

by setting

$$p'(\{(e, y)\}) = p(e)$$

then we get a bundle $((E \times Y)/G, X, Y, G, p')$.

Let $\beta = (E, X, J, J, p)$ be a principal J-bundle where J is a closed subgroup of a Lie group G. Let $i: J \to G$ denote the injection map. One defines a left action of J on G in the obvious way. Using the construction of the preceding paragraph we get a bundle $((E \times G)/J, X, G, J, p')$. It is easy to see that this bundle initially defined as a J-bundle, can also be viewed as a principal G-bundle $((E \times G)/J, X, G, G, p')$. (Just observe that G acts on $(E \times G)/J$ on the right by acting on the second factor.) We will denote this latter bundle by $i_+(\beta)$. Thus $i_+(\beta)$ is a G-bundle which, from its definition, is clearly reducible to a bundle with respect to the subgroup $J \leq G$.

Now let $\xi = (E, X, G, G, p)$ be a principal G-bundle, G a Lie group. Let J again denote a closed subgroup of G, $i: J \to G$ the injection. Since J acts on E on the right, we can form E/J. One checks that $\beta = (E, E/J, J, J, \sigma)$ is a principal J-bundle over E/J where the projection σ is defined by

$$\sigma(e) = \{e\}.$$

In the preceding paragraph, we have defined the G-bundle $i_+(\beta)$. Let $\{e\} \in E/J$ and let e, e' be 2 representatives of $\{e\}$. Then e and e' are equivalent modulo J and therefore, a fortiori, equivalent modulo G. Hence the map

$$\varrho: E/J \to X$$

given by

$$\varrho(\{e\}) = p(e)$$

is well-defined and the diagram

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[dr, "p"'] & E/J \arrow[d, "\varrho"] \\ & X \end{tikzcd}$$

is commutative. It is clear that the collection $(E/J, X, G/J, G, \varrho)$ is a bundle. On the other hand, the map ϱ induces a principal G-bundle $\varrho^\dagger(\xi)$ over the base E/J. The total space of $\varrho^\dagger(\xi)$, as we have seen in Chapter 4, consists of all pairs $(\{e\}, e')$ (with $\{e\} \in E/J$; $e, e' \in E$) such that $\varrho(\{e\}) = p(e)$. Let $\bar{E}$ denote this total space and let $\bar{p}$ be the corresponding projection

$$\bar{p}\,((\{e\}, e')) = \{e\}.$$

The following theorem now identifies the induced bundle $\varrho^\dagger(\xi)$ with the reducible bundle $i_+(\beta)$.

Theorem (6): $\varrho^\dagger(\xi)$ is equivalent to $i_+(\beta)$.

Proof: We shall construct a map j' such that the diagram

$$\begin{tikzcd} (E \times G)/J \arrow[r, dotted, "j'"] \arrow[dr, "p'"'] & \bar{E} \subseteq E/J \times E \arrow[d, "\bar{p}"] & E \arrow[d, "p"] \\ & E/J \arrow[r, "\varrho"] & X \end{tikzcd}$$

becomes commutative, this is done as follows. We first define a map

$$j: E \times G \to (E/J) \times E$$

by setting
$$j((e, g)) = (\sigma(e), e \cdot g).$$

Using the facts that $\varrho\sigma = p$, $p(e \cdot h) = p(e)$ for any $h \in G$, and $\sigma(e \cdot k) = \sigma(e)$ for any $k \in J$, we see easily that j induces a map
$$j' : (E \times G)/J \to \bar{E}.$$

It is readily seen that j' is $1-1$ and onto.
It is also clear that j' covers the identity map of E/J into itself, i.e. $p' = \bar{p} \circ j'$. This proves the theorem.

Taking G to be the unitary group U_n and J a "maximal torus" T_n of U_n (e.g. the (closed) subgroup of all diagonal matrices in U_n), we get the following theorem.

Theorem (7): Let $\xi = (E, X, U_n, U_n, p_n)$ be a principal U_n-bundle. Let T_n be the subgroup of diagonal matrices of U_n, $i_n : T_n \to U_n$ the inclusion map. Let $\beta = (E, E/T_n, T_n, T_n, \sigma_n)$, $(i_n)_+(\beta) = ((E \times U_n)/T_n, E/T_n, U_n, U_n, p_n')$ and $\varrho_n^\dagger(\xi) = (\bar{E}, E/T_n, U_n, U_n, \bar{p}_n)$ be as above (where $\sigma_n, p_n \bar{p}_n$ have the same meaning as $\sigma, p', \bar{p}$ above). Then $(i_n)_+(\beta) \cong \varrho_n^\dagger(\xi)$. Moreover, E/T_n is the total space of a U_n-bundle over X with fiber U_n/T_n and projection ϱ_n; the diagram

$$\begin{tikzcd}
E \arrow[r, "\sigma"] \arrow[dr, "p_n"'] & E/T_n \arrow[d, "\varrho_n"] \\
 & X
\end{tikzcd}$$

is commutative.

Recalling the definition of bundle reducibility (see Chapter 4, Definition 3), we have

Corollary (1): The U_n-bundle $\varrho_n^\dagger(\xi)$ is reducible to T_n.

Since $T_n = U_1 \times U_1 \times \cdots \times U_1$ (n factors), we have

Corollary (2): $\varrho_n^\dagger(\xi)$ is a (Whitney) sum of n U_1-bundles.

This completes the first step of the argument outlined in a), b) above. Next we specialize still further, taking E in Theorem 7 to be a classifying bundle $V_{n,\infty}$, and studying the relation between $H^*(G_{n,\infty})$ and $H^*(V_{n,\infty}/T_n)$.

Let $\alpha_m = (V_{m,\infty}, G_{m,\infty}, U_m, U_m, \varrho_m)$ be the universal U_m-bundle (see Chapter 6, Definition 5). We set $\alpha_1 = \alpha$. Recall that $G_{1,\infty} = P_\infty$, infinite dimensional (complex) projective space, while $V_{1,\infty} = S^\infty$, the infinite dimensional (complex) sphere. We define

$$\alpha^n = (S^\infty \times \cdots \times S^\infty, P_\infty \times \cdots \times P_\infty,$$
$$U_1 \times \cdots \times U_1, U_1 \times \cdots \times U_1, p_1 \times \cdots \times p_1),$$

where each Cartesian product has n factors. Since the space $S^\infty \times \cdots \times S^\infty$ is homotopically trivial in all dimensions and since $U_1 \times \cdots \times U_1 = T_n$, it follows from Chapter 6, Theorem 3 that α^n can be considered as a universal T_n-bundle. Thus, the space $P_\infty \times \cdots \times P_\infty$ is a universal classifying space for T_n-bundles. Instead of the cumbersome notation above, we write

$$\alpha^n = (E_{T_n}, X_{T_n}, T_n, T_n, p_{T_n}).$$

Recall the result proved in Chapter 2:

$$H^*(X_{T_n}; Z) = H^*(P_\infty \times \cdots \times P_\infty; Z) \cong Z(t_1, \ldots, t_n),$$

the polynomial ring in n indeterminates.

We now apply Theorem 7 to the U_n-bundle α_n, setting

$$\beta = (V_{n,\infty}, V_{n,\infty}/T_n, T_n, T_n, \sigma_n)$$

in Theorem 7. It follows that there exists a communitative diagram

$$\begin{tikzcd}
V_{n,\infty} \arrow[r, "\sigma_n"] \arrow[dr, "p_n"'] & V_{n,\infty}/T_n \arrow[d, "\varrho_n"] \\
& G_{n,\infty}
\end{tikzcd}$$

and that $(i_n)_+(\beta) \cong \varrho_n^\dagger(\alpha_n)$. Since $V_{n,\infty}$ is homotopically trivial in all dimensions, we can consider β as a universal T_n-bundle with base $V_{n,\infty}/T_n$. In what follows, we write β_n for this universal bundle β. By the universality property, the spaces $V_{n,\infty}/T_n$ and X_{T_n} are homotopically equivalent and we get

$$H^*(V_{n,\infty}/T_n; Z) \cong H^*(X_{T_n}) \cong Z[t_1, \ldots, t_n].$$

Again by Theorem 7, $V_{n,\infty}/T_n$ is the total space of a U_n-bundle over $G_{n,\infty}$ with fiber U_n/T_n. We then have the following theorem, which we state without proof. See Husseini—*Topics in the Algebraic Topology of the Classical Groups,* University of Wisconsin lecture notes, Chapter 2, page 14.

Theorem (8): The sequence of graded rings

$$1 \longrightarrow H^*(G_{n,\infty}) \xrightarrow{\varrho_n^*} H^*(V_{n,\infty}/T_n) \xrightarrow{j_n^*} H^*(U_n/T_n) \longrightarrow 1$$

is exact. Here, $j_n\colon U_n/T_n \to V_{n,\infty}/T_n$ is the inclusion of the fiber into the total space.

Now let $\xi = (E, X, U_n, U_n, p)$ be any principal U_n-bundle. We have then bundles $\beta = (E, E/T_n, T_n, T_n, \sigma_n)$ and $(E/T_n, X, U_n/T_n, U_n, \varrho)$. By the Classification Theorem, there exists a map

$$f\colon X \to G_{n,\infty}$$

such that $f^\dagger(\alpha_n) = \xi$. If we let $\bar{f}$ be the map

$$\bar{f}\colon E/T_n \to V_{n,\infty}/T_n$$

induced by the bundle map $\tilde{f}\colon E \to V_{n,\infty}$ covering the map f, then it is easily seen from the above definitions that $\bar{f}^\dagger(\beta_n) = \beta$. The following diagram is then seen to be commutative, where j and j_n are inclusions of fibers in total spaces:

$$\begin{tikzcd}
 & & & U_n/T_n \arrow[dl, "j"'] \arrow[dr, "j_n"] & & & \\
E \arrow[rr, "\sigma"] \arrow[drr, "p"'] & & E/T_n \arrow[rr, "\bar{f}"] \arrow[d, "\varrho"] & & V_{n,\infty}/T_n \arrow[d, "\varrho_n"] & & V_{n,\infty} \arrow[ll, "\sigma_n"'] \arrow[dll, "p_n"] \\
 & & X \arrow[rr, "f"] & & G_{n,\infty} & &
\end{tikzcd} \qquad (*)$$

We look at the top triangle of diagram (*) and take its induced cohomology diagram:

$$\begin{tikzcd}
 & H^*(U_n/T_n) & \\
H^*(E/T_n) \arrow[ur, "j^*"] & & H^*(V_{n,\infty}/T_n) \arrow[ul, "j_n^*"'] \arrow[ll, "\bar{f}^*"']
\end{tikzcd}$$

Theorem 8 tells us that j_n^* is surjective. It follows immediately that j^* is also surjective. A nontrivial argument (see Husseini, loc. cit.) then establishes the following result.

Theorem (9): The map $\varrho^*\colon H^*(X) \to H^*(E/T_n)$ is injective.

After these preliminaries we are prepared to give the definition and derive the properties of the Chern character:

Let $c_1, c_2, \ldots, c_n$ be the Chern classes of $G_{n,\infty}$, i.e. the elements of $H^*(G_{n,\infty})$ corresponding to the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_n$ on n indeterminates $t_1, t_2, \ldots, t_n$. The Chern classes of ξ (see diagram (*)) are $c_i(\xi) = f^*(c_i), i = 1, \ldots, n$. The total Chern class of ξ is $c(\xi) = 1 + c_1(\xi) + \cdots + c_n(\xi)$.

The function $G(t_1, \ldots, t_n) = \sum_{i=1}^{n} e^{t_i}$ is a power series in the indeterminates t_i, with rational coefficients and symmetrical in these indeterminates. Thus we may write $G(t_1, \ldots, t_n) = R(\sigma_1, \ldots, \sigma_n)$, where R is a power series in the elementary symmetric functions $\sigma_1, \ldots, \sigma_n$. We define the Chern character of ξ by $ch(\xi) = R(c_1(\xi), \ldots, c_n(\xi))$. Since any sufficiently high power of a

Chern class (for a bundle over a finite complex X as base) vanishes, the expression $R\,(c_1(\xi), \ldots, c_n(\xi))$ reduces to a finite sum, and $ch\,(\xi)$ is a well-defined element of $H^*(X) * \mathbb{Q} = H^*(X; \mathbb{Q})$.

