PLASMA ELECTRODYNAMICS

VOLUME 1 LINEAR THEORY

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Preface

THE properties of plasmas as a specific state of matter are to an important extent determined by the fact that there are between the particles which constitute the plasma electromagnetic forces which act over macroscopic distances. Processes occurring in a plasma are therefore as a rule accompanied by the excitation of electromagnetic fields which play a fundamental role in the way these processes develop.

The electromagnetic interactions which extend over macroscopic distances show up first of all in the occurrence in the plasma of collective oscillations in which a large number of particles takes part simultaneously. The existence of these specific collective electromagnetic oscillations is just as much a characteristic of a plasma as a specific state of matter as, for instance, the crystalline ordering is for the solid state of matter.

This explains the place occupied in plasma physics by plasma electrodynamics, that is, the theory of electromagnetic fields in a plasma—and, in the first instance, the theory of electromagnetic oscillations of a plasma—and the theory of macroscopic electrical and magnetic properties of a plasma.

Such problems as the theory of magnetic traps, the problem of plasma heating by external fields or currents, and the theory of instabilities in a non-uniform plasma belong also to the field of plasma electrodynamics in its widest sense. We shall not consider these problems in the present book—not because they are not important; to the contrary, they are of great importance. We restrict ourselves to an exposition of the theory of the electromagnetic properties of a uniform plasma as this theory is the basis of the whole of plasma electrodynamics.

Although there are several monographs (see, for example, Alfvén, 1950; Artsimovich, 1963; Akhiezer, Akhiezer, Polovin, Sitenko, and Stepanov, 1967; Vedenov, 1965; Ginzburg, 1970; Cowling, 1957; Kulikovskiĭ and Lyubimov, 1962; Leontovich, 1965, 1966, 1967, 1968, 1970; Silin and Rukhadze, 1961; Spitzer, 1956; Stix, 1962; Tsytovich, 1970) devoted to the problems of plasma electrodynamics, we decided all the same to write yet another book on this topic having in mind to give the theory of both low- and high-frequency oscillations—without restricting ourselves to small amplitude oscillations only—from a unified point of view and to give the basic fundamental applications of this theory.

The book starts with an exposition of the general methods of describing a plasma. Chapter 1 is devoted to this problem; in this chapter we construct the BBGKY-hierarchy of kinetic equations, introduce the self-consistent field, and introduce the Vlasov kinetic equation and the Landau collision integral. We give an account of Boltzmann's H-theorem as applied to a plasma and study the problem of the relaxation of a plasma. Finally, in that chapter we elucidate the transition from a kinetic to a hydrodynamic description of a plasma and derive the equations of magneto-hydrodynamics.

The methods for describing a plasma which we have discussed allow us then to start a detailed study of both low- and high-frequency plasma oscillations.

We start with the theory of low-frequency oscillations in the case of frequent collisions when the concise hydrodynamic description of a plasma suffices. Chapters 2 and 3 are devoted to the low-frequency oscillations.

We give in Chapter 2 the linear theory of magneto-hydrodynamic waves. We define there phase velocities, damping, and polarization of different waves and study the conic refraction of magneto-hydrodynamic waves and the excitation of these waves, as well as the problem of the formation of lacunae when two-dimensional excitations propagate from a point source. Finally, we study the characteristics of magneto-hydrodynamic flow.

After that we turn to non-linear magneto-hydrodynamic waves, both simple waves and shock waves (Chapter 3). Here we study the distortion of the profile of a simple wave leading to the formation of discontinuities. We integrate the equations for simple waves and, in particular, we evaluate the Riemann invariants.

Then follows an exposition of the theory of shock waves. We prove the Zemplén theorem and study simple and shock waves in relativistic magneto-hydrodynamics. We put the problem of the evolutionarity and structure of shock waves. Finally, we solve the problem of the formation and splitting-up of an arbitrary discontinuity in magneto-hydrodynamics.

Having studied magneto-hydrodynamic waves for the case of frequent collisions we turn to a consideration of another limiting case—oscillations in a collisionless plasma. Chapters 4 and 5 deal with this problem. In the first of these chapters we give the theory of oscillations in an unmagnetized plasma, and in the second one the theory of oscillations in a collisionless plasma in an external magnetic field.

Chapter 4 starts with an exposition of the theory of oscillations in a collisionless plasma in the hydrodynamic approximation and then these oscillations are studied using a kinetic equation. The spectra of both the high- and the low-frequency oscillations (Langmuir waves and ion-sound waves) are studied in detail. We consider the collisionless (Landau) damping of the oscillations and we solve the problem of the anomalous skin effect.

In Chapter 5 we study in detail the spectra and damping of oscillations in a collisionless magneto-active plasma. At the start of the chapter we consider oscillations in a "cold" magneto-active plasma. Then we determine the dielectric tensor of a magneto-active plasma, using a kinetic equation, and we introduce a dispersion relation for electromagnetic waves, taking spatial dispersion, caused by the thermal motion of the electrons and ions in the plasma, into account. We find the frequencies and damping rates (Cherenkov and cyclotron damping) of practically all branches of the oscillations which can propagate in a magneto-active plasma with a Maxwell particle velocity distribution — the ordinary, fast and slow extra-ordinary, fast magneto-sound, and Alfvén waves, fast and slow ion-sound oscillations, electron-sound oscillations in a non-isothermal plasma, and different branches of electron and ion cyclotron waves.

Having studied the oscillation spectra in an equilibrium plasma, we turn to the study of oscillations in a non-equilibrium, uniform plasma (Chapter 6).

First of all we study the interaction of a beam of charged particles with the oscillations of an unmagnetized plasma and show that the plasma-beam system is unstable, that is, that the interaction between the beam particles and the plasma oscillations leads to an exponential growth in time of a small initial perturbation. We then find the growth rates for different kinds of oscillations, consider the problem of the stability of a plasma in an electric field, and study the excitation of non-potential (electromagnetic) waves in a plasma with anisotropic particle velocity distributions. We study the interaction of charged particle currents with slow waves in a magneto-active plasma (the particles in the currents are characterized either by isotropic or by anisotropic distribution functions). Finally, we consider the excitation of electromagnetic waves in a plasma by currents of relativistic particles.

Having studied the interaction of charged particle currents with the plasma, we elucidate the general criteria of the stability of different particle distributions in a plasma. We consider separately an unmagnetized plasma and a plasma in an external magnetic field. We solve the problem of the two-beam instability.

Concluding Chapter 6 we study the general problem of the nature of the instability, give a definition of absolute and convective instabilities, and establish criteria for those two kinds of instability. We also establish criteria for the amplification and blocking of waves and, finally, consider the global instability caused by the reflection of waves from the system boundaries.

The problem of the interaction between charged particle currents and the plasma is related to the problem of the behaviour of a partially ionized plasma in an external electric field. As the stationary states of such a plasma are characterized by a directed motion of the electrons relative to the ions there can arise in such a plasma an instability analogous to the beam instability of a collisionless plasma.

Having studied the interaction of charged particle currents with the plasma we consider the oscillations of a partially ionized plasma in an external electric field (both with and without an external magnetic field). This problem is treated in Chapter 7. We derive there the kinetic equation describing the electron component of a partially ionized plasma in external electric and magnetic fields and we determine the stationary electron distribution function in such a plasma (Druyvesteyn–Davydov distribution). We then study high-frequency (transverse electromagnetic and Langmuir), ion-sound, and magneto-sound oscillations and show that ion-sound and magneto-sound oscillations in an external electric field turn out to be growing oscillations. Finally, we study in Chapter 7 the peculiar oscillations of an partially ionized plasma—the ionization-recombination oscillations in which not only the charged particle density, but also their total number changes.

In Chapter 8 we turn again to the study of a completely ionized plasma. In this chapter we study the non-linear oscillations in such a plasma (in contrast to Chapters 4 to 6 in which we restricted our considerations to oscillations with a small amplitude). We discuss here non-linear high-frequency waves in a cold plasma, Langmuir waves in a non-relativistic plasma, and longitudinal, transverse, and coupled longitudinal-transverse waves in a relativistic plasma. We study non-linear waves in a plasma in which the average electron energy appreciably exceeds the average ion energy (ion-sound and magneto-sound waves of finite amplitude) and consider both simple (Riemann) waves and waves with a stationary profile (periodic, isolated, and quasi-shock waves with an oscillatory structure). We show that the nature of simple and stationary waves depends greatly on the electron velocity distribution. Finally, we study in Chapter 8 non-linear low-frequency waves in a cold magneto-active plasma.

Chapter 9 is devoted to a study of oscillations in the quasi-linear approximation in which the simplest non-linear effect is taken into account—the influence of oscillations on resonance particles. We first consider the interaction between resonance particles with longitudinal oscillations of an unmagnetized plasma and we give the derivation of the basic equation of a quasi-linear theory—the particle diffusion equation in velocity space. We then consider the quasi-linear relaxation process which leads to the formation of a plateau on the distribution function of resonance particles and study the quasi-linear wave transformation. In the same chapter we consider the quasi-linear theory of the interaction between resonance particles and the oscillations of a magneto-active plasma and study the problem of the quasi-linear relaxation of wave packets for the cases of cyclotron and Cherenkov resonance. Finally, we consider the influence of collisions on the quasi-linear relaxation process and on the damping of oscillations.

Having expounded the quasi-linear theory, which describes the effects of the first approximation in terms of the plasma wave energy, we turn to a study of the processes of higher order in the energy of the oscillations: the interaction between waves and waves and the non-linear interaction of waves and particles. This is the subject of Chapter 10 in which we obtain a kinetic equation for waves which takes into account three-wave processes and the non-linear interaction between waves and particles (sometimes called the non-linear Landau damping). We then study turbulent processes in which Langmuir waves take part: their interaction with ion sound and the decay instability and non-linear damping of Langmuir waves, and we study in detail ion-sound turbulence which occurs in a plasma with a directed motion of electrons relative to ions. Finally, we consider the interaction between Alfvén and magneto-sound waves in a magneto-active plasma.

The last three chapters of the book deal with the theory of fluctuations and of the wave and particle-scattering processes in a plasma caused by the fluctuations.

We give in Chapter 11 the theory of electromagnetic fluctuations in a plasma. We start with the derivation of the general fluctuation-dissipation relation which establishes a connection between the spectral distribution of the fluctuations and the energy dissipation in the medium; we use this relation to determine the fluctuations first in an equilibrium and then in a two-temperature plasma, both for an unmagnetized plasma and for a plasma in a magnetic field.

We then develop the theory of fluctuations in a non-equilibrium plasma and a kinetic theory of fluctuations; we find the fluctuations in the particle-distribution function and consider the critical fluctuations near the instability limits of the plasma and study the fluctuations in the plasma–beam system. We elucidate how one can proceed to a hydrodynamic theory of fluctuations and, finally, we study fluctuations in a partially ionized plasma in an electric field.

Chapter 12 is devoted to the theory of scattering processes and the transformation of waves in a plasma. We study here the scattering of electromagnetic waves in an unmagnetized plasma and determine the spectral distribution of the scattered radiation. We consider critical opalescence connected with the scattering of waves in a plasma near the limits of instability, and we study the transformation of transverse and longitudinal waves in an unmagnetized plasma and also the spontaneous emission in an non-equilibrium plasma. We give the theory of incoherent reflection of electromagnetic waves from a plasma. We study scattering and transformation of waves in a magnetoactive plasma, in a partially ionized plasma in an external electric field, and in a turbulent plasma. Finally, we discuss echo effects in a plasma; these are connected with undamped oscillations of particle distribution functions in a plasma.

In Chapter 13 we study the scattering of charged particles in a plasma. We determine here the polarization energy losses when charged particles move in a plasma; we find the energy losses caused by the fluctuations of the field in the plasma, and we determine the coefficients of dynamic friction in diffusion. We also study the propagation of charged particles through a magneto-active plasma and the interaction of charged particles with a non-equilibrium plasma, as well as the scattering of particles by critical fluctuations and the interaction between particles and a turbulent plasma.

We are well aware that the problems considered by us do not cover the complete theory, even of a uniform plasma, and that we have not given equal weight to the different problems. However, this is apparently unavoidable when writing a relatively large book. A very apt quotation comes from one of the best books on elementary particle theory (Bernstein, 1968): "No doubt another physicist writing the same book would have emphasized different aspects of the subject or would have treated the same aspects differently. One of the few pleasures in writing such a book is that the author can present the subject as he would like to see it presented ... and if this encourages someone else to write a better book, then the present author will be among its most enthusiastic readers."

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Preface to the English Edition

Plasma Electrodynamics, which is here brought before the English-speaking public, is devoted to the theory of collective oscillations in a plasma—strictly speaking, of a uniform plasma.

We have endeavoured to collect here the most important aspects of the theory of plasma oscillations and decided therefore to present not only the theory of oscillations in a collisionless plasma, but also the theory of magneto-hydrodynamic waves. Of course, our considerations include both linear oscillations and large amplitude oscillations.

The book is an expanded and extended version of our booklet *Collective Oscillations in a Plasma*, the English edition of which appeared in 1967.

Although we restricted our considerations solely to a uniform plasma, nevertheless the material referring to the electromagnetic properties of such a plasma is so extensive that we considered it appropriate to split the English edition into two volumes. The first volume contains the theory of magneto-hydrodynamical waves and the theory of linear oscillations of a collisionless plasma. The second volume contains the theory of non-linear waves in a collisionless plasma, including the quasi-linear theory, the theory of plasma turbulence, and the theory of electromagnetic fluctuations in a plasma.

The publication of an English edition of our book would have been impossible without the active participation of D. ter Haar: not only was it his initiative which led to the publication in England, but he also undertook the translation of the book, which—as far as we can judge with our knowledge of the English language—is excellent. Both for this and for his many useful comments we want to thank Professor ter Haar most sincerely.

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CHAPTER 1

Kinetic and Hydrodynamic Methods of Describing a Plasma

1.1. Kinetic Equations Hierarchy

1.1.1. SCREENED COULOMB INTERACTION AND THE EXISTENCE OF PLASMA OSCILLATIONS

A completely or partially ionized gas, which is, however, electrically neutral on average is called a plasma. The electromagnetic interaction between the particles in a plasma is therefore of the utmost importance for the plasma. If the plasma is totally ionized—and we shall in what follows be mainly concerned with such a plasma—the interaction between its particles is basically only electromagnetic; if the plasma is, moreover, non-relativistic, we may assume this interaction to be purely electrostatic.

The electrostatic interaction between any two plasma particles is, however, not described by the usual Coulomb law, since the other particles always screen the Coulomb interaction between the two particles considered. Due to the complicated motion of the particles in the plasma this screening has a complicated, dynamic character. However, one can obtain some ideas about it even when starting from a static picture.

In this very simple picture we consider the static electron–ion cloud in the vicinity of an arbitrary particle in the plasma and determine the electrostatic potential φ inside the cloud. It clearly satisfies the Poisson equation

$$\nabla^2\varphi = -4\pi \sum_i e_i n_i ,$$

where n_i is the density of particles of kind i (with charge e_i) in the cloud. If the plasma is in a state of thermodynamic equilibrium, the n_i will be determined by the Boltzmann formula

$$n_i = n_{i0} \exp\left(-e_i\varphi/T\right),$$

where T is the plasma temperature (in energy units) while n_{i0} is the density of particles of kind i far from the point considered, where $\varphi = 0$. These densities must satisfy the neutrality condition

$$\sum_i e_i n_{i0} = 0.$$

Substituting the expression for n_i into the Poisson equation and assuming that

$$e_i\varphi/T \ll 1,$$

we obtain the following equation to determine φ:

$$\nabla^2\varphi - \frac{\varphi}{r_D^2} = 0, \qquad (1.1.1.1)$$

where r_D is the so-called Debye radius,

$$r_D = \sqrt{\left(\frac{T}{4\pi \sum_i n_{i0}e_i^2}\right)}. \qquad (1.1.1.2)$$

We are interested in the spherically symmetric solution of eqn. (1.1.1.1):

$$\varphi \propto \frac{\exp(-r/r_D)}{r},$$

which corresponds to the following interaction energy between two charges e_1 and e_2 at a distance apart r_{12}:

$$U_{12} = \frac{e_1 e_2}{r_{12}} \exp(-r_{12}/r_D). \qquad (1.1.1.3)$$

We see that due to the screening the expression for the usual Coulomb interaction potential e_1e_2/r_{12} is multiplied by the screening factor $\exp(-r_{12}/r_D)$.

As we have mentioned already, eqn. (1.1.1.3) describes the screening in a plasma in thermodynamic equilibrium. It is valid provided [see formula (1.2.1.2)]

$$n_{i0} \ll \left(\frac{T}{e_i^2}\right)^3. \qquad (1.1.1.4)$$

In reality the screening is dynamic rather than static in character so that eqn. (1.1.1.3) describes only the average value of the interaction energy or potential in the plasma.

Because of the dynamic nature of the screening the potential φ in the cloud will necessarily be a function of time; it will be an oscillating function. This can be explained by the fact that the moving plasma particles when screening "overrun" their respective equilibrium positions rather than "come to a stop". The dynamic screening is thus closely connected with the possibility that there appear collective oscillations in a plasma.

To elucidate this very important fact we shall assume that only the electrons in the plasma are moving in the screening process; we consider a plane-parallel plasma layer of thickness d (see Fig. 1.1.1). Let the layer of electrons be displaced over a small distance x. There will then arise in the plasma an electric field E,

$$E = 4\pi enx,$$

where n is the electron density. Under the action of this field the electrons will start to move and we have the following equation of motion for the quantity x:

$$m_e\ddot{x} = -eE = -4\pi e^2 nx,$$

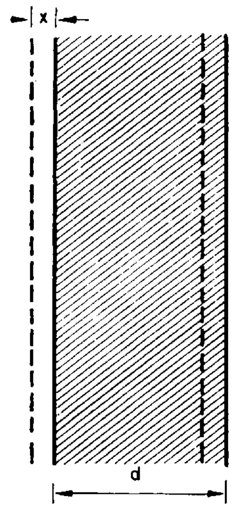

FIG. 1.1.1. The appearance of oscillations in a plasma; d: thickness of the plasma layer, x: displacement of the layer.

where m_e is the electron mass and $-e$ the electron charge. We obtained here the equation of a harmonic oscillator with frequency

$$\omega_p = \sqrt{\left(\frac{4\pi e^2 n}{m_e}\right)}. \tag{1.1.1.5}$$

We see thus that the appearance of oscillations is, indeed, possible in a plasma and these oscillations are connected with the effect of the screening of the interaction. Equation (1.1.1.5) determines the frequency of the simplest plasma oscillations in which only the electrons take part and in which their thermal motion is neglected; these oscillations are called *Langmuir* oscillations.

In the general case the dynamic screening in the plasma is determined by its dielectric constant tensor and the eigen oscillations of the plasma are also determined by the same tensor. This connection between the dynamic screening and the occurrence of eigen oscillations can be illustrated by means of the well-known formula for the dielectric constant of a plasma in the high-frequency region,

$$\varepsilon = 1 - \frac{4\pi n e^2}{m_e \omega^2}.$$

From the condition that the normal component of the electric induction vector must be continuous one concludes easily that the eigen oscillations of the plasma correspond to a vanishing of this component at the boundary of the plasma, that is, to a vanishing of the dielectric constant. However, the zero of ε is just the Langmuir frequency ω_p, determined by eqn. (1.1.1.5).

The present book is devoted to a study of the various collective oscillations of either an unmagnetized plasma or of a plasma in an external electromagnetic field, as well as of different processes connected with these oscillations. Before we start this study we must, however, expound the basic methods for describing a plasma.

1.1.2. MANY-PARTICLE DISTRIBUTION FUNCTIONS AND CORRELATION FUNCTIONS

A plasma is a many-particle system and for its description it is thus natural to use methods from statistical physics. To do this we introduce the *phase space*, spanned by the coordinates and momenta of all particles in the plasma. To simplify the notation we shall denote the set of Cartesian coordinates and momentum components of the ith particle simply by $i : i \equiv \boldsymbol{r}^{(i)}, \boldsymbol{p}^{(i)}$, where $\boldsymbol{r}^{(i)}$ is the radius vector of the ith particle and $\boldsymbol{p}^{(i)}$ its momentum.

We shall further consider the probability density for phase points at time t, $\mathcal{D}(1, 2, \ldots, N; t)$. Its physical meaning lies in the fact that the quantity

$$dw = \mathcal{D}\, d1\, d2 \ldots dN \qquad (1.1.2.1)$$

is the probability that at time t the plasma particles are, respectively, within the volume elements of their own phase space $d1 \equiv d^3\boldsymbol{r}^{(1)}\, d^3\boldsymbol{p}^{(1)}$, $d2 \equiv d^3\boldsymbol{r}^{(2)}\, d^3\boldsymbol{p}^{(2)}, \ldots$ (N is the total number of particles in the plasma).

We shall not dwell in detail here on the definition of the probability concept but remind ourselves none the less that the probability description assumes the introduction of an ensemble of identical systems—in our case, plasmas—and that the relative number of these systems with given values of dynamic characteristics just determines the probability concept.

The probability interpretation of the density function $\mathcal{D}$ assumes that it is normalized as follows:

$$\int \mathcal{D}(1, 2, \ldots, N; t)\, d1\, d2 \ldots dN = 1. \qquad (1.1.2.2)$$

The description of the plasma by means of the function $\mathcal{D}$ is essentially a complete one that is, it is the most detailed possible microscopic description of a many-body system. It is clear that as $N \to \infty$ it loses all practical value.

The actual description of a plasma can not be based upon knowing the probability to find each of the plasma particles in a given volume element of their own phase space, but upon a knowledge of the probability to find one, or two, or at the most a few, of the plasma particles in given elementary volumes of the appropriate phase spaces, irrespective of where the remainder of the plasma particles can be found.

These probabilities can clearly be found by integrating $\mathcal{D}$ over all variables bar those which refer to one, two, or the chosen few, particles. As a result we obtain the so-called single-particle, two-particle, and in general many-particle, *distribution functions*. The single-particle distribution function $f_1(1)$ is thus defined by the integral

$$f_1(1) = N \int \mathcal{D}\, d2\, d3 \ldots dN, \qquad (1.1.2.3)$$

the two-particle distribution function $f_2(1, 2)$ by the integral

$$f_2(1, 2) = N^2 \int \mathcal{D}\, d3 \dots dN,$$

and, in general, the s-particle distribution function by the integral

$$f_s(1, 2, \dots, s) = N^s \int \mathcal{D}\, d(s+1) \dots dN. \tag{1.1.2.4}$$

We have here for the sake of simplicity assumed that the plasma consisted of one kind of particles, for instance, electrons; we did not consider the distribution of the particles of other kinds assuming that their role can be reduced to merely producing a neutralizing charge. It is clear that one can easily relax this assumption by introducing an additional index characterizing the kind of particle one is dealing with.

The distribution functions satisfy the normalization conditions

$$\begin{aligned} \int f_1(1)\, d1 &= N, \\ \int f_2(1, 2)\, d1\, d2 &= N^2, \\ &\dots, \end{aligned} \tag{1.1.2.5}$$

while there is the following obvious connection between them:

$$\int f_{s+1}(1, 2, \dots, s+1)\, d(s+1) = N f_s(1, 2, \dots, s). \tag{1.1.2.6}$$

This last formula shows that the senior distribution functions contain all the information carried by the junior distribution functions. For instance, if we know $f_2(1, 2)$, we can obtain $f_1(1)$. This leads to the consequence that on increasing the number s the functions $f_s(1, 2, \dots, s)$ become increasingly more complicated.

Apart from the distribution functions we introduce the *correlation functions* $C_1(1)$, $C_2(1, 2)$, $C_3(1, 2, 3)$, ... which are connected with the distribution functions through the equations

$$\begin{aligned} f_1(1) &= C_1(1), \\ f_2(1, 2) &= C_1(1) C_1(2) + C_2(1, 2), \\ f_3(1, 2, 3) &= C_1(1) C_1(2) C_1(3) + \Sigma C_1(1) C_2(2, 3) + C_3(1, 2, 3), \\ &\dots, \end{aligned} \tag{1.1.2.7}$$

where the summation is over all possible permutations of the particles which lead to a different result.

One can easily show that the correlation functions $C_s(1, 2, \dots, s)$ defined in this way vanish as soon as at least one pair of the set of particles $1, 2, \dots, s$ is statistically independent.

It is clear that the distribution functions and the correlation functions of identical particles are symmetric under a permutation of the particles.

We shall assume that at $t = -\infty$ and when the particles are sufficiently far from one another, the particles are statistically independent. In other words, we shall assume that the correlation functions C_2 satisfy the following condition

$$\lim_{\substack{t \to -\infty \\ r_{12} \to \infty}} C_2(1, 2) = 0. \tag{1.1.2.8}$$

This condition is called the condition of correlation weakening (Bogolyubov, 1946, 1962).

We note that, in general, the correlation functions which satisfy condition (1.1.2.8) will not vanish as $t \to +\infty$, $r_{12} \to \infty$ although in both cases there are no interactions between the particles. This means between particles which were statistically independent at $t = -\infty$, $r_{12} = \infty$, a correlation has arisen at $t = +\infty$, $r_{12} = \infty$.

1.1.3. CHAIN OF EQUATIONS FOR MANY-PARTICLE FUNCTIONS

We shall now obtain equations to be satisfied by the many-particle distribution functions. To find these we use Liouville's theorem according to which the density function $\mathcal{D}(1, 2, \ldots, N; t)$ satisfies a continuity equation in phase space:

$$\frac{\partial \mathcal{D}}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(w_i \mathcal{D}) = 0, \tag{1.1.3.1}$$

where $x_i \equiv \boldsymbol{r}_i, \boldsymbol{p}_i$, $w_i \equiv \dot{x}_i$.

Using the Hamiltonian equations

$$\dot{\boldsymbol{r}}_i = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i}, \quad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i},$$

where $\mathcal{H}$ is the plasma Hamiltonian, we can clearly write the continuity eqn. (1.1.3.1) in the form

$$\frac{\partial \mathcal{D}}{\partial t} + \{\mathcal{H}, \mathcal{D}\} = 0, \tag{1.1.3.2}$$

where $\{\mathcal{H}, \mathcal{D}\}$ is a Poisson bracket,

$$\{\mathcal{H}, \mathcal{D}\} = \sum_{i=1}^{N} \left(\frac{\partial \mathcal{D}}{\partial \boldsymbol{p}_i} \frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i} - \frac{\partial \mathcal{D}}{\partial \boldsymbol{r}_i} \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}_i} \right).$$

If the plasma is in an external electrostatic field, the Hamiltonian $\mathcal{H}$ has the form

$$\mathcal{H} = \sum_i \mathcal{H}_i + \sum_{i<j} \frac{e^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}, \tag{1.1.3.3}$$

where

$$\mathcal{H}_i = \frac{p_i^2}{2m} + e\varphi_i,$$

$\varphi_i \equiv \varphi_i(\boldsymbol{r}_i, t)$ is the potential of the external field while m and e are the mass and charge of the plasma particles; if there are various kinds of particles we must in $\mathcal{H}$ replace m by m_i and e^2 by $e_i e_j$.

In order to obtain the equations for the many-particle distribution functions we integrate the continuity equation in the form (1.1.3.2) over the variables of particles $s+1, s+2, \ldots, N$:

$$\frac{\partial f_s}{\partial t} + N^s \sum_{i=1}^{N} \int \{\mathcal{H}_i, \mathcal{D}\}\, d(s+1) \ldots dN$$
$$+ N^s \sum_{1 \le i < j \le N} \int \{\varphi_{ij}, \mathcal{D}\}\, d(s+1) \ldots dN = 0, \tag{1.1.3.4}$$

where

$$\varphi_{ij} = \frac{e^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}.$$

If we then use the relations

$$\int \{\mathcal{H}_i, \mathcal{D}\}\, di = 0, \quad \int \{\varphi_{ij}, \mathcal{D}\}\, di\, dj = 0,$$

we can write eqn. (1.1.3.4) in the form

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \{\mathcal{H}_i, f_s\} + \sum_{1 \le i < j \le s} \{\varphi_{ij}, f_s\}$$
$$+ N^s \sum_{\substack{1 \le i \le s \\ s+1 \le j \le N}} \int \{\varphi_{ij}, \mathcal{D}\}\, d(s+1) \ldots dN = 0. \qquad (1.1.3.4')$$

Because of the symmetry of the probability density $\mathcal{D}$ the last term in this equation equals

$$N^s \frac{N-s}{N^{s+1}} \sum_{i=1}^{s} \int \{\varphi_{i,s+1}, f_{s+1}\} d(s+1).$$

We now let the number of particles N and the volume V occupied by the plasma tend to infinity assuming, however, that the particle density, $n = N/V$, remains fixed and finite. Equation (1.1.3.4′) then takes the form

$$\frac{\partial f_s}{\partial t} + \sum_{i=1}^{s} \{\mathcal{H}_i, f_s\} + \sum_{1 \le i \le j \le s} \{\varphi_{ij}, f_s\}$$
$$+ \sum_{i=1}^{s} \int \{\varphi_{i,s+1}, f_{s+1}\}\, d(s+1) = 0, \qquad (1.1.3.5)$$

or, in expanded form,

$$\left[\frac{\partial}{\partial t} + \mathcal{L}(1)\right] f_1(1) - \int \left(\frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_2(1,2)}{\partial \boldsymbol{p}_1}\right) d2 = 0,$$

$$\left\{\frac{\partial}{\partial t} + \mathcal{L}(1) + \mathcal{L}(2) - \left(\frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial}{\partial \boldsymbol{p}_1}\right) - \left(\frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial}{\partial \boldsymbol{p}_2}\right)\right\} f_2(1,2) \qquad (1.1.3.6)$$
$$- \int \left[\left(\frac{\partial \varphi_{13}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial f_3(1,2,3)}{\partial \boldsymbol{p}_1}\right) + \left(\frac{\partial \varphi_{23}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial f_3(1,2,3)}{\partial \boldsymbol{p}_2}\right)\right] d(3) = 0,$$
$$\ldots,$$

where

$$\mathcal{L}(i) = \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{r}_i}\right) + e\left(\boldsymbol{E}^{(e)}(\boldsymbol{r}_i) \cdot \frac{\partial}{\partial \boldsymbol{p}_i}\right),$$

while $\boldsymbol{E}^{(e)}$ is the external electric field and $\boldsymbol{v}_i$ the velocity of the ith particle.

When deriving eqn. (1.1.3.6) we assumed for the sake of simplicity that the plasma was in an external electrostatic field. One can easily show that in the general case when the plasma is in an arbitrary external electromagnetic field the many-particle functions also satisfy equations of the form (1.1.3.6), but one must take for $\mathscr{L}(i)$ the operator

$$\mathscr{L}(i) = \left(\boldsymbol{v}_i \cdot \frac{\partial}{\partial \boldsymbol{r}_i}\right) + e\left(\left\{\boldsymbol{E}^{(e)}(\boldsymbol{r}_i) + \left[\frac{\boldsymbol{v}_i}{c} \wedge \boldsymbol{B}^{(e)}(\boldsymbol{r}_i)\right]\right\} \cdot \frac{\partial}{\partial \boldsymbol{p}_i}\right), \tag{1.1.3.7}$$

where $\boldsymbol{E}^{(e)}$ and $\boldsymbol{B}^{(e)}$ are the external electric field and magnetic induction.

Equations (1.1.3.6) form an infinite set of integro-differential equations which connect the successive many-particle distribution functions f_s and f_{s+1}. This "coupled" set of equations is called the Bogolyubov–Born–Green–Kirkwood–Yvon or BBGKY hierarchy of equations (Bogolyubov, 1946, 1962; Born and Green, 1947; Kirkwood, 1946; Yvon, 1935); these equations are the starting point for the kinetic description of a plasma.†

1.2. The Vlasov Equation

1.2.1. THE PLASMA PARAMETER

The infinite set of eqns. (1.1.3.6) in itself does not contain any more information than the continuity eqn. (1.1.3.1) for the probability density $\mathscr{D}$. However, the usefulness of the BBGKY-hierarchy of equations lies in the fact that one can introduce some kind of physical approximation which allows us to cut off the chain at some stage and make clear the error which occurs due to the cut-off.

The termination of the chain occupies a very delicate position in kinetic theory as it can not be performed in a unique and universal way for all cases; in particular, in the case of a plasma the termination is closely connected with the physical state of the plasma.

It is clear that the termination of the chain corresponds to a diminution of our information about the state of the plasma as instead of a detailed microscopic description by means of the probability density $\mathscr{D}$ we obtain a description by means of a few of the first many-particle distribution functions (the probability density $\mathscr{D}$ corresponds to an infinite number of distribution functions).

Let us now consider a concrete procedure for terminating the chain of kinetic equations. We note first of all that any termination requires the existence of at least one small parameter in the theory. For those plasmas which we shall study in what follows we can use as such a small parameter the ratio of the average interaction energy of two particles to their average kinetic energy; that is, we have

$$\left\langle \frac{e^2}{r_{12}} \right\rangle \Big/ \left\langle \frac{1}{2} m v_1^2 \right\rangle \ll 1, \tag{1.2.1.1}$$

where the pointed brackets $\langle \ldots \rangle$ indicate some kind of average. If we bear in mind that $\langle \frac{1}{2} m v_1^2 \rangle \sim T$ and $\langle r_{12} \rangle \sim n^{-1/3}$, we can write the criterion (1.2.1.1) in the form

$$\frac{e^2 n^{1/3}}{T} \ll 1, \tag{1.2.1.2}$$

which indicates that the plasma must be sufficiently hot and sufficiently rarefied. Bearing

† Klimontovich (1967) has given a different derivation of the Bogolyubov chain of equations.

in mind the definition (1.1.1.2) of the Debye screening radius r_D we can write condition (1.2.1.2) in the equivalent form

$$g \equiv \frac{1}{nr_D^3} \ll 1, \tag{1.2.1.3}$$

which means that the number of plasma particles in a Debye volume—that is a sphere with radius r_D—must be large.

We shall thus assume that the *plasma parameter* g is small. The correlation effects in the plasma will then as a rule be small. This means that the single-particle distribution function, or what amounts to the same the single-particle correlation function $C_1(1)$, can be assumed to be of order unity, the pair correlation function $C_2(1, 2)$ of order g, the three-particle correlation function $C_3(1, 2, 3)$ of order g^2, and so on. We shall check in Section 1.3.1 that this statement is correct for a plasma which is not far removed from a state of statistical equilibrium.

1.2.2. THE SELF-CONSISTENT FIELD

Let us consider the limiting case as $g \to 0$ in some more detail. As the correlations are then small, we can put

$$f_s(1, 2, \ldots, s) = \prod_{i=1}^{s} F(i), \quad F(i) \equiv f_1(i): \tag{1.2.2.1}$$

this means that we may assume the many-particle distribution functions to be equal to products of the appropriate single-particle functions.

The single-particle distribution function $F(1)$ satisfies then, according to (1.1.3.6), the equation

$$\left\{\frac{\partial}{\partial t} + \left(v_1 \cdot \frac{\partial}{\partial r_1}\right) + e\left(\left\{E_1^{(e)} + \frac{1}{c}\,[v_1 \wedge B_1^{(e)}]\right\} \cdot \frac{\partial}{\partial p_1}\right) - \left(\int \left[\frac{\partial}{\partial r_1}\,\frac{e^2}{|r_1 - r_2|}\right] F(2)\,d2 \cdot \frac{\partial}{\partial p_1}\right)\right\} F(1) = 0. \tag{1.2.2.2}$$

This equation can be easily interpreted physically; the integral which occurs here and which contains the function $F(2)$ determines clearly the electrostatic field $E^{(p)}(r_1)$ produced by the particles in the plasma itself at the point r_1,

$$E^{(p)}(r_1, t) = -\frac{\partial}{\partial r_1} \int \frac{eF(2)\,d2}{|r_1 - r_2|},$$

as the quantity $\varrho^{(p)}(r_2, t) = e\int F(2) d^3p_2$ is the electric charge density in the plasma at the point r_2. We can thus say that in eqn. (1.2.2.2), which determines the change in the single-particle distribution function $F(1)$, we have not just the external electric field $E^{(e)}$, but the total electric field,

$$E(r, t) = E^{(e)}(r, t) + E^{(p)}(r, t),$$

which is equal to the sum of the external field $E^{(e)}$ and the internal field $E^{(p)}$ which is produced by the particles in the plasma itself. The latter is usually called the *self-consistent* electric field as it not only affects the single-particle distribution function F, but is itself determined by this function:

$$\operatorname{div} \boldsymbol{E}^{(p)} = 4\pi\varrho^{(p)}, \quad \varrho^{(p)}(\boldsymbol{r}, t) = e \int F(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{p}. \tag{1.2.2.3}$$

As far as the magnetic field is concerned, only the external magnetic field $\boldsymbol{B}^{(e)}$ occurs in the kinetic eqn. (1.2.2.2). However, in actual fact the change in the single-particle distribution function is determined not only by the external magnetic field, but by the total magnetic field, consisting of the external field $\boldsymbol{B}^{(e)}$ and the internal plasma field $\boldsymbol{B}^{(p)}$ produced by the separate, moving plasma particles, that is, by the current density

$$\boldsymbol{j}^{(p)}(\boldsymbol{r}, t) = e \int \boldsymbol{v} F(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{p}.$$

This latter field, like the field $\boldsymbol{E}^{(p)}$, is called a *self-consistent field*, as it both affects the distribution function F and is determined by it.

The reason why the self-consistent magnetic field $\boldsymbol{B}^{(p)}$ did not occur in eqn. (1.2.2.2) was that in deriving that equation we used for the plasma Hamiltonian expression (1.1.3.3) which contained only the electrostatic interaction between plasma particles and neglected their magnetic interaction. If the latter is taken into account we have in the kinetic equation for the single-particle distribution function instead of the external magnetic field $\boldsymbol{B}^{(e)}$ the total field $\boldsymbol{B}$,

$$\boldsymbol{B} = \boldsymbol{B}^{(e)} + \boldsymbol{B}^{(p)}.$$

The quantity $\boldsymbol{B}$ is, of course, called the magnetic induction.

The equation for the single-particle distribution function $F(\boldsymbol{r}, \boldsymbol{p}, t)$ has thus the following form, when we neglect correlation effects ($g \to 0$):

$$\left\{\frac{\partial}{\partial t} + \left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) + e\left(\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\right\} F(\boldsymbol{r}, \boldsymbol{p}, t) = 0, \tag{1.2.2.4}$$

where $\boldsymbol{E} \equiv \boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B} \equiv \boldsymbol{B}(\boldsymbol{r}, t)$ are the total electric field and the total magnetic induction, containing both the external and the self-consistent terms, the latter being produced by the plasma particles. The electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ satisfy the Maxwell equations,

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \operatorname{div} \boldsymbol{B} = 0, \quad \operatorname{div} \boldsymbol{E} = 4\pi(\varrho^{(e)} + \varrho^{(p)}),$$

$$\operatorname{curl} \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}(\boldsymbol{j}^{(e)} + \boldsymbol{j}^{(p)}), \tag{1.2.2.5}$$

where $\varrho^{(e)}$ and $\boldsymbol{j}^{(e)}$ are the external charge and current densities and $\varrho^{(p)}$ and $\boldsymbol{j}^{(p)}$ the charge and current densities produced by the particles in the plasma itself. These quantities are connected with the single-particle distribution function through the equations

$$\varrho^{(p)}(\boldsymbol{r}, t) = e \int F(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{p}, \quad \boldsymbol{j}^{(p)}(\boldsymbol{r}, t) = e \int \boldsymbol{v} F(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{p}. \tag{1.2.2.6}$$

Equation (1.2.2.4), which was first written down by Vlasov (1938), is called the *Vlasov equation*.

1.2.3. SET OF KINETIC EQUATIONS WITH SELF-CONSISTENT FIELDS FOR A MULTI-COMPONENT PLASMA

So far we have assumed for the sake of simplicity that the plasma contained only one kind of particles. This assumption is, however, not important. If the plasma contains different kinds of particles we must introduce instead of a single one-particle distribution function several single-particle distribution functions—one for each kind of particles. Denoting these functions by $F^{(i)}(\boldsymbol{r}, \boldsymbol{p}, t)$, where the index i denotes the kind of particle, we get instead of (1.2.2.4) the equations

$$\left\{\frac{\partial}{\partial t}+\left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)+e_i\left(\left\{\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right)\right\}F^{(i)} = 0, \tag{1.2.3.1}$$

where $\boldsymbol{v} = \boldsymbol{p}/m_i$, e_i and m_i are the charge and mass of the particles of the ith kind, while $\boldsymbol{E}$ and $\boldsymbol{B}$ are the fields which satisfy the Maxwell equations (1.2.2.5) in which we must now take for $\varrho^{(p)}$ and $\boldsymbol{j}^{(p)}$ the expressions

$$\left.\begin{aligned}\varrho^{(p)}(\boldsymbol{r}, t) &= \sum_i e_i \int F^{(i)}(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{p},\\ \boldsymbol{j}^{(p)}(\boldsymbol{r}, t) &= \sum_i e_i \int \boldsymbol{v}F^{(i)}(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{p}.\end{aligned}\right\} \tag{1.2.3.2}$$

The distribution functions $F^{(i)}$ are now normalized through the condition

$$\int F^{(i)}(\boldsymbol{r}, \boldsymbol{p}, t)\, d^3\boldsymbol{r}\, d^3\boldsymbol{p} = N_i, \tag{1.2.3.3}$$

where N_i is the total number of particles of the ith kind.

The Vlasov equation (1.2.3.1) forms together with the definitions (1.2.3.2) of the self-consistent charge and current densities the basis for the theory of oscillations and waves in a plasma with a small parameter g and with small higher-order correlation effects.

We have especially emphasized here the condition that the higher-order correlation effects must be small, separately from the criterion that the plasma parameter g be small, although we have a moment ago derived the smallness of higher-order correlation effects starting from the fact that g was small. Such a derivation is valid for a plasma which is not far from a state of thermodynamic equilibrium or, as the saying goes, a quiescent plasma. It might, however, not be valid for a plasma which is far from an equilibrium state. In such a strongly non-equilibrium and unstable plasma different kinds of oscillations are strongly excited and the higher-order correlation functions may be of the same order of magnitude as the single-particle function F (Ichimaru and Nakano, 1968). In that case we speak of a turbulent plasma; we shall consider it in Chapter 10.

1.3. The Pair Correlation Function of an Equilibrium Plasma and the Landau Collision Integral

1.3.1. THE PAIR CORRELATION FUNCTION

Let us now elucidate the role of the higher correlation effects in the case when they are small, that is, in the case of a quiescent plasma with a small parameter g. To do this we consider the chain of eqns. (1.1.3.6).

If we carry along the pair correlation function—in contrast to what we did when we derived equation (1.2.2.4) with the self-consistent field—we can rewrite the first of the equations of the chain (1.1.3.6) in the form

$$\left\{\frac{\partial}{\partial t}+\left(v_1\cdot\frac{\partial}{\partial r_1}\right)+e\left(\left\{E_1+\frac{1}{c}[v_1\wedge B_1]\right\}\cdot\frac{\partial}{\partial p_1}\right\}F(1) = e^2\int\left(\frac{\partial}{\partial r_1}\frac{1}{|r_1-r_2|}\cdot\frac{\partial}{\partial p_1}\right)G(1,2)\,d2, \tag{1.3.1.1}$$

where $E_1 \equiv E(r_1, t)$, $B_1 \equiv B_1(r_1, t)$, while $G(1, 2) \equiv C_2(1, 2)$ is the pair correlation function.

We should add to this equation an equation for the function $G(1, 2)$ which may be obtained from the second equation of the chain (1.1.3.6), but that equation will contain the three-particle distribution function, that is, the three-particle correlation function $C_3(1, 2, 3)$; the equation for C_3 will contain the correlation function $C_4(1, 2, 3, 4)$, and so on. We shall assume that if $g \ll 1$ we can in the case of a quiescent plasma cut this chain off at the third member, that is, we shall assume that

$$C_3(1, 2, 3) = 0.$$

Under that assumption the equation for the pair correlation function $G(1, 2)$ takes the form

$$\left\{\frac{\partial}{\partial t}+\mathcal{L}(1)+\mathcal{L}(2)-[V(1,2)+V(2,1)]\right\}G(1,2)-\int V(1,3)\,F(1)\,G(2,3)\,d3$$
$$-\int V(2,3)\,F(2)\,G(1,3)\,d3 = F(2)\,V(1,2)\,F(1)+F(1)\,V(2,1)\,F(2), \tag{1.3.1.2}$$

where

$$V(i,j) = \left(\frac{\partial\varphi_{ij}}{\partial r_i}\cdot\frac{\partial}{\partial p_i}\right),$$

and $\varphi_{ij} \equiv \varphi_{ij}(r_i, r_j)$ is the energy of the interaction between the ith and the jth particles, while $\mathcal{L}(i)$ is defined by eqn. (1.1.3.7).

Equations (1.3.1.1) and (1.3.1.2) form a closed set of equations to determine the single-particle distribution function $F(1)$ and the pair correlation function $G(1, 2)$. To find a unique solution of these equations we still need a boundary condition for the function G. We choose for this the boundary condition (1.1.2.8) which determines the asymptotic behaviour of the correlation function. This condition of the correlation weakening as $t \to -\infty$ appears in the kinetic theory of irreversible phenomena.

We shall first of all determine the pair correlation function of an equilibrium plasma when $g \ll 1$. The velocity distribution function of the plasma particles is in that case a Maxwellian one and the correlation occurs only between the spatial positions of the particles and not between their velocities. The pair correlation function therefore will be of the form

$$G(1, 2) = F_M(p_1)\,F_M(p_2)\,\psi(r),$$

where $F_M(p)$ is the Maxwell distribution,

$$F_M(p) = \frac{n}{(2\pi mT)^{3/2}}\exp\,[-p^2/2mT],$$

and $\psi(\boldsymbol{r})$ a function of the distance between the particles, $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 = \boldsymbol{r}_{12}$. To find that function let us estimate the various terms in equation (1.3.1.2). Bearing in mind that $\mathcal{L} \sim T^{1/2}/rm^{1/2}$, $V \sim \varphi/rT^{1/2}m^{1/2}$, $F^2 \sim G/\psi$, $\int F\, d3 \sim nr^3$, and dropping the time-derivative in (1.3.1.2) we have, as to order of magnitude

$$\mathcal{L}G : VG : \int VFG\, d3 : FVF = 1 : \frac{\varphi}{T} : \frac{\varphi}{T} nr^3 : \frac{\varphi}{T}\,\frac{1}{\psi}.$$

As $\varphi \sim e^2/r$ we see that if $g \equiv 1/nr_{\mathrm{D}}^3 \ll 1$ and $\psi \ll 1$ and $r \gg \bar{r} = n^{-1/3}$ we can neglect in (1.3.1.2) the term $[V(1,2)+V(2,1)]G(1,2)$ so that the equation for ψ becomes

$$\left([\boldsymbol{v}_1 - \boldsymbol{v}_2]\cdot\frac{\partial\psi(\boldsymbol{r}_{12})}{\partial\boldsymbol{r}_1}\right) + \frac{n}{T}\int\left(\frac{\partial\varphi_{13}}{\partial\boldsymbol{r}_1}\cdot\boldsymbol{v}_1\right)\psi(\boldsymbol{r}_{23})\, d^3\boldsymbol{r}_3 + \frac{n}{T}\int\left(\frac{\partial\varphi_{23}}{\partial\boldsymbol{r}_2}\cdot\boldsymbol{v}_2\right)\psi(\boldsymbol{r}_{13})\, d^3\boldsymbol{r}_3$$
$$= -\frac{1}{T}\left([\boldsymbol{v}_1 - \boldsymbol{v}_2]\cdot\frac{\partial\varphi_{12}}{\partial\boldsymbol{r}_1}\right).$$

Fourier transforming,

$$\varphi(\boldsymbol{r}) = \int \varphi_k\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3\boldsymbol{k}, \qquad \psi(\boldsymbol{r}) = \int \psi_k\, e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3\boldsymbol{k},$$

and bearing in mind that

$$\varphi_k = \frac{e^2}{2\pi^2 k^2},$$

we find

$$\psi_k = -\frac{e^2 r_{\mathrm{D}}^2}{2\pi^2(1+k^2 r_{\mathrm{D}}^2)T},$$

and hence

$$\psi(\boldsymbol{r}) = -\frac{e^2}{Tr}\exp\left(-\frac{r}{r_{\mathrm{D}}}\right), \qquad r \gg \bar{r}.$$

This formula is valid when $g \ll 1$ as we dropped terms of order g^2 in its derivation; in fact, the neglected terms are of order $g^2 \ln g$ (O'Neil and Rostoker, 1965).

One can show that at small distances, $r \ll \bar{r}$, the function ψ has the form

$$\psi(\boldsymbol{r}) = -1 + \exp(-e^2/Tr),$$

in the same approximation.

If we add under the exponential sign a factor $\exp(-r/r_{\mathrm{D}})$ we obtain an interpolation formula for ψ which is applicable for all distances:

$$\psi(\boldsymbol{r}) = -1 + \exp\left[-\frac{e^2}{Tr}e^{-r/r_{\mathrm{D}}}\right].$$

The following expression for the two-particle distribution function of an equilibrium plasma corresponds to this expression:

$$F(1,2) = \frac{n^2}{(2\pi mT)^3}\, e^{-W_{12}/T}, \tag{1.3.1.2'}$$

where

$$W_{12} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \Phi_{12}(\boldsymbol{r})$$

and

$$\Phi_{12}(\boldsymbol{r}) = \frac{e^2}{r} \exp\left(-\frac{r}{r_D}\right).$$

We obtain here, as we should have expected a Boltzmann factor for two particles in which the screened Coulomb potential occurs for the interaction energy.

We note that as $r \to 0$ the two-particle distribution functions tend to zero; this is connected with the mutual repulsion of the particles.

Calculations similar to the ones given here show that the three-particle correlation function of an equilibrium plasma is of order $g^2 \ln g$ (O'Neil and Rostoker, 1965).

Let us now determine the pair correlation function for a non-equilibrium plasma, but which is not far removed from equilibrium, that is, a quiescent plasma. In that case we can for an estimate of the different terms in eqn. (1.3.1.2) use the results obtained a moment ago for an equilibrium plasma. We shall as before assume that $g \ll 1$. In that case we can drop in eqn. (1.3.1.2) the term $[V(1, 2) + V(2, 1)]\, G(1, 2)$ for distances $r \gg \bar{r}$. If, moreover, the distance between the particles is appreciably less than the Debye radius r_D, so that r lies in the interval $\bar{r} \ll r \ll r_D$, we can also neglect in eqn. (1.3.1.2) the terms

$$\int V(1, 3)\, F(1)\, G(2, 3)\, d3 \quad \text{and} \quad \int V(2, 3)\, F(2)\, G(1, 3)\, d3.$$

In other words, we can use for the determination of the pair correlation function in the interval $\bar{r} \ll r \ll r_D$ the following equation:

$$\left\{\frac{\partial}{\partial t} + \mathcal{L}(1) + \mathcal{L}(2)\right\} G(1, 2) = F(2)\, V(1, 2)\, F(1) + F(1)\, V(2, 1)\, F(2). \qquad (1.3.1.3)$$

We shall consider in somewhat more detail the case when the external field is rather weak. In that case eqn. (1.3.1.3) becomes

$$\left\{\frac{\partial}{\partial t} + \left(\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{r}_1}\right) + \left(\boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{r}_2}\right)\right\} G(1, 2)$$
$$= F(2)\left(\frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_1} \cdot \frac{\partial F(1)}{\partial \boldsymbol{p}_1}\right) + F(1)\left(\frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial F(2)}{\partial \boldsymbol{p}_2}\right). \qquad (1.3.1.4)$$

We draw attention to the fact that in this equation we have only the derivatives with respect to the time and the spatial coordinates of the correlation function; the equation does not contain derivatives of G with respect to the particle momenta. In other words, under the given assumption that the interaction is weak and that the external field is weak the momentum dependence of the correlation function arises simply because the particle momenta occur as parameters on the right-hand side of eqn. (1.3.1.3). One may say that the neglect of the terms containing the interaction energy of the particles in eqn. (1.3.1.2) corresponds to taking only long-distance collisions into account in which the particle momenta are changed only little.

It is convenient to bring the inhomogeneous eqn. (1.3.1.4) first to a homogeneous form in order to find its solution. To do this we write the solution of eqn. (1.3.1.4) formally in the form

$$\Phi(t, \boldsymbol{r}_1, \boldsymbol{r}_2, G) = 0,$$

whence we have

$$\frac{\partial G}{\partial t} = -\frac{\partial \Phi}{\partial t}\Big/\frac{\partial \Phi}{\partial G}, \quad \frac{\partial G}{\partial \boldsymbol{r}_i} = -\frac{\partial \Phi}{\partial \boldsymbol{r}_i}\Big/\frac{\partial \Phi}{\partial G},$$

and hence

$$\frac{\partial \Phi}{\partial t} + \left(\boldsymbol{v}_1 \cdot \frac{\partial \Phi}{\partial \boldsymbol{r}_1}\right) + \left(\boldsymbol{v}_2 \cdot \frac{\partial \Phi}{\partial \boldsymbol{r}_2}\right) + \left(\left\{\frac{\partial F(1)}{\partial \boldsymbol{p}_1} F(2) - F(1)\frac{\partial F(2)}{\partial \boldsymbol{p}_2}\right\} \cdot \frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_1}\right)\frac{\partial \Phi}{\partial G} = 0. \quad (1.3.1.5)$$

We have thus obtained a homogeneous linear equation for the function Φ and we can find its solution by means of the method of characteristics. The characteristics equation has, clearly, the form

$$\frac{d\boldsymbol{r}_i}{dt} = \boldsymbol{v}_i, \quad \frac{dG}{dt} = \left(\left\{\frac{\partial F(1)}{\partial \boldsymbol{p}_1} F(2) - F(1)\frac{\partial F(2)}{\partial \boldsymbol{p}_2}\right\} \cdot \frac{\partial \varphi_{12}}{\partial \boldsymbol{r}_1}\right), \quad (1.3.1.6)$$

where the $\boldsymbol{v}_i$ must be considered to be constant parameters.

Integrating the first of the characteristics equations,

$$\boldsymbol{r}_i = \boldsymbol{r}_i^{(0)} + \boldsymbol{v}_i t,$$

and substituting the result into the second equation, we find

$$\frac{dG}{dt} = \left(\left\{\frac{\partial F(1)}{\partial \boldsymbol{p}_1} F(2) - F(1)\frac{\partial F(2)}{\partial \boldsymbol{p}_2}\right\} \cdot \frac{\partial}{\partial \boldsymbol{r}_1^{(0)}}\right) \varphi_{12}(\boldsymbol{r}_1^{(0)} + \boldsymbol{v}_1 t, \boldsymbol{r}_2^{(0)} + \boldsymbol{v}_2 t). \quad (1.3.1.7)$$

When we integrate this equation we shall assume that the single-particle distribution function is a slowly changing function of time. Using that assumption we find for the function G which satisfies the boundary condition (1.1.2.8)

$$G(\boldsymbol{r}_1^{(0)}, \boldsymbol{r}_2^{(0)}, \boldsymbol{p}_1, \boldsymbol{p}_2, 0)$$
$$= \left(\left\{\frac{\partial F(1)}{\partial \boldsymbol{p}_1} F(2) - F(1)\frac{\partial F(2)}{\partial \boldsymbol{p}_2}\right\} \cdot \int_{-\infty}^{0} \frac{\partial}{\partial \boldsymbol{r}_1^{(0)}} \varphi_{12}\left(\boldsymbol{r}_1^{(0)} + \frac{\boldsymbol{p}_1}{m} t, \boldsymbol{r}_2^{(0)} + \frac{\boldsymbol{p}_2}{m} t\right) dt\right). \quad (1.3.1.8)$$

This equation is valid in the range $\bar{r} \ll r \ll r_D$, provided the external field is sufficiently weak and provided the time τ_F, over which the single-particle distribution function F changes appreciably, is considerably longer than the time τ_G over which the correlation function G changes appreciably,

$$\tau_F \gg \tau_G. \quad (1.3.1.9)$$

If there are no varying fields,

$$\tau_F \sim \frac{l}{\bar{v}},$$

where l is the particle mean free path in the plasma and $\bar{v}$ the particle thermal velocity. On the other hand, the quantity τ_G is of the order of

$$\tau_G \sim \frac{r_D}{\bar{v}}.$$

As $l \gg r_D$ (see Section 1.4), we have $\tau_F \gg \tau_G$. We note that in fast-varying fields with frequencies ω of the order of or larger than ω_p($\omega_p = \bar{v}/r_D$ is the plasma frequency) condition (1.3.1.9) is no longer satisfied and eqn. (1.3.1.8) ceases to be valid. However, eqn. (1.3.1.7) remains valid even when $\omega \gtrsim \omega_p$.

1.3.2. LANDAU COLLISION INTEGRAL

Using eqn. (1.3.1.8) to express the pair correlation function G in eqn. (1.3.1.1) in terms of the single-particle distribution function we get the following kinetic equation which contains only the single-particle function:

$$\frac{\partial F(1)}{\partial t} + \left(\boldsymbol{v}_1 \cdot \frac{\partial F(1)}{\partial \boldsymbol{r}_1}\right) + e\left(\left\{\boldsymbol{E}^{(e)}(\boldsymbol{r}_1) + \frac{1}{c}[\boldsymbol{v}_1 \wedge \boldsymbol{B}^{(e)}(\boldsymbol{r}_1)]\right\} \cdot \frac{\partial F(1)}{\partial \boldsymbol{p}_1}\right) = \left(\frac{\partial F(1)}{\partial t}\right)_c, \quad (1.3.2.1)$$

where

$$\left(\frac{\partial F(1)}{\partial t}\right)_c = \sum_{\substack{\alpha,\beta \\ =x,y,z}} \frac{\partial}{\partial p_{1\beta}} \int d^3\boldsymbol{p}_2 \left[\frac{\partial F(1)}{\partial p_{1\alpha}} F(2) - F(1)\frac{\partial F(2)}{\partial p_{2\alpha}}\right] I_{\alpha\beta}(\boldsymbol{v}),$$

$$\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2, \quad \boldsymbol{v}_i = \boldsymbol{p}_i/m, \quad (1.3.2.2)$$

and

$$I_{\alpha\beta}(\boldsymbol{v}) = \int_{-\infty}^{0} dt \int d^3\boldsymbol{r}_2 \frac{\partial}{\partial r_{1\alpha}^{(0)}} \varphi_{12}(\boldsymbol{r}^{(0)} + \boldsymbol{v}t) \frac{\partial}{\partial r_{1\beta}^{(0)}} \varphi_{12}(\boldsymbol{r}^{(0)}),$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \quad \boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2. \quad (1.3.2.3)$$

The quantity $(\partial F/\partial t)_c$ is called the collision integral.

The kinetic equation (1.3.2.1) is valid, as we made clear earlier, provided the distribution function $F(1)$ varies sufficiently slowly with time ($\tau_F \gg \tau_G$) and the external fields are sufficiently weak.

Let us transform expression (1.3.2.3). If we write the interaction potential in the form of a Fourier integral,

$$\varphi_{12}(\boldsymbol{r}) = \int \varphi_k e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3\boldsymbol{k}, \qquad \varphi_k = \frac{1}{(2\pi)^3}\int \varphi_{12}(\boldsymbol{r}) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3\boldsymbol{r},$$

one shows easily that

$$I_{\alpha\beta}(\boldsymbol{v}) = 8\pi^4 \int k_\alpha k_\beta |\varphi_k|^2 \delta_-[(\boldsymbol{k}\cdot\boldsymbol{v})]\, d^3\boldsymbol{k},$$

where

$$\delta_-(x) = \frac{1}{\pi}\int_0^\infty e^{-ikx}\, dk.$$

We noted earlier that φ_k is a function of k only—as $\varphi_{12}(\boldsymbol{r})$ is a function of the distance between the particles $|\mathbf{r}| = |\mathbf{r}_1 - \mathbf{r}_2|$—and using the relation

$$\delta_-(x) = \delta(x) - \frac{i}{\pi} \mathcal{P} \frac{1}{x},$$

where $\mathcal{P}$ is the principal value symbol, we can write $I_{\alpha\beta}(v)$ in the form

$$I_{\alpha\beta}(v) = 8\pi^5 \frac{\delta_{\alpha\beta} v^2 - v_\alpha v_\beta}{v^3} \int_0^\infty k^3 \, |\varphi_k|^2 \, dk.$$

Let us evaluate the integral

$$J = \int_0^\infty k^3 \, |\varphi_k|^2 \, dk$$

which appears here. In the case of a pure Coulomb interaction between the particles,

$$\varphi_{12}(\mathbf{r}) = \frac{e_1 e_2}{r},$$

the Fourier component φ_k is equal to

$$\varphi_k = \frac{e_1 e_2}{2\pi^2 k^2},$$

and the integral J diverges logarithmically both at the lower and at the upper limit. This divergence has a simple physical meaning. As small values of k correspond to large, and large values of k to small interparticle distances, these divergences are connected with the nature of the interaction between the particles and its behaviour at large and small distances. At large distances, the screening of the Coulomb interaction occurs so that we should take for the lower limit of the integral J not infinity, but a quantity of the order of the inverse Debye radius,

$$k_{\min} \sim \frac{1}{r_D} = \sqrt{\left(\frac{4\pi n e^2}{T}\right)}.$$

As far as the upper limit is concerned, which corresponds to small distances, we must bear in mind that we can no longer assume the trajectories of the particles to be straight lines, as we did in the characteristics method used earlier. If we used in our calculations the true, that is, hyperbolic, trajectories, the divergence at the lower limit would have disappeared. On the other hand, the logarithm is not very sensitive to small changes in its argument, so that we can take for the upper limit

$$k_{\max} \sim \frac{1}{r_{\min}},$$

where $r_{\min}$ is the distance at which the interaction energy of the particles is equal to their average kinetic energy,

$$r_{\min} = \frac{e^2}{T}.$$

Evaluating J with these limits we get

$$I_{\alpha\beta}(v) = 2\pi e_1^2 e_2^2 \frac{\delta_{\alpha\beta}v^2 - v_\alpha v_\beta}{v^3} L,$$

where

$$L = \ln \frac{r_D}{r_{min}} = \ln \left(\frac{T}{e^2} \sqrt{\left(\frac{T}{4\pi n e^2} \right)} \right); \tag{1.3.2.4}$$

this quantity is called the Coulomb logarithm.

Using eqn. (1.3.2.2) we can write the collision integral in the form

$$\left(\frac{\partial F(1)}{\partial t} \right)_c = -\sum_\alpha \frac{\partial j_\alpha}{\partial p_{1\alpha}}, \tag{1.3.2.5}$$

where

$$j_\alpha(1) = \pi e^4 L \sum_\beta \int \left[F(1) \frac{\partial F(2)}{\partial p_\beta^{(2)}} - F(2) \frac{\partial F(1)}{\partial p_\beta^{(1)}} \right] \left(\delta_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^2} \right) \frac{d^3 p^{(2)}}{u}. \tag{1.3.2.6}$$

So far we have assumed for the sake of simplicity that the plasma contained only one kind of particles. In the general case when the plasma contains several kinds of particles the collision integral has the form

$$\left[\frac{\partial F(a)}{\partial t} \right]_c = -\sum_\alpha \frac{\partial j_\alpha(a)}{\partial p_\alpha^{(a)}},$$

$$j_\alpha(a) = \pi e_a^2 \sum_b e_b^2 L_{ab} \sum_\beta \int \left[F(a) \frac{\partial F(b)}{\partial p_\beta^{(b)}} - F(b) \frac{\partial F(a)}{\partial p_\beta^{a}} \right] \left(\delta_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^2} \right) \frac{d^3 p^{(b)}}{u}, \tag{1.3.2.7}$$

where $u \equiv u_{ab}$ is the relative velocity of the particles a and b—we use the indexes a and b to indicate the different kinds of particles; the summation is over the various kinds of particles, and

$$L_{ab} = \ln \frac{T_a + T_b}{e_a e_b \left[8\pi \left(\frac{e_a^2 n_a}{T_a} + \frac{e_b^2 n_b}{T_b} \right) \right]^{1/2}}. \tag{1.3.2.8}$$

We see that the collision integral has the form of a divergence in momentum space of a vector $j_\alpha(a)$. This vector is a functional of the single-particle distribution function and its derivatives with respect to the momentum components. The collision integral itself therefore contains only the first and second derivatives of the distribution function with respect to the momentum components, that is, the kinetic equation for the single-particle distribution function is for the case when it varies slowly with time a Fokker–Planck type equation.

Landau (1937) was the first to obtain eqn. (1.3.2.7) for the collision integral and this form is therefore called the *Landau collision integral*. Landau started from the Boltzmann collision integral,

$$\left[\frac{\partial F(a)}{\partial t} \right]_c = \int w(a, b \to a', b') [F(a') F(b') - F(a) F(b)] \, da' \, db \, db', \tag{1.3.2.9}$$

where $w(a, b \to a', b')$ is the probability for a transition $(a, b) \to (a', b')$ per unit time, and applied it to Coulomb collisions.

We draw attention to the fact that in deriving the expression for the Landau collision integral we needed to take into account the screening of the Coulomb field—as otherwise we would not have obtained a finite expression for $[\partial F(1)/\partial t]_c$. This means that all particles affect in fact binary collisions. In the case considered by us of a plasma close to thermodynamic equilibrium this influence is described by a single parameter—the Debye screening radius which occurs in the Coulomb logarithm L which appears as a factor in the collision integral. The situation is very much more complicated for states far from thermodynamic equilibrium: the binary collisions are affected by the interaction of the particles with waves which may propagate in the plasma, and the collision integral therefore turns out to depend on the dielectric constant of the plasma. We shall, however, not give here the corresponding expressions for the collision integral (see in this connection work by Silin (1967), Ginzburg (1970), and Golant (1963)).

1.4. Relaxation of a Plasma

1.4.1. RELAXATION TIME OF A PLASMA

The Landau collision integral has an important property: it vanishes when the particles have a statistical equilibrium *Maxwell distribution,*

$$F_M(\boldsymbol{p}) = \frac{n}{(2\pi mT)^{3/2}} e^{-p^2/2mT}, \qquad (1.4.1.1)$$

that is,

$$\left[\frac{\partial F_M(\boldsymbol{p})}{\partial t}\right]_c = 0. \qquad (1.4.1.2)$$

For other distribution functions the collision integral does not vanish.

In connection with this property of the collision integral and of the Maxwell distribution the problem arises about the asymptotic behaviour, as $t \to \infty$, of the solution of the kinetic eqn. (1.3.2.1) for the case of an unmagnetized plasma, that is, a plasma for the case when there are no external fields. The answer to this problem tells us that independent of the initial value of the distribution function $F(\boldsymbol{p}, 0)$, the solution of eqn. (1.3.2.1) for the case when $\boldsymbol{E}^{(e)} = \boldsymbol{B}^{(e)} = 0$ tends asymptotically as $t \to \infty$ to the Maxwell distribution:

$$\lim_{t \to \infty} F(\boldsymbol{p}, t) = F_M(\boldsymbol{p}). \qquad (1.4.1.3)$$

If we had neglected the collision integral in the kinetic eqn. (1.3.2.1), the asymptotic behaviour (1.4.1.3) would not have been true. One can thus say that the process of the approach of the distribution function to the Maxwell distribution—which is called the *relaxation* process—occurs thanks to the binary collisions of the plasma particles. A more general statement is that the higher-order correlation effects are responsible for the relaxation in the plasma.

One can easily estimate the order of magnitude of the *relaxation time*, that is, the time after which the function $F(\boldsymbol{p}, t)$ will differ little from the Maxwellian distribution function $F_M(\boldsymbol{p})$. To do this let us first estimate the mean free path l of a plasma particle. This can be done by dimensional analysis. Indeed, in principle the quantity l can depend on the following quantities: e, n, T, and m, that is

$$l = l(e, n, T, m).$$

However, in the Landau collision integral the charge e and the density n appear in the combination e^4nL, where L is the Coulomb logarithm which occurs in the collision integral,

$$L = \ln\left[\frac{T}{e^2}\sqrt{\left(\frac{T}{4\pi ne^2}\right)}\right].$$

This means that l can depend only on e^4nL, T, and m:

$$l = l(e^4nL, T, m).$$

There is only one way to construct from these quantities a quantity with the dimensions of a length, namely T^2/e^4nL, and therefore

$$l \sim \frac{T^2}{e^4nL}. \tag{1.4.1.4}$$

We see that the particle *mean free path* is proportional to the square of the temperature and inversely proportional to the plasma density, but independent of the particle mass.

Knowing l we can estimate the relaxation time τ: to do this we must clearly divide l by the average thermal velocity of the particles, $\bar{v} = (T/m)^{1/2}$, where m is the particle mass:

$$\tau \sim \frac{l}{\bar{v}} \sim \frac{T^{3/2}}{e^4nL}m^{1/2}. \tag{1.4.1.5}$$

The relaxation time is thus proportional to the 3/2 power of the temperature, inversely proportional to the density, and proportional to the square root of the particle mass. In contrast to the mean free path the relaxation time depends on the particle mass, that is, it is different for different kinds of particle.

At high temperatures and low densities the mean free path and the relaxation time may become very large. For instance, for $T = 10^7$ °K and $n = 10^{12}$ cm^{-3} the mean free path becomes $l \sim 2\times10^6$ cm while the values of the relaxation time for electrons and protons are, respectively, $\tau_e \sim 3\times10^{-4}$ s and $\tau_p \sim 10^{-2}$ s.

The quantity which is the inverse of τ determines the *collision frequency*, that is, the average number of collisions of a given particle per unit time.

Using the fact that the collision integral vanishes for the case of the Maxwell distribution function we can assume that

$$\left[\frac{\partial F(\boldsymbol{p})}{\partial t}\right]_c \sim -\frac{1}{\tau}[F(\boldsymbol{p}, t)-F_M(\boldsymbol{p})], \tag{1.4.1.6}$$

if we want to estimate the collision integral. The kinetic eqn. (1.3.2.1) for an unmagnetized and uniform plasma then becomes

$$\frac{\partial F(\boldsymbol{p}, t)}{\partial t} = -\frac{1}{\tau}[F(\boldsymbol{p}, t) - F_{\mathrm{M}}(\boldsymbol{p})], \qquad (1.4.1.7)$$

and hence

$$F(\boldsymbol{p}, t) = F_{\mathrm{M}}(\boldsymbol{p}) + [F(\boldsymbol{p}, 0) - F_{\mathrm{M}}(\boldsymbol{p})]e^{-t/\tau}. \qquad (1.4.1.8)$$

This formula shows, in accordance with what we have stated earlier, that the distribution function $F(\boldsymbol{p}, t)$, independent of its initial value $F(\boldsymbol{p}, 0)$, asymptotically tends to the Maxwell distribution $F_{\mathrm{M}}(\boldsymbol{p})$ as $t \to \infty$ and that after a time $t \sim \tau$ the function $F(\boldsymbol{p}, t)$ is already close to $F_{\mathrm{M}}(\boldsymbol{p})$. We must nevertheless bear in mind that the relaxation process is only described in its overall behaviour by this formula—the fine structure of the relaxation is not reflected by this formula. In particular, the relaxation proceeds according to this formula uniformly for all velocities while in actual fact the relaxation is slower in the high velocity region than in the low-velocity region. This is connected with the nature of the Coulomb interaction and caused by the way the particle scattering cross-section depends on the velocity; according to the Rutherford formula this cross-section decreases with increasing energy inversely proportional to the square of the energy and as a result there is non-uniformity in the relaxation process.

The simplest model (1.4.1.6) of the collision integral does not describe this non-uniformity, but the use of the exact expression (1.3.2.6) for the collision integral leads to a very complicated integro-differential equation, the solution of which can not be given analytically. One has therefore applied for the solution of the kinetic equation with the Landau collision integral numerical methods which, as one would expect, have led to the result that the "tail" of the Maxwell distribution is established after its main part (MacDonald, Rosenbluth, and Chuck, 1957).

We have already shown that the mean free path of the plasma particles is independent of their mass while the relaxation time is proportional to the square root of the particle mass. As a result light particles relax faster than heavy particles. On the other hand, due to the large difference in mass between an electron and an ion, exchange of energy between like particles is much faster than the exchange of energy between unlike particles. Therefore, complete equilibrium between electrons and ions characterized by the same temperature for both will be established only after thermodynamic equilibrium distributions have been set up for the electrons and the ions with, in general, different temperatures, and the electron distribution will be established before the ion distribution.

1.4.2. THE EQUALIZATION OF THE ELECTRON AND ION TEMPERATURES

In order to describe the equalization process we shall determine the change in the ion energy density $\mathcal{E}^{(i)}$ per unit time due to collisions between ions and electrons.

The change per unit time in the number of ions per unit volume with a given momentum due to collisions with electrons is according to (1.3.2.2) equal to

$$\left[\frac{\partial F^{(i)}(\boldsymbol{p}^{(i)})}{\partial t}\right]_{c}^{e} = -\sum_{\alpha} \frac{\partial j_{\alpha}^{ie}(\boldsymbol{p}^{(i)})}{\partial p_{\alpha}^{(i)}},$$

where

$$j_\alpha^{ie}(p^{(i)}) = \pi e^4 Z^2 L \sum_\beta \left\{ \int \left[F_M^{(i)}(p_M^{(i)}) \frac{\partial F_M^{(e)}(p^{(e)})}{\partial p_\beta^{(e)}} - F_M^{(e)}(p^{(e)}) \frac{\partial F_M^{(i)}(p^{(i)})}{\partial p_\beta^{(i)}} \right] \left(\delta_{\alpha\beta} - \frac{u_\alpha u_\beta}{u^2} \right) \frac{1}{u}\, d^3p^{(e)} \right\},$$

while $F_M^{(i)(e)}$ are the ion and electron Maxwell distribution functions with temperatures T_i and T_e which satisfy the normalization condition

$$\int F_M(p)\, d^3p = n,$$

Ze is the ion charge, and $u = v^{(e)} - v^{(i)} \approx v^{(e)}$; the indices i and e throughout indicate ions and electrons.

From this equation we can easily find the quantity $\dot{\mathscr{E}}^{(i)}$. We have

$$\dot{\mathscr{E}}^{(i)} = -\sum_\alpha \int \frac{\partial j_\alpha^{ie}(p^{(i)})}{\partial p_\alpha^{(i)}} \varepsilon^{(i)}\, d^3p^{(i)} = \sum_\alpha \int j_\alpha^{ie} v_\alpha^{(i)}\, d^3p^{(i)},$$

where $\varepsilon^{(i)}$ is the energy of an ion with momentum $p^{(i)}$ and $v^{(i)} = \partial\varepsilon^{(i)}/\partial p^{(i)}$ is its velocity. Substituting here the expression for $j_\alpha^{ie}(p^{(i)})$ which we gave a moment ago, we get

$$\dot{\mathscr{E}}^{(i)} = \frac{2n^2 e^4 Z^2 (2\pi m_e)^{1/2}}{m_i} L \frac{T_e - T_i}{T_e^{3/2}}. \tag{1.4.2.1}$$

If we equate the rate of change of the ion energy with the rate of change of the electron energy which is clearly equal to $-\frac{3}{2} n \dot{T}_e$, we easily find the *rate of change of the electron temperature*:

$$\dot{T}_e = \frac{4}{3} L \frac{e^4 n Z^2 \sqrt{(2\pi m_e)}}{m_i} \frac{T_i - T_e}{T_e^{3/2}}. \tag{1.4.2.2}$$

This formula is valid when $m_i T_e \gg m_e T_i$.

1.4.3. BOLTZMANN'S *H*-THEOREM FOR A QUIESCENT PLASMA

The relaxation process is accompanied by an increase in the entropy of the plasma. We shall verify this for the example of a quiescent plasma with a small plasma parameter.

We can consider the plasma in this case to be a weakly imperfect gas and neglect in the definition of the entropy higher-order correlation effects. In other words, we can define the *entropy* of a gas using the classical Boltzmann formula

$$S = -\sum \int F(i) \ln F(i)\, d^3p^{(i)}\, d^3r^{(i)}, \tag{1.4.3.1}$$

where $F(i)$ is the distribution function of the ith kind of particles which is normalized by the condition

$$\int F(i)\, d^3p^{(i)} = n_i,$$

and where the summation is over all different kinds of particles.

It follows from this definition that

$$\dot{S} = -\sum \int \frac{\partial F(i)}{\partial t} [1+\ln F(i)]\, di, \tag{1.4.3.2}$$

where $di \equiv d^3p^{(i)}\, d^3r^{(i)}$ while $\partial F/\partial t$ is determined by the kinetic eqn. (1.3.2.1),

$$\frac{\partial F(i)}{\partial t} = -\left\{\left(v^{(i)}\cdot\frac{\partial}{\partial r^{(i)}}\right)+e_i\left(\left\{E^{(i)}+\frac{1}{c}[v^{(i)}\wedge B^{(i)}]\right\}\cdot\frac{\partial}{\partial p^{(i)}}\right)\right\}F(i)+\left[\frac{\partial F(i)}{\partial t}\right]_c, \tag{1.4.3.2'}$$

and $[\partial F(i)/\partial t]_c$ is the collision integral (1.3.2.2).

Substituting this expression into (1.4.3.2) we get

$$\dot{S} = \sum\int [1+\ln F(i)]\left\{\left(v^{(i)}\cdot\frac{\partial}{\partial r^{(i)}}\right)+e_i\left(\left\{E^{(i)}+\frac{1}{c}[v^{(i)}\wedge B^{(i)}]\right\}\cdot\frac{\partial}{\partial p^{(i)}}\right)\right\}F(i)\, di$$
$$-\sum\int [1+\ln F(i)]\left[\frac{\partial F(i)}{\partial t}\right]_c di.$$

If we integrate by parts in the first term and drop the integrated parts, we can write $\dot{S}$ in the form

$$\dot{S} = -\sum\int \left\{\left(v^{(i)}\cdot\frac{\partial}{\partial r^{(i)}}\right)+e_i\left(\left\{E^{(i)}+\frac{1}{c}[v^{(i)}\wedge B^{(i)}]\right\}\cdot\frac{\partial}{\partial p^{(i)}}\right)\right\}F(i)\, di$$
$$-\sum\int [1+\ln F(i)]\left[\frac{\partial F(i)}{\partial t}\right]_c di. \tag{1.4.3.3}$$

The integrand in the first sum is nothing but $[\partial F(i)/\partial t]_c - \partial F(i)/\partial t$, and as the total number of particles in the plasma is conserved while also the total number of particles of each kind is unchanged in the collisions, we have

$$\int \frac{\partial F(i)}{\partial t}\, di = 0, \qquad \int \left[\frac{\partial F(i)}{\partial t}\right]_c di = 0$$

and hence

$$\dot{S} = -\sum\int \ln F(i)\left[\frac{\partial F(i)}{\partial t}\right]_c di.$$

Substituting here expression (1.3.2.9) for $[\partial F(i)/\partial t]_c$ we find

$$\dot{S} = -\sum\int \ln F(i) w\, [F(i')\,F(j') - F(i)\,F(j)]\, di\, di'\, dj\, dj'. \tag{1.4.3.4}$$

Bearing in mind that the expression uw is invariant under the transformations

$$ij \to ji,\quad i'j' \to i'j'; \qquad ij \to i'j',\quad i'j' \to ij; \qquad ij \to ij,\quad i'j' \to j'i';$$

we can rewrite eqn. (1.4.3.4) in the form

$$\dot{S} = \frac{1}{4}\sum\int \ln\left(\frac{F(i')\,F(j')}{F(i)\,F(j)}\right)\left[\frac{F(i')\,F(j')}{F(i)\,F(j)} - 1\right]F(i)\,F(j)\, w\, di\, di'\, dj\, dj'. \tag{1.4.3.5}$$

On the other hand, the function

$$\psi(x) = (x-1) \ln x$$

is non-negative so that we get from eqn. (1.4.3.5) the inequality

$$\dot{S} \geqslant 0: \tag{1.4.3.6}$$

The entropy of the plasma particle gas therefore does not decrease. Equation (1.4.3.6) is the *Boltzmann H-theorem* for a quiescent plasma.

We draw attention to the fact that neither the external nor the internal self-consistent field influences the change in the plasma entropy. This conclusion is connected with the assumption that the plasma is quiescent.

In the general case of a turbulent plasma such a conclusion is not valid as the higher-order correlation effects are important for a turbulent plasma and these have not been taken into account in the Boltzmann definition (1.4.3.1) of the entropy which contains only the single-particle distribution function, that is, the first correlation function. We shall, however, not consider here the general definition of entropy which contains higher-order correlation functions.

We noted earlier that the collision integral (1.3.2.2) is practically contained in the pair correlation function $G(1, 2)$ if we impose on it the condition (1.1.2.8) of correlation weakening. As, on the other hand, the existence of the collision integral leads to the H-theorem, that is, to irreversibility, one could say that irreversibility arises in kinetic theory through the requirement of correlation weakening.

We note that binary collisions are not the only mechanism for relaxation, even in a quiescent plasma. We note here another possible relaxation mechanism which acts when a strong external magnetic field is present, namely, *radiative relaxation* caused by the emission and absorption of electromagnetic waves by the plasma particles (electrons; Akhiezer, Aleksin, Baryakhtar, and Peletminskiĭ, 1962).

Radiative relaxation occurs over a time of the order of the ratio of the average electron energy to the average intensity of the emission by an electron in the magnetic field. In the non-relativistic case this time is of the order of magnitude of

$$\tau^{(r)} \sim \frac{c}{r_0 \omega_{Be}^2}, \tag{1.4.3.7}$$

where c is the velocity of light, $r_0 = e^2/m_e c^2$ the electron radius, and $\omega_{Be} = eB/m_e c$ is the electron Larmor frequency in a magnetic field B.

We first of all bear in mind that these equations are valid for a quiescent—not for a turbulent—plasma in the case of a small plasma parameter g when higher-order correlation effects do not play an important role. The self-consistent field is then an effect of order 1 while binary collisions are an effect of order g^1. From this it follows at once that kinetic equations for the single-particle distribution function without a collision integral can be used for studying high-frequency properties of quiescent plasmas. The frequencies ω of the plasma oscillations must then satisfy the condition

$$\omega\tau \gg 1, \tag{1.4.3.8}$$

where τ is the relaxation time defined by eqn. (1.4.1.5) and, furthermore, the condition

$$k\bar{v}\tau \gg 1, \tag{1.4.3.8'}$$

where k is the wavenumber of the excitation and $\bar{v}$ the thermal velocity, must be satisfied. We could have obtained these conditions by using the qualitative expression (1.4.1.6) for the collision integral and noting that the quantities $\partial F/\partial t$ and $(\boldsymbol{v}\cdot\partial F/\partial \boldsymbol{r})$ occurring in the kinetic equation can be replaced by ωF and $k\bar{v}F$. When conditions (1.4.3.8) and (1.4.3.8′) are satisfied, we talk about oscillations in a collisionless plasma.

As far as the kinetic eqn. (1.3.2.1) with the collision integral is concerned, as the collision integral leads to an increase in entropy, that is, to irreversibility, we can use this equation to study irreversible processes in a quiescent plasma and, in particular, to determine its transport coefficients such as electrical conductivity, diffusion constant, and thermal conductivity.

To estimate these coefficients we can use the well-known formulae from elementary kinetic theory of gases. To estimate the static *electrical conductivity of a plasma*, for instance, we can use the formula

$$\sigma \sim \frac{e^2 n}{m_e}\tau.$$

Substituting there for τ expression (1.4.1.5) we get

$$\sigma \sim \frac{T^{3/2}}{m_e^{1/2}e^2L}.$$

The exact expression for σ has the form

$$\sigma = \frac{4\sqrt{2}}{\pi\sqrt{\pi}}\,\frac{T^{3/2}}{m_e^{1/2}e^2L}. \tag{1.4.3.9}$$

1.5. The Hydrodynamical Description of a Plasma

1.5.1. THE HYDRODYNAMICAL DESCRIPTION

We showed in the preceding section that one can adequately describe high-frequency oscillatory processes in a quiescent plasma by using an equation with a self-consistent field without a collision integral together with the vacuum Maxwell equations for the electromagnetic field.

If the condition $\omega\tau \gg 1$ which is necessary for that description is not satisfied, but in contrast the following condition holds:

$$\omega\tau \ll 1, \tag{1.5.1.1}$$

where τ is the average time between binary collisions between the particles and ω is the frequency of the oscillations, we must take higher-order correlation effects into consideration and, first of all, the binary collision integral. In other words, in principle we can use for a study of oscillatory processes in a plasma when condition (1.5.1.1) is satisfied the kinetic eqn.

(1.3.2.1) with a self-consistent field and a collision integral together with the Maxwell equations.

However, such a method is very complicated and, in fact, unnecessary. Indeed, if we consider condition (1.5.1.1) as a condition on the relaxation time, it means nothing but the limit $\tau \to 0$ which entails the limit $l \to 0$ for the mean free path, $l \sim \bar{v}\tau$. This kind of limit means, however, that the plasma can be considered to be a continuous medium, and that we can neglect its molecular structure. We therefore do not need to use for the description of the plasma such concepts as particle distribution functions or correlation functions, but it is sufficient to use the more compact hydrodynamical description in which the basic concepts are the hydrodynamical velocity of the medium, its density, and its pressure. All these quantities, which we shall call macroscopic quantities, can change as functions of time and space, but their change is a slow one. In fact, if we denote by ω and L a frequency and a length characteristic for appreciable changes in the macroscopic quantities, the following conditions will be satisfied:

$$\omega\tau \ll 1, \quad L \gg l. \tag{1.5.1.2}$$

It is reasonable to use a hydrodynamical description if the physical problem is such that a local quasi-equilibrium particle distribution is fast established. The macroscopic quantities determine these distributions and they change according to laws which we shall call hydrodynamical laws. In principle one can formulate these laws starting from the kinetic description.

We showed in section 1.4 that due to the large mass difference between electrons and ions we must distinguish in the case of a plasma three relaxation times: the relaxation time τ_{ee} in the electron system, the relaxation time τ_{ii} in the ion system, and the relaxation time τ_{ei} for the establishment of equilibrium between the electrons and the ions. It follows from (1.4.1.5) and (1.4.2.2) that these relaxation times are interconnected through the relations

$$\tau_{ii} \sim \sqrt{\frac{m_i}{m_e}}\,\tau_{ee}, \qquad \tau_{ei} \sim \frac{m_i}{m_e}\,\tau_{ee},$$

where m_i and m_e are, respectively the ion and electron masses, so that the electron equilibrium (or rather quasi-equilibrium) is established first, then the ion equilibrium, and finally equilibrium between electrons and ions is reached.

The quasi-equilibrium electron and ion distributions are characterized by different temperatures and by different average (hydrodynamical) particle velocities so that we can in this case speak of a two-component—or a multi-component—plasma. After equilibrium is established between the electrons and the ions they have a common temperature and a common hydrodynamical particle velocity so that then we can consider the plasma to be a single hydrodynamical medium.

It is clear that the hydrodynamical velocities of different plasma components and their temperatures, and also the common temperature and the common hydrodynamic velocity, can change in time and space, but these changes are slow, in accordance with the conditions (1.5.1.2).

It follows from the picture we have just painted of the plasma that if the frequency ω of the oscillations satisfies the inequalities

$$\omega\tau_{ee} \ll 1, \quad \omega\tau_{ii} \ll 1, \quad \omega\tau_{ei} \gg 1,$$

we can, when studying such oscillations, consider the plasma to be a two-component, or multi-component, medium; if, however, $\omega\tau_{ei} \ll 1$, we can consider the plasma to be a single hydrodynamical medium.

So far we have connected the possibility of a hydrodynamical description with the smallness of the relaxation time, that is, with the case of "frequent" collisions between the particles. In the case of "rare" collisions when the condition $\omega\tau \gg 1$ is satisfied we might have assumed that only the kinetic description would be possible. However, for the case of "rare" collisions, or as one says differently for a "collisionless" plasma, one can also use a hydrodynamical description, and just in the case when the thermal spread in particle velocities is very small, that is, in the case of a sufficiently cold plasma. In that case also the particle velocity is the hydrodynamical velocity of the medium.

We shall now turn to a formulation of the hydrodynamical laws and we shall start with the case when the plasma can be considered to be a single hydrodynamical medium.

1.5.2. THE EQUATIONS OF MAGNETO-HYDRODYNAMICS

When we consider the plasma as a single hydrodynamical medium, the basic concepts are the hydrodynamical plasma velocity $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{r}, t)$ and its density $\varrho_m \equiv \varrho_m(\boldsymbol{r}, t)$, and the basic equations are the continuity equation,

$$\frac{\partial \varrho_m}{\partial t} + \operatorname{div} \varrho_m \boldsymbol{u} = 0, \tag{1.5.2.1}$$

and the Navier–Stokes equation,

$$\varrho_m \frac{d\boldsymbol{u}}{dt} \equiv \varrho_m \left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \right) = -\nabla p + \boldsymbol{f}_e + \boldsymbol{f}_v, \tag{1.5.2.2}$$

where p is the pressure and $\boldsymbol{f}_e$ and $\boldsymbol{f}_v$ are, respectively, the ponderomotive force density caused by the electromagnetic interaction and the force density caused by the viscosity of the medium. These forces are determined by the well-known equations

$$\boldsymbol{f}_e = \varrho_e \boldsymbol{E} + \frac{1}{c} [\boldsymbol{j} \wedge \boldsymbol{B}], \quad \boldsymbol{f}_v = \eta \nabla^2 \boldsymbol{u} + \left(\zeta + \frac{1}{3} \eta \right) \operatorname{grad} \operatorname{div} \boldsymbol{u}, \tag{1.5.2.3}$$

where $\boldsymbol{E}$ is the electrical field strength, $\boldsymbol{B}$ the magnetic induction, $\boldsymbol{j}$ the conduction current density, ϱ_e the charge density, and η and ζ the viscosity coefficients of the plasma.

For sufficiently slow motions and low frequencies the conduction current is determined by Ohm's law, taking the convective current into account:

$$\boldsymbol{j} = \sigma \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{u} \wedge \boldsymbol{B}] \right\}, \tag{1.5.2.4}$$

where σ is the electrical conductivity of the plasma for a static field. Strictly speaking, this formula is valid for not too large magnetic fields, as otherwise the conductivity becomes anisotropic. We note that in equation (1.5.2.4) we find not just the electrical field, but the quantity

$$E^* = E + \frac{1}{c}[u \wedge B],$$

which is the electrical field in a reference frame fixed to the moving plasma element.

We must add to the hydrodynamical equations (1.5.2.1) to (1.5.2.3) the Maxwell equations;

$$\text{curl}\, E = -\frac{1}{c}\frac{\partial B}{\partial t}, \qquad \text{div}\, B = 0,$$

$$\text{curl}\, H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} j, \quad \text{div}\, E = 4\pi\varrho_e, \tag{1.5.2.5}$$

where $H = B - 4\pi M$ is the magnetic field strength (M is the magnetization vector), the equation of state of the plasma,

$$p = p(\varrho_m, \theta), \tag{1.5.2.6}$$

and finally the equation for heat transfer which connects the entropy density, $s \equiv s(\varrho_m, \theta)$, with the energy density q which is dissipated per unit time in the plasma,

$$\varrho_m \theta \frac{ds}{dt} = q \equiv \sum_{\alpha, \beta} \pi_{\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta} + \text{div}\,(\varkappa \nabla\theta) + \frac{j^2}{\sigma}, \tag{1.5.2.7}$$

where θ is the plasma temperature, $\pi_{\alpha\beta}$ the viscous stress tensor,

$$\pi_{\alpha\beta} = \eta\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta}\sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma}\right) + \zeta\delta_{\alpha\beta}\sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma},$$

and $\varkappa$ the plasma thermal conductivity coefficient. The different terms on the right-hand side of eqn. (1.5.2.7) are, respectively, the amounts of heat produced per unit time and unit volume of the plasma due to viscosity, thermal conductivity, and electrical conductivity. In an ionized plasma the magnetization vector M vanishes so that the vectors B and H are the same.†

In the case of sufficiently slow motions and low frequencies, when the inequalities

$$\frac{\omega}{\sigma} \ll 1, \quad \frac{V}{\sigma L} \ll 1, \quad \frac{V}{c} \ll 1, \tag{1.5.2.8}$$

where V is a characteristic hydrodynamic velocity and L and ω are a characteristic length and a characteristic frequency of the changes in the macroscopic quantities, are satisfied one can make a number of simplifications, and especially the displacement current turns out to be small compared to the conduction current in which we can neglect the convective current $\varrho_e u$; finally we can neglect in the ponderomotive force the purely electrical force $\varrho_e E$.

To see this we note first of all that we have, as to order of magnitude,

$$\frac{1}{c}\left|\frac{\partial E}{\partial t}\right| \sim \frac{E}{cT},$$

† We refer to Landau and Lifshitz (1960) for a discussion of the separation of the magnetic induction into the magnetic field strength and the magnetization.

where $T(\sim \omega^{-1})$ is a characteristic time, and thus, by virtue of (1.5.2.8),

$$\frac{1}{c}\left|\frac{\partial E}{\partial t}\right| \ll \frac{4\pi}{c}\sigma |E|.$$

Moreover, we get from the first and fourth of eqn. (1.5.2.5) the following estimates:

$$E \sim \frac{L}{cT} B, \quad \varrho_e \sim \frac{E}{L}; \tag{1.5.2.9}$$

hence, we find

$$\frac{|\varrho_e u|}{|\sigma E|} \sim \frac{V}{\sigma L} \ll 1.$$

Finally, using (1.5.2.4) and putting $E \sim |[u \wedge B]/c|$, we get

$$\frac{|\varrho_e E|}{\left|\frac{1}{c}[j \wedge B]\right|} \sim \frac{V}{\sigma L}.$$

Thus, if conditions (1.5.2.8) are satisfied, and we shall in what follows assume that this is the case (moreover, the condition $\omega\tau_{ei} \ll 1$ must, of course, be satisfied; see Subsection 1.5.1) our basic equations take the following form:

$$\left.\begin{aligned} &\varrho_m \frac{du}{dt} = -\nabla p + \frac{1}{c}[j \wedge B] + f_v, \\ &\frac{\partial \varrho_m}{\partial t} + \operatorname{div} \varrho_m u = 0, \\ &\varrho_m \theta \frac{ds}{dt} = q, \\ &\operatorname{curl} B = \frac{4\pi}{c} j, \\ &j = \sigma\left\{E + \frac{1}{c}[u \wedge B]\right\}, \\ &\operatorname{div} B = 0, \\ &\operatorname{curl} E = -\frac{1}{c}\frac{\partial B}{\partial t}. \end{aligned}\right\} \tag{1.5.2.10}$$

These equations are, together with the equation of state, called the equations of magneto-hydrodynamics.†

One can easily obtain the equation satisfied by the magnetic field. To do this we express the electrical field in terms of the magnetic field:

$$E = -\frac{1}{c}[u \wedge B] + \frac{1}{\sigma} j = -\frac{1}{c}[u \wedge B] + \frac{c}{4\pi\sigma} \operatorname{curl} B. \tag{1.5.2.11}$$

† Here and henceforth we shall not distinguish between the vectors B and H and use the notation B.

Substituting now this equation into the last of eqns. (1.5.2.10), we find

$$\frac{\partial \boldsymbol{B}}{\partial t} = \text{curl}\,[\boldsymbol{u}\wedge\boldsymbol{B}] - \text{curl}\,(\nu_m\,\text{curl}\,\boldsymbol{B}), \tag{1.5.2.12}$$

where

$$\nu_m = \frac{c^2}{4\pi\sigma}.$$

This is now the required equation, satisfied by the magnetic field. For given $\boldsymbol{u}(\boldsymbol{r}, t)$ this equation, together with the equation div $\boldsymbol{B} = 0$, uniquely determines $\boldsymbol{B}$.

We note that the quantity ν_m has the same dimensions as the kinematic viscosity $\nu = \eta/\varrho_m$; it is called the *magnetic viscosity*.

If the conductivity σ does not depend on the coordinates, eqn. (1.5.2.12) becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \text{curl}\,[\boldsymbol{u}\wedge\boldsymbol{B}] + \nu_m \nabla^2\boldsymbol{B}. \tag{1.5.2.13}$$

When the conductivity is sufficiently large we can neglect the second term on the right-hand side of this equation. More exactly, to do this, one needs the following inequality to be satisfied:

$$R_\sigma \gg 1, \tag{1.5.2.14}$$

where

$$R_\sigma = \frac{LV}{\nu_m}. \tag{1.5.2.15}$$

This quantity, which is the analogue of the Reynolds number,

$$R_\nu = \frac{LV}{\nu},$$

is called the *magnetic Reynolds number* or the *Lundquist number* (Lundquist, 1952).

When $R_\sigma \gg 1$, the magnetic field satisfies the equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \text{curl}\,[\boldsymbol{u}\wedge\boldsymbol{B}], \quad R_\sigma \gg 1. \tag{1.5.2.16}$$

Substituting here

$$\text{curl}\,[\boldsymbol{u}\wedge\boldsymbol{B}] = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} - \boldsymbol{B}\,\text{div}\,\boldsymbol{u},$$

and using the relation

$$\text{div}\,\boldsymbol{u} = -\frac{1}{\varrho_m}\frac{\partial\varrho_m}{\partial t} - \left(\frac{\boldsymbol{u}}{\varrho_m}\cdot\nabla\varrho_m\right),$$

one shows easily that

$$\frac{d}{dt}\frac{\boldsymbol{B}}{\varrho_m} \equiv \left\{\frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\nabla)\right\}\frac{\boldsymbol{B}}{\varrho_m} = \left(\frac{\boldsymbol{B}}{\varrho_m}\cdot\nabla\right)\boldsymbol{u}. \tag{1.5.2.17}$$

On the other hand, the change of an element $\delta \boldsymbol{l}$ of some "fluid line" satisfies the same equation,

$$\frac{d}{dt}\,\delta \boldsymbol{l} = (\delta \boldsymbol{l}\cdot\nabla)\boldsymbol{u}.$$

One can thus say that when $R_\sigma \gg 1$ the magnetic field lines move together with the plasma particles which are situated on them: the magnetic field lines are so to speak "glued" to the plasma (Alfvén, 1950).

The condition $R_\sigma \gg 1$ means clearly that the energy dissipation caused by the conductivity of the medium is small. Similarly the condition $R_\nu \gg 1$ means that the energy dissipation caused by the viscosity of the medium is small. Finally, if the number

$$R_\chi = \frac{LV}{\chi},$$

where $\chi = \varkappa/c_p$ with $\varkappa$ the thermal conductivity coefficient and c_p the specific heat per unit volume, is large, the energy dissipation caused by the thermal conductivity is small.

We shall assume in what follows that the following inequalities hold:

$$R_\sigma \gg 1, \quad R_\nu \gg 1, \quad R_\chi \gg 1, \tag{1.5.2.18}$$

and we shall therefore neglect wherever possible energy dissipation. We note that these conditions have a different meaning for different problems. Without going into this question in detail we merely note that the quantity V for stationary flow has the meaning of the flow velocity and for wave motions the meaning of the phase velocity. In Subsection 2.1.4 we shall study the effect of energy dissipation on the damping of magneto-hydrodynamic waves.

When conditions (1.5.2.18) are satisfied one talks about an ideal medium. For the sake of convenience we once more write down the equations of magnetohydrodynamics for the special case of an ideal medium:†

$$\begin{aligned}
&\frac{\partial \varrho_m}{\partial t} + \operatorname{div}(\varrho_m \boldsymbol{u}) = 0,\\
&\frac{ds}{dt} = 0,\\
&\varrho_m \frac{d\boldsymbol{u}}{dt} = -\nabla p + \frac{1}{4\pi}[\operatorname{curl} \boldsymbol{B}\wedge \boldsymbol{B}],\\
&\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}[\boldsymbol{u}\wedge \boldsymbol{B}],\\
&\operatorname{div} \boldsymbol{B} = 0.
\end{aligned} \tag{1.5.2.19}$$

The absence of energy dissipation together with the assumption that the macroscopic processes proceed sufficiently slowly—this assumption is necessary in order that we can

† We note that these equations can be obtained from a variational principle (Polovin and Akhiezer, 1959; Seliger and Whitham, 1968).

apply the hydrodynamical description—finds its expression in the equation $ds/dt = 0$, indicating the *adiabaticity of the processes*:

$$s = \text{constant}.$$

We note that the condition $R_\sigma \gg 1$ in the non-relativistic case leads in turn to the first two conditions (1.5.2.8). Indeed, writing the condition $R_\sigma \gg 1$ in the form

$$\frac{\sigma L}{V} \cdot \frac{V^2}{c^2} \gg 1,$$

we have

$$\frac{\sigma L}{V} \gg \frac{c^2}{V^2} \gg 1.$$

Furthermore, from the condition $R_\sigma \gg 1$ we find the inequality

$$\sigma \gg \frac{c^2}{LV},$$

and as $L \sim V/\omega$, we have

$$\sigma \gg \frac{c^2\omega}{V^2} \gg \omega.$$

In the case of an ideal medium the first two conditions (1.5.2.8) are thus contained in (1.5.2.14).

1.5.3. TRANSITION FROM THE KINETIC TO THE HYDRODYNAMICAL DESCRIPTION

Let us now elucidates the idea of deriving the hydrodynamical equations from the kinetic equations. We shall first of all consider the simplest case when the system consists of one kind of particles which are subject to the Boltzmann equation

$$\mathscr{L}[f] \equiv \left\{\frac{\partial}{\partial t} + \left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) + \left(\frac{\boldsymbol{F}}{m}\cdot\frac{\partial}{\partial \boldsymbol{v}}\right)\right\} f = \mathscr{J}[f,f], \tag{1.5.3.1}$$

where $\boldsymbol{F}$ is the force acting upon a particle for which we shall take the Lorentz force,

$$\boldsymbol{F} = e\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\},$$

while $\mathscr{J}[f,f]$ is the collision integral (1.3.2.2) which is a quadratic functional of the distribution function $f \equiv f(\boldsymbol{v}, \boldsymbol{r}, t)$.

As the number of particles, the total momentum, and the total energy of the particles are conserved in the collisions, the collision integral satisfies the conditions

$$\begin{aligned} &\int \mathscr{J}[f,f]\, d^3\boldsymbol{v} = 0, \\ &\int m\boldsymbol{v}\mathscr{J}[f,f]\, d^3\boldsymbol{v} = 0, \\ &\int \tfrac{1}{2}mv^2\mathscr{J}[f,f]\, d^3\boldsymbol{v} = 0. \end{aligned} \tag{1.5.3.2}$$

From the kinetic equation (1.5.3.1) it then follows that we also have the relations

$$\begin{aligned}&\int \mathcal{L}[f]\, d^3v = 0,\\ &\int m\boldsymbol{v}\mathcal{L}[f]\, d^3v = 0,\\ &\int \tfrac{1}{2}mv^2\mathcal{L}[f]\, d^3v = 0.\end{aligned} \tag{1.5.3.3}$$

From these relations we can obtain the hydrodynamical equations.

Let us first of all consider the first of equations (1.5.3.3):

$$\frac{\partial}{\partial t}\int f\, d^3v + \left(\frac{\partial}{\partial \boldsymbol{r}}\cdot\int \boldsymbol{v}f\, d^3v\right) + \int\left(\frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right) d^3v = 0. \tag{1.5.3.4}$$

One sees easily that the last term vanishes. Indeed, writing it in the form

$$\int\left(\frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right) d^3v = \frac{e}{m}\left(\boldsymbol{E}\cdot\int\frac{\partial f}{\partial \boldsymbol{v}}\, d^3v\right) + \frac{e}{mc}\sum_\alpha\int[\boldsymbol{v}\wedge\boldsymbol{B}]_\alpha\frac{\partial f}{\partial v_\alpha}\, d^3v,$$

we find after integrating the second term by parts and applying Gauss's theorem to the first term:

$$\int\left(\frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}}\right) d^3v = \frac{e}{m}\left(\boldsymbol{E}\cdot\int \boldsymbol{n}f\, d^2\omega\right) - \frac{e}{mc}\sum_\alpha\int f\frac{\partial}{\partial v_\alpha}[\boldsymbol{v}\wedge\boldsymbol{B}]_\alpha\, d^3v,$$

where $\boldsymbol{n}$ is a unit vector normal to the surface of integration. However, the second integral is identically equal to zero while the first one vanishes because f decreases exponentially as $|\boldsymbol{v}| \to \infty$. We thus have

$$\frac{\partial}{\partial t}\int f\, d^3v + \left(\frac{\partial}{\partial \boldsymbol{r}}\cdot\int \boldsymbol{v}f\, d^3v\right) = 0,$$

or

$$\frac{\partial n}{\partial t} + \operatorname{div} n\boldsymbol{u} = 0, \tag{1.5.3.5}$$

where

$$n \equiv n(\boldsymbol{r}, t) = \int f\, d^3v, \qquad \boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{r}, t) = \frac{\int \boldsymbol{v}f\, d^3v}{\int f\, d^3v}. \tag{1.5.3.6}$$

The quantities $n(\boldsymbol{r}, t)$ and $\boldsymbol{u}(\boldsymbol{r}, t)$ are the particle density and the hydrodynamical velocity of the medium, and eqn. (1.5.3.5) is the well-known continuity equation.

Let us now consider the second equation (1.5.3.3):

$$\frac{\partial}{\partial t}\int mv_\alpha f\, d^3v + \sum_\beta\frac{\partial}{\partial x_\beta}\int mv_\alpha v_\beta f\, d^3v + \sum_\beta\int v_\alpha F_\beta\frac{\partial f}{\partial v_\beta}\, d^3v = 0. \tag{1.5.3.7}$$

According to what we did earlier we can write the first and third terms in the form

$$\frac{\partial}{\partial t}\int m v_\alpha f\, d^3v = \frac{\partial}{\partial t}\varrho_m u_\alpha,$$

$$\sum_\beta \int v_\alpha F_\beta \frac{\partial f}{\partial v_\beta}\, d^3v = -e\sum_\beta E_\beta \int f \frac{\partial v_\alpha}{\partial v_\beta}\, d^3v - \frac{e}{c}\sum_\beta \int f \frac{\partial}{\partial v_\beta} v_\alpha [\boldsymbol{v}\wedge \boldsymbol{B}]_\beta\, d^3v$$

$$= -en\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}\wedge\boldsymbol{B}]\right\}_\alpha,$$

where $\varrho_m = mn$.

Introducing now the tensor $\Pi_{\alpha\beta}$,

$$\Pi_{\alpha\beta} = \varrho_m u_\alpha u_\beta - \sigma_{\alpha\beta}, \tag{1.5.3.8}$$

where

$$\sigma_{\alpha\beta} = -m\int w_\alpha w_\beta f\, d^3v, \quad \boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u},$$

we can write the second term of (1.5.3.7) in the form

$$\sum_\beta \frac{\partial}{\partial x_\beta}\int m v_\alpha v_\beta f\, d^3v = \sum_\beta \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta}.$$

Equation (1.5.3.7) thus becomes

$$\frac{\partial}{\partial t}\varrho_m u_\alpha + \sum_\beta \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = en\left\{E_\alpha + \frac{1}{c}[\boldsymbol{u}\wedge\boldsymbol{B}]_\alpha\right\}.$$

Using the continuity equation we then get

$$\varrho_m \frac{du_\alpha}{dt} = \frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + en\left\{E_\alpha + \frac{1}{c}[\boldsymbol{u}\wedge\boldsymbol{B}]_\alpha\right\}, \tag{1.5.3.9}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_\beta u_\beta \frac{\partial}{\partial x_\beta}.$$

This equation is the hydrodynamical Navier–Stokes equation and $\sigma_{\alpha\beta}$ is nothing but the stress tensor.

We shall, finally, consider the third eqn. (1.5.3.3). Introducing the quantities

$$\varepsilon = \frac{\int \frac{1}{2} w^2 f\, d^3v}{\int f\, d^3v}, \qquad \boldsymbol{q} = \int m\boldsymbol{w}\,\tfrac{1}{2}w^2 f\, d^3v, \tag{1.5.3.9'}$$

which are the internal energy per unit mass and the energy flux density, and proceeding as before, we get

$$\frac{\partial}{\partial t}\left\{\varrho_m\left(\frac{1}{2}u^2+\varepsilon\right)\right\} + \sum_\beta \frac{\partial}{\partial x_\beta}\left\{\varrho_m u_\beta\left(\frac{1}{2}u^2+\varepsilon\right) - \sum_\alpha u_\alpha \sigma_{\alpha\beta}\right\} + \sum_\beta \frac{\partial q_\beta}{\partial x_\beta} = en\sum_\alpha E_\alpha u_\alpha. \tag{1.5.3.10}$$

This equation expresses the conservation of energy in hydrodynamics.

Using the continuity and Navier–Stokes equations we can rewrite eqn. (1.5.3.10) in the form

$$\varrho_m \frac{d\varepsilon}{dt} = \sum_{\alpha,\beta} u_{\alpha\beta}\sigma_{\alpha\beta} - \sum_{\beta} \frac{\partial q_\beta}{\partial x_\beta}, \tag{1.5.3.11}$$

where $u_{\alpha\beta}$ is the velocity deformation tensor,

$$u_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right).$$

Equations (1.5.3.5), (1.5.3.9), and (1.5.3.10) acquire an actual physical content only after the kinetic eqn. (1.5.3.1) has been solved and eqns. (1.5.3.6) and (1.5.3.8) have been used to evaluate the hydrodynamical quantities n, $\boldsymbol{u}$, and $\sigma_{\alpha\beta}$. We shall therefore elucidate the nature of the solutions of the kinetic eqn. (1.5.3.1) when the conditions for the applicability of the hydrodynamical description, $\omega\tau \ll 1$ and $L \gg l$, are fulfilled. When these conditions are satisfied we can look for the particle distribution function $f(\boldsymbol{v}, \boldsymbol{r}, t)$ as a sum of a local Maxwell distribution,

$$f^{(0)}(\boldsymbol{v}, \boldsymbol{r}, t) = n\left(\frac{m}{2\pi\theta}\right)^{3/2} \exp\left[-\frac{m(\boldsymbol{v}-\boldsymbol{u})^2}{2\theta}\right], \tag{1.5.3.12}$$

where $n = n(\boldsymbol{r}, t)$, $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}, t)$, and $\theta = \theta(\boldsymbol{r}, t)$ are slowly changing functions of the coordinates and the time, and a small correction, $f^{(1)}(\boldsymbol{v}, \boldsymbol{r}, t)$:

$$f(\boldsymbol{v}, \boldsymbol{r}, t) = f^{(0)}(\boldsymbol{v}, \boldsymbol{r}, t) + f^{(1)}(\boldsymbol{v}, \boldsymbol{r}, t), \quad |f^{(1)}| \ll f^{(0)}. \tag{1.5.3.13}$$

To check this we substitute expression (1.5.3.13) into the kinetic eqn. (1.5.3.1). Bearing in mind that the Maxwell distribution makes the collision integral vanish,

$$\mathcal{J}[f^{(0)}, f^{(0)}] = 0,$$

and that the collision integral $\mathcal{J}[f, f]$ is a quadratic functional of f, we introduce the functional $\mathcal{J}[f^{(1)}, f^{(0)}] = \mathcal{J}[f^{(0)}, f^{(1)}]$ which is linear in $f^{(1)}$, and we neglect terms quadratic in $f^{(1)}$,

$$2\mathcal{J}[f^{(1)}, f^{(0)}] \cong \mathcal{J}[f^{(0)}+f^{(1)}, f^{(0)}+f^{(1)}].$$

The kinetic equation then takes the form

$$\mathcal{L}[f^{(1)}+f^{(0)}] = 2\mathcal{J}[f^{(1)}, f^{(0)}].$$

Dropping on the left-hand side the small correction $f^{(1)}$ we obtain an equation from which, in principle, we can determine $f^{(1)}$:

$$2\mathcal{J}[f^{(1)}, f^{(0)}] = \mathcal{L}[f^{(0)}]. \tag{1.5.3.14}$$

In the general case this equation is a complicated integro-differential equation.

To obtain order of magnitude estimates we shall assume that

$$2\mathcal{J}[f^{(1)}, f^{(0)}] \sim \frac{f^{(1)}}{\tau},$$

and we then get

$$f^{(1)} \sim \tau \mathscr{L}[f^{(0)}]. \tag{1.5.3.15}$$

Clearly, however,

$$\mathscr{L}[f^{(0)}] \sim \left(\omega + \frac{l}{\tau L}\right) f^{(0)}$$

and hence

$$\frac{|f^{(1)}|}{f^{(0)}} \sim \omega\tau + \frac{l}{L} \ll 1,$$

as we had stated earlier.

We draw attention to the fact that eqn. (1.5.3.14) which is used to find the function $f^{(1)}$ does not determine it unambiguously. This is connected with the fact that if we put $f^{(1)} = \Phi$, where

$$\Phi = [c_0 + (\boldsymbol{c}_1 \cdot m\boldsymbol{v}) + c_2 \tfrac{1}{2} m v^2] f^{(0)}, \tag{1.5.3.16}$$

where c_0, $\boldsymbol{c}_1$, and c_2 are arbitrary constants, the functional $\mathscr{I}[f^{(1)}, f^{(0)}]$ vanishes:

$$\mathscr{I}[\Phi, f^{(0)}] = 0. \tag{1.5.3.17}$$

We must therefore formulate three more conditions to determine the function $f^{(1)}$ unambiguously. We can choose for these conditions the relations

$$\begin{aligned} \int f^{(1)}\, d^3v &= 0, \\ \int m\boldsymbol{v} f^{(1)}\, d^3v &= 0, \\ \int \tfrac{1}{2} m v^2 f^{(1)}\, d^3v &= 0. \end{aligned} \tag{1.5.3.18}$$

These conditions mean that the particle density, their average velocity, and their average energy are completely determined by the function $f^{(0)}$.

We shall call $f^{(0)}$ the zeroth and $f^{(1)}$ the first approximation distribution functions. The zeroth approximation corresponds to a quasi-equilibrium state and the first approximation takes into account small deviations from an equilibrium state.

We shall now turn to the evaluation of the hydrodynamic quantities defined by eqns. (1.5.3.6), (1.5.3.8), and (1.5.3.9′). Taking into account that the correction $f^{(1)}$ is small we can to begin with perform the calculations in the zeroth approximation. The density and hydrodynamical velocity will then clearly be the same as the parameters n and $\boldsymbol{u}$ occurring in the Maxwell distribution (1.5.3.12).

Moreover, we get in this approximation for the stress tensor $\sigma_{\alpha\beta}$ the expression

$$\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}, \tag{1.5.3.19}$$

where

$$p = \tfrac{1}{3} m \int w^2 f^{(0)}\, d^3v = n\theta.$$

This quantity is clearly the mechanical pressure. Substituting (1.5.3.19) into (1.5.3.8) we get the following expression for the tensor $\Pi_{\alpha\beta}$, which is called the *momentum flux density* *tensor*:

$$\Pi_{\alpha\beta} = \varrho_m u_\alpha u_\beta + p\delta_{\alpha\beta}. \tag{1.5.3.20}$$

Finally, we get for ε the expression

$$\varepsilon = \frac{3}{2m}\theta, \tag{1.5.3.21}$$

while the energy flux vector vanishes:

$$\boldsymbol{q}^{(0)} = 0.$$

Substitution of expressions (1.5.3.19) to (1.5.3.21) for $\sigma_{\alpha\beta}$, ε, and $\boldsymbol{q}$ into (1.5.3.9) and (1.5.3.11) leads to the equations

$$\begin{aligned} &\frac{\partial \varrho_m}{\partial t} + \operatorname{div} \varrho_m \boldsymbol{u} = 0, \\ &\varrho_m \frac{d\boldsymbol{u}}{dt} = -\nabla p + en\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}\wedge\boldsymbol{B}]\right\}, \\ &\frac{d\theta}{dt} + \frac{\theta}{c_v}\operatorname{div} u = 0, \end{aligned} \tag{1.5.3.22}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\nabla),$$

and $c_v = \frac{3}{2}$ is the specific heat per particle.

These relations are a set of hydrodynamical equations for a perfect gas and do not contain viscous forces or heat currents caused by viscosity or thermal conductivity. In other words, the eqns. (1.5.3.22) describe only reversible gas flow. This fact has a deep significance and is connected with the nature of the approximations made by us: we have taken for the distribution functions used to evaluate the hydrodynamical quantities the quasi-equilibrium Maxwell distribution which corresponds to the assumption that all processes are reversible.

The zeroth approximation distribution functions are insufficient to describe irreversible processes and we necessarily need to take the first approximation into account. It is just the function $f^{(1)}$ determining the deviation from the local equilibrium distribution which determines also both viscous forces and energy dissipation.

It follows from (1.5.3.15) that the irreversible fluxes evaluated by using $f^{(1)}$ will be proportional to the relaxation time, as should be the case according to the elementary kinetic theory. We shall, however, not perform here these calculations—one can find detailed calculations of irreversible fluxes in the literature (see, for instance, Chapman and Cowling, 1953, or Huang, 1963)—but turn to an elucidation of those complications which occur in a real plasma as compared to the simplest scheme which we presented here.

1.5.4. TWO-COMPONENT HYDRODYNAMICS

Thanks to the large difference in mass between an electron and an ion and the consequent difficulty of exchanging energy between electrons and ions, relaxation in a plasma takes place, as we saw in Section 1.4, in three stages: first, the quasi-equilibrium electron distribution is set up, then the quasi-equilibrium ion distribution and, finally, equilibrium is established between the electrons and the ions. The respective relaxation times τ_{ee}, τ_{ii}, and τ_{ei} are interconnected through the relations $\tau_{ii} \sim \sqrt{(m_i/m_e)}\,\tau_{ee}$, $\tau_{ei} \sim (m_i/m_e)\,\tau_{ee}$.

This nature of the relaxation makes it possible for us to consider the plasma, when we use a hydrodynamical description, as a collection of several "fluids"—as many as there are plasma components—with different hydrodynamical velocities and densities, and also with different temperatures. We can obtain the equations for this multi-component hydrodynamics in principle in the same way as in the case of systems with one kind of particle, but the initial kinetic equations are now not a single one, but several.

We shall elucidate the method for obtaining these equations by considering the case of a two-component plasma with one kind of ions and electrons. We write the kinetic equations for the distribution functions for the electrons, $f_e(\boldsymbol{v}^{(e)}, \boldsymbol{r}, t)$, and for the ions, $f_i(\boldsymbol{v}^{(i)}, \boldsymbol{r}, t)$ in the form

$$\begin{aligned} \mathcal{L}_e[f_e] &= \left[\frac{\partial f_e}{\partial t}\right]_c \equiv \mathcal{J}_{ee}[f_e, f_e] + \mathcal{J}_{ei}[f_e, f_i], \\ \mathcal{L}_i[f_i] &= \left[\frac{\partial f_i}{\partial t}\right]_c \equiv \mathcal{J}_{ie}[f_i, f_e] + \mathcal{J}_{ii}[f_i, f_i], \end{aligned} \tag{1.5.4.1}$$

where

$$\mathcal{L}_a = \frac{\partial}{\partial t} + \left(\boldsymbol{v}^{(a)} \cdot \frac{\partial}{\partial \boldsymbol{r}}\right) + \frac{e_a}{m_a}\left(\left\{\boldsymbol{E} + \frac{1}{c}\,[\boldsymbol{v}^{(a)} \wedge \boldsymbol{B}]\right\} \cdot \frac{\partial}{\partial \boldsymbol{v}^{(a)}}\right),$$

$a \equiv$ e, i and $\mathcal{J}_{ab}[f_a, f_b]$ is that part of the collision integral for particles of kind a which describes collisions between particles of kind a and particles of kind b.

From the laws of conservation of the numbers of particles of each kind, of the total momentum, and of the total energy, we have the following equations:

$$\int \left[\frac{\partial f_e}{\partial t}\right]_c d^3\boldsymbol{v}^{(e)} = 0, \qquad \int \left[\frac{\partial f_i}{\partial t}\right]_c d^3\boldsymbol{v}^{(i)} = 0, \tag{1.5.4.2}$$

$$\int m_e \boldsymbol{v}^{(e)} \left[\frac{\partial f_e}{\partial t}\right]_c d^3\boldsymbol{v}^{(e)} + \int m_i \boldsymbol{v}^{(i)} \left[\frac{\partial f_i}{\partial t}\right]_c d^3\boldsymbol{v}^{(i)} = 0, \tag{1.5.4.3}$$

$$\int \frac{1}{2} m_e v^{(e)2} \left[\frac{\partial f_e}{\partial t}\right]_c d^3\boldsymbol{v}^{(e)} + \int \frac{1}{2} m_i v^{(i)2} \left[\frac{\partial f_i}{\partial t}\right]_c d^3\boldsymbol{v}^{(i)} = 0. \tag{1.5.4.4}$$

Note that equations such as (1.5.4.3) and (1.5.4.4) are not valid for each kind of particle separately.

We shall assume that the conditions for the applicability of the hydrodynamical description,

$$\tau \ll T, \qquad l \ll L,$$

are valid and we shall thus assume that the state of the plasma differs little from a state of local equilibrium. In other words, we shall assume that the particle distribution functions have the form

$$\begin{aligned} f_e(\boldsymbol{v}^{(e)}, \boldsymbol{r}, t) &= f_e^{(0)}(\boldsymbol{v}^{(e)}, \boldsymbol{r}, t) + f_e^{(1)}(\boldsymbol{v}^{(e)}, \boldsymbol{r}, t), \\ f_i(\boldsymbol{v}^{(i)}, \boldsymbol{r}, t) &= f_i^{(0)}(\boldsymbol{v}^{(i)}, \boldsymbol{r}, t) + f_i^{(1)}(\boldsymbol{v}^{(i)}, \boldsymbol{r}, t), \end{aligned} \tag{1.5.4.5}$$

where $f_{e,i}^{(0)}$ are the local particle Maxwell distributions and $f_{e,i}^{(1)}$ small corrections to them, $|f_{e,i}^{(1)}| \ll f_{e,i}^{(0)}$.

Because of the multi-stage nature of the relaxation in a plasma we can assume that the local electron and ion Maxwell distributions correspond to different hydrodynamical velocities $\boldsymbol{u}^{(e)} \equiv \boldsymbol{u}^{(e)}(\boldsymbol{r}, t)$, $\boldsymbol{u}^{(i)} \equiv \boldsymbol{u}^{(i)}(\boldsymbol{r}, t)$ and to different temperatures $\theta_e \equiv \theta_e(\boldsymbol{r}, t)$, $\theta_i = \theta_i(\boldsymbol{r}, t)$, that is,

$$
\begin{aligned}
f_e^{(0)}(\boldsymbol{v}^{(e)}, \boldsymbol{r}, t) &= n_e\left(\frac{m_e}{2\pi\theta_e}\right)^{3/2} \exp\left\{-\frac{m_e(\boldsymbol{v}^{(e)}-\boldsymbol{u}^{(e)})^2}{2\theta_e}\right\},\\
f_i^{(0)}(\boldsymbol{v}^{(i)}, \boldsymbol{r}, t) &= n_i\left(\frac{m_i}{2\pi\theta_i}\right)^{3/2} \exp\left\{-\frac{m_i(\boldsymbol{v}^{(i)}-\boldsymbol{u}^{(i)})^2}{2\theta_i}\right\}
\end{aligned}
\tag{1.5.4.6}
$$

where $n_e \equiv n_e(\boldsymbol{r}, t)$ and $n_i \equiv n_i(\boldsymbol{r}, t)$ are the local electron and ion densities.

In contrast to the case considered in the preceding subsection, the functions $f_{e,i}^{(0)}$ do not lead to the vanishing of the collision integrals; they only make those parts of the collision integrals vanish which describe collisions between the same kind of particle,

$$
\mathcal{J}_{ee}[f_e^{(0)}, f_e^{(0)}] = 0, \quad \mathcal{J}_{ii}[f_i^{(0)}, f_i^{(0)}] = 0. \tag{1.5.4.7}
$$

The first approximation functions will now therefore be determined from the equations

$$
\begin{aligned}
2\mathcal{J}_{ee}[f_e^{(1)}, f_e^{(0)}]+\mathcal{J}_{ei}[f_e^{(1)}, f_i^{(0)}]+\mathcal{J}_{ei}[f_e^{(0)}, f_i^{(1)}] &= \mathcal{L}_e[f_e^{(0)}]-\mathcal{J}_{ei}[f_e^{(0)}, f_i^{(0)}],\\
2\mathcal{J}_{ii}[f_i^{(1)}, f_i^{(0)}]+\mathcal{J}_{ie}[f_i^{(1)}, f_e^{(0)}]+\mathcal{J}_{ie}[f_i^{(0)}, f_e^{(1)}] &= \mathcal{L}_i[f_i^{(0)}]-\mathcal{J}_{ie}[f_i^{(0)}, f_e^{(0)}],
\end{aligned}
\tag{1.5.4.8}
$$

to which we must add conditions, similar to (1.5.3.18):

$$
\begin{aligned}
&\int f_e^{(1)}\, d^3v^{(e)} = 0, \quad \int m_e \boldsymbol{v}^{(e)} f_e^{(1)}\, d^3v^{(e)} = 0, \quad \int \tfrac{1}{2} m_e v^{(e)2} f_e^{(1)}\, d^3v^{(e)} = 0,\\
&\int f_i^{(1)}\, d^3v^{(i)} = 0, \quad \int m_e \boldsymbol{v}^{(i)} f_i^{(1)}\, d^3v^{(i)} = 0, \quad \int \tfrac{1}{2} m_i v^{(i)2} f_i^{(1)}\, d^3v^{(i)} = 0.
\end{aligned}
\tag{1.5.4.9}
$$

If we assume that we know the solution of these equations we can then construct the hydrodynamical equations. To do that, let us consider eqns. (1.5.4.2) to (1.5.4.4). Equation (1.5.4.2) is the same as the first of eqns. (1.5.3.2). We therefore get immediately from it the separate continuity equations for the electron and ion components of the plasma:

$$
\begin{aligned}
\frac{\partial n_e}{\partial t} + \operatorname{div} n_e \boldsymbol{u}^{(e)} &= 0,\\
\frac{\partial n_i}{\partial t} + \operatorname{div} n_i \boldsymbol{u}^{(i)} &= 0,
\end{aligned}
\tag{1.5.4.10}
$$

where

$$
n_{e,i} = \int f_{e,i}\, d^3v, \quad \boldsymbol{u}^{e,i} = \frac{\int \boldsymbol{v} f_{e,i}\, d^3v}{\int f_{e,i}\, d^3v}.
$$

When there are two components, the second of eqns. (1.5.3.2) is satisfied for a single component only when we consider collisions between particles of the same kind:

$$
\begin{aligned}
&\int m_e \boldsymbol{v}^{(e)} \mathcal{J}_{ee}[f_e, f_e]\, d^3v^{(e)} = 0,\\
&\int m_i \boldsymbol{v}^{(i)} \mathcal{J}_{ii}[f_i, f_i]\, d^3v^{(i)} = 0.
\end{aligned}
$$

If we replace in these equations $\mathcal{J}_{ee}[f_e, f_e]$ by $\mathcal{L}_e[f_e] - \mathcal{J}_{ei}[f_e, f_i]$ and $\mathcal{J}_{ii}[f_i, f_i]$ by $\mathcal{L}_i[f_i] - \mathcal{J}_{ie}[f_i, f_e]$, we get

$$\begin{aligned} \int m_e \boldsymbol{v}^{(e)} \mathcal{L}_e[f_e]\, d^3v^{(e)} &= \int m_e \boldsymbol{v}^{(e)} \mathcal{J}_{ei}[f_e, f_i]\, d^3v^{(e)}, \\ \int m_i \boldsymbol{v}^{(i)} \mathcal{L}_i[f_i]\, d^3v^{(i)} &= \int m_i \boldsymbol{v}^{(i)} \mathcal{J}_{ie}[f_i, f_e]\, d^3v^{(i)}. \end{aligned} \tag{1.5.4.11}$$

We can transform the left-hand sides of these relations in the same way as in the preceding subsection $(d_{e,i}/dt \equiv \partial/\partial t + (\boldsymbol{u}_{e,i} \cdot \nabla))$

$$\int m_e v_\alpha^{(e)} \mathcal{L}_e[f_e]\, d^3v^{(e)} = m_e n_e \frac{d_e u_\alpha^{(e)}}{dt} - \sum_\beta \frac{\partial \sigma_{\alpha\beta}^{(e)}}{\partial x_\beta} + e n_e \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{u}^{(e)} \wedge \boldsymbol{B}] \right\}_\alpha ,$$

$$\int m_i v_\alpha^{(i)} \mathcal{L}_i[f_i]\, d^3v^{(i)} = m_i n_i \frac{d_i u_\alpha^{(i)}}{dt} - \sum_\beta \frac{\partial \sigma_{\alpha\beta}^{(i)}}{\partial x_\beta} - e n_i \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{u}^{(i)} \wedge \boldsymbol{B}] \right\}_\alpha .$$

Equations (1.5.4.11) therefore become

$$m_e n_e \frac{d_e u_\alpha^{(e)}}{dt} = \sum_\beta \frac{\partial \sigma_{\alpha\beta}^{(e)}}{\partial x_\beta} - e n_e \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{u}^{(e)} \wedge \boldsymbol{B}] \right\}_\alpha + R_\alpha^{(e)}, \tag{1.5.4.12}$$

$$m_i n_i \frac{d_i u_\alpha^{(i)}}{dt} = \sum_\beta \frac{\partial \sigma_{\alpha\beta}^{(i)}}{\partial x_\beta} + e n_i \left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{u}^{(i)} \wedge \boldsymbol{B}] \right\}_\alpha + R_\alpha^{(i)}, \tag{1.5.4.13}$$

where

$$\begin{aligned} \boldsymbol{R}^{(e)} &= \int m_e \boldsymbol{v}^{(e)} \mathcal{J}_{ei}[f_e, f_i]\, d^3v^{(e)}, \\ \boldsymbol{R}^{(i)} &= \int m_i \boldsymbol{v}^{(i)} \mathcal{J}_{ie}[f_i, f_e]\, d^3v^{(i)}, \\ \sigma_{\alpha\beta}^{(e)} &= -m_e \int w_\alpha^{(e)} w_\beta^{(e)} f_e\, d^3v^{(e)}, \\ \sigma_{\alpha\beta}^{(i)} &= -m_i \int w_\alpha^{(i)} w_\beta^{(i)} f_i\, d^3v^{(i)}, \\ \boldsymbol{w}^{(e)} &= \boldsymbol{v}^{(e)} - \boldsymbol{u}^{(e)}, \quad \boldsymbol{w}^{(i)} = \boldsymbol{v}^{(i)} - \boldsymbol{u}^{(i)}. \end{aligned}$$

We have obtained the Navier–Stokes equations in which there are, when compared with (1.5.3.9), the extra terms $\boldsymbol{R}^{(e)}$ and $\boldsymbol{R}^{(i)}$. These terms have a simple physical meaning: they determine the forces which act upon the electrons as a result of collisions with ions and upon the ions as a result of collisions with electrons. It follows from the conservation of momentum that

$$\boldsymbol{R}^{(i)} = -\boldsymbol{R}^{(e)}. \tag{1.5.4.14}$$

Let us finally consider the third eqn. (1.5.4.4). For particles of one kind we have

$$\begin{aligned} \int \tfrac{1}{2} m_e v^{(e)2} \mathcal{J}_{ee}[f_e, f_e]\, d^3v^{(e)} &= 0, \\ \int \tfrac{1}{2} m_i v^{(i)2} \mathcal{J}_{ii}[f_i, f_i]\, d^3v^{(i)} &= 0. \end{aligned}$$

From these equations it follows that

$$\begin{aligned} \int \tfrac{1}{2} m_e v^{(e)2} \mathcal{L}_e[f_e]\, d^3v^{(e)} &= \int \tfrac{1}{2} m_e v^{(e)2} \mathcal{J}_{ei}[f_e, f_i]\, d^3v^{(e)}, \\ \int \tfrac{1}{2} m_i v^{(i)2} \mathcal{L}_i[f_i]\, d^3v^{(i)} &= \int \tfrac{1}{2} m_i v^{(i)2} \mathcal{J}_{ie}[f_i, f_e]\, d^3v^{(i)}. \end{aligned}$$

Transforming the left-hand sides of these equations in the same way as was done in the preceding subsection and using the continuity equations and the Navier-Stokes equations we find

$$\frac{3}{2}\frac{\partial}{\partial t}n_e\theta_e+\frac{3}{2}\sum_\alpha\frac{\partial}{\partial x_\alpha}(n_e\theta_e u_\alpha^{(e)})+n_e\theta_e\sum_\alpha\frac{\partial u_\alpha^{(e)}}{\partial x_\alpha}-\sum_{\alpha,\beta}\pi_{\alpha\beta}^{(e)}u_{\alpha\beta}^{(e)}+\sum_\alpha\frac{\partial q_\alpha^{(e)}}{\partial x_\alpha}=Q_e, \quad (1.5.4.15)$$

$$\frac{3}{2}\frac{\partial}{\partial t}n_i\theta_i+\frac{3}{2}\sum_\alpha\frac{\partial}{\partial x_\alpha}(n_i\theta_i u_\alpha^{(i)})+n_i\theta_i\sum_\alpha\frac{\partial u_\alpha^{(i)}}{\partial x_\alpha}-\sum_{\alpha,\beta}\pi_{\alpha\beta}^{(i)}u_{\alpha\beta}^{(i)}+\sum_\alpha\frac{\partial q_\alpha^{(i)}}{\partial x_\alpha}=Q_i, \quad (1.5.4.16)$$

where $\pi_{\alpha\beta}^{(e;\,i)}$ is the viscous stress tensor,

$$\pi_{\alpha\beta}^{(e,\,i)}=\sigma_{\alpha\beta}^{(e,\,i)}+n_{e,\,i}\theta_{e,\,i}\delta_{\alpha\beta},$$
$$u_{\alpha\beta}^{(e,\,i)}=\frac{1}{2}\left[\frac{\partial u_\alpha^{(e,\,i)}}{\partial x_\beta}+\frac{\partial u_\beta^{(e,\,i)}}{\partial x_\alpha}\right],$$

and

$$q^{(e)}=\frac{1}{n_e}\int w^{(e)}\frac{1}{2}w^{(e)2}f_e^{(1)}\,d^3v^{(e)},$$
$$q^{(i)}=\frac{1}{n_i}\int w^{(i)}\frac{1}{2}w^{(i)2}f_i^{(1)}\,d^3v^{(i)},$$
$$Q_e=\int\tfrac{1}{2}m_e v^{(e)2}\mathcal{J}_{ei}[f_e,f_i]\,d^3v^{(e)},$$
$$Q_i=\int\tfrac{1}{2}m_i v^{(i)2}\mathcal{J}_{ie}[f_i,f_e]\,d^3v^{(i)}.$$

As compared to eqns. (1.5.3.11), eqns. (1.5.4.15) and (1.5.4.16) contain the extra terms Q_e and Q_i. The first of these is the amount of heat obtained by the electrons per unit time as a result of their collisions with ions, and the second one the analogous quantity for ions. It follows from the conservation of energy that

$$Q_i=-Q_e.$$

Once we have obtained the solution of eqns. (1.5.4.8), that is, once we know $f_e^{(1)}$ and $f_i^{(1)}$, we can evaluate R, $\sigma_{\alpha\beta}$, q, and Q which occur in the hydrodynamical eqns. (1.5.4.12) to (1.5.4.16). We shall not do this here but only give a few final results.† If $n_e=n_i=n$, we have

$$R^{(e)}=R^{(j)}+R^{(T)},$$
$$R^{(j)}=en\left[\frac{j_\parallel}{\sigma_\parallel}+\frac{j_\perp}{\sigma_\perp}\right], \quad (1.5.4.17)$$
$$R^{(T)}=-b_\parallel\nabla_\parallel\theta_e-b_\perp\nabla_\perp\theta_e-b_B[b\wedge\nabla\theta_e],$$

where $j_\parallel$ and $j_\perp$ are the components of the current density which are parallel and perpendicular to the magnetic field,

$$j=ne(u^{(i)}-u^{(e)}),$$

† The equations given here, as well as formulae for q, Q, and $\Pi_{\alpha\beta}^{(e;\,i)}$ were obtained by Braginskiĭ (1965).