The fundamental fact that $ch\,(\xi)$ induces a homomorphism of the ring $K(X)$ into $H^*(X, \mathbb{Q})$ may be deduced as follows. We showed above that

$$H^*(V_{n,\infty}/T_n) \cong \mathbb{Z}\,[t_1, \ldots, t_n].$$

Let us regard the indeterminates t_i as being the ring generators of $H^*(V_{n,\infty}/T_n)$; $t_i \in H^2(V_{n,\infty}/T_n)$. We set

$$\gamma_i = f^*(t_i), \quad (i \leqq i \leqq n).$$

Then $\gamma_i \in H^2(E/T_n)$ and are called the *roots* of the principal U_n-bundle ξ. Now, setting $c_0(\xi) = 1$, we have

$$\varrho^*c\,(\xi) = \varrho^*\left(\sum_{i=0}^{n} c_i(\xi)\right) = \varrho^* f^*\left(\sum_{i=0}^{n} c_i\right)$$

By the commutativity of the square in (*), we deduce

$$\varrho^*c\,(\xi) = f^*\varrho_n^*\left(\sum_{i=0}^{n} c_i\right).$$

By Theorem 9, $\varrho_n^* : H^*(G_{n,\infty}) \to H^*(V_{n,\infty}/T_n)$ is a $1-1$ map of the subring of $\mathbb{Z}\,[t_1, \ldots, t_n]$ consisting of all symmetric polynomials into the full ring $\mathbb{Z}\,[t_1, \ldots, t_n]$. Thus,

$$\varrho_n^*\left(\sum_{i=0}^{n} c_i\right) = 1 + \sum_{i=1}^{n} \sigma_i.$$

But by the definition of elementary symmetric polynomials,

$$1 + \sum_{i=1}^{n} \sigma_i = \prod_{i=1}^{n} (1 + t_i).$$

Therefore,

$$\varrho^*c\,(\xi) = f^*\left(\prod_{i=1}^{n} (1 + t_i)\right) = \prod_{i=1}^{n} (1 + \gamma_i).$$

Now, $\sum_{i=1}^{n} e^{\gamma_i}$ is an element of the ring $H^*(E/T_n : \mathbb{Q})$. (Note again that at this point, we must assume that for some integer m_0, the cohomology groups $H^m(E/T_n)$ are 0 for $m \geqq m_0$; of course, this follows automatically if E/T_n is a finite dimensional complex. If this condition is not satisfied then the power series $\sum_{i=1}^{n} e^{\gamma_i}$ may not degenerate into a finite sum and will then represent an element in $H^{**}(E/T_n; \mathbb{Q}) = \prod_{j \geq 0} H^j(E/T_n; \mathbb{Q})$, the unrestricted direct product.) Moreover, $\sum_{i=1}^{n} e^{\gamma_i}$ is a symmetric polynomial in the roots γ_i, and is therefore a polynomial in the elementary symmetric functions on the roots γ_i. In other words,

$$\sum_{i=1}^{n} e^{\gamma_i} = R\,(f^*(\sigma_1), \ldots, f^*(\sigma_n)),$$

where R is as above.

Notation: If z is an indeterminate, we define the *Chern polynomial* by

$$c\,(\xi, z) = 1 + c_1(\xi)\,z + \cdots + c_n(\xi)\,z^n.$$

Recalling the equation $\varrho^* c\,(\xi) = \prod_{i=1}^{n} (1 + \gamma_i)$, we will allow ourselves to write formally

$$c\,(\xi, z) = \prod_{i=1}^{n} (1 + \gamma_i z).$$

If $\xi \cong \eta$, then evidently $ch\,(\xi) = ch\,(\eta)$. Thus, the map

$$(\xi) \to ch\,(\xi)$$

is a well defined map. (Recall that (ξ) is the equivalence class of ξ.) Let K_F and K_F^+ be as in the paragraph following the proof of Lemma 2, Chapter 8. We may now extend by linearity to get a homomorphism

$$ch : K_F(X) \to H^*(X; \mathbb{Q}).$$

We shall now investigate the behavior under Whitney addition of ch on $K_F^+(X)$, and also its behavior for tensor products. We have

Theorem (10): If ξ is a principal U_n-bundle on X and η is a principal U_m-bundle on X, then

(a) $$ch\,(\xi \oplus \eta) = ch\,(\xi) + ch\,(\eta).$$

(b) $$ch\,(\xi \otimes \eta) = ch\,(\xi) \cdot ch\,(\eta).$$

Proof:

(a) It follows from the Whitney Product Theorem that

$$c\,(\xi \oplus \eta, z) = c\,(\xi, z) \cdot c\,(\eta, z).$$

But this means that if $\{\gamma_1, \ldots, \gamma_n\}$ are the roots of ξ and $\{\gamma'_1, \ldots, \gamma'_m\}$ are the roots of η, then $\{\gamma_1, \ldots, \gamma_n\} \cup \{\gamma'_1, \ldots, \gamma'_m\}$ are the roots of $\xi \oplus \eta$.

Hence,

$$\varrho^* ch(\xi \oplus \eta) = \sum_{i=1}^{n} e^{\gamma_i} + \sum_{j=1}^{m} e^{\gamma'_j} = \varrho^* ch(\xi) + \varrho^* ch(\eta),$$

from which (a) follows immediately.

(b) Since

$$\left(\sum_{i=1}^{n} e^{\gamma_i}\right) \cdot \left(\sum_{j=1}^{m} e^{\gamma'_j}\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} e^{\gamma_i + \gamma'_j},$$

it is sufficient, in order to establish (b), to show that the set of roots of $\xi \otimes \eta$ consists of $\{\gamma_i + \gamma'_j;\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\}$. We shall now prove this:

Case 1: ξ and η are U_1-bundles.

Proof: We have $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$. The proof of this fact is left as an exercise. Since $c(\xi \otimes \eta, z) = 1 + c_1(\xi \otimes \eta) \cdot z$, $c(\xi, z) = 1 + c_1(\xi) z$, $c(\eta, z) = 1 + c_1(\eta) \cdot z$, the result follows immediately from the definition of the quantities γ_i.

Case 2: $\xi = \xi_1 \oplus \cdots \oplus \xi_n$, $\eta = \eta_1 \oplus \cdots \oplus \eta_m$, where the ξ_i's and η_j's are U_1-bundles.

Proof: We have

$$\xi \otimes \eta = \sum_{i,j} \xi_i \otimes \eta_j.$$

Hence, by the Whitney Product Theorem,

$$c(\xi, z) = \prod_{i=1}^{n} (1 + c_1(\xi_i) z)$$

$$c(\eta, z) = \prod_{j=1}^{m} (1 + c_1(\eta_j) z)$$

$$c(\xi \otimes \eta, z) = \prod_{i,j} (1 + c_1(\xi_i \otimes \eta_j) z) = \prod_{i,j} [1 + \{c_1(\xi_1) + c_1(\eta_j)\} z].$$

Hence, the assertion follows in Case 2.

Case 3: ξ and η are arbitrary.

Proof: This will follow from

Lemma (5): Any algebraic relation between Chern classes that holds for all sums of U_1-bundles holds generally.

Lemma 5 will in turn be a consequence of

Lemma (6): Let $\xi_1, \ldots, \xi_r$ be principal unitary bundles over X (not necessarily of the same dimension). Then there exists a space Y and a map

$$\phi : Y \to X$$

such that the induced map

$$\phi^*: H^*(X) \to H^*(Y)$$

is injective and such that each of the induced bundles $\phi^\dagger(\xi_i)$ $(1 \leqq i \leqq r)$ over Y is a sum of U_1-bundles.

Proof: The proof is divided into 2 parts.

(1) $r = 1$: Let $\xi_1 = (E_1, X, U_{m_1}, U_{m_1}, p_1)$. We define Y to be the space E_1/T_{m_1} and ϕ to be the map ϱ (see Theorem 7). The fact that ϕ^* is injective follows from Theorem 9 and the fact that $\phi^\dagger(\xi_1)$ is a sum of U_1-bundles follows from Corollary 2 of Theorem 7.

(2) $r > 1$: Using part (1), we find a space Y_1 and a map

$$\phi_1: Y_1 \to X$$

such that ϕ_1^* is injective and such that $\phi_1^\dagger(\xi_1)$ is a sum of U_1-bundles. Again by part (1), we find a space Y_2 and a map

$$\phi_2: Y_2 \to Y_1$$

such that ϕ_2^* is injective and such that $\phi_2^\dagger\phi_1^\dagger(\xi_2)$ is a sum of U_1-bundles. Observe also that $\phi_2^\dagger\phi_1^\dagger(\xi_1)$ is still a sum of U_1-bundles. By defining

$$\psi: Y_2 \to X$$

as $\psi = \phi_1 \circ \phi_2$, we see that

$$\psi^* = \phi_2^* \circ \phi_1^*: H^*(X) \to H^*(Y_2)$$

is injective and that $\psi^\dagger(\xi_1)$ and $\psi^\dagger(\xi_2)$ are sums of U_1-bundles. This proves the lemma in case $r = 2$. The general case is done inductively using the above procedure.

Thus, Lemma 9 and with it Lemma 8 and Theorem 10 are proved. Q.E.D.

We now note an important corollary of Theorem 10.

Corollary: Let K_F and K_F^+ be as in the paragraph following the proof of Lemma 2, Chapter VIII. $ch|K_F^+(X) = 0$. Thus, $ch: K_F(X) \to H^*(X; Q)$ induces a hommorphism

$$ch: K(X) \to H^*(X; Q),$$

which is a ring homomorphism.

In the situation of Theorem 10, we dealt with bundles ξ, η over the same space X. More generally, we may suppose ξ is a bundle over X and η is a bundle over another space Y. Then, statements analogous to those of Theorem 10 and its corollary hold. For example, we have a commutative diagram

$$\begin{tikzcd}
K(X) \otimes K(Y) \arrow[r, "\mu"] \arrow[d, "ch \otimes ch"] & K(X \times Y) \arrow[d, "ch"] \\
H^*(X; \mathbb{Q}) \otimes H^*(Y; \mathbb{Q}) \arrow[r] & H^*(X \times Y; \mathbb{Q})
\end{tikzcd}$$

The map on the bottom line is just the Künneth isomorphism.

If we look at $ch|\tilde{K}(X)$, we obtain the following commutative diagram:

$$\begin{tikzcd}
\tilde{K}(X) \otimes \tilde{K}(Y) \arrow[r, "\tilde{\mu}"] \arrow[d, "ch \otimes ch"] & \tilde{K}(X \wedge Y) \arrow[d, "ch"] \\
\tilde{H}^*(X; \mathbb{Q}) \otimes \tilde{H}^*(Y; \mathbb{Q}) \arrow[r] & \tilde{H}^*(X \wedge Y)
\end{tikzcd}$$

Here $\tilde{H}^*(X; \mathbb{Q})$ means reduced cohomology, i.e. $\tilde{H}^*(X; \mathbb{Q}) = H^*(X, x_0; \mathbb{Q})$. The map in the bottom line is again an isomorphism by the Künneth tensor formula.

We introduce additional notation as follows: First, let $H_G^*(X)$ denote the ordinary cohomology ring of X with coefficients in G. We set

$$H_G^e(X) = \sum_{i \geqq 0} H^{2i}(X; G), \quad H_G^o(X) = \sum_{i \geqq 0} H^{2i+1}(X; G).$$

Then evidently,

$$H_G^*(X) = H_G^e(X) \oplus H_G^o(X).$$

We have the following

Lemma (7): *ch* maps $K(X)$ into $H_{\mathbb{Q}}^e(X) \subseteq H_{\mathbb{Q}}^*(X)$. Moreover, if X is connected, *ch* maps $\tilde{K}(X)$ into $\tilde{H}_{\mathbb{Q}}^e = \sum_{i \geqq 1} H^{2i}(X; \mathbb{Q})$.

Proof: The proof follows easily from the fact that the roots of a bundle are 2-dimensional cohomology classes.

We shall now define a map

$$ch: K^{-n}(X, X_0) \to H^*(X, X_0; \mathbb{Q}).$$

To do this, we begin by defining the suspension isomorphism

$$\sigma^n: \tilde{H}_{\mathbb{Q}}^*(W) \to \tilde{H}_{\mathbb{Q}}^*(S^n(W)).$$

We put

$$\sigma^n(x) = x \otimes s^n,$$

where s^n is the canonical generator of $H^n(S^n; Z)$ and where W is any complex. The fact that σ^n is an isomorphism follows from the Künneth tensor formula. Now let $a \in K^{-n}(X, X_0) = \tilde K(S^n(X/X_0))$. Then, $ch(a) \in \tilde H^*_{\mathbb{Q}}(S^n(X/X_0))$ by Lemma 7 and $(\sigma^n)^{-1}\, ch(a) \in \tilde H^*_{\mathbb{Q}}(X/X_0) \cong H^*_{\mathbb{Q}}(X, X_0)$. The map

$$a \to (\sigma^n)^{-1}\, ch(a)$$

is then the desired map (by abuse of notation)

$$ch : K^{-n}(X, X_0) \to H^*_{\mathbb{Q}}(X, X_0).$$

To stress the analogy with Theorem 2, we shall set

$$s^2 = \hat g, \quad \sigma^2 = \hat\beta.$$

Thus, we have an isomorphism

$$\hat\beta : \tilde H^*_{\mathbb{Q}}(X/X_0) \to \tilde H^*_{\mathbb{Q}}(S^2(X/X_0))$$

given by

$$\beta(a) = \hat a \otimes \hat g.$$

Lemma (8): The diagram

$$\begin{tikzcd} \tilde K(X) \arrow[rr, "\beta"] \arrow[dr, "ch"'] & & \tilde K(S^2(X)) \arrow[dl, "ch"] \\ & \tilde H^*_{\mathbb{Q}}(X) & \end{tikzcd}$$

is commutative.