$\nabla_{\parallel}$ and $\nabla_{\perp}$ are the analogous components of the ∇-operator, $\sigma_{\parallel}$ and $\sigma_{\perp}$ are the longitudinal and transverse plasma conductivity,

$$\sigma_{\parallel} = 2\frac{ne^2\tau_{ee}}{m_e},$$

$$\sigma_{\perp} = \begin{cases} \sigma_{\parallel}, & \text{if} \quad \omega_{Be}\tau_{ee} \ll 1, \\ \sigma_{\parallel}/2, & \text{if} \quad \omega_{Be}\tau_{ee} \gg 1, \end{cases} \qquad \omega_{Be} = \frac{eB}{m_e c},$$

$\boldsymbol{b} = \boldsymbol{B}/B$, and, finally,

$$b_{\parallel} = 0.71n,$$

$$b_{\perp} = \begin{cases} b_{\parallel}, & \text{if} \quad \omega_{Be}\tau_{ee} \ll 1, \\ 0, & \text{if} \quad \omega_{Be}\tau_{ee} \gg 1, \end{cases}$$

$$b_B = \begin{cases} 0, & \text{if} \quad \omega_{Be}\tau_{ee} \ll 1, \\ \dfrac{3}{2}\cdot\dfrac{n}{\omega_e\tau_{ee}}, & \text{if} \quad \omega_{Be}\tau_{ee} \gg 1. \end{cases}$$

The formulae given here are valid for any values of the velocities $\boldsymbol{u}^{(e)}$ and $\boldsymbol{u}^{(i)}$. The quantities $\sigma_{\alpha\beta}^{(e,\,i)}$, however, depend in an essential way on the ratio of the average electron and ion velocities. We shall give here the expressions for $\sigma_{\alpha\beta}^{(e,\,i)}$ only for the case when these velocities differ little from one another† and when the magnetic field is sufficiently weak ($\omega_{Be}\tau_{ee} \ll 1$, $\omega_{Bi}\tau_{ii} \ll 1$):

$$\sigma_{\alpha\beta}^{(e,\,i)} = \eta_{e,\,i}u_{\alpha\beta}^{(e,\,i)},$$

where

$$\eta_e = 2n_e\theta_e\tau_{ee}, \qquad \eta_i = 2n_i\theta_i\tau_{ii}.$$

1.5.5. GENERALIZED OHM LAW

We shall now show how to change from the equations of two-component hydrodynamics to the equations of magneto-hydrodynamics. In magneto-hydrodynamics we are dealing with an average density ϱ_m of the medium and with an average hydrodynamical velocity $\boldsymbol{u}$. These quantities are clearly connected with the quantities n_e, n_i, $\boldsymbol{u}^{(e)}$, and $\boldsymbol{u}^{(i)}$ which appear in the equations of two-component hydrodynamics through the relations

$$\varrho_m = m_e n_e + m_i n_i, \qquad \boldsymbol{u} = \frac{m_e n_e \boldsymbol{u}^{(e)} + m_i n_i \boldsymbol{u}^{(i)}}{m_e n_e + m_i n_i}. \tag{1.5.5.1}$$

From this it is clear that to obtain the equations to be satisfied by the quantities ϱ_m and $\boldsymbol{u}$ we must, first of all, add the continuity equations (1.5.4.10) for the electron and the ion component and, secondly, add the equations of motion (1.5.4.12) and (1.5.4.13) for these components. As a result we obtain the continuity equation for the whole plasma:

$$\frac{\partial \varrho_m}{\partial t} + \operatorname{div} \varrho_m \boldsymbol{u} = 0, \tag{1.5.5.2}$$

† Kiriĭ and Silin (1969) have given expressions for $\sigma_{\alpha\beta}$ taking terms quadratic in $\boldsymbol{u}^{(e)} - \boldsymbol{u}^{(i)}$ into account.

and the equation of motion for the average plasma velocity $\boldsymbol{u}$:

$$\frac{\partial}{\partial t}\varrho_m u_\alpha + \sum_\beta \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = \varrho_e E_\alpha + \frac{1}{c}[\boldsymbol{j}\wedge\boldsymbol{B}]_\alpha . \tag{1.5.5.3}$$

Here $\Pi_{\alpha\beta}$ is the total momentum current density tensor which is equal to

$$\Pi_{\alpha\beta} = \Pi^{(e)}_{\alpha\beta} + \Pi^{(i)}_{\alpha\beta},$$

ϱ_e is the electric charge density,

$$\varrho_e = e(n_i - n_e),$$

and $\boldsymbol{j}$ is the conduction current density,

$$\boldsymbol{j} = en_i \boldsymbol{u}^{(i)} - en_e \boldsymbol{u}^{(e)}.$$

(We note that if the additional conditions (1.5.4.9) are satisfied, the conduction current density is determined solely by the Maxwell distribution of the particles.)

We shall now transform the expression for the total momentum current density tensor. We shall assume that the quasi-neutrality condition is satisfied for the plasma, $n_i \approx n_e = n$. Up to terms of order m_e/m_i we have then the following relations:

$$\boldsymbol{u}^{(i)} = \boldsymbol{u} + \frac{m_e}{m_i ne}\boldsymbol{j},$$

$$\boldsymbol{u}^{(e)} = \boldsymbol{u} - \frac{\boldsymbol{j}}{ne},$$

and the momentum current density tensor will have the form

$$\Pi_{\alpha\beta} = \varrho_m u_\alpha u_\beta + p\delta_{\alpha\beta} - \pi_{\alpha\beta} + \frac{m_e}{ne^2} j_\alpha j_\beta,$$

$$p = p_e + p_i, \quad \pi_{\alpha\beta} = \pi^{(e)}_{\alpha\beta} + \pi^{(i)}_{\alpha\beta}.$$

One easily checks that in the expression for $\Pi_{\alpha\beta}$ one can drop, firstly, the viscous stress tensor $\pi^{(e)}_{\alpha\beta}$ and, secondly, the quantity $m_e j_\alpha j_\beta/ne^2$. Indeed, the ratio of the viscosity coefficients η_e and η_i of the electron and the ion components is, as to order of magnitude, equal to $\eta_e/\eta_i \sim \sqrt{(m_e/m_i)}$. Therefore, $|\pi^{(e)}_{\alpha\beta}| \ll |\pi^{(i)}_{\alpha\beta}|$. As to the expression $m_e j_\alpha j_\beta/ne^2$, it is of the order of magnitude of $m_e c^2 B^2/ne^2 L^2$. Therefore we have

$$\frac{\left|\sum_\beta \frac{\partial}{\partial x_\beta}\frac{m_e}{ne^2} j_\alpha j_\beta\right|}{\left|\frac{1}{c}[\boldsymbol{j}\wedge\boldsymbol{B}]\right|} \sim \frac{m_e c^2}{ne^2 L^2}.$$

If the condition

$$\xi \equiv \frac{ne^2 L^2}{m_i c^2} \gg 1$$

is satisfied—we shall make clear in what follows the relation between this condition and the criteria $\omega\tau_{ei} \ll 1$, $R_\sigma \gg 1$—we can neglect the term $m_e j_\alpha j_\beta / ne^2$.

Noting that

$$\sum_\beta \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = \sum_\beta \varrho_m u_\beta \frac{\partial u_\alpha}{\partial x_\beta} - u_\alpha \frac{\partial \varrho_m}{\partial t} + \frac{\partial p}{\partial x_\alpha} - \sum_\beta \frac{\partial \pi^{(i)}_{\alpha\beta}}{\partial x_\beta},$$

we get from (1.5.5.3) the Navier–Stokes equation,

$$\varrho_m \frac{du_\alpha}{dt} = -\frac{\partial p}{\partial x_\alpha} + \sum_\beta \frac{\partial \pi^{(i)}_{\alpha\beta}}{\partial x_\beta} + \left\{ \varrho_e \boldsymbol{E} + \frac{1}{c}[\boldsymbol{j} \wedge \boldsymbol{B}] \right\}_\alpha . \qquad (1.5.5.4)$$

The viscous stress tensor $\pi^{(i)}_{\alpha\beta}$ is here expressed in terms of the average plasma velocity $\boldsymbol{u}$ in the usual way:

$$\pi^{(i)}_{\alpha\beta} = \eta_i \left[\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} - \frac{2}{3}\delta_{\alpha\beta} \sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma} \right] + \zeta \delta_{\alpha\beta} \sum_\gamma \frac{\partial u_\gamma}{\partial x_\gamma},$$

as we can assume that when $m_e \ll m_i$ the velocity of the ion component is the same as the average plasma velocity, $\boldsymbol{u}^{(i)} \approx \boldsymbol{u}$ and that the viscosity coefficients η_i and ζ_i are determined by the ion component.

We saw in Subsection 1.5.2 that Ohm's law was one of the basic relations of magnetohydrodynamics. We must therefore still make clear what is the relation between the formulation of Ohm's law, given in Subsection 1.5.2, with the equations of two-component hydrodynamics.

To do this we turn to the equation of motion (1.5.4.12) for the electron component in which we neglect the viscous terms and in which we shall assume that $n_e = n_i = n$:

$$-nm_e \frac{d_e \boldsymbol{u}^{(e)}}{dt} - \nabla p_e + en\left\{ \boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}^{(e)} \wedge \boldsymbol{B}] \right\} + \boldsymbol{R}^{(e)} = 0. \qquad (1.5.5.5)$$

Assuming that the the condition

$$\omega \ll |\omega_{Be}| \equiv \frac{eB}{m_e c}$$

is satisfied, we can clearly neglect on the left-hand side of (1.5.5.5) the first term, $-m_e n \, d_e \boldsymbol{u}^{(e)}/dt$. Substituting for $\boldsymbol{R}^{(e)}$ expression (1.5.4.17) we obtain the equation

$$\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u} \wedge \boldsymbol{B}] + \boldsymbol{E}^{(p)} + \boldsymbol{E}^{(T)} = \frac{\boldsymbol{j}_{\parallel}}{\sigma_{\parallel}} + \frac{\boldsymbol{j}_{\perp}}{\sigma_{\perp}} - \frac{B}{enc}[\boldsymbol{b} \wedge \boldsymbol{j}], \quad \boldsymbol{b} = \frac{\boldsymbol{B}}{B}, \qquad (1.5.5.6)$$

where

$$\boldsymbol{E}^{(p)} = -\frac{1}{ne}\nabla p_e,$$

$$\boldsymbol{E}^{(T)} = \boldsymbol{E}^{(T)}_{\parallel} + \boldsymbol{E}^{(T)}_{\perp} + \boldsymbol{E}^{(T)}_{B},$$

$$\boldsymbol{E}^{(T)}_{\parallel} = \frac{0.71}{e}\nabla_{\parallel}\theta_e,$$

$$E_{\perp}^{(T)} = \begin{cases} \dfrac{0.71}{e} \nabla_{\perp}\theta_e, & \text{if} \quad \omega_{Be}\tau_{ee} \ll 1, \\ 0, & \text{if} \quad \omega_{Be}\tau_{ee} \gg 1, \end{cases}$$

$$E_{B}^{(T)} = \begin{cases} 0, & \text{if} \quad \omega_{Be}\tau_{ee} \ll 1, \\ \dfrac{3}{2}\dfrac{1}{e\omega_e\tau_{ee}}[\boldsymbol{b}\wedge\nabla\theta_e], & \text{if} \quad \omega_{Be}\tau_{ee} \gg 1. \end{cases}$$

Equation (1.5.5.6) is the generalized Ohm law for the plasma.

We emphasize that there occur in this equation a number of terms which did not appear in eqn. (1.5.2.4). These terms are connected with pressure and temperature gradients (the terms $\boldsymbol{E}^{(p)}$ and $\boldsymbol{E}^{(T)}$).† Moreover, we have here the term $-B[\boldsymbol{b}\wedge\boldsymbol{j}]/enc$ which determines the Hall field. Finally, instead of the scalar σ in the Ohm law we have here two quantities $\sigma_{\parallel}$ and $\sigma_{\perp}$ which are the longitudinal and transverse plasma conductivities. If the condition $\omega_{Be}\tau_{ee} \ll 1$ is satisfied, these quantities are the same, but if $\omega_{Be}\tau_{ee} \gg 1$, we have $\sigma_{\parallel} = 1.96\sigma_{\perp}$.

We shall show that in eqn. (1.5.5.6) we can neglect the terms $\boldsymbol{E}^{(p)}$ and $\boldsymbol{E}^{(T)}$ (they are the same order of magnitude), provided the frequency ω of the macroscopic quantities satisfies the condition

$$\omega \ll \omega_{Bi},$$

where

$$\omega_{Bi} = \frac{eB}{m_i c}.$$

To see this, we consider first of all the case when the quantity

$$\beta = \frac{8\pi p}{B^2}$$

is small, $\beta \lesssim 1$. In that case $|\nabla p| \lesssim |[\boldsymbol{j}\wedge\boldsymbol{B}]/c|$ and hence $|nm_i\, du/dt| \sim |[\boldsymbol{j}\wedge\boldsymbol{B}]/c|$, which means $m_i n\omega u \sim jB/c$. Therefore

$$\frac{|\boldsymbol{E}^{(p)}|}{|[\boldsymbol{j}\wedge\boldsymbol{B}]/c|} \sim \frac{|\nabla p|}{enuB/c} \lesssim \frac{jB/c}{enuB/c} \sim \frac{m_i n\omega u}{enuB/c} = \frac{\omega}{\omega_{Bi}} \ll 1.$$

If, however, $\beta \gtrsim 1$, we have $|\nabla p| \gtrsim jB/c$ and $m_i n\omega u \sim \nabla p$. We have therefore again

$$\frac{|\nabla p|}{enuB/c} \sim \frac{\omega}{\omega_{Bi}} \ll 1.$$

Let us also elucidate under what conditions we can drop in Ohm's law the Hall term $-B[\boldsymbol{b}\wedge\boldsymbol{j}]/enc$. Noting that

$$\sigma \sim \frac{ne^2\tau_{ee}}{m_e},$$

† They describe thermoelectric effects (Landau and Lifshitz, 1960). Various authors (Samokhin, 1963 a, b; Malik and Trehan, 1965; Demetriades and Argyropoulos, 1966; Polovin and Cherkasova, 1962a) have evaluated these terms when there are neutral particles present.

we have

$$\frac{|B[\boldsymbol{b}\wedge \boldsymbol{j}]/enc|}{|\boldsymbol{j}/\sigma|} \sim \omega_{Be}\tau_{ee}.$$

We can thus drop the Hall term, if the condition

$$|\omega_{Be}|\,\tau_{ee} \ll 1$$

is satisfied.

We can clearly use the equations of magneto-hydrodynamics, if $\tau_{ei} \ll T$. One sees easily that if that condition is satisfied and also the inequality $R_\sigma = LV/\nu \gtrsim 1$ holds, the condition

$$\xi \equiv \frac{ne^2L^2}{m_i c^2} \gg 1 \tag{1.5.5.7}$$

will also be satisfied. Indeed, using the relations

$$\tau_{ei} \sim \frac{m_i}{m_e}\tau_{ee}, \quad \sigma \sim \frac{ne^2}{m_e}\tau_{ee}, \quad V = \frac{L}{T},$$

we have $T \gg m_i\tau_{ee}/m_e$, $T \lesssim L^2e^2n\tau_{ee}/m_ec^2$, whence follows (1.5.5.7).

We have earlier used the quasi-neutrality condition,

$$\frac{|n_e - n_i|}{n_e} \ll 1.$$

We shall show that this condition is satisfied, if

$$\xi \gg 1, \quad nm_ic^2 \gg \frac{V^2}{c^2}B^2. \tag{1.5.5.8}$$

To do this we show that

$$\varrho_e = e(n_i - n_e) \sim \frac{E}{L},$$

and as $E \sim (V/c)B$, we have

$$\frac{|n_e - n_i|}{n_e} \sim \frac{VB}{encL}.$$

The quasi-neutrality condition is thus equivalent to the inequality

$$L \gg \frac{V}{c}\,\frac{B}{en},$$

which one easily sees to be a consequence of inequalities (1.5.5.8).

We noted in Subsection 1.1.1 that the Coulomb interaction is screened at a distance of the order of the Debye radius r_D. For quasi-neutrality it is therefore necessary that

$$L \gg r_D.$$

One easily checks that this condition follows from the condition $\xi \gg 1$ provided the ions have a non-relativistic temperature, $\theta_i \ll m_i c^2$.

In concluding this section we give in Table 1.5.5.1 some data about the parameters in some different plasmas (Polovin and Cherkasova, 1963; Alfvén and Fälthammer, 1963; Lehnert, 1959).

TABLE 1.5.5.1. *Parameters of different plasmas*

	Interstellar plasma	Solar corona	Sunspot	Ionosphere
Characteristic macroscopic length L (cm)	10^{11}	10^{9}	10^{8}	10^{6}
Magnetic field (Oersted)	10^{-5}	10^{2}	10^{3}	1
Electron density (cm^{-3})	1	10^{7}	10^{12}	10^{5}
Plasma mass density (g/cm^3)	10^{-24}	10^{-17}	10^{-12}	10^{-13}
Electron temperature (°K)	10^{4}	10^{6}	10^{4}	10^{3}
Electrical conductivity (s^{-1})	10^{13}	10^{16}	10^{13}	10^{10}
Viscosity coefficient (cm^2/s)	10^{18}	10^{16}	10^{6}	10^{10}
Ratio of characteristic velocity to velocity of light	10^{-4}	10^{-1}	10^{-2}	10^{-5}
Parameter $\beta = 8\pi p/B^2$	10^{-1}	10^{-6}	10^{-5}	10^{-3}
Reynolds number, $R_\nu = LV/\nu$	10^{3}	10^{2}	10^{9}	10^{2}
Lundquist number	10^{10}	10^{15}	10^{9}	10^{2}
Ratio of the frequency of a magneto-hydrodynamical wave to the ion cyclotron frequency	10^{-4}	10^{-5}	10^{-7}	10^{-1}
Parameter $\lvert\omega_{Be}\rvert\tau_{ee}$	10^{7}	10^{10}	10^{3}	10^{4}

CHAPTER 2

Small Amplitude Magneto-hydrodynamic Waves

2.1. Magneto-sound and Alfvén Waves

2.1.1. PHASE VELOCITIES AND POLARIZATION

We shall now consider systematically the oscillations which can be excited and which can propagate in a plasma. We start with a study of low-frequency oscillations and we shall use for their description the hydrodynamical method. To begin with, we shall assume that there is no energy dissipation so that the magneto-hydrodynamics equations have the form (1.5.2.19).

We shall consider one-dimensional waves propagating along the z-axis for which all magneto-hydrodynamical quantities depend solely on z and t. The set of eqns. (1.5.2.19) takes in that case the form

$$\begin{aligned}
\frac{\partial \varrho}{\partial t}+u_z\frac{\partial \varrho}{\partial z}+\varrho\frac{\partial u_z}{\partial z} &= 0,\\
\frac{\partial s}{\partial t}+u_z\frac{\partial s}{\partial z} &= 0,\\
\frac{\partial u_x}{\partial t}+u_z\frac{\partial u_x}{\partial z}-\frac{B_z}{4\pi\varrho}\frac{\partial B_x}{\partial z} &= 0,\\
\frac{\partial u_y}{\partial t}+u_z\frac{\partial u_y}{\partial z}-\frac{B_z}{4\pi\varrho}\frac{\partial B_y}{\partial z} &= 0,\\
\frac{\partial u_z}{\partial t}+\frac{p_\varrho}{\varrho}\frac{\partial \varrho}{\partial z}+\frac{p_s}{\varrho}\frac{\partial s}{\partial z}+u_z\frac{\partial u_z}{\partial z}+\frac{B_x}{4\pi\varrho}\frac{\partial B_x}{\partial z}+\frac{B_y}{4\pi\varrho}\frac{\partial B_y}{\partial z} &= 0,\\
\frac{\partial B_x}{\partial t}-B_z\frac{\partial u_x}{\partial z}+B_x\frac{\partial u_z}{\partial z}+u_z\frac{\partial B_x}{\partial z} &= 0,\\
\frac{\partial B_y}{\partial t}-B_z\frac{\partial u_y}{\partial z}+B_y\frac{\partial u_z}{\partial z}+u_z\frac{\partial B_y}{\partial z} &= 0,\\
\frac{\partial B_z}{\partial t}=\frac{\partial B_z}{\partial z} &= 0.
\end{aligned} \tag{2.1.1.1}$$

Here $p_\varrho = (\partial p/\partial \varrho)_s$, $p_s = (\partial p/\partial s)_\varrho$ and we omit here and henceforth the index m of the density ϱ_m.

From the last of eqns. (2.1.1.1) it follows that

$$B_z = \text{constant}.$$

In a one-dimensional wave there are thus seven magneto-hydrodynamic variables: ϱ, s, u_x, u_y, u_z, B_x, and B_y.

It is convenient to write the set of eqns. (2.1.1.1) in matrix form (Bohachevsky, 1962):

$$\frac{\partial a_i}{\partial t}+\sum_{j=1}^{7} Z_{ij}(a)\,\frac{\partial a_j}{\partial z} = 0, \tag{2.1.1.2}$$

where a_i indicates the *magneto-hydrodynamical state "vector"* of the plasma, that is, the set of the magneto-hydrodynamical quantities

$$a_i = \begin{bmatrix} \varrho \\ s \\ u_x \\ u_y \\ u_z \\ B_x \\ B_y \end{bmatrix}$$

(the index i can here take on the values 1, 2, ..., 7 corresponding to the seven magneto-hydrodynamical quantities in the column a_i), and $Z_{ij}(a)$ is the matrix

$$Z_{ij} = \begin{bmatrix} u_z & 0 & 0 & 0 & \varrho & 0 & 0 \\ 0 & u_z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_z & 0 & 0 & -\dfrac{B_z}{4\pi\varrho} & 0 \\ 0 & 0 & 0 & u_z & 0 & 0 & -\dfrac{B_z}{4\pi\varrho} \\ \dfrac{p_\varrho}{\varrho} & \dfrac{p_s}{\varrho} & 0 & 0 & u_z & \dfrac{B_x}{4\pi\varrho} & \dfrac{B_y}{4\pi\varrho} \\ 0 & 0 & -B_z & 0 & B_x & u_z & 0 \\ 0 & 0 & 0 & -B_z & B_y & 0 & u_z \end{bmatrix}. \tag{2.1.1.3}$$

We shall first of all consider small oscillations and therefore linearize the set (2.1.1.2) Putting $a_i = a_i^{(0)}+a_i^{(1)}$ where the quantities $a_i^{(0)}$ correspond to the uniform equilibrium state of the plasma, and neglecting all but the linear terms in the $a_i^{(1)}$, we get

$$\frac{\partial a_i^{(1)}}{\partial t}+\sum_{j=1}^{7} Z_{ij}(a^{(0)})\frac{\partial a_j^{(1)}}{\partial z} = 0. \tag{2.1.1.4}$$

We now choose such a system of coordinates that the matrix Z_{ij} has the simplest possible form. If the frame of reference moves with the plasma velocity, we have $\boldsymbol{u}^{(0)} = 0$. By rotating the system of coordinates around the z-axis, we can make $B_y^{(0)} = 0$. The matrix Z_{ij}

then has the form

$$Z_{ij}=\begin{bmatrix} 0 & 0 & 0 & 0 & \varrho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{B_z}{4\pi\varrho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{B_z}{4\pi\varrho} \\ \frac{p_\varrho}{\varrho} & \frac{p_s}{\varrho} & 0 & 0 & 0 & \frac{B_x}{4\pi\varrho} & 0 \\ 0 & 0 & -B_z & 0 & B_x & 0 & 0 \\ 0 & 0 & 0 & -B_z & 0 & 0 & 0 \end{bmatrix}. \tag{2.1.1.5}$$

We shall look for a solution of equations (2.1.1.4) in the form of a *plane wave:*

$$a_i^{(1)} = A_i e^{i(kz-\omega t)}. \tag{2.1.1.6}$$

Substituting this expression into (2.1.1.4) leads to

$$\sum_{j=1}^{7} Z_{ij}A_j = VA_i, \tag{2.1.1.7}$$

where $V = \omega/k$ is the *phase velocity of the wave.*

Equations (2.1.1.7) show that the amplitudes of the different magneto-hydrodynamical quantities in the plane wave (2.1.1.6) are the components of a column eigenvector r_i of the matrix Z_{ij} while the phase velocity V of the wave is an eigenvalue of that matrix:

$$\sum_{j=1}^{7} Z_{ij}r_j = Vr_i. \tag{2.1.1.8}$$

The phase velocities of the different waves are thus the roots of the seventh degree algebraic equation

$$\text{Det}\,|Z_{ij}-V\delta_{ij}| = 0, \tag{2.1.1.9}$$

and there are therefore seven kinds of plane waves which can propagate in a plasma.

We can find the phase velocities of these waves from eqns. (2.1.1.5) and (2.1.1.9) (Herlofson, 1950):

$$\begin{aligned} V_{1,2} &= \varepsilon v_A\,|\cos\theta|, \\ V_{3,4} &= \varepsilon v_+, \\ V_{5,6} &= \varepsilon v_- \\ V_7 &= 0, \end{aligned} \tag{2.1.1.10}$$

where

$$\begin{aligned} v_A &= \frac{B}{\sqrt{(4\pi\varrho)}}, \\ v_\pm &= \left[\tfrac{1}{2}\left\{v_A^2+c_s^2\pm[(v_A^2+c_s^2)^2-4v_A^2c_s^2\cos^2\theta]^{1/2}\right\}\right]^{1/2}, \end{aligned} \tag{2.1.1.10'}$$

$c_s = \sqrt{p_\varrho}$ is the sound velocity, θ the angle between the direction of the constant magnetic field and the direction of the wave propagation, $\varepsilon = +1$ for a wave propagating in the direction of the positive z-axis, and $\varepsilon = -1$ for a wave propagating in the opposite direction.

The quantity v_A is called the *Alfvén velocity* and the quantities $V_{1,2}$ are the phase velocities of two *Alfvén waves*, propagating at an angle θ to the magnetic field—one in the direction of the positive z-axis and the other in the opposite direction.

The quantities $v_\pm$ are the phase velocities of two *magneto-sound waves*—the fast one (v_+) and the slow one (v_-)—and finally, $V_7 = 0$ is the phase velocity of the *entropy wave*.

All these quantities refer to a frame of reference in which the plasma is at rest. To find the phase velocities V' of the waves in a plasma which moves with a velocity $\boldsymbol{u}^{(0)}$ we must add to the expressions (2.1.1.10) the longitudinal component, $u_z^{(0)}$, of the plasma velocity:

$$\begin{aligned} V'_{1,2} &= u_z^{(0)} + v_A |\cos\theta|, \\ V'_{3,4} &= u_z^{(0)} + v_+, \\ V'_{5,6} &= u_z^{(0)} + v_-, \\ V'_7 &= u_z^{(0)}. \end{aligned} \tag{2.1.1.11}$$

We can easily find the column *eigenvectors* (sometimes called right eigenvectors) of the matrix Z_{ij} from eqns. (2.1.1.8) (Jeffrey and Taniuti, 1964):

$$\begin{aligned} r^{(1,2)} &= [0, 0, 0, 1, 0, 0, -\varepsilon\sqrt{(4\pi\varrho)}\,\mathrm{sgn}\,B_z], \\ r^{(3,4)} &= \left[\varrho, 0, \frac{\varepsilon v_A^2 v_+ \sin\theta\cos\theta}{v_A^2\cos^2\theta - v_+^2}, 0, \varepsilon v_+, \frac{v_+^2 B\sin\theta}{v_+^2 - v_A^2\cos^2\theta}, 0\right], \\ r^{(5,6)} &= \left[\varrho, 0, \frac{\varepsilon v_A^2 v_- \sin\theta\cos\theta}{v_A^2\cos^2\theta - v_-^2}, 0, \varepsilon v_-, \frac{v_-^2 B\sin\theta}{v_-^2 - v_A^2\cos^2\theta}, 0\right], \\ r^{(7)} &= [-p_s, p_\varrho, 0, 0, 0, 0, 0]. \end{aligned} \tag{2.1.1.12}$$

The components of these vectors determine the perturbations of the various magneto-hydrodynamical quantities.

The vector $r^{(7)}$ describes an *entropy wave* in which only the density and the entropy are perturbed, while the velocity and the magnetic field are unperturbed:

$$\boldsymbol{u}^{(1)} = 0, \quad \boldsymbol{B}^{(1)} = 0, \quad \varrho^{(1)} \neq 0, \quad s^{(1)} \neq 0.$$

In an *Alfvén wave*—these waves were first studied by Alfvén (1950)—which is described by the vectors $r^{(1)}$ and $r^{(2)}$ there are no perturbations of the density, of the entropy, and of those components of the velocity and of the magnetic field which lie in the plane through the unperturbed magnetic field $\boldsymbol{B}^{(0)}$ and the direction of the wave propagation:

$$\varrho^{(1)} = s^{(1)} = u_x^{(1)} = u_z^{(1)} = B_x^{(1)} = 0, \quad u_y^{(1)} \neq 0, \quad B_y^{(1)} \neq 0.$$

Finally, in the *magneto-sound waves*, described by the vectors $r^{(3)}$, $r^{(4)}$ (fast waves) and by the vectors $r^{(5)}$, $r^{(6)}$ (slow waves), there are no perturbations of the entropy and of the components of the velocity and of the magnetic field at right angles to both the direction of wave propagation and $\boldsymbol{B}^{(0)}$:

$$s^{(1)} = u_y^{(1)} = B_y^{(1)} = 0, \quad \varrho^{(1)} \neq 0, \quad u_x^{(1)} \neq 0, \quad u_z^{(1)} \neq 0, \quad B_x^{(1)} \neq 0.$$

In other words, the magneto-sound waves are plane-polarized and isentropic.

We shall also need in what follows the row eigenvectors (sometimes called left eigenvectors) l_i of the matrix Z_{ij} which satisfy the set of equations.†

$$\sum_{i=1}^{7} l_i Z_{ij} = l_j V, \qquad j = 1, 2, \ldots, 7. \tag{2.1.1.13}$$

The row eigenvectors correspond to the same eigenvalues as the column eigenvectors and they have the form

$$\begin{aligned}
l^{(1,2)} &= [0, 0, 0, \sqrt{(4\pi\varrho)}, 0, 0, -\varepsilon \operatorname{sgn} B],\\
l^{(3,4)} &= \left[p_\varrho, p_s, \frac{\varepsilon v_A^2 v_+ \varrho \sin\theta\cos\theta}{v_A^2\cos^2\theta - v_+^2}, 0, \varepsilon v_+ \varrho, \frac{B}{4\pi}\frac{v_+^2 \sin\theta}{v_+^2 - v_A^2\cos^2\theta}, 0\right],\\
l^{(5,6)} &= \left[p_\varrho, p_s, \frac{\varepsilon v_A^2 v_- \varrho \sin\theta\cos\theta}{v_A^2\cos^2\theta - v_-^2}, 0, \varepsilon v_- \varrho, \frac{B}{4\pi}\frac{v_-^2 \sin\theta}{v_-^2 - v_A^2\cos^2\theta}, 0\right],\\
l^{(7)} &= [0, 1, 0, 0, 0, 0, 0].
\end{aligned} \tag{2.1.1.14}$$

An important property of the column and row eigenvectors is their mutual *orthogonality:*

$$\sum_{k=1}^{7} l_k^{(i)} r_k^{(j)} = 0, \quad \text{when} \quad i \neq j. \tag{2.1.1.15}$$

This property can be used to write an arbitrary small perturbation of the magneto-hydrodynamic quantities $a_j^{(1)}(z, t)$ as a superposition of seven magneto-hydrodynamic waves. To do this, we write $a_j^{(1)}(z, 0)$ as a Fourier integral:

$$a_j^{(1)}(z, 0) = \int f_j(k) e^{ikz}\, dk,$$

and we put

$$f_j(k) = \sum_{i=1}^{7} C^{(i)} r_j^{(i)}. \tag{2.1.1.16}$$

From the orthogonality property (2.1.1.15) it follows that the expansion coefficients $C^{(i)}$ from (2.1.1.16) are equal to

$$C^{(i)} = \frac{\sum_{j=1}^{7} l_j^{(i)} f_j(k)}{\sum_{j=1}^{7} l_j^{(i)} r_j^{(i)}}.$$

† We use r_i and l_i for the column and row eigenvectors because they operate from the right and from the left on the matrix Z_{ij}.

Using this formula we can find the coordinate and time dependence of any perturbation:

$$a_j^{(1)}(z, t) = \sum_{i=1}^{7} \int C^{(i)} r_j^{(i)} e^{i(kz-\omega_i t)}\, dk, \tag{2.1.1.17}$$

where $\omega_i \equiv \omega_i(k)$ are the frequencies of the magneto-hydrodynamic waves.

2.1.2. POLARS

We showed in the preceding subsection that the phase velocities of the (Alfvén and magneto-sound) magneto-hydrodynamic waves depend on the angle θ between the direction of the wave propagation and the direction of the constant magnetic field. The curves depicting this angular dependence are called *phase polars.*

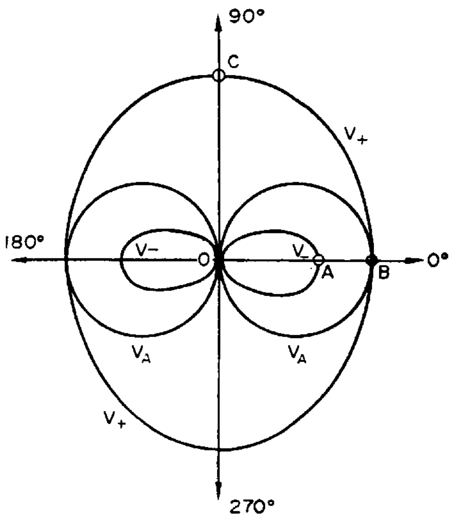

FIG. 2.1.1. Phase polars. v_+: fast phase polar; v_-: slow phase polar; v_A: Alfvén phase polar. The magnetic field is along *the abscissa axis.*

We have given in Fig. 2.1.1 the phase polars (Kulikovskiĭ and Lyubimov, 1962; Sears, 1960) for a fast magneto-sound wave (v_+), a slow magneto-sound wave (v_-), and an Alfvén wave (v_A) for the case when $v_A > c_s$. If $v_A < c_s$, the polar for the Alfvén wave will go through the point A rather than through the point B.

The sections OA, OB, and OC are, according to (2.1.1.10′), respectively equal to

$$OA = \text{Min}\,(v_A, c_s), \quad OB = \text{Max}\,(v_A, c_s), \quad OC = \sqrt{(v_A^2 + c_s^2)}.$$

We note that it follows directly from (2.1.1.10′) that for $0 \leqslant \theta \leqslant \pi/2$ the phase velocity of the fast magneto-sound wave is an increasing function of θ while the phase velocity of the slow wave is a decreasing function of θ. It is clear from Fig. 2.1.1 that

$$v_+(\theta) \geqslant \text{Max}\,(v_A, c_s), \quad v_-(\theta) \leqslant \text{Min}\,(v_A, c_s).$$

In the limiting cases when one of the two velocities v_A and c_s is much larger than the other, the phase polars become circles. When $v_A \gg c_s$, we have

$$v_+ \approx v_A, \quad v_- \approx c_s \,|\cos\theta|,$$

and when $c_s \gg v_A$, we find

$$v_+ \approx c_s, \quad v_- \approx v_A \,|\cos\theta|.$$

An arbitrary, small perturbation of the magneto-hydrodynamic state of the medium can be written, as we noted in Subsection 2.1.1, in the form of a superposition of seven magneto-hydrodynamic waves, two Alfvén, four magneto-sound, and one entropy wave.

Let us consider in somewhat more detail a *wavepacket* of one kind of magneto-hydrodynamic waves, that is, a superposition of waves of one type with a small spread in wavevector:

$$a_i(\mathbf{r}, t) = \int A_i(\mathbf{k}) \exp\{i(\mathbf{k}\cdot\mathbf{r}) - i\omega_j(\mathbf{k})\,t\}\, d^3k,$$

where $\omega_j(\mathbf{k})$ and $A_i(\mathbf{k})$ are the frequency and amplitude of the magneto-hydrodynamic wave of the kind considered with wavevector $\mathbf{k}$, while the integration is over a small volume in $\mathbf{k}$-space near the point $\mathbf{k} = \mathbf{k}_0$. Putting

$$\omega_j(\mathbf{k}) = \omega_j(\mathbf{k}_0) + (\mathbf{U}_j\cdot\{\mathbf{k}-\mathbf{k}_0\}), \quad \mathbf{U}_j = \left[\frac{\partial\omega_j}{\partial\mathbf{k}}\right]_{\mathbf{k}=\mathbf{k}_0},$$

and removing A_i from under the integral sign, we get

$$a_i(\mathbf{r}, t) = A_i(\mathbf{k}_0)\frac{\sin\xi_x}{\xi_x}\,\frac{\sin\xi_y}{\xi_y}\,\frac{\sin\xi_z}{\xi_z}\,\Delta k_x\,\Delta k_y\,\Delta k_z \exp\,[i(\mathbf{k}_0\cdot\mathbf{r}) - i\omega_j(\mathbf{k}_0)\,t],$$

where

$$\xi_x = (x - U_{jx}t)\,\Delta k_x, \quad \xi_y = (y - U_{jy}t)\,\Delta k_y, \quad \xi_z = (z - U_{jz}t)\,\Delta k_z,$$

while Δk_x, Δk_y, and Δk_z are the intervals in which the wavevector components lie.

It follows from this formula that the wavepacket is appreciably different from zero only when the quantities ξ_x, ξ_y, and ξ_z are small ($\xi \lesssim 1$). From this it follows in turn that the wavepacket moves with the velocity

$$\mathbf{U}_j = \frac{\partial\omega_j(\mathbf{k})}{\partial\mathbf{k}}, \quad j = 1, 2, \ldots, 7. \tag{2.1.2.1}$$

This velocity is called the *group velocity* of the jth kind of wave.

It is clear that the group and phase velocities of one-dimensional wavepackets are the same because the dispersion laws of the magneto-hydrodynamic waves are linear. However, the group and phase velocities of two- or three-dimensional packets are no longer the same.

Let us first of all consider a two-dimensional packet. It follows from (2.1.2.1) that

$$U_x = \frac{\partial\omega}{\partial k_x}, \quad U_y = \frac{\partial\omega}{\partial k_y}.$$

Substituting here $\omega = V(\theta)k$, where θ is the angle between the direction of the wave propagation and the direction of the constant magnetic field which we shall take as the x-axis, and bearing in mind that $\theta = \arctan(k_y/k_x)$, $k = \sqrt{(k_x^2+k_y^2)}$, we get

$$U_x = V(\theta)\cos\theta - V'(\theta)\sin\theta,$$
$$U_y = V(\theta)\sin\theta + V'(\theta)\cos\theta, \tag{2.1.2.2}$$

where $V'(\theta)$ is the derivative of $V(\theta)$ with respect to θ. These relations can be represented graphically in a U_x, U_y-plane. There will in this plane be a point for each angle θ. The locus of these points is called the *group polar*.

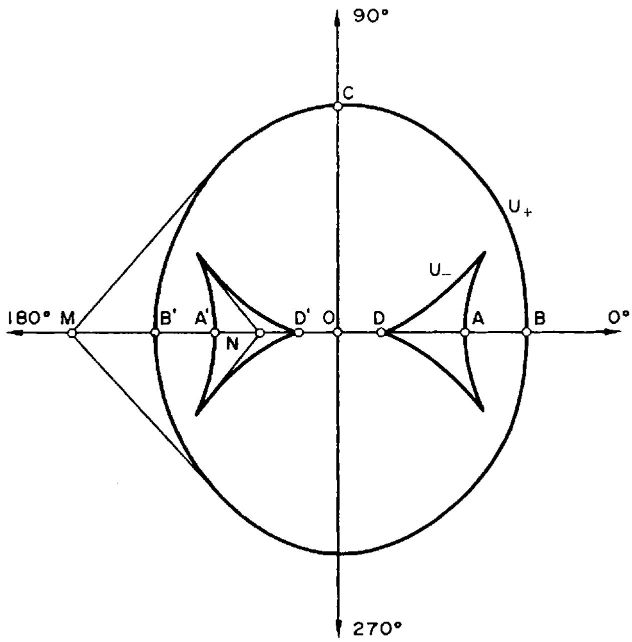

FIG. 2.1.2. Group polars. U_+: fast group polar; U_-: slow group polar.

We depict in Fig. 2.1.2 the group velocity polars (Lynn, 1962) for fast (U_+) and slow (U_-) magneto-hydrodynamical waves (they are called the fast and slow group polars). The sections OA, OB, OC, and OD are equal to

$$\mathrm{OA} = \mathrm{Min}\,(v_A, c_s), \quad \mathrm{OB} = \mathrm{Max}\,(v_A, c_s), \quad \mathrm{OC} = \sqrt{(v_A^2+c_s^2)}, \quad \mathrm{OD} = \frac{v_A c_s}{\sqrt{(v_A^2+c_s^2)}}. \tag{2.1.2.3}$$

The group polar for an Alfvén wave clearly degenerates into two points: B and B′ (see Fig. 2.1.2) when $v_A > c_s$ and A and A′ when $v_A < c_s$. The group polar for the entropy wave degenerates into the point 0.

We have considered two-dimensional wavepackets. In the case of three-dimensional wavepackets the end of the group velocity vector describes a surface when the propagation direction changes. One can obtain this surface by rotating the group polar diagram around the x-axis, that is, around the direction of the constant magnetic field.

2.1.3. CONICAL REFRACTION

When deriving equations (2.1.2.2) which connect the group and the phase velocities we tacitly assumed that there exists for each angle θ a unique derivative $V'_{\pm}(\theta)$. However, the phase polar can have points where it intersects itself and in those points the value of $V'_{\pm}(\theta)$ becomes undetermined.

We shall show that points of self-intersection indeed occur in phase polars, namely just when the Alfvén velocity v_A is the same as the sound velocity c_s, $v_A = c_s$. It follows from (2.1.1.10′) that in that case

$$v_{\pm}(\theta) = c_s \sqrt{(1 \pm \sin\theta)}, \tag{2.1.3.1}$$

and the equations for the magneto-sound phase polars have therefore the following form in a Cartesian system of coordinates:

$$\begin{aligned} x &\equiv \frac{v_{\pm}\cos\theta}{c_s} = \cos\theta\sqrt{(1\pm\sin\theta)}, \\ y &\equiv \frac{v_{\pm}\sin\theta}{c_s} = \sin\theta\sqrt{(1\pm\sin\theta)}. \end{aligned} \tag{2.1.3.2}$$

Eliminating θ we get a single equation for the fast and the slow magneto-sound polars

$$y^2 = (x^2+y^2-1)^2(x^2+y^2). \tag{2.1.3.3}$$

These polars are shown in Fig. 2.1.3. Differentiating (2.1.3.3) with respect to x we get

$$\frac{\partial y}{\partial x} = \frac{x(x^2+y^2-1)(3x^2+3y^2-1)}{y[1-(x^2+y^2-1)(3x^2+3y^2-1)]}$$

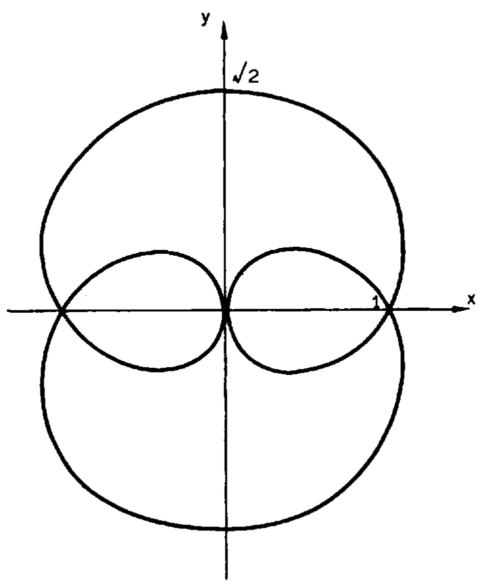

FIG. 2.1.3. Degenerate phase polars.

This expression loses its meaning when the denominator and the numerator vanish at the same time:

$$x(x^2+y^2-1)(3x^2+3y^2-1) = 0,$$
$$y[1-(x+{}^2y^2-1)(3x^2+3y^2-1)] = 0.$$

We see that the singular points of the polars are the points $x = \pm 1$, $y = 0$. Putting $x = \pm 1+\xi$, $y = \eta$ and retaining in (2.1.3.3) only terms up to the second degree in ξ and η, we find that in the vicinity of the points $x = \pm 1$, $y = 0$, the equations of the phase polars take the form

$$\eta^2 = 4\xi^2. \tag{2.1.3.4}$$

Hence it follows that the points $x = \pm 1$, $y = 0$ are points of self-intersection and that in those points there are no unique derivatives $v'_\pm(\theta)$.

To determine the group velocity when $v_A = c_s$ we shall treat the degenerate case $v_A = c_s$ as the limit of an ordinary case when $v_A \neq c_s$ as $v_A \to c_s$. The phase polars for v_A close to c_s are shown in Fig. 2.1.4. The fast and the slow polars in that case do not have points

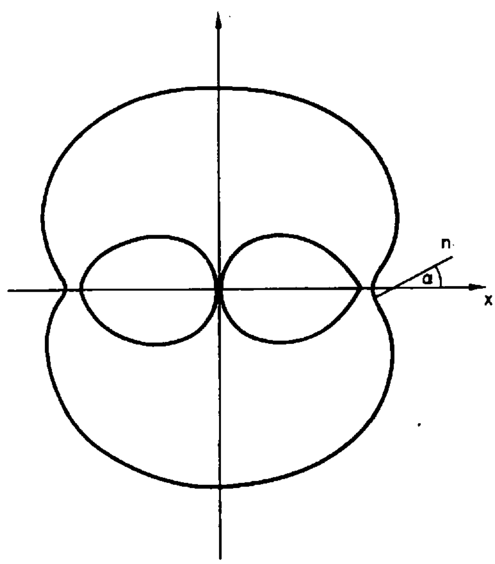

FIG. 2.1.4. Phase polars near degeneracy; *n* is the normal to the phase polar and α the angle between the normal and the abscissa axis.

in common although they approach each other very closely near the points $x = \pm 1$, $y = 0$. There is a unique derivative $v'_\pm(\theta)$ in all points of the phase polars and hence in all points the direction of the normal *n* to the polar is well defined (see Fig. 2.1.4) although near the points $x = \pm 1$, $y = 0$ the normal changes its direction extremely fast.

If the difference $v_A - c_s$ is infinitesimally small, the angle α between the normal *n* and the x-axis will change by a finite amount in an infinitesimally small region near the points $x = \pm 1$, $y = 0$. It is clear from eqn. (2.1.3.4) that the tangent of that angle takes on all values in the range $(-\frac{1}{2}, \frac{1}{2})$:

$$-\tfrac{1}{2} \leq \tan\alpha \leq \tfrac{1}{2}. \tag{2.1.3.5}$$

On the other hand, $\tan\alpha$ is connected with $v'_{\pm}(\theta)$ through the relation

$$\tan\alpha = v'_{\pm}(\theta)/v_{\pm}(\theta),$$

so that we can write eqn. (2.1.2.2) for the group velocity in the form

$$U_x = v_{\pm}(\theta)(\cos\theta - \tan\alpha\sin\theta),$$
$$U_y = v_{\pm}(\theta)(\sin\theta + \tan\alpha\cos\theta).$$

Substituting in these equations $\theta = 0$ and $v_A = c_s$ (in this case $v_{\pm}(0) = c_s$) we get

$$U_x = c_s, \quad U_y = c_s\tan\alpha.$$

Taking (2.1.3.5) into account we see that when $v_A = c_s$ the angle $\theta = 0$ corresponds not to a single group velocity vector, but to a continuum of values:

$$U_x = c_s, \quad -\tfrac{1}{2}c_s \leqslant U_y \leqslant \tfrac{1}{2}c_s. \tag{2.1.3.6}$$

In the three-dimensional case the group velocity components satisfy the relations

$$U_x = c_s, \quad -\tfrac{1}{2}c_s \leqslant U_y \leqslant \tfrac{1}{2}c_s, \quad -\tfrac{1}{2}c_s \leqslant U_z \leqslant \tfrac{1}{2}c_s,$$

that is, they fill the interior of a cone with its axis along the direction of the magnetic field and with semi-vertical angle equal to $\arctan\frac{1}{2} = 26°34'$.

An important consequence follows from this. If the direction of the magnetic field is at right angles to the surface of the plasma and if from the outside a sound or electromagnetic wave falls onto the plasma at right angles to its surface (fast and slow), magneto-sound waves will be excited in the plasma when $v_A \neq c_s$ which propagate along the normal to the plasma surface. However, when $v_A = c_s$, there will occur a wavepacket of magneto-sound waves and the different wavevectors in this wavepacket fill a cone with semi-vertical angle equal to 26°34′.

This effect may be called *conical refraction* (Sivukhin, 1966) by analogy with the analogous phenomenon in crystal optics (Landau and Lifshitz, 1960). Conical refraction in crystal optics consists of the following effect. If the direction of a light ray which is incident from the outside on the crystal hits an ordinary point of the phase velocity surface, the ray will produce a small spot on a photographic plate. If, however, the light ray hits a singular point of the phase velocity surface, we get instead of a point a circle on the photographic plate (Ludwig, 1961).

It follows from the existence of conical refraction that the intensity of a wave which is registered by a detector placed on the path along which the wave propagates will show a steep drop for that value of the magnetic field for which the condition $v_A = c_s$ is satisfied, that is, when

$$\frac{B^2}{4\pi} = \gamma nT,$$

where B is the magnetic field, n the plasma density, T the temperature, and $\gamma = c_p/c_v$, where c_p and c_v are the specific heats at constant pressure and constant volume, respectively. This effect can be used for plasma diagnostics (Polovin and Cherkasova, 1967).

2.1.4. DAMPING OF MAGNETO-HYDRODYNAMIC WAVES

So far we have when studying the magneto-hydrodynamic waves used the equations for the magneto-hydrodynamics of an ideal medium, that is, we have neglected the effects which lead to a dissipation of energy. These effects lead, in particular, to the damping of the magneto-hydrodynamic waves, and we shall study this in the present subsection.

The equations of magneto-hydrodynamics when dissipation is taken into account have according to Section 1.5 the following form (Syrovatskiĭ, 1957; Landau and Lifshitz, 1960):

$$\begin{aligned}
&\frac{\partial \varrho}{\partial t}+\operatorname{div}\,\varrho \boldsymbol{u} = 0,\\
&\varrho T\frac{ds}{dt} = \sum_{\alpha,\beta}\pi_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}+\operatorname{div}\,(\varkappa\,\nabla T),\\
&\frac{d\boldsymbol{u}}{dt} = -\frac{1}{\varrho}\,\nabla p+\frac{1}{4\pi\varrho}\,[\operatorname{curl}\,\boldsymbol{B}\wedge\boldsymbol{B}]+\nu_1\,\nabla^2\boldsymbol{u}+\left(\nu_2+\frac{1}{3}\,\nu_1\right)\operatorname{grad}\,\operatorname{div}\,\boldsymbol{u},\\
&\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl}\,[\boldsymbol{u}\wedge\boldsymbol{B}]+\nu_m\,\nabla^2\boldsymbol{B},\\
&\operatorname{div}\,\boldsymbol{B} = 0,
\end{aligned} \tag{2.1.4.1}$$

where $\pi_{\alpha\beta}$ is the viscous stress tensor

$$\pi_{\alpha\beta} = \eta\left[\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}-\frac{2}{3}\,\delta_{\alpha\beta}\sum_\gamma\frac{\partial u_\gamma}{\partial x_\gamma}\right]+\zeta\delta_{\alpha\beta}\sum_\gamma\frac{\partial u_\gamma}{\partial x_\gamma},$$

η and ζ are the viscosity coefficients, $\nu_1 = \eta/\varrho$, $\nu_2 = \zeta/\varrho$, $\varkappa$ the thermal conductivity coefficient, $\nu_m = c^2/4\pi\sigma$ the magnetic viscosity, and σ the electrical conductivity of the medium.

We shall assume that all quantities depend on z and t only and we shall introduce the magneto-hydrodynamical state vector, $a_i(\varrho, s, u_x, u_y, u_z, B_x, B_y)$, as we did in Subsection 2.1.1. For small perturbations we then have instead of eqn. (2.1.1.4) the equation

$$\frac{\partial a_i}{\partial t}+\sum_{j=1}^{7}Z_{ij}\frac{\partial a_j}{\partial z} = \sum_{j=1}^{7}D_{ij}\frac{\partial^2 a_j}{\partial z^2}, \tag{2.1.4.2}$$

where Z_{ij} is the matrix defined by eqn. (2.1.1.5),

$$D_{ij} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0\\
\dfrac{\varkappa T_\varrho}{\varrho T} & \dfrac{\varkappa T_s}{\varrho T} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \nu_1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \nu_1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{4}{3}\nu_1+\nu_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \nu_m & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \nu_m
\end{bmatrix} \tag{2.1.4.3}$$

and

$$T_\varrho = \left(\frac{\partial T}{\partial \varrho}\right)_s, \quad T_s = \left(\frac{\partial T}{\partial s}\right)_\varrho.$$

We shall consider the solution of these equations which has the form of plane monochromatic waves:

$$a_i = A_i e^{i(kz-\omega t)}.$$

Substituting this expression into eqn. (2.1.4.2) we get

$$-i\omega A_i + ik \sum_j Z_{ij} A_j = -k^2 \sum_j D_{ij} A_j. \tag{2.1.4.4}$$

We shall assume that the frequency is given and that the wavenumber k is a function of the frequency. It is clear that the latter will be complex, due to energy dissipation,

$$k = k_1 + ik_2,$$

where k_1 and k_2 are real.

If the matrix elements D_{ij} are sufficiently small, we can solve eqn. (2.1.4.2) by the method of successive approximations. In first approximation we can neglect the terms containing D_{ij} and we get then eqn. (2.1.1.7) from which we find

$$A_i^{(1)} = r_i, \quad k_1 = \omega/V, \tag{2.1.4.5}$$

where the r_i are the column eigenvectors of the matrix Z_{ij} determined by eqns. (2.1.1.12) and V is the phase velocity determined by eqns. (2.1.1.10).

In second approximation we put $A_i = A_i^{(1)} + A_i^{(2)}$, $k = k_1 + ik_2$, where $A_i^{(2)}$ and k_2 are small quantities of the order of D_{ij}/Z_{ij}. Substituting these expressions into (2.1.4.4) and using the fact that

$$\sum_j Z_{ij} r_j = V r_i,$$

we find

$$-i\omega A_i^{(2)} - k_2 \sum_j Z_{ij} r_j + \frac{i\omega}{V} \sum_j Z_{ij} A_j^{(2)} = -k^2 \sum_j D_{ij} r_j.$$

Multiplying this equation by the row eigenvector l_i of the matrix Z_{ij}, defined by

$$\sum_i l_i Z_{ij} = V l_j,$$

and summing over i from 1 to 7, we find up to first order in the D_{ij}

$$k_2 = \frac{\omega^2}{V^3} \frac{\sum_{i,j} l_i D_{ij} r_j}{\sum_i l_i r_i}. \tag{2.1.4.6}$$

One can easily show that

$$\frac{\sum_{ij} l_i D_{ij} r_j}{\sum_i l_i r_i} \geqslant 0, \tag{2.1.4.7}$$

so that the quantity k_2 will be positive when $V > 0$ and negative when $V < 0$. It follows thus that the wave amplitude when it propagates in the positive x-direction changes as $\exp(-|k_2|x)$, where $x > 0$. In other words, the wave is damped when it propagates. For this reason the quantity k_2 is called the *damping coefficient*.

Using the actual form of the column and row eigenvectors we can evaluate the damping coefficient. It is especially simple for Alfvén waves (Alfvén, 1950):

$$k_2 = \frac{\omega^2}{2v_A^3}(\nu_1 + \nu_m). \tag{2.1.4.8}$$

In the case of magneto-sound waves, we use the relations

$$\frac{v_\pm^2 v_A^2 \sin^2\theta}{v_\pm^2 - v_A^2\cos^2\theta} = v_\pm^2 - c_s^2,$$

and (Landau and Lifshitz, 1969)

$$p_s T_\varrho = -\mathcal{V}^2\left(\frac{\partial p}{\partial s}\right)_{\mathcal{V}}\left(\frac{\partial T}{\partial \mathcal{V}}\right)_s = Tc_s^2\left[\frac{1}{c_v} - \frac{1}{c_p}\right],$$

where $\mathcal{V} = 1/\varrho$ is the specific volume and c_v and c_p are the specific heats at constant volume and constant pressure, respectively, and we find the following expression for the damping coefficient (Sirotina and Syrovatskiĭ, 1961):

$$k_2 = \frac{\omega^2}{2v_\pm^3(v_\pm^4 - c_s^2 v_A^2\cos^2\theta)}\left\{(v_\pm^2 - v_A^2\cos^2\theta)\left[\frac{\varkappa c_s^2}{\varrho}\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right.\right.$$
$$\left.\left. + v_\pm^2\left(\frac{4}{3}\nu_1 + \nu_2\right)\right] + (v_\pm^2 - c_s^2)(\nu_1 v_A^2\cos^2\theta + \nu_m v_\pm^2)\right\}. \tag{2.1.4.9}$$

This formula can be greatly simplified in the case when $c_s \ll v_A$; in that case we have for the fast wave

$$k_2 = \frac{\omega^2}{2v_A^3}\nu_m,$$

and for the slow wave

$$k_2 = \frac{\omega^2 \nu_m \sin^2\theta}{2v_A^2 c_s |\cos\theta|^3}.$$

2.1.5. EXCITATION OF MAGNETO-HYDRODYNAMICAL WAVES

So far we have studied the dispersion and polarization properties as well as the damping of magneto-hydrodynamical waves. We now turn to the problem of their excitation.

Magneto-hydrodynamical waves can be excited either mechanically, for instance, using a rotating disc (Lundquist, 1949) or by external variable currents (Akhiezer and Sitenko, 1960; see also Sitenko and Kirochkin, 1960). The latter method is of most interest for a plasma and we shall discuss it here.

We shall first of all assume that the medium is perfectly conducting and neglect all dissipative processes. We can then base our discussion upon the following linearized equations for the velocity of the medium, $\boldsymbol{u}$, the deviation ϱ_1 of the density of the medium from its equilibrium density ϱ_0, and the deviation $\boldsymbol{B}_1$ of the magnetic field from the constant, uniform field $\boldsymbol{B}_0$:

$$\begin{aligned}\varrho_0 \frac{\partial \boldsymbol{u}}{\partial t} &= -c_s^2 \nabla \varrho_1 + \frac{1}{c}[\boldsymbol{j}^{(i)} \wedge \boldsymbol{B}_0],\\ \frac{\partial \varrho_1}{\partial t} + \varrho_0 \operatorname{div} \boldsymbol{u} &= 0,\\ \frac{\partial \boldsymbol{B}_1}{\partial t} &= \operatorname{curl}[\boldsymbol{u} \wedge \boldsymbol{B}_0],\\ \operatorname{curl} \boldsymbol{B}_1 &= \frac{4\pi}{c}\boldsymbol{j}^{(i)} + \frac{4\pi}{c}\boldsymbol{j}^{(e)}.\end{aligned} \tag{2.1.5.1}$$

Here $\boldsymbol{j}^{(i)} = en(\boldsymbol{u}_i - \boldsymbol{u}_e)$ is the "intrinsic" conduction current ($\boldsymbol{u}_i$ and $\boldsymbol{u}_e$ are, respectively, the ion and electron velocities and n is the density of the particles of one kind), $\boldsymbol{j}^{(e)}$ is the "external" current which produces the magnetic field, and c_s is the sound velocity. We have dropped in the first of these equations the term $[\boldsymbol{j}^{(e)} \wedge \boldsymbol{B}]/c$, as it is a ponderomotive force acting upon the "carrier" current and not upon the plasma.

Using the relation

$$\boldsymbol{j}^{(i)} = \frac{c}{4\pi} \operatorname{curl} \boldsymbol{B}_1 - \boldsymbol{j}^{(e)}$$

to eliminate the intrinsic current from these equations, and differentiating the first equation with respect to the time, we get

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - c_s^2 \operatorname{grad} \operatorname{div} \boldsymbol{u} = \boldsymbol{u} - [\{\operatorname{curl} \operatorname{curl} [\boldsymbol{u} \wedge \boldsymbol{v}_A]\} \wedge \boldsymbol{v}_A] = \frac{1}{c\varrho_0}\left[\boldsymbol{B}_0 \wedge \frac{\partial \boldsymbol{j}^{(e)}}{\partial t}\right], \tag{2.1.5.2}$$

where $\boldsymbol{v}_A = \boldsymbol{B}_0/\sqrt{(4\pi\varrho_0)}$ is the Alfvén velocity.

This equation enables us to find the velocity of the medium $\boldsymbol{u}$ for a given external current $\boldsymbol{j}^{(e)}$. If we then use the second and third of eqn. (2.1.5.1) we can find ϱ_1 and $\boldsymbol{B}_1$.

The simplest way to solve eqn. (2.1.5.2) is to use Fourier transforms. Let us, for instance, consider the case when the current is a harmonic function of the time:

$$\boldsymbol{j}^{(e)}(\boldsymbol{r}, t) = e^{-i\omega t}\boldsymbol{j}^{(e)}(\boldsymbol{r}).$$

We can then look for $\boldsymbol{u}(\boldsymbol{r}, t)$ in the form

$$\boldsymbol{u}(\boldsymbol{r}, t) = e^{-i\omega t}\boldsymbol{u}(\boldsymbol{r}).$$

Expanding $\boldsymbol{j}^{(e)}(\boldsymbol{r})$ and $\boldsymbol{u}(\boldsymbol{r})$ in Fourier integrals,

$$\begin{aligned}\boldsymbol{j}^{(e)}(\boldsymbol{r}) &= \int \boldsymbol{j}_k e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3k,\\ \boldsymbol{u}(\boldsymbol{r}) &= \int \boldsymbol{u}_k e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3k,\end{aligned} \tag{2.1.5.3}$$

and substituting these expressions into (2.1.5.2) we get the following algebraic equation to determine $\boldsymbol{u}_k$:

$$\begin{aligned}\{\omega^2 - (\boldsymbol{k}\cdot\boldsymbol{v}_A)^2\}\boldsymbol{u}_k - \{(c_s^2 + v_A^2)(\boldsymbol{k}\cdot\boldsymbol{u}_k) - (\boldsymbol{u}_k\cdot\boldsymbol{v}_A)(\boldsymbol{k}\cdot\boldsymbol{v}_A)\}\boldsymbol{k}\\ + \boldsymbol{v}_A(\boldsymbol{k}\cdot\boldsymbol{v}_A)(\boldsymbol{k}\cdot\boldsymbol{u}_k) = \frac{i\omega}{c\varrho_0}[\boldsymbol{B}_0 \wedge \boldsymbol{j}_k].\end{aligned} \tag{2.1.5.4}$$

Once we know $\boldsymbol{u}$ we can determine the level of excitation of the magneto-hydrodynamic waves. To do this we determine the time derivative of the total energy of the medium:

$$W = \int \left(\frac{1}{2} \varrho u^2 + \varrho \varepsilon + \frac{B^2}{8\pi} \right) d^3\boldsymbol{r},$$

where $\varrho = \varrho_0 + \varrho_1$ is the density of the medium, ε the internal energy per unit mass, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1$ the magnetic field, and $\boldsymbol{u}$ the velocity of the medium. Differentiating this expression with respect to time and using both the exact (unlinearized) magneto-hydrodynamics eqns. (1.5.2.19) and the relations (w is the enthalpy per unit mass)

$$\boldsymbol{j}^{(i)} = \frac{c}{4\pi} \operatorname{curl} \boldsymbol{B} - \boldsymbol{j}^{(e)}, \quad d\varepsilon = \frac{p}{\varrho^2} d\varrho, \quad dp = \varrho\, dw,$$

we find, after dropping terms which have the form of divergences, the following expression for the level of excitation of the magneto-hydrodynamic waves:

$$I \equiv \frac{\partial W}{\partial t} = \frac{1}{c} \int (\boldsymbol{u} \cdot [\boldsymbol{B} \wedge \boldsymbol{j}^{(e)}])\, d^3\boldsymbol{r}. \tag{2.1.5.5}$$

Noting that in the case of infinite conductivity considered by us

$$\boldsymbol{E} + \frac{1}{c} [\boldsymbol{u} \wedge \boldsymbol{B}] = 0,$$

we can rewrite eqn. (2.1.5.5) for the level of excitation of magneto-hydrodynamic waves in the form

$$I = -\int (\boldsymbol{E} \cdot \boldsymbol{j}^{(e)})\, d^3\boldsymbol{r}.$$

Equation (2.1.5.5) is exact. As we are interested in the level of excitation of small amplitude waves we can rewrite it in the form

$$I = \frac{1}{c} \int (\boldsymbol{u} \cdot [\boldsymbol{B}_0 \wedge \boldsymbol{j}^{(e)}])\, d^3\boldsymbol{r}. \tag{2.1.5.6}$$

If the current is a harmonic function of the time, we can use the expansions (2.1.5.3) to obtain the following expression for the time-averaged level of excitation of magneto-hydrodynamic waves:

$$\bar{I} = \left(\frac{\boldsymbol{B}_0 V}{4c} \cdot \int \{ [\boldsymbol{j}_k \wedge \boldsymbol{u}^*_{-k}] + [\boldsymbol{j}^*_k \wedge \boldsymbol{u}_{-k}] \}\, d^3\boldsymbol{k} \right),$$

where V is here the volume of the system.

Let us consider in more detail the excitation of magneto-hydrodynamical waves by a surface current flowing in the xy-plane at right angles to the direction of the magnetic field $\boldsymbol{B}_0$:

$$\boldsymbol{j}^{(e)} = \boldsymbol{j}_S \delta(z) e^{-i\omega t}.$$

Here $\boldsymbol{j}_S$ is a constant vector which we shall assume to be parallel to the x-axis.

Expanding $\delta(z)$ in a Fourier integral,

$$\delta(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikz}\, dk,$$

and substituting this expansion into (2.1.5.4), we get

$$u_{ky} = \frac{i\omega B_0 j_S}{2\pi c \varrho_0 (\omega^2 - k^2 v_A^2)}, \qquad u_{kx} = u_{kz} = 0. \tag{2.1.5.7}$$

From these equations we can find the velocity $\boldsymbol{u}(0, u_y, 0)$:

$$u_y = \frac{-i\omega B_0 j_S}{2\pi c \varrho_0 v_A^2} \int_{-\infty}^{+\infty} \frac{e^{ikz}}{k^2 - (\omega^2/v_A^2)}\, dk.$$

To evaluate the integral which occurs here we must remember how to go round the poles $k = \pm\omega/v_A$: one must go around the poles using the causality principle (Lighthill, 1960), that is, assume that Im $\omega = -\varepsilon$, where ε is an infinitesimal positive quantity, $\varepsilon \to +0$. As a result we get from integrating

$$u_y = \frac{B_0 j_S}{2c\varrho_0 v_A} \exp\left(-\frac{i\omega |z|}{v_A}\right). \tag{2.1.5.8}$$

We can now find the average level of excitation of the magneto-hydrodynamic waves. To do this we substitute (2.1.5.8) into (2.1.5.6):

$$\bar{I} = \frac{\pi j_S^2}{c^2} v_A \int \exp\left(\frac{i\omega |z|}{v_A}\right) \delta(z)\, d^3\boldsymbol{r}.$$

We see that the average excitation level per unit surface area is equal to

$$\bar{I}_S = \frac{\pi j_S^2}{c^2} v_A. \tag{2.1.5.9}$$

Let us now consider the excitation of magneto-hydrodynamic waves by a current which has the form of a wave moving along the z-axis,

$$\boldsymbol{j}^{(e)} = \boldsymbol{J} \exp\left[i\omega\left(t - \frac{z}{V}\right)\right].$$

We cannot directly use in this case the formulae obtained earlier as resonance will occur when the velocity of the current wave is the same as any one of the phase velocities of the magneto-hydrodynamic waves; to study this resonance we must take the dissipative processes into account. We shall here only take the dissipation into account which is connected with a finite electrical conductivity of the medium. In that case only the third equation of the set (2.1.5.1) will change, that is, the equation for the "freezing-in" of the magnetic field lines, and this equation now becomes

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \text{curl}\,[\boldsymbol{u} \wedge \boldsymbol{B}_0] + \nu_m \nabla^2 \boldsymbol{B}_1,$$

where $\nu_m = c^2/4\pi\sigma$ is again the magnetic viscosity coefficient. Choosing the y-axis at right angles to the direction of the wave propagation (the z-axis) and to the direction of the magnetic field $\boldsymbol{B}_0$ we get the following equations to determine the quantities ϱ_1, $\boldsymbol{u}$, and $\boldsymbol{B}_1$:

$$\begin{gathered}
\frac{\partial \varrho_1}{\partial t} + \varrho_0 \frac{\partial u_z}{\partial z} = 0\\
\frac{\partial u_x}{\partial t} - \frac{B_{0z}}{4\pi\varrho_0}\frac{\partial B_{1x}}{\partial z} = -\frac{j_y^{(e)} B_{0z}}{\varrho_0 c},\\
\frac{\partial u_y}{\partial t} - \frac{B_{0z}}{4\pi\varrho_0}\frac{\partial B_{1y}}{\partial z} = \frac{-j_z^{(e)} B_{0x} + j_x^{(e)} B_{0z}}{\varrho_0 c},\\
\frac{\partial u_z}{\partial t} + \frac{c_s^2}{\varrho_0}\frac{\partial \varrho_1}{\partial z} + \frac{B_{0x}}{4\pi\varrho_0}\frac{\partial B_{1x}}{\partial z} = \frac{j_y^{(e)} B_{x0}}{\varrho_0 z},\\
\frac{\partial B_{1x}}{\partial t} - B_{0z}\frac{\partial u_x}{\partial z} + B_{0x}\frac{\partial u_z}{\partial z} = \nu_m \frac{\partial^2 B_{1x}}{\partial z^2},\\
\frac{\partial B_{1y}}{\partial t} - B_{0z}\frac{\partial u_y}{\partial z} = \nu_m \frac{\partial^2 B_{1y}}{\partial z^2}.
\end{gathered} \tag{2.1.5.10}$$

These equations split into two sets of equations: one set containing the quantities u_y and B_{1y}, referring to an Alfvén wave, and another set containing the quantities ϱ_1, u_x, u_z, and B_{1x}, referring to magneto-sound waves.