Proof: Let g, β be as in Theorem 2, and let $\hat g$, $\hat\beta$ be as above. The result would follow immediately from the second commutative diagram in the discussion after the corollary to Theorem 10 if we knew that $ch(g) = \hat g$. But $g = [\eta - 1]$ where η is the canonical line bundle over $P_1^{\mathbb{C}}$, i.e. η is induced by the inclusion map $P_1^{\mathbb{C}} \xrightarrow{i_1} P_\infty^{\mathbb{C}}$. Thus,

$$c_1(\eta) = i_1^*(c_1),$$

i.e. $c_1(\eta)$ is a generator of $H^2(P_1^{\mathbb{C}})$. Now,

$$ch(\eta) = e^{c_1(\eta)} = 1 + c_1(\eta) + \tfrac{1}{2}c_1^2(\eta) + \cdots$$

But $c_1^2(\eta) = c_1^3(\eta) = \cdots = 0$ and therefore

$$ch(g) = ch(\eta) - ch(1) = 1 + c_1(\eta) - 1 = c_1(\eta).$$

Q.E.D.

If now $a \in K^n(X, X_0)$, we define

$$ch(a) = ch(\beta^{-m} a)$$

where m is so chosen that $n - 2m \leqq 0$. (Note again the abuse of notation.) Lemma 8 shows that this definition is legitimate. We have thus defined

$$ch : K^n(X, X_0) \to H^*_{\mathbb{Q}}(X, X_0)$$

for all integers n. Moreover (see Lemma 7), it follows that

(a) ch maps $K^0(X, X_0)$ into $H^e_{\mathbb{Q}}(X, X_0)$,
(b) ch maps $K^1(X, X_0)$ into $H^\sigma_{\mathbb{Q}}(X, X_0)$.

To complete the picture, recall that in Theorem 4, we constructed an exact "K-hexagon". Similarly, we have an exact "H-hexagon" as follows:

$$\begin{tikzcd}
 & H^\sigma_{\mathbb{Q}}(X, X_0) \arrow[r] & \tilde{H}^\sigma_{\mathbb{Q}}(X) \arrow[dr] & \\
\tilde{H}^e_{\mathbb{Q}}(X_0) \arrow[ur] & & & \tilde{H}^\sigma_{\mathbb{Q}}(X_0) \arrow[dl] \\
 & \tilde{H}^e_{\mathbb{Q}}(X) \arrow[ul] & H^e_{\mathbb{Q}}(X, X_0) \arrow[l] &
\end{tikzcd}$$

Exercise: Prove that ch is a natural transformation of the K-hexagon to the H-hexagon.

(Hint: By virtue of Theorem 4, it suffices to show that ch is a $\mathbb{Q}$-cohomology invariant of bundles, i.e. $ch \circ f^\dagger = f^* \circ ch$, where f is a continuous map of spaces. The latter is most conveniently proved by first proving it for sums of U_1-bundles, in which case it is practically trivial, and then extending to the general case by using the line of argument established above.)

Remark: When we defined ch on $K^{-n}(X, X_0)$ and $K^n(X, X_0)$ above, we required the image to lie in $H^*_{\mathbb{Q}}(X, X_0)$. Alternatively, we could have proceeded as follows. Define

$$ch: K^{-n}(X, X_0) \to H^*_{\mathbb{Q}}(S^n(X/X_0))$$

in the obvious way. Then put

$$\mathscr{H}^{-n}(X, X_0) = H^e_{\mathbb{Q}}(S^n(X/X_0)).$$

We claim that the groups $\mathscr{H}^{-n}(X, X_0)$ form a generalized cohomology theory. In fact, for any integer n, there exists a space $\mathscr{E}_n$ (a so-called Eilenberg-MacLane space; the usual notation found in the literature for such a space is $K(\mathbb{Q}, n)$) such that for any finite complex X,

$$H^n_{\mathbb{Q}}(X) \cong (\mathscr{E}_n^X).$$

We define

$$\mathscr{E} = \mathscr{E}_2 \times \mathscr{E}_4 \times \mathscr{E}_6 \times \cdots$$

Then clearly

$$(\mathscr{E}^X) \cong \sum_{i \geqq 1} H^{2i}(X; \mathbb{Q}) = \tilde{H}^e_{\mathbb{Q}}(X).$$

This shows that $\mathscr{E}$ is a test space and our contention that the groups $\mathscr{H}^{-n}(X, X_0)$ give a generalized cohomology theory is proved. Finally, we can construct an

"$\mathscr{H}$-hexagon" just as for K-Theory (using $\hat{\beta}$ instead of β) and then show that ch is a natural transformation from the K-hexagon to the $\mathscr{H}$-hexagon.

4. The Atiyah-Hirzebruch Spectral Sequence

In this section, we describe a spectral sequence relating the ordinary cohomology of a cellular pair with its K-cohomology. The spectral sequence will allow us to obtain some useful information about the K-groups. The spectral sequence in question is essentially that derived in Chapter 5. We shall, however, reformulate and generalize the spectral sequence described in that chapter, casting the results derived into the general algebraic form set forth in Cartan and Eilenberg—*Homological Algebra,* Chapter 15. This generalized formulation is as follows.

We first define the notion of a *group table.* By a *pair* we shall mean a pair of integers (p, q) with $-\infty \leqq p \leqq q \leqq +\infty$; by a *triple* we shall mean a triple of integers (p, q, r) with $-\infty \leqq p \leqq q \leqq r \leqq +\infty$. A group table is then a collection as follows:

(a) For each pair (p, q), there is an abelian group $H(p, q)$.

(b) Given 2 pairs such that $(p, q) \leqq (p', q')$ (i.e. $p \leqq p'$, $q \leqq q'$), there is a homomorphism $H(p', q') \to H(p, q)$.

(c) For each triple (p, q, r), there is a homomorphism $H(p, q) \xrightarrow{\delta} H(q, r)$.

These groups and homomorphisms will be required to satisfy the following set of axioms:

(0) $H(p, q) = 0$.

(1) $H(p, q) \to H(p, q)$ is the identity map.

(2) If $(p, q) \leqq (p', q') \leqq (p'', q'')$, then the triangle

$$\begin{tikzcd} H(p'', q'') \arrow[rr] \arrow[dr] & & H(p, q) \\ & H(p', q') \arrow[ur] & \end{tikzcd}$$

is commutative.

(3) If $(p, q, r) \leqq (p', q', r')$, then the square

$$\begin{tikzcd} H(p', q') \arrow[r, "\delta"] \arrow[d] & H(q', r') \arrow[d] \\ H(p, q) \arrow[r, "\delta"] & H(q, r) \end{tikzcd}$$

is commutative.

(4) For each triple (p, q, r), the triangle

$$\begin{tikzcd}
H(p,q) \arrow[rr, "\delta"] & & H(q,r) \arrow[dl] \\
& H(p,r) \arrow[ul] &
\end{tikzcd}$$

is exact.

(5) The groups $H(p, q)$ stabilize for small values of p or for large values of q.

Examples:

(i) Let (X, A) be a pair of spaces and let

$$X = X_N \supseteq X_{N-1} \supseteq \cdots \supseteq X_0 = A$$

be a finite filtration of (X, A). We define X_m for arbitrary m by setting

$$X_m = A \quad \text{if} \quad m \leqq 0, \qquad X_m = X \quad \text{if} \quad m \geqq N.$$

We define a group table by putting

$$H(p, q) = \sum_{n \geqq 0} T^{-n}(X_q, X_p),$$

for an arbitrary T-cohomology theory (T grouplike). The homomorphism $H(p, q) \to H(p', q')$ is just the homomorphism

$$\sum_{n \geqq 0} T^{-n}(X_{q'}, X_{p'}) \to \sum_{n \geqq 0} T^{-n}(X_q, X_p),$$

induced by the inclusion map

$$(X_q, X_p) \subseteq (X_{q'}, X_{p'}).$$

The homomorphism $H(p, q) \to H(q, r)$ required in (c) above is the composition

$$\sum_{n \geqq 0} T^{-n}(X_q, X_p) \longrightarrow \sum_{n \geqq 0} T^{-n}(X_q) \xrightarrow{\delta} \sum_{n \geqq 0} T^{-n}(X_r, X_q).$$

The axioms are all seen to be verified.

(ii) As a particular case of (i), we may replace $\sum_{n \geqq 0} T^{-n}(X_q, X_p)$ by $K^*(X_q, X_p)$ (see Definition 1), or $H_Q^*(X_q, X_p)$ (with its usual Z-grading or with the Z_2-grading defined above).

(iii) As a special case of (i) in another sense, let (X, A) be a pair of finite complexes. Then we obtain a canonical filtration of (X, A) by using the skeleta of X. In other words, let X_p be defined as the union of A with the p-skeleton of X.

The most important fact about group tables is the following theorem.

Theorem (11): Given a group table, there exists a spectral sequence $\{E_r^p\}$, $r \geqq 1$, such that (a) $E_1^p = H(p, p+1)$, (b) the first differential ∂^1 coincides with $H(p, p+1) \xrightarrow{\delta} H(p+1, p+2)$, (c) the groups E_∞^p are the factors of a filtration of $H(-\infty, \infty)$.

This theorem has essentially been proved in Chapter 5 of these notes. In fact, what we proved there was the existence of a spectral sequence with the desired properties in case the group table is the one described in Example (i) above. However, we observed that the construction of the spectral sequence does not depend in an essential way on the topological structure of (X, A). In fact, the same proof goes through for any group table.

We shall now restrict our attention to the situation described in Example (iii) above. In particular, we want to analyze the terms E_1 and E_2 in more detail in this case. Let us make the abbreviation

$$T^*(X, A) = \sum_{n \geqq 0} T^{-n}(X, A).$$

Recall that we have

$$E_1^p = H(p, p+1) = T^*(X_{p+1}, X_p).$$

Notice that X_{p+1}/X_p is a grape bunch (or a bouquet of spheres), that is,

$$X_{p+1}/X_p = S_1^{p+1} \vee S_2^{p+1} \vee \cdots \vee S_{\alpha_{p+1}}^{p+1}$$

where α_{p+1} is the number of $(p+1)$-cells in (X, A). Let

$$f_i : (D_i^{p+1}, \partial D_i^{p+1}) \to (X_{p+1}, X_p) \qquad (1 \leqq i \leqq \alpha_{p+1})$$

be the attaching maps for the $(p+1)$-cells of (X, A) (see Chapter 2). Define

$$I_{p+1} : T^*(X_{p+1}, X_p) \to \sum_{i=1}^{\alpha_{p+1}} T^*(D_i^{p+1}, \partial D_i^{p+1})$$

by setting

$$I_{p+1}(a) = (f_1^*(a), \ldots, f_{\alpha_{p+1}}^*(a)).$$

I_{p+1} is evidently an isomorphism. We shall describe the inverse isomorphism I_{p+1}^{-1} explicitly. We set

$$\theta_i^{p+1} = f_i\left(\mathring{D}_i^{p+1}\right), \qquad (1 \leqq i \leqq \alpha_{p+1}).$$

Evidently,

$$\theta_i^{p+1} \subseteq X_{p+1} - X_p,$$

or equivalently

$$X_p \subseteq X_{p+1} - \theta_i^{p+1}.$$

Consider the composition map g_i,

$$X_{p+1}/X_p \to X_{p+1}/X_{p+1} - \theta_i^{p+1} \to D_i^{p+1}/\partial D_i^{p+1},$$

defined as follows. The first map is induced by the inclusion

$$(X_{p+1}, X_p) \subseteq (X_{p+1}, X_{p+1} - \theta_i^{p+1}).$$

The second map can be described by taking θ_i^{p+1} to $\mathring{D}_i^{p+1}$ by means of f_i^{-1} and by taking $X_{p+1} - \theta_i^{p+1}$ to a point. We then have a map

$$J_{p+1}: \sum_{i=1}^{\alpha_{p+1}} T^*(D_i^{p+1}/\partial D_i^{p+1}) \to T^*(X_{p+1}/X_p)$$

given by

$$J_{p+1}(a_1, \ldots, a_{\alpha_{p+1}}) = g_1^*(a_1) + \cdots + g_{\alpha_{p+1}}^*(a_{\alpha_{p+1}})$$