The solution of the first set has the form

$$\begin{aligned}
B_{1y} &= \text{Re}\,\frac{i(J_z B_{0x} - J_x B_{0z})\, B_{0z} V}{\omega\varrho_0 c[(V^2 - v_A^2) - i\omega\nu_m]},\\
u_{1y} &= \text{Re}\,\frac{-V^2 + i\omega\nu_m}{V B_{0z}} B_{1y}.
\end{aligned} \tag{2.1.5.11}$$

The solution of the second set looks very similar. We give here only the expression for B_{1x}:

$$B_{1x} = \text{Re}\,\frac{iB_0^2 V J_y (V^2 - c_s^2 \cos^2\theta)}{\omega\varrho_0 c[-(V^2 - v_+^2)(V^2 - v_-^2) + i\omega\nu_m(V^2 - c_s^2)]}, \tag{2.1.5.12}$$

where θ is the angle between the z-axis and the direction of the magnetic field.

The formulae obtained here show that when the velocity of the current wave is the same as the Alfvén velocity or one of the magneto-sound velocities there is resonance and the amplitude of the corresponding magneto-hydrodynamic wave becomes inversely proportional to the magnetic viscosity.

2.1.6. EVOLUTION OF A PERTURBATION

Let us now consider the problem of the evolution of a perturbation in a magneto-hydrodynamic medium. If we exclude an entropy perturbation,

$$s \neq 0, \quad \varrho_1 \neq 0, \quad \boldsymbol{u}_1 = \boldsymbol{B}_1 = 0,$$

we can assume that the remaining perturbation is isentropic,

$$dp = c_s^2 \, d\varrho, \tag{2.1.6.1}$$

where c_s is the sound velocity.

If we linearize the eqns. (1.5.2.19) of magneto-hydrodynamics of an ideal medium, take the time-derivative of the equation of motion,

$$\frac{\partial \boldsymbol{u}}{\partial t} = -\frac{c_s^2}{\varrho_0} \nabla \varrho + \frac{1}{4\pi\varrho_0} [\text{curl } \boldsymbol{B} \wedge \boldsymbol{B}_0], \tag{2.1.6.2}$$

and use the continuity equation

$$\frac{\partial \varrho}{\partial t} = -\varrho_0 \, \text{div } \boldsymbol{u}, \tag{2.1.6.3}$$

and the freezing-in equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \text{curl } [\boldsymbol{u} \wedge \boldsymbol{B}_0], \tag{2.1.6.4}$$

we get the following equation to determine the velocity:

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = c_s^2 \, \nabla \, \text{div } \boldsymbol{u} + \frac{1}{4\pi\varrho_0} [\{\text{curl curl } [\boldsymbol{u} \wedge \boldsymbol{B}_0]\} \wedge \boldsymbol{B}_0]. \tag{2.1.6.5}$$

We are interested in the problem of finding solutions of eqns. (2.1.6.2) to (2.1.6.4) for given initial values of the velocity $\boldsymbol{u}$, the magnetic field $\boldsymbol{B}$ and the density ϱ in the whole of space. If we consider only eqn. (2.1.6.5) we can solve it once $\boldsymbol{u}$ and $\partial \boldsymbol{u}/\partial t$ are given at one time in the whole of space. (The value of $\partial \boldsymbol{u}/\partial t$ at a given time is, according to eqn. (2.1.6.2), determined by the initial values of ϱ and $\boldsymbol{B}$.)

Turning to the solution of eqn. (2.1.6.5) we introduce the quantities

$$w_1 = \frac{\partial u_x}{\partial x}, \quad w_2 = \text{div } \boldsymbol{u}, \quad \xi = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z},$$

and we have chosen the x-axis along the direction of the magnetic field $\boldsymbol{B}_0$. Equation (2.1.6.5) then splits into two equations (Lighthill, 1960) for ξ and w_1:

$$\frac{\partial^2 \xi}{\partial t^2} = v_A^2 \frac{\partial^2 \xi}{\partial x^2}, \quad v_A = \frac{B_0}{\sqrt{(4\pi\varrho_0)}}, \tag{2.1.6.6}$$

$$\frac{\partial^4 w_1}{\partial t^4} - (v_A^2 + c_s^2) \frac{\partial^2}{\partial t^2} \nabla^2 w_1 + v_A^2 c_s^2 \frac{\partial^2}{\partial x^2} \nabla^2 w_1 = 0. \tag{2.1.6.7}$$

The function w_2 is connected with w_1 through the relation

$$\frac{\partial^2 w_2}{\partial t^2} = c_s^2 \frac{\partial^2 w_1}{\partial x^2}. \tag{2.1.6.8}$$

Equation (2.1.6.6) describes Alfvén waves and eqn. (2.1.6.7) magneto-sound waves. To solve eqn. (2.1.6.6) we need to give the values of ξ and $\partial\xi/\partial t$ at $t = 0$ and to solve eqn. (2.1.6.7) the values of w_1, $\partial w_1/\partial t$, $\partial^2 w_1/\partial t^2$, and $\partial^3 w_1/\partial t^3$ at $t = 0$. After finding w_1 we can use eqn. (2.1.6.8) to determine w_2. Knowing ξ, w_1, and w_2 for $t > 0$ we can easily determine the components u_x, u_y, and u_z of the velocity vector: the component u_x is found by integrating w_1 over x; knowing w_1 and w_2 we can then find $(\partial u_y/\partial y)+(\partial u_z/\partial z)$, that is, the two-dimensional divergence of the velocity vector which together with the two-dimensional curl, ξ, is sufficient to determine u_y and u_z.

Let us now study the evolution of the Alfvén perturbations described by eqn. (2.1.6.6). For Alfvén waves we have

$$w_1 = 0, \quad w_2 = 0,$$

and, moreover, $B_x = 0$, as follows from (2.1.6.4). As eqn. (2.1.6.6) contains only derivatives with respect to a single space coordinate we can restrict our discussion to one-dimensional perturbations $\xi(x, t)$ which depend merely on t and x. We have noted already that the boundary conditions for eqn. (2.1.6.6) are of the form

$$\left.\begin{aligned} \xi\Big|_{t=0} &= f_0(x), \\ \frac{\partial\xi}{\partial t}\Big|_{t=0} &= f_1(x), \end{aligned}\right\} \tag{2.1.6.9}$$

where f_0 and f_1 are given functions of x.

It is convenient for the solution of eqn. (2.1.6.6) with boundary conditions (2.1.6.9) to introduce a *Green function* $g(x, t)$ which is a solution of eqn. (2.1.6.6) satisfying the boundary conditions

$$g\Big|_{t=0} = 0, \quad \frac{\partial g}{\partial t}\Big|_{t=0} = \delta(x). \tag{2.1.6.9'}$$

Using eqn. (2.1.6.6) one can easily verify that the time-derivative of the Green function $\partial g/\partial t$ satisfies eqn. (2.1.6.6) with the boundary conditions

$$\frac{\partial g}{\partial t}\Big|_{t=0} = \delta(x), \quad \frac{\partial}{\partial t}\frac{\partial g}{\partial t}\Big|_{t=0} = 0.$$

The solution $\xi(x, t)$ which satisfies the boundary conditions (2.1.6.9) therefore is of the form

$$\xi(x, t) = \int_{-\infty}^{+\infty} g(x', t) f_1(x-x')\,dx' + \int_{-\infty}^{+\infty} \frac{\partial g(x', t)}{\partial t} f_0(x-x')\,dx'. \tag{2.1.6.10}$$

Consider now an initial perturbation which is concentrated near the point $x = y = z = 0$:

$$a_i(\mathbf{r}, t)\Big|_{t=0} = A_i\delta(\mathbf{r}), \tag{2.1.6.11}$$

where the A_i are constants. As the quantity ξ is obtained from the velocity components by differentiating with respect to y and z, the initial velocity perturbations of the form (2.1.6.11) correspond to an initial value of ξ of the form

$$\xi\Big|_{t=0} = B\delta(x), \tag{2.1.6.12}$$

where B is independent of x; B can be a function of y and z, but this is unimportant for us as the differential equation does not contain derivatives with respect to y and z. Using eqns. (2.1.6.2) and (2.1.6.11) we can show that

$$\frac{\partial \xi}{\partial t}\Big|_{t=0} = D\delta'(x). \tag{2.1.6.13}$$

From (2.1.6.10) it follows that the solution of eqn. (2.1.6.6) which satisfies the boundary conditions (2.1.6.12) and (2.1.6.13) is of the following form:

$$\xi(x, t) = B\frac{\partial g(x, t)}{\partial t} - D\frac{\partial g(x, t)}{\partial x}. \tag{2.1.6.14}$$

We shall look for the Green function of eqn. (2.1.6.6) as a superposition of plane waves (John, 1955):

$$g(x, t) = \sum_{i=1}^{2} C_i f(x - V_i t), \tag{2.1.6.15}$$

where C_i and V_i are constants. Substituting (2.1.6.15) into (2.1.6.6) we find that the last equation is satisfied, provided

$$V_i^2 = v_A^2,$$

whence

$$V_1 = v_A, \quad V_2 = -v_A.$$

The constants C_1 and C_2 follow from the boundary conditions (2.1.6.9′):

$$C_1 + C_2 = 0,$$

$$-C_1 v_A f'(x) + C_2 v_A f'(x) = \delta(x).$$

Assuming that the condition

$$-C_1 v_A + C_2 v_A = 1$$

is fulfilled, we find

$$f(x) = \theta[x] \equiv \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x < 0, \end{cases} \tag{2.1.6.15′}$$

and

$$g(x, t) = \frac{1}{2v_A}\theta[x + v_A t] - \frac{1}{2v_A}\theta[x - v_A t],$$

whence we find from eqn. (2.1.6.14)

$$\xi(x, t) = \frac{1}{2} B[\delta(x-v_A t)+\delta(x+v_A t)]+\frac{D}{2v_A}[\delta(x-v_A t)-\delta(x+v_A t)], \quad (2.1.6.16)$$

where B and D are the constants occurring in eqns. (2.1.6.12) and (2.1.6.13).

This equation shows that the quantity ξ is non-zero only when $x = \pm v_A t$. In other words, a perturbation of a point source, acting at $t = 0$ is non-zero for $t > 0$ only at the wavefront surface, that is, for $x = \pm v_A t$; in front of the wavefront ($|x| > v_A t$) and behind it ($|x| < v_A t$) there is no perturbation. When this situation occurs we shall speak of a *Huygens effect*.

We must distinguish the Huygens effect from the *Huygens principle* according to which each point of any fixed fictitious surface can be considered to be the source of secondary waves propagating from it in all directions (Landau and Lifshitz, 1971). In that case the perturbation vanishes in front of the wavefront, but is, in general, non-zero behind it.

It follows from our formulation of the Huygens principle that the wavefront at a time $t > 0$ is the envelope of all wavefronts emitted from wavefront points at the time $t = 0$ (Landau and Lifshitz, 1959).

The Huygens principle is valid for any hyperbolic system of differential equations—that is, for any wave process in which there is no dissipation—and, in particular, for the equations of the magneto-hydrodynamics of an ideal system. As far as the Huygens effect is concerned, it does not necessarily occur for a hyperbolic system of equations. For instance, transverse oscillations of a string are also described by an equation of the kind (2.1.6.6) but in that case the Huygens effect does not occur (Morse and Feshbach, 1953). This is connected with the fact that in the string problem when we have a perturbation at a point, the boundary conditions have the form

$$\xi\Big|_{t=0} = B\,\delta(x), \quad \frac{\partial \xi}{\partial t}\Big|_{t=0} = D\,\delta(x),$$

and the solution of equation (2.1.6.6) is thus the function

$$\xi(x, t) = \frac{1}{2} B[\delta(x-v_A t)+\delta(x+v_A t)]-\frac{D}{2v_A}[\theta[x-v_A t]-\theta[x+v_A t]],$$

where $\theta[x]$ is defined by formula (2.1.6.15′).

One may say that the Huygens effect occurs for Alfvén waves due to the boundary conditions

$$\xi\Big|_{t=0} = B\,\delta(x), \quad \frac{\partial \xi}{\partial t}\Big|_{t=0} = D\,\delta'(x).$$

The Huygens effect does not occur for eqn. (2.1.6.7) in the two- and three-dimensional, cases. However, it does occur in the one-dimensional case (Polovin and Cherkasova, 1966a). We note that in ordinary hydrodynamics the Huygens effect occurs for one- and three-dimensional sound (that is, longitudinal) waves, but not for two-dimensional waves.

We have considered the problem of the evolution of a one-dimensional perturbation, that is, a perturbation which at the initial time occurred in the $x = 0$ plane (the boundary conditions (2.1.6.11) contained $\delta(x)$). The perturbation then split up in seven waves: an entropy wave, at rest in the medium, two Alfvén waves, moving with velocities $\pm v_A$, two fast magneto-sound waves, moving with velocities $\pm v_+$ and, finally, two slow magneto-sound waves, moving with velocities $\pm v_-$. Such a division of the perturbation into the simplest magneto-hydrodynamic waves does not always occur, but only in the particular case of one-dimensional perturbations. If the initial perturbation is not one-dimensional, but, for instance, two-dimensional, that is, if it contained $\delta(x)\,\delta(y)$ instead of $\delta(x)$, we can, as before, split off from the total perturbation the entropy wave, and also two Alfvén waves. As far as the magneto-sound waves are concerned, they cannot be split into fast and slow waves: the front of any magneto-sound perturbation moves with the fast group velocity $U_+(\theta)$. In other words, in the region B bounded by the fast group polar U_+ (see Fig. 2.1.5) perturbations corresponding both to the fast and to the slow magneto-sound waves are

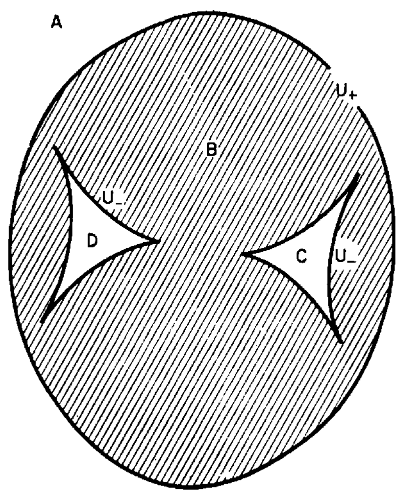

FIG. 2.1.5. Two-dimensional magneto-hydrodynamical perturbation from a point source. U_+ and U_- are the fast and the slow group polars. A is the region which is not yet reached by the perturbation, C and D are lacunae. The region where there is a perturbation is shaded.

non-vanishing. The perturbation vanishes in the region A which lies outside the fast group polar. The perturbation of the magneto-hydrodynamical quantities vanishes when $t > 0$ (Weitzner, 1961 a, b; Polovin and Cherkasova, 1966a) in the two triangular regions C and D which are bounded by the slow group polar U_- (see Fig. 2.1.5). Such regions of space in which the perturbation remains constantly zero will be called *lacunae*. Lacunae occur in magneto-hydrodynamics in the case of two-dimensional perturbations, but not in the case of three-dimensional perturbations, that is, in the case when the initial perturbation contains a factor $\delta(x)\,\delta(y)\,\delta(z)$. In the case of three-dimensional perturbations the surface of the slow group polars is a *weak discontinuity surface*, that is, a surface along which the quantities $a_i(x, y, z, t)$ are continuous, while their derivatives suffer a discontinuity when one crosses the surface.

2.2. Characteristics of the Magneto-hydrodynamical Equations

2.2.1. CHARACTERISTIC LINES

The magneto-hydrodynamical eqns. (1.5.2.19) are a set of partial differential equations for which we can pose the following problem (the *Cauchy problem*). Let the magneto-hydrodynamical state vector $a_i \equiv a_i(x, y, z, t)$ be given on some three-dimensional hypersurface,

$$f(x, y, z, t) = 0. \tag{2.2.1.1}$$

We then need to determine the vector a_i outside this hypersurface. In particular, the vector a_i may be given on the hyperplane $t = 0$, that is, in the whole of x, y, z-space at $t = 0$; the Cauchy problem then consists in finding a_i for $t \neq 0$.

We shall first of all make clear that the Cauchy problem does not always have a unique solution. To do this let us consider the simplest case of one differential equation involving two independent variables,

$$\frac{\partial a}{\partial t}+Z\frac{\partial a}{\partial z} = 0, \tag{2.2.1.2}$$

where $Z \equiv Z(z, t, a)$ is a function of z, t and a; we now pose the problem of finding a solution of this equation which takes the value a_0 on a certain curve L in the z, t-plane:

$$a(z, t) = a_0(z, t) \quad \text{when} \quad f(z, t) = 0,$$

where $f(z, t) = 0$ is the equation of the curve L. This condition can, clearly, also be written in the form

$$a(z, t)\Big|_L = b(s), \tag{2.2.1.3}$$

where s is the length of the arc along the curve L, measured from some point 0 (see Fig. 2.2.1).

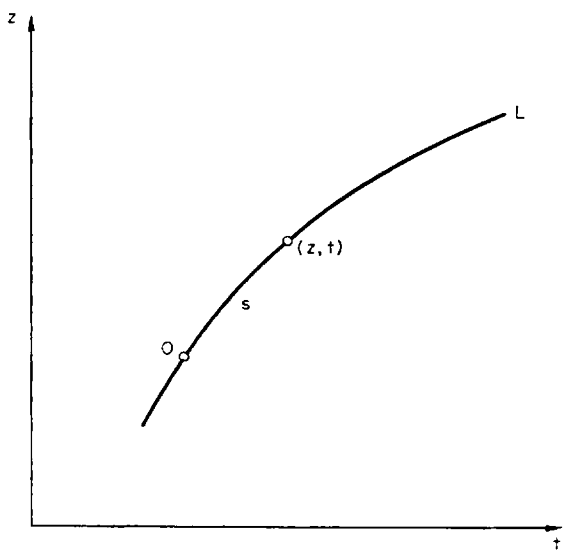

FIG. 2.2.1. Characteristics of one-dimensional waves.

In order to determine, starting from the condition (2.2.1.3), the value of the function $a(z, t)$ outside the curve L we must first of all find the derivative $\partial a/\partial n$ of a with respect to the normal n to that curve. To find this derivative we express the quantities $\partial a/\partial t$ and $\partial a/\partial z$ on the curve L in terms of $\partial a/\partial n$ and derivative $\partial a/\partial s$ of a along the tangent (Polovin, 1965a). Introducing the notation

$$V = \left.\frac{dz}{dt}\right|_{\mathrm{L}},$$

we have, clearly, on the curve L:

$$\frac{\partial a}{\partial t} = \frac{1}{\sqrt{(V^2+1)}}\frac{\partial a}{\partial s} - \frac{V}{\sqrt{(V^2+1)}}\frac{\partial a}{\partial n},$$

$$\frac{\partial a}{\partial z} = \frac{V}{\sqrt{(V^2+1)}}\frac{\partial a}{\partial s} + \frac{V}{\sqrt{(V^2+1)}}\frac{\partial a}{\partial n}.$$

Substituting these expressions into (2.2.1.2), we get

$$(V-Z)\frac{\partial a}{\partial n} = (1+ZV)\frac{\partial a}{\partial s}. \tag{2.2.1.4}$$

As $\partial a/\partial s$ is known, this equation enables us to determine $\partial a/\partial n$, provided $V \neq Z$. In that case, the Cauchy problem has thus a solution, and a unique one.

If, however, $V = Z$, that is, if

$$\frac{dz}{dt} = Z(z, t, a), \tag{2.2.1.5}$$

eqn. (2.2.1.4) does not enable us to find $\partial a/\partial n$. In that case the Cauchy problem either has no solution at all (when $\partial a/\partial s \neq 0$) or has an infinite set of solutions (when $\partial a/\partial s = 0$).

Equation (2.2.1.5) determines a family of curves which are called the *characteristic curves* or *characteristics* of the differential eqn. (2.2.1.2).

We have thus seen that by giving the function $a(z, t)$ arbitrary values $a_0(z, t)$ along any curve which is not a characteristic we can find a unique solution of the eqn. (2.2.1.2) outside that curve. On the other hand, giving the values of $a(z, t)$ on a characteristic does not determine the solution of (2.2.1.2) outside the characteristic. Moreover, the values of $a(z, t)$ on a characteristic can not be given arbitrarily; they must satisfy the additional condition $\partial a/\partial s = 0$.

One can say that the characteristics are the curves which divide different kinds of solution. In particular, if at some time $t = t_0$ the function $a(z, t)$ is non-zero only in some interval $a \leqslant z \leqslant b$ of the z-axis—we shall say in that case that at time $t = t_0$ there is a perturbation on the section (a, b)—the function $a(z, t)$, that is, the perturbation, will for $t > t_0$ be non-zero in that part of the z, t-plane which is bounded by the straight line $t = t_0$ and the two characteristics passing through the points A and B (shaded region in Fig. 2.2.2).

Let us now study the set of the magneto-hydrodynamical equations,

$$\frac{\partial a_i}{\partial t} + \sum_{j=1}^{7} Z_{ij}\frac{\partial a_j}{\partial z} = 0, \tag{2.2.1.6}$$

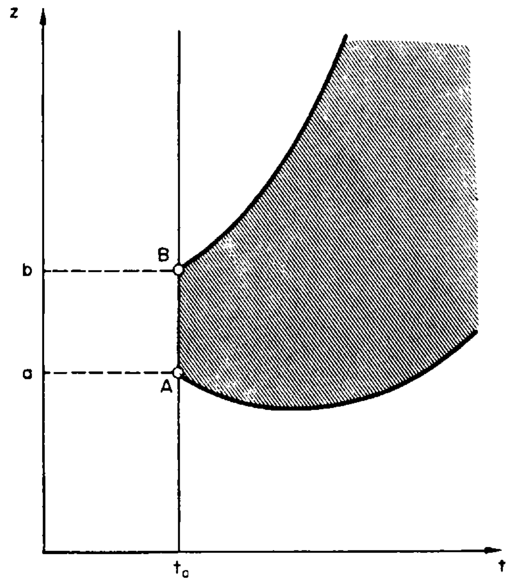

FIG. 2.2.2. Regions of influence.

for the case of one-dimensional perturbations, that is, perturbations which depend on a single space coordinate, z.

Assuming again that the functions a_i are given on a curve L and changing to the derivatives of a_i along the tangent to the curve L and along the normal to that curve, we get instead of (2.2.1.4) the following set of equations to determine the $\partial a_j/\partial n$:

$$\sum_{j=1}^{7} (Z_{ij} - V\delta_{ij}) \frac{\partial a_j}{\partial n} = -\sum_{j=1}^{7} (VZ_{ij} + \delta_{ij}) \frac{\partial a_j}{\partial s}. \tag{2.2.1.7}$$

In order that this set has a unique solution, it is necessary and sufficient that the determinant of the system,

$$\Delta \equiv \operatorname{Det} | Z_{ij} - V\delta_{ij} |,$$

is non-vanishing.

If the determinant Δ vanishes,

$$\operatorname{Det} | Z_{ij} - V\delta_{ij} | = 0, \tag{2.2.1.8}$$

the set (2.2.1.7) either has no solutions at all, or it has an infinite set of solutions. In the first case the Cauchy problem has no solutions and in the second case in order that there do exist solutions it is necessary that the values of the functions a_i on the curve L satisfy certain well-defined relations.

Equation (2.2.1.8), which is a polynomial of the seventh degree in V, defines in each point of the z, t-plane seven values of V, $V_i \equiv V_i(z, t)$, $i = 1, 2, \ldots, 7$. We get thus seven differential equations

$$\frac{dz}{dt} = V_i, \qquad i = 1, 2, \ldots, 7,$$

which determine seven families of curves such that when the functions a_i are given on these curves this does not determine them outside the curves. These curves are called the *characteristics*.

In the general case, the quantities V_i depend on z, t, and the a_j. In the case of the magneto-hydrodynamical equations the quantities Z_{ij} are independent of z and t. The V_i, therefore, do not depend explicitly on z and t and are only functions of the a_j,

$$V_i \equiv V_i(a_j).$$

We can easily see the physical meaning of the quantities V_i. As eqn. (2.2.1.8) is the same as eqn. (2.1.1.9) which determines the phase velocities of the magneto-hydrodynamical waves, the V_i are the phase velocities of small amplitude magneto-hydrodynamical waves when the initial state of the magneto-hydrodynamical medium is described by the state vector a_i. In other words, if we "freeze in" the magneto-hydrodynamical state, that is, assume that $a_i(z, t) = a_i(z_0, t_0)$, the quantities $V_j[a_i(z_0, t_0)]$ will be the phase velocities of the small amplitude hydrodynamical waves in a magneto-hydrodynamical medium with a constant state vector $a_i(z_0, t_0)$.

As there are seven kinds of small amplitude magneto-hydrodynamical waves—two fast magneto-sound, two slow magneto-sound, two Alfvén (each pair unites waves propagating in opposite directions), and one entropy wave—through each point of the z, t-plane there pass seven characteristics and the tangents of the angle they make with the t-axis are equal to the appropriate phase velocities of the small amplitude waves.

Along the characteristics the components of the magneto-hydrodynamical state vector a_i can change so that also the slope of the characteristic changes, that is, generally speaking, the characteristics are complicated curves.

Using the concept of characteristics we can write the magneto-hydrodynamics eqn. (2.2.1.6) in a simpler form (Lax, 1957; Rozhdestvenskiĭ, 1960). To do this we introduce the derivative along the jth characteristic:

$$\frac{d}{dj} \equiv \frac{\partial}{\partial t}+V_j\frac{\partial}{\partial z}, \qquad j = 1, 2, \ldots, 7.$$

The magneto-hydrodynamics eqns. (2.2.1.6) can then be written in the form

$$\sum_{i=1}^{7} l_i^{(j)} \frac{da_i}{dj} = 0, \tag{2.2.1.9}$$

where $l_i^{(j)}$ is the ith component of the jth row eigenvector. To check that (2.2.1.9) is correct, we multiply eqn. (2.2.1.6) by $l_i^{(j)}$ and sum over i. Using the fact that the row eigenvectors satisfy the equation

$$\sum_{i=1}^{7} l_i^{(j)} Z_{ik} = V_j l_k^{(j)},$$

we are led to eqn. (2.2.1.9).

2.2.2. CHARACTERISTIC SURFACES

In the preceding subsection we considered a magneto-hydrodynamical state depending on a single space coordinate. Let us now consider the case when the magneto-hydrodynamical state vector a_i depends on two space coordinates, $a_i = a_i(x, y, t)$. The magneto-hydrodynamics eqns. (1.5.2.19) have in that case the following structure:

$$\frac{\partial a_i}{\partial t}+\sum_{j=1}^{7}\left[X_{ij}\frac{\partial a_j}{\partial x}+Y_{ij}\frac{\partial a_j}{\partial y}\right] = 0, \tag{2.2.2.1}$$

where the X_{ij} and Y_{ij} are functions of the a_k.

The Cauchy problem can now be formulated as follows. We must find a solution of eqns. (2.2.2.1) when the values of the state vector a_i on some surface S in x, y, t-space are given,

$$a_i = a_i^{(0)}(x, y, t), \quad \text{when} \quad t = \psi(x, y), \tag{2.2.2.2}$$

where $t = \psi(x, y)$ is the equation of the surface S.

Introducing a quantity β through the equation

$$\beta = t-\psi(x, y)$$

and writing the equation for S in a parametric form,

$$x = x(\alpha_1, \alpha_2), \quad y = y(\alpha_1, \alpha_2), \quad t = t(\alpha_1, \alpha_2),$$

we can change in eqns. (2.2.2.1) from the variables x, y, t to the variables α_1, α_2, β:

$$\sum_j\left[\delta_{ij}\frac{\partial\beta}{\partial t}+X_{ij}\frac{\partial\beta}{\partial x}+Y_{ij}\frac{\partial\beta}{\partial y}\right]\frac{\partial a_j}{\partial\beta} = -\sum_{j,\nu}\left[\delta_{ij}\frac{\partial\alpha_\nu}{\partial t}+X_{ij}\frac{\partial\alpha_\nu}{\partial x}+Y_{ij}\frac{\partial\alpha_\nu}{\partial y}\right]\frac{\partial a_j}{\partial\alpha_\nu},$$

$$j = 1, 2, \ldots, 7, \quad \nu = 1, 2. \tag{2.2.2.3}$$

If the surface S is given and if the value of the functions a_i on it are given, we know the right-hand sides of these equations. In order that we can determine $\partial a_j/\partial\beta$ on the surface S by means of these equations it is necessary that the determinant

$$\Delta \equiv \text{Det}\left|\delta_{ij}\frac{\partial\beta}{\partial t}+X_{ij}\frac{\partial\beta}{\partial x}+Y_{ij}\frac{\partial\beta}{\partial y}\right|$$

is non-vanishing. If, however, $\Delta = 0$, we can not find the values of the a_i close to the surface S if their values are given on the surface. The surface $t = \psi(x, y)$ is in that case a *characteristic*.

Since $\beta = t-\psi(x, y)$, the equation to determine the characteristic surfaces has the form

$$\text{Det}\left|\delta_{ij}-X_{ij}\frac{\partial\psi}{\partial x}-Y_{ij}\frac{\partial\psi}{\partial y}\right| = 0. \tag{2.2.2.4}$$

As in the case of one-dimensional perturbations one can say that the characteristic surface is a surface which separates different kinds of solution of eqns. (2.2.2.1). In particular, this surface separates the region where there is a perturbation from the region where the medium is at rest, so that the *wavefront* is a characteristic surface (Courant, 1962).

We note that eqn. (2.2.2.4) for the characteristic surface is an equation containing first-order partial derivatives so that to find its solution we must give a line through which the characteristic surface passes. This means that there is an infinite set of wavefronts.

If at time $t = t_0$ the perturbation is non-vanishing on part of the plane $t = t_0$, bounded by the curve L, the wavefront at t will be the surface $t = \psi(x, y)$ which is a solution of eqn. (2.2.2.4) and passes through the curve L.

If at time $t = t_0$ the perturbation is non-vanishing only in a single point (x_0, y_0) the curve L degenerates into a point. If in that case the coefficients X_{ij} and Y_{ij} are constant, he characteristic surface will be a cone with its vertex at the point x_0, y_0, t_0. This is called the *ray cone*. If X_{ij} and Y_{ij} are not constant, the characteristic surface will not be a cone—in that case it is called a characteristic conoid—but on it the point x_0, y_0, t_0 will be conical. The cone which is tangent to the characteristic surface at that point is called the *Monge cone*.

In the present section we restrict our considerations to small perturbations, that is, to the case where the coefficients X_{ij} and Y_{ij} are constant. If the magneto-hydrodynamical perturbation at time $t = 0$ occurred in the point $x = 0$, $y = 0$, the perturbation will in this case at a later time $t > 0$ be concentrated inside the ray cone with its vertex in the point $x = 0$, $y = 0$, $t = 0$.

The generatrices of the ray cone, x/t = constant, y/t = constant are called *rays*. The curve where the ray cone intersects the plane $t = 1$ is called the *ray curve*. This curve is determined by the equation

$$\psi(x, y) = 1, \tag{2.2.2.5}$$

where the function $\psi(x, y)$ satisfies eqn. (2.2.2.4).

A perturbation starting at the point $x = 0$, $y = 0$, called the centre of the ray curve, at time $t = 0$ will at time $t = 1$ be in that part of the x, y-plane which is bounded by the ray curve.

To make clear the physical meaning of the equation for the ray curve, (2.2.2.5), we shall consider plane, two-dimensional, small amplitude magneto-hydrodynamical waves,

$$a_j^{(1)} = r_j \exp\{i(k_x x + k_z y - \omega t)\}.$$

Putting $a_j = a_j^{(0)} + a_j^{(1)}$ in the set (2.2.2.1) and linearizing it with respect to the $a_j^{(1)}$ we get a dispersion relation connecting ω with k_x and k_y:

$$\text{Det}\,|X_{ij}k_x + Y_{ij}k_y - \delta_{ij}\omega| = 0; \tag{2.2.2.6}$$

we have assumed that the X_{ij} and Y_{ij} are constant. Now introducing the phase velocity $V(\theta)$ of the wave,

$$V(\theta) = \omega/k,$$

where θ is the angle between the direction of the wave propagation and the direction of the magnetic field, we can write eqn. (2.2.2.6) in the form

$$\text{Det}\left| -X_{ij}\frac{\cos\theta}{V(\theta)} - Y_{ij}\frac{\sin\theta}{V(\theta)} + \delta_{ij} \right| = 0. \tag{2.2.2.7}$$

Comparison of this equation and eqn. (2.2.2.4) shows that

$$\frac{\partial \psi}{\partial x} = \frac{\cos \theta}{V(\theta)}, \quad \frac{\partial \psi}{\partial y} = \frac{\sin \theta}{V(\theta)}, \tag{2.2.2.8}$$

or

$$\frac{\partial \psi}{\partial x} = -\xi, \quad \frac{\partial \psi}{\partial y} = -\eta, \tag{2.2.2.9}$$

where

$$\xi = -r \cos \theta, \quad \eta = -r \sin \theta, \quad r = 1/V(\theta).$$

Hence it follows that if we have in the ξ, η-plane the curve

$$r \equiv \sqrt{(\xi^2+\eta^2)} = \frac{1}{V(\theta+\pi)}, \tag{2.2.2.10}$$

which is called the *curve of normals*† (Bazer and Fleischman, 1959)—we can as follows construct the ray curve. We construct a cone with its vertex at the origin, $x = 0$, $y = 0$, $t = 0$, and which passes through the curve of normals—it is called the *cone of normals* Each point ξ, η, 1 of the curve of normals (2.2.2.10) determines a generatrix of the cone of normals,

$$\frac{x}{\xi} = \frac{y}{\eta} = t. \tag{2.2.2.11}$$

The straight line (2.2.2.11) is a normal to the plane

$$t = -\xi x - \eta y. \tag{2.2.2.12}$$

Equation (2.2.2.9) shows that the plane (2.2.2.12) is tangent to the ray cone $t = \psi(x, y)$. The generatrices of the cone of normals are thus normals to the ray cone.

It is clear that this connection between the cone of normals and the ray cone is a reciprocal one: the generatrices of the ray cone are normals to the cone of normals.

Putting $t = 1$ in equations (2.2.2.11) and (2.2.2.12) we see that each point ξ, η of the curve of normals correspond to a tangent,

$$\xi x + \eta y + 1 = 0, \tag{2.2.2.13}$$

to the ray curve.

The transformation (2.2.2.13) is called a *polar* one, the point ξ, η is called the *pole*, and the straight line (2.2.2.13) the polar line corresponding to this pole.

By virtue of what we have said a moment ago about the reciprocity between the ray cone and the cone of normals the polar connection (2.2.2.13) between the curve of normals and the ray curve is also a reciprocal one: Each ξ, η point of the ray curve corresponds to the tangent (2.2.2.13) to the curve of normals.

If we have the phase polar (Fig. 2.1.1) we can use eqns. (2.2.2.9) and (2.2.2.13) to construct the curve of normals.

† We note that a different terminology (Holweger, 1963) is used in crystal optics (Landau and Lifshitz, 1960) and in plasma physics (Heald and Wharton, 1965), where the phase polar is called the curve of normals.

We note that the phase polar in magneto-hydrodynamics is symmetrical with respect to the centre so that we can write the eqn. (2.2.2.10) of the curve of normals in the form

$$r = \frac{1}{V(\theta)}. \qquad (2.2.2.14)$$

Using the transformation (2.2.2.13) we can easily find the equation for the ray curve. Indeed, according to (2.2.2.9) we can write the equation for the curve of normals in parametric form:

$$\xi = -\frac{\cos\theta}{V(\theta)}, \quad \eta = -\frac{\sin\theta}{V(\theta)}.$$

Performing the transformation (2.2.2.13) we find that the ray curve is in the xy-plane the envelope of the family of straight lines (Jeffrey and Taniuti, 1964)

$$x\cos\theta - y\sin\theta - V(\theta) = 0. \qquad (2.2.2.15)$$

To find the envelope we differentiate this equation with respect to θ:

$$-x\sin\theta - y\cos\theta - V'(\theta) = 0. \qquad (2.2.2.16)$$

Solving eqns. (2.2.2.15) and (2.2.2.16) for x and y we find the equation for the ray curve in the parametric form (2.1.2.2). The ray curve is thus the same as the group polar. The fast and slow group polars correspond now to the fast and slow ray cones.

The wavefront, that is, the ray curve in a moving plasma, is simply connected with the wavefront in a medium at rest. In fact, if the medium moves with a velocity u along the x-axis and if $t_0 = \psi_0(x_0, y_0)$ is the equation of the wavefront in the medium at rest, the equation of the wavefront in the moving medium will be $t = \psi_0(x-ut, y)$, that is, one obtains the wavefront in the moving medium from the wavefront in the medium at rest by a simple translation over a distance ut; in other words, the perturbation is carried along with the flow velocity.

In the three-dimensional case the wavefront surface is obtained by rotating the ray curve round the direction of the magnetic field—this surface is called the *ray surface*; the surface of normals is defined similarly.†

2.2.3 CHARACTERISTICS OF STATIONARY FLOW

In the previous subsections we have studied the propagation of a perturbation in a magneto-hydrodynamic medium and we have shown that the region where there is a perturbation is separated from the region where the medium is unperturbed by a characteristic surface. We shall now show that the characteristic surfaces play an essential role also when we study stationary flow in a magneto-hydrodynamic medium. The differential equations in that case do not contain time derivatives and we are dealing with the characteristic surfaces—or characteristic lines—of differential equations which contain only spatial deriv-

† We draw attention to the fact that Landau and Lifshitz (1960) call the surface obtained by rotating the phase polar around the direction of the magnetic field the surface of normals.

atives. The differential equations in that case do not necessarily have characteristics—in contrast to the case where the differential equations contain time derivatives when there always are characteristics.

If there are no characteristics, the perturbation will extend over the whole of space when there is a current flowing around some body. If, however, there are characteristics, the picture of the flow is completely different: the region where there is a perturbation will be separated by characteristics from the rest of space where there is no perturbation.

In ordinary hydrodynamics there is an essential difference between supersonic and subsonic flow. If a subsonic gas flow meets on its path some obstacle, for instance, flows around some body, the presence of this obstacle changes the motion in the whole of space both upstream and downstream; the effect of the body around the gas flows vanishes only asymptotically when one moves away from the body. On the other hand, supersonic flow hits an obstacle "blindly"; the effect of the obstacle extends only to the downstream region while in the whole of the remainder of space, upstream, the gas moves as if there were no obstacle at all (Landau and Lifshitz, 1959). This is connected with the fact that the set of partial differential equations which in ordinary hydrodynamics describes stationary flow is elliptic if the velocity of the medium is less than the sound velocity and hyperbolic for a supersonic velocity.

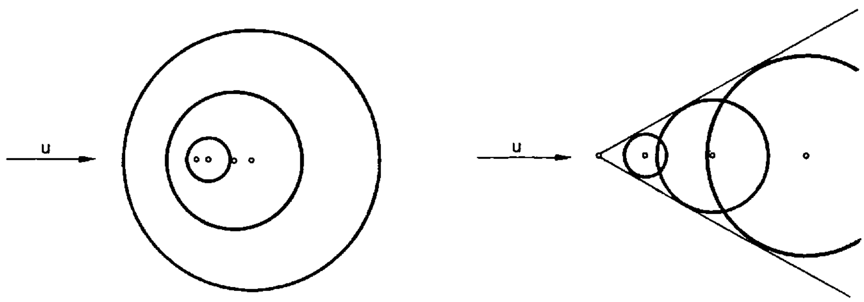

FIG. 2.2.3. Perturbation fronts at different times for a moving particle; u is the particle velocity. (a) Subsonic flow; (b) supersonic flow.

The difference between subsonic and supersonic flow is clarified by Fig. 2.2.3. In Fig. 2.2.3 the circles depict the perturbation fronts at different times when a particle is moving through a hydrodynamic medium. If the particle velocity u is less than the sound velocity c_s, the perturbation propagates as shown in Fig. 2.2.3a. If, however, $u > c_s$, a situation arises which is shown in Fig. 2.2.3b. There appear then two characteristics: they are the envelopes of the family of circles and the perturbation occurs only between them.

We note that the inclination of the characteristics to the flow direction is determined by the ratio c_s/u. As this quantity is different in different points of the xy-plane, the characteristics will, in general, be complicated curves. When we talk about straight characteristics, we have in mind an infinitesimal vicinity of the point considered.

To clarify the problem of whether or not there are characteristics, we can proceed as follows, as is clear from Fig. 2.2.3. We construct a circle with radius c_s and plot from its centre the vector $-\boldsymbol{u}$, where $\boldsymbol{u}$ is the velocity of the medium relative to the obstacle around which it is flowing. If it is impossible to draw tangents from the end of this vector (the point O in Fig. 2.2.4) to the circle, there are no characteristics. If, however, one can draw tangents, there are characteristics and they are the same as those tangents (Grad, 1960).

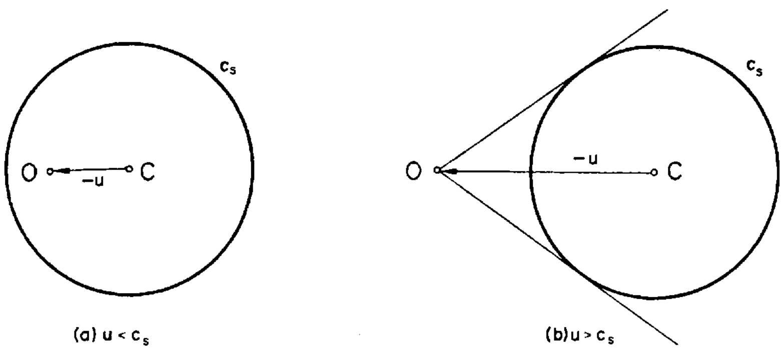

FIG. 2.2.4. Construction of the characteristics for stationary flow. (a) Subsonic flow: there are no characteristics; (b) supersonic flow: two characteristics, directed downstream, can be drawn from each point.

So far we have talked only about ordinary hydrodynamics. The situation is much more complicated in magneto-hydrodynamics as there are two types of magneto-sound waves —the fast and the slow ones—and their propagation speed depends on the angle between the direction of propagation and the direction of the magnetic field.

Moreover, there are also Alfvén and entropy waves, but there are always characteristics connected with those waves and they can be found trivially: the Alfvén characteristic is the same as the magnetic field line and the entropy characteristic the same as the current line.

To clarify the problem of the existence of the magneto-sound characteristics we consider the case of a plane flow in which the flow velocity all the time lies in the same plane—the xy-plane—while we shall assume that the magnetic field is either at right angles to that plane or lies in the same plane.

If the magnetic field is at right angles to the plane of the flow, the magneto-hydrodynamics equations reduce to the equations of ordinary hydrodynamics (Kaplan and Stanyukovich, 1954) with one difference, namely, that the quantity $\sqrt{(v_A^2+c_s^2)}$ plays the role of the sound velocity. The region in which the magneto-hydrodynamics equations are hyperbolic is thus determined by the condition (Kato and Taniuti, 1959)

$$u > \sqrt{(v_A^2+c_s^2)}. \tag{2.2.3.1}$$

As in ordinary hydrodynamics there are now two characteristics starting from each point and being directed downstream.

Let us now consider the case when the magnetic field lies in the xy-plane, that is, the plane of the flow. To find the characteristics we must construct the wavefronts which are produced by the emission of the obstacle—which we shall assume to be a point obstacle—at different times, and we must make clear whether these fronts have an envelope. However, in contrast to the case of ordinary hydrodynamics where, in the plane case, the wavefront is a circle, the wavefront in magneto-hydrodynamics is the group velocity polar diagram.

We must thus find the envelope of the family of curves

$$r = U(\theta)t + ut,$$

where the parameter t takes on all values from zero to infinity, and where $U(\theta)$ is the group velocity and u the flow velocity.

One sees easily that this envelope is the set of tangents drawn from the end of the vector $-u$, plotted from the centre of the group polar, to the group polar (Grad, 1960).

For the sake of simplicity we shall restrict our discussion to the case when the obstacle is a a perfect conductor—and we assume that the magneto-hydrodynamic medium is all the time perfectly conducting—so that at its surface the tangential component of the electrical field vanishes. This means for the case of two-dimensional flow in the xy-plane that $E_z = 0$. If we use the Ohm law for the case of infinite electrical conductivity,

$$E + \frac{1}{c}[u \wedge B] = 0,$$

we find that the vectors u and B are parallel:

$$\frac{u_x}{B_x} = \frac{u_y}{B_y}.$$

If the velocity of the medium is parallel to the magnetic field, the segment $-u$ plotted from the centre of the polar diagram is directed along the ray OB′ (see Fig. 2.1.2). If the end of the vector $-u$ in this case lies outside the fast group polar (the point M in Fig. 2.1.2) there are two characteristics directed downstream. If the end of this vector lies inside the slow group polar (the point N in Fig. 2.1.2), there are also two characteristics, directed upstream (Kogan, 1959a). If, however, the end of the vector $-u$ lies between the fast and the slow group polars (the sections B′A′ and D′O in Fig. 2.1.2) there are no characteristics.

Using equations (2.1.2.3) we find that the set of magneto-hydrodynamics equations is hyperbolic, that is, that there exist characteristics, if the velocity of the medium lies either in the range

$$\frac{c_s v_A}{\sqrt{(v_A^2 + c_s^2)}} < u < \text{Min}\,(v_A, c_s), \tag{2.2.3.2}$$

or satisfies the inequality

$$u > \text{Max}\,(v_A, c_s) \tag{2.2.3.3}$$

(Syrovatskiĭ, 1956; Kogan, 1959a; Taniuti, 1958b; Imai, 1960; Germain, 1960b). We note that the conditions (2.2.3.2) and (2.2.3.3) are the same as the conditions for the Cherenkov emission of magneto-sound waves (Morozov, 1958).

We have constructed the characteristics for the case of an obstacle of infinitesimally small dimensions. In the case of finite obstacles the lower half-plane in Fig. 2.1.2 is occupied by the body. When conditions (2.2.3.2) or (2.2.3.3) are satisfied there is thus only one cha racteristic starting from each point of the surface of the obstacle. The fast characteristic is then directed downstream and the slow one upstream. We emphasize that all these relations refer to the case where the velocity of the medium is parallel to the magnetic field.

We have considered the case where the obstacle was perfectly conducting. In the case of non-conducting bodies past which a perfectly conducting fluid is flowing the situation is possible where the vectors $\boldsymbol{u}$ and $\boldsymbol{B}$ lie in the plane of the flow, the xy-plane, but at some angle to one another. In that case the conditions (2.2.3.2) and (2.2.3.3) for the hyperbolic behaviour are changed (Kato and Taniuti, 1959; Kogan, 1960a; McCune and Resler, 1960; Polovin, 1961a). If in the case where the vectors $\boldsymbol{u}$ and $\boldsymbol{B}$ are parallel there would be at most one characteristic starting from each point of the surface of the obstacle, in the case where $\boldsymbol{u}$ and $\boldsymbol{B}$ are no longer parallel two characteristics—a fast one and a slow one or two slow ones—may start from each point of the surface. Moreover, the fast characteristic may be directed upstream and the slow one downstream.

CHAPTER 3

Simple Waves and Shock Waves in Magneto-hydrodynamics

3.1. Simple Waves

3.1.1. THE CONNECTION BETWEEN SIMPLE WAVES AND SMALL AMPLITUDE WAVES

In the preceding chapter we have studied small amplitude magneto-hydrodynamical waves. We now turn to a study of large amplitude magneto-hydrodynamical waves and we start with a study of the so-called *simple waves*, that is, those solutions of the magneto-hydrodynamics solution which have the form of a travelling wave. We shall restrict our considerations to the simplest case of plane simple waves propagating along the z-axis; in that case all components of the magneto-hydrodynamical state vector a_i are functions of the single combination $z-Vt$:

$$a_i = f_i(z-Vt), \tag{3.1.1.1}$$

where V is the wave velocity which, in turn, is a function of the a_i:

$$V = V(a_1, a_2, \ldots).$$

It follows from (3.1.1.1) that in a simple wave all magneto-hydrodynamical quantities are functions of one of them, for instance, the density ϱ, which itself depends on the coordinates and the time in the combination $z-Vt$.

It also follows from (3.1.1.1) that

$$z-V(a_1, a_2, \ldots)t = g_i(a_i), \tag{3.1.1.1'}$$

where g_i is the inverse function of f_i. In the particular case where $g_i(a_i) = 0$, we find

$$a_i = h_i(z/t),$$

where the h_i are functions of z/t. In this case one says that the simple wave is *self-similar.*†
Self-similar waves occur in all those cases where the initial and boundary conditions do not contain parameters which have the dimensions of a length or a time.

† Various authors (Yavorskaya, 1958, 1959; Kulikovskiĭ, 1958 a, b; Korobeĭnikov, 1958, 1960; Korobeĭnikov, and Ryazanov, 1959, 1960; Kochina, 1959; Ryazanov, 1959 a, b; Sharikadze, 1959; Berezin, 1961; Kozlov, 1960; Severnyĭ, 1961; Malyshev, 1961; and Greifinger and Cole, 1961) have studied a more general class of self-similar magneto-hydrodynamical waves.

For a study of simple waves we write the magneto-hydrodynamics equations of an ideal medium in the form

$$\frac{\partial a_i}{\partial t}+\sum_{k=1}^{7} Z_{ik}(a_1, a_2, \ldots)\frac{\partial a_k}{\partial z} = 0. \tag{3.1.1.2}$$

We shall consider one-dimensional motions.‡ In the case of simple waves all a_k are functions of one of them, say a_1, and we can write eqns. (3.1.1.2) in the form

$$\sum_k \left[\delta_{ik}\frac{\partial a_1}{\partial t}+Z_{ik}\frac{\partial a_1}{\partial z}\right]\frac{da_k}{da_1} = 0. \tag{3.1.1.3}$$

These equations are a set of seven homogeneous algebraic equations for the quantities da_k/da_1 which are not all at the same time equal to zero. The determinant of the set (3.1.1.3) must therefore vanish:

$$\text{Det}\left|\delta_{ik}\frac{\partial a_1}{\partial t}+Z_{ik}\frac{\partial a_1}{\partial z}\right| = 0.$$

If we denote the roots of the equation, which are all real because of the hyperbolic nature of the original set of equations,

$$\text{Det}\,|Z_{ik}(a)-V\delta_{ik}| = 0 \tag{3.1.1.4}$$

by $V_j(a_i)$, $j = 1, 2, \ldots, 7$ we can say that the function a_1 satisfies one of the equations

$$\frac{\partial a_1}{\partial t}+V_j\frac{\partial a_1}{\partial z} = 0, \qquad j = 1, 2, \ldots, 7. \tag{3.1.1.5}$$

There are thus seven kinds of simple waves corresponding to the different values of the quantities $V_j(a_i)$ which are clearly the velocities with which the values of the magneto-hydrodynamical state vector are "transferred".

Equation (3.1.1.4) is the same as eqn. (2.1.1.9) which determines the phase velocities of the small amplitude magneto-hydrodynamical waves. One can thus state that the quantities $V_j(a_1, a_2, \ldots)$ are the same as the phase velocities of the small amplitude magneto-hydrodynamical waves if the initial state is the one with the given values a_i of the magneto-hydrodynamical state vector—these values are, so to speak, "frozen in".

Substituting (3.1.1.5) into (3.1.1.3) we get a set of ordinary differential equations to determine the da_k/da_1:

$$\sum_k (Z_{ik}-V\delta_{ik})\frac{da_k}{da_1} = 0,$$

whence we have

$$\frac{da_k}{da_1} = A_k(a_1, a_2, \ldots, a_7), \tag{3.1.1.6}$$

where the A_k are functions of the a_i. It is important that it is not necessary to solve the non-linear magneto-hydrodynamical problem to find these functions, but that it is sufficient

‡ Yanenko (1956) and Komarovskiĭ (1961) have studied simple waves in the case when the number of independent variables is larger than two.

to use the solutions of the set of magneto-hydrodynamical eqns. (3.1.1.2) in the linear approximation. Indeed, considering the exact relation (3.1.1.6) in the linear approximation,

$$\frac{d\delta a_k}{d\delta a_1} = A_k(a_1^\circ, a_2^\circ, \ldots, a_7^\circ), \qquad a_k = a_k^\circ + \delta a_k,$$

we get

$$\frac{\delta a_k}{\delta a_1} = A_k(a_1^\circ, a_2^\circ, \ldots, a_7^\circ). \tag{3.1.1.7}$$

The ratios of the amplitudes δa_k in the linear approximation therefore directly give us the functions $A_k(a_1, a_2, \ldots, a_7)$.

From eqns. (3.1.1.7) and (2.1.1.8) it follows that the differential equations for simple waves can be written in the form

$$\frac{da_1}{r_1} = \frac{da_2}{r_2} = \ldots = \frac{da_7}{r_7}, \tag{3.1.1.8}$$

where the r_l are the components of the column eigenvector, which are determined by eqns. (2.1.1.12).

3.1.2. KINDS OF SIMPLE WAVES

We now turn to finding the different kinds of simple waves (Akhiezer, Lyubarskiĭ, and Polovin, 1960). We start with the Alfvén waves. As it is impossible to split the magnetic field into two terms—the unperturbed field and the perturbation—in a non-linear wave, we can not immediately use eqns. (2.1.1.12) which were derived under the assumption that the unperturbed magnetic field lies in the xz-plane, that is, that $B_y^{(0)} = 0$. For the case of an arbitrary orientation of the magnetic field the first of eqns. (2.1.1.12) has the form

$$r^{(1,2)} = \left[0, 0, -\frac{B_y}{B_x}, 1, 0, \frac{\varepsilon B_y}{B_x}\sqrt{(4\pi\varrho)}\operatorname{sgn} B_z, \ -\varepsilon\sqrt{(4\pi\varrho)}\operatorname{sgn} B_z\right],$$

where $\varepsilon = +1$ if the wave propagates in the direction of the positive z-axis and $\varepsilon = -1$ if the wave propagates in the opposite direction.

Using eqns. (3.1.1.8) and (2.1.1.12) and choosing for a_1 the quantity u_y, we get

$$\frac{du_x}{du_y} = -\frac{B_y}{B_x}, \quad \frac{dB_x}{du_y} = \frac{\varepsilon B_y}{B_x}\sqrt{(4\pi\varrho)}\operatorname{sgn} B_z,$$

$$\frac{dB_y}{du_y} = -\varepsilon\sqrt{(4\pi\varrho)}\operatorname{sgn} B_y, \quad d\varrho = ds = du_z = 0,$$

whence we find

$$B_x^2 + B_y^2 = \text{constant}, \quad u_{x,y} = -\varepsilon\frac{B_{x,y}}{\sqrt{(4\pi\varrho)}}\operatorname{sgn} B_z + \text{constant}. \tag{3.1.2.1}$$

We find thus that in an *Alfvén* simple wave the quantities ϱ, s, u_z, and $|\boldsymbol{B}|$ do not change.

In the case considered eqn. (3.1.1.1′) becomes

$$z - \left[u_z + \frac{\varepsilon|B_z|}{\sqrt{(4\pi\varrho)}}\right]t = g(u_y). \tag{3.1.2.2}$$

We now turn to the determination of the *magneto-sound simple* waves. In the case where the directions of the vectors $\boldsymbol{B}$ and $\boldsymbol{u}$ are arbitrary the second and third eqns. (2.1.1.12) have the form

$$r^{(3,4,5,6)} = \begin{bmatrix} 1 \\ 0 \\ \dfrac{\varepsilon B_x B_z v_\pm^2}{4\pi\varrho^2(v_{Az}^2 - v_\pm^2)} \\ \dfrac{\varepsilon B_y B_z v_\pm^2}{4\pi\varrho^2(v_{Az}^2 - v_\pm^2)} \\ \dfrac{\varepsilon v_\pm}{\varrho} \\ \dfrac{B_x v_\pm}{\varrho(v_\pm^2 - v_{Az}^2)} \\ \dfrac{B_y v_\pm}{\varrho(v_\pm^2 - v_{Az}^2)} \end{bmatrix}$$

where $v_A = B\sqrt{(4\pi\varrho)}$. The upper sign on $v_\pm$ corresponds to the *fast* magneto-sound wave and the lower sign to the *slow* wave. It follows from (3.1.1.8) that

$$\begin{aligned} \frac{ds}{d\varrho} &= 0, \\ \frac{du_x}{d\varrho} &= \frac{\varepsilon B_x B_z v_\pm}{4\pi\varrho^2(v_{Az}^2 - v_\pm^2)}, \\ \frac{du_y}{d\varrho} &= \frac{\varepsilon B_y B_z v_\pm}{4\pi\varrho^2(v_{Az}^2 - v_\pm^2)}, \\ \frac{du_z}{d\varrho} &= \varepsilon \frac{v_\pm}{\varrho}, \\ \frac{dB_x}{d\varrho} &= \frac{B_x v_\pm^2}{\varrho(v_\pm^2 - v_{Az}^2)}, \\ \frac{dB_y}{d\varrho} &= \frac{B_y v_\pm^2}{\varrho(v_\pm^2 - v_{Az}^2)}. \end{aligned} \tag{3.1.2.3}$$

From the last four eqns. (3.1.2.3) we get

$$\frac{B_y}{B_x} = \text{constant}, \quad u_y - \frac{B_y}{B_x} u_x = \text{constant}. \tag{3.1.2.3'}$$

These equations mean that the magneto-sound simple waves are plane polarized, that is, the vector $\boldsymbol{B}$ lies for them in a well-defined plane, the xz-plane. One can then choose the system of coordinates in such a way that the vector $\boldsymbol{u}$ also lies in that plane, so that

$$u_y \equiv 0, \quad B_y \equiv 0.$$

In the case of magneto-sound waves eqn (3.1.1.1′) takes the following form:

$$z-(u_z+\varepsilon v_\pm)t = g(\varrho). \tag{3.1.2.4}$$

It follows from this formula that at $t = 0$ the condition $g'(\varrho) < 0$ is satisfied along compression regions where $\partial\varrho/\partial z < 0$ (the wave propagates in the direction of the positive z-axis), while we have $g'(\varrho) > 0$ along the rarefaction regions ($\partial\varrho/\partial z > 0$).

If we use the relation

$$v_\pm^4-(v_A^2+c_s^2)v_\pm^2+c_s^2 v_{Az}^2 = 0, \tag{3.1.2.5}$$

as well as eqns. (3.1.2.3) we can write the velocity $v_\pm$ of the magneto-sound waves in a form which is analogous to the form in ordinary hydrodynamics (Jeffrey and Taniuti, 1964):

$$v_\pm = \sqrt{\frac{dp^*}{d\varrho}}, \tag{3.1.2.6}$$

where

$$p^* = p+\frac{B^2}{8\pi}.$$

Let us, finally, consider the *entropy* simple wave. It follows from eqns. (3.1.1.8) and (2.1.1.12) that in this wave the quantities $\boldsymbol{u}$, $\boldsymbol{B}$, and ϱ remain constant while the entropy is an arbitrary function of $z-\varepsilon u_z t$.

Let us now dwell upon a few qualitative conclusions which follow directly from the differential equations for the simple magneto-sound waves. First of all we note that the pressure changes in the same direction as the density; the transverse magnetic field $|B_x|$ changes in a fast magneto-sound wave in the same direction as the density, but in a slow magneto-sound wave in the opposite direction.

3.1.3. DISTORTION OF THE PROFILE OF A SIMPLE WAVE

It turns out that in ordinary hydrodynamics (Landau and Lifshitz, 1959) in a simple wave, points with a larger density move faster than points with a lower density, provided the condition

$$\left(\frac{\partial^2 1/\varrho}{\partial p^2}\right)_s > 0 \tag{3.1.3.1}$$

is satisfied, that is, provided the specific volume $1/\varrho$ is a concave function of the pressure, for constant entropy. We note that for a perfect gas $p/\varrho^\gamma =$ const. so that

$$\left(\frac{\partial^2 1/\varrho}{\partial p^2}\right)_s = \frac{1+\gamma}{\gamma^2}\frac{1}{\varrho p^2},$$

so that condition (3.1.3.1) is satisfied.

We shall now show that a similar situation occurs for magneto-sound simple waves in magneto-hydrodynamics (Lyubarskiĭ and Polovin, 1959a; Kulikovskiĭ, 1959). For the two other kinds of magneto-hydrodynamical simple waves—the Alfvén and entropy waves—the density is constant, and they propagate without changing their form.

To fix the ideas we assume that the wave moves in the direction of the positive z-axis. It then follows from eqn. (3.1.2.3) that

$$\frac{dV}{d\varrho} = \frac{1}{\varrho}\,\frac{d}{d\varrho}(\varrho v_{\pm}), \tag{3.1.3.2}$$

where $V = u_z + v_{\pm}$. To evaluate the derivative $d(\varrho v_{\pm})/d\varrho$ we multiply eqn. (3.1.2.5) by ϱ^4 and differentiate it with respect to ϱ; bearing in mind that ϱv_{Az}^2 = constant, we find

$$4v_{\pm}\left(v_{\pm}^2 - \frac{1}{2}[v_A^2 + c_s^2]\right)\frac{d(\varrho v_{\pm})}{d\varrho} = 2c_s\frac{d(\varrho c_s)}{d\varrho}(v_{\pm}^2 - v_{Az}^2) + (v_{\pm}^2 v_A^2 - c_s^2 v_{Az}^2) + \frac{v_{\pm}^2}{4\pi}\,\frac{dB_x^2}{d\varrho}.$$

Using now eqns. (3.1.2.3) and the relation (Landau and Lifshitz, 1959)

$$\frac{d(\varrho c_s)}{d\varrho} = \frac{1}{2}c_s^5\left(\frac{\partial^2 1/\varrho}{\partial p^2}\right)_s,$$

we find (Lyubarskiĭ and Polovin, 1958)

$$4v_{\pm}\frac{d(\varrho v_{\pm})}{d\varrho} = \varrho^3 c_s^6\left(\frac{\partial^2(1/\varrho)}{\partial p^2}\right)_s \frac{v_{\pm}^2 - v_{Az}^2}{v_{\pm}^2 - \frac{1}{2}(v_A^2 + c_s^2)} + \frac{v_{\pm}^2 v_A^2 - c_s^2 v_{Az}^2}{v_{\pm}^2 - \frac{1}{2}(v_A^2 + c_s^2)}$$
$$+\frac{v_{\pm}^4 B_x^2}{2\pi\varrho(v_{\pm}^2 - v_{Az}^2)[v_{\pm}^2 - \frac{1}{2}(v_A^2 + c_s^2)]}. \tag{3.1.3.3}$$

All terms on the right-hand side of eqn. (3.1.3.3) are positive both for fast and for slow magneto-sound waves. This follows from condition (3.1.3.1) and the fact that we have for a fast magneto-sound wave the inequalities

$$v_+ \geqslant v_{Az}, \quad v_+^2 > \frac{1}{2}(v_A^2 + c_s^2), \quad v_+^2 > \frac{c_s^2 v_{Az}^2}{v_A^2}, \tag{3.1.3.4}$$

and for slow magneto-sound waves

$$v_- \leqslant v_{Az}, \quad v_-^2 < \frac{1}{2}(v_A^2 + c_s^2), \quad v_-^2 < \frac{c_s^2 v_{Az}^2}{v_A^2}. \tag{3.1.3.5}$$

We find thus that $d(\varrho v_{\pm})/d\varrho > 0$, and hence

$$\frac{dV}{d\varrho} > 0, \quad V = u_z + v_{\pm}. \tag{3.1.3.6}$$

Inequality (3.1.3.6) means that when a simple wave propagates, points with a relative large density move faster than points with a relative low density (Kulikovskiĭ and Lyubimov, 1962).

The profile of a simple wave is thus distorted when it propagates. Those parts of the wave for which $\partial\varrho/\partial z > 0$—that is, parts where there is rarefaction since the wave moves in the direction of the positive z-axis—will when time evolves be stretched out, while the parts where $\partial\varrho/\partial z < 0$ will be compressed (see Fig. 3.1.1a) (Kulikovskiĭ, 1959; Lyubarskiĭ and

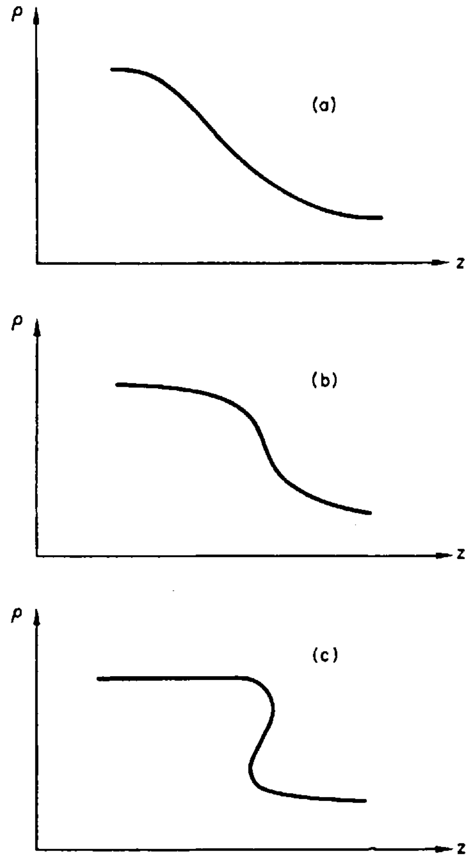

FIG. 3.1.1. Distortion of the profile of a magneto-sound simple wave. (a) Initial wave profile; (b) increase in the steepness of the profile; (c) unrealizable wave profile in which the density is a multi-valued function of the coordinate.

Polovin, 1958; Bazer, 1958).† We shall now show that this distortion of the wave profile leads to the formation of discontinuities.

For a more detailed study of this problem we use eqn. (3.1.1.1′) with $a_1 = \varrho$ and $\varepsilon = +1$. Differentiating this equation with respect to z for constant t we get

$$[tV'(\varrho)+g'(\varrho)]\frac{\partial \varrho}{\partial z} = 1. \qquad (3.1.3.7)$$

Using the equation of continuity (1.5.2.1) it follows from eqns. (3.1.2.3) that

$$\frac{\partial \varrho}{\partial z} = \frac{\varrho}{v_\pm}\frac{\partial u_z}{\partial z} = -\frac{1}{v_\pm}\frac{d\varrho}{dt}.$$

† Kaplan and Stanyukovich (1954), Segre (1958), and Taniuti (1958a) have studied similarly the case when the magnetic field is at right angles to the direction of the wave propagation. The same result has also been obtained for the case of a plasma with anisotropic pressure (Sagdeev, 1958a, 1959; Akhiezer, Polovin, and Tsintsadze, 1960; Vedenov, Velikhov, and Sagdeev, 1961a) and for a collisionless plasma in the case when $p \ll B^2/8\pi$ (Lundquist, 1952; Montgomery, 1959).