Clearly, J_{p+1} is the inverse of I_{p+1}. Using the fact that $E_1^p = H(p, p+1)$, we obtain easily

$$E_1^p = C^p\Big(X, A; \sum_{n \geqq 0} T^{-n-p-1}(x_0)\Big),$$

where C^p is the cochain group. We now examine the differential ∂^1, i.e. the coboundary map

$$\delta: \sum_{n \geqq 0} T^{-n}(X_{p+1}, X_p) \to \sum_{n \geqq 0} T^{-n}(X_{p+2}, X_{p+1}).$$

We shall define a map

$$h^*: \sum_{i=1}^{\alpha_{p+1}} T^*(D_i^{p+1}/\partial D_i^{p+1}) \to \sum_{j=1}^{\alpha_{p+2}} T^*(\partial D_j^{p+2}),$$

where α_{p+2} is, of course, the number of $(p+2)$-cells of (X, A). To do this, we consider the $(p+2)$-cells of (X, A) and the attaching maps

$$f_j: (D_j^{p+2}, \partial D_j^{p+2}) \to (X_{p+2}, X_{p+1}), \qquad (1 \leqq j \leqq \alpha_{p+2}).$$

(Note that we are overworking the symbol f_j.) For any i such that $1 \leqq i \leqq \alpha_{p+1}$, we have the composed map

$$\partial D_j^{p+2} \xrightarrow{f_j} X_{p+1} \longrightarrow X_{p+1}/X_{p+1} - \theta_i^{p+1} \longrightarrow D_i^{p+1}/\partial D_i^{p+1},$$

where the latter two maps have been defined previously; we call this composed map h_{ij}. We then have the induced map

$$h_{ij}^*: T^*(D_i^{p+1}/\partial D_i^{p+1}) \to T^*(\partial D_j^{p+2}).$$

We now define a map

$$h_j^*: \sum_{i=1}^{\alpha_{p+1}} T^*(D_i^{p+1}/\partial D_i^{p+1}) \to T^*(\partial D_j^{p+2})$$

by setting

$$h_j^*(a_1, \ldots, a_{\alpha_{p+1}}) = h_{1j}^*(a_1) + \cdots + h_{\alpha_{p+1},j}^*(a_{\alpha_{p+1}}).$$

Finally, we define the desired map h^* by

$$h^*(a_1, \ldots, a_{\alpha_{p+1}}) = (h_1^*(a_1, \ldots, a_{\alpha_{p+1}}), \ldots, h^*_{\alpha_{p+2}}(a_1, \ldots, a_{\alpha_{p+1}})).$$

We see easily that the following diagram is commutative:

$$\begin{tikzcd}
 & \sum_{i=1}^{\alpha_{p+1}} T^*(D_i^{p+1}/\partial D_i^{p+1}) \arrow[r, "I_{p+1}^{-1}"] \arrow[dl, "h^*"'] & T^*(X_{p+1}/X_p) \arrow[dd, "\delta"] \\
\sum_{j=1}^{\alpha_{p+2}} T^*(\partial D_j^{p+2}) \arrow[dr, "\delta"'] & & \\
 & \sum_{j=1}^{\alpha_{p+2}} T^*(D_j^{p+2}/\partial D_j^{p+2}) & T^*(X_{p+2}/X_{p+1}) \arrow[l, "I_{p+2}"']
\end{tikzcd}$$

We have at this point almost shown that the differential ∂^1 is precisely the "formal coboundary" operator $\tilde{\delta}$ of the cell complex (X, A) (cf. Chapter 2, Lemma 11). To complete the proof that $\partial^1 = \tilde{\delta}$, we make use of the following statement: The map h_{ij} defined above can be viewed as a map of a $(p + 1)$-sphere into itself. As such, h_{ij} belongs to a homotopy class characterized by its degree, say D_{ij}. If we can show that

$$h_{ij}^*(a) = D_{ij} \cdot a$$

for any $a \in T^*(S^{p+1})$, then we will have completed the proof that $\tilde{\delta} = \partial^1$. We state the needed assertion as

Lemma (9)**:** Consider an arbitrary generalized cohomology theory, based on a test space T. Let

$$\phi : S^m \to S^m$$

be a map of degree D. Then the induced map

$$\phi^* : T^{-n}(S^m) \to T^{-n}(S^m)$$

takes an element $a \in T^{-n}(S^m)$ into the element $D \cdot a$.

Proof: a is represented by some map

$$f: S^n \wedge S^m \to T,$$

or, by means of the usual identification (see the proof of Lemma 1 above), by a map

$$f' : S^m \to \Omega^n T.$$

But then $\phi^*(a)$ is represented by the map

$$f' \circ \phi : S^m \to \Omega^n T.$$

If I^m is the m-cube and we identify S^m with $I^m/\partial I^m$, then ϕ is homotopic to the map $S^m \to S^m$ obtained by setting n copies of I^m next to each other to make a "pile" J, and taking the composite map $I^m \to J \to I^m$, where the map $I^m \to J$ multiplies a coordinate of $x \in I^m$ by n, and $J \to I^m$ maps this stretched coordinate back to $[0, 1]$ by periodicity. Since precisely the same construction is used to define addition of homotopy classes, it is clear that $[f' \circ \phi] = D \cdot [f']$. Q.E.D.

Since ∂^1 is the formal coboundary operator, it follows that

$$E_2^p = H^p\Big(X, A; \sum_{n \geq 0} T^{-n-p-1}(x_0)\Big),$$

where H^p is the formal cohomology group. By virtue of Lemma 11 of Chapter 2 (more precisely, by virtue of the dual assertion of this lemma), H^p is isomorphic to the ordinary cohomology group of (X, A).

Remark: The proof of Lemma 11 was not given in Chapter 2. However, the proof of the lemma can easily be obtained by using the results of Chapter 5. Indeed, let us consider the spectral sequence of Chapter 5, with $H(X, A)$, etc. taken as ordinary homology. By looking at the definition of the differential ∂^r (see the first line of the proof of Theorem 1 of Chapter 5), we see immediately that $\partial^r = 0$ for $r \geq 2$. Thus, it follows that

$$E_2^j \cong E_3^j \cong \cdots \cong E_\infty^j.$$

But, we have

$$E_\infty^j = \frac{\operatorname{Im}[H(X_j, A) \to H(X, A)]}{\operatorname{Im}[H(X_{j-1}, A) \to H(X, A)]}$$

and this quotient group is readily seen to be $H_j(X, A)$, the ordinary j-homology group of (X, A).

Note: The fact that the differentials ∂^r $(r \geq 2)$ are zero in case T^* is ordinary cohomology theory will be of great importance to us in what follows.

Let us sum up the results we have obtained so far.

Theorem (12): Let (X, A) be a cellular pair. There exists a spectral sequence such that

$$E_1^p = C^p\Big(X, A; \sum_{n \geq 0} T^{-n-p-1}(x_0)\Big)$$

$$E_2^p = H^p\Big(X, A; \sum_{n \geq 0} T^{-n-p-1}(x_0)\Big)$$

and such that the E_∞^p are the graded groups of a suitable filtration of the generalised cohomology group $T^*(X, A)$.

Corollary: Let (X, A) be a cellular pair. There exists a spectral sequence such that

$$E_1^p = C^p(X, A; \mathbb{Z})$$

$$E_2^p = H^p(X, A; \mathbb{Z})$$

and such that the E_∞^p are the graded groups of a suitable filtration of the group $K^*(X, A)$.

Proof: We have only to use the fact that

$$K^*(x_0) = K^0(x_0) \oplus K^1(x_0) \cong \mathbb{Z} \oplus 0. \text{ Q.E.D.}$$

We have now the following situation. We have a cellular pair (X, A) and two group tables associated with it. Namely, we have

$$H(p, q) = K^*(X_q, X_p) \tag{1}$$

$$\hat{H}(p, q) = H_{\mathbb{Q}}^{\$}(X_q, X_p). \tag{2}$$

We use the notation

$$H_{\mathbb{Q}}^{\$}(X_q, X_p) = H_{\mathbb{Q}}^{e}(X_q, X_p) + H_{\mathbb{Q}}^{\sigma}(X_q, X_p)$$

to distinguish from the usual $\mathbb{Z}$-grading

$$H_{\mathbb{Q}}^{*}(X_q, X_p) = \sum_n H_{\mathbb{Q}}^{n}(X_q, X_p).$$

We have shown in Section 3 the existence of a natural transformation, namely ch, from the K^*-Theory to the $H_{\mathbb{Q}}^{\$}$-Theory. In other words, we have a natural transformation from one group table to the other. We propose to study the relationships between the resulting spectral sequences E_r^p, $\hat{E}_r^p$ which can be obtained by virtue of this natural transformation. We look first at the abstract algebraic situation.

Lemma (10): Let $\{H(p, q)\}$, $\{\hat{H}(p, q)\}$ be two group tables and let $\phi : \{H(p, q)\} \to \{\hat{H}(p, q)\}$ be a natural transformation which is a homomorphism, i.e. ϕ consists of a collection of homomorphisms

$$\phi^{p,q} : H(p, q) \to \hat{H}(p, q)$$

which commute in an obvious way with induced homomorphisms and coboundary homomorphisms of the group tables. Then there exists a collection $\Phi = \{\Phi_r^p\}$ of homomorphisms of the spectral sequence $\{E_r^p, \partial^r\}$ into the spectral sequence $\{\hat{E}_r^p, \hat{\partial}^r\}$. In other words, we have homomorphisms

$$\Phi_r^p : E_r^p \to \hat{E}_r^p$$

such that $\Phi_r^{p-r} \circ \partial^r = \hat{\partial}^r \circ \Phi_r^p$. Moreover, the mapping

$$\Phi_1^p : E_1^p \to \hat{E}_1^p$$

coincides with the mapping

$$\phi^{p,p+1} : H(p, p+1) \to \hat{H}(p, p+1).$$

Proof: The intermediate groups E_r^p are defined by

$$E_r^p = \frac{\operatorname{Im}[H(p-1, p+r-1) \to H(p-r, p+r-1)]}{\operatorname{Im}[H(p, p+r-1) \to H(p-r, p+r-1)]},$$

and similarly for $\hat{E}_r^p$. (In fact, we have observed that the case of a general group table is essentially no different than the case of Example (i) above, treated in Chapter 5 where essentially the above formula is given.) Since the ϕ's commute with induced homomorphisms, it is clear that they induce a homomorphism from $\operatorname{Im}[H(p-r, p+r-1) \to \hat{H}(p-r, p+r-1)]$ into $\operatorname{Im}[\hat{H}(p-1, p+r-1) \to \hat{H}(p-r, p+r-1)]$, and similarly for the "denominators" of E_r^p and $\hat{E}_r^p$. We then have the required induced map from E_r^p to $\hat{E}_r^p$ which we denote by Φ_r^p. The fact that the Φ's commute with the differentials of the spectral sequences follows from the fact that the differentials are defined by using induced homomorphisms and coboundary homomorphisms and the ϕ's commute with these. The assertion about Φ_1^p is plain. Q.E.D.

Lemma (11): Suppose that Φ is a collection of homomorphisms of $\{E_r^p, \partial^r\}$ into $\{\hat{E}_r^p, \hat{\partial}^r\}$ as above. Suppose that for some r_0,

$$\Phi_{r_0}^p : E_{r_0}^p \to \hat{E}_{r_0}^p$$

is injective for all p. If the $\hat{\partial}^r$ are zero for all $r \geqq r_0$, then also the ∂^r are zero for all $r \geqq r_0$ and

$$\Phi_r^p : E_r^p \to \hat{E}_r^p$$

is injective for all $r \geqq r_0$. If, in addition, $\Phi_{r_0}^p$ is surjective, then all the Φ_r^p $(r \geqq r_0)$ are surjective.

Proof: We show that $\partial^{r_0} = 0$. In fact, let $a \in E_{r_0}^p$. Then

$$\Phi_{r_0}^{p-r_0}(\partial^{r_0} a) = \hat{\partial}^{r_0} \Phi_{r_0}^p(a) = 0$$

since $\hat{\partial}^{r_0} = 0$. As $\Phi_{r_0}^{p-r_0}$ is injective, $\partial^{r_0} a = 0$; whence $\partial^{r_0} = 0$. But then, by the Leray relation, $E_{r_0+1}^p = E_{r_0}^p$, $\hat{E}_{r_0+1}^p = \hat{E}_{r_0}^p$ and the map $\Phi_{r_0+1}^p : E_{r_0+1}^p \to \hat{E}_{r_0+1}^p$ coincides with the map $\Phi_{r_0}^p : E_{r_0}^p \to \hat{E}_{r_0}^p$. In particular, $\Phi_{r_0+1}^p$ is injective and we may use induction to complete the proof.

We now return to the concrete group tables $H(p, q) = K^*(X_q, X_p)$, $\hat{H}(p, q) = H_{\mathbb{Q}}^{\$}(X_q, X_p)$. We have

Theorem (13): Let (X, A) be a cellular pair such that $H^*(X, A; \mathbb{Z})$ is torsion-free. Then,

$$ch: K^*(X, A) \to H_{\mathbb{Q}}^{\$}(X, A)$$

is injective. Consequenty $K^*(X, A)$ is free abelian of finite rank.

Proof: Letting $\phi = ch$ and using Lemma 10, we find homomorphisms Φ of $\{E_r^p, \partial^r\}$ into $\{\hat{E}_r^p, \hat{\partial}^r\}$. For $r = 1$,

$$\Phi_1^p : E_1^p \to \hat{E}_1^p$$

coincides with

$$ch: K^*(X_{p+1}, X_p) \to H_{\mathbb{Q}}^{\$}(X_{p+1}, X_p),$$

again by Lemma 10. But we have calculated $K^*(X_{p+1}, X_p)$ and $H_{\mathbb{Q}}^{\$}(X_{p+1}, X_p)$, and we have

$$ch: C^p(X, A; \mathbb{Z}) \to C^p(X, A; \mathbb{Q})$$

(see Theorem 12 and its corollary) where ch is just the map induced by the coefficient homomorphism $\mathbb{Z} \to \mathbb{Q}$. For $r = 2$, we have

$$\Phi_2^p : H^p(X, A; \mathbb{Z}) \to H^p(X, A; \mathbb{Q})$$

and Φ_2^p is again the map induced by the homomorphism $\mathbb{Z} \to \mathbb{Q}$ (because ∂^1 coincides with the formal coboundary operator and the Φ's commute with the differentials). Since $H^*(X, A; \mathbb{Z})$ is torsion-free, Φ_2^p is injective. Moreover, we have observed that the differentials $\hat{\partial}^r$ ($r \geqq 2$) vanish. By Lemma 11,

$$\Phi_r^p : E_r^p \to \hat{E}_r^p$$

is injective for $r \geqq 2$. But for r large enough, say $r \geqq R$, $E_r^p = E_\infty^p$ (because the skeleta of (X, A) stabilize) and hence

$$\Phi_\infty^p : E_\infty^p \to \hat{E}_\infty^p$$

is injective. Using the remark following Lemma 9 (applied to cohomology rather than homology), we see finally that

$$ch: K^*(X, A) \to H_{\mathbb{Q}}^{\$}(X, A)$$

is injective. This proves Theorem 13.

Theorem (14): Let (X, A) be any cellular pair. Then

$$ch \otimes 1 : K^*(X, A) \otimes \mathbb{Q} \to H_{\mathbb{Q}}^{\$}(X, A) \otimes \mathbb{Q} \cong H_{\mathbb{Q}}^{\$}(X, A)$$

is an isomorphism and maps $K^0(X, A) \otimes \mathcal{Q}$ into $H^e_{\mathcal{Q}}(X, A)$ and $K^1(X, A) \otimes \mathcal{Q}$ onto $H^{\sigma}_{\mathcal{Q}}(X, A)$.

Proof: In place of the group table $H(p, q) = K^*(X_q, X_p)$, we consider the group table $\bar{H}(p, q) = K^*(X_q, X_p) \otimes \mathcal{Q}$, and argue as in Theorem 13. The Chern character induces a homomorphism Ψ of $\{\bar{E}^p_r, \bar{\partial}^r\}$ into $\{\hat{E}^p_r, \hat{\partial}^r\}$ and

$$\Psi^p_1 : \bar{E}^p_1 \to \hat{E}^p_1$$

coincides with the identity homomorphism

$$ch : C^p(X, A; \mathcal{Q}) \to C^p(X, A; \mathcal{Q}).$$

By Lemma 11, Ψ^p is an isomorphism for all r and the theorem is proved. Q.E.D.

Corollary: Let (X, A) be any cellular pair. Then $K^0(X, A)$ (resp. $K^1(X, A)$) is a finitely generated abelian group whose rank is equal to the sum of the even-dimensional (resp. odd-dimensional) Betti numbers of (X, A).

5. Cohomology Operations in K-Theory

In this section, we shall define a sequence of maps

$$\Psi^k : K(X) \to K(X).$$

These maps will be ring homomorphisms which commute with induced homomorphisms, i.e. cohomology operations. The operations Ψ^k play a fundamental role in the proof of Adams' Theorem on vector fields. See Adams—*Vector Fields on Spheres*, Ann. of Meth., Vol. 75, 1962.

We now begin the construction of the Ψ^k. First, we need some preliminaries on representations of compact topological groups. As a reference for unproved results, see Pontryagin—*Topological Groups*. Recall that a unitary representation ϱ of a compact group G is a continuous homomorphism

$$\varrho : G \to U^n$$

for some n; n is called the degree of ϱ. (Of course, the concept of representation can be defined for any topological group, but our only interest later will be in compact groups G.) Representations ϱ_1 and ϱ_2 of G, having the same degree n, are called equivalent (we write $\varrho_1 \sim \varrho_2$) if there exists a $u \in U(n)$ such that $\varrho_1(g) = u^{-1}\varrho_2(g)\,u$ for all $g \in G$. We denote the equivalence class of ϱ by (ϱ). The notions of direct sum and tensor product of representations are defined in the evident manner. Note that if $\deg(\varrho_i) = n_i$ $(i = 1, 2)$, then $\deg(\varrho_1 \oplus \varrho_2) = n_1 + n_2$ and $\deg(\varrho_1 \otimes \varrho_2) = n_1 \cdot n_2$. We say that a representation ϱ of degree n is reducible if the representation space E^n (on which $U(n)$ acts in the usual way) has a proper subspace left invariant by all the transformations $\varrho(g)$, $g \in G$. By classical properties of the unitary group, any unitary representation is completely reducible, i.e. can be written as a direct sum of irreducible representations. The character of a representation ϱ is the complex-valued function

$$\chi_\varrho(g) = tr\,(\varrho(g)).$$

Clearly, $\varrho_1 \sim \varrho_2$ implies $\chi_{\varrho_1} = \chi_{\varrho_2}$. Moreover, we have

$$\chi_{\varrho_1 \oplus \varrho_2} = \chi_{\varrho_1} + \chi_{\varrho_2}, \qquad \chi_{\varrho_1 \otimes \varrho_2} = \chi_{\varrho_1} \cdot \chi_{\varrho_2}.$$

Finally, we shall need a few deeper results of representation theory. We can define a group-invariant integral (the Haar integral) on G. With respect to this integral, the notion of orthogonality of functions may be defined. If

$$f_1, f_2 : G \to \mathbb{C}$$

are complex-valued functions, f_1 is said to be orthogonal to f_2 if

$$\int_G f_1(g)\,\overline{f_2(g)}\,dg = 0.$$

We then have the following propositions:

1) Let ϱ_1, ϱ_2 be two nonequivalent unitary irreducible representations of G. Then the characters χ_{ϱ_1} and χ_{ϱ_2} are orthogonal functions. If ϱ is a unitary irreducible representation of G, then χ_ϱ has length, i.e.

$$\int_G \chi_\varrho(g)\,\overline{\chi_\varrho(g)}\,dg = 1.$$

As an immediate corollary of 1), we have

2) Two unitary representations of a compact group G are equivalent if and only if their characters are equal.

We now wish to define an analogue of the Atiyah-Hirzebruch-Grothendieck functior $K(X)$ for compact groups G. We begin by defining $\hat{K}_F(G)$ as the free abelian group generated by equivalence classes of representations of G. (By representation we will always mean unitary representation of some degree.) We denote by $\hat{K}_F^+(G)$ the subgroup of $\hat{K}_F(G)$ generated by elements of the form $(\varrho_1 \oplus \varrho_2) - (\varrho_1) - (\varrho_2)$ and we put

$$\hat{K}(G) = \hat{K}_F(G)/\hat{K}_F^+(G).$$

We denote by $[\varrho]$ the class of ϱ in $\hat{K}(G)$. The elements of $\hat{K}(G)$ may be called *virtual representations.* (Similarly, the elements of $K(X)$ may be called *virtual bundles.*) Observe that an arbitrary element θ of $\hat{K}(G)$ may be written as

$$\theta = \Sigma\, n_i[\varrho_i]$$

with the representatives ϱ_i of $[\varrho_i]$ being irreducible. This follows from the fact that all representations are completely reducible and we are dividing out by $\hat{K}_F^+(G)$. By analogy with the dimension homomorphism of $K(X)$ onto Z, we may define the degree homomorphism

$$\deg : \hat{K}(G) \to Z$$

in the obvious way. Thus $K(G)$ is a commutative ring with unit (the product in $\hat{K}(G)$ being induced by tensor product of representations, the unit being induced by the trivial representation of degree 1) together with an augmentation deg. A continuous homomorphism $\varrho : G \to H$, with H another compact group, gives rise in a natural way to a map

$$\varrho^* : \hat{K}(H) \to \hat{K}(G)$$

which has evident functorial properties.

Let ξ be a G-bundle structure on X. Thus ξ is determined by an open covering $\{U\}$ of X together with functions

$$g_{UV} : U \cap V \to G$$

which satisfy certain properties (see Chapter 4). If ϱ is a continuous homomorphism from G into H, we define an H-bundle structure $\varrho(\xi)$ on X as follows. The open covering $\{U\}$ of X remains the same. We define the functions

$$h_{UV} : U \cap V \to H$$

by setting

$$h_{UV}(x) = \varrho\,(g_{UV}(x)), \qquad x \in U \cap V.$$

The function h_{UV} satisfy the requirements for a bundle structure, as is easily seen.

Let now θ be a virtual representation of $U(n)$, so that

$$\theta = \Sigma\, n_i[\varrho_i].$$

If ξ is a $U(n)$-bundle on X, we define a virtual bundle $\theta(\xi)$ by setting

$$\theta(\xi) = \Sigma\, n_i\,[\varrho_i(\xi)],$$

where $\varrho_i(\xi)$ has the meaning described in the last paragraph.

Suppose that for each n we have an element $\theta_n \in \tilde{K}(U(n))$. We shall call $\Theta = \{\theta_i\} = \{\theta_1, \theta_2, \ldots,\}$ simply a *sequence*. Let

$$\pi_1 : U(n) \times U(m) \to U(n),$$

$$\pi_2 : U(n) \times U(m) \to U(m)$$

be projection maps.

Definition (2): Θ is called an additive sequence if

$$\theta_{n+m} (\pi_1 \oplus \pi_2) = (\theta_n \circ \pi_1) + (\theta_m \circ \pi_2)$$

for all n, m. In other words, for all $u \in U(n)$, $v \in U(m)$, we have

$$\theta_{n+m} (u \oplus v) = \theta_n(u) + \theta_m(v).$$

Lemma (12): Let ξ be a $U(n)$-bundle on X, η a $U(m)$-bundle on X and Θ an additive sequence. Then

$$\theta_{n+m} (\xi \oplus \eta) = \theta_n(\xi) + \theta_m(\eta).$$

Proof: The proof is easy and its details are left to the reader.

Definition (3): Let Θ be an additive sequence and $a \in K(X)$. We write a as

$$a = \Sigma\, n_i[\xi_i]$$

and we define

$$\Theta(a) = \Sigma\, n_i \theta_{d_i} (\xi_i),$$

where ξ_i is a $U(d_i)$-bundle. It follows from Lemma 12 that this definition is consistent.

Definition (4): Let Θ be an additive sequence. Θ is called multiplicative if

$$\theta_{nm} \circ (\pi_1 \otimes \pi_2) = (\theta_n \circ \pi_1) \cdot (\theta_m \circ \pi_2)$$

for all n, m. In other words, for all $u \in U(n)$, $v \in U(m)$, we have

$$\theta_{nm} (u \otimes v) = \theta_n(u) \cdot \theta_m(v).$$

Lemma (13): Let ξ be a $U(n)$-bundle on X, η a $U(m)$-bundle on X and Θ a multiplicative sequence. Then

$$\theta_{nm} (\xi \otimes \eta) = \theta_n(\xi) \cdot \theta_m(\eta).$$

Proof: The proof is similar to that of Lemma 12 and is also omitted.

Theorem (15): If Θ is a multiplicative sequence, then

$$\Theta : K(X) \to K(X)$$

(see Definiton 3) is a ring homomorphism which is natural with respect to maps of X, i.e. Θ is a cohomology operation.