We have thus

$$-\frac{1}{v_{\pm}(\varrho)}[tV'(\varrho)+g'(\varrho)]\frac{d\varrho}{dt} = 1. \tag{3.1.3.8}$$

From this equation it follows in turn that if $tV'+g' > 0$, the derivative $d\varrho/dt$ is negative, that is, we have rarefaction; when $tV'+g' < 0$ we have compression. Because of inequality (3.1.3.6) those regions of the medium which initially were rarefaction regions ($g' > 0$) remain rarefaction regions at any later time.

In regions where $g' < 0$ we have compression as long as the inequality $tV'+g' < 0$ is satisfied. When $tV'+g'$ becomes zero, the derivative $\partial\varrho/\partial z$ tends to infinity, according to (3.1.3.7). This happens at the time

$$t = -\frac{g'(\varrho)}{V'(\varrho)}. \tag{3.1.3.9}$$

Different points z, that is, different values of ϱ, correspond according to (3.1.3.9) to different moments when $\partial\varrho/\partial z$ becomes infinite. The first time when $\partial\varrho/\partial z$ becomes infinite is when

$$t = \min_{\varrho}\left[-\frac{g'(\varrho)}{V'(\varrho)}\right]. \tag{3.1.3.9'}$$

From the condition for a minimum,

$$\left(\frac{g'}{V'}\right)' = 0,$$

it follows that in that point the following equation holds:

$$\begin{vmatrix} V' & g' \\ V'' & g'' \end{vmatrix} = 0. \tag{3.1.3.10}$$

On the other hand, by differentiating the equation

$$z - V(\varrho)t = g(\varrho)$$

with respect to ϱ we find

$$\left(\frac{\partial z}{\partial\varrho}\right)_t - V't = g',$$
$$\left(\frac{\partial^2 z}{\partial\varrho^2}\right)_t - V''t = g''. \tag{3.1.3.11}$$

Using eqns. (3.1.3.9) and (3.1.3.10) we find

$$\left(\frac{\partial z}{\partial\varrho}\right)_t = 0, \quad \left(\frac{\partial^2 z}{\partial\varrho^2}\right)_t = 0. \tag{3.1.3.12}$$

These equations shows that at the moment when $\partial\varrho/\partial z$ becomes infinite there is a point of inflexion on the profile of the simple wave with a vertical tangent (see Fig. 3.1.1b). As points with a greater density move faster the density must after that become a multi-valued function of the coordinate (see Fig. 3.1.1c). This is, however, impossible; there thus occurs a discontinuity, that is, a *shock wave* is formed at the time t determined by eqn. (3.1.3.9′).

We have already noted that a self-similar wave corresponds to the vanishing of the function $g(\varrho)$. It then follows from eqn. (3.1.1.1′) that

$$V(\varrho) = z/t. \tag{3.1.3.13}$$

This equation shows that the propagation velocity V of a self-similar wave at a given point z decreases with increasing time t. As the propagation velocities of Alfvén and entropy simple waves are constant, these waves can not be self-similar.

It follows from eqns. (3.1.3.6) and (3.1.3.7) that self-similar magneto-sound waves are rarefaction waves: $\partial\varrho/\partial z > 0$, $d\varrho/dt < 0$.

Noting that the differential equation (3.1.3.8) is linear in the function $t(\varrho)$ and using the equations $V = v_\pm + u_z$, $du_z/d\varrho = v_\pm/\varrho$, we can find the time t it takes the magneto-hydrodynamical medium to change from a density ϱ_0 to a density ϱ:

$$t = \frac{1}{\varrho v_\pm(\varrho)} \int_\varrho^{\varrho_0} g'(\varrho)\varrho \, d\varrho. \tag{3.1.3.14}$$

3.1.4. INTEGRATION OF THE EQUATIONS FOR SIMPLE WAVES

Friedrichs (see Bazer, 1958) has shown that if the magneto-hydrodynamical medium is described by the equation of state of a perfect gas,

$$p\varrho^{-\gamma} = \text{constant},$$

one can, reduce the integration of the eqns. (3.1.2.3) for magneto-sound simple waves to quadratures. Indeed, if we introduce the dimensionless quantities

$$r = \frac{c_s^2}{v_{Az}^2} \equiv \frac{4\pi\gamma p}{B_z^2}, \quad q_\pm = \frac{v_\pm^2}{c_s^2},$$

it follows that

$$dr = \frac{4\pi\gamma}{B_z^2} dp, \tag{3.1.4.1}$$

since B_z = constant. We then choose our system of coordinates such that $u_y \equiv 0$, $B_y \equiv 0$. The equation

$$v_\pm^4 - (v_A^2 + c_s^2) v_\pm^2 + c_s^2 v_{Az}^2 = 0$$

can then be written in the form

$$v_{Ax}^2 v_\pm^2 = (v_\pm^2 - c_s^2)(v_\pm^2 - v_{Az}^2), \tag{3.1.4.2.}$$

or, in terms of the variables r and $q_\pm$,

$$\frac{B_x^2}{B_z^2} = \frac{(q_\pm - 1)(rq_\pm - 1)}{q_\pm}. \tag{3.1.4.3}$$

Using eqn. (3.1.4.2) we transform the penultimate eqn. (3.1.2.3) to

$$\frac{dB_x}{d\varrho} = \frac{4\pi(v_\pm^2 - c_s^2)}{B_x},$$

whence we get, using (3.1.4.1),

$$\frac{dB_x^2}{dr} = \frac{2}{\gamma} B_z^2(q_\pm - 1).$$

Substituting this expression into eqn. (3.1.4.3) we get (Jeffrey and Taniuti, 1964)

$$\frac{dr}{dq_\pm} = \theta \frac{rq_\pm^2 - 1}{q_\pm^2(q_\pm - 1)}, \tag{3.1.4.4}$$

where

$$\theta = \frac{\gamma}{2-\gamma}.$$

As the quantity γ lies within the range (Taub, 1948) $1 < \gamma \leqslant \frac{5}{3}$, we have

$$1 < \theta \leqslant 5.$$

One can easily integrate the linear eqn. (3.1.4.4) (Bohachevsky, 1962):

$$\frac{r}{(q_\pm - 1)^\theta} + \theta \int \frac{dq_\pm}{q_\pm^2(q_\pm - 1)^{1+\theta}} = \text{constant}. \tag{3.1.4.5}$$

Using this equation to determine $q_\pm$ as function of r and using (3.1.4.3) to find the magnetic field component B_x we can determine the velocity components from eqns. (3.1.2.3) which in terms of the dimensionless variables have the form

$$\begin{aligned} \frac{du_z}{dr} &= \varepsilon \frac{c_s \sqrt{q_\pm}}{\gamma r}, \\ \frac{du_x}{dr} &= \mp \frac{\varepsilon c_s}{\gamma r} \sqrt{\left(\frac{q_\pm - 1}{rq_\pm - 1}\right)} \operatorname{sgn}(B_x B_z). \end{aligned} \tag{3.1.4.6}$$

We have thus reduced the problem of integrating the set of ordinary differential equations which describe magneto-sound simple waves to quadratures.

Let us now make clear how the magneto-hydrodynamical quantities vary in magneto-sound simple waves. It follows from the definition of the quantities r and $q_\pm$ that in a fast magneto-sound wave the following inequalities are satisfied (Polovin, 1961b):

$$rq_+ \geqslant 1, \quad q_+ > 1, \quad rq_+^2 > 1, \tag{3.1.4.7}$$

while for a slow magneto-sound wave we have

$$rq_- \leqslant 1, \quad q_- < 1, \quad rq_-^2 < 1. \tag{3.1.4.8}$$

From inequalities (3.1.4.7) and (3.1.4.8) and eqn. (3.1.4.4) it follows that $dr/dq_\pm > 0$. As the quantity r, which is a dimensionless pressure, changes according to (3.1.4.1) in the same direction as the density, the quantity $q_{\pm}$ also changes in the same direction as the density. In particular, the quantities r and q decrease in self-similar waves, both in fast and in slow magneto-sound waves.

If we turn the first of inequalities (3.1.4.7) or (3.1.4.8) into an equality:

$$rq_{+} = 1 \quad \text{or} \quad rq_{-} = 1,$$

it follows from (3.1.4.3) that the transverse magnetic field vanishes. As the quantities r and q decrease in self-similar waves, the above equality can occur only behind a fast wave or in front of a slow wave. The corresponding waves are called *peculiar waves*; the peculiar fast wave is also called a *switch-off wave* and the peculiar slow wave a *switch-on wave* (Bohachevsky, 1962).

When $r = 1$, $q = 1$ the differential eqn. (3.1.4.4) has a singular point. In the vicinity of this point the equation of the integrated curve has the form (Jeffrey and Taniuti, 1964; Polovin, 1965b)

$$r = 1 - \frac{\gamma(q-1)}{\gamma-1} + C(q-1)^{\theta}, \tag{3.1.4.9}$$

where C is an integration constant. All integrated curves have a common tangent in the singular point with slope

$$\frac{dr}{dq} = -\frac{\gamma}{\gamma-1}.$$

The singular point $r = 1$, $q = 1$ is thus a nodal point.

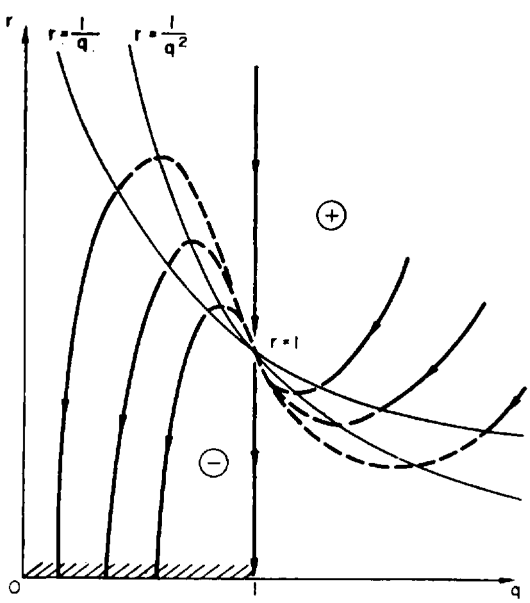

FIG. 3.1.2. Variation of the quantities r and q in self-similar magneto-sound waves. The point q, r moves along one of the integrated curves (full-drawn lines) in the direction indicated by arrows. The dashed parts of the curves are the non-existing ones. The region indicated by the plus sign corresponds to a fast self-similar wave and the region indicated by a minus sign to a slow magneto-sound wave. The section 0,1 along the abscissa axis which corresponds to cavitation is indicated by shading.

The general form of the integrated curves of equation (3.1.4.4) is shown in Fig. 3.1.2. The non-existing parts of the integrated curves along which conditions (3.1.4.7) and (3.1.4.8) are violated are indicated by dashed lines. The arrows indicate the direction in which r and q change at a given point in space when the self-similar wave propagates. The plus sign indicates fast magneto-sound waves ($q > 1$) and the minus sign slow magneto-sound waves ($q < 1$). Points situated behind the fast and in front of the slow wave fall on the curve $r = 1/q$. It is clear from Fig. 3.1.2 that in those points $r < 1$ for a fast and $r > 1$ for a slow wave.

The pressure vanishes on the $r = 0$ axis: *cavitation* sets in. There the wave has the largest possible amplitude compatible with the given initial conditions. It is clear from Fig. 3.1.2 that cavitation can occur only in a slow magneto-sound wave. Cavitation is impossible in a fast magneto-sound wave as its amplitude is limited by the condition $rq_+ \geqslant 1$.

A special case of fast and slow magneto-sound waves are the *degenerate* waves in which all the time the equality $q_\pm = 1$ holds—the quantity r decreases in such waves. It follows from eqn. (3.1.4.3) that the transverse magnetic field B_x remains zero in degenerate waves. Degenerate waves do not differ at all from the simple waves in ordinary hydrodynamics, when there is no magnetic field present. In the region $r > 1$ such a wave must be numbered among the fast and in the region $r < 1$ among the slow magneto-sound waves.

As an example (Polovin, 1961b) we shall find the condition for cavitation in a slow self-similar wave for the case when the Alfvén velocity v_{1A} in front of the wave is much smaller than the sound velocity c_{1s}. According to the definition of r this means that the condition $r_1 \gg 1$ is satisfied in front of the wave. The quantity q_- is then equal to $q_- = 1/r_1$. Because of the decrease in pressure in a self-similar wave the inequality $r_1 \gg 1$ is valid only in front of the wave. However, as the quantity q_- decreases, the inequality $q_- \ll 1$ will be valid in the whole of the slow wave. From eqn. (3.1.4.4) we find when $q \ll 1$

$$r = \frac{1+\theta}{q_1} - \frac{\theta}{q}, \tag{3.1.4.10}$$

where the index 1 refers to a region of space in front of the wave. We see from (3.1.4.10) that at the point where cavitation starts ($r_2 = 0$), the quantity q is determined by the formula

$$q_2 = \frac{\gamma q_1}{2}, \tag{3.1.4.11}$$

where the index 2 refers to a region of space behind the wave.

Using eqns. (3.1.4.6) and expressing q in terms of r by means of (3.1.4.10) we get the jumps in the velocity components in a slow wave when there is cavitation:

$$\Delta_- u_z = -\frac{\varepsilon\,|v_{1Az}|}{\sqrt{(2\gamma)}} \int_0^1 \frac{\sigma^{(-\gamma-1)/2\gamma}}{\sqrt{(1-[\sigma/(\theta+1)])}}\, d\sigma, \tag{3.1.4.12}$$

$$\Delta_- u_x = -\frac{\varepsilon c_{1s}}{\gamma} \operatorname{sgn}(B_{1x}B_z) \int_0^1 \sqrt{\left(\frac{1-(\sigma/[\theta+1])}{1-\sigma}\right)}\sigma^{(-\gamma-1)/2\gamma}\, d\sigma, \tag{3.1.4.13}$$

where $\sigma = r/r_1$. Expanding the integrands in (3.1.4.12) and (3.1.4.13) in power series we find for the case $\gamma = \frac{5}{3}$

$$\frac{1}{\sqrt{(2\gamma)}}\int_0^1 \frac{\sigma^{-(\gamma+1)/2\gamma}}{\sqrt{(1-[\sigma/(\theta+1)])}}\,d\sigma = 2.78\ \ldots, \tag{3.1.4.14}$$

$$\frac{1}{\gamma}\int_0^1 \sqrt{\left(\frac{1-(\sigma/[\theta+1])}{1-\sigma}\right)}\sigma^{-(\gamma+1)/2\gamma}\,d\sigma = 3.67\ \ldots\ . \tag{3.1.4.15}$$

3.1.5. RIEMANN INVARIANTS

A function R of the hydrodynamical quantities $a_1, a_2, \ldots, a_n$ which remains constant along a characteristic is in ordinary hydrodynamics called a Riemann invariant. If we use that definition there are no Riemann invariants in magneto-hydrodynamics (Rozhdestvenskiĭ, 1960).† Indeed, as a Riemann invariant is constant along a characteristic, we have

$$\frac{\partial R}{\partial t}+V(a_1, \ldots, a_n)\frac{\partial R}{\partial z} = 0, \tag{3.1.5.1}$$

where V is the tangent of the angle the characteristic makes with the z-axis. Equation (3.1.5.1) must be a consequence of eqns. (3.1.1.2). There must thus exist such functions $\mu_i(a_1, \ldots, a_n)$ that the equation

$$\sum_{i=1}^{n} \mu_i \frac{\partial a_i}{\partial t}+\sum_{i,k=1}^{n} \mu_i Z_{ik}\frac{\partial a_k}{\partial z} = 0 \tag{3.1.5.2}$$

is identically the same as eqn. (3.1.5.1). If we write that equation in the form

$$\sum_i \frac{\partial R}{\partial a_i}\frac{\partial a_i}{\partial t}+\sum_k V\frac{\partial R}{\partial a_k}\frac{\partial a_k}{\partial z} = 0, \tag{3.1.5.3}$$

and compare (3.1.5.2) with (3.1.5.3) we find that

$$\mu_i = \frac{\partial R}{\partial a_i}, \tag{3.1.5.4}$$

$$\sum_i \mu_i Z_{ik} = V\frac{\partial R}{\partial a_k} = V\mu_k. \tag{3.1.5.5}$$

From eqns. (3.1.5.4) we get the $\frac{1}{2}n(n-1)$ equations

$$\frac{\partial \mu_i}{\partial a_k} = \frac{\partial \mu_k}{\partial a_i},$$

† We note that when $B_z = 0$ the magneto-hydrodynamics equations have the same form as the equations of ordinary hydrodynamics (Kaplan and Stanyukovich, 1954) if we introduce instead of the ordinary pressure p and the ordinary energy per unit mass ε the "total" pressure $p^* = p+B^2/8\pi$ and the "total" energy per unit mass $\varepsilon^* = \varepsilon+B^2/4\pi\varrho$. In that case there are the same Riemann invariants as in ordinary hydrodynamics (Golitsyn, 1959; Saltanov and Tkalich, 1961; Greifinger, 1960; Mitchner, 1959). We note also that there exist Riemann invariants for any linear system of equations among which we have the linearized magneto-hydrodynamics equations (Grad, 1960).

which the functions μ must satisfy. The set (3.1.5.5) contains another $n-1$ independent equations. The n functions $\mu_1, \ldots, \mu_n$ must thus satisfy $(n-1)+\frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n+2)$ equations which, in general, is impossible when $n > 2$.† Hence it follows that the definition of the Riemann invariants which we gave earlier can not be taken over into magneto-hydrodynamics.

However, one can change the definition of a Riemann invariant in such a way (Lax, 1957) that it makes sense also in magneto-hydrodynamics. We shall call a function $R(a_1, \ldots, a_n)$ of the magneto-hydrodynamical quantities a *Riemann invariant* if it remains constant in a simple wave. The two definitions are equivalent in ordinary hydrodynamics.

As simple waves are described by a set of n first-order ordinary differential equations, there are $n-1$ integrals of this set which are just the Riemann invariants. Each simple wave corresponds thus to $n-1$ Riemann invariants.

According to eqns. (3.1.4.5), (3.1.4.6), and (3.1.2.3′) we have six Riemann invariants for a simple magneto-sound wave (we do not use here the earlier chosen system of coordinates in which $u_y \equiv 0$ and $B_y \equiv 0$):

$$\left.\begin{aligned}
R_1 &= \frac{r}{(q_\pm-1)^\theta}+\theta\int\frac{dq_\pm}{q_\pm^2(q_\pm-1)^{\theta+1}},\\
R_2 &= u_n-\frac{\varepsilon}{\gamma}\int\frac{c_s(r)\sqrt{q_\pm(r)}}{r}\,dr,\\
R_3 &= u_x\pm\frac{\varepsilon}{\gamma}\,\frac{B_x}{B_t}\,\mathrm{sgn}\,(B_nB_x)\int\frac{c_s(r)}{r}\sqrt{\left[\frac{q_\pm(r)-1}{rq_\pm(r)-1}\right]}\,dr,\\
R_4 &= u_y\pm\frac{\varepsilon}{\gamma}\,\frac{B_y}{B_t}\,\mathrm{sgn}\,(B_nB_y)\int\frac{c_s(r)}{r}\sqrt{\left[\frac{q_\pm(r)-1}{rq_\pm(r)-1}\right]}\,dr,\\
R_5 &= \frac{p}{\varrho^\gamma},\\
R_6 &= \frac{B_y}{B_x},
\end{aligned}\right\} \tag{3.1.5.6}$$

where the indexes n and t refer to components of vectors along the direction of wave propagation and to the direction at right angles to it. The functions $q_\pm(r)$ in the second, third, and fourth Riemann invariant are determined by using the first Riemann invariant; the function $c_s(r)$ by using the fifth Riemann invariant. The fifth eqn. (3.1.5.6) shows that the entropy is one of the Riemann invariants.

The invariant R_1 can for the case when $\gamma = \frac{5}{3}$ be expressed in terms of elementary functions:

$$R_1 = \frac{r-1}{(q_\pm-1)^5}+\frac{5}{2(q_\pm-1)^4}-\frac{5}{(q_\pm-1)^3}+\frac{10}{(q_\pm-1)^2}-\frac{25}{q_\pm-1}-\frac{5}{q_\pm}+30\ln\frac{q_\pm}{|q_\pm-1|}. \tag{3.1.5.7}$$

† Equations (3.1.5.4) and (3.1.5.5) might be compatible through an accidental degeneracy. One can easily show that such a degeneracy does not occur in the case of magneto-hydrodynamics and that the eqns. (3.1.5.4) and (3.1.5.5) are, indeed, incompatible.

One can use the invariants R_2 and R_3 to determine the increase in the quantities u_z and u_x in a simple wave:

$$\Delta u_z = -\frac{\varepsilon}{\gamma}\int_{r_2}^{r_1} \frac{c_s(r)\sqrt{q_\pm(r)}}{r}\,dr, \tag{3.1.5.8}$$

$$\Delta u_x = \pm\frac{\varepsilon}{\gamma}\,\mathrm{sgn}\,(B_zB_x)\int_{r_2}^{r_1} \frac{c_s(r)}{r}\sqrt{\left[\frac{q_\pm(r)-1}{rq_\pm(r)-1}\right]}\,dr, \tag{3.1.5.9}$$

where the index 1 refers to the region in front of the wave and the index 2 to the region behind the wave; we have used here a system of coordinates in which $B_y = 0$.

We note that the $n-1$ Riemann invariants $R_1^{(k)}, \ldots, R_{n-1}^{(k)}$ which correspond to the kth simple wave satisfy the same differential equation. To obtain this equation we differentiate the relation

$$R_i^{(k)}(a_1, \ldots, a_n) = \text{constant}, \qquad i = 1, \ldots, n-1,$$

and find

$$\sum_{j=1}^{n} \frac{\partial R_i}{\partial a_j}\,da_j = 0.$$

Using now eqns. (3.1.1.8) which the magneto-hydrodynamical quantities a_i satisfy in the kth simple wave, we get the required equation:

$$\sum_{j=1}^{n} \frac{\partial R_i^{(k)}}{\partial a_j}\,r_j^{(k)} = 0. \tag{3.1.5.10}$$

3.1.6. FRIEDRICHS' THEOREM

The special role played by the simple waves in magneto-hydrodynamics is connected with the fact that when there are no discontinuities only they can occur near a region of constant flow, where a_i = constant. This theorem—Friedrichs' theorem (Lax, 1957; Lyubarskiĭ and Polovin, 1961)—has in magneto-hydrodynamics a slightly smaller domain of applicability than in ordinary hydrodynamics. In fact, when a piston is moved in ordinary hydrodynamics into a region of constant flow along the characteristic $dz/dt = c_s$ a simple wave is bounded which reaches up to the piston (see Fig. 3.1.3a). On the other hand, if a piston moves in a magneto-hydrodynamical medium (see Subsection 3.4.3) there start two characteristics from the origin $z = 0$, $t = 0$: a "fast" one, $dz/dt = u_z+v_+$ and a "slow" one, $dz/dt = u_z+v_-$, where v_+ and v_- are the phase velocities of the fast and slow magnetosound waves (see Fig. 3.1.3b). In the region between the two characteristics a simple wave appears, but in the region between the slow characteristic and the piston, the motion of the medium is, in general, not described by a simple wave.†

† This is already clear from the fact that a simple wave is characterized by a single arbitrary function, while at the surface of the piston one must satisfy boundary conditions which are characterized by two arbitrary functions, $u_x(t)$ and $u_z(t)$, when the piston moves in the xz-plane in which the magnetic field (B_x, B_z) is also situated.

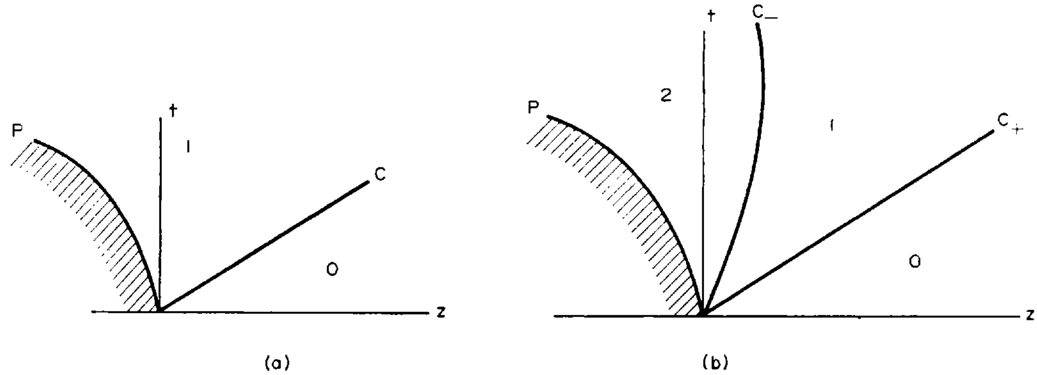

FIG. 3.1.3. Waves formed when a piston is moved: (a) in ordinary hydrodynamics; (b) in magneto-hydrodynamics. P is the line of motion of the piston, C the characteristic in ordinary hydrodynamics, $dz/dt = u_z + v_-$, C_+ the "fast" characteristic in magneto-hydrodynamics, $dz/dt = u_z + v_+$, C_- the "slow" characteristic in magneto-hydrodynamics, $dz/dt = u_z + v_-$; 0 is the unperturbed region, 1 the simple wave region, and 2 the complex wave region.

Turning now to a proof of Friedrich's theorem we shall assume that the region of constant flow is limited by some variable flow along the kth characteristic. It follows from equation (3.1.5.10) that the $n-1$ vectors

$$\text{grad } R_i^{(k)} \equiv \left(\frac{\partial R_i^{(k)}}{\partial a_1}, \ldots, \frac{\partial R_i^{(k)}}{\partial a_n}\right), \qquad i = 1, 2, \ldots, n-1,$$

are orthogonal to the column eigenvector $r^{(k)} = (r_1^{(k)}, \ldots, r_n^{(k)})$. Since, according to (2.1.1.15) any row eigenvector $l^{(j)}$ with $j \neq k$ is also orthogonal to $r^{(k)}$, the vector $l^{(j)}$ must lie in the sub-space spanned by the $n-1$ vectors grad $R_1^{(k)}, \ldots,$ grad $R_{n-1}^{(k)}$. In other words,

$$l^{(j)} = \sum_{l=1}^{n-1} \alpha_{jl} \frac{\partial R_l^{(k)}}{\partial a_i}. \tag{3.1.6.1}$$

Substituting (3.1.6.1) into (2.2.1.9) we get

$$\sum_{i=1}^{n} \sum_{l=1}^{n-1} \alpha_{jl} \frac{\partial R_l^{(k)}}{\partial a_i} \frac{da_i}{dj} = 0,$$

or

$$\sum_{l=1}^{n-1} \alpha_{jl} \frac{dR_l^{(k)}}{dj} = 0, \qquad j \neq k. \tag{3.1.6.2}$$

Equations (3.1.6.2) form a set of n—1 equations in the characteristic form (2.2.1.9) where the components of the column eigenvector $l_i^{(j)}$ are the quantities α_{ji} while the Riemann invariants $R_i^{(k)}$ occur instead of the a_i corresponding to the kth wave. As $j \neq k$ the characteristics of the set (3.1.6.2) are all characteristics of the original set (2.2.1.9) except the kth one. In other words, the kth characteristic of the original set (2.2.1.9) is for the set (3.1.6.2) an ordinary curve, along which none of the derivatives of the $R_i^{(k)}$ can have a discontinuity. This means that in the flow considered all Riemann invariants $R_1^{(k)}, \ldots, R_{n-1}^{(k)}$ are constants.

From this statement it follows that all magneto-hydrodynamical quantities $a_1, a_2, \ldots, a_n$ can be expressed in terms of one of them, that is, that the region of constant flow is bounded by a simple wave.

3.2 Discontinuities

3.2.1. BOUNDARY CONDITIONS

We showed in Subsection 3.1.3 that if there is a compression region when there is a simple magneto-sound wave present this will lead to the formation of discontinuities. Discontinuities in the magneto-hydrodynamical quantities occur also as a consequence of discontinuous initial or boundary conditions. For instance, if a piston, originally at rest, is suddenly brought into motion the velocity of the piston, as function of time, has a discontinuity.

One sees easily that discontinuities in the macroscopic quantities are possible only when the corresponding differential equations which describe the state of the medium have characteristics. Indeed, if there are no characteristics, it is possible analytically to continue the quantities describing the flow, that is, in that case the flow must be continuous.

As along the lines of discontinuity the differential eqns. (3.1.1.2) lose their meaning they do not determine the resultant jumps in the magneto-hydrodynamical quantities. We must thus add to the magneto-hydrodynamics eqns. (3.1.1.2) seven *boundary conditions*, that is, as many as there are magneto-hydrodynamical variables, on each surface (line) of discontinuity, if we want to find a unique solution for magneto-hydrodynamical problems.

To establish these boundary conditions (Hoffmann and Teller, 1950) we introduce a system of coordinates which is connected with an element of surface area of the discontinuity surface and use the general conservation laws. First of all, it is clear that on the discontinuity surface the component of the current density vector $\varrho \boldsymbol{u}$ normal to the discontinuity surface must be continuous, that is,

$$(\varrho u_{\mathrm{n}})_1 = (\varrho u_{\mathrm{n}})_2, \tag{3.2.1.1}$$

where the index n indicates the normal component while the indexes 1 and 2 indicate quantities on both sides of the discontinuity surface. We shall further use the notation

$$\Delta A = A_2 - A_1$$

and can then rewrite (3.2.1.1) in the form

$$\Delta j = 0, \tag{3.2.1.2}$$

where $j = \varrho u_{\mathrm{n}}$.

We then use the momentum conservation law. From the condition that the momentum flux is continuous follows the boundary condition

$$\Delta \Pi_{iz} = 0, \tag{3.2.1.3}$$

where we have taken the z-axis along the normal to the discontinuity surface and where

Π_{ij} is the momentum flux density tensor ($i, j = x, y, z$),

$$\Pi_{ij} = \varrho u_i u_j + p\delta_{ij} - \frac{1}{4\pi}\left(B_i B_j - \frac{1}{2}B^2\delta_{ij}\right) \tag{3.2.1.4}$$

The first two terms in this expression determine the momentum flux density tensor in ordinary hydrodynamics [compare eqn. (1.5.3.20)] while the last two terms are the Maxwell stress tensor of the magnetic field.

Substituting (3.2.1.4) into (3.2.1.3) we get

$$\begin{aligned} \Delta\left\{p + \varrho u_n^2 + \frac{1}{8\pi}(B_t^2 - B_n^2)\right\} &= 0, \\ \Delta\left\{\varrho u_n u_t - \frac{B_n B_t}{4\pi}\right\} &= 0, \end{aligned} \tag{3.2.1.5}$$

where the index t indicates vector components tangential to the discontinuity surface.

The energy flux must also be continuous on the discontinuity surface. According to (1.5.3.10) the energy flux density in ordinary hydrodynamics is equal to

$$\boldsymbol{q}^{(\varepsilon)} = \varrho \boldsymbol{u}(\tfrac{1}{2}u^2 + w),$$

where $w = \varepsilon + p/\varrho$ is the enthalpy of the fluid per unit mass. To this expression we must add the electromagnetic energy flux density

$$\boldsymbol{\pi} = \frac{c}{4\pi}[\boldsymbol{E}\wedge\boldsymbol{B}],$$

where $\boldsymbol{E}$ is the electrical field which in the case of a perfectly conducting medium is connected with the magnetic field through the relation

$$\boldsymbol{E} = -\frac{1}{c}[\boldsymbol{u}\wedge\boldsymbol{B}].$$

The total energy flux density is thus determined by the equation

$$\boldsymbol{Q} = \boldsymbol{q}^{(\varepsilon)} + \boldsymbol{\pi} = \varrho\boldsymbol{u}\left(\frac{1}{2}u^2 + w\right) + \frac{1}{4\pi}[\boldsymbol{u}B^2 - \boldsymbol{B}(\boldsymbol{u}\cdot\boldsymbol{B})]. \tag{3.2.1.6}$$

The normal component of this vector must be continuous on the discontinuity surface:

$$\Delta Q_n = 0$$

or

$$\Delta\left\{\varrho u_n\left(\frac{1}{2}u^2 + w\right) + \frac{1}{4\pi}[u_n B^2 - B_n(\boldsymbol{u}\cdot\boldsymbol{B})]\right\} = 0. \tag{3.2.1.7}$$

Finally, the normal component of the magnetic field and the tangential component of the electrical field must be continuous, that is,

$$\begin{aligned} \Delta B_n &= 0, \\ \Delta(B_n \boldsymbol{u}_t - \boldsymbol{B}_t u_n) &= 0. \end{aligned} \tag{3.2.1.8}$$

As we have already noted, the boundary conditions (3.2.1.2), (3.2.1.5), (3.2.1.7) and (3.2.1.8) refer to a system of reference moving with the discontinuity. One sees easily that we can add to the vector $\boldsymbol{u}_t$ any constant vector, that is, the coordinate system considered is fixed only as far as its velocity along the normal to the discontinuity surface is concerned.

Taking this fact into account we can arrange it in such a way that the vectors $\boldsymbol{u}$ and $\boldsymbol{B}$ are parallel at one side of the discontinuity surface. It then follows from the second eqn. (3.2.1.8) that they will also be parallel at the other side of the discontinuity. In the reference system chosen in this way the boundary condition (3.2.1.7) can be appreciably simplified (such a simplification is only possible provided $B_n \neq 0$):

$$\Delta(\tfrac{1}{2}u^2+w) = 0. \tag{3.2.1.7'}$$

It is, however, not always convenient to work with a system of reference moving with the discontinuity. To obtain the appropriate boundary conditions in the laboratory frame of reference in which the discontinuity moves with a velocity U, we must replace u_n by $u_n - U$ in eqns. (3.2.1.5), (3.2.1.7), and (3.2.1.8).

We have formulated the boundary conditions, starting from the continuity of the flux densities of a number of quantities—the mass, the momentum, and the energy. This continuity is, in turn, a consequence of the integral conservation laws of matter, momentum, and energy. At first sight one might think that the connection with the integral conservation laws is here unimportant and that it would be sufficient for the derivation of the boundary conditions at the discontinuity surface to use the equations of motion themselves; in fact, this is not correct as by using only the differential equations one can in fact obtain inaccurate boundary conditions. To clarify this fact we shall as an example study the discontinuous solutions of the differential equation

$$\frac{\partial u}{\partial t}+u\,\frac{\partial u}{\partial z} = 0. \tag{3.2.1.9}$$

One can write this equation in the form of a "differential conservation law":

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial z}\,\frac{1}{2}\,u^2 = 0, \tag{3.2.1.10}$$

and integrating it over some region $\mathcal{D}$ in the z, t-plane we get an "integral conservation law":

$$\iint\limits_{\mathcal{D}} \left[\frac{\partial u}{\partial t}+\frac{\partial}{\partial z}\,\frac{1}{2}\,u^2\right] dz\,dt = 0. \tag{3.2.1.11}$$

Transforming the double integral over the region $\mathcal{D}$ into a line integral along the boundary Γ of the region,

$$\int\limits_{\Gamma} (u\,dz-\tfrac{1}{2}u^2\,dt) = 0,$$

and integrating over a narrow loop which encircles the discontinuity line, we get the boundary condition

$$U\,\Delta u = \Delta\tfrac{1}{2}u^2, \tag{3.2.1.12}$$

where $U = dz/dt$ is the tangent of the slope of the discontinuity line with respect to the t-axis.

On the other hand, multiplying eqn. (3.2.1.9) by u we get another "differential conservation law":

$$\frac{\partial}{\partial t} u^2 + \frac{\partial}{\partial z} \frac{2}{3} u^3 = 0, \tag{3.2.1.13}$$

which corresponds to another "integral conservation law"

$$\iint_{\mathcal{D}} \left[\frac{\partial}{\partial t} u^2 + \frac{\partial}{\partial z} \frac{2}{3} u^3 \right] dz\, dt = 0. \tag{3.2.1.14}$$

This one leads to the boundary condition

$$U \Delta u^2 = \Delta \tfrac{2}{3} u^3,$$

which contradicts the boundary condition (3.2.1.12). This apparent contradiction is explained by the fact that we started from the conservation laws in the differential form (3.2.1.10) and (3.2.1.13) rather than in the integral form (3.2.1.11) and (3.2.1.14) (Sedov, 1967). The conservation laws (3.2.1.10) and (3.2.1.13) in the differential form are equivalent to one another, while the conservation laws in integral form, (3.2.1.11) and (3.2.1.14), are not equivalent. The boundary conditions follow uniquely from the conservation laws in integral form (Oleĭnik, 1957).

3.2.2. CLASSIFICATION OF DISCONTINUITIES

We now turn to consider discontinuities in magneto-hydrodynamics. We first of all remind ourselves that there are three types of discontinuities in ordinary hydrodynamics: shock waves, tangential discontinuities, and contact discontinuities.

In shock waves the pressure, density (and entropy), and normal velocity component are discontinuous, while the tangential velocity component is continuous:

$$\Delta p \neq 0, \quad \Delta \varrho \neq 0, \quad \Delta s \neq 0, \quad \Delta u_n \neq 0, \quad \Delta \boldsymbol{u}_t = 0.$$

In tangential discontinuities only the tangential velocity component is discontinuous:

$$\Delta \boldsymbol{u}_t \neq 0, \quad \Delta p = \Delta \varrho = \Delta s = \Delta u_n = 0.$$

In contact discontinuities the density and the entropy are discontinuous, and the pressure and velocity are continuous:

$$\Delta \varrho \neq 0, \quad \Delta s \neq 0, \quad \Delta p = 0, \quad \Delta \boldsymbol{u} = 0.$$

We have already noted that the existence of discontinuities is connected with the existence of characteristics for the hydrodynamic equations. Here tangential and contact discontinuities occur along the characteristic surfaces (which are composed of flow lines). As far as the shock waves are concerned, the front of these waves is not the same as the characteristic surfaces and the wavefront goes over into a characteristic surface only in the limiting case of waves of infinitesimally small intensity.

In the general case when $B_n \neq 0$ there are the following kinds of discontinuities in magneto-hydrodynamics:

Shock waves, for which all magneto-hydrodynamic quantities are discontinuous:

$$\Delta p \neq 0, \quad \Delta \varrho \neq 0, \quad \Delta s \neq 0, \quad \Delta \boldsymbol{u} \neq 0, \quad \Delta \boldsymbol{B}_t \neq 0;$$

Alfvén discontinuities, where the velocity and the magnetic field are discontinuous while the density, pressure, and entropy are continuous:

$$\Delta \boldsymbol{u} \neq 0, \quad \Delta \boldsymbol{B}_t \neq 0, \quad \Delta \varrho = \Delta p = \Delta s = 0;$$

Contact discontinuities, where the density and entropy are discontinuous and the pressure, velocity, and magnetic field continuous:

$$\Delta \varrho \neq 0, \quad \Delta s \neq 0, \quad \Delta p = \Delta \boldsymbol{u} = \Delta \boldsymbol{B}_t = 0.$$

All these discontinuities are possible, as in ordinary hydrodynamics, because of the existence of characteristics, and the Alfvén and contact discontinuities occur on the characteristic surfaces while the shock-wave front does not coincide with a characteristic surface, and only in the case of a shock wave of infinitesimal intensity is there such a coincidence.

In the present subsection we shall show that only the above-mentioned kinds of discontinuities exist and we shall elucidate their general properties. Turning to a proof of these statements we first of all split off discontinuities which are moving with respect to the medium for which

$$j \neq 0.$$

(we shall immediately show that these discontinuities are shock waves and Alfvén discontinuities). We noted in the preceding subsection that when $B_n \neq 0$ there exists a frame of reference in which the vectors $\boldsymbol{u}$ and $\boldsymbol{B}$ are parallel on both sides of the discontinuity,

$$\boldsymbol{u}_t = \frac{u_n}{B_n} \boldsymbol{B}_t. \tag{3.2.2.1}$$

The second eqn. (3.2.1.8) then becomes an identity and the second eqn. (3.2.1.5) can be written in the form

$$\left[\frac{1}{\varrho_1} - \frac{B_n^2}{4\pi j^2}\right] \boldsymbol{B}_{1t} = \left[\frac{1}{\varrho_2} - \frac{B_n^2}{4\pi j^2}\right] \boldsymbol{B}_{2t}. \tag{3.2.2.2}$$

In a shock wave $\varrho_1 \neq \varrho_2$ so that it follows from this equation that the vectors $\boldsymbol{B}_{1t}$ and $\boldsymbol{B}_{2t}$ are parallel. This means that the vectors $\boldsymbol{B}_{1t}$ and $\boldsymbol{B}_{2t}$ and the normal to the discontinuity surface lie in one plane; in other words, shock waves are plane-polarized. Using eqn. (3.2.2.1) we can choose our frame of reference in such a way that the normal to the discontinuity surface lies along the z-axis and that the relations

$$u_y = 0, \quad B_y = 0$$

are satisfied on both sides of the shock wave. Given the quantities ϱ_1, p_1, B_{1x} and B_z in front of the shock wave and also one of these quantities, say ϱ_2, behind the shock wave we can determine the quantities p_2, B_{2x}, u_{1z}, u_{2z} from the four boundary conditions (3.2.1.2), (3.2.1.5) and (3.2.1.7) after which the velocity components u_{1x} and u_{2x} are determined using eqn. (3.2.2.1). The component u_{1x} is clearly equal to the propagation velocity of the shock wave which is determined not only by the parameters p_1, ϱ_1 and $\boldsymbol{B}_1$ in front of the shock wave, but also by its intensity $\Delta\varrho$.

We shall dwell in some detail on three particular cases of magneto-hydrodynamic shock waves: parallel waves, perpendicular waves, and peculiar waves.

In a *parallel shock wave* the magnetic field is parallel to the normal on both sides of the discontinuity surface:

$$\boldsymbol{B}_{1t} = \boldsymbol{B}_{2t} = 0.$$

Changing to a frame of reference in which the velocity of the medium on both sides of the discontinuity is parallel to the magnetic field we achieve that the velocity of the medium will be parallel to the normal,

$$\boldsymbol{u}_{1t} = \boldsymbol{u}_{2t} = 0.$$

The boundary conditions then differ not at all from the boundary conditions in ordinary hydrodynamics (Landau and Lifshitz, 1959). However, we shall see in Subsection 3.4.3 that the presence of a magnetic field parallel to the normal leads in a number of cases to a splitting of the shock waves.

In a *perpendicular shock wave* the magnetic field is at right angles to the normal, that is, it lies in the discontinuity plane,

$$B_n = 0.$$

It then follows from eqn. (3.2.2.1) that there is no frame of reference in which the velocity of the medium is parallel to the magnetic field (and the discontinuity is at rest). On the other hand, it follows for a perpendicular shock wave from the second boundary condition (3.2.1.5) that $\Delta\boldsymbol{u}_t = 0$. We can thus choose our frame of reference such that the velocity of the medium on both sides of the discontinuity is along the normal,†

$$\boldsymbol{u}_{1t} = \boldsymbol{u}_{2t} = 0.$$

One verifies easily that one-dimensional motion in magneto-hydrodynamics is the same as one-dimensional motion in ordinary hydrodynamics (Kaplan and Stanyukovich, 1954) provided we replace the ordinary pressure p by the effective pressure

$$p^* = p + \frac{B^2}{8\pi},$$

and take into account that in that case the magnetic field $\boldsymbol{B} = \boldsymbol{B}_t$ is proportional to the density,

$$\frac{\boldsymbol{B}}{\varrho} = \text{constant}.$$

† We can choose such a frame of reference for parallel and perpendicular waves. Parallel and perpendicular shock waves are therefore often called *longitudinal shock waves*.

Finally, in a *peculiar shock wave* the tangential component of the magnetic field on one side of the discontinuity surface vanishes.‡ It follows from eqn. (3.2.2.2) that if $\boldsymbol{B}_t = 0$ behind the peculiar shock wave, the velocity of the medium in front of the wave (with respect to the discontinuity) is equal to the Alfvén velocity,

$$\boldsymbol{B}_{2t} = 0, \quad u_{1n} = v_{1An}.$$

We find from eqn. (3.2.2.1) that the transverse component of the velocity of the medium in front of the wave is then equal to the transverse component of the Alfvén velocity,

$$\boldsymbol{u}_{1t} = \boldsymbol{v}_{1At}.$$

If, however, the transverse component of the magnetic field vanishes in front of the peculiar shock wave, the velocity of the medium behind the discontinuity is equal to the Alfvén velocity,

$$\boldsymbol{B}_{1t} = 0, \quad \boldsymbol{u}_2 = \boldsymbol{v}_{2A}.$$

We turn now to the Alfvén discontinuities for which $j \neq 0$, $\varrho_1 = \varrho_2$. If

$$\frac{1}{\varrho} \neq \frac{B_n^2}{4\pi j^2},$$

it then follows from eqn. (3.2.2.2) that $\Delta \boldsymbol{B}_t = 0$. It follows in that case from eqns. (3.2.1.2), (3.2.2.1), (3.2.1.5), and (3.2.1.7) that $\Delta \boldsymbol{u} = 0$, $\Delta p = 0$, and $\Delta w = 0$, that is, there is no discontinuity. In an Alfvén discontinuity the following relation must therefore hold:

$$\frac{1}{\varrho_1} = \frac{1}{\varrho_2} = \frac{B_n^2}{4\pi j^2},$$

from which it follows that the velocity of propagation of an Alfvén discontinuity is equal to the Alfvén velocity:

$$u_{1n} = -\varepsilon \frac{|B_n|}{\sqrt{(4\pi\varrho)}} \tag{3.2.2.3}$$

($\varepsilon = 1$, if the discontinuity moves in the direction of the positive z-axis and $\varepsilon = -1$, if the discontinuity moves in the opposite direction). Substituting eqn. (3.2.2.3) into the second eqn. (3.2.1.5) we find

$$\Delta \boldsymbol{u}_t = -\varepsilon \operatorname{sgn} B_n \frac{\Delta \boldsymbol{B}_t}{\sqrt{(4\pi\varrho)}}. \tag{3.2.2.4}$$

For Alfvén discontinuities the first boundary condition (3.2.1.5) simplifies:

$$\Delta\left(p + \frac{B_t^2}{8\pi}\right) = 0. \tag{3.2.2.5}$$

As in Alfvén discontinuities $\Delta\varrho = 0$, it follows from the boundary condition (3.2.1.2) that $\Delta u_n = 0$.

‡ A peculiar shock wave for which the tangential component of the magnetic field vanishes in front of the discontinuity is called a *switch-on shock*, and a peculiar shock wave for which $B_t = 0$ behind the discontinuity a *switch-off shock*.

Using eqns. (3.2.2.1) and (3.2.2.3) we can write the boundary condition (3.2.1.7′) in the form

$$\Delta\left(w+\frac{B_t^2}{8\pi\varrho}\right) = 0.$$

If we use eqn. (3.2.2.5) and the thermodynamic identity $w = \varepsilon+p/\varrho$, it follows from this equation that

$$\Delta\varepsilon = 0.$$

As one can choose any two thermodynamic quantities as independent variables (Landau and Lifshitz, 1969), the relations $\Delta\varrho = 0$ and $\Delta\varepsilon = 0$ mean that also

$$\Delta p = 0.$$

We then find from eqns. (3.2.2.5) and (3.2.1.7′)

$$\Delta B_t^2 = 0, \quad \Delta u_t^2 = 0.$$

These equations show that the magnitude of the transverse components of the magnetic field and of the velocity do not change in an Alfvén discontinuity. In other words, the vectors $\boldsymbol{u}$ and $\boldsymbol{B}$ only turn over an angle on the discontinuity surface.† We note that the rotation of the velocity vector occurs only in the specially chosen frame of reference in which the velocity of the medium is parallel to the magnetic field. If the state in front of the Alfvén discontinuity is given, the state behind the discontinuity is determined by a single parameter, the jump $\Delta\boldsymbol{B}_t$. Normally, Alfvén discontinuities are not plane-polarized. An exception is the case when the magnetic field turns over 180° (Polovin and Cherkasova, 1966b). We can choose for 180° Alfvén discontinuities a frame of reference such that at both sides of the discontinuity the condition $u_y = 0$, $B_y = 0$ is satisfied.

We finally consider discontinuities which do not move relative to the medium, that is,

$$j = 0, \quad u_n = 0.$$

Assuming as before that $B_n \neq 0$ we find from the boundary conditions (3.2.1.5) and (3.2.1.8) that

$$\Delta\boldsymbol{u} = 0, \quad \Delta\boldsymbol{B} = 0, \quad \Delta p = 0.$$

As far as the boundary condition (3.2.1.7) is concerned, it becomes now an identity. Such discontinuities are called *contact discontinuities.*

A contact discontinuity is characterized by a jump in any thermodynamic quantity, except the pressure. One usually chooses for this quantity the entropy.

Let us now consider the degenerate case when the normal component of the magnetic field vanishes,

$$B_n = 0.$$

For shock waves which move relative to the medium ($j \neq 0$) this does not lead to essential

† Because of this, Alfvén discontinuities are sometimes called *rotational discontinuities*. As in these discontinuities only the transverse components of the velocity and the magnetic field change, Alfvén discontinuities are also called *transverse discontinuities*.

changes. As far as Alfvén discontinuities are concerned, for them it follows from eqn. (3.2.2.3) that $j = 0$. This means that when $B_n = 0$ the Alfvén discontinuities merge into a single type of discontinuity with the contact discontinuities.

When the relations $B_n = 0$, $j = 0$ are satisfied, the boundary conditions (3.2.1.2), (3.2.1.7), (3.2.1.8), and the second boundary condition (3.2.1.5) are satisfied identically. As far as the first boundary condition (3.2.1.5) is concerned, in the case considered it becomes

$$\Delta\left(p+\frac{B_t^2}{8\pi}\right) = 0. \tag{3.2.2.6}$$

Such discontinuities are called *tangential discontinuities*.† As only two boundary conditions, (3.2.1.2) and (3.2.2.6), come into play for a tangential discontinuity, it is characterized by five parameters, for which we can take the jumps in any five magneto-hydrodynamic quantities (the meaning of these parameters will be made clear in Subsection 3.4.2).

3.2.3. ZEMPLÉN'S THEOREM

When in ordinary hydrodynamics the conditions

$$\left(\frac{\partial^2(1/\varrho)}{\partial p^2}\right)_s > 0, \tag{3.2.3.1}$$

$$\left(\frac{\partial s}{\partial p}\right)_\varrho > 0 \tag{3.2.3.2}$$

are satisfied, where s is the entropy per unit mass, we have Zemplén's theorem (Landau and Lifshitz, 1959), according to which in a shock wave the pressure and the density increase, $p_2 > p_1$, $\varrho_2 > \varrho_1$,‡ or, in other words, shock waves are compression waves.

We shall show that if conditions (3.2.3.1) and (3.2.3.2) are satisfied, Zemplén's theorem also holds in magneto-hydrodynamics.§ To do this we first introduce Hugoniot's equation, that is, the equation which determines the jumps in the magneto-hydrodynamic quantities in a shock wave.

Using (3.2.1.8) we get from the second eqn. (3.2.1.5)

$$\boldsymbol{B}_{2t} = \frac{\dfrac{1}{\varrho_1}-\dfrac{B_n^2}{4\pi j^2}}{\dfrac{1}{\varrho_2}-\dfrac{B_n^2}{4\pi j^2}}\boldsymbol{B}_{1t}. \tag{3.2.3.3}$$

† Syrovatskiĭ (1953) was the first to give the classification of magneto-hydrodynamic discontinuities discussed here.

‡ We remind ourselves that the index 1 refers to the region in front of the shock wave and the index 2 to the region behind the shock wave.

§ Landau and Lifshitz (1957) proved Zemplén's theorem for small amplitude magneto-hydrodynamic waves and Iordanskiĭ (1959) and Polovin and Lyubarskiĭ (1958, 1959) proved it for shock waves with arbitrary amplitude. The inverse theorem is also true: the entropy increases in compression shock waves (Iordanskiĭ, 1959; Ericson and Bazer, 1960).

Writing $w_2 - w_1 = \Delta w$ we can transform the boundary condition (3.2.1.7′) to read

$$\Delta w = j^2 \overline{\left(\frac{1}{\varrho}\right)} \frac{\Delta\varrho}{\varrho_1\varrho_2} = \frac{j^2}{2B_n^2}\left[\frac{B_{1t}^2}{\varrho_1^2} - \frac{B_{2t}^2}{\varrho_2^2}\right], \tag{3.2.3.4}$$

where

$$\overline{\left(\frac{1}{\varrho}\right)} = \frac{1}{2}\left(\frac{1}{\varrho_1} + \frac{1}{\varrho_2}\right).$$

The quantity j^2 occurring on the right-hand side of eqn. (3.2.3.4) can be determined by using the first eqn. (3.2.1.5):

$$j^2 \frac{\Delta\varrho}{\varrho_1\varrho_2} = \Delta p + \frac{B_{2t}^2 - B_{1t}^2}{8\pi}. \tag{3.2.3.5}$$

Substituting now expression (3.2.3.5) into eqn. (3.2.3.4) and using relation (3.2.3.3), we get the Hugoniot equation:

$$\Delta w - \overline{\left(\frac{1}{\varrho}\right)} \Delta p = \frac{(B_{2t} - B_{1t})^2}{16\pi\varrho_1\varrho_2} \Delta\varrho \equiv Q. \tag{3.2.3.6}$$

We note that the Hugoniot equation becomes the shock adiabat of ordinary hydrodynamics when $\boldsymbol{B} = 0$.

We have already noted that in ordinary hydrodynamics shock waves are always compression waves. This is connected with the fact that if a rarefaction shock wave existed, $\Delta\varrho < 0$, the entropy would decrease on it. An analogous situation occurs also in magnetohydrodynamics where we proved that in the case of a rarefaction shock wave the presence of a magnetic field would lead to an even larger decrease in entropy than in the case where there is no field.

To check the validity of this statement we turn to the Hugoniot equation and assume that there exists a shock wave with $\varrho_2 < \varrho_1$. The quantity Q in eqn. (3.2.3.6) is then negative when $\boldsymbol{B} \neq 0$ and vanishes when $\boldsymbol{B} = 0$. Differentiating Q with respect to p_2 for constant ϱ_2,

$$\left(\frac{\partial Q}{\partial p_2}\right)_{\varrho_2} = \left(\frac{\partial w}{\partial p_2}\right)_{\varrho_2} - \overline{\left(\frac{1}{\varrho}\right)},$$

and bearing in mind that

$$\left(\frac{\partial w}{\partial p_2}\right)_{\varrho_2} = \frac{1}{\varrho_2} + T_2\left(\frac{\partial s_2}{\partial p_2}\right)_{\varrho_2},$$

we find

$$\left(\frac{\partial Q}{\partial p_2}\right)_{\varrho_2} = T_2\left(\frac{\partial s_2}{\partial p_2}\right)_{\varrho_2} + \frac{\varrho_1 - \varrho_2}{2\varrho_1\varrho_2}.$$

This quantity is according to (3.2.3.2) positive when $\varrho_2 < \varrho_1$. The fact that the derivative $(\partial Q/\partial p_2)_{\varrho_2}$ is positive means that for fixed ϱ_2 a decrease in Q causes a decrease in p_2. However, a decrease in the pressure at constant density leads according to (3.2.3.2) to a decrease in entropy. In a rarefaction shock wave we must thus have the inequality $s_2 < s_1$, which is impossible.

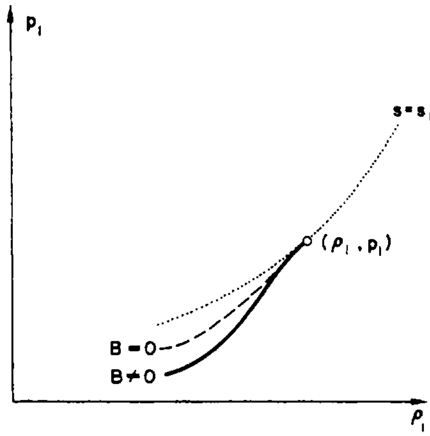

FIG. 3.2.1. The relative position of the line of constant entropy (dotted), the Hugoniot line when there is no magnetic field (dashed) and the Hugoniot line when there is a magnetic field present (full-drawn).

In Fig. 3.2.1 we show the relative position of the Hugoniot lines in ordinary and magneto-hydrodynamics and the line of constant entropy. In accordance with Zemplén's theorem, the Hugoniot line in ordinary hydrodynamics lies in the region $\varrho_2 < \varrho_1$ below the line of constant entropy $s_2 = s_1$. As we have just shown, the Hugoniot line in magneto-hydrodynamics lies even lower.

3.2.4. SIMPLE AND SHOCK WAVES IN RELATIVISTIC MAGNETO-HYDRODYNAMICS

So far we have studied magneto-hydrodynamic waves, neglecting relativistic effects. The results obtained are therefore, in particular, valid only when the phase velocities of the waves are appreciably less than the velocity of light. We shall now take relativistic effects into account. To do this we must start from the equations of relativistic magneto-hydrodynamics.

As before we shall assume the medium to be ideal—that is, we assume the viscosity and thermal conductivity coefficients to be zero and the electrical conductivity to be infinite—and we can then write the equations of motion in the form

$$\sum_k \frac{\partial T^{ik}}{\partial x^k} = 0, \qquad i, k = 0, 1, 2, 3, \tag{3.2.4.1}$$

and the equation of continuity in the form

$$\sum_k \frac{\partial n U^k}{\partial x^k} = 0, \tag{3.2.4.2}$$

where n is the density of the system (in the eigen frame of reference), U^i the four-vector of its hydrodynamic velocity,

$$U^i = \left(\frac{1}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}}, \frac{u/c}{\sqrt{\left(1-\frac{u^2}{c^2}\right)}} \right)$$

($\boldsymbol{u}$ is the normal three-dimensional particle velocity),

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z,$$

and T^{ik} is the *energy-momentum tensor* (Landau and Lifshitz, 1971). This tensor is the sum of the energy-momentum tensor of the hydrodynamic medium, T_f^{ik}, and the energy-momentum tensor of the electromagnetic field, T_e^{ik}:

$$T^{ik} = T_f^{ik} + T_e^{ik},$$

where

$$T_f^{ik} = WU^iU^k - pg^{ik},$$

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$T_e^{00} = \frac{1}{8\pi}(E^2 + B^2),$$

$$T_e^{0\alpha} = T_e^{\alpha 0} = \frac{1}{4\pi}[\boldsymbol{E} \wedge \boldsymbol{B}]_\alpha,$$

$$T_e^{\alpha\beta} = \frac{1}{4\pi}\left\{-E_\alpha E_\beta - B_\alpha B_\beta + \frac{1}{2}\delta_{\alpha\beta}(E^2 + B^2)\right\}, \qquad \alpha, \beta = 1, 2, 3,$$

p is the pressure, and W the enthalpy per unit volume in the eigen frame of reference. The electrical field $\boldsymbol{E}$ occurring in these formulae is connected with the magnetic field $\boldsymbol{B}$ and the hydrodynamic velocity $\boldsymbol{u}$ through the relation

$$\boldsymbol{E} = -\frac{1}{c}[\boldsymbol{u} \wedge \boldsymbol{B}],$$

and satisfies with $\boldsymbol{B}$ the Maxwell equations

$$\text{curl}\, \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \text{div}\, \boldsymbol{B} = 0.$$

We note that as in the relativistic case $|\boldsymbol{u}| \sim c$, the electrical field is of the same order o, magnitude as the magnetic field $\boldsymbol{B}$. (In non-relativistic magneto-hydrodynamics the electrical field is appreciably smaller then the magnetic field.)

When studying discontinuous solutions of these equations we must add to them boundary conditions at the discontinuity surface. These can in relativistic magneto-hydrodynamicsf as in ordinary magneto-hydrodynamics, be obtained starting from the integral conservation laws (Hoffmann and Teller, 1950):

$$\Delta T^{nn} \equiv \Delta\left[\frac{W\dfrac{u_n^2}{c^2}}{1-\dfrac{u^2}{c^2}}+p+\frac{B_t^2-B_n^2+E_t^2-E_n^2}{8\pi}\right] = 0,$$

$$\Delta T^{tn} \equiv \Delta\left[\frac{W\dfrac{u_n \boldsymbol{u}_t}{c^2}}{1-\dfrac{u^2}{c^2}}-\frac{B_n \boldsymbol{B}_t}{4\pi}-\frac{E_n \boldsymbol{E}_t}{4\pi}\right] = 0,$$

$$\Delta c T^{0n} \equiv \Delta\left[\frac{W u_n}{1-\dfrac{u^2}{c^2}}-\frac{B_n(\boldsymbol{u}_t \boldsymbol{B}_t)-u_n B_t^2}{4\pi}\right] = 0, \qquad (3.2.4.3)$$

$$\Delta(u_n \boldsymbol{B}_t - B_n \boldsymbol{u}_t) = 0,$$

$$\Delta \frac{nu}{\sqrt{\left(1-\dfrac{u^2}{c^2}\right)}} = 0,$$

$$\Delta B_n = 0.$$

Let us now study small amplitude waves in relativistic magneto-hydrodynamics. To do this we apply a method (Hoffmann and Teller, 1950) which differs slightly from the one used in Subsection 2.1.1. We shall now start from the set of boundary conditions (3.2.4.3) and consider the jumps in them to be small, that is, we shall neglect products of jumps of different quantities.

In small amplitude waves we can split off perturbations in the entropy. In the remaining magneto-hydrodynamic waves the entropy does not change. This enables us to simplify somewhat the set of eqns. (3.2.4.3) by dropping the penultimate equation and replacing it by the adiabaticity condition

$$\Delta p = \frac{a^2}{c^2}\,\Delta W,$$

where a is a quantity which has a simple connection with the relativistic sound velocity c_s:

$$a = \frac{c_s}{\sqrt{\left(1+\dfrac{c_s^2}{c^2}\right)}}, \qquad c_s = c\sqrt{\left(\frac{\partial p}{\partial \varepsilon}\right)_s}$$

(ε and s are the internal energy and entropy per unit mass of the medium in its eigen frame of reference). Linearizing the set of eqns. (3.2.4.3) with respect to the seven small jumps

ΔW, Δp, $\Delta \boldsymbol{u}$, and $\Delta \boldsymbol{B}_t$, we get

$$\frac{\frac{u^2}{c^2}+\frac{a^2}{c^2}\left(1-\frac{u^2}{c^2}\right)}{1-\frac{u^2}{c^2}}\Delta W+\frac{\frac{2wu}{c^2}\Delta u_z}{\left(1-\frac{u^2}{c^2}\right)^2}+\frac{B_x}{4\pi}\Delta B_x = 0,$$

$$\frac{\frac{wu}{c^2}\Delta u_x}{1-\frac{u^2}{c^2}}-\frac{B_z}{4\pi}\Delta B_x = 0,$$

$$\left[\frac{w}{1-\frac{u^2}{c^2}}+\frac{B_x^2}{4\pi}\right]\frac{u\,\Delta u_y}{c^2}-\frac{B_z\,\Delta B_y}{4\pi} = 0, \tag{3.2.4.4}$$

$$\frac{u\,\Delta W}{1-\frac{u^2}{c^2}}+\left[w\frac{1+\frac{u^2}{c^2}}{\left(1-\frac{u^2}{c^2}\right)^2}+\frac{B_x^2}{4\pi}\right]\Delta u_z-\frac{B_xB_z}{4\pi}\Delta u_x+\frac{uB_x}{2\pi}\Delta B_x = 0,$$

$$u\,\Delta B_x+B_x\,\Delta u_z-B_z\,\Delta u = 0,$$

$$u\,\Delta B_y-B_z\,\Delta u_y = 0.$$

(We have used a frame of reference in which the z-axis is along the normal to the discontinuity surface and in which $B_y = u_x = u_y = 0$.)

We note that it is impossible to put the quantity $u = u_z$ equal to zero as in the chosen frame of reference the discontinuity is at rest. In the laboratory frame of reference, in which the medium is at rest, the quantity u is, clearly, the phase velocity of the propagation of small perturbations, that is, the velocity of the magneto-hydrodynamic waves.

The set of eqns. (3.2.4.4) splits into two subsets: four equations for the jumps ΔW, Δu_x, Δu_z, ΔB_x, and two equations for the jumps Δu_y and ΔB_y. Each of these subsets is a homogeneous system of linear algebraic equations. Putting the determinant of these systems equal to zero we get relations which connect the phase velocity of the propagation of small perturbations, u, with the magnetic field and with the enthalpy W.

Adding to the waves which we have found the entropy wave which is at rest relative to the medium we get seven magneto-hydrodynamic small amplitude waves: two fast magneto-sound waves (propagating in opposite directions), two slow magneto-sound waves, two Alfvén waves, and one entropy wave. In the magneto-sound waves the perturbations ΔW, Δu_x Δu_z, and ΔB_x are non-vanishing, in the Alfvén waves the perturbations Δu_y and ΔB_y are non-zero, and in the entropy waves only the quantity Δs differs from zero.

One can easily show that the velocities of the Alfvén and the magneto-sound waves are determined by the following equations:†

$$v_{Az} = \frac{U_{Az}}{\sqrt{\left(1+\frac{U_A^2}{c^2}\right)}}, \tag{3.2.4.5}$$

† These equations were obtained by Khalatnikov (1957), Zumino (1957), and Harris (1957).

$$v_{\pm} = \left[\frac{c_s^2\left(1+\frac{U_{Az}^2}{c^2}\right)+U_A^2 \pm \left\{\left[c_s^2\left(1+\frac{U_{Az}^2}{c^2}\right)-U_A^2\right]^2+4c_s^2\frac{U_{Ax}^2}{c^2}\right\}^{1/2}}{2\left(1+\frac{U_A^2}{c^2}\right)}\right]^{1/2}, \quad (3.2.4.6)$$

where

$$U_A = \frac{cB_0}{\sqrt{(4\pi W)}},$$

and B_0 is the magnetic field in the eigen frame of reference:

$$B_{0z} = B_z, \quad B_{0x} = B_x \sqrt{\left(1-\frac{u^2}{c^2}\right)}.$$

In the non-relativistic case when the Alfvén velocity and the sound velocity are small compared with the velocity of light, these formulae go over into eqns. (2.1.1.10′) which we have already seen.

We showed in Subsection 3.1.1 that knowing the relations between the amplitudes of small perturbations, we can establish differential equations for simple waves. The same is true also in relativistic magneto-hydrodynamics. We must then in eqns. (3.2.4.4) replace Δa_i by da_i. In relativistic, as in non-relativistic, magneto-hydrodynamics there are magneto-sound, Alfvén, and entropy simple waves. The Alfvén and entropy waves propagate without distortion of the profile, while the profile of the magneto-sound waves is distorted.

In order to determine how the shape of a magneto-sound wave changes, we must evaluate the derivative dV/dW, where V is the phase velocity for the propagation of the wave in a moving medium,

$$V = u_z + v_{\pm}.$$

Similar to the derivation of eqn. (3.1.3.3) we find

$$\frac{dV}{dW} = \frac{1}{4v_{\pm}W\left(1+\frac{c_s^2}{c^2}A\right)}\left[(v_{\pm}^2-c_s^2)\,v_{\pm}^2\left(1-\frac{c_s^2}{c^2}\right)+2(v_{\pm}^2U_A^2-c_s^2U_{Az}^2)\right.$$
$$\left.+2Bc^4+BWn^2c_s^6\left(\frac{\partial W/n^2}{\partial p^2}\right)_s\right], \quad (3.2.4.7)$$

where

$$A = v_{\pm}^2\left(1+\frac{U_A^2}{c^2}\right)-\frac{U_A^2+c_s^2\left(1+\frac{U_{Az}^2}{c^2}\right)}{2}, \qquad B = v_{\pm}^2\left(1+\frac{U_{Az}^2}{c^2}\right)-U_{Az}^2.$$

For fast magneto-sound waves we have the inequalities

$$v_+ > c_s, \quad A > 0, \quad B > 0, \quad v_+ > \frac{c_s U_{Az}}{U_A},$$

and for slow magneto-sound waves the inequalities

$$v_- < c_s, \quad A < 0, \quad B < 0, \quad v_- < \frac{c_s U_{Az}}{U_A}.$$

For a relativistic gas we usually have the following inequality:

$$\left(\frac{\partial^2 W/n^2}{\partial p^2}\right)_s > 0, \tag{3.2.4.8}$$

Which is the relativistic analogue of condition (3.1.3.1) in non-relativistic hydrodynamics when condition (3.2.4.8) is satisfied, it follows from eqn. (3.2.4.7) that (Akhiezer and. Polovin, 1959; Stanyukovich, 1955a, b)

$$\frac{dV}{dW} > 0. \tag{3.2.4.9}$$

This means that points of greater density move faster.† This leads to the result that along sections with compression the density gradient increases while it decreases along sections with rarefaction.

The further study of simple waves in relativistic mangeto-hydrodynamics is the same as the corresponding study in non-relativistic magneto-hydrodynamics (see Subsection 2.2.3): discontinuities arise along compression sections, self-similar waves are rarefaction waves, and the transverse magnetic field decreases in a fast self-similar wave and increases in a slow self-similar wave.

An in ordinary magneto-hydrodynamics, there are shock waves, Alfvén discontinuities, and contact discontinuities in relativistic magneto-hydrodynamics.‡ One difference between relativistic magneto-hydrodynamics shock waves and the corresponding waves in ordinary magneto-hydrodynamics consists in the fact that there does not always exist a frame of reference in which the velocity of the medium is parallel to the magnetic field. In order that such a frame of reference exists it is necessary and sufficient that

$$\left|\frac{u_z B}{B_z}\right| < c.$$

Zemplén' s theorem holds also in relativistic magneto-hydrodynamics (Akhiezer and Polovin, (1959. For the validity of this theorem it is sufficient that condition (3.2.4.8) is satisfied and that

$$\left(\frac{\partial s}{\partial p}\right)_{W/n} > 0. \tag{3.2.4.10}$$

These conditions guarantee the validity of Zemplén's theorem also in relativistic magneto-hydrodynamics when there is no magnetic field.§ The proof is analogous to the corresponding proof on non-relativistic magneto-hydrodynamics (see Subsection 3.1.3).

† By virtue of the thermodynamic identity $(\partial W/\partial n)_s > 0$, an increase in W corresponds to an increase in the density.

‡ Khalatnikov (1957)and Akhiezer and Polovin (1959) have studied discontinuous solutions in relativistic magneto-hydrodynamics. Stanyukovich (1959) has studied discontinuities in relavistic magneto-hydrodynamics for the case where the longitudinal magnetic field B_z vanishes.

§ Khalatnikov (1954) obtained this result for small intensity shock waves and Polovin (1959) for the case of shock of arbitrary intensity. We note that in relativistic hydrodynamics not only the pressure p and the density n but also the quantity n^2/W increases on the shock wave (Israel, 1960).

The Hugoniot eqn. (3.2.3.6) has in the relativistic case the form

$$\frac{W_2^2}{n_2^2}-\frac{W_1^2}{n_1^2}-(p_2-p_1)\left(\frac{W_1}{n_1^2}+\frac{W_2}{n_2^2}\right)=Q,$$

$$Q=\frac{B_{1x}^2\left(\frac{W_2}{n_2^2}\right)^2\left(\frac{u_{1z}}{u_{2z}}-1\right)^2\left(\frac{W_1}{n_1^2}-\frac{W_2}{n_2^2}\right)}{8\pi\left(\frac{W_2}{n_2^2}-\frac{B_z^2}{4\pi j^2}\right)^2}, \tag{3.2.4.11}$$

$$j=\frac{n_1u_{1z}}{\sqrt{\left(1-\frac{u_1^2}{c^2}\right)}}.$$

The remainder of the proof is similar to the what was done in Subsection 3.2.3. In relativistic magneto-hydrodynamics we must then replace the density ϱ by the quantity n^2/W.

3.3. Stability and Structure of Shock Waves

3.3.1. EVOLUTIONARITY OF SHOCK WAVES

Giving the boundary conditions at the discontinuity is insufficient to determine uniquely the discontinuous solution. This difficulty is also encountered in ordinary hydrodynamics. For instance, when a piston moves out of a pipe there are formally two possible solutions: a self-similar rarefaction wave, and a rarefaction shock wave. In ordinary hydrodynamics one can forget about the second solution as rarefaction in a shock wave leads to a decrease in entropy.

We showed earlier that rarefaction shock waves are impossible in magneto-hydrodynamics as in them also the entropy decreases. However, in magneto-hydrodynamics there is too large a set of compression shock waves and the problems of the motion of the medium for given initial and boundary conditions often have several solutions. For instance, we shall show in Subsection 3.4.3 that if a perfectly conducting piston moves into a magneto-hydrodynamical medium at rest in which there is a magnetic field directed along the normal to the piston, there are two possible solutions:

1. The same compression shock wave as when there is no magnetic field (parallel magneto-hydrodynamic shock wave; see Subsection 3.2.2),
2. Two peculiar magneto-hydrodynamic compression shock waves (see Subsection 3.2.2). As the entropy increases in compression shock waves, the condition that the entropy must increase—which is used in ordinary hydrodynamics to help us to exclude "superfluous" discontinuities—turns out to be excessively weak in magneto-hydrodynamics.

However, in actual fact not all shock waves for which the boundary conditions are satisfied and in which the entropy increases can occur. It is necessary for the existence of a solution that it is stable. The study of stability usually proceeds as follows. One imposes onto the unperturbed values of the density, velocity, magnetic field, and so on infinitesimally small perturbations δa_1, δa_2, ... After linearization one obtains a set of differential equations with constant coefficients, the solution of which is a superposition of plane waves $e^{i(kz-\omega t)}$. The set of differential equations then reduces to a homogeneous system of linear algebraic equations:

$$\begin{aligned} A_{11}\,\delta a_1 + \dots + A_{1n}\,\delta a_n &= 0, \\ \dots \\ A_{n1}\,\delta a_1 + \dots + A_{nn}\,\delta a_n &= 0, \end{aligned} \qquad (3.3.1.1)$$

where the A_{ik} are functions of ω and k. The set (3.3.1.1) has a non-vanishing solution for $\delta a_1, \dots, \delta a_n$ only provided its determinant $\mathcal{D}(\omega, k)$ vanishes, that is, provided ω and k are connected through the dispersion relation

$$\mathcal{D}(\omega, k) = 0. \qquad (3.3.1.2)$$

If we give a real value for k—that is, if we give the wavelength $\lambda = 2\pi/k$ of the perturbation—we can find from the dispersion relation (3.3.1.2) the corresponding value of ω. Real ω mean that the solution is stable with respect to perturbations with the given wavelength λ. Complex ω (with a positive imaginary part) indicate an exponential growth of the perturbation with time, that is, the instability of the original solution.†

In a number of cases the scheme we have just described for studying stability is inapplicable since it may happen that the number of equations in the set (3.3.1.1) is not equal to the number of unknowns.‡ In that case there is either no solution, or there exists an infinite set of solutions (Landau and Lifshitz, 1959; Courant and Friedrichs, 1948).

On the other hand, both in ordinary and in magneto-hydrodynamics the Cauchy problem, that is, the problem of finding the magneto-hydrodynamical quantities at $t > 0$, if they are known at $t = 0$, always has a solution and, moreover, a unique one. The absence or non-uniqueness of a solution would be a violation of the causality principle. As the only assumption made in deriving the set (3.3.1.1) was the assumption that it was possible to linearize the equations, it follows that perturbations which were infinitesimally small at $t = 0$ at once become large.

For instance, if the starting solution were a shock wave with the profile shown in Fig. 3.3.1a, it would split into two shock waves under an infinitesimally small perturbation

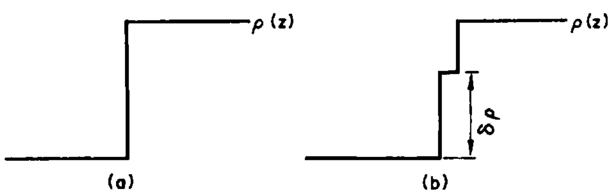

FIG. 3.3.1. The splitting of a shock wave. (a) Initial wave; (b) the wave after the splitting; $\delta\varrho$ is the perturbation in the density. The initial shock wave was moving from right to left.

† Gardner and Kruskal (1964) and Lessen and Deshpande (1967) have studied the stability of magneto-hydrodynamic shock waves.

‡ The number of equations is equal to the number of independent boundary conditions at the discontinuity surface, while the number of unknowns is equal to the number of infinitesimally small amplitude waves which leave the two sides of the discontinuity surface.

(Polovin and Cherkosova, 1966b). The perturbation $\delta\varrho$ here at once becomes large even though for small values of t this perturbation is localized only in a small region of space (Fig. 3.3.1b).

Such solutions in which infinitesimally small perturbations cause a finite change in the solution will be called *non-evolutionary* solutions (Gel'fand, 1959), or, in other words, solutions which are unstable against the appearance of new discontinuities.

The study of the evolutionarity is appreciably simpler than the study of ordinary stability as it simply reduces to counting the number of emerging waves. At the same time the conditions for evolutionarity enable us to elucidate from a unified point of view the non-existence of a number of solutions of the equations of ordinary hydrodynamics and to predict the non-existence of a number of solutions of the equations of magneto-hydrodynamics. Without using the conditions for evolutionarity it would be impossible to solve the problem of magneto-hydrodynamics when there are shock waves present.

We now turn to a study of the conditions for evolutionarity of magneto-hydrodynamic shock waves.† If we consider one-dimensional waves, moving along the z-axis the quantity B_z remains constant, so that there are seven magneto-hydrodynamic perturbations, δp, $\delta\varrho$, δu_x, δu_y, δu_z, δB_x, and δB_y, on each side of the discontinuity surface. After linearizing with respect to these perturbations we find the boundary conditions (3.2.1.2), (3.2.1.5), (3.2.1.7), and (3.2.1.8) at the shock front in the following form (we have chosen a frame of reference in which the unperturbed magnetic field and the unperturbed velocity lie in the xz-plane: $B_y = 0$, $u_y = 0$):

$$\Delta\left(\varrho u_z\,\delta u_y - \frac{B_z\,\delta B_y}{4\pi}\right) = 0, \tag{3.3.1.3}$$

$$\Delta(u_z\,\delta B_y - B_z\,\delta u_y) = 0, \tag{3.3.1.4}$$

$$\Delta[\varrho(\delta u_z - \delta U) + u_z\,\delta\varrho] = 0, \tag{3.3.1.5}$$

$$\Delta\left[\delta p + 2\varrho u_z(\delta u_z - \delta U) + u_z^2\,\delta\varrho + \frac{B_x\,\delta B_x}{4\pi}\right] = 0, \tag{3.3.1.6}$$

$$\Delta\left(\varrho u_z\,\delta u_x - \frac{B_z\,\delta B_x}{4\pi}\right) = 0, \tag{3.3.1.7}$$

$$\Delta[B_z\,\delta u_x - B_x(\delta u_z - \delta U) - u_z\,\delta B_x] = 0, \tag{3.3.1.8}$$

$$\Delta\left\{\varrho u_z[u_z(\delta u_z - \delta U) + u_x\,\delta u_x + \delta w] + \frac{1}{4\pi}(u_z B_x - u_x B_z)\,\delta B_x\right\} = 0, \tag{3.3.1.9}$$

where δU is the perturbation of the shock wave velocity. The number of boundary conditions (3.3.1.3) to (3.3.1.9) equals seven. However, these boundary conditions are not independent as they contain the perturbation δU of the wave velocity. After eliminating δU we obtain a set of six independent boundary conditions. It is thus necessary for the evolutionarity of a magneto-hydrodynamic shock wave that the number of waves emerging at both sides of the discontinuity surface be equal to six (Akhiezer, Lyubarskiĭ, and Polovin,

† Akhiezer, Lyubarskiĭ, and Polovin (1959) solved the problem of the evolutionarity of magneto-hydrodynamic shock waves.

1959). In the case of an arbitrary set of equations the total number of emerging waves must be less than the number of boundary conditions by unity (Lax, 1957; Babenko and Gel'fand, 1958).

In magneto-hydrodynamics there are fourteen different phase velocities for the propagation of small amplitude waves (on both sides of the shock wave front):

$$u_{1z}+v_{1Az},\quad u_{1z}-v_{1Az},\quad u_{1z}+v_{1+},\quad u_{1z}-v_{1+},\quad u_{1z}+v_{1-},\quad u_{1z}-v_{1-},\quad u_{1z},$$
$$u_{2z}+v_{2Az},\quad u_{2z}-v_{2Az},\quad u_{2z}+v_{2+},\quad u_{2z}-v_{2+},\quad u_{1z}+v_{2-},\quad u_{1z}-v_{2-},\quad u_{2z}$$

(the index "1" refers to the region $z < 0$ in front of the shock wave and the index "2" to the region $z > 0$ behind the shock wave; the coordinate systems was chosen such that the discontinuity was at rest in it and was lying in the $z = 0$ plane; the direction of the z-axis was chosen such that the component u_z of the velocity of the medium was positive along that axis). A negative phase velocity corresponds to waves emerging into the region in front of the shock wave, and a positive phase velocity to the waves emerging behind the shock wave.

Of the fourteen phase velocities enumerated here four velocities correspond to incoming waves:

$$u_{1z}+v_{1+},\quad u_{1z}+v_{1-},\quad u_{1z}+v_{1Az},\quad u_{1z},$$

and four to outgoing waves:

$$u_{2z}+v_{2+},\quad u_{2z}+v_{2-},\quad u_{2z}+v_{2Az},\quad u_{2z}$$

(all these phase velocities are necessarily positive). The remaining waves will be incoming or outgoing depending on the ratios of the quantities u_z and v_+, v_-, v_{Az}. In Fig. 3.3.2 we show the total number of outgoing waves for different values of u_{1z} and u_{2z} (Polovin, 1961c). The region of the u_{1z}, u_{2z}-plane in which the total number of outgoing waves is six corresponds to evolutionary waves.

However, the fact that the number of independent boundary conditions and the number of emerging waves are equal is not yet sufficient to ensure the existence and uniqueness of the solution. It might happen that the equations serving to determine the amplitudes of the

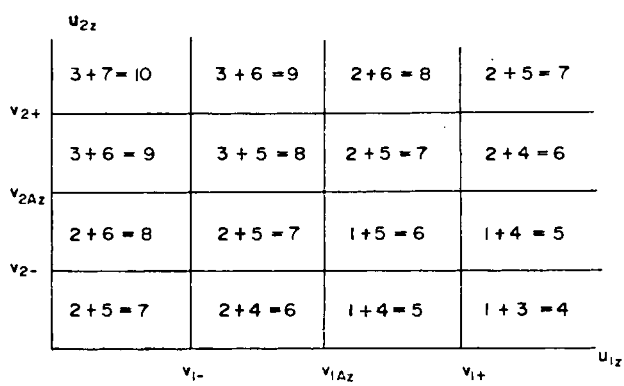

FIG. 3.3.2. Number of waves which are outgoing waves at the discontinuity surface. The first number is the number of Alfvén waves and the second number that of magneto-sound and entropy waves.

emerging waves and the boundary conditions split into a few independent groups. In that case the conditions for evolutionarity—that is, the equality of the number of outgoing waves and the number of independent boundary conditions—must be satisfied not only for the whole set of variables, but also for each isolated group separately.

Such a splitting of the equations and the boundary conditions into two independent groups occurs in magneto-hydrodynamics for waves propagating at right angles to the discontinuity surface (Syrovatskii, 1959). Indeed, only the quantities δu_y and δB_y (the xz-plane is orientated in such a way that $B_y = 0$) are non-vanishing in small amplitude Alfvén waves; in magneto-sound and entropy waves the quantities $\delta \varrho$, δp, δu_x, δu_z, and δB_x are non-vanishing. The boundary conditions (3.3.1.3) to (3.3.1.9) split also into the same groups:

1. Alfvén perturbations: (3.3.1.3) and (3.3.1.4);
2. magneto-sound and entropy perturbations: (3.3.1.5) to (3.3.1.9).

The boundary conditions (3.3.1.3) and (3.3.1.4) do not contain δU. They are thus independent and the number of outgoing Alfvén waves must thus equal two (Syrovatskiĭ, 1959). The boundary conditions (3.3.1.5) to (3.3.1.9) contain the perturbation δU of the shock wave velocity; after eliminating this there remain four independent boundary conditions. Hence, the number of outgoing magneto-sound and entropy waves must equal four.

We show in Fig. 3.3.3 the Hugoniot lines in the u_{1z}, u_{2z}-plane. This figure is a combination of two diagrams:† the section of the Hugoniot line 1–2–3–4–5–6–1, corresponding to a slow shock wave, constructed for given values of v_{1Az}, v_{1-}, v_{2Az}, and v_{2-}, and a section of the Hugoniot line 7–8–9 corresponding to a fast wave—for given v_{1Az}, v_{1+}, v_{2Az}, and v_{2+}.‡ The sections 4–5–6–1 and 7–8 correspond to rarefaction shock waves: $u_{1z} < u_{2z}$.

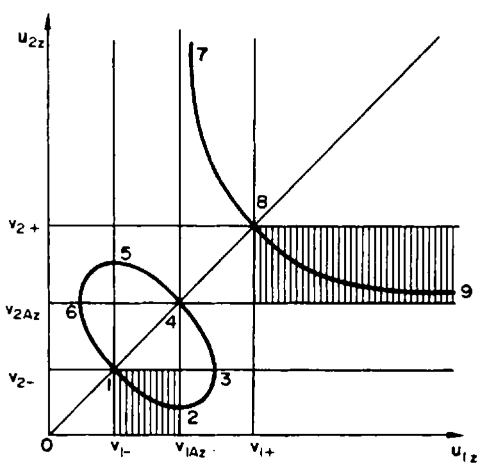

FIG. 3.3.3. The regions where shock waves are evolutionary (shaded). 1-2-3-4-5-6-1: slow shock wave; 7-8-9: fast shock wave.

† The occurrence of additional sections of the Hugoniot line in Fig. 2.11 of Anderson's book (1963) is connected with the neglect of this fact.

‡ If we fix other quantities, for instance, $N_1 = v_{1A}/c_{1s}$ and $M_1 = u_1/c_{1s}$ (Kogan, 1960b), the Hugoniot line will be qualitatively different from the one depicted in Fig. 3.3.3 (Polovin, 1963a).

In such waves the entropy decreases (see Subsection 3.2.3). The sections 1–2–3–4 and 8–9 correspond to compression shock waves: $u_{1z} > u_{2z}$. In such waves the entropy increases.

It is clear from Fig. 3.3.3 that there are two regions where the shock waves are evolutionary (Akhiezer, Lyubarskiĭ and Polovin, 1959); these regions are shaded in Fig. 3.3.3:

1. *Fast shock waves* (see section 8–9 in Fig. 3.3.3), for which

$$v_{1+} < u_{1z}, \quad v_{2Az} < u_{2z} < v_{2+}. \tag{3.3.1.10}$$

2. *Slow shock waves* (see section 1–2 in Fig. 3.3.3), for which

$$v_{1-} < u_{1z} < v_{1Az}, \quad u_{2z} < v_{2-}. \tag{3.3.1.11}$$

It is clear from Fig. 3.3.3 that the fast compression shock wave is evolutionary on the whole of section 8–9, that is, for any amplitude. On the other hand, the slow compression shock wave is evolutionary only on the section 1–2. On the section 2–3–4† the slow compression shock wave is non-evolutionary and cannot exist, notwithstanding the fact that the entropy increases in such a wave.

So far we have considered the conditions for evolutionarity with regard to perturbations depending only on z and t. Consideration of perturbations of a general form (depending also on x and y) leads to the same conditions (3.3.1.10) and (3.3.1.11) for evolutionarity (Kontorovich, 1959). This is connected with the fact that the front of waves leaving the discontinuity surface (which along a small section can be approximated by a plane) is the plane, tangential to the ray surface (the centre of which lies on the discontinuity surface). As the straight line (2.2.2.15) is the tangent to the ray line, the distance from wavefront to the discontinuity surface is equal to the phase velocity in the direction of the normal.

We note that the condition for evolutionarity is stronger than the condition that the entropy must increase (Taniuti, 1962): on evolutionary shock waves the entropy always increases (Jeffrey and Taniuti, 1964).

We must emphasize the essential difference between non-evolutionarity and instability. Unstable states can arise when the magneto-hydrodynamical medium moves under the action of internal causes. They exist for some time, until fluctuations reach a critical value after which the unstable state is destroyed. As far as non-evolutionary discontinuities are concerned, they cannot arise spontaneously. Non-evolutionary discontinuities can be formed only under the action of external factors, such as, for instance, the collision of gas masses. They can exist only for a moment as discontinuities in initial conditions, after which the non-evolutionary discontinuities quickly split into several shock waves or self-similar waves. Such a splitting up of a non-evolutionary shock wave will be considered in Subsection 3.4.5.‡

So far we have tacitly assumed in the formulation of the conditions for evolutionarity of magneto-hydrodynamic waves that there is no dispersion for small amplitude waves,

† The point $u_{1z} = v_{1+}$, $u_{2z} = v_{2-}$ can for some values of the parameters lie to the left of the line 2-3-4 (Shercliff, 1960).

‡ If a shock wave is considered not as a discontinuity, but as a fast continuous transition from one state to another, one can study the evolution of non-stationary perturbations only by means of electronic computers. Such calculations confirm the conclusions reached earlier about the splitting-up of non-evolutionary shock waves (Todd, 1964, 1965; Chu and Taussig, 1967).

that is, that the velocity of these waves is independent of frequency. In principle the phase velocities can depend on the frequency, for instance, in two-fluid magneto-hydrodynamics. In that case we must, when formulating the conditions for the evolutionarity, understand by the velocities of small amplitude waves the phase velocities of low-frequency oscillations (there is no dispersion in the low-frequency limit). This is connected with the fact that the main part of the perturbation moves with the phase velocity of the low-frequency waves (Whitham, 1959). Neglect of this fact can lead to an apparent non-evolutionarity of the shock waves (Karpman and Sagdeev, 1964).§

We note in this connection that the sound velocity in a gas, which occurs in eqn. (2.1.1.10′) must be determined from the equation

$$c_s = \sqrt{\left[\frac{(f+2)T}{f}\right]},$$

where T is the temperature and f the total number of degrees of freedom of the molecules. One must distinguish this quantity, which is usually called the "equilibrium sound" velocity, from the "frozen-in sound" velocity

$$c_\infty = \sqrt{\tfrac{5}{3}T},$$

which corresponds to excluding the internal degrees of freedom of the molecule.

We emphasize that in the evolutionarity conditions the equilibrium sound velocity enters (which is at most equal to the frozen-in sound velocity).

3.3.2. STRUCTURE OF THE SHOCK WAVES

We tacitly assumed in our study of the discontinuities in magneto-hydrodynamics that the magneto-hydrodynamical medium is ideal. This assumption is a matter of principle, as in a real medium the flow is always continuous and can only approximately be considered to be discontinuous. Such a difference between ideal and real media is connected with the fact that they are described by essentially different differential equations. Indeed, the equations of magneto-hydrodynamics of an ideal medium are obtained by dropping terms describing dissipative effects, that is, viscosity, thermal conductivity, and Joule heating. As these terms contain higher derivatives with respect to the coordinates, the order of the differential equations is lowered when they are dropped. This leads to a decrease in the number of integration constants and as a result it becomes impossible to satisfy the boundary conditions. In order to satisfy them we must consider discontinuous solutions (Friedrichs, 1955).

Let us elucidate this by an example. Let us try to find the solution of the differential equation

$$\varepsilon\frac{d^2w}{dz^2}+w\frac{dw}{dz} = 0, \quad \varepsilon > 0, \tag{3.3.2.1}$$

satisfying the boundary conditions

$$w(-\infty) = -1, \quad w(+\infty) = 1. \tag{3.3.2.2}$$

§ We refer to a paper by Lur'e (1964) for a discussion of errors connected with the neglect of the "sound" velocity dispersion in magneto-hydrodynamics.

The solution of this problem is

$$w = \tanh \frac{z}{2\varepsilon}. \qquad (3.3.2.3)$$

On the other hand, it is natural to drop the first term in eqn. (3.3.2.1) when ε is small and to start from the simplified equation

$$w \frac{dw}{dz} = 0, \quad \varepsilon \to 0. \qquad (3.3.2.4)$$

The solution of the simplified eqn. (3.3.2.4)—corresponding to the equation of the ideal medium—has the form

$$w = C = \text{constant}, \qquad (3.3.2.5)$$

and, of course, cannot satisfy the two boundary conditions (3.3.2.2). However, we see at once that the solution (3.3.2.5) satisfies eqn. (3.3.2.1) when $|z| \gg \varepsilon$. Indeed, taking the limit as $\varepsilon \to 0$ in the exact solution (3.3.2.1) we get

$$w = \begin{cases} 1, & \text{when} \quad z > 0, \\ -1, & \text{when} \quad z < 0. \end{cases} \qquad (3.3.2.6)$$

This solution corresponds in the region $z > 0$ to the solution (3.3.2.5) with $C = 1$, and in the region $z < 0$ to the solution (3.3.2.5) with $C = -1$. The continuous solution (3.3.2.3) thus tends to the discontinuous solution (3.3.2.6) as $\varepsilon \to 0$.

A similar situation occurs in ordinary and magneto-hydrodynamics when we change from the equations for a real medium, which take into account the effects connected with energy dissipation, to the equations for an ideal medium which neglect these effects: we get then from the continuous solutions of the differential equations for a real medium the discontinuous solutions of the differential equations for an ideal medium. It is important that the continuous solutions are everywhere practically the same as the discontinuous solutions, except for a narrow layer close to the discontinuity surface.

We see thus that one can talk about the problem of the *structure of the discontinuities* but that one must, when solving this problem, clearly take dissipative effects into account.

Let us determine the structure of an established magneto-hydrodynamical shock wave of small intensity. To do this we must turn to the set (2.1.4.2) of differential equations. However, in order not to obscure the basic idea, we shall consider a mathematical model, based upon using a single differential equation

$$\frac{\partial a}{\partial t} + Z(a) \frac{\partial a}{\partial z} = \mathcal{D} \frac{\partial^2 a}{\partial z^2} \qquad (3.3.2.7)$$

for a single magneto-hydrodynamical variable a.†

We first of all elucidate the meaning of the coefficients $Z(a)$ and $\mathcal{D}$. Considering the propagation of small perturbations of the form $a - a_0 = Ae^{i(kz-\omega t)}$ one can easily verify (see

† Polovin (1965a) has considered the problem of how to find the shock-wave structure for the case of a set of equations.

Subsection 2.1.1) that Z is the same as the phase velocity $V = \omega/k$ of small perturbations:

$$Z(a) = V(a).$$

The quantity $\mathcal{D}$ describes clearly the energy dissipation and as due to this dissipation the amplitude of the oscillations decreases, $\mathcal{D}$ must be positive, $\mathcal{D} > 0$.

Let us now consider a shock wave moving relative to the medium along the z-axis with a velocity U. If the shock wave is established, that is, if its profile does not change with time, the quantity a will not depend separately on z and t, but only in the combination

$$Ut - z \equiv \xi$$

so that

$$\frac{\partial a}{\partial t} = U\frac{da}{d\xi}, \quad \frac{\partial a}{\partial z} = -\frac{da}{d\xi} \tag{3.3.2.8}$$

the region of space in front of the shock wave corresponds to $\xi = -\infty$ and the region behind the wave to $\xi = +\infty$). Substitution of expressions (3.3.2.8) into (3.3.2.7) gives

$$U\frac{da}{d\xi} - V(a)\frac{da}{d\xi} = \mathcal{D}\frac{d^2a}{d\xi^2}.$$

Integrating this equation, we find

$$U\eta - \int_0^\eta V(\eta)\,d\eta = \mathcal{D}\frac{d\eta}{d\xi}, \tag{3.3.2.9}$$

where $\eta = a - a_1$ while a_1 is the value of the quantity a in front of the shock wave—that is, in the unperturbed region—where $da/d\xi$ vanishes.

As we are considering small intensity shock waves, we can expand the function $V(\eta)$ in a power series in η and restict ourselves to the first two terms of the expansion.

$$V(\eta) = V(0) \pm \eta V'(0).$$

Substituting this expression into eqn. (3.3.2.9) and integrating it, we find

$$\eta = \frac{1}{2}\eta_0\left[1 + \tanh\frac{\xi}{L}\right],$$

where

$$L = \frac{2\mathcal{D}}{U - V(0)}, \quad \eta_0 = \frac{2[U - V(0)]}{V'(0)}$$

(we assume that the shock wave corresponds to values of η).

The solution considered by us has clearly a meaning, provided

$$U > V(0).$$

This condition means that the shock wave moves relative to the medium with a velocity which is larger than the propagation velocity of the perturbations. The difference $U - V(0)$ can then serve as a measure for the shock-wave strength.

Equation (3.3.2.10) shows that the profile of a small intensity shock wave has the form of a hyperbolic tangent (Polovin, 1965a). Such a profile is characteristic for shock waves in ordinary hydrodynamics (Landau and Lifshitz, 1959) and also in magneto-hydrodynamics [both in the non-relativistic (Sirotina and Syrovatskiĭ, 1961) and in the relativistic case (Deutsch, 1963)] and also in two-component hydrodynamics (Hu, 1966).

We note that according to the law (3.3.2.10) not all macroscopic quantities, which characterize the shock wave, change, but only those which are not Riemann invariants. As far as the Riemann invariants are concerned, they vary as (Polovin, 1965a)

$$R_i = R_{i0} + \Delta R_i \cosh^{-2}(\xi/L),$$

where R_{i0} and ΔR_i are independent of ξ.

In particular, the entropy is a Riemann invariant and, if in the expansion of $V(\eta)$ we take the two first terms into account it does not change in a shock wave, like all Riemann invariants. Taking the third term into account leads, however—as should be the case—to an increase of the entropy (Landau and Lifshitz, 1959).

The quantity L occurring in eqn. (3.3.2.10) is clearly the width of the shock wave. It follows from (3.3.2.11) that the width of the shock wave is proportional to the quantity $\mathcal{D}$ which determines the energy dissipation, and inversely proportional to the quantity $U - V(0)$ which is a measure of the shock wave intensity.

Using eqn. (2.1.4.3) for the matrix $\mathcal{D}_{ij}$ and noting that we can replace $\mathcal{D}$—as to order of magnitude—by the trace of the matrix $\mathcal{D}_{ij}$, $\mathcal{D} \sim \sum_i \mathcal{D}_{ii}$, we get for L the expression

$$L = \frac{\nu + \nu_m + \dfrac{\varkappa}{\varrho T}\left(\dfrac{\partial T}{\partial s}\right)_\varrho}{U - V(0)},$$

where ν is the viscosity coefficient, $\varkappa$ the coefficient of thermal conductivity, and ν_m the coefficient of magnetic viscosity. As (Landau and Lifshitz, 1959)

$$\nu \sim l v_i, \quad \frac{\varkappa}{\varrho T}\left(\frac{\partial T}{\partial s}\right)_\varrho \sim l v_e, \quad \nu_m = \frac{c^2}{4\pi\sigma},$$

where l is the mean free path of the plasma particles which, according to (1.4.1.4), is independent of their kind, v_i and v_e the ion and electron thermal velocities, and $\sigma \sim e^2 nl/mv_e$ is the plasma electrical conductivity, we have

$$L \sim l' \frac{v_i}{U - V(0)}, \tag{3.3.2.12}$$

where (m_i being the ion mass)

$$l' = l + \frac{d^2}{l}, \quad d^2 = \frac{m_i c^2}{4\pi n e^2}.$$

For a small intensity shock wave $U - V(0) \ll v_i$ and hence

$$L \gg l',$$

that is, the width of the small intensity shock wave is appreciably larger than the particle mean free path l.

We note that in the case of a rarefied high-temperature plasma when the mean free path is very large

$$l' \sim l.$$

On the other hand, for dense low-temperature magneto-hydrodynamic media with particles which undergo frequent collisions (Marshall, 1955; Sen, 1956)

$$l' \sim d^2/l.$$

So far we have considered a small intensity shock wave, corresponding to a power series expansion in η of the function $V(\eta)$ in eqn. (3.3.2.9) and retention in that expansion of the first two terms. One can try to study a large intensity shock wave by not expanding the function $V(\eta)$ in a power series in η. Without entering into a detailed discussion of such a study we shall merely present its main results.

First of all, if we take higher-order terms in the expansion of $V(\eta)$ into account, the profile of the shock wave will no longer be described by a hyperbolic tangent and will be of a more complicated type. Furthermore, it turns out that macroscopic quantities can change non-monotonically in the shock wave (even when there are no external fields; see Subsection 3.3.3). In particular, in a sufficiently strong shock wave the temperature always has a maximum (Zel'dovich and Raĭzer, 1966).

While in a small intensity wave different dissipative processes affect the structure of the shock wave to the same extent, the situation is completely changed in the case of a large intensity shock wave. For instance, if we take only the viscosity into account, we get a continuous structure of the shock wave. If, however, we take only the thermal conductivity into account and neglect the viscosity, there is no continuous structure of the wave in the case of a large intensity shock wave (Landau and Lifshitz, 1959; see also Zel'dovich, 1957; Shafranov, 1957; Cohen and Clarke, 1965; Kulikovskiĭ and Lyubimov, 1962; Whitham, 1959).

We note that the existence of a continuous structure of a shock wave is closely connected with its evolutionarity. In fact, if a shock wave is evolutionary, there is a unique continuous structure of the shock wave for any non-vanishing values of all dissipative coefficients. On the other hand, if the shock wave is not evolutionary, the hydrodynamical equations, taking the dissipative processes into account, do not lead to a unique shock-wave structure (Germain, 1960a; Kulikovskiĭ and Lyubimov, 1961, 1962; Lyubarskiĭ, 1962a).

All these results which refer to large intensity shock waves were obtained under the assumption that the Navier–Stokes equation—or other differential equations similar to it—correctly describe the shock wave structure, however large the intensity (Kulikovskiĭ and Lyubimov, 1962). However, when the intensity of a shock wave increases, its width decreases and when $L \sim l$ the condition for the applicability of the hydrodynamic description is violated. The use of the Navier–Stokes is thus valid only when $L \gg l$. If, on the other hand, the shock wave intensity is so large that L becomes of the order of l, the Navier–Stokes equation can give at best only qualitatively correct results. It is in that case necessary to use instead of the hydrodynamic the kinetic description which is based on the use of the integral kinetic equations rather than of differential equations.

The statement of the problem of the shock-wave structure in kinetic theory consists of the following. We give the kinetic equations for the particle distribution functions $F_\alpha \equiv F_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$:

$$\frac{\partial F_\alpha}{\partial t} + (\boldsymbol{v}\cdot\nabla)F_\alpha + \frac{e_\alpha}{m_\alpha}\left(\left\{\boldsymbol{E} + \frac{1}{c}\,[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial F_\alpha}{\partial \boldsymbol{v}}\right) = \mathcal{J}_\alpha\{F_\beta\},$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the self-consistent electrical and magnetic fields and $\mathcal{J}_\alpha$ the particle collision integrals (the index α numbers the particles of different kinds). The moments of the distribution functions describe different macroscopic quantities; for instance, the density ϱ and the macroscopic velocity $\boldsymbol{u}$ are determined by the equations

$$\varrho = \sum_\alpha m_\alpha \int F_\alpha\, d^3v, \qquad \boldsymbol{u} = \frac{1}{\varrho}\sum_\alpha m_\alpha \int \boldsymbol{v} F_\alpha\, d^3v.$$

We need to find those stationary—that is, time-independent—solutions of the kinetic equations which would lead to the given jumps in the macroscopic quantities

$$\Delta\varrho \equiv \varrho\Big|_{z=+\infty} - \varrho\Big|_{z=-\infty}, \qquad \Delta\boldsymbol{u} \equiv \boldsymbol{u}\Big|_{z=+\infty} - \boldsymbol{u}\Big|_{z=-\infty},$$

where the z-axis is along the direction of propagation of the shock wave. Strictly speaking, these jumps must be connected with one another through the boundary conditions established in Subsection 3.2.1.

In this general form this problem is too complex and its solution is not yet known. There are some simplifications if we use instead of the exact expression for the collision integral an expression of the form (Bhatnagar, Gross and Krook, 1954)

$$\mathcal{J}\{F\} = -\frac{F - F_0}{\tau},$$

where τ is a velocity-independent constant (the average relaxation time), F_0 the local equilibrium (Maxwell) distribution function,

$$F_0 = \left[\frac{m}{2\pi T(\boldsymbol{r})}\right]^{3/2} n(\boldsymbol{r}) \exp\left\{-\frac{m[\boldsymbol{v} - \boldsymbol{u}(\boldsymbol{r})]^2}{2T(\boldsymbol{r})}\right\}$$

and n, $\boldsymbol{u}$, and T the local density, velocity and temperature. However, even with such a simplified model of the collision integral the problem still remains very complex so that one can only obtain its solution for the case of a small intensity shock wave when there are no electromagnetic fields (Lyubarskiĭ, 1962b).

A peculiarity of the solution obtained (Lyubarskiĭ, 1961; Barantsev, 1962) is that far from the shock-wave front the macroscopic quantities vary as $\exp(-|z|/L)^{2/3}$, whereas this variation behaves as $\exp(-|z|/L)$ in the hydrodynamic theory.

Another approach to studying shock-wave structure in the framework of the kinetic theory consists in postulating the shape of the distribution function (Mott-Smith, 1951; Tidman, 1958; Greenberg, Sen and Treve, 1960; Comisar, 1962; Krook, 1959; Muckenfuss, 1960; Gustafson, 1960); in fact, one assumes that the distribution function has the form of a superposition of two local equilibrium Maxwell distributions,

$$F = \left[\frac{m}{2\pi T_1(\boldsymbol{r})}\right]^{3/2} n_1(\boldsymbol{r}) \exp\left\{-\frac{m[\boldsymbol{u}-\boldsymbol{v}_1(\boldsymbol{r})]^2}{2T_1(\boldsymbol{r})}\right\} + \left[\frac{m}{2\pi T_2(\boldsymbol{r})}\right]^{3/2} n_2(\boldsymbol{r}) \exp\left\{-\frac{m[\boldsymbol{u}-\boldsymbol{v}_2(\boldsymbol{r})]^2}{2T_2(\boldsymbol{r})}\right\},$$

where n_1, n_2, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, T_1 and T_2 are functions of the coordinates; using the kinetic equations —which, strictly speaking, are not satisfied by these functions—one finds the moments of the distribution function, that is, relations between the functions n_1, n_2, $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, T_1 and T_2. We shall, however, not consider this approach here in detail.

3.3.3. OSCILLATORY STRUCTURE OF A SHOCK WAVE WHEN THERE IS A MAGNETIC FIELD PRESENT

To explain how qualitatively new features must arise in the shock-wave structure when its intensity increases we consider the simplest case when the thermal spread in the particle velocities is very small and when there are two groups of particles—ions and electrons—moving with different velocities. In that case the ion and electron velocities $\boldsymbol{u}_i$ and $\boldsymbol{u}_e$ satisfy the equations of two-component hydrodynamics:

$$\frac{\partial n_\alpha}{\partial t} + \operatorname{div}(n_\alpha \boldsymbol{u}_\alpha) = 0,$$

$$m_\alpha n_\alpha \frac{\partial \boldsymbol{u}_\alpha}{\partial t} + m_\alpha n_\alpha(\boldsymbol{u}_\alpha \cdot \nabla)\boldsymbol{u}_\alpha = e_\alpha n_\alpha\left(\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}_\alpha \wedge \boldsymbol{B}]\right) + \boldsymbol{R}_\alpha, \tag{3.3.3.1}$$

where n_α is the density of the particles of kind $\alpha(\alpha \equiv \mathrm{e, i})$, $\boldsymbol{R}_\alpha$ the friction force felt by particles of kind α from the particles of the other kind,

$$\boldsymbol{R}_e = -\boldsymbol{R}_i = -m_e n_e \nu \boldsymbol{u}_e$$

(ν is the friction coefficient which is connected with the electrical conductivity through the relation $\sigma = e^2 n_e / m_e \nu$), and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electrical and magnetic fields which satisfy the Maxwell equation

$$\operatorname{curl} \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}. \tag{3.3.3.2}$$

(In eqns. (3.3.3.1) there are no hydrodynamic pressure or viscosity forces by virtue of the assumption that the thermal spread in particle velocities is small.)

Let us now use these equations to consider a shock wave propagating in the direction of the negative z-axis with a velocity u_1. Let the state of the plasma in front of the wave front ($z = -\infty$) be characterized by the following values of the variables:

$$n_e = n_i \equiv n_1, \quad \boldsymbol{u}_e = \boldsymbol{u}_i = 0, \quad B_x = B_1, \quad B_y = B_z = 0, \quad \boldsymbol{E} = 0.$$

It then follows from eqns. (3.3.3.1) and (3.3.3.2) that the quantities E_x, E_z, B_y, B_z, u_{ex} and u_{ix} will vanish not only in front of the shock wave, but also over the whole of its extent ($-\infty < z < +\infty$):

$$E_x = E_z = B_y = B_z = u_{ex} = u_{ix} = 0.$$

To find the state of the plasma in the shock wave we change to a frame of reference which moves with the velocity u_1 of the wave front. In that frame the state of the plasma is time-independent and, moreover, all quantities are independent of x and y. It is clear that there will exist in that frame a constant electrical field

$$E_y \equiv E = -u_1 B_1/c, \quad E_x = E_z = 0. \tag{3.3.3.3}$$

We shall assume that everywhere the quasi-neutrality condition,

$$n_e = n_i \equiv n,$$

is satisfied and we neglect the transverse motion of the ions. Under these assumption the following equation clearly holds:

$$u_{ez} = u_{iz} \equiv u_z .$$

It follows from the Maxwell eqn. (3.3.3.2) that

$$j_y = \frac{c}{4\pi} \frac{dB_x}{dz},$$

and as $j_y = -enu_{ey}$ ($|u_{iy}| \ll |u_{ey}|$) we have

$$u_{ey} = -\frac{c}{4\pi ne} \frac{dB_x}{dz}. \tag{3.3.3.4}$$

Combining eqns. (3.3.3.1) for the electron and ion components, we get

$$m_i n u_z \frac{du_z}{dz} = -\frac{d}{dz} \frac{B_x^2}{8\pi},$$

whence

$$u_z = \frac{B_1^2 - B_x^2}{8\pi m_i n_i u_1} + u_1 . \tag{3.3.3.5}$$

Taking now the y-component of the second eqn. (3.3.3.1) for electrons,

$$m_e n u_z \frac{du_{ey}}{dz} = -enE_y - \frac{en}{c} u_z B_x - m_e n \nu u_{ey},$$

and using eqns. (3.3.3.3) to (3.3.3.5) we get the following equation for $B_x \equiv B$ (Sagdeev, 1962):

$$\frac{d^2B}{d\tau^2} + \nu \frac{dB}{d\tau} + \Phi(B) = 0, \tag{3.3.3.6}$$

where $d\tau = dz/u_z$ and

$$\Phi(B) = \frac{\omega_{Be} u_1^2}{c^2} \left[\frac{B(B^2 - B_1^2)}{8\pi m_i n_i u_1^2} - (B - B_1) \right],$$

where ω_{Be} is the electron Langmuir frequency. This non-linear equation determines the magnetic field in the shock wave.

We shall first of all determine the magnetic field behind the shock wave, that is, for $\tau = +\infty$ (the wave propagates in the direction of the negative z-axis). In that region $dB/d\tau = d^2B/d\tau^2 = 0$ and thus

$$\Phi(B) = 0. \tag{3.3.3.7}$$

This equation has three solutions:

$$B^{(1)} = B_1, \quad B^{(2,3)} = -\tfrac{1}{2}B_1 \pm \sqrt{[\tfrac{1}{4}B_1^2 + 8\pi m_i n_i u_1^2]}. \tag{3.3.3.8}$$

The solution $B^{(1)}$ clearly does not correspond to a shock wave which is characterized by a difference between the initial and final constants,

$$B\Big|_{z=-\infty} \neq B\Big|_{z=+\infty},$$

but a *solitary wave* (or *soliton*) for which the final state is the same as the initial one:

$$B\Big|_{z=-\infty} = B\Big|_{z=+\infty}.$$

Of the remaining two solutions $B^{(2)}$ and $B^{(3)}$ the solution with the plus sign in front of the radical

$$B^{(2)} = -\tfrac{1}{2}B_1 + \sqrt{[\tfrac{1}{4}B_1^2 + 8\pi m_i n_i u_1^2]} \equiv B_2.$$

corresponds to a shock wave. However, $B^{(3)}$ does not have a meaning as we shall show in Subsection 3.4.1 that the transverse magnetic field cannot change its sign when going through a shock-wave front. Assuming that $B_1 > 0$, we must thus take in eqn. (3.3.3.8) the square root with the plus sign.

Let us now elucidate how the magnetic field behaves as $\tau = \pm\infty$. Assuming that at $\tau = -\infty$

$$B = B_1 + b_1, \quad |b_1| \ll B_1,$$

and linearizing eqn. (3.3.3.6) with respect to b_1, we get

$$\frac{d^2b_1}{d\tau^2} + \nu\frac{db_1}{d\tau} + \omega_h^2(M_1^2 - 1)\,b_1 = 0, \tag{3.3.3.9}$$

where ω_h is the hybrid frequency,

$$\omega_h = \frac{eB_1}{c\sqrt{(m_e m_i)}},$$

and M_1 the Mach number in front of the wave ($\tau = -\infty$),

$$M_1 = \frac{u_1\sqrt{(4\pi m_i n_i)}}{B_1}.$$

The solution of eqn. (3.3.3.9) which vanishes as $\tau \to -\infty$ has the form

$$b_1 = C_1 e^{\mu_1 \tau}, \quad \text{Re}\,\mu_1 > 0,$$

where C_1 is a constant and

$$\mu_1 = -\tfrac{1}{2}\nu + \sqrt{[\tfrac{1}{4}\nu^2 + (M_1^2 - 1)\omega_h^2]}.$$

As $\tau \to -\infty$, the magnetic field thus approaches B_1 monotonically.

Let us now consider how the magnetic field behaves behind the shock wave ($\tau = +\infty$). Putting

$$B = B_2 + b_2, \quad |b_2| \ll B_2,$$

and again linearizing eqn. (3.3.3.6), we get

$$\frac{d^2 b_2}{d\tau^2} + \nu \frac{db_2}{d\tau} + \frac{\omega_{Be} u_1^2}{c^2}\left[\frac{3B_2^2 - B_1^2}{8\pi m_i n_1 u_1^2} - 1\right] b_2 = 0,$$

and hence

$$b_2 = C_2 e^{\mu_2 \tau}, \quad \mathrm{Re}\, \mu_2 < 0,$$

where C_2 is a constant and

$$\mu_2 = -\frac{1}{2}\nu \pm \sqrt{\left[\frac{1}{4}\nu^2 - \frac{\omega_{Be} u_1^2}{c^2}\left(\frac{3B_2^2 - B_1^2}{8\pi m_i n_1 u_1^2} - 1\right)\right]}.$$

We see that for sufficiently small ν,

$$\nu < \frac{2\omega_{Be} u_1}{c} \sqrt{\left[\frac{3B_2^2 - B_1^2}{8\pi m_i n_1 u_1^2} - 1\right]}, \tag{3.3.3.10}$$

the quantity μ_2 will be complex, that is, the shock wave will have an oscillatory structure.

One sees easily that the spatial period of the oscillation is equal to

$$\lambda = \frac{2\pi u_1}{|\mathrm{Im}\, \mu_2|} = \frac{2\pi u_1}{\sqrt{\left[\frac{\omega_{Be} u_1^2}{c^2}\left(\frac{3B_2^2 - B_1^2}{8\pi m_i n_1 u_1^2} - 1\right) - \frac{1}{4}\nu^2\right]}},$$

while the width of the shock wave, that is, the distance over which the oscillation gets damped, is equal to

$$L \equiv \frac{2\pi u_1}{|\mathrm{Re}\, \mu_2|} = \frac{4\pi u_1}{\nu}.$$

For large magnetic fields and $M_1 \sim 1$ the period of the oscillation is of the order of magnitude of

$$\lambda \sim c/\Omega.$$

If the damping ν is sufficiently large,

$$\nu > \frac{2\omega_{Be} u_1}{c} \sqrt{\left[\frac{3B_2^2 - B_1^2}{8\pi m_i n_1 u_1^2} - 1\right]},$$

the quantity μ_2 will be real and hence the magnetic field changes monotonically from B_1 to B_2. In that case the width of the shock wave is equal to

$$L' \equiv \frac{2\pi u_1}{|\mathrm{Re}\, \mu_2|} = \frac{2\pi u_1}{\frac{1}{2}\nu - \sqrt{\left[\frac{1}{4}\nu^2 - \frac{\Omega_{Be} u_1^2}{c^2}\left(\frac{3B_2^2 - B_1^2}{8\pi m_i n_1 u_1^2} - 1\right)\right]}}.$$

This quantity is clearly larger than the width of the shock wave in the previous ("oscillatory") case.

We draw attention to the fact that in the case of a small intensity shock wave the quantity B_2 is close to B_1 and u_1 close to $B_1/\sqrt{(4\pi m_i n_i)}$; therefore the right-hand side of the inequality (3.3.3.10) tends to zero so that that inequality cannot be satisfied. Small intensity shock waves can therefore not have an oscillatory structure. One sees easily that in that case the profile of a shock wave has the form of a hyperbolic tangent (see eqn. (3.3.2.10)).

Equation (3.3.3.6) has the form of the equation of a one-dimensional anharmonic oscillator with potential energy

$$V(B) = \int_{B_1}^{B} \Phi(\xi)\, d\xi = \frac{1}{2}(B-B_1)^2\left[\frac{(B+B_1)^2}{16\pi m_i n_1 u_1^2}-1\right].$$

The role of the coordinate is here played by the magnetic field; the mass of the oscillator is equal to unity. We show in Fig. 3.3.4 the potential energy V as function of B. If there were no "friction ($\nu = 0$) the oscillator would move between the points A and B. The point A corresponds to the initial state of the oscillator for which $dB/d\tau = d^2B/d\tau^2 = 0$, $B = B_1$.

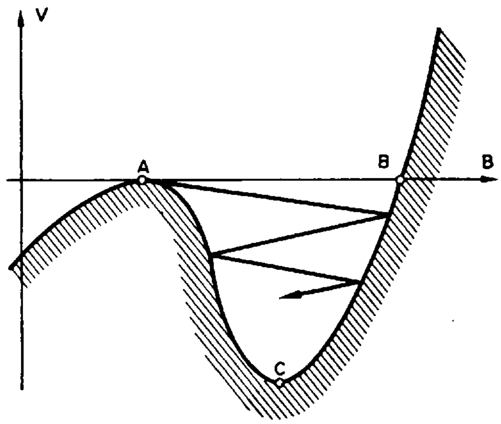

FIG. 3.3.4. Oscillator potential energy $V(B)$. A is the state in front of the shock wave, B the state at the soliton vertex. The line with the arrow shows the direction of the change in the magneto-hydrodynamic quantities in the case of an oscillatory structure.

Because of friction the point depicting the state of the oscillator will show zigzag motion and approach the equilibrium point C. The point C corresponds to the final state of the oscillator for which $dB/d\tau = d^2B/d\tau^2 = 0$, $B = B_2$. We draw attention to the fact that if $\nu \neq 0$, but $\nu \to 0$, the final state of the oscillator will, as before, correspond to the position C and not to A.

The picture of the oscillator motion becomes clearer if we change to the B, $\dot{B}$-phase plane ($\dot{B} = dB/d\tau$, see Fig. 3.3.5). One sees easily that the point A is a saddle point and the point C is either a centre (when $\nu = 0$) or a stable focus (when $\nu \neq 0$). The closed line ABA is a separatrix inside which are included the trajectories corresponding to a finite motion and outside ABA there are the trajectories corresponding to infinite motion (this region is shaded in Fig. 3.3.5).

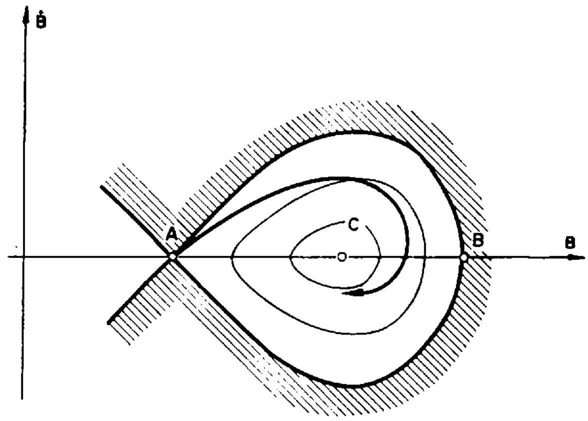

FIG. 3.3.5. Phase trajectories corresponding to the variation of the magnetic field in a shock wave. The inaccessible region is shaded. The letters A, B, and C have the same meaning as in Fig. 3.3.4.

Multiplying eqn. (3.3.3.6) by $dB/d\tau$ we get the energy-conservation law:

$$\frac{d}{d\tau}\left[\frac{1}{2}\dot{B}^2+V(B)\right] = -\nu\dot{B} < 0. \tag{3.3.3.11}$$

When $\nu = 0$ the point depicting the magnetic field in the shock wave moves along the separatrix

$$\tfrac{1}{2}\dot{B}^2+V(B) = \text{constant}$$

(the line ABA in Fig. 3.3.5). If, however, $\nu \neq 0$, the phase trajectories starting from the point A will finish in the point C.

3.3.4. CASES OF DEGENERACY

Let us now consider in somewhat more detail some important degenerate cases of magneto-hydrodynamical discontinuities.

We shall consider first of all the parallel shock wave for which the boundary conditions have the same form as when there is no magnetic field. The presence of a magnetic field changes only the condition for evolutionarity which leads in a number of cases to a splitting of this "hydrodynamic" shock wave. We shall then consider magneto-hydrodynamical peculiar shock waves, the existence of which was put in doubt as they are non-evolutionary in the linear theory (Syrovatskiĭ, 1959; Kemp, Germain, and Grad, 1960; Sarason, 1965), and, finally we shall consider Alfvén discontinuities for which the existence problem also arises in connection with the fact that these discontinuities do not have a stationary structure (Sirotina and Syrovatskiĭ, 1961; Shercliff, 1960).

We start with the consideration of a parallel magneto-hydrodynamical shock wave. If in front of such a wave the Alfvén velocity is less than the sound velocity,

$$v_{1Az} < c_{1s},$$

the velocities of the slow and of the fast magneto-sound waves are equal to the Alfvén velocity and the sound velocity, respectively,

$$v_{1-} = v_{1Az}, \quad v_{1+} = c_{1s}. \tag{3.3.4.1}$$

As the density and the temperature increase in a shock wave and, moreover, the Alfvén velocity is a decreasing function of the density while the sound velocity is an increasing function of the temperature, the Alfvén velocity will remain less than the sound velocity behind the shock wave,

$$v_{2Az} < c_{2s};$$

in that case behind the shock wave conditions similar to (3.3.4.1) are satisfied:

$$v_{2-} = v_{2Az}, \quad v_{2+} = c_{2s}. \tag{3.3.4.2}$$

The first of these relations means that the conditions for evolutionarity of a slow parallel shock wave cannot be satisfied when $v_{1Az} < c_{1s}$. As far as the conditions for evolutionarity of a fast parallel shock wave are concerned, they take the form, according to (3.3.4.1) and (3.3.4.2),

$$c_{1s} < u_{1z}, \quad v_{2Az} < u_{2z} < c_{2s}. \tag{3.3.4.3}$$

We show in Fig. 3.3.6 the Hugoniot line $u_{2z} \equiv u_{2z}(u_{1z})$ and also the graphs of the functions $v_{2Az}(u_{1z})$ and $c_{2s}(u_{1z})$ for a parallel magneto-hydrodynamic shock wave for the case

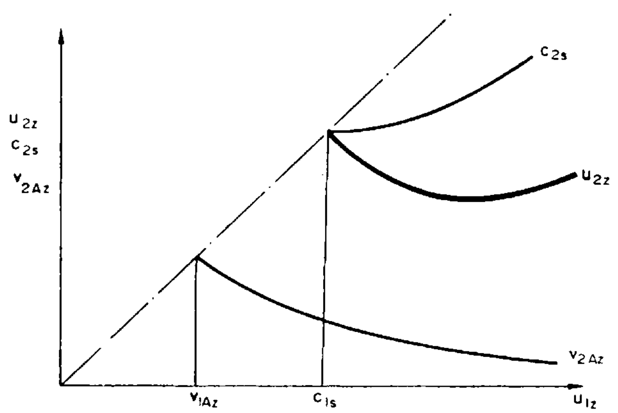

FIG. 3.3.6. Parallel shock wave for $v_{1Az} < c_{1s}$. Along the abscissa axis we have u_{1z}. The thick line gives the dependence of u_{2z} on u_{1z} and the thin lines the dependence of c_{2s} and v_{2Az} on u_{1z}. There is no region of non-evolutionarity.

$v_{1Az} < c_{1s}$ (we assume that the magneto-hydrodynamic medium is described by the equation of state of an ideal gas). It is clear from Fig. 3.3.6 that the conditions for evolutionarity (3.3.4.3) for a fast parallel magneto-hydrodynamica shock wave are satisfied for any wave intensity.

The picture is more complicated for the case of a parallel shock wave if in front of the wave the Alfvén velocity is larger than the sound velocity,

$$v_{1Az} > c_{1s}.$$

As on the shock wave the Alfvén velocity v_{Az} decreases, while the sound velocity increases, for small intensities we shall have the following inequality behind the shock wave:

$$v_{2Az} > c_{2s},$$

while for large intensities the opposite inequality holds:

$$v_{2Az} < c_{2s}.$$

In other words, the graphs of the functions $v_{2Az}(u_{1z})$ and $c_{2s}(u_{1z})$ intersect (see Fig. 3.3.7).

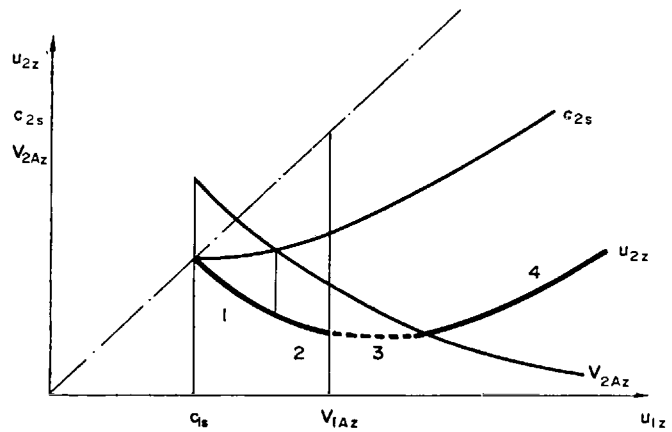

FIG. 3.3.7. The region of the evolutionarity of a parallel shock wave when $v_{1Az} > c_{1s}$. Along the abscissa axis we have u_{1z}. The thick line gives the dependence of u_{2z} on u_{1z}, the thin lines that of c_{2s} and v_{2Az} on u_{1z} (the region of evolutionarity is indicated by a thick line, that of non-evolutionarity by a dotted line).

Let us study this problem in more detail. In the case of a parallel shock wave the boundary conditions (3.2.1.2), (3.2.1.3), (3.2.1.5), (3.2.1.7), and (3.2.1.8) take the same form as in ordinary hydrodynamics:

$$\begin{aligned} \varrho_1 u_{1z} &= \varrho_2 u_{2z}, \\ p_1 + \varrho_1 u_{1z}^2 &= p_2 + \varrho_2 u_{2z}^2, \\ \frac{1}{2} u_{1z}^2 + \frac{5}{2} \frac{p_1}{\varrho_1} &= \frac{1}{2} u_{2z}^2 + \frac{5}{2} \frac{p_2}{\varrho_2} \end{aligned} \tag{3.3.4.4}$$

(we use the equation of state of a perfect gas with adiabatic index $\gamma = \frac{5}{3}$ and we assume therefore the enthalpy to be equal to $w = \frac{5}{2} p/\varrho$). From eqns. (3.3.4.4) we find

$$\begin{aligned} \varrho_2 &= \frac{\varrho_1 u_{1z}}{u_{2z}}, \\ p_2 &= p_1 + \varrho_1 u_{1z}(u_{1z} - u_{2z}), \\ u_{2z} &= \frac{u_{1z}^2 + 3c_{1s}^2}{4u_{1z}}, \end{aligned} \tag{3.3.4.5}$$

where $c_{1s} = \sqrt{(5p_1/3\varrho_1)}$ is the sound velocity. The last eqn. (3.3.4.5) also gives the Hugoniot line.

To determine the regions of evolutionarity of the shock wave we must still find v_{2Az} and c_{2s} as functions of u_{1z}. For these we find

$$\begin{aligned} v_{2Az} &= \frac{\sqrt{(u_{1z}^2 + 3c_{1s}^2)}}{2u_{1z}}, \\ c_{2s} &= \frac{[(5u_{1z}^2 - c_{1s}^2)(u_{1z}^2 + 3c_{1s}^2)]^{1/2}}{4u_{1z}}. \end{aligned} \tag{3.3.4.6}$$

In Fig. 3.3.7 we have depicted the functions $u_{2z}(u_{1z})$ and (3.3.4.6). The numbers 1, 2, 3, 4 in Fig. 3.3.7 indicate the following sections:

1. $c_{1s} < u_{1z} < v_{1Az}, \quad u_{2z} < c_{2s} < v_{2Az};$
2. $c_{1s} < u_{1z} < v_{1Az}, \quad u_{2z} < v_{2Az} < c_{2s};$
3. $c_{1s} < v_{1Az} < u_{1z}, \quad u_{2z} < v_{2Az} < c_{2s};$
4. $c_{1s} < v_{1Az} < u_{1z}, \quad v_{2Az} < u_{2z} < c_{2s}.$

Repeating the discussion given in Subsection 3.3.1 one can show that the sections 1, 2, and 4 are evolutionary, but that section 3 is non-evolutionary.

It follows from eqns. (3.3.4.5) and (3.3.4.6) that the abscissa of the point where the lines $u_{2z} \equiv u_{2z}(u_{1z})$ and $v_{2Az} \equiv v_{2Az}(u_{1z})$ intersect is equal to $u_{1z} = \sqrt{(4v_{1Az}^2 - 3c_{1s}^2)}$. The parallel shock wave will thus be non-evolutionary when the following condition is satisfied:

$$v_{1Az} < u_{1z} < \sqrt{(4v_{1Az}^2 - 3c_{1s}^2)}. \tag{3.3.4.7}$$

We now turn to a consideration of peculiar shock waves. As the transverse magnetic field $|\boldsymbol{B}_t|$ increases in fast shock waves, and decreases in slow shock waves, a peculiar shock wave for which $\boldsymbol{B}_{1t} = 0$ (switch-on shock) is a fast wave and a wave for which $\boldsymbol{B}_{2t} = 0$ (switch-off shock) is a slow wave.

To fix the ideas we restrict ourselves to the case of a fast peculiar wave (switch-on shock) for which we showed in Subsection 3.2.2 that

$$u_{2z} - v_{2Az} = 0.$$

As in a fast shock wave

$$u_{1z} - v_{1Az} > 0,$$

from the four Alfvén waves at the two sides of the discontinuity with velocities $u_z \pm v_{Az}$, only one is an outgoing wave which has the phase velocity $u_{2z} + v_{2Az}$. The wave with the zero-phase velocity $u_{2z} - v_{2Az}$ is not an outgoing one (Akhiezer, Lyubarskiĭ, and Polovin, 1959).

According to the conditions for evolutionarity the number of outgoing Alfvén waves must be equal to two so that at first sight it appears that peculiar shock waves are non-evolutionary. On the other hand, peculiar shock waves necessarily occur when one solves the problem of a piston if the magnetic field was initially at right angles to the surface of the piston (see Subsection 3.4.3; Polovin, 1965a). In this connection the problem of the evolutionarity of peculiar shock waves needs a special discussion.

As, generally speaking, it is possible to have transverse components of the magnetic field, we shall consider a peculiar shock wave as the limiting case of a non-peculiar shock wave when the angle between the magnetic field vector and the normal to the discontinuity surface tends to zero. We shall limit ourselves to a study of a fast peculiar shock wave.

If the transverse magnetic field in front of the discontinuity vanishes, $B_{1x} = 0$, either a parallel shock wave (if $B_{2x} = 0$) or a peculiar shock wave (if $B_{2x} \neq 0$) is possible. We shall determine the Hugoniot equation corresponding to this shock wave. The Hugoniot line corresponding to a parallel shock wave is determined in the u_{1z}, u_{2z}-plane by the last eqn. (3.3.4.5). The evolutionary sections corresponding to a parallel shock wave are shown

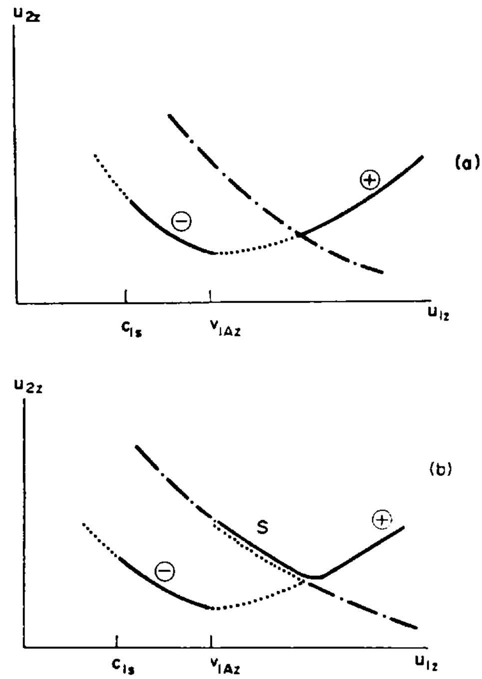

FIG. 3.3.8. The Hugoniot line in the variables u_{1z}, u_{2z} for the case $v_{1Az} > c_{1s}$. (a) Parallel shock wave; (b) parallel and peculiar shock wave. Full-drawn line: evolutionary parts of the Hugoniot line. The plus sign indicates the fast and the minus sign the slow parallel shock wave, and S the fast peculiar shock wave.

by full-drawn lines in Fig. 3.3.8a and the non-evolutionary sections by dotted lines (to fix the ideas we have assumed that $v_{1Az} > c_{1s}$).

We see that the Hugoniot line of a parallel shock wave consists of two parts, corresponding to the slow and the fast wave, which are indicated in Fig. 3.3.8a by a plus and a minus sign, respectively.

We have also indicated in Fig. 3.3.8a the Hugoniot line of the fast peculiar shock wave —by a dash-dot line; its equation follows from eqn. (3.2.1.1) and the first of eqns. (3.2.1.8) and has the form

$$u_{2z} = v_{1Az}^2/u_{1z}. \tag{3.3.4.8}$$

However, the peculiar shock wave does not exist for all values of u_{1z} since for sufficiently large values of u_{1z} the transverse magnetic field B_{2x} behind the peculiar shock wave becomes imaginary. Indeed, from the boundary conditions it follows for the case of an equation of state of a perfect gas with the adiabat index $\gamma = \frac{5}{3}$ that (Lyubarskiĭ and Polovin, 1959b)

$$\frac{3B_x^2}{8\pi} = \frac{\varrho_1(u_{1z}^2 - v_{1Az}^2)(4v_{1Az}^2 - 3c_{1s}^2 - u_{1z}^2)}{v_{1Az}^2}. \tag{3.3.4.9}$$

As the left-hand side of this equation is positive and as by virtue of the evolutionarity conditions we have the inequality

$$u_{1z} > v_{1Az}, \tag{3.3.4.10}$$

the condition that the right-hand side of (3.3.4.9) be positive leads to

$$u_{1z} < \sqrt{(4v_{1Az}^2 - 3c_{1s}^2)}. \qquad (3.3.4.11)$$

The inequalities (3.3.4.10) and (3.3.4.11) bound the finite stretches (see Fig. 3.3.8b) corresponding to the fast peculiar shock wave (full-drawn line indicated by S). In the point $u_{1z} = \sqrt{(4v_{1Az}^2 - 3c_{1s}^2)}$ the fast peculiar shock wave joins up with the fast parallel shock wave.