Proof: The first part follows from Lemmas 12 and 13. The second part is easy to prove and its details are left to the reader.

We are now ready to define the cohomology operations Ψ^k. Let V be a (complex) vector space, V^* the dual space of V. Let V_r be the set of multilinear maps

$$V^* \times \cdots \times V^* \to \mathbb{C} \qquad (r \text{ factors of } V^*)$$

which are skew-symmetric; V_r is then the Grassmann space associated with V^*. A unitary map

$$u : V \to V$$

induces a unitary map

$$u^{(r)} : V_r \to V_r$$

defined by

$$u^{(r)}(v_1 \wedge \cdots \wedge v_r) = uv_1 \wedge \cdots \wedge uv_r.$$

The map

$$u \xrightarrow{K_V^{(r)}} u^{(r)}$$

is a unitary representation of the group of unitary transformations on V. If $V = E^n$, a Euclidean space, we write

$$K_{E^n}^{(r)} = K_n^{(r)}.$$

Thus $K_n^{(r)}$ is a unitary representation of the unitary group $U(n)$.

Let $x_1, \ldots, x_n$ be n formal variables. $\sum_{i=1}^{n} x_i^k$ is a symmetric function of the variables and can therefore be written as $Q_n^k(\sigma_1^{(n)}, \ldots, \sigma_n^{(n)})$, where Q_n^k is a polynomial with integral coefficients and where $\sigma_j^{(n)}$ is the j-th elementary symmetric function of the variables. We set

$$\psi_n^k = Q_n^k(K_n^{(1)}, K_n^{(2)}, \ldots, K_n^{(n)}),$$

where addition and multiplication are to be taken in the sense of direct sum and tensor product of representations. Thus, ψ_n^k is a virtual representation of $U(n)$ for each k, and we can form the sequence

$$\Psi^k = \{\psi_1^k, \psi_2^k, \ldots\}.$$

Lemma (14): Ψ^k is a multiplicative sequence for each k.

Before proving this, we establish the following

Formula: If $u \in U(n)$, then $\chi_{\psi_n^k}(u) = tr(u^k)$.

Proof: With respect to a suitable basis, u is in diagonal form. Let the eigenvalues of u be $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenvectors be $e_1, \ldots, e_n$. The vectors $e_{i_1} \wedge \cdots \wedge e_{i_r}$ $(1 \leqq i_1 < i_2 < \cdots < i_r \leqq n)$ form a basis for the space E_n^r. These vectors are eigenvectors of $u^{(r)}$ with corresponding eigenvalues $\lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_r}$. By the classical trace formula,

$$\chi_{K_n^{(r)}}(u) = \sum_{1 \leqq i_1 < \cdots < i_r \leqq n} \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_r} = \sigma_r^{(n)}\left(\vec{\lambda}\right),$$

where $\sigma_r^{(n)}\left(\vec{\lambda}\right)$ is the r-th elementary symmetric function of the numbers $\lambda_1, \ldots, \lambda_n$. By the way characters behave with respect to direct sums and tensor products, we obtain

$$\begin{aligned}\chi_{\psi_n^k}(u) &= Q_n^k\left(\sigma_1^{(n)}\left(\vec{\lambda}\right), \ldots, \sigma_n^{(n)}\left(\vec{\lambda}\right)\right) \\ &= \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k = tr\,(u^k).\end{aligned}$$

Q.E.D.

Proof of Lemma 14: We have to prove two things:

(a) $$\psi_{n+m}^k\,(u_1 \oplus u_2) = \psi_n^k(u_1) + \psi_m^k(u_2),$$

(b) $$\psi_{nm}^k\,(u_1 \otimes u_2) = \psi_n^k(u_1) \cdot \psi_m^k(u_2).$$

Proof of (a): We shall show that the trace of both sides of (a) are equal. Applying proposition 2) above will yield the result. We have, using the formula just established, $tr\,(\psi_{n+m}^k(u_1 \oplus u_2)) = tr\,((u_1 \oplus u_2)^k)) = tr\,(u_1^k) + tr\,(u_2^k) = tr\,(\psi_n^k u_1) + tr\,(\psi_m^k u_2)$.

(b) is proved similarly.

Theorem (16): $\Psi^k : K(X) \to K(X)$ is a cohomology operation for each positive integer.

Proof: This follows directly from Theorem 15.

If Θ and Θ' are additive sequences, we shall define a composed sequence $\Theta \circ \Theta'$ as follows. The n-th term $(\Theta \circ \Theta')_n$ of the sequence $\Theta \circ \Theta'$ shall be the virtual representation $\Theta(\theta_n')$. This latter element is defined in a manner completely analogous to the situation in Definition 3. We have then the following theorem.

Theorem (17): $\Psi^k \circ \Psi^l = \Psi^{kl}$.

Proof: As in Lemma 14, it will suffice to show that

$$tr\,(\Psi^k \circ \psi_n^l(u)) = tr\,(\psi_n^{kl}(u))$$

for all $u \in U(n)$. Using the formula established above,

$$tr\,(\Psi^k \circ \psi_n^l(u)) = tr\,(u^{kl}) = tr\,(\psi_n^{kl}(u)). \quad \text{Q.E.D.}$$

Let $a \in K(X)$ and let $ch_q(a)$ denote the $2q$-dimensional component of $ch\,(a)$. We have then the following

Lemma (15): $ch_q\,(\Psi^k a) = k^q \cdot ch_q\,(a)$.

Proof: First of all, let z be a complex number of modulus 1, regarded as a unitary matrix of degree 1. Then $tr\,(\psi_1^k(z)) = z^k$. It follows that if ζ is a $U(1)$-bundle, then

$$\Psi^k(\zeta) = \zeta^k.$$

Let $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ be a sum of $U(1)$-bundles. Then

$$ch\,(\xi) = e^{c_1(\xi_1)} + \cdots + e^{c_n(\xi_n)}.$$

Since $\Psi^k(\xi) = \Psi^k(\xi_1) + \cdots + \Psi^k(\xi_n)$, it follows that

$$ch\,(\Psi^k(\xi)) = ch\,\Psi^k(\xi_1)) + \cdots + ch\,(\Psi^k(\xi_n)) = e^{c_1(\xi_1^k)} + \cdots + e^{c_n(\xi_n^k)}.$$

Looking at the proof of part (b) of Theorem 10, we see that

$$c_1(\xi_i^k) = k \cdot c_1(\xi_i), \qquad 1 \leqq i \leqq n.$$

Therefore,

$$ch\,(\Psi^k(\xi)) = e^{k \cdot c_1(\xi_1)} + \cdots + e^{k \cdot c_1(\xi_n)}.$$

This, together with the formula for $ch\,(\xi)$, implies the result for ξ. The result for general bundles now follows from Lemma 5. Finally, the passage from bundles to elements of $K(X)$ offers no difficulties.

Let us observe now that the virtual representation ψ_n^k has degree n. This follows directly from

$$\sum_{i=1}^{n} x_i^k = Q_n^k\,(\sigma_1^{(n)}, \ldots, \sigma_n^{(n)})$$

upon making the substitution $x_1 = 1, x_2 = 1, \ldots, x_n = 1$.

If X is a space with basepoint x_0, then

$$K(X) \cong K(x_0) \oplus \tilde{K}(X).$$

It follows from the preceding remark that the cohomology operations Ψ^k respect this direct sum decomposition. In fact, $\tilde{K}(X)$ is characterized by the condition that its elements have dimension zero and Ψ^k acts like the identity on $K(x_0)$. (Just look at what Ψ^k does to the line bundle over x_0.) We have now the following

Theorem (18): The operation $\Psi^k : \tilde{K}(S^{2q}) \to \tilde{K}(S^{2q})$ is given by

$$\Psi^k(a) = k^q \cdot a.$$

Proof: Recall that $\tilde{K}(S^{2q}) = K^0(S^{2q})$. Since $H^*(S^{2q}; Z)$ is torsion-free, it follows from Theorem 13 that

$$ch : K^0(S^{2q}) \to \tilde{H}^e_Q(S^{2q})$$

is injective. It even follows that ch maps $K^0(S^{2q})$ isomorphically onto $\tilde{H}^e(S^{2q}; Z) \subseteq \tilde{H}^e_Q(S^{2q})$. But

$$\tilde{H}^e(S^{2q}; Z) = H^{2q}(S^{2q})$$

so that actually

$$ch_q : K^0(S^{2q}) \to H^{2q}_Q(S^{2q})$$

is an injective map, taking $K^0(S^{2q})$ isomorphically onto $H^{2q}(S^{2q}; Z) \subseteq H^{2q}_Q(S^{2q})$. Applying ch_q to $\Psi^k(a)$, we get, by Lemma 15,

$$ch_q(\Psi^k a) = k^q \cdot ch_q(a).$$

The theorem follows because of the injectivity of ch_q.

Corollary: Let $\beta : \tilde{K}(X) \to \tilde{K}(S^2X)$ be the Bott isomorphism. Then

$$\Psi^k \circ \beta = k \cdot (\beta \circ \Psi^k).$$

Proof: Let g be the generator of $\tilde{K}(S^2)$. By Theorem 18,

$$\Psi^k(g) = k \cdot g.$$

Therefore,

$$\Psi^k \circ \beta(a) = \Psi^k(a \cdot g) = \Psi^k(a) \cdot \Psi^k(g) = \Psi^k(a) \cdot kg = k \cdot \beta \circ \Psi^k(a)$$

or any $a \in \tilde{K}(X)$. This proves the corollary.

CHAPTER X

Vector Fields on Spheres

In this chapter, we shall outline the determination of the maximum number of linearly independent vector fields on the sphere S^{n-1}. We shall denote this integer by $M(n-1)$. For every integer $n \geqq 1$, we define an integer $\varrho(n)$ as follows: Write n as a product of a power of 2 and an odd number

$$n = (2a(n) + 1) \cdot 2^{b(n)}$$

and then divide $b(n)$ by 4 to obtain

$$b(n) = c(n) + 4d(n).$$

Thus, $a(n), b(n), c(n), d(n) \geqq 0$ and $0 \leqq c(n) \leqq 3$. Finally, set

$$\varrho(n) = 2^{c(n)} + 8d(n).$$

We shall then outline the proof of the following theorem.

Theorem: $M(n-1) = \varrho(n) - 1$.

A manifold M^n is called *parallelizable* if it admits n linearly independent vector fields. As an immediate application of the above theorem, we have the

Corollary: The only parallelizable spheres are S^1, S^3, S^7.

The inequality $M(n-1) \geqq \varrho(n) - 1$ is established in Part A of this chapter. This part of the proof involves no algebraic topology whatsoever; it is due to Hurwitz, Radon and Eckmann. See Eckmann—*Gruppentheoretischer Beweis des Satzes von Hurwitz-Radon über die Komposition quadratischer Formen*, Comment. Math. Helvet., Vol. 15, 1942. The inequality $M(n-1) \leqq \varrho(n) - 1$ is sketched in part B. The proof makes use of K-Theory, as well as some recent work of Atiyah and James; it is due to J.F.Adams. See Adams—*Vector Fields on Spheres*, Ann. of Meth., Vol. 75, 1962.

Part A: $M(n-1) \geqq \varrho(n) - 1$

The desired result follows from the following theorem. See Eckmann, loc. cit.

Theorem (1): Let G_p be the abstract group generated by p elements $\varepsilon, a_1, a_2, \ldots, a_{p-1}$ and subject to the defining relations $\varepsilon^2 = 1$, $a_j^2 = \varepsilon$ ($j = 1, \ldots, p-1$), $a_k a_l = \varepsilon a_l a_k$ ($k \neq l$; $1 \leqq k, l \leqq p-1$). Then if and only if $p \leqq \mathrm{p}(n)$ does there exist an orthogonal representation

$$\phi : G_p \to O_n$$

such that $\phi(\varepsilon) = -I$.

Corollary: $M(n-1) \geqq \varrho(n) - 1$.

Proof of Corollary: Let $\phi : G_{\varrho(n)} \to O_n$ have the properties described in Theorem 1. Put $A_i = \phi(a_i)$ $(1 \leqq i \leqq \varrho(n) - 1)$. Plainly, $A_k^2 = -I$, $A_k A_l = -A_l A_k$ ($k \neq l$). Now, to define $\varrho(n) - 1$ linearly independent vector fields on S^{n-1}, it suffices to define maps $V_i : S^{n-1} \to R^n$ $(1 \leqq i \leqq \varrho(n) - 1)$ such that

(a) $(V_i(x), x) = 0$ $\quad (1 \geqq i \geqq \varrho(n) - 1$; all $x \in S^{n-1})$.

(b) $(V_i(x), V_j(x)) = \delta_{ij}$ $\quad (1 \leqq i, j \leqq \varrho(n) - 1$; all $x \in S^{n-1})$.