Let there now be an infinitesimally small transverse magnetic field component B_{1x}. The Hugoniot at eqn. (3.3.4.5) corresponding to the parallel wave is then changed infinitesimally. As far as the Hugoniot eqn. (3.3.4.8) corresponding to the peculiar wave is concerned it will have two infinitesimally different solutions (Polovin and Demutskiĭ, 1960):

$$\begin{aligned} u_{2z}^{(1)} &= \frac{v_{1Az}^2}{u_{1z}} - \frac{v_{1Ax}v_{1Az}}{u_{1z}}\left[\frac{3}{2}\,\frac{u_{1z}^2 - v_{1Az}^2}{4v_{1Az}^2 - 3c_{1s}^2 - u_{1z}^2}\right]^{1/2}, \\ u_{2z}^{(2)} &= \frac{v_{1Az}^2}{u_{1z}} + \frac{v_{1Ax}v_{1Az}}{u_{1z}}\left[\frac{3}{2}\,\frac{u_{1z}^2 - v_{1Az}^2}{4v_{1Az}^2 - 3c_{1s}^2 - u_{1z}^2}\right]^{1/2}. \end{aligned} \qquad (3.3.4.12)$$

The first of these corresponds to a non-evolutionary discontinuity (the line cd in Fig. 3.3.9) and the second to an evolutionary shock wave (the line ef in Fig. 3.3.9).

We see thus that infinitesimally close to the peculiar shock wave there is an evolutionary branch of a non-peculiar shock wave. One must thus consider the peculiar shock wave to be evolutionary.

We can study the problem of the evolutionarity of peculiar shock waves also in a different way, by studying the change in the shock-wave structure with time. Such calculations which

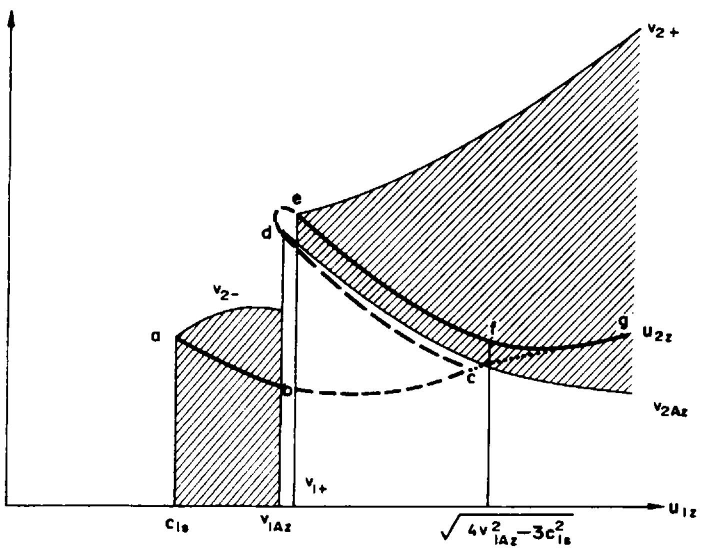

FIG. 3.3.9. Region of evolutionarity of the Hugoniot line when $|B_{1t}| \ll |B_{1z}|$ and $v_{1Az} > c_{1s}$. The region of evolutionarity has been shaded. ab is a shock wave which is nearly a slow parallel shock wave, bc the non-evolutionary part of the parallel shock wave, cd the non-evolutionary part of the Hugoniot line close to the peculiar shock wave, ef the fast shock wave, close to the peculiar wave, and fg the shock wave close to the fast parallel shock wave.

can be performed only by means of electronic computers also lead to the conclusion that the peculiar shock waves are evolutionary (Todd, 1966; Chu, 1967; Chu and Taussig, 1967).

From similar considerations one must conclude that the Alfvén discontinuity in which the magnetic field rotates over 180° and the tangential discontinuity in which the condition

$$\frac{\boldsymbol{B}_{1t}}{\varrho_1} = \frac{\boldsymbol{B}_{2t}}{\varrho_2}$$

is satisfied must also be reckoned among the evolutionary discontinuities (Polovin, 1963a) notwithstanding the fact that both these discontinuities are formally non-evolutionary (Jeffrey and Taniuti, 1964).

We finally turn to the problem of the existence of Alfvén discontinuities. As the propagation velocity of an Alfvén discontinuity is independent of its amplitude (see Subsection 3.2.2), the quantity U in eqn. (3.3.2.11) will be equal to $V(0)$. The denominator in (3.3.2.11) then vanishes which indicates the absence of a stationary structure for Alfvén discontinuities. This follows also from the fact that the kinetic, the internal, and the magnetic energy of the medium on both sides of an Alfvén discontinuity are the same so that there is no energy source to counteract the dissipation connected with the rotation of the velocity vector and of the magnetic force lines.

On the other hand, in the case of some well-defined values of the parameters in the solution of the piston problem (see Subsection 3.4.3) there occurs necessarily an Alfvén discontinuity. The Alfvén discontinuity must thus have a non-stationary structure. One can show (Landau and Lifshitz, 1960; Turcotte and Chu, 1966) that the width of the Alfvén discontinuity L increases with time as follows:

$$L \sim \{(\nu+\nu_m)t\}^{1/2},$$

where ν and ν_m are the hydrodynamic and magnetic viscosities. For small values of ν and ν_m the Alfvén discontinuity may exist for a very long time.

3.3.5. EXOTHERMIC AND ENDOTHERMIC DISCONTINUITIES

So far we have not taken possible energy emission or absorption into account in our considerations of different discontinuities. However, such effects may occur for very different reasons; for instance, because of the occurrence of chemical reactions and condensation processes, recombination (Feldman, 1958) and absorption of radiation (Axford, 1961; Goldsworthy, 1958) energy may be released and energy may be absorbed because of ionization (Baum, Kaplan, and Stanyukovich, 1958; Gross, 1965), dissociation (Zel'dovich and Raĭzer, 1957), and emission.†

We now turn to a study of discontinuities accompanied by the emission or absorption of energy—in the first case we talk about *exothermic* and in the second case of *endothermic* discontinuities.

† We have in mind discontinuities in which the process of the emission of radiant energy is not compensated by the absorption in neighbouring layers of the medium. In the opposite case there is no change in energy and the presence of radiation only influences the form of the equation of state (Sachs, 1946).

We start by considering such discontinuities in ordinary hydrodynamics. In that case the boundary conditions

$$\Delta \varrho u_n = 0, \quad \Delta(p+\varrho u_n^2) = 0, \quad \Delta \boldsymbol{u}_t = 0, \tag{3.3.5.1}$$

which follow from the mass and momentum conservation laws, must be satisfied at the exothermic and endothermic discontinuities, just as at discontinuities without energy emission or absorption; as far as the boundary condition following from the energy conservation law is concerned, this must now be formulated as follows:

$$\Delta(w+\tfrac{1}{2}u^2) = q, \tag{3.3.5.2}$$

where q is the energy per unit mass of the matter liberated at the discontinuity; in that case $q > 0$ for exothermic and $q < 0$ for endothermic discontinuities.

Eliminating from eqns. (3.3.5.1) and (3.3.5.2) the components of the velocities $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, we get the Hugoniot equation for the case of a discontinuity with $q \neq 0$, which connects the pressure p_2 behind the discontinuity with the specific volume $1/\varrho_2$. We show the corresponding Hugoniot lines in Fig. 3.3.10. The point 1 corresponds to the initial state $1/\varrho_1, p_1$, the

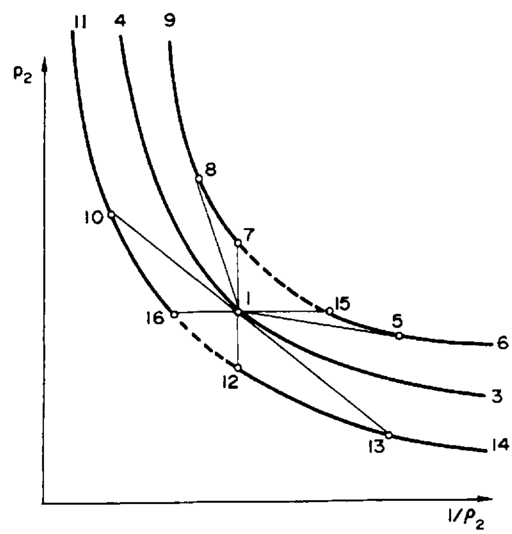

FIG. 3.3.10. Hugoniot lines in the variables $1/\varrho$, p in ordinary hydrodynamics. 1 is the state in front of the shock wave ($1/\varrho_1$, p_1), 3-1-4 the Hugoniot line without change in energy, 6-5-15 and 7-8-9 Hugoniot lines with energy emission, 14-13-12 and 16-10-11 Hugoniot lines with energy absorption. 1-4 a shock wave, 8-9 supercompression detonation, 8 Chapman–Jouguet detonation, 7-8 supersonic combustion, 5-15 subsonic combustion, 10-11 ionization in a shock wave.

line 4–1–3 corresponds to $q = 0$; the sections 9–8–7 and 15–5–6 correspond to $q > 0$, and, finally, the sections 11–10–16 and 12–13–14 correspond to $q < 0$. Along the sections 7–15 and 16–12, bounded by the vertical straight line 7–1–12 and the horizontal straight line 16–1–15, the mass flux density $\varrho_1 u_{1z}$ becomes imaginary so that these sections cannot be realized.

It is convenient when determining the conditions for evolutionarity to use the Hugoniot lines in the M_1, M_2-plane, where M_1 is the Mach number u_{1z}/c_{1s} in front of the disconti-

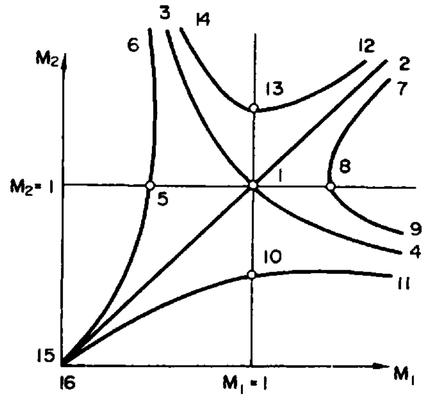

FIG. 3.3.11. Hugoniot lines in the variables M_1, M_2 in ordinary hydrodynamics. M_1 and M_2 are the Mach numbers in front of and behind the discontinuity. The numbers refer to the same points as in Fig. 3.3.10.

nuity and M_2 the Mach number behind the discontinuity. We show in Fig. 3.3.11 the Hugoniot lines in the M_1, M_2-plane (Shapiro, Hawthorne, and Edelman, 1947). The numbers refer to the same points as in Fig. 3.3.10. The points 15 and 16 of Fig. 3.3.10 correspond to the origin in Fig. 3.3.11. In the points 8 and 5, $M_2 = 1$; in these points the straight lines 1–8 and 1–5 in Fig. 3.3.10 touch (Landau and Lifshitz, 1959) the Hugoniot lines 9–7 and 15–6. In the points 10 and 13, $M_1 = 1$; in these the tangent 10–1–13 (see Fig. 3.3.10) to the Hugoniot line 4–1–3 intersects the Hugoniot line 11–14.

We now turn to a determination of the conditions for evolutionarity of different kinds of discontinuities and we start with shock waves. The propagation velocity of a shock wave depends on its amplitude. The condition for the evolutionarity of a shock wave therefore has in ordinary hydrodynamics the form (Landau and Lifshitz, 1959)

$$M_1 > 1, \quad M_2 \leqslant 1. \tag{3.3.5.3}$$

These conditions can be realized both when $q = 0$ (section 1–4 in Figs. 3.3.10 and 3.3.11) and when $q \neq 0$ (section 8–9 along which $q > 0$ and section 10–11 along which $q < 0$).

A shock wave in which the medium is heated to the ignition temperature which is accompanied by combustion is called a *detonation wave*. Such a wave corresponds to the section 8–9. The point 8 corresponds to the so-called *Chapman–Jouguet detonation* in which the velocity of the medium behind the wave relative to the discontinuity is equal to the local sound velocity ($M_2 = 1$).

The section 8–9 corresponds to a so-called *supercompression detonation wave*† for which

† The heating of the medium in a detonation in the Chapman–Jouguet regime occurs because of the liberation of reaction energy. The amplitude of the detonation wave is then independent of the kinetic energy of the moving gas and is determined by the properties of the medium. In the case of a supercompressed detonation the medium is heated both due to the liberation of reaction energy and because of the kinetic energy of the moving gas. The amplitude of the detonation wave will then be larger than in the Chapman–Jouguet regime.

A detonation in the Chapman–Jouguet regime is stable in the sense that it changes into a supercompressed detonation when the parameters of the medium are slightly changed. A change in the parameters of the medium causes a change in the amplitudes of the hydrodynamic waves which accompany a Chapman–Jouguet detonation.

Zel'dovich and Kompaneets (1955) give a more detailed discussion of the properties of detonation and combustion waves.

$M_2 < 1$. The section 10–11 corresponds to a shock wave in which the heating of the medium is accompanied by energy absorption due to dissociation or ionization. This section also corresponds to a shock wave in a vapour which is accompanied by evaporation. The section 7–8 corresponds to the so-called "weak detonation" which is non-evolutionary and hence unrealizable (Polovin, 1965b).‡

In contrast to detonation waves, in combustion waves the heating of the medium to the ignition temperature occurs thanks to the thermal conductivity. It follows from this that the propagation velocity of a combustion wave is completely determined by the state of the medium in front of the wave and is independent of the wave amplitude. As then the perturbation of the propagation velocity of a combustion wave vanishes, the conditions for evolutionarity of a combustion wave are determined not by the inequalities (3.3.4.3) but by the relations (Landau and Lifshitz, 1959)

$$M_1 < 1, \quad M_2 < 1 \quad \text{or} \quad M_1 > 1, \quad M_2 > 1. \tag{3.3.5.4}$$

Such waves can, when $q = 0$, be realized in the trivial case when there is no discontinuity: $M_1 = M_2$ (the line 15–1–2 in Fig. 3.3.11). When $q > 0$ the waves (3.3.5.4) correspond to the sections 15–5 and 7–8 in Figs. 3.3.10 and 3.3.11. The section 15–5 corresponds to a normal combustion wave in which the propagation velocity of the front is less than the sound velocity, $M_1 < 1$.

The section 7–8 corresponds to a so-called *supersonic combustion* wave. In ordinary combustion the velocity with which the reaction front moves is appreciably less than the sound velocity (Landau and Lifshitz, 1959) so that supersonic combustion is impossible. However, supersonic combustion can be realized in thermonuclear reactions when the heating of the medium to the ignition temperature does not occur due to collisions, as in ordinary combustion, but due to radiative thermal conductivity.

The section 7–8 of the Hugoniot line can also be realized when there is no combustion, if in a supersonic flow there is a narrow region—discontinuity surface—in which energy is liberated, for instance, due to recombination or condensation—the so-called *condensation discontinuities* (Landau and Lifshitz, 1959; Belen'kiĭ, 1945; Gross and Oppenheim, 1959; Hayes, 1958b; Reed, 1952).

Energy liberation can also occur in the neutral interstellar gas when stellar radiation is absorbed; the gas is then ionized and the excess of the photon energy over the ionization energy is liberated in the form of heat—the so-called photo-ionization discontinuities (Axford, 1961; Goldsworthy, 1958); the propagation velocity of a photo-ionization discontinuity can be either less than or more than the sound velocity.

We showed earlier that supersonic combustion waves are impossible in ordinary combustion. However, one often uses for a study of combustion the so-called hydraulic approximation, averaging all hydrodynamic quantities over the cross-section of a pipe in which a

‡ One can reach the same conclusion also without using the evolutionarity conditions by starting from the fact that a weak detonation wave has no structure (Erpenbeck, 1964). However, because of the complexity of the equations which describe the structure of detonation waves, a detailed study has not always been strictly carried out. An approximate study of the structure of a detonation waves may lead to incorrect conclusions (Hayes, 1958a; Erpenbeck, 1964) about the possibility of a weak detonation, if the combustion process consists of several chemical reactions.

chemical reaction takes place. In the hydraulic approximation an oblique two-dimensional detonation wave (see Fig. 3.3.12) emerges as a one-dimensional supersonic combustion wave (Fletcher, Dorsch, and Allen, 1960; McLafferty, 1960; Dunlap, Brehm, and Nicolls, 1958; Gross, 1959); in Fig. 3.3.12 $u_{2n} = c_{2s}$, while $u_1 > c_{1s}$.

For all discontinuities enumerated here the sections 16–10, 5–6, 1–3, 13–14, and 12–13 are non-evolutionary and can therefore not be realized.

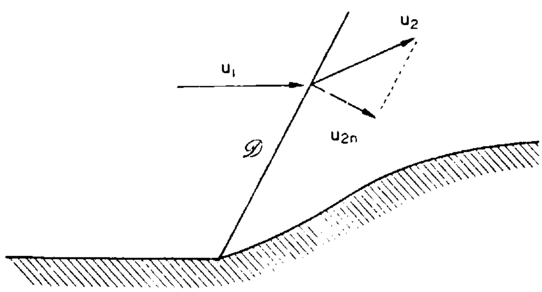

FIG. 3.3.12. An oblique detonation wave $\mathcal{D}$, simulating supersonic combustion.

We now turn to exothermic and endothermic discontinuities in magneto-hydrodynamics. Magneto-hydrodynamic detonation and combustion waves are exothermic. An ionization wave, in which the ionization of the medium occurs due to heating in a shock wave, is endothermic.

If an ionization shock wave propagates in a non-conducting medium, in front of it there may move an electromagnetic wave which does not interact at all with the hydrodynamic waves. On the other hand, in the region behind the ionization wave the medium is conducting and due to the interaction with hydrodynamic waves the electromagnetic waves combine into magneto-sound and Alfvén waves. As the amplitude of the electromagnetic wave which moves in front of the ionization wave is independent of the parameters, the problem has an infinite number of solutions (Cowley, 1967) if we neglect dissipative effects, that is, assume that the shock wave is infinitely thin. To determine the amplitude of the electromagnetic wave we must consider the structure of the ionization shock wave. Such a study shows that the amplitude of the electromagnetic wave depends on the ratios of the dissipation coefficients (Kulikovskiĭ and Lyubimov, 1959, 1960; Zhilin, 1960; Chu, 1964; Butler, 1965).

If, however, the medium in front of the shock wave initially was ionized and there is merely an increase in the degree of ionization in the shock wave, the electromagnetic wave considered a moment ago will not be present and one must solve the problem without studying the structure of the shock wave. We shall understand by an ionization shock wave in what follows, only such a wave.

In the case of a combustion wave, its propagation velocity is independent of its amplitude and it remains constant when we impose a small perturbation. Repeating the discussion of Subsection 3.3.1 we can show that the number of outgoing small amplitude magneto-hydrodynamic waves must equal seven in the case of combustion waves. There are then four possible magneto-hydrodynamic combustion regimes (Polovin and Demutskiĭ, 1961; Barmin, 1961):

1. "Slow" combustion,

$$u_{1z} < v_{1-}, \quad u_{2z} < v_{2-}; \tag{3.3.5.5}$$

2. "Sub-Alfvénic" combustion,

$$v_{1-} < u_{1z} < v_{1Az}, \quad v_{2-} < u_{1z} < v_{2Az}; \tag{3.3.5.6}$$

3. "Super-Alfvénic" combustion,

$$v_{1Az} < u_{1z} < v_{1+}, \quad v_{2Az} < u_{2z} < v_{2+}; \tag{3.3.5.7}$$

4. "Fast" combustion,

$$v_{1+} < u_{1z}, \quad v_{2+} < u_{2z}. \tag{3.3.5.8}$$

The difference between the four magneto-hydrodynamic combustion regimes consists in the different directions in which the density, magnetic field, and velocity of the medium change (see Subsection 3.4.1).

As the propagation velocity of a combustion front is appreciably smaller than the sound velocity (Landau and Lifshitz, 1959) we can have, for $v_{1Az} \ll c_{1s}$ and for temperatures which are normal for chemical reactions, slow, sub-Alfvénic, and super-Alfvénic combustion. Fast combustion is possible only for thermonuclear reactions when the reaction front moves with supersonic velocity thanks to the radiative thermal conductivity.

As the propagation velocities of detonation waves and of ionization shock waves depend on their amplitude, the conditions for evolutionarity of such discontinuities are the same as the conditions for evolutionarity of magneto-hydrodynamic shock waves without emission or absorption of energy.

The two propagation velocities of small perturbations—the fast and slow magneto-sound velocities—correspond to two detonation waves:† a fast one for which

$$v_{1+} < u_{1z}, \quad v_{2Az} < u_{2z} \leqslant v_{2+}, \tag{3.3.5.9}$$

and a slow one, for which

$$v_{1-} < u_{1z} < v_{1Az}, \quad u_{2z} \leqslant v_{2-}. \tag{3.3.5.10}$$

The third velocity—the Alfvén velocity—cannot correspond to a detonation wave as there is no heating of the medium in an Alfvén discontinuity.

As in ordinary hydrodynamics, the *Chapman–Jouguet* detonation region plays a peculiar role; in it the velocity of the detonation wave relative to the reaction products is equal to the propagation velocity of small perturbations. The equal signs in eqns. (3.3.5.9) and (3.3.5.10) correspond to this regime.

† In the case when the normal component B_n of the magnetic field vanishes, there is one detonation wave (Larish and Shekhtman, 1958; Lyubimov, 1959a, b; Gross, Chinitz, and Rivlin, 1960), the evolutionarity condition of which has the form

$$u_{1z} > \sqrt{v_{1A}^2 + c_{1s}^2}, \quad u_{2z} \leqslant \sqrt{(v_{2A}^2 + c_{2s}^2)},$$

and one ionization wave (Kulikovskiĭ and Lyubimov, 1959, 1960; Lyubimov, 1959c).

Two kinds of Chapman–Jouguet detonations correspond to two types of shock waves (Demutskiĭ and Polovin, 1961; Barmin, 1961): the fast Chapman–Jouguet detonation,

$$v_{1+} < u_{1z}, \quad v_{2Az} < u_{2z} = v_{2+}, \tag{3.3.5.11}$$

and the slow Chapman–Jouguet detonation,

$$v_{1-} < u_{1z} < v_{1Az}, \quad v_{2-} = u_{2z} < v_{2Az}. \tag{3.3.5.12}$$

We shall call the detonation regime, in which the velocity of the detonation wave relative to the reaction products is less than the propagation velocity of the corresponding small amplitude magneto-sound wave, a *supercompressed detonation*. The inequality signs in eqns. (3.3.5.9) and (3.3.5.10) correspond to that regime.

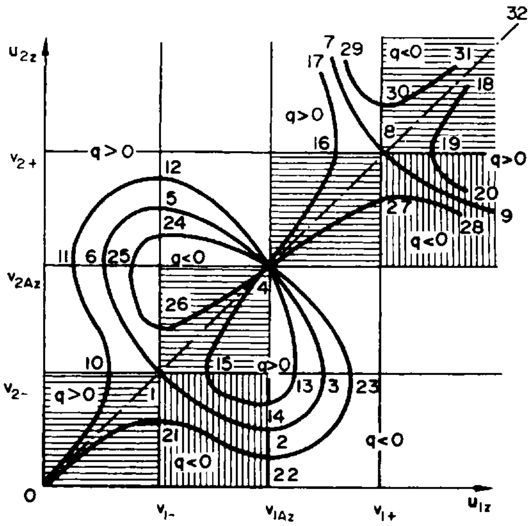

FIG. 3.3.13. Evolutionarity regions of magneto-hydrodynamic shock, detonation, and ionization waves and combustion waves. 1–2–3–4–5–6–1 and 7–8–9 are Hugoniot lines corresponding to energy conservation, 0–10–11–12–4–13–14–15–4–16–17 and 18–19–20 Hugoniot lines corresponding to energy liberation, and 0–21–22–23–4–24–25–26–4–27–28 and 29–30–31 Hugoniot lines corresponding to energy absorption. Vertical hatching indicates the evolutionarity regions of shock, detonation, and ionization waves; horizontal hatching indicates the evolutionarity regions of combustion waves.

To distinguish the actually occurring evolutionary waves from the non-realizable, that is, the non-evolutionary waves, we use (Polovin and Demutskiĭ, 1963) the u_{1z}, u_{2z}-plane (see Fig. 3.3.13). The lines 1–2–3–4–5–6–1 and 7–8–9 correspond to shock waves (Bazer and Ericson, 1959; Ericson and Bazer, 1960). The region of evolutionarity of shock waves is hatched vertically in Fig. 3.3.13.

We shall now consider discontinuities with energy liberation ($q > 0$). Such discontinuities are (for $v_A \ll c_s$, $q \ll c_s^2$) depicted in Fig. 3.3.13 by the lines 0–10–11–12–4–13–14–15–4–16–17 and 18–19–20. The section 0–10 corresponds to slow combustion (rarefaction wave), the section 15–4 to sub-Alfvénic combustion (compression wave), the section 4–16 to super-Alfvénic combustion (rarefaction wave), and, finally, the section 18–19 to fast combustion (compression wave). The sections 14–15 and 19–20 correspond, respectively, to detonations in the slow and the fast wave (both compression waves). The points 15 and 19 correspond to Chapman–Jouguet detonations in the slow and the fast wave. The sections 10–11–12–4–13–14 and 16–17 correspond to non-evolutionary discontinuities and can therefore not be realized.

Discontinuities with energy absorption ($q < 0$) correspond to the lines 0–21–22–23–4–24–25–26–4–27–28 and 29–30–31. The section 21–22 corresponds to ionization in a slow shock wave and section 27–28 to ionization in a fast shock wave. The sections 0–21, 4–26, 4–27, and 30–31 could correspond to evolutionary discontinuities with energy absorption with propagation velocities, independent of amplitude. We do, however, not know actual physical mechanisms which could lead to a realization of such discontinuities

We note that if the reaction energy is sufficiently large, the slow detonation, and also the sub-Alfvénic and super-Alfvénic combustions, are impossible (Barmin, 1961).

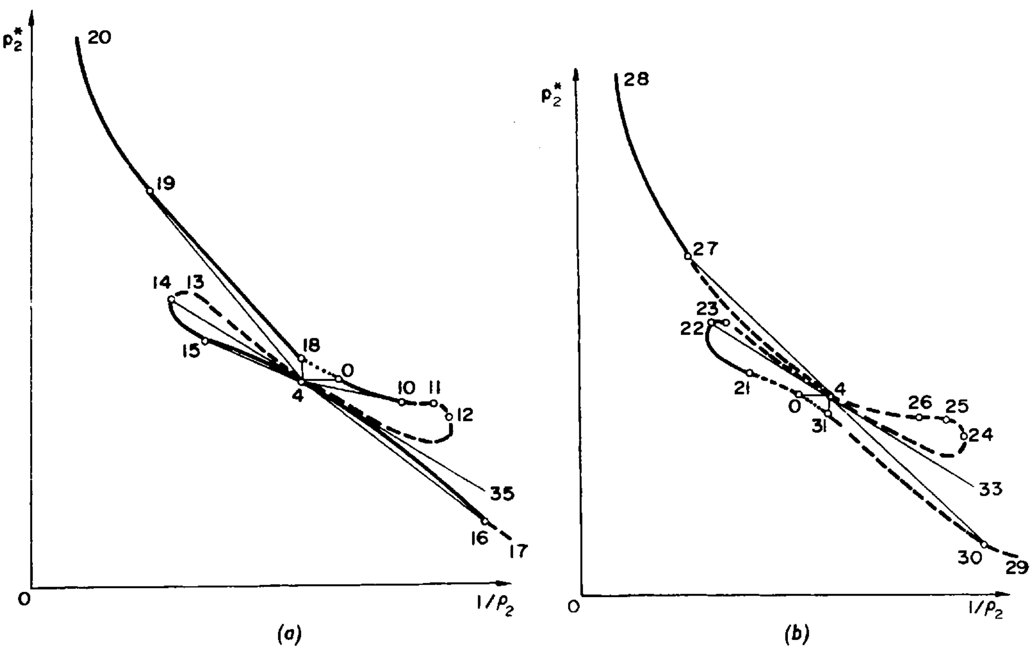

FIG. 3.3.14. Hugoniot lines in the $1/\varrho_2, p_2^*$-plane. (a) Exothermic discontinuities, (b) endothermic discontinuities. The numbers indicate the same points as in Fig. 3.3.13. The full-drawn lines indicate evolutionary sections and the broken lines non-evolutionary sections.

The Hugoniot lines in the u_{1z}, u_{2z}-plane, depicted in Fig. 3.3.13, correspond to the Hugoniot lines in the $1/\varrho_2$, p_2^*-plane ($p_2^* = p_2 + B_2^2/8\pi$) depicted in Fig. 3.3.14a (exothermic discontinuities) and Fig. 3.3.14b (endothermic discontinuities). The numbering in Fig. 3.3.14 is the same as in Fig. 3.3.13. The initial state $1/\varrho_1$, p_1^* in front of the discontinuity is depicted by the point 4. The vertical and horizontal straight lines through the point 4 cut from the Hugoniot lines a section which cannot be realized (dotted in Figs. 3.3.14 a, b) as the mass flux density $\varrho_1 u_{1z}$ on it becomes imaginary.

The straight lines 4–19, 4–10, 4–15, and 4–16 are tangents to the Hugoniot line at the points 19, 10, 15, and 16, in which the velocity of the discontinuity relative to the reaction products is equal to the propagation velocity of small perturbations. Point 19 corresponds to the Chapman–Jouguet detonation on the fast wave and separates the section 19–20 of the fast supercompressed detonation from the section 18–19 of fast combustion. The point 15 corresponds to the Chapman–Jouguet detonation on the slow wave and separates the section 15–14 of the slow supercompressed detonation from the section 15–4 of sub-Alfvénic combustion. The point 10 is the limit of the slow combustion section 0–10, and the point 16 that of the super-Alfvénic combustion section 4–16.

We have also given in Fig. 3.3.14a the straight line 4–35 with a slope $-\varrho_1^2 v_{1Az}^2$. This line cuts the Hugoniot line in the point 14 where the propagation velocity of the discontinuity in the medium at rest equals the Alfvén velocity. The point 14 is the limit of the slow supercompressed detonation section 15–14. The straight lines 4–11 and 4–13, through the point 4, are tangent to the Hugoniot line at the points 11 and 13 (these lines are not shown in Fig. 3.3.14a). The point 12 determines the intersection of the straight line 4–12, through the point 4 with a slope $-\varrho_1^2 v_{1-}^2$, and the Hugoniot line.

We finally consider the Hugoniot line for discontinuities with energy absorption (Fig. 3.3.14b). In that figure, the straight line 21–4–26–24, through the point 4 with a slope $-\varrho_1^2 v_{1-}^2$, intersects the Hugoniot line in the points 21, 26, and 24 while the straight line 4–33 with the slope $-\varrho_1^2 v_{1Az}^2$ intersects the Hugoniot line in the point 22.

The section 21–22 corresponds to ionization occurring in a slow shock wave. The straight line with slope $-\varrho_1^2 v_{1+}^2$ intersects the Hugoniot line in the point 27. The point 27 is the limit section 27–28 of the Hugoniot line corresponding to ionization in a fast shock wave. The lines 4–23 and 4–25 (not shown in the figure) which go through the point 4 are tangent to the Hugoniot line in the points 23 and 25.

3.3.6. SWEEPING-OUT CONDITIONS

We saw earlier that the evolutionarity conditions enable us to exclude from our considerations non-realizable shock waves on which all boundary conditions are satisfied and on which the entropy increases, but which are unstable with respect to splitting up.

However, satisfying the evolutionarity conditions is, generally speaking, not yet sufficient that a shock wave is indeed realized. For instance, if we start only from the conservation laws and the evolutionarity conditions, there may arise shock waves, when there is flow around a wedge, starting from the wedgetip—the so-called *oblique shock waves*. Notwithstanding the fact that these waves are evolutionary, they may not be realizable, as in a number of cases an infinitesimally small perturbation may cause separation of these waves from the edge of the wedge. In order that oblique shock waves occur, one must satisfy additional conditions—the sweeping-out conditions which we shall formulate in the present subsection.

Let us consider a supersonic symmetric flow past a wedge in ordinary hydrodynamics. In such flow there always arise shock waves and depending on the opening of the wedge two flow regimes are possible: when the opening angle is smaller than some critical value, two symmetrically arranged oblique shock waves, starting from the edge of the wedge (see Fig. 3.3.15a), are possible, while for angles larger than this critical angle there occurs a single detached curved shock wave (see Fig. 3.3.15b).

Let us consider in more detail the oblique shock wave starting from the edge of the wedge.

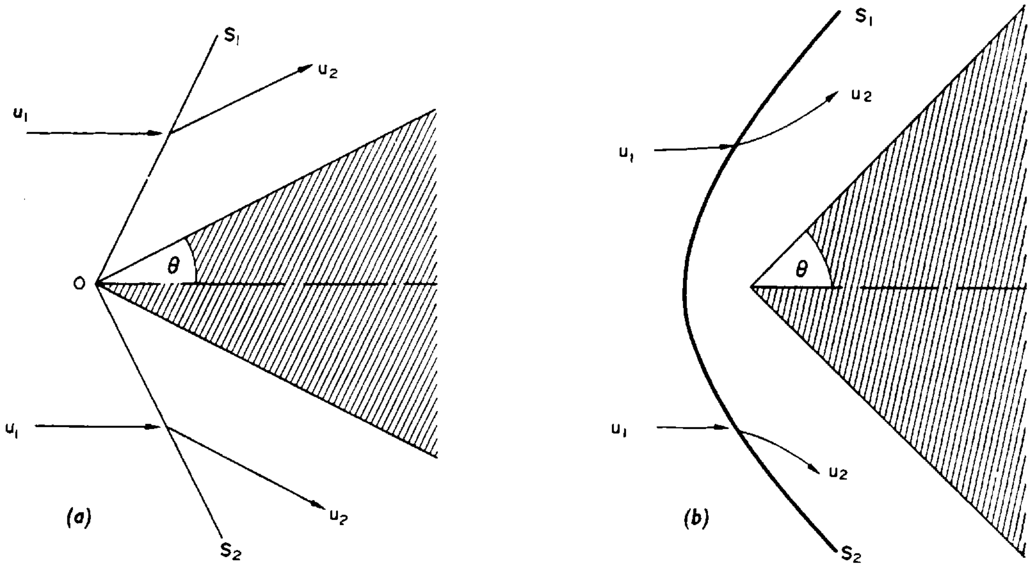

FIG. 3.3.15. Supersonic flow past a wedge. (a) Two oblique shock waves OS_1 and OS_2 starting from the edge of the wedge; (b) detached curved shock wave S_1S_2.

We denote by u_1 the velocity of the medium in front of the shock wave and by u_2 the velocity of the medium behind it. In the case of an oblique shock wave starting from the edge of the wedge, which we are discussing, the angle between the vectours u_1 and u_2 is equal to the semi-opening angle of the wedge, θ. One can easily determine u_2 as function of u_1, starting from the boundary conditions on the shock wave surface. We give in Fig. 3.3.16 the shock polar, that is, the locus of the end points of the vector u_2 for different θ and given u_1, directed along the x-axis. We see that, if $\theta < \theta_0$, where θ_0 is the angle between the tangent OC_0 and the x-axis, two solutions are possible, corresponding to two velocity vectors u_2, equal to $\overrightarrow{OS}$ and $\overrightarrow{OW}$. The shock wave with $u_2 = \overrightarrow{OS}$ is said to belong to the *strong family* of shocks and the shock wave with $u_2 = \overrightarrow{OW}$ to the *weak family*. The point

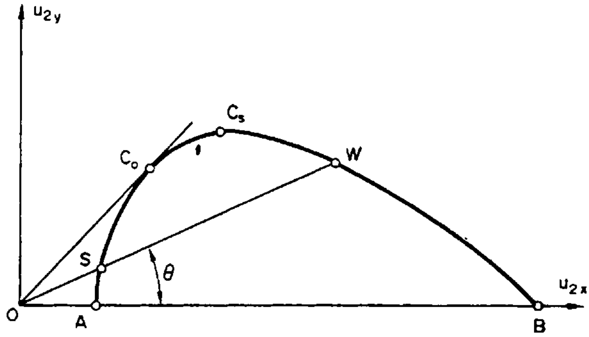

FIG. 3.3.16. Shock polar. A: Normal shock wave, B: oblique shock wave of infinitesimally small amplitude; S and W are oblique shock waves of the strong and weak families, θ is the semi-opening angle of the wedge, C_0 the point of contact of the tangent from the origin to the shock polar, and C_s the point where the velocity of the medium behind the shock wave equals the local sound velocity.

of contact C_0 of the tangent from the origin O to the shock polar separates the waves of the strong family, corresponding to the points on the arc AC_0, from those of the weak family, corresponding to the arc C_0B. If the semi-opening angle of the wedge is larger than θ_0, corresponding to the point C_0, oblique shock waves cannot be realized.

To the right of the point C_0 there is a point C_s for which the flow velocity behind the shock wave equals the local sound velocity c_{2s}, $u_2 \equiv \overrightarrow{OC_s} = c_{2s}$. We shall show that only those shock waves of the weak family can be realized which correspond to the section C_sB of the shock polar. As far as oblique shock waves of the strong family as well as oblique shock waves of the weak family corresponding to the arc C_0C_s of the shock polar are concerned, they cannot be realized.

To prove this statement, we note that the section AC_s of the shock polar corresponds to shock waves for which the velocity of the medium behind the wave is subsonic, $u_2 < c_{2s}$, and the section C_sB to shock waves for which $u_2 > c_{2s}$ (Belen'kiĭ, 1958). Imagine now that we impose a small perturbation on the flow. As the perturbation propagates with the sound velocity while the flow velocity in front of the shock wave exceeds the sound velocity, the perturbation will in front of the shock wave be "removed" from the shock wave.

As far as the region behind the shock wave is concerned, in the case when $u_2 < c_{2s}$ the perturbation will not be removed (see Fig. 3.3.17a). On the other hand, as the perturbation is carried away in front of the shock wave, it cannot change the shock wave. The angle of rotation of the velocity is also unchanged. The perturbation will thus be "detached" also from the region behind the shock wave. However, such a situation can only occur in the case when $u_2 > c_{2s}$ (see Fig. 3.3.17b). In other words, shock waves corresponding to the section AC_s of the shock polar cannot be realized (Polovin, 1965c; Polovin and Cherkasova, 1966b).

We see thus that for shock waves occurring when there is flow around a wedge the following inequality must hold:

$$u_2 > c_{2s}, \tag{3.3.6.1}$$

and, moreover, the evolutionarity conditions must, of course, be satisfied:

$$u_{1n} > c_{1s}, \quad u_{2n} < c_{2s}. \tag{3.3.6.2}$$

(These inequalities are necessary conditions for the existence of a shock wave.) We shall call inequality (3.3.6.1) the *sweeping-out condition.*

Using the evolutionarity conditions we can write the sweeping-out condition (3.3.6.1) in the symmetric form:

$$u_1 > c_{2s}, \quad u_2 > c_{2s}. \tag{3.3.6.3}$$

We are thus led to the conclusion that although all shock waves, both from the weak and from the strong family, are evolutionary, because of the "sweeping-out" effect for perturbations only the shock waves of the weak family from the section C_sB of the shock polar can be realized.

The presence of the additional sweeping-out condition besides the evolutionarity conditions which separate off the realizable shock waves is connected with the fact that in the problem considered the shock waves are formed near a wedge which exerts an important

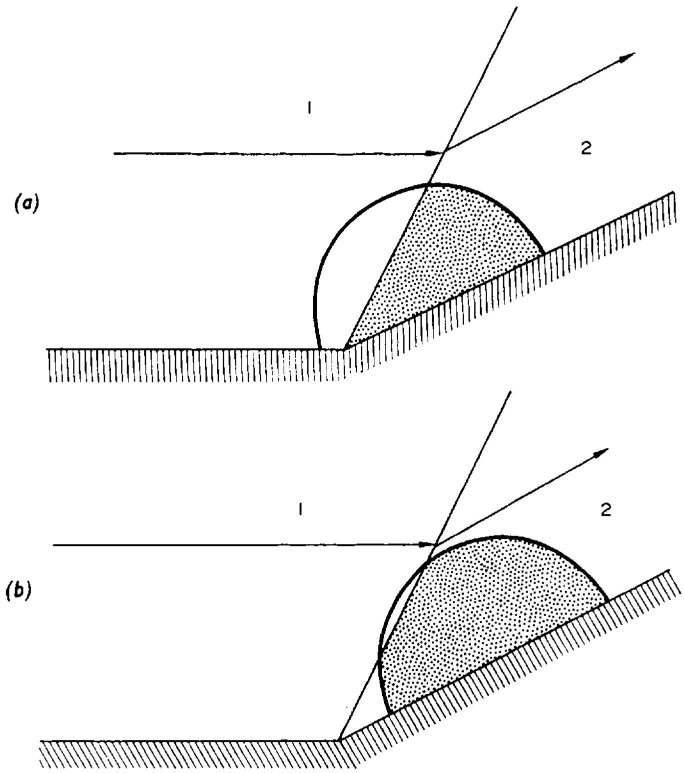

FIG. 3.3.17. Sweeping-out conditions. (a) $u_2 < c_{2s}$, the perturbation is not swept out from the edge; (b) $u_2 > c_{2s}$, the perturbation is swept out. The perturbed region is indicated by spots.

influence on their existence. When the evolutionarity conditions were formulated we did not take the role of solid bodies into account when gases or liquids flow around them.

We note that the discussion given here refers only to two-dimensional flow and is not valid for three-dimensional flow. For instance, it is invalid for shock waves occurring when there is supersonic flow around a cone. In that case there is also an attached shock wave (Busemann, 1942)—which is called a conical shock wave—but in contrast to the oblique shock waves the flow behind a conical shock wave can be subsonic (Taylor and MacColl, 1933). This is connected with the following fact: in the above discussion we tacitly assumed that the Huygens phenomenon did not occur. This is valid for two-dimensional flow, but invalid for three-dimensional flow (see Subsection 2.1.6). One can, however, show that in the case of supersonic flow around a cone again only shock waves of the weak family can be realized. The proof is based upon a study of the perturbation of the body around which there is the flow (Cabannes and Stael, 1961).

So far we have considered the flow around a wedge in ordinary hydrodynamics. The picture is considerably more complicated for the case of magneto-hydrodynamic flow around a wedge. We shall consider the simplest case of flow around a perfectly conducting wedge when the velocity of the medium and the magnetic field on one side of the shock wave are parallel and these four vectors on the two sides of the shock wave lie in one plane. In that case the sweeping-out condition can be written in the form (Polovin, 1961d; Polovin and Cherkasova, 1966b)

$$u_1 > \text{Max}\,(v_{1A}, c_{1s}), \quad u_2 > \text{Max}\,(v_{2A}, c_{2s}), \tag{3.3.6.4}$$

for a fast shock wave, when the evolutionarity conditions are taken into account, and for a slow shock wave in the form (Polovin, 1965d)

$$\frac{v_{1A}\,c_{1s}}{\sqrt{(v_{1A}^2+c_{1s}^2)}} < u_1 < \text{Min}\,(v_{1A}, c_{1s}), \quad \frac{v_{2A}\,c_{2s}}{\sqrt{(v_{2A}^2+c_{2s}^2)}} < u_2 < \text{Min}\,(v_{2A}, c_{2s}). \tag{3.3.6.5}$$

We finally discuss the sweeping-out conditions for conical shock waves in magneto-hydrodynamics. In contrast to ordinary hydrodynamics the Huygens phenomenon does not occur in three-dimensional magneto-hydrodynamical flow (see Subsection 2.1.6). It follows from this that the sweeping-out condition (3.3.6.4) must also be satisfied for fast conical shock waves.

As to the sweeping-out conditions (3.3.6.5) for slow conical shock waves, as there are no lacunae in three-dimensional magneto-hydrodynamical flow (see Subsection 2.1.6) there are no additional conditions, apart from the evolutionarity conditions which must be satisfied. It is therefore not necessary to satisfy for slow conical shock waves all inequalities (3.3.6.5), but only those which are a consequence of the evolutionarity conditions:

$$\frac{v_{1A}\,c_{1s}}{\sqrt{(v_{1A}^2+c_{1s}^2)}} < u_1 < \text{Min}\,(v_{1A}, c_{1s}), \quad u_2 < \text{Min}\,(v_{2A}, c_{2s}).$$

As in ordinary hydrodynamics the flow behind a magneto-hydrodynamical conical shock wave is not uniform Such discontinuous magneto-hydrodynamical flow has been studied for the case of symmetrical flow around a cone with a small opening angle (Bausset, 1963).

3.4. Study of Discontinuities

3.4.1. DISCONTINUITIES IN VARIOUS QUANTITIES

When two shock waves collide or when a piston is suddenly set in motion there arises a surface of discontinuity of magneto-hydrodynamical quantities. Generally speaking, the conservation laws, that is, the boundary conditions (3.2.1.2), (3.2.1.3), (3.2.1.5), (3.2.1.7), and (3.2.1.8), are not satisfied on this surface. Such a discontinuity can therefore not exist and it splits at once into a number of magneto-hydrodynamical shock and simple waves. If we neglect dissipation, there will in the problem of the splitting-up of such a discontinuity not be a parameter with the dimensions of length and as a result all simple waves will be self-similar (Landau and Lifshitz, 1959).

One can use each of the shock and self-similar waves which occur when the discontinuity splits up to find in accordance with the conservation laws the values of the magneto-hydrodynamic quantities on both sides of the discontinuity surface.

To elucidate the necessity that some wave or other appears we must, first of all, qualitatively study the discontinuities of the magneto-hydrodynamical quantities in different kinds of waves. We start with simple self-similar waves.

Using eqns. (3.1.2.3) we can reach well-determined conclusions about the direction in which various magneto-hydrodynamical quantities change in fast and slow self-similar waves (Polovin and Cherkasova, 1966b). Denoting by Δa_i the change in an appropriate magneto-hydrodynamical quantity, we have (see Subsection 3.1.3)

$$\Delta \varrho < 0,$$

that is, self-similar waves are rarefaction waves. From the relation $dp = c_s^2\, d\varrho$ it follows that

$$\Delta p < 0.$$

We now turn to a study of the direction in which the quantities ΔB_x, Δu_x, and Δu_z change, and we assume for the sake of simplicity that $\varepsilon = +1$, $B_x > 0$, and $B_z > 0$. Otherwise we must replace ΔB_x, Δu_x, and Δu_z by $(\operatorname{sgn} B_x)\Delta B_x$, $\varepsilon \operatorname{sgn}(B_xB_z)\Delta u_x$, and $\varepsilon\,\Delta u_z$. From eqns. (3.1.2.3) we find that in fast waves

$$\Delta B_x < 0, \quad \Delta u_x > 0, \quad \Delta u_z < 0.$$

We give in Table 3.4.1 the signs of the changes in the quantities in rarefaction waves and also in other waves. R⁺ indicates a fast (self-similar) rarefaction wave.

TABLE 3.4.1

			ΔB_x	Δu_x	Δu_z			
D⁺	I⁺	C_f	+	—	+	S⁺		
		C_{sup}	—	+	—	R⁺		
			—	+	0		A	
D⁻	I⁻	C_{sub}	—	+	+			S⁻
		C_{sl}	+	—	—			R⁻

Change in B_x, u_x, and u_z in different kinds of waves.

D⁺ and D⁻ are (fast and slow) detonation waves, I⁺ and I⁻ (fast and slow) ionization waves, C_f, C_{sup}, C_{sub}, and C_{sl} (fast, super-Alfvénic, sub-Alfvénic, and slow) combustion waves, S⁺ and R⁺ fast shock waves and fast rarefaction waves, A a 180° Alfvén discontinuity, and S⁻ and R⁻ slow shock waves and slow rarefaction waves. The z-axis is along the normal to the discontinuity surface and the wave moves in the direction of the positive z-axis while $B_zB_x > 0$, $B_z > 0$.

In slow self-similar waves we find similarly

$$\Delta B_x > 0, \quad \Delta u_x < 0, \quad \Delta u_z < 0$$

(the last row in Table 3.4.1, indicated by R⁻ on the right-hand side of the Table).

Let us now consider a 180° Alfvén discontinuity—in which the magnetic field turns over 180°. The quantity u_z does not change in such a discontinuity, $\Delta u_z = 0$, and the direction of the component of the magnetic field B_x at right angles to the direction of propagation of the discontinuity changes into the opposite one: when $B_x > 0$, we have $\Delta B_x < 0$. Moreover, it follows from eqn. (3.1.2.1) that for $\varepsilon = +1$ we have $\Delta u_x > 0$ (see the third row of Table 3.4.1, indicated by A on the right-hand side of the Table).

We now turn to shock waves. It follows from the evolutionarity conditions that in the case of fast shock waves the enumerator and the denominator in eqn. (3.2.3.3) are positive, while in the case of slow waves the enumerator and denominator are negative. Hence it follows that the transverse magnetic field does not change its sign in evolutionary shock wave (Polovin and Lyubarskiĭ, 1959).†

Moreover, from Zemplén's theorem ($\varrho_2 > \varrho_1$) and formula (3.2.3.3) it follows that in fast shock waves the transverse magnetic field B_x increases, when $B_x > 0$, while it decreases in slow shock waves. It also follows from Zemplén's theorem that the absolute magnitude of the quantity u_z decreases in shock waves. In waves moving to the right u_z is negative so that $\Delta u_z > 0$. It follows from the second eqn. (3.2.1.5) that for waves which move to the right ($j \equiv \varrho u_z < 0$) the quantities Δu_z and ΔB_x have the opposite sign when $B_z > 0$. We have thus $\Delta u_x < 0$ for fast shock waves and $\Delta u_x > 0$ for slow ones (see the first and fourth rows of Table 3.4.1, indicated by S^+ and S^- on the right-hand side of the table).

The signs of ΔB_x, Δu_x, and Δu_z can be determined in a similar way for the remaining waves: detonation waves (indicated by D^+ and D^- in Table 3.4.1), fast and slow ionization slow waves (I^+ and I^-), and the four combustion regimes (C_f, C_{sup}, C_{sub}, and C_{sl}).

We note that the transverse magnetic field does not change its sign, not only in evolutionary shock waves, but also in the other kinds (except 180° Alfvén discontinuities) of evolutionary discontinuities (combustion, detonation, and ionization waves). This is connected with the fact that, according to eqn. (3.2.3.3), the transverse magnetic field B_{2t} vanishes when $u_{1z} = v_{1Az}$, that is, on the boundaries of the evolutionarity region.

The above indicated increase of the transverse magnetic field in fast shock waves and its decrease in slow shock waves plays a determining role in magneto-hydrodynamical turbulence. In fact, small magnetic fields ($H_z^2/8\pi < \frac{1}{2}\varrho u_{1z}^2$) increase when a shock wave passes through, while large magnetic fields ($H_z^2/8\pi > \frac{1}{2}\varrho u_{1z}^2$) weaken. This leads to the result that when a large number of random shock waves pass through the medium, statistical equilibrium is attained when the magnetic and the kinetic energy are equal (Polovin, 1961c)

$$\frac{B_z'^2}{8\pi} = \frac{1}{2}\varrho u_z^2.$$

This form of the energy equipartition law is characteristic for magneto-hydrodynamics. A number of authors (Alfvén, 1949; Batchelor, 1950; Ferraro, 1956; Kantrovich and Petchek, 1958; Woltjer, 1959; Kautzleben, 1958; Ginzburg, 1954) have obtained it from other considerations.

We now turn to a more detailed study of the discontinuities in various magneto-hydrodynamical quantities in shock waves. The jumps in the pressure and entropy are monotonically increasing functions of the jump in the density in fast and slow shock waves (along the evolutionary section).

When the intensity of the fast shock wave is large, $p_2 \gg p_1 + B_1^2/8\pi$, the magnetic pressure behind the wave becomes appreciably smaller than the hydrostatic pressure (Lüst, 1955;

† Taking this fact into account considerably lowers the number of possible configurations when studying magneto-hydrodynamic shock waves in the circumterrestrial space (Rosenkilde, 1965).

Baum, Kaplan, and Stanyukovich, 1958):

$$\frac{B_2^2}{8\pi} \ll p_2.$$

The magnetic field in front of the wave can then be neglected compared with the kinetic energy of the medium,

$$\frac{B_1^2}{8\pi} \ll \frac{1}{2}\varrho_1 u_1^2.$$

The largest compression which can be attained in the shock wave is thus the same as in ordinary gas dynamics (Bazer and Ericson, 1959; Kaplan, 1965):

$$\frac{\varrho_2}{\varrho_1} = \frac{\gamma+1}{\gamma-1},$$

where γ is the adiabatic index.

The dependence of the jump in the magnetic field, $\Delta B_x \equiv B_{2x} - B_{1x}$, on the jump in the density can be of two kinds in a fast wave (Jeffrey and Taniuti, 1964; Bazer and Ericson, 1959). In the first kind of waves, which occur when

$$\sin^2\theta_1 \geqslant \frac{(\gamma-1)(1-r_1)}{\gamma}, \tag{3.4.1.1}$$

where $r_1 \equiv c_{1s}^2/v_{1Az}^2 = 4\pi\gamma p_1/B_1^2$ and θ_1 the angle between the direction of the magnetic field $\boldsymbol{B}_1$ and the normal to the discontinuity surface, the jump in the magnetic field increases monotonically from zero to its maximum value which is equal to

$$(\Delta B_x)_{\max} = \frac{2B_{1x}}{\gamma-1}, \tag{3.4.1.2}$$

when the jump in the density increases. In waves of the second type which exist when condition (3.4.1.1) is not satisfied there occurs a non-monotonic dependence of the jump in the magnetic field on the jump in density: when the jump in density increases, the jump in the magnetic field initially increases from zero to some maximum value and then decreases to the value (3.4.1.2).

On the evolutionary section of the slow shock wave the jump in the magnetic field always increases with increasing density jump (Polovin, 1961a).

In the limiting case as $\theta_1 \to 0$, $r_1 > 1$ the fast shock wave is a parallel one, that is, the same as when there is no magnetic field while the slow wave has an infinitesimally small amplitude. In the case $\theta_1 \to 0$, $r_1 < 1$ the fast wave belongs to the second kind. When the intensity of the fast wave is sufficiently small,

$$\frac{\varrho_2}{\varrho_1} < \frac{\gamma+1-2r_1}{\gamma-1},$$

the transverse magnetic field behind the shock wave is non-vanishing and the wave is a peculiar wave. If, however,

$$\frac{\varrho_2}{\varrho_1} > \frac{\gamma+1-2r_1}{\gamma-1},$$

the transverse magnetic field B_{2x} behind the wave vanishes and the shock wave becomes a parallel wave (Jeffrey and Taniuti, 1964). The slow shock wave will on the evolutionary section in the case $\theta_1 \to 0$, $r_1 < 1$ be a parallel one, that is, the same as when there is no magnetic field.

In the limiting case $\theta_1 \to \pi/2$ the fast shock wave is a first kind wave and the slow shock wave turns into a tangential discontinuity.

The presence of a magnetic field enhances the pressure jump for a given density jump (Bazer and Ericson, 1959; Golitsyn, 1959).

A slow shock wave cannot have an arbitrarily large intensity. Therefore, if $p_2 \gg p_1 + B_1^2/8\pi$, only one shock wave—the fast one—exists (Iordanskiĭ, 1959; Lüst, 1955).

We saw in Subsection 3.2.1 that if the velocity of the medium is parallel to the magnetic field on one side of any of the magneto-hydrodynamical discontinuities, it will be parallel to the magnetic field also on the other side (in the case when the component of the magnetic field normal to the discontinuity is non-vanishing). However, if there are self-similar waves as well as discontinuities, this parallel behaviour is violated. Indeed, if the vectors $\boldsymbol{u}$ and $\boldsymbol{B}$ lie in the x, z-plane and are parallel, we have

$$u_z B_x - u_x B_z = 0. \quad (3.4.1.3)$$

We now determine the change in the left-hand side of eqn. (3.4.1.3) in simple magneto-hydrodynamic waves. Using eqns. (3.1.2.3) we find

$$\frac{d}{d\varrho}(u_z B_x - u_x B_z) = \frac{(u_z + \varepsilon v_\pm) B_x v_\pm^2}{\varrho(v_\pm^2 - v_{Az}^2)},$$

rom which it is clear that equation (3.4.1.3) cannot hold in a finite magnitude magneto-sound wave.

3.4.2. ORDER OF SEQUENCE OF WAVES

So far we have studied isolated discontinuities and simple waves. They arise when a solid body moves through the medium (piston problem) and also when gas masses collide. However, as a rule in such cases there occurs not a single discontinuity, but several discontinuities of different types. We shall now show that not more than one (discontinuity or self-similar) wave of each type can be excited and that they must follow in a strictly defined sequence.

We shall first consider ordinary hydrodynamics. We know that in that case a shock wave moves with supersonic velocity relative to the medium in front of it and with subsonic velocity relative to the medium behind it. If, therefore, two shock waves with velocities U_1 and U_2 (we assume the first wave to move in front) move in the same direction the following inequalities must hold:

$$U_1 < c_s, \qquad U_2 > c_s,$$

where c_s is the sound velocity in the space between the two waves. From these inequalities it follows that

$$U_2 > U_1.$$

In other words, the second wave overtakes the first one, and this means that the waves considered do not have a common source. In turn it follows from this that two shock waves which are formed from the same source cannot move in the same direction.

We now consider self-similar waves. Their velocity is the same as the local sound velocity —both in front and behind the wave. From this it follows that two self-similar waves cannot move in the same direction and that the same is true for one shock wave and one self-similar wave. In other words, if a single source produces shock and self-similar waves, not more than one wave can move in the same direction.

Let us now consider magneto-hydrodynamic waves. Repeating the previous discussion and using the evolutionarity condition we can easily show that in that case a single source cannot produce two fast or two slow shock (or self-similar) waves moving in the same direction (Akhiezer, Lyubarskiĭ, and Polovin, 1959).

As to waves of different types, Alfvén discontinuities overtake a slow (shock or self-similar) wave, and a fast shock wave overtakes all kinds of discontinuities or self-similar waves. Finally, a shock wave overtakes a self-similar wave of the same type as the shock wave or of a slower type; on the other hand, a self-similar wave overtakes a shock wave of the same type or a shock wave of a slower type. Thus, if magneto-hydrodynamic waves are formed in a single source not more than three waves can move in the same direction: in front a fast (shock or self-similar) wave, behind it follows an Alfvén discontinuity and, finally, behind them there moves a slow (shock or self-similar) wave. Of course, some of the waves enumerated here may not be present.

When two magneto-hydrodynamical shock waves collide a discontinuity is formed in which the boundary conditions are not satisfied. Therefore, such a discontinuity splits into a number of discontinuities or self-similar waves (see Subsection 3.4.4). It follows from the foregoing that the number of waves which is formed in that case equals seven: three waves (fast, Alfvén, and slow waves) moving to the right, also three waves moving to the left, and in between a contact discontinuity, at rest relative to the medium. The fast and slow waves can be either shock waves or self-similar waves.

When elucidating the picture of the splitting-up of a discontinuity we can study the problem of transitions between various magneto-hydrodynamical discontinuities when the external parameters (such as the velocity or the magnetic field) change. It follows, first of all, from the evolutionarity conditions that a continuous transition from a shock wave into an Alfvén discontinuity is impossible. Indeed, an Alfvén discontinuity can coincide with a shock wave only when the magnetic field on both sides of the discontinuity lies in a plane through the normal, that is, if the magnetic field in the Alfvén discontinuity turns through 180°. However, in such a discontinuity the transverse component of the magnetic field changes sign and we saw in the preceding subsection that this contradicts the evolutionarity condition for a shock wave.†

A continuous transition between fast and slow shock waves is also impossible. This

† Neglect of the evolutionarity conditions leads to an incorrect picture of transitions between magneto-hydrodynamical discontinuities (Syrovatskiĭ, 1956).

follows from the fact that the evolutionarity regions for fast and slow shock waves have no points of contact (see Fig. 3.3.3).

A fast shock wave cannot continuously change into a tangential discontinuity as this would violate the evolutionarity condition $u_{1z} > v_{1+}$.

The only transitions possible are thus between tangential and contact discontinuities, between tangential and Alfvén discontinuities, and between a tangential discontinuity and a slow shock wave. These discontinuities change to a tangential discontinuity in the limiting case as $B_z \to 0$.

The meaning of the transitions between magneto-hydrodynamical discontinuities becomes clearer when we consider the problem of the splitting-up of a discontinuity when the initial conditions are such that the conservation laws are not satisfied (Polovin, 1961c). If in that case the normal magnetic field B_z is non-vanishing, the discontinuity splits into seven waves, each of which is characterized by one parameter.

If, however, the normal magnetic field B_z vanishes, the initial discontinuity in the $z = 0$ plane splits into three waves: a fast shock wave moving to the right, a fast shock wave moving to the left, and a tangential discontinuity in between. Each of the shock waves is characterized by a single parameter, and the tangential discontinuity by five parameters. The total number of parameters equals seven, that is, the number of discontinuities of the magneto-hydrodynamical quantities in the original discontinuity.

The tangential discontinuity is thus a merger of five discontinuities: two slow shock waves, two Alfvén discontinuities, and a contact discontinuity. We give in Fig. 3.4.1 the scheme of the transitions between magneto-hydrodynamic discontinuities.

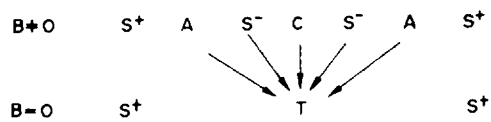

FIG. 3.4.1. Scheme of transitions between magneto-hydrodynamic discontinuities. S^+, S^-, A, C, and T indicate, respectively, fast and slow shock waves, Alfvén, contact, and tangential discontinuities.

We shall now consider waves accompanying exothermic and endothermic discontinuities. In ordinary hydrodynamics there can be no waves moving in front of supersonic and behind subsonic combustion waves. At the same time a shock wave or a self-similar wave can move behind a supersonic combustion wave. Exactly similarly a shock or self-similar wave can move in front of a subsonic combustion wave.

We shall now determine what kind of magneto-hydrodynamic waves can accompany detonation and ionization waves (Demutskiĭ and Polovin, 1961). As the evolutionarity conditions for detonation waves and supercompressed detonations are the same as the evolutionarity conditions for shock waves, a fast ionization wave or supercompressed detonation cannot be accompanied by a fast shock wave or a fast self-similar wave—and analogously for slow waves. Therefore, when there is ionization in a shock wave or supercompressed detonation, not more than three waves can move in the same direction.

If ionization or supercompressed detonation occurs in a fast wave the sequence of the waves will be as follows: in front will move the supercompressed detonation (or ionization) wave, behind it an Alfvén discontinuity and, finally, a slow (shock or self-similar) wave.

In the case of Chapman–Jouguet detonation the velocity of the wave relative to the reaction products is equal to the velocity of propagation of small perturbations so that behind a fast detonation wave in the Chapman–Jouguet regime there can follow a fast self-similar wave—and analogously for the case of a slow wave. Four waves can thus move in the same direction.

If the Chapman–Jouguet detonation occurs in a fast wave, the sequence of the waves will be the following: in front moves a fast detonation wave in the Chapman–Jouguet regime, behind it a fast (self-similar) rarefaction wave, then an Alfvén discontinuity and finally a slow (shock or self-similar) wave. Some of the waves enumerated here may not be present. The presence or absence of these waves is determined by the boundary conditions (see Subsection 3.4.5).

We now turn to a determination of the possible kinds of magneto-hydrodynamic waves which can accompany combustion waves (Polovin and Demutskiĭ, 1961). It follows from the evolutionarity conditions that a fast combustion wave moves in front of all kinds of magneto-hydrodynamic waves, a super-Alfvénic combustion wave moves between the fast (shock or self-similar) and the Alfvén waves, a sub-Alfvénic combustion wave between Alfvén and slow waves and, finally, a slow combustion wave moves behind all magneto-hydrodynamic waves. For instance, in slow combustion we may have a fast (shock or self-similar) wave moving in front, behind it an Alfvén wave, then a slow (shock or self-similar) wave, and finally a slow combustion wave.

3.4.3. THE PISTON PROBLEM

We have noted earlier that a parallel magneto-hydrodynamic shock wave, for which the magnetic field on both sides of the discontinuity plane is parallel to the normal, is non-evolutionary and thus cannot be realized in a number of cases. In order to elucidate the fate of a non-evolutionary shock wave we shall consider the following concrete problem: the piston problem. Let the half-space $z > 0$ be filled with a perfectly conducting medium which is in a magnetic field B_z ($B_x = B_y = 0$) and which is at rest at $t = 0$. The state of the medium is characterized by the density ϱ_0 and the pressure p_0. To the left the medium is bounded by a perfectly conducting piston which lies in the $z = 0$ plane. At $t = 0$ the piston starts to move with a constant velocity $\boldsymbol{u}$. We must now find the state of the medium at $t > 0$.

We first of all consider the case when the piston moves along the z-axis which coincides with the normal to its surface and with the direction of the magnetic field. There will then arise in front of the piston a shock wave which will be a parallel one, that is, the same as in ordinary hydrodynamics. We shall first of all find out whether this solution is unique, that is, whether it would not be possible that there would move in front of the piston a number of magneto-hydrodynamic waves rather than only the parallel shock wave. It is clear that these waves will move in the same direction as the original wave.

As the piston is perfectly conducting, the magnetic field lines are frozen-in in the piston, that is, the transverse magnetic field component B_x vanishes at its surface. The quantity B_x must thus vanish in front of a fast wave if it is formed as well as behind a slow wave. On the other hand, according to Table 3.4.1, the transverse magnetic field decreases in a fast self-similar wave and increases in a slow one. Neither the slow nor the fast waves can therefore be self-similar waves and hence must be shock waves. As B_x vanishes in front of a fast and behind a slow peculiar shock wave, a parallel wave can split only into two peculiar shock waves. (The direction of the transverse magnetic field in the region between two peculiar shock waves coincides with the direction of the fluctuating transverse magnetic field in front of a fast peculiar wave (Cherkasova, 1965).) On the other hand, the conditions for the existence of a peculiar shock wave, (3.3.4.10) and (3.3.4.11), are the same as the non-evolutionarity conditions (3.3.4.7) for a parallel shock wave. The evolutionarity conditions for a parallel shock wave thus coincide with the conditions for its stability relative to its splitting-up—into two peculiar shock waves.

Let us now determine the dependence of the various magneto-hydrodynamic quantities on the piston velocity u (Polovin, 1960). Let the piston velocity be such that a parallel shock wave arising in front of it does not split up. If the velocity of the shock wave in the unperturbed medium equals U, the velocities u_{0z} and u_{2z} of the medium on both sides of the shock wave in the frame of reference in which the wave is at rest are equal to

$$u_{0z} = -U, \quad u_{2z} = u-U \tag{3.4.3.1}$$

(behind the shock wave the velocity of the medium in the laboratory frame is equal to the piston velocity).

Substituting eqns. (3.4.3.1) into the last eqn. (3.3.4.5) and replacing c_{1s}, by c_{0s}, where c_{0s} is the sound velocity in the region 0 in front of the wave, we find the velocity of the shock wave:

$$U = \tfrac{2}{3}u + \sqrt{\{(\tfrac{2}{3}u)^2 + c_{0s}^2\}}. \tag{3.4.3.2}$$

We have assumed here that the medium is described by the equation of state of a perfect gas with adiabatic index $\gamma = 5/3$.