We define $V_i(x) = A_i x$. A direct calculation, using the above properties of the A_i, shows that (a) and (b) are indeed satisfied.

Q.E.D.

In order to carry out the proof of Theorem 1, we shall quote a series of results on the theory of representations of finite groups. We remark that statements 1), 2), 3), 5) below generalize to arbitrary compact groups. For proofs, see Hall—*The Theory of Groups* and Pontryagin—*Topological Groups*.

By representation, we mean representation in $Gl(n, C)$ (or $Gl(n, R)$).

1) Every representation ϱ of G is equivalent (notation: $\sim$) to a direct sum of irreducible unitary representations. Moreover, the irreducible parts of ϱ are unique up to $\sim$.

2) Let ϱ_1, ϱ_2 be representations of G. Define $\chi_{\varrho_1} : G \to C$ by $\chi_{\varrho_1}(g) = tr(\varrho_1(g))$. Then $\varrho_1 \sim \varrho_2$ iff $\chi_{\varrho_1} = \chi_{\varrho_2}$. If, moreover, ϱ_1 and ϱ_2 are irreducible, then $\varrho_1 \sim \varrho_2$ iff $\Sigma_{g \in G}\, \chi_{\varrho_1}(g)\, \overline{\chi_{\varrho_2}(g)} \neq 0$.

3) Let $g \in G$ be an arbitrary element, different from 1. Then there exists an irreducible representations R_g of G such that $R_g(g) \neq I$.

4) The number of inequivalent irreducible representations of G is equal to the number of conjugacy classes in G.

5) If G is abelian, all the irreducible representations of G are one-dimensional.

6) $\Sigma_R \{\deg R\}^2 = \# (G)$, where $\# (G)$ is the order of G and where we sum over the set of all inequivalent irreducible representations of G.

7) For any irreducible representation R of G, $\deg R \mid \# (G)$.

8) (Schur's Criterion): Let R be an irreducible representation of G and let $\bar{R}$ be the representation of G defined by $\bar{R}(g) = \overline{R(g)}$ (i.e. the complex conjugate of the matrix $R(g)$). Then, (i) R is $\sim$ to a real representation of G iff $\Sigma_{g \in G} \chi_R(g^2) > 0$; (ii) $R \sim \bar{R}$ but R is *not* $\sim$ to a real representation of G iff $\Sigma_{g \in G} \chi_R(g^2) < 0$; (iii) R is not $\sim \bar{R}$ (in which case R cannot be $\sim$ to a real representation of G) iff $\Sigma_{g \in G} \chi_R(g^2) = 0$.

9) (Special case of Schur's Lemma): Let Ω be an irreducible set of $r \times r$ matrices and let B be an $r \times r$ matrix which commutes with each element of Ω. Then $B = \beta I$, where $\beta \in \mathcal{C}$.

We now proceed to the proof of Theorem 1: We have

$$G_p = (\varepsilon, a_1, \ldots, a_{p-1}; \varepsilon^2 = 1, a_j^2 = \varepsilon, a_k a_l = \varepsilon a_l a_k \, (k \neq l)).$$

Using the defining relations, we see immediately that every element in G can be written as either $a_{i_1} \cdots a_{i_s}$ or $\varepsilon a_{i_1} \cdots a_{i_s}$ $(1 \leqq i_1 < \cdots < i_s \leqq p-1)$. In particular, $\#(G) = 2^p$. Let H be the subgroup of G_p generated by ε. As ε lies in $\zeta(G_p)$, the center of G_p, H is normal. Clearly, G_p/H is a direct sum of $p-1$ copies of Z_2 and is thus abelian. Hence, $H \supseteq [G_p, G_p]$ and it follows that for all g, h in G, $ghg^{-1}h^{-1} = 1$ or ε, i.e. $ghg^{-1} = h$ or εh. Therefore, the conjugacy class of h consists of h and εh unless h lies in $\zeta(G_p)$.

We now find $\zeta(G_p)$. If $\{i_1, i_2, \ldots, i_s\} \neq \{1, 2, \ldots, p-1\}$, (i.e. if $s < p-1$; recall that $i_1, i_2, \ldots, i_s$ are arranged in ascending order), pick an index j not in $\{i_1, i_2, \ldots, i_s\}$. Then, $a_j^{-1}(a_{i_1} \cdots a_{i_s})\, a_j = \varepsilon^s a_{i_1} \cdots a_{i_s}$. Now pick j in $\{i_1, \ldots, i_s\}$. Then, $a_j^{-1}(a_{i_1} \cdots a_{i_s})\, a_j = \varepsilon^{s-1} a_{i_1} \cdots a_{i_s}$. Thus, since either ε^s or $\varepsilon^{s-1} \neq 1$, the element $a_{i_1} \cdots a_{i_s}$ is not in $\zeta(G_p)$ if $i_1, \ldots, i_s \neq \{1, \ldots, p-1\}$. Moreover, $a_j\,(a_1 a_2 \cdots a_{p-1})\, a_j^{-1} = \varepsilon^{p-2} a_1 a_2 \cdots a_{p-1}$. Hence, $a_1 a_2 \cdots a_{p-1}$ lies in $\zeta(G_p)$ iff p is even. Similarly, $\varepsilon a_{i_1} \cdots a_{i_s}$ lies in $\zeta(G_p)$ iff $\{i_1, \ldots, i_s\} = \{1, \ldots, p-1\}$ and p is even. We have then $\zeta(G_p) = \{1, \varepsilon\}$ if p odd, $\zeta(G_p) = \{1, \varepsilon, a_1 a_2 \cdots a_{p-1}, \varepsilon a_1 \cdots a_{p-1}\}$ if p even.

Now, the number of conjugacy classes in G_p is equal to the number of elements not in $\zeta(G_p)$ divided by 2, plus the number of elements in $\zeta(G_p)$. Using 4) above, we find:

(1–o) For odd p, the number of inequivalent irreducible representations of G_p is $\dfrac{2^p - 2}{2} + 2 = 2^{p-1} + 1$.

(1–e) For even p, this number is $\dfrac{2^p - 4}{2} + 4 = 2^{p-1} + 2$.

Using 5) above, we see that G_p/H has 2^{p-1} one-dimensional irreducible representations. Hence G_p has at least 2^{p-1} one-dimensional irreducible representations. In fact, if $R: G_p/H \to U_1$, then $R \circ \pi : G_p \to U_1$ (where $\pi : G_p \to G_p/H$ is the natural map) and R not $\sim R$ implies $R \circ \pi$ not $\sim R' \circ \pi$. Thus, there are at least 2^{p-1} irreducible representations $R_1, \ldots, R_{2^{p-1}}$ of G with $\varepsilon \to +I$. We now find, using (1–o) and (1–e):

(2–0) For odd p, there is exactly one irreducible R^* not included in the collection $R_1, \ldots, R_{2^{p-1}}$. Using 6) above, we have $2^{p-1} \cdot 1^2 + (\deg R^*)^2 = 2^p$; then $\deg R^* = 2^{(p-1)/2}$.

(2–e) For even p, there are exactly 2 irreducible R_1^*, R_2^* not included in the collection $R_1, \ldots, R_{2^{p-1}}$. Again by 6), $2^{p-1} \cdot 1^2 + (\deg R_1^*)^2 + (\deg R_2^*)^2 = 2^p$, i.e. $(\deg R_1^*)^2 + (\deg R_2^*)^2 = 2^{p-1}$. By 7), $\deg R_i^* = 2^{\alpha_i}$, $\alpha_i \leqq p$ $(i = 1, 2)$. In order for $(2^{\alpha_1})^2 + (2^{\alpha_2})^2 = 2^{p-1}$, we must have $\alpha_1 = \alpha_2$. It follows that $\deg R_1^* = \deg R_2^* = 2^{(p-2)/2}$.

By the irreducibility of R^*, R_1^*, R_2^* and by 1), 3), 9) above, we find

(3–0) R^* maps $\varepsilon \to \beta I$, where $\beta \in \mathcal{C}$, $|\beta| = 1$ and $\beta \neq 1$.

(3–e) Either R_1^* or R_2^* (or both) maps $\varepsilon \to \beta I$, where β has the same property as in (3–0). For definiteness, assume $R_1^*(\varepsilon) = \beta I$.

Recall that Theorem 1 asks for the existence of *real* orthogonal representations of G_p. Up to now, we may only assert, by virtue of 1) above, that R^*, R_1^* are complex unitary representations of G_p. But now, in general, if R is a complex irreducible representations of a finite group G and if R is not $\sim$ to a real representation, then $R \oplus \bar{R}$ is $\sim$ to a real irreducible representation. Thus, if necessary, we may replace R^*, R_1^* by $R^* \oplus \bar{R}^*$, $R_1^* \oplus \bar{R}_1^*$ respectively. Using 8) above, we shall now find the exact values of p for which R^* and R_1^* are $\sim$ to real representations. Let $g \in G_p$ be arbitrary; then $g = a_{i_1} \cdots a_{i_r}$ or $\varepsilon a_{i_1} \cdots a_{i_r}$ as we have already observed. In either case, $g^2 = \varepsilon^r \cdot \varepsilon^{r-1} \cdot \cdots \cdot \varepsilon = \varepsilon^{r(r+1)/2}$. Thus, $g^2 = 1$ if $r \equiv 0, 3 \pmod 4$ and $g^2 = \varepsilon$ if $r \equiv 1, 2 \pmod 4$. Now, let $R = R_1$ if p is odd, $R = R_1^*$ if p is even and let $d = \deg R^* = 2^{(p-1)/2}$ if p is odd, $d = \deg R_1^* = 2^{(p-2)/2}$ if p is even. Then $\chi_R(g^2) = d$ if $r \equiv 0, 3 \pmod 4$ and $\chi_R(g^2) = -d$ if $r \equiv 1, 2 \pmod 4$. Summing over G_p, we get

$$\Sigma_{g \in G_p} \chi_R(g^2) = 2d \left\{ \binom{p-1}{0} - \binom{p-1}{1} - \binom{p-1}{2} + \binom{p-1}{3} - \cdots \right\}. \tag{†}$$

Let $(1-i)^{p-1} = x + iy$. Expanding by the binomial theorem, we see that $x + y$ is equal to the expression in the brackets on the right hand side of (†). It now remains to determine the sign of $x + y$ for the various values of p. An easy check, (using the Argand Diagram), shows that $x + y > 0$ if $p \equiv 0, 1, 7 \pmod 8$, $x + y = 0$ if $p \equiv 2, 6 \pmod 8$, $x + y < 0$ if $p \equiv 3, 4, 5 \pmod 8$.

Put $\hat{R} = R$ if p is such that $x + y > 0$ and put $\hat{R} = R \oplus \bar{R}$ otherwise. We define $k(p)$ by $\deg \hat{R} = 2^{k(p)}$. Then it follows from (2–o), (2–e) and the results of the last paragraph that

$$k(p) = \begin{cases} \dfrac{p-2}{2} & \text{if } p \equiv 0 \pmod 8 \\[2ex] \dfrac{p-1}{2} & \text{if } p \equiv 1,7 \pmod 8 \\[2ex] \dfrac{p}{2} & \text{if } p \equiv 2, 4, 6 \pmod 8 \\[2ex] \dfrac{p+1}{2} & \text{if } p \equiv 3, 5 \pmod 8). \end{cases}$$

It follows from (3–0), (3–e), 8) that $\hat{R}$ is (up to $\sim$) a real orthogonal representation of G taking $\varepsilon \to -I$ (i.e. $\beta = -1$).

Now let $n \geqq 1$ be given. Using 1), (2–0), (2–e), we see that there exists a real orthogonal (not necessarily irreducible) representation $\phi : G_p \to O_n$ iff $2^{k(p)} | n$. Writing $n = (2a(n) + 1) \cdot 2^{b(n)}$, we see that $2^{k(p)} | 2^{b(n)}$. Theorem 1 will be proved if we can show that the largest value of p satisfying $k(p) \leqq b(n)$ is just $\varrho(n)$. Having calculated $k(p)$ in the last paragraph, we have thus reduced Theorem 1 to a purely number-theoretic verification. Substituting for $k(p)$, we see that we must find the largest p for which

$$p \leqq \begin{cases} 2b(n) + 2 & \text{if} \quad p \equiv 0 \pmod 8) \\ 2b(n) + 1 & \text{if} \quad p \equiv 1, 7 \pmod 8) \\ 2b(n) & \text{if} \quad p \equiv 2, 4, 6 \pmod 8) \\ 2b(n) - 1 & \text{if} \quad p \equiv 3, 5 \pmod 8). \end{cases}$$

From this calculation, it is clear that the maximum p is of the form $2b(n) + \delta$, where δ depends only on the congruence class of $b(n)$ mod 4. We calculate δ explicitly as follows: Consider the table

Value of $b(n)$	Value of $2b(n) + \delta$			
0	2	①	0	-1
1	4	3	②	1
2	6	5	④	3
3	⑧	7	6	5
	$p \equiv 0$ (mod 8)	$p \equiv 1, 7$ (mod 8)	$p \equiv 2, 4, 6$ (mod 8)	$p \equiv 3, 5$ (mod 8)

In each of the four rows of the table, we have encircled the largest number consistent with the restriction on p on the bottom. Thus, in the first row, 2 is greater than 1, but $2 \not\equiv 0 \pmod 8$ while $1 \equiv 1 \pmod 8$. We have thus found that the maximum p is of the form

$$p = \begin{cases} 2b(n) + 1 & \text{if } b(n) \equiv 0 \pmod 4 \\ 2b(n) + 2 & \text{if } b(n) \equiv 1 \pmod 4 \\ 2b(n) + 4 & \text{if } b(n) \equiv 2 \pmod 4 \\ 2b(n) + 8 & \text{if } b(n) \equiv 3 \pmod 4. \end{cases}$$

Finally, to see that these values give $\varrho(n)$, write $b(n) = c(n) + 4d(n)$ and note that by definition, $c(n) = i$ iff $b(n) \equiv i \pmod 4$, $0 \leqq i \leqq 3$.

Part B: $M(n-1) \leqq \varrho(n) - 1$

Lack of time prevents us from entering into the details of the proof. We shall therefore content ourselves with giving an outline of the main ideas of the proof, referring the reader to Adams' very lucid paper (hereafter denoted by (*VF*)) for most of the details.

The starting point for Adams' investigation is the following theorem.

Theorem (2): Let $n \geqq 1$. If there exist $\varrho(n)$ linearly independent vector fields on S^{n-1}, then there exists an integer $m \geqq 1$ with $\varrho(m) = \varrho(n)$ such that the truncated real projective space $P^R_{m+\varrho(m)}/P^R_{m-1}$ has the following property: there exists a map

$$f\colon P^R_{m+\varrho(m)}/P^R_{m-1} \to S^m$$

such that the composite map

$$S^m = P^R_m/P^R_{m-1} \xrightarrow{i} P^R_{m+\varrho(m)}/P^R_{m-1} \xrightarrow{f} S^m$$

is of degree 1.

Note: For any integers $p > q$, there is a natural embedding of P^R_q into P^R_p. P^R_p/P^R_q is therefore a well-defined object. The mapping i in the above theorem is of course induced by the mapping of pairs

$$(P^R_m, P^R_{m-1}) \to (P^R_{m+\varrho(m)}, P^R_{m-1}).$$

Adams proves Theorem 2 by making extensive use of James' work on the homotopy of Stiefel manifolds and Atiyah's work on the so-called Thom complexes. The ideas used in proving this theorem require the introduction of a whole collection of concepts, including the Spanier-Whitehead S-Theory. We will not discuss these concepts here, referring to the papers of James and Atiyah cited in (*VF*).

By virtue of Theorem 2, the inequality $M(n-1) \leqq \varrho(n) - 1$ is reduced to the following theorem.

Theorem (3): There does not exist a map

$$f\colon P^R_{m+\varrho(m)}/P^R_{m-1} \to S^m$$

satisfying the condition mentioned in Theorem 2.

Thus, the problem of proving the nonexistence of a certain number of vector fields is reduced to the problem of proving the nonexistence of a certain map. There is a familiar method in algebraic topology of attacking such a problem. Namely, if $g: X \to Y$ is a hypothetical map whose existence we are trying to disprove, then g induces a map $g^*: H^*(Y) \to H^*(X)$ in cohomology. If g is assumed to posses some topological property which we wish to show is impossible, we attempt to deduce that g^* has some simple algebraic property incompatible with the properties of the rings $H^*(X)$, $H^*(Y)$, thus contradicting the existence of g^*, hence of g. (For simple examples illustrating this method, see Hu—*Homotopy Theory*, Chapter 1.) Even if the ring structure of the cohomology rings is insufficient to contradict the existence of g^*, it may be possible to contradict the existence of g^* by introducing cohomology operations into the cohomology rings of spaces. For an account of this theory, together with several applications (including a partial solution to the vector field problem), we refer the reader to Epstein and Steenrod—*Cohomology Operations.*

In (*VF*), Adams uses K-theory instead of ordinary cohomology theory in order to contradict the existence of f (see Theorem 3). That is, he considers the induced map

$$f^* : K_{\mathbf{R}}(S^m) \to K_{\mathbf{R}}\,(P^{\mathbf{R}}_{m+\varrho(m)}/P^{\mathbf{R}}_{m-1}).$$

Here, $K_{\mathbf{R}}$ is the functor, analogous to K of Chapter 9, but with real vector bundles in place of complex vector bundles. (We should then write $K = K_{\mathbf{C}}$.) Using the notation of Theorem 2, we see that $f \circ i \simeq$ ident. (the topological property of f). It follows that $i^* \circ f^* =$ ident. (the algebraic property of f^*). Thus, $K_{\mathbf{R}}\,(P^{\mathbf{R}}_{m+\varrho(m)}/P^{\mathbf{R}}_{m-1})$ splits as a direct sum $\ker i^* \oplus \operatorname{im} f^*$. Although the ring structure of $K_{\mathbf{R}}\,(P^{\mathbf{R}}_{m+\varrho(m)}/P^{\mathbf{R}}_{m-1})$ does not rule out the possibility of such a splitting, it turns out that the operations Ψ^k discussed in Chapter 9 do in fact disallow such a splitting.

The work to be done therefore consists of explicit calculations of the rings $K_{\mathbf{R}}\,(P^{\mathbf{R}}_p/P^{\mathbf{R}}_q)\,(p > q)$ and of the operations $\Psi^k : K_{\mathbf{R}}\,(P^{\mathbf{R}}_p/P^{\mathbf{R}}_q) \to K_{\mathbf{R}}\,(P^{\mathbf{R}}_p/P^{\mathbf{R}}_q)$. Actually, Adams calculates the rings $K_\Lambda(X)$ in the following order:

a) $\Lambda = \mathbf{C}, \quad X = P^{\mathbf{C}}_p/P^{\mathbf{C}}_q$

b) $\Lambda = \mathbf{C}, \quad X = P^{\mathbf{R}}_p/P^{\mathbf{R}}_q$

c) $\Lambda = \mathbf{R}, \quad X = P^{\mathbf{R}}_p/P^{\mathbf{R}}_q.$

Since the spaces P^Λ_p/P^Λ_q are quite simple cell complexes, the necessary calculations, using the methods of Chapter 9, are very manageable. (In particular, the ordinary cohomology groups of these spaces can be calculated trivially, using the method of Chapter 2.)

Simply for the purpose of illustrating the results of Chapter 9, we shall close this chapter by carrying out step a) above. Before stating the result, we introduce a bit of notation. First of all, we allow ourselves to write K, P_p instead of $K_{\mathbf{C}}$, $P^{\mathbf{C}}_p$ in the next few paragraphs. Let η be the canonical line bundle over P_p, i.e. over each projective class $[z_1, z_2, \ldots, z_{p+1}] \in P_p$, the fiber of η is the complex line consisting of all $(p+1)$-tuples $(\lambda z_1, \lambda z_2, \ldots, \lambda z_{p+1})$, $\lambda \in \mathbf{C}$: (cf. Chapter 9, Theorem 2). If 1 is the trivial line bundle over P_p, we put $\mu = \eta - 1$. Then $[\mu] \in \tilde{K}(P_p)$.

We now observe that the Chern class $c_1(\eta)$ is a generator y of the cohomology group $H^2(P_p; \mathbf{Z})$. In fact, the standard inclusion

$$i : P_p \to P_\infty$$

serves as a classifying map for the bundle η and therefore, if $c_1 \in H^2(P_\infty)$ is the universal Chern class,

$$c_1(\eta) = i^*(c_1),$$

from which our assertion follows immediately.

We now prove the following result.

Theorem (4): The ring $K(P_p)$ is a truncated integral polynomial ring with a single generator $[\mu]$ and a single defining relation $[\mu]^{p+1} = 0$. The natural projection $P_p \to P_p/P_q$ sends $\tilde{K}(P_p/P_q)$ isomorphically onto the ideal of $K(P_p)$ generated by $[\mu]^{q+1}$. Finally, the cohomology operations Ψ^k satisfy the relation $\Psi^k([\mu]^s) = ((1 + [\mu])^k - 1)^s$.

Proof: The additive structure of $K(P_p)$ is easily deduced from Chapter 9, Theorem 13 and corollary to Theorem 14. In fact, since $H^*(P_p; Z)$ is torsion-free, it follows that (with the notation of Chapter 9) $K^r(P_p) \cong Z^p$ if r is even, $\cong 0$ if r is odd (where, in general, Z^f means direct sum of f copies of Z). In particular, $\tilde{K}(P_p) = K^0(P_p) \cong Z^p$ and therefore

$$K(P_p) \cong \tilde{K}(P_p) \oplus Z \cong Z^{p+1}.$$

We claim that the elements $1, [\mu], [\mu]^2, \ldots, [\mu]^p$ actually form a set of group generators for $K(P_p)$ (from which it follows that $[\mu]$ is a ring generator for $K(P_p)$). We shall show that the elements $1, [\mu], [\mu]^2, \ldots, [\mu]^p$ form a Q-basis for the Q-linear space $K(P_p) \otimes Q$. To see this, we observe first that

$$ch(\eta) = e^{c_1(\eta)},$$

and since $c_1(\eta)$ has been identified with a generator y of

$$H^*(P_p; Z) \cong Z[y]/(y^{p+1}),$$

$$ch(\mu) = ch(\eta) - 1 = e^y - 1 = y + \frac{y^2}{2!} + \cdots. \qquad (*)$$

To prove the Q-basis property of the powers of $[\mu]$, is clearly enough to show that the elements $1, [\mu], [\mu]^2, \ldots, [\mu]^p$ are Q-linearly independent. Suppose then that we have a linear relation

$$a_0 \cdot 1 + a_1 \cdot [\mu] + \cdots + a_p \cdot [\mu]^p = 0.$$

Applying the Chern character to this equation, we get

$$a_0 + a_1\left(y + \frac{y^2}{2!} + \cdots\right) + a_2\left(y + \frac{y^2}{2!} + \cdots\right)^2 + \cdots = 0.$$

But it follows from the known properties of y that then

$$a_0 = a_1 = \cdots = a_p = 0,$$

which is what we wanted. We omit the additional calculations needed to show that $\{1, [\mu], [\mu]^2, \ldots, [\mu]^p\}$ actually forms a "Z-basis" of $K(P_p)$) (cf. Atiyah and Hirzebruch, *Vector Bundles and Homogeneous Spaces*, p. 19, Corrollary 3, (iii)).

Since $y^{p+1} = 0$, it follows easily that

$$ch(\mu^{p+1}) = 0.$$

But ch is injective (see Chapter 9, Theorem 13) whence $\mu^{p+1} = 0$. Thus, we have established the first assertion of Theorem 4. To obtain the second assertion, we consider the exact sequence of the pair (P_p, P_q):

$$\cdots \longrightarrow K^{-1}(P_q) \xrightarrow{\delta} K^0(P_p, P_q) \xrightarrow{j^*} K^0(P_p) \xrightarrow{i^*} K^0(P_q)$$

(Here, i^* and j^* are induced by the standard inclusions.) We have observed above that $K^r(P_q) = 0$ if r is odd. Hence, j^* embeds $\tilde{K}(P_p/P_q) = K^0(P_p, P_q)$ into $K^0(P_p)$. By exactness, $\operatorname{im} j^* = \ker i^*$, and $\ker i^*$ is clearly generated by $[\mu]^{q+1}, \ldots, [\mu]^p$. This establishes the second assertion.

Finally, we calculate the operations Ψ^k in $K(P_p)$. Since η is a line bundle, we have

$$\Psi^k(\eta) = \eta^k$$

(see the first paragraph of the proof of Lemma 15, Chapter 9). By the definition of $[\mu]$,

$$\Psi^k(1 + [\mu]) = (1 + [\mu])^k.$$

Since Ψ^k is a group homomorphism,

$$\Psi^k([\mu]) = (1 + [\mu])^k - 1.$$

Finally, since Ψ^k is a ring homomorphism,

$$\Psi^k([\mu]^s) = ((1 + [\mu])^k - 1)^s.$$

Q.E.D.

The calculations corresponding to b) and c) above may be carried by similar but technically slightly move complicated methods. Finally, as indicated above, the structural properties of these rings can be shown to preclude the existence of the map f of Theorem 3.

Subject Index