We can determine the compression of the medium in the shock wave from the first two eqns. (3.3.4.5),

$$\frac{\varrho_2}{\varrho_1} = \frac{U}{U-u}, \tag{3.4.3.3}$$

as well as the pressure behind the shock wave

$$p_2 = p_0 + \varrho_0 u U. \tag{3.4.3.4}$$

The values of the velocity of the medium $|u_{0z} - U| = U$ relative to the shock wave, which are equal to v_{0Az} and $\sqrt{(4v_{0Az}^2 - 3c_{0s}^2)}$, at which there is a transition from a parallel shock wave to a peculiar shock wave, correspond to two critical piston velocities:

$$u_- = \frac{3}{4}\,\frac{v_{0Az}^2 - c_{0s}^2}{v_{0Az}}, \quad u_+ = 3\,\frac{v_{0Az}^2 - c_{0s}^2}{\sqrt{(4v_{0Az}^2 - 3c_{0s}^2)}}. \tag{3.4.3.5}$$

It is clear from Fig. 3.3.7 that one must take the parallel shock wave to be a slow wave when $u < u_-$ and to be a fast wave when $u > u_+$. The jump in the velocity therefore takes place in a slow shock wave ($\Delta_- u_z = u$) when $u < u_-$, while the jump in velocity vanishes in the fast shock wave ($\Delta_+ u_z = 0$). Hence, when $u < u_-$ the velocity of the fast shock wave is equal to the propagation velocity of infinitesimally small perturbations while the velocity of the slow shock wave is determined by eqn. (3.4.3.2) (see Fig. 3.4.2):

$$U_+ = v_{0Az}, \quad U_- = \tfrac{2}{3}u + \sqrt{\{(\tfrac{2}{3}u)^2 + c_{0s}^2\}}. \tag{3.4.3.6}$$

When $u > u_+$ the jump in velocity occurs in the fast shock wave and the velocity of the slow wave equals the propagation velocity of sonic perturbations of infintesimally small amplitude in a medium with pressure p_2 and density ϱ_2, moving with the piston velocity u:

$$\Delta_+ u_z = u, \quad \Delta_- u_z = 0, \quad U_+ = \frac{2}{3}u + \sqrt{\left\{\left(\frac{2}{3}u\right)^2 + c_{0s}^2\right\}}, \quad U = u + \sqrt{\left\{\frac{5}{3}\frac{p_2}{\varrho_2}\right\}}, \tag{3.4.3.7}$$

where p_2 and ϱ_2 are determined by the first two eqns. (3.3.4.5). The magnetic field B_{1x} between the fast and the slow peculiar waves is determined by eqn. (3.3.4.9) (see Fig. 3.4.3).

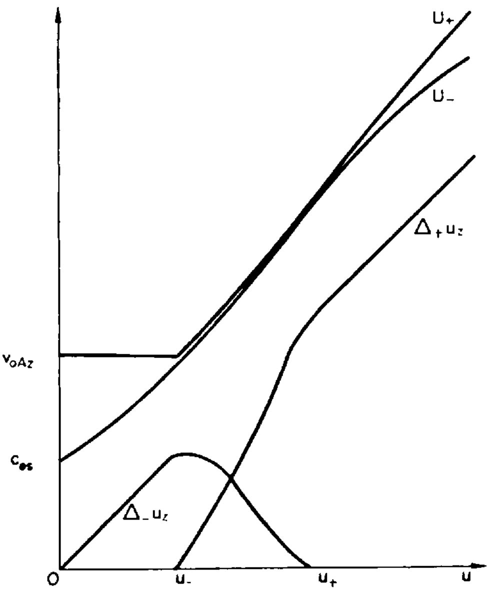

FIG. 3.4.2. Propagation velocities of shock waves formed when a piston moves along the magnetic field, and the jumps in the velocity of the medium in these waves. U_+ and U_- are the velocities of the fast and slow shock waves, $\Delta_+ u_z$ and $\Delta_- u_z$ the jumps in the velocity on the fast and the slow shock wave. The piston velocity u is plotted along the abscissa axis.

When the piston velocity changes from u_- to u_+ the amplitude of the fast shock wave $\Delta_+ u_z$ changes from zero to u_+, while the amplitude of the slow wave, $\Delta_- u_z$ changes from u_- to zero (see Fig. 3.4.2). The dependence of the compression of the medium ϱ_1/ϱ_0 and ϱ_2/ϱ_1 in the fast and the slow peculiar shock waves on the piston velocity is shown in Fig. 3.4.3.

Let us now consider the motion of the piston in the transverse direction, when the piston velocity u_x is at right angles to the normal to its surface and lies in the xz-plane which passes through the magnetic field vector and the normal to the piston surface. We shall first of all determine qualitatively the nature of the waves which then appear (Akhiezer and Polovin, 1960); the magnetic field can make an arbitrary angle with the plane of the piston.

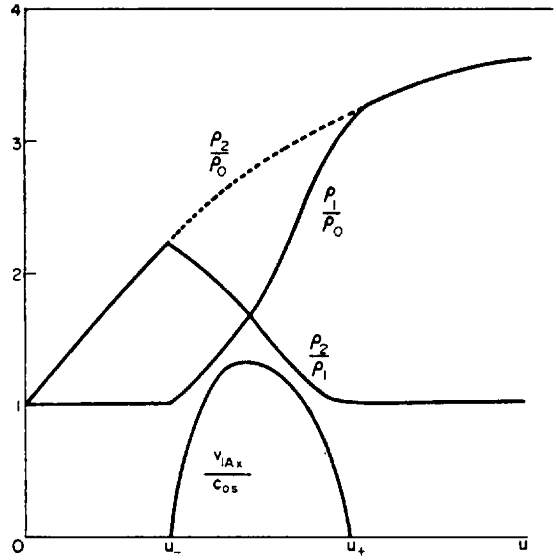

FIG. 3.4.3. Change in density and in magnetic field in the peculiar shock waves formed when a parallel shock wave splits up; ϱ_0, ϱ_1, and ϱ_2 are the densities of the unperturbed medium, the medium between the peculiar waves, and the medium behind both the peculiar waves. v_{1Ax}/c_{0s} is a dimensionless quantity characterizing the transverse magnetic field between the peculiar waves.

As the magnetic field lines are "glued" to the particles of the medium and to the piston, the magnetic field line will be deformed as shown in Fig. 3.4.4a when $u_x < 0$ (we have chosen the x-axis such that $B_x > 0$). The bending of the magnetic field line leads to the appearance of elastic tension forces which are directed in the direction where the field lines are concave.

When the piston moves, fast and slow (shock or self-similar) magneto-hydrodynamic waves will arise, which move away from the piston. As near the piston and at infinity $u_z' = 0$, there will be in front a compression (shock) wave and behind it a rarefaction (self-similar) wave (see Fig. 3.4.4a where these waves are indicated by S^+ and R^-) when the elastic force is directed as shown in Fig. 3.4.4a. There does not arise in this case an Alfvén wave, as B_x near the piston and at infinity has the same sign. We bear in mind that the sign of B_x does not change in shock or self-similar waves, but that it changes in a 180° Alfvén discontinuity.

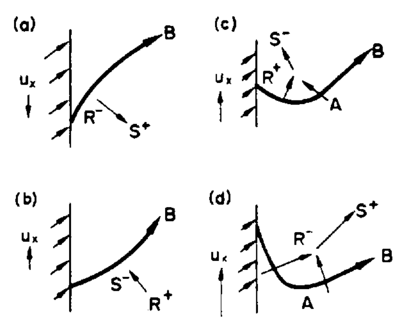

FIG. 3.4.4. Formation of a shock wave (S), Alfvén discontinuity (A), and rarefaction wave (R) due to the tension of the magnetic field lines when a piston moves in a transverse direction.

A similar discussion shows that in the case shown in Fig. 3.4.4b there appear a fast rarefaction and a slow shock wave (R^+, S^-).

When the piston velocity u_x increases the sign of B_x near the piston changes and this leads to the appearance of an Alfvén wave as well as a fast rarefaction and a slow shock wave (R^+, A, S^-; see Fig. 3.4.4c). When u_x increases further the value of $|B_x|$ near the piston becomes larger than the value of B_x at infinity; the resultant tension force is then directed away from the piston (Fig. 3.3.4d) which corresponds to the appearance of the waves S^+, A, and R^-.

For sufficiently large values of $|u_x|$ the amplitude of the slow rarefaction wave becomes so large that the density of the medium behind the wave vanishes: *cavitation* sets in.

The picture of the motion of the medium when $u_z \neq 0$, $u_x \neq 0$ can be obtained by starting from the case when $u_z = 0$ which we have just considered. When u_z increases, the amplitude of the rarefaction wave decreases and that of the compression wave increases. For some critical value of u_z, which is a function of u_x, the rarefaction wave becomes a compression wave, that is, a shock wave. Similarly, a decrease in u_z leads to the change of a shock wave into a rarefaction wave. A further reduction in the magnitude of u_z leads to the appearance of cavitation.

We show in Fig. 3.4.5 for the case $u_y = 0$ what kind of waves can occur when the piston moves with an arbitrary velocity u_x, u_z. The topological picture of the different regions in this

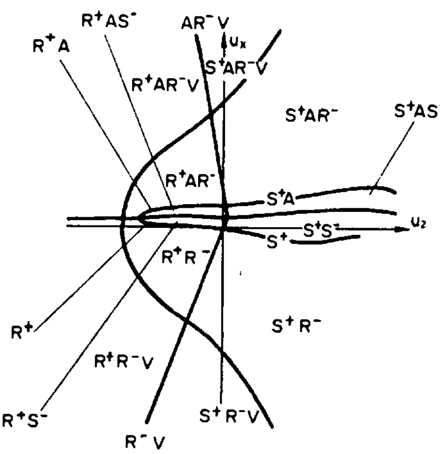

FIG. 3.4.5. Waves appearing when a piston is moved. The abscissa and the ordinate correspond, respectively, to the longitudinal and the transverse components of the piston velocity, u_z and u_x. The letters S^+, S^-, R^+, R^-, A and V indicate, respectively, the presence of fast and slow shock waves, fast and slow (self-similar) rarefaction waves, an Alfvén discontinuity, and the formation of a vacuum (cavitation).

figure is independent of the magnitude of the magnetic field (Akhiezer and Polovin, 1960). When B_x and B_z change there only occurs a deformation of the lines which bound the different regions. Figure 3.4.5 corresponds to the case when the Alfvén velocity is appreciably less than the sound velocity (Polovin, 1960). The four cases depicted in Fig. 3.4.4 correspond to points on the ordinate axis in Fig. 3.4.5.

As an example we shall determine the moment when for the case of a transverse motion of the piston cavitation sets in (Bazer, 1958), assuming that the transverse component of the magnetic field B_x vanishes. In that case the fast wave will only be a shock wave and the slow wave will thus be a self-similar rarefaction wave.

One sees easily that in that case the fast shock wave has an infinitesimally small amplitude. Indeed, the density and temperature increase in the shock wave and the condition $v_A \ll c_s$ will thus be valid also in the region between the fast shock wave and the slow rarefaction wave,

$$v_{1A} \ll c_{1s}.$$

It follows from eqns. (3.1.4.12) and (3.1.4.13) that the quantity Δu_z in the slow rarefaction wave can be neglected. As the longitudinal component u_z of the piston velocity vanishes this means that

$$u_{1z} = 0.$$

On the other hand, the velocity of the medium vanishes, $u_{0z} = 0$, in the unperturbed medium in front of the shock wave. The fast shock wave has thus, as we mentioned earlier, an infinitesimally small amplitude and can be neglected.

Using this fact we get from eqns. (3.1.4.13) and (3.1.4.15) the condition for cavitation, in the case when $\gamma = 5/3$ (Bazer, 1958):

$$|u| > 3.67c_{0s}.$$

So far the problem of the piston motion has only been solved qualitatively in the general case (Barmin and Gogosov, 1961; Gogosov, 1961a; Lyubarskiĭ and Polovin, 1960).

3.4.4. SPLITTING-UP OF A DISCONTINUITY

Let us now turn to a study of the splitting-up of a discontinuity in which the boundary conditions (3.2.1.2), (3.2.1.3), (3.2.1.5), (3.2.1.7), and (3.2.1.8) are not satisfied. This problem is the first step on the path to a proof of the existence and uniqueness of the solution of the Cauchy problem in the class of evolutionary discontinuities (Rozhdestvenskiĭ, 1960; Gel'fand 1959). The problem of the collision of magneto-hydrodynamic waves with one another and with a wall (Gogosov, 1961b, 1962a; Grebenshchikov, Raĭzer, Rukhadze, and Frank, 1961) also reduces to the problem of the splitting-up of discontinuities.

If the values of the magneto-hydrodynamical quantities are constant (in space and time) on both sides of the discontinuity, the waves appearing in the splitting-up of the discontinuity can only be discontinuities or self-similar waves.

Let us first consider the splitting-up of a small intensity discontinuity (Lax, 1957). In the first approximation in the amplitudes of the waves formed, the relations between the jumps in the magneto-hydrodynamical quantities are the same in the shock and the self-similar waves. The difference between a shock wave and a self-similar wave consists in that approximation only in the fact that the density increases in a shock wave and decreases in a self-similar wave.

If we denote by Δa_i the initial jumps in the magneto-hydrodynamical quantities and by $\Delta_j a_i$ the jumps in the quantities a_i in the jth type wave which is formed, we have, clearly,

$$\Delta a_i = \sum_{j=1}^{7} \Delta_j a_i \tag{3.4.4.1}$$

(by Δa_i we understand in this formula the difference $a_i(z+0)-a_i(z-0)$; the z-axis is along the normal to the initial discontinuity).

According to (2.1.1.7) we can write the jump $\Delta_j a_i$ in the form

$$\Delta_j a_i = \eta_j r_i^{(j)}, \tag{3.4.4.2}$$

where η_j is the amplitude of the jth wave and $r^{(j)}$ the column eigenvector corresponding to the jth wave. We have thus

$$\sum_{j=1}^{7} \eta_j r_i^{(j)} = \Delta a_i. \tag{3.4.4.3}$$

As the vectors $r^{(1)}$, $r^{(2)}$, ..., $r^{(7)}$ are linearly independent we can uniquely determine all amplitudes η_j from the set of eqns. (3.4.4.3).

As we consider a small intensity discontinuity, the sign of the quantity η_j determines the character of the jth wave: compression waves are shock waves and rarefaction waves are self-similar waves.

We shall give now the expressions for the jumps in the density in magneto-sound waves (Lyubarskiĭ and Polovin, 1959c):

$$\Delta_{\pm}^{(\varepsilon)}\varrho = \pm\frac{1}{2R}\left\{\varepsilon\frac{c_s^2 v_{\mathrm{At}}^2\left[\Delta\varrho-\left(\frac{\partial\varrho}{\partial s}\right)_p\Delta s\right]}{v_{\mp}^2-v_{\mathrm{A}}^2}-\varepsilon\frac{(\boldsymbol{B}_{\mathrm{t}}\cdot\Delta\boldsymbol{B}_{\mathrm{t}})}{4\pi}+\frac{\varrho v_{\mathrm{A}z}}{v_{\pm}}\left[\frac{(\boldsymbol{B}_{\mathrm{t}}\cdot\Delta\boldsymbol{u}_{\mathrm{t}})}{B_z}+\frac{v_{\mathrm{At}}^2\,\Delta u_z}{v_{\mp}^2-v_{\mathrm{A}z}^2}\right]\right\}. \tag{3.4.4.4}$$

Here

$$R = \sqrt{\{(v_{\mathrm{A}}^2+c_s^2)^2-4v_s^2v_{\mathrm{A}z}^2\}},$$

and $\varepsilon=+1$ if the wave propagates in the direction of the positive z-axis while $\varepsilon=-1$ if the wave propagates in the opposite direction. The upper (plus) and lower (minus) signs refer, respectively, to the fast and the slow magneto-sound waves. The index t is used to indicate the tangential components. A shock wave corresponds to $\Delta_{\pm}^{(\varepsilon)}\varrho > 0$ and a self-similar wave to $\Delta_{\pm}^{(\varepsilon)}\varrho < 0$.

For instance, when two gas masses with the same values of ϱ, s, $\boldsymbol{B}_{\mathrm{t}}$, and $\boldsymbol{u}_{\mathrm{t}}$ collide, we have

$$\Delta\varrho = \Delta s = \Delta\boldsymbol{B}_{\mathrm{t}} = \Delta\boldsymbol{u}_{\mathrm{t}} = 0, \quad \Delta u_z < 0.$$

Using the inequalities $v_+^2 > v_{\mathrm{A}z}^2$, $v_-^2 < v_{\mathrm{A}z}^2$, we get $\Delta_{\pm}^{(\varepsilon)}\varrho > 0$. This means that all four magneto-sound waves emerging from both sides will be shock waves. This result agrees with the fact that in magneto-hydrodynamics when a piston moves along the normal to its surface there are always formed two shock waves (Lyubarskiĭ and Polovin, 1960).

Equation (3.4.4.4) refers to the case of small amplitude discontinuities. The general problem has been studied qualitatively by Gogosov (1961c, 1962 b, c).†

† For the case of a perfect gas it has been proved (Gogosov, 1961c) that the solution of the problem of the splitting-up of a discontinuity of arbitrary intensity exists, but up to the present the proof of its uniqueness has not been given. Such a proof is based in ordinary hydrodynamics on the "convexity" of the set of equations (Rozhdestvenskiĭ, 1960).

So far we have assumed that the quantities Δa_i are independent. If, however, the quantities Δa_i are interconnected through the relations

$$\frac{\Delta a_1}{r_1^{(k)}} = \frac{\Delta a_2}{r_2^{(k)}} = \ldots = \frac{\Delta a_7}{r_7^{(k)}}, \tag{3.4.4.5}$$

which is true for a shock wave of the kth type, all terms, bar the one with $j = k$, will be absent from the superposition of waves (3.4.4.3). If, however, the quantities in the discontinuity do not satisfy relation (3.4.4.5) there will be a number of terms present in formula (3.4.4.3). This will mean that the shock wave is unstable against splitting-up. In particular, eqns. (3.4.4.5) will not be satisfied for a non-evolutionary shock wave and such a shock wave will therefore split up into a number of discontinuities or self-similar waves (Lyubarskiĭ and Polovin, 1959b; Polovin and Cherkasova, 1962b; Cherkasova, 1961).

3.4.5. THE CHAPMAN-JOUGUET THEOREM

We considered in Subsection 3.3.5 possible detonation, combustion, and ionization regimes in a shock wave without imposing well-defined boundary conditions, that is, essentially in an infinite medium. Because of that we obtained not one solution, but a whole family of solutions (Hugoniot lines). We shall now study different exothermic and endothermic discontinuitites occurring in a tube which is closed on one side by a perfectly conducting piston.

It is well known (Landau and Lifshitz, 1959) that if there occurs a detonation in a tube bounded by a non-moving wall, the velocity of the reaction products relative to the detonation front is equal to the local sound velocity (Chapman–Jouguet theorem).† We shall show that the Chapman–Jouguet theorem is also valid in magneto-hydrodynamics, where it is valid both for detonation (Polovin, 1965b) and for ionization (Taussig, 1967) shock waves; we assume that the tube is closed by a non-moving perfectly conducting wall.

As there are two shock waves—the fast and the slow one—in magneto-hydrodynamics, two detonation regimes are possible for which the velocities of the reaction products, relative to the detonation front, are equal to the velocities of the fast and the slow magneto-sound waves.

To prove the Chapman–Jouguet theorem we assume that the velocity of the reaction products is different from the velocity of the magneto-sound waves, that is, we assume that the detonation is supercompressed. We have shown in Subsection 3.4.2 that in that case not more than three waves—a fast wave, an Alfvén discontinuity, and a slow wave—can move in the same direction (in this case away from the wall); the fast and the slow wave can here be either supercompressed detonation waves, or shock waves, or self-similar waves. If the detonation occurred in the Chapman–Jouguet regime four waves could propagate in the same direction, for instance, a fast detonation wave, a fast self-similar wave, an Alfvén discontinuity and, finally, a slow shock or self-similar wave.

† This theorem was proved by Zel'dovich (1940; see also von Neumann, 1943; Döring, 1943; Erpenbeck, 1961; Soloukhin, 1960; Hirschfelder and Curtiss, 1958; Linder, Curtiss, and Hirschfelder, 1958; Penney, 1951; Ribaud, 1959; Adamson, 1960).

We shall first show that in the supercompressed detonation regime and in the case of a wall which is at rest the Alfvén discontinuity cannot be realized. To fix ideas we shall assume that the supercompressed detonation occurs in a fast wave. We shall indicate the magneto-hydrodynamic quantities in front of the fast wave, between the fast and the Alfvén wave, between the Alfvén and the slow wave, and, finally, between the slow wave and the wall by the indexes 0, 1, 2, and 3, respectively. As in front of the fast wave the medium is at rest, we have $\boldsymbol{u}_0 = 0$. From the "glueing" of the magnetic field lines to the particles and to the perfectly conducting wall it follows that $\boldsymbol{u}_3 = 0$.

We first of all determine the relation between the magneto-hydrodynamic quantities in the regions 1 and 2 which adjoin the Alfvén discontinuity. As the direction of the magnetic field $\boldsymbol{B}_3$ near the wall is not given, we can assume without losing generality that the Alfvén discontinuity turns the magnetic field over 180°.

From the second boundary condition (3.2.1.8) we find

$$u_{2x} - u_{1x} = -(v_{2Ax} - v_{1Ax}),$$

where the wave moves in the direction of the positive z-axis while the x-axis is directed along the tangential component $\boldsymbol{B}_{1t}$ of the magnetic field. We choose this axis in such a way that $B_{1x} > 0$ and, using the relation $v_{2Ax} = -v_{1Ax}$, we find (Polovin, 1965b)

$$\left(\frac{u_{1x}}{v_{1Ax}} + 1\right) + \left(\frac{u_{2x}}{v_{2Ax}} + 1\right) = 0. \tag{3.4.5.1}$$

However, this equation can not be satisfied as both expressions in brackets are positive. Indeed, let us, for instance, assume that the fast wave is a supercompressed detonation wave and the slow one a self-similar wave. We then find from the boundary condition

$$-\varrho_1 |u_{1z} - U_+| u_{1x} = \frac{B_z(B_{1x} - B_{0x})}{4\pi},$$

where U_+ is the propagation velocity of the fast detonation wave in the laboratory frame (to fix the ideas we assume that $B_z > 0$) and from the evolutionarity condition

$$|u_{1z} - U_+| > v_{1Az}$$

that

$$-1 < \frac{u_{1x}}{v_{1Ax}} < 0, \tag{3.4.5.2}$$

that is, the first bracket in eqn. (3.4.5.1) is positive. For a slow self-similar wave it follows from the equation $\boldsymbol{u}_3 = 0$ and Table 3.4.1 that

$$\frac{u_{2x}}{v_{2Ax}} > 0,$$

that is, the second bracket in eqn. (3.4.5.1) is also positive. An Alfvén discontinuity is thus impossible.

One can similarly prove the impossibility of an Alfvén discontinuity in those cases where a slow shock wave follows behind a fast supercompressed detonation wave. Hence either a slow shock wave or a slow self-similar wave must follow behind a fast supercompressed detonation wave.

However, a slow self-similar wave is also impossible. Indeed, according to (3.4.5.2) the quantity u_{1x}/v_{1Ax} must be negative in the region 1, which lies between the fast and the slow waves, that is, behind the fast detonation wave. If we take the direction of the x-axis such that $v_{1Ax} > 0$ and use Table 3.4.1 we find that behind the slow self-similar wave the inequality $u_x > 0$ must hold. This violates, however, the boundary condition $u_x = 0$ at the perfectly conducting wall which is at rest.

On the other hand, a slow shock wave cannot follow behind a fast supercompressed detonation wave. This is immediately clear from Table 3.4.1: the longitudinal component u_z of the velocity increases in shock and detonation waves, while at the same time $u_z = 0$ in the unperturbed region and at the surface of the wall.

It is clear that a single supercompressed detonation wave, unaccompanied by other magneto-hydrodynamic waves, is also impossible.

In the case where a tube is bounded on one side by a perfectly conducting wall the detonation therefore always proceeds in the Chapman–Jouguet regime. In other words, the velocity of the reaction products relative to the detonation front is equal to the phase velocity of the propagation of small perturbations—in the direction of the normal to the surface of the discontinuity:

$$|u_z - U_\pm| = v_\pm \,. \tag{3.4.5.3}$$

We emphasize that the conclusion we have reached about the impossibility of a supercompressed detonation is valid only for the case of a wall at rest. When the wall is moving (a magneto-hydrodynamical piston) the detonation can also be supercompressed. This depends on the longitudinal and transverse components (u_z and u_x) of the piston velocity

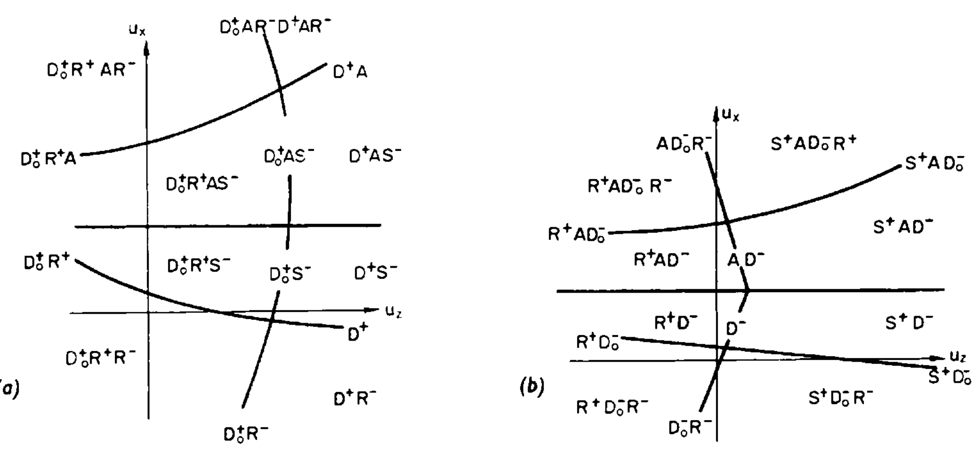

FIG. 3.4.6. Magneto-hydrodynamic waves accompanying a detonation. (a) Detonation in a fast wave; (b) detonation in a slow wave.

(Demutskiĭ and Polovin, 1961). We show in Fig. 3.4.6 the different possibilities in that case: the letters S, R, A, and D indicate, respectively, the presence of a shock wave, of a (self-similar) rarefaction wave, of an Alfvén discontinuity, and of a detonation wave; the plus-sign refers to a fast and the minus-sign to a slow wave, the index zero of the letter D indicates that the detonation occurs in the Chapman–Jouguet regime and the absence of that index the presence of supercompressed detonation. We note that a supercompressed detonation in a fast wave occurs only if the piston moves sufficiently fast ($u_z > 0$). A supercompressed detonation in a slow wave occurs if the piston velocity has the same direction as the magnetic field in the unperturbed medium ($u_x > 0$) and lies within certain finite limits. These limits correspond to a sufficiently large intensity of the slow shock wave which forms the detonation wave when merging with a combustion wave.

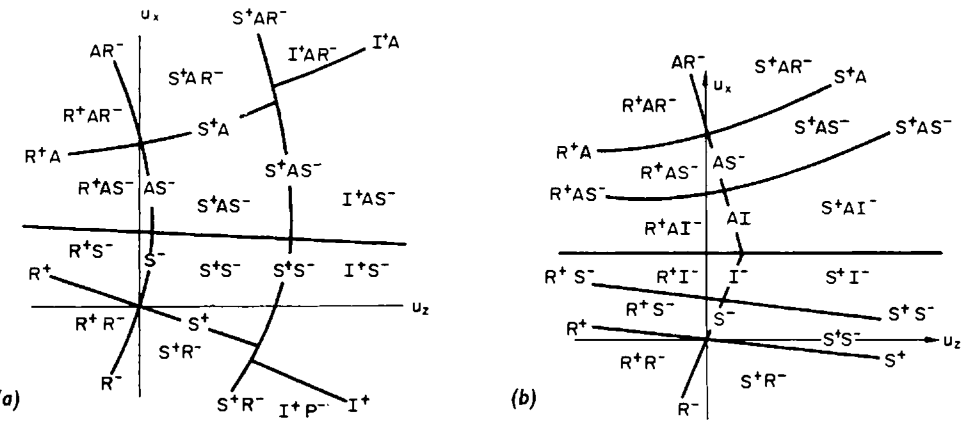

FIG. 3.4.7. Magneto-hydrodynamical waves accompanying ionization in a shock wave. (a) Ionization in a fast shock wave; (b) ionization in a slow shock wave.

We show in Fig. 3.4.7 the magneto-hydrodynamic waves accompanying ionization (indicated by the letter I) in a fast or a slow shock wave in dependence on the piston velocity u_z, u_x (Demutskiĭ and Polovin, 1961). As ionization is possible only for a sufficiently large shock wave amplitude, ionization in a fast wave occurs when u_z is sufficiently large and ionization in a slow wave when u_x lies between well-defined finite limits.

In Fig. 3.4.8 we show the magneto-hydrodynamic waves (Polovin and Demutskiĭ, 1961) which accompany different combustion waves (indicated by C).

3.4.6. OBLIQUE SHOCK WAVES

We showed in Subsection 3.3.6 that apart from the evolutionarity conditions another condition, the sweeping-out condition, must be satisfied for oblique shock waves which are attached to the edge of a wedge around which the medium is flowing. However, we did not obtain in Subsection 3.3.6 the consequences of the evolutionarity conditions, in particular, we did not prove that in the case when the velocity of the medium is parallel to the magnetic field the fast shock waves are always directed downstream and the slow shock waves upstream; moreover, we did not show that not more than one shock wave can be attached to the

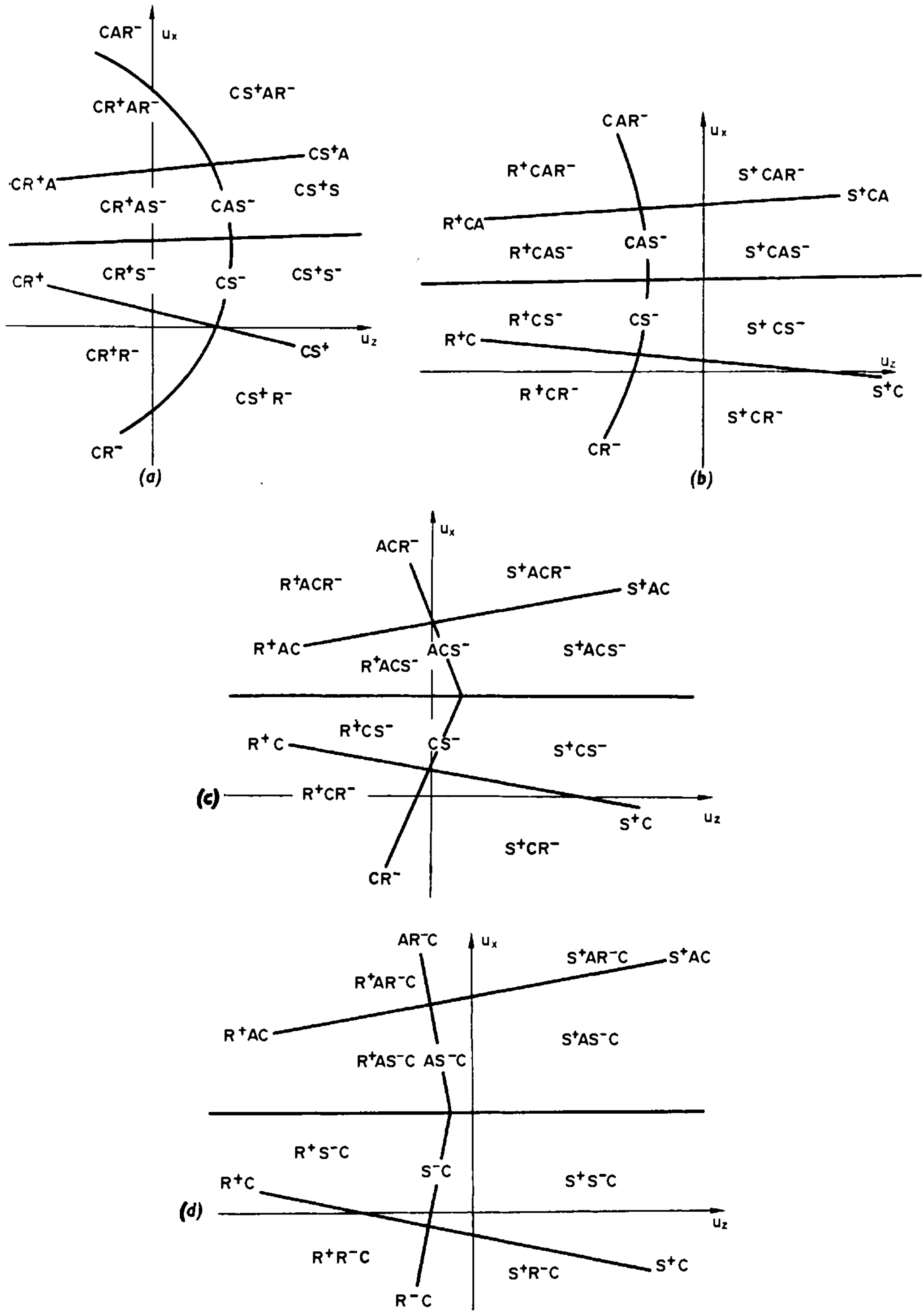

FIG. 3.4.8. Magneto-hydrodynamic waves accompanying different combustion regimes. (a) Fast combustion; (b) super-Alfvénic combustion; (c) sub-Alfvénic combustion; (d) slow combustion.

edge of a wedge around which the medium flows. The present subsection is devoted to a proof of these statements.

We first of all make clear at which angle to a body the oblique magneto-hydrodynamic shock waves are moving when they flow around that body.† It follows from the first boundary condition (3.2.1.8) at the surface of the shock wave that when the medium moves through the shock wave, that is, in a frame of reference fixed in the shock wave the normal component of the magnetic field remains unchanged. On the other hand, it is clear from Table 3.4.1 that the tangential component of the magnetic field increases in a fast shock wave and decreases in a slow one. Moreover, we showed in Subsection 3.4.1 that the tangential component of the magnetic field does not change sign when the medium passes through a shock wave. Therefore, the angle between the magnetic field and the normal to the discontinuity surface increases in fast shock waves and decreases in slow ones. As the velocity vector is parallel to the magnetic field, it is refracted in the same way in a shock wave. If the flow is around a concave angle, the discontinuity line can thus only lie in the sector shown in Fig. 3.4.9 (Polovin and Cherkasova, 1966b). It is clear from that figure that the fast shock wave is directed downstream and the slow one upstream (Kogan, 1959b). The limits of the regions

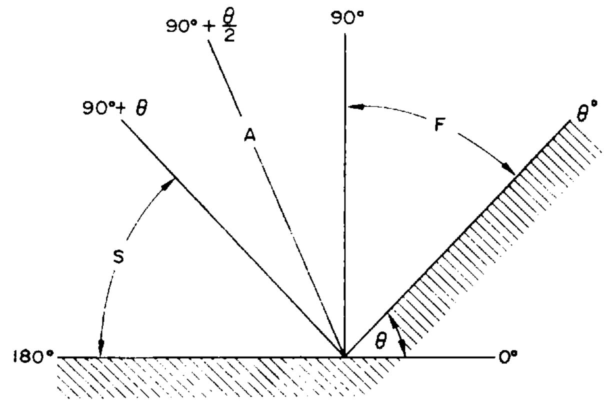

FIG. 3.4.9. Sectors in which a discontinuity line is possible for flow around a concave angle. F: sector in which a fast shock wave is possible; S: sector in which a slow shock wave is possible; A: direction of the Alfvén discontinuity. The medium moves in the unhatched region from left to right and the velocity vector is parallel to the magnetic field.

F and S—corresponding to inclinations of the discontinuity lines of 90° and $90°+\theta$, where θ is the angle over which the velocity turns—are reached in the fast and slow peculiar shock waves. We note that it has no sense to consider an Alfvén discontinuity in the case of stationary flow, as this discontinuity has no stationary structure.

Let us now consider a few oblique shock waves attached to a single edge. It follows from the inequalities (3.3.6.4) and (3.3.6.5) that we cannot have a slow wave behind a fast wave. Similarly we find that there cannot be a fast wave behind a slow one. We see thus that shock waves of different kinds cannot be attached to a single edge.

† Kogan (1959b; see also Kiselev and Kolosnitsyn, 1960; Cabannes, 1957, 1960 a, b; Kogan, 1960c; Tamada, 1962; Geffen, 1963) has studied magneto-hydrodynamic shock waves. Taking the evolutionarity conditions into account considerably restricts the region of possible solutions (Kogan, 1960b, 1962; Cabannes, 1963).

We shall now show that it is also impossible to attach several shock waves of the same kind to a single edge (Polovin, 1965d; Walsh, Shreffler, and Willig, 1953). Let us, for example, consider two fast shock waves OM and ON, attached to the vertex O (see Fig. 3.4.10). Let $\overrightarrow{AB}$ be the velocity vector in region 2 which lies between the two shock waves and $\overrightarrow{AC}$ and $\overrightarrow{AD}$ be the components of the velocity $\overrightarrow{AB}$, which are normal to, respectively, the shock waves

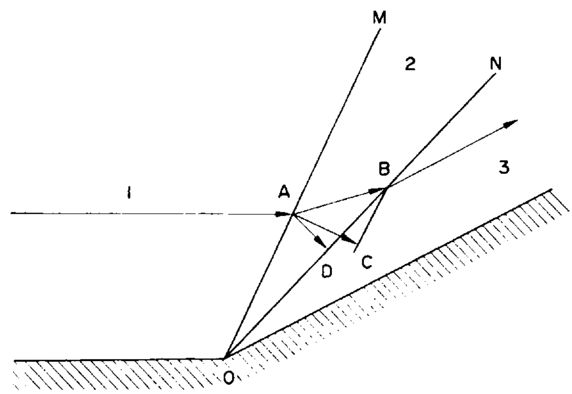

FIG. 3.4.10. Impossibility of the existence of two fast magneto-hydrodynamic shock waves, attached to the same vertex.

OM and ON. As AC = AB cos BAC, AD = AB cos BAD and the angle BAC is smaller than the angle BAD, we have AC > AD. On the other hand, AC is the normal component of the velocity of the medium behind the fast shock wave OM and it follows from the evolutionarity conditions that its magnitude must be less than the phase velocity v_+(AC) of the fast magneto-sound wave in the direction of AC, that is AC < v_+(AC). Furthermore, AD is the normal component of the velocity of the medium in front of the fast shock wave ON. It must thus be larger than the velocity of the fast magneto-sound wave, that is, AD > v_+(AD). Since, according to (2.1.1.10′), the phase velocity of the fast magneto-sound wave increases with increasing angle between the direction of the wave propagation and the direction of the magnetic field, we have v_+(AD) > v_+(AC). Combining this inequality with the two earlier ones we get AC < AD, which contradicts the earlier obtained inequality. One can similarly prove that it is impossible that two slow oblique shock waves can be attached to the same vertex (Polovin, 1965d).

We note that the discussion given here refers only to the case of two-dimensional flow around a perfectly conducting body when the velocity of the medium is parallel to the magnetic field. In the case of two-dimensional flow around non-conducting bodies the velocity of the medium cannot be parallel to the magnetic field and in that case it is possible that two shock waves can be attached to the vertex of the angle around which the flow takes place (Mimura, 1963). This is in accordance with the position of the characteristics noted in Subsection 2.2.3. Indeed, the line of a discontinuity of infinitesimally small intensity coincides with a characteristic. Therefore, if we consider oblique shock waves with sufficiently small intensity, two shock waves—a fast and a slow one, or two slow ones—can be attached to a single vertex when we consider flow around a non-conducting body; moreover, in a number of cases the fast shock wave may be directed upstream.

CHAPTER 4

High-frequency Oscillations in an Unmagnetized Plasma

4.1. Hydrodynamical Theory of High-frequency Oscillations of an Unmagnetized Plasma

4.1.1. ELECTROMAGNETIC WAVES IN A PLASMA

In the preceding two chapters we studied low-frequency magneto-hydrodynamic waves with frequencies which were low compared with the particle-collision frequency. We now turn to a study of waves which propagate in a plasma with frequencies high compared with the frequency of binary collisions between electrons and ions. In that case we can in general neglect the binary collisions, that is, start from the concept of a collisionless plasma. In the present chapter we shall study the oscillations of an unmagnetized collisionless plasma, that is, a collisionless plasma which is not acted upon by external fields.

We shall begin with a consideration of electromagnetic waves in a "cold" plasma, when we can neglect the influence of the thermal motion of the particles on the wave propagation. To do this it is necessary that the phase velocity of the waves is appreciably larger than the average thermal velocity of the particles. The state of the plasma can in this case be described hydrodynamically, by giving the average particle velocities and densities. When studying the propagation of electromagnetic waves we shall neglect the influence of the ions and consider the plasma to be a "cold" electron gas.

The electron velocity $\boldsymbol{u}_e \equiv \boldsymbol{u}_e(\boldsymbol{r}, t)$ and their density $n_e \equiv n_e(\boldsymbol{r}, t)$ satisfy the continuity equation,

$$\frac{\partial n_e}{\partial t} + \operatorname{div} n_e \boldsymbol{u}_e = 0, \tag{4.1.1.1}$$

and the equation of motion,

$$\frac{d_e \boldsymbol{u}_e}{dt} = -\frac{e}{m_e}\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{u}_e \wedge \boldsymbol{B}]\right\}, \tag{4.1.1.2}$$

where m_e and $-e$ are the electron mass and charge,

$$\frac{d_e}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{u}_e \cdot \nabla),$$

and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electrical and magnetic field of the propagating wave; we assume the plasma to be non-relativistic, that is, $u_e \ll c$. The fields satisfy the Maxwell equations:

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \quad \operatorname{div} \boldsymbol{B} = 0, \quad \operatorname{curl} \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} + \frac{4\pi}{c}\boldsymbol{j}, \quad \operatorname{div} \boldsymbol{E} = 4\pi\varrho, \tag{4.1.1.3}$$

where j and ϱ are the current and charge densities produced by the electrons,

$$j = j_e = -en_e u_e, \quad \varrho = -e(n_e - n_0), \tag{4.1.1.4}$$

with n_0 the equilibrium electron density which equals the ion density.

As the magnetic permeability of the plasma differs little from unity we shall in what follows not distinguish between the vectors $\boldsymbol{B}$ and $\boldsymbol{H}$.

Equations (4.1.1.1) to (4.1.1.4) form a complete set of equations describing the propagation of electromagnetic waves in a cold plasma in the hydrodynamic approximation.

We shall now assume that the amplitude of the electromagnetic wave is sufficiently small. In that case we can use the linearized equation of motion for the electrons:

$$\frac{\partial \boldsymbol{u}_e}{\partial t} = -\frac{e}{m_e}\boldsymbol{E}. \tag{4.1.1.5}$$

From this equation it follows for harmonic oscillations that

$$\boldsymbol{u}_e = -\frac{ie}{m_e\omega}\boldsymbol{E}, \tag{4.1.1.6}$$

where ω is the frequency of the oscillations; we take the time-dependence of all variables to be of the form $\propto e^{-i\omega t}$.

As $\boldsymbol{u}_e$ is a small quantity we can in the linear approximation use for the current density the expression

$$\boldsymbol{j} = -en_0\boldsymbol{u}_e \equiv \sigma\boldsymbol{E}, \tag{4.1.1.7}$$

where

$$\sigma \equiv \sigma(\omega) = \frac{ie^2 n_0}{m_e\omega}. \tag{4.1.1.8}$$

This quantity is called the *high-frequency plasma conductivity*.

We now introduce the electrical induction $\boldsymbol{D}$ through the equation

$$\frac{\partial \boldsymbol{D}}{\partial t} \equiv \frac{\partial \boldsymbol{E}}{\partial t} + 4\pi \boldsymbol{j}. \tag{4.1.1.8'}$$

For a small amplitude monochromatic wave we get from this the relation

$$\boldsymbol{D} = \varepsilon(\omega)\boldsymbol{E}, \tag{4.1.1.9}$$

where

$$\varepsilon(\omega) = 1 + \frac{4\pi i\sigma(\omega)}{\omega}. \tag{4.1.1.10}$$

This quantity is called the *plasma dielectric constant* or *plasma dielectric permittivity*.

Using (4.1.1.8) we get the following expression for the plasma dielectric constant:

$$\varepsilon(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}, \tag{4.1.1.11}$$

where

$$\omega_{pe} = \sqrt{\left(\frac{4\pi e^2 n_e}{m_e}\right)}.$$

This quantity which has the dimensions of a frequency is called the *plasma frequency* or the *Langmuir frequency*.

We note that in deriving eqn. (4.1.1.11) for the dielectric constant we did not make any assumptions about the homogeneity of the plasma. It is thus valid both for a homogeneous and for an inhomogeneous plasma. Hence it follows that the Maxwell equations,

$$\text{curl}\, \boldsymbol{E} = \frac{i\omega \boldsymbol{B}}{c}, \quad \text{curl}\, \boldsymbol{B} = -\frac{i\omega}{c}\varepsilon(\omega)\boldsymbol{E}, \tag{4.1.1.12}$$

describe the propagation of small amplitude monochromatic electromagnetic waves in an inhomogeneous cold plasma, when there is no external magnetic field.

The propagation of plane monochromatic waves,

$$\boldsymbol{E}, \boldsymbol{B} \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega t},$$

is possible in a homogeneous plasma. For such waves the magnetic field is, according to (4.1.1.12), connected as follows with the electrical field:

$$\boldsymbol{B} = \frac{c}{\omega}[\boldsymbol{k}\wedge\boldsymbol{E}], \tag{4.1.1.13}$$

that is, the magnetic field is at right angles to the electrical field and to the wave vector.

Substituting this expression into the second eqn. (4.1.1.12) we get

$$[\boldsymbol{k}\wedge[\boldsymbol{k}\wedge\boldsymbol{E}]]+\frac{\omega^2}{c^2}\varepsilon\boldsymbol{E} = 0,$$

or

$$\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})-\left(k^2-\frac{\omega^2}{c^2}\varepsilon\right)\boldsymbol{E} = 0. \tag{4.1.1.14}$$

Taking the scalar product of this equation with $\boldsymbol{k}$, we get

$$\varepsilon(\omega)(\boldsymbol{k}\cdot\boldsymbol{E}) = 0. \tag{4.1.1.15}$$

Therefore, if $\varepsilon(\omega) \neq 0$, we have $(\boldsymbol{k}\cdot\boldsymbol{E}) = 0$, that is, the electrical field in a plane monochromatic wave is at right angles to the direction of wave propagation. Using this fact, we get from (4.1.1.14)

$$k^2 = \frac{\omega^2}{c^2}\varepsilon(\omega). \tag{4.1.1.15'}$$

We see that electromagnetic waves can propagate in a plasma only if $\varepsilon(\omega) > 0$, that is, provided the frequency of the wave exceeds the plasma frequency,

$$\omega > \omega_{pe}.$$

The phase velocity of the wave,

$$v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon}}, \tag{4.1.1.16}$$

will then be larger than the velocity of light,

$$v_{ph} > c,$$

and the refractive index n,

$$n = \frac{c}{v_{ph}} = \sqrt{\varepsilon}$$

will be less than unity, $n < 1$.

According to (4.1.1.15′) the frequency of the wave ω will for a given wave vector be equal to

$$\omega(k) = \sqrt{(k^2c^2 + \omega_{pe}^2)}. \tag{4.1.1.17}$$

In Fig. 4.1.1 we show the wavenumber dependence of the frequency and in Fig. 4.1.2 the frequency dependence of the refractive index.

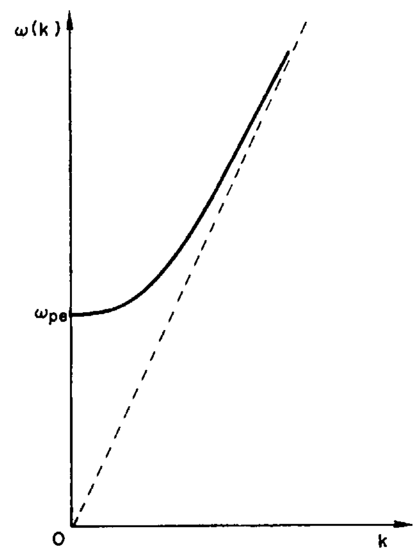

FIG. 4.1.1. Wavenumber dependence of the frequency of an electromagnetic wave in an isotropic plasma.

The dielectric constant is negative for frequencies below the plasma frequency and the wavevector, and hence also the index of refraction, will, according to (4.1.1.15′), be purely imaginary. This means that the wave will be damped exponentially when moving away from its sources, penetrating into the plasma to a distance

$$l = \frac{1}{|k|} = \frac{c}{\omega\sqrt{|\varepsilon(\omega)|}}.$$

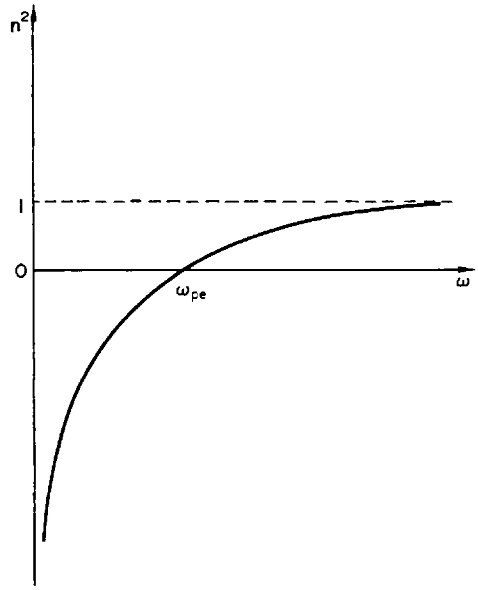

FIG. 4.1.2. Frequency dependence of the square of the refractive index of an electromagnetic wave.

When $\omega \ll \omega_{pe}$ the penetration depth is frequency-independent and equal to

$$l_0 = \frac{c}{\omega_{pe}};$$

this quantity is called the *skin depth*.

So far we have not touched upon the problem of changes in the electron density when an electromagnetic wave passes through the plasma. To elucidate this problem we turn to the equation of continuity and substitute into it

$$n_e = n_0 + n_e',$$

where n_0 is the equilibrium electron density and n_e' a small correction to it, $|n_e'| \ll n_0$. Linearizing the equation of continuity we get for the case of a plane monochromatic wave

$$n_e' = n_0 \frac{(\boldsymbol{k} \cdot \boldsymbol{u}_e)}{\omega},$$

or, after substituting for $\boldsymbol{u}_e$ expression (4.1.1.6),

$$n_e' = -\frac{ien_0}{m_e \omega^2} (\boldsymbol{k} \cdot \boldsymbol{E}). \tag{4.1.1.18}$$

This is a general formula which connects the varying part of the electron density with the varying electrical field in the plasma for the case when all varying quantities change as $e^{i(\boldsymbol{k} \cdot \boldsymbol{r}) - i\omega t}$.

When electromagnetic waves propagate through the plasma the electrical field of the wave is, as we saw, at right angles to the wavevector, that is, $(\boldsymbol{k} \cdot \boldsymbol{E}) = 0$, and hence $n_e' = 0$.

To conclude this subsection we shall estimate the contribution from the ions to the plasma dielectric constant. Assuming the ions to be cold, like the electrons, we can write down the linearized equations of motion for the ion velocity $\boldsymbol{u}_i$ which differ from eqns. (4.1.1.5) and (4.1.1.6) in the sign of the charge and in that the electron mass m_e must be replaced by the ion mass m_i. Knowing the total current density of electrons and ions, $\boldsymbol{j} = en_0(\boldsymbol{u}_i - \boldsymbol{u}_e)$, we can easily evaluate the plasma dielectric constant, using (4.1.1.10):

$$\varepsilon(\omega) = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{m_e}{m_i}\right). \qquad (4.1.1.19)$$

We see that the role of the ions is reduced to replacing the term $-\omega_{pe}^2/\omega^2$ by $-(\omega_{pe}^2/\omega^2) \times \{1 + (m_e/m_i)\}$ in the plasma dielectric constant. The difference between these terms is very small, as $m_e/m_i \ll 1$.

4.1.2. LANGMUIR OSCILLATIONS

Apart from the transverse electromagnetic oscillations one can also excite in the plasma quasi-electrostatic longitudinal oscillations which are connected with oscillations in the electron density. These oscillations are called *plasma oscillations* or *Langmuir oscillations*; the mechanism of their production is, as we saw in Subsection 1.1.1, closely connected with the problem of charge screening in a plasma. Indeed, if in the plasma a non-equilibrium charge distribution is produced through the displacement of a charged layer d (see Fig. 1.1.1) over a small distance x, this layer will act as a capacitor with a surface charge density $n_e ex$. There arises thus in the layer an electrical field $E = 4\pi e n_e x$ which tends to establish electrical neutrality and leads to the oscillations of the electrons.

We now turn to a detailed study of the longitudinal oscillations in a plasma. We start with a consideration of small oscillations of a cold plasma. We start in that case from the linearized continuity equation and equations of motion for the electrons,

$$\frac{\partial \boldsymbol{u}_e}{\partial t} = -\frac{e}{m_e}\boldsymbol{E}, \quad \frac{\partial n_e}{\partial t} + n_0 \operatorname{div} \boldsymbol{u}_e = 0, \qquad (4.1.2.1)$$

and also the equations of electrostatics,

$$\operatorname{curl} \boldsymbol{E} = 0, \quad \operatorname{div} \boldsymbol{E} = -4\pi e(n_e - n_0). \qquad (4.1.2.2)$$

One can easily establish the general time-dependence of all quantities occurring in these equations without making any assumptions about their dependence on the spatial coordinates. To do this we take the time-derivative of the equation of continuity and use the equation of motion. As a result we get

$$\frac{\partial^2 n_e'}{\partial t^2} - \frac{en_0}{m_e} \operatorname{div} \boldsymbol{E} = 0.$$

Using then the second eqn. (4.1.2.2) we are led to the following equation for the varying part of the electron density, $n_e' = n_e - n_0$:

$$\frac{\partial^2 n_e'}{\partial t^2} + \frac{4\pi e^2 n_0}{m_e} n_e' = 0. \qquad (4.1.2.3)$$

One sees easily that the electron velocity satisfies the same equation:

$$\frac{\partial^2 \boldsymbol{u}_e}{\partial t^2}+\frac{4\pi e^2 n_0}{m_e}\boldsymbol{u}_e = 0. \tag{4.1.2.4}$$

The general solution of eqn. (4.1.2.3) has clearly the form

$$n_e'(\boldsymbol{r}, t) = \nu_1(\boldsymbol{r})\cos\omega_{pe}t+\nu_2(\boldsymbol{r})\sin\omega_{pe}t, \tag{4.1.2.5}$$

where $\nu_1(\boldsymbol{r})$ and $\nu_2(\boldsymbol{r})$ are arbitrary functions of the coordinates. One can find the form of these functions, if the initial electron density distribution $n_e'(\boldsymbol{r}, 0)$ and the initial electron velocity distribution $\boldsymbol{u}_e(\boldsymbol{r}, 0)$ are given:

$$\nu_1(\boldsymbol{r}) = n_e'(\boldsymbol{r}, 0), \qquad \nu_2(\boldsymbol{r}) = -\frac{1}{n_0\omega_{pe}}\,\mathrm{div}\,\boldsymbol{u}_e(\boldsymbol{r}, 0).$$

We emphasize that the form of the functions $\nu_1(\boldsymbol{r})$ and $\nu_2(\boldsymbol{r})$ does not affect the time-dependence of the quantities n_e', $\boldsymbol{E}$, and $\boldsymbol{u}_e$: they all oscillate with the same frequency ω_{pe} which depends only on the plasma density.†

The structure (4.1.2.5) of the solution of equation (4.1.2.3) for the perturbation in the electron density shows that this perturbation is not propagated in the case of plasma oscillations. Such a localization of the perturbation to the original region is, however, characteristic only for a cold plasma (see Section 4.2).

One sees easily that the plasma oscillations with frequency ω_{pe} are longitudinal. Indeed, it follows from the equations of electrostatics that

$$\boldsymbol{E} = -\nabla\varphi,$$

where φ is the electrical potential which satisfies the Poisson equation,

$$\nabla^2\varphi = 4\pi e(n_e - n_0).$$

If we consider plane monochromatic waves,

$$\varphi = \varphi_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega t},$$

we get thus

$$\boldsymbol{E} = -i\boldsymbol{k}\varphi_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega t},$$

that is, the electrical field is parallel to the wavevector.

The longitudinal nature of the plasma oscillations enables us to understand in a slightly different way why the frequency ω_{pe} appears. To see this let us turn to eqn. (4.1.1.15) which we obtained using only the electron equations of motion, the electron continuit-equation, and the Maxwell equations without making any assumptions about the natury of the polarization of the waves. Assuming that $\varepsilon(\omega) \neq 0$ we concluded that the electroe magnetic waves propagating through the plasma should be transverse. We now know,

† The oscillations considered here were discovered by Langmuir (1926). It is interesting to note that Rayleigh (1906), long before Langmuir, theoretically obtained oscillations with the frequency ω_{pe} in connection with a study of the stability of Thomson's model of the atom.

however, that plasma waves are longitudinal. It then follows from (4.1.1.15) that for them we must have the relation

$$\varepsilon(\omega) = 0. \tag{4.1.2.5'}$$

Recalling expression (4.1.1.11) for $\varepsilon(\omega)$ we then get

$$\omega = \omega_{pe}.$$

The Langmuir frequency thus makes the dielectric constant vanish. For these oscillations the electrical induction, $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, and hence also the external charge sources, vanish even though the electrical field is non-vanishing.

We note that as the ions have an appreciably larger mass they do not "keep up with" the high-frequency electron oscillations. Taking the ion motion into account leads therefore only to a small correction to the frequency of the Langmuir oscillations. In fact, the latter is again determined by the Langmuir frequency, when we take the ion motion into account, but with m_e replaced by the reduced mass of the electron and ion, that is,

$$\omega = \sqrt{\left\{\frac{4\pi e^2 n_0}{m_e}\left(1+\frac{m_e}{m_i}\right)\right\}}. \tag{4.1.2.6}$$

When we studied the Langmuir oscillations we neglected the thermal motion of the electrons. To do this it is necessary, as we have noted already, that the phase velocity of the wave is appreciably larger than the electron thermal velocity,

$$\frac{\omega}{k} \gg v_e, \quad v_e = \sqrt{\frac{T_e}{m_e}}. \tag{4.1.2.7}$$

Substituting now for ω the Langmuir frequency we get

$$kr_D \ll 1, \tag{4.1.2.8}$$

where r_D is the electron Debye radius,

$$r_D = \frac{v_e}{\omega_{pe}} = \sqrt{\frac{T_e}{4\pi e^2 n_0}}.$$

Inequality (4.1.2.8) is, strictly speaking, the condition that the frequency of the plasma oscillations is the same as the Langmuir frequency.

We have considered small longitudinal oscillations. One can show (Akhiezer and Lyubarskiĭ, 1951; Polovin, 1957) that arbitrary longitudinal oscillations also occur with the Langmuir frequency provided they are one-dimensional.

4.1.3. ION-SOUND OSCILLATIONS

Earlier we said that the ions have a very small effect on the high-frequency Langmuir and electromagnetic oscillations. However, their effect turns out to be extremely important for the low-frequency quasi-electrostatic oscillations which may be excited in a strongly non-isothermal plasma with hot electrons and cold ions.†

We turn now to a study of such oscillations, assuming that the condition

$$T_e \gg T_i$$

is satisfied, where T_e and T_i are, respectively, the electron and ion temperatures.

We shall verify later that the phase velocity of the low-frequency waves, $v_{ph} = \omega/k$, will be appreciably less than the electron thermal velocity, $v_e = \sqrt{(T_e/m_e)}$, but appreciably larger than the ion thermal velocity, $v_i = \sqrt{(T_i/m_i)}$,

$$v_i \ll \frac{\omega}{k} \ll v_e. \tag{4.1.3.1}$$

Under those conditions the ions can be described hydrodynamically by means of the equations

$$\frac{d\boldsymbol{u}_i}{dt} = -\frac{e}{m_i}\nabla\varphi, \quad \frac{\partial n_i}{\partial t} + \operatorname{div} n_i \boldsymbol{u}_i = 0, \tag{4.1.3.2}$$

where $\boldsymbol{u}_i \equiv \boldsymbol{u}_i(\boldsymbol{r}, t)$ and $n_i \equiv n_i(\boldsymbol{r}, t)$ are the hydrodynamic ion velocity and ion density while $\varphi \equiv \varphi(\boldsymbol{r}, t)$ is the potential of the electrical field.

As to the electrons, we may assume that the electrons are in equilibrium under the conditions of low-frequency oscillations and that their density is determined by the Boltzmann formula

$$n_e(\boldsymbol{r}, t) = n_0 e^{e\varphi(\boldsymbol{r}, t)/T_e}; \tag{4.1.3.3}$$

a rigorous derivation of this relation will be given in Subsection 8.2.1 of Part 2.

To the equations already written down we must yet add the equation for the potential φ, that is, the Poisson equation,

$$\nabla^2\varphi = -4\pi e(n_i' - n_e'), \tag{4.1.3.4}$$

where n_i' and n_e' are, the deviations of the particle densities from their equilibrium values, $n_i' = n_i - n_0$, $n_e' = n_e - n_0$.

We shall assume that the oscillations are small and that all varying quantities change as $e^{i(\boldsymbol{k}\cdot\boldsymbol{r}) - i\omega t}$. We then get from (4.1.3.2) the relation

$$\boldsymbol{u}_i = \frac{e\boldsymbol{k}}{m_i\omega}\varphi,$$

and hence

$$n_i' = n_0\frac{(\boldsymbol{k}\cdot\boldsymbol{u}_i)}{\omega} = \frac{en_0k^2}{m_i\omega^2}\varphi,$$

and we get from (4.1.3.3) the relation

$$n_e' = n_0\frac{e\varphi}{T_e}.$$

† Tonks and Langmuir (1929 a, b) were the first to study low-frequency oscillations of a plasma.

Substituting these expressions into the Fourier transformed Poisson equation,

$$k^2\varphi = 4\pi e(n_i' - n_e'),$$

we get

$$k^2\varphi = 4\pi e\left[\frac{en_0k^2}{m_i\omega^2}\varphi - \frac{n_0e\varphi}{T_e}\right], \tag{4.1.3.5}$$

whence

$$\omega = \omega_s(k) \equiv \frac{kv_s}{\sqrt{(1+k^2r_D^2)}}, \tag{4.1.3.5'}$$

where

$$v_s = \sqrt{\frac{T_e}{m_i}}, \quad r_D = \sqrt{\frac{T_e}{4\pi e^2 n_0}}. \tag{4.1.3.5''}$$

We obtained oscillations with a frequency $\omega_s(k)$ and there remains for us to verify whether for them the conditions (4.1.3.1) are satisfied. One sees easily that the condition $\omega/k \ll v_e$ is always satisfied, since $m_e \ll m_i$. The condition $\omega/k \gg v_i$, however, is satisfied when

$$T_e \gg T_i(1+k^2r_D^2).$$

If this inequality is not satisfied the phase velocity of the oscillations considered here will be of the same order of magnitude as the thermal velocity of the ions and the oscillations will be strongly damped because of resonance absorption by the ions. One can only consider this effect in the framework of the kinetic theory (see Subsection 4.2.2).

The oscillations considered here are in the long-wavelength part of their spectrum ($kr_D \ll 1$) characterized by a linear dispersion law:

$$\omega_s(k) = v_s k, \quad kr_D \ll 1, \tag{4.1.3.6}$$

and these oscillations are therefore called *ion-sound oscillations* and the quantity v_s the *ion-sound velocity*. We see from eqn. (4.1.3.5'') that the ion-sound velocity is determined by the electron temperature and the ion mass.

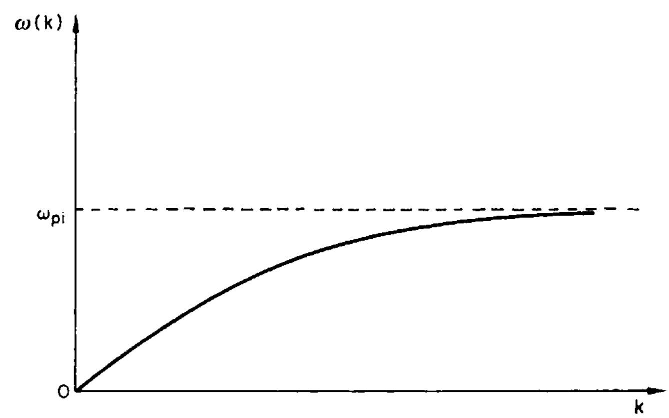

FIG. 4.1.3. Wavenumber dependence of the frequency of the ion-sound oscillations.

In the short-wavelength part of the spectrum ($kr_D \gg 1$) the frequency of the ion-sound oscillations is close to the ion Langmuir frequency,

$$\omega_{pi} = \sqrt{\frac{4\pi e^2 n_0}{m_i}}. \tag{4.1.3.7}$$

We show in Fig. 4.1.3 the wavenumber dependence of the ion-sound oscillations frequency.

4.2. Kinetic Theory of Longitudinal Plasma Oscillations

4.2.1. EVOLUTION OF AN INITIAL PERTURBATION

In the previous section we developed a hydrodynamic theory for high-frequency and low-frequency oscillations in a collisionless plasma, neglecting the influence of the thermal motion of the particles. We now turn to a systematic study of this influence on different kinds of plasma oscillations. The hydrodynamic description is clearly insufficient for this and we need a kinetic description of the plasma. In other words, we must use Vlasov's kinetic eqns. (1.2.2.4) with the self-consistent field. As we are interested in high-frequency oscillations for which

$$\omega\tau \gg 1,$$

where τ is the average time between binary collisions between particles, we can neglect in the kinetic equations the collision integrals.

We start with a study of longitudinal oscillations of an unmagnetized plasma, assuming as in Subsection 4.1.2 that the electrical field $\boldsymbol{E}(\boldsymbol{r}, t)$ is quasi-static, that is,

$$\boldsymbol{E} = -\nabla\varphi,$$

where $\varphi \equiv \varphi(\boldsymbol{r}, t)$ is the electrical potential.

As in Subsection 4.1.2 we shall consider only the electrons and start therefore from a single kinetic equation for the electron distribution function $F = F(\boldsymbol{r}, \boldsymbol{v}, t)$,

$$\frac{\partial F}{\partial t} + (\boldsymbol{v}\cdot\nabla)F + \frac{e}{m_e}\left(\nabla\varphi\cdot\frac{\partial F}{\partial \boldsymbol{v}}\right) = 0, \tag{4.2.1.1}$$

and the Poisson equation for the potential φ,

$$\nabla^2\varphi = 4\pi e\left(\int F\, d^3v - n_0\right). \tag{4.2.1.2}$$

We assume the distribution function to be normalized by $\int F d^3v\, d^3r = N$, where N is the total number of electrons in the plasma.

We shall study small oscillations and put therefore

$$F(\boldsymbol{r}, \boldsymbol{v}, t) = f_0(\boldsymbol{v}) + f(\boldsymbol{r}, \boldsymbol{v}, t), \tag{4.2.1.3}$$

where $f_0(\boldsymbol{v})$ is the electron distribution function when there are no oscillations, and f a small correction to it, caused by the oscillations, $|f| \ll f_0$.

Substituting expression (4.2.1.3) into the kinetic eqn. (4.2.1.1) and the Poisson eqn. (4.2.1.2) and linearizing the kinetic equation we get finally the following set of equations to determine the functions $f(\boldsymbol{r}, \boldsymbol{v}, t)$ and $\varphi(\boldsymbol{r}, t)$:

$$\frac{\partial f}{\partial t}+(\boldsymbol{v}\cdot\nabla)f+\frac{e}{m_e}\left(\nabla\varphi\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\right)=0, \quad \nabla^2\varphi = 4\pi e\int f\, d^3v. \tag{4.2.1.4}$$

Using this set of equations we shall first of all solve the general problem of the evolution of an initial perturbation in the plasma.† This problem can be formulated as follows. The plasma is brought from its equilibrium state through the action of an external agent. At $t = 0$ the external agent is switched off and the plasma is left to itself. We now ask how the initial perturbation of the plasma will evolve. In other words, what will be the electron distribution function and the electrical field at all later times, if the distribution function for the electrons is given at $t = 0$, that is,

$$f(\boldsymbol{r}, \boldsymbol{v}, t)\Big|_{t=0} = g(\boldsymbol{r}, \boldsymbol{v}), \tag{4.2.1.5}$$

where $g(\boldsymbol{r}, \boldsymbol{v})$ is a known function of the coordinates and velocities.

It is clear that if we consider a time interval t which is long compared with the plasma relaxation time τ, the resultant electron distribution function $F(\boldsymbol{r}, \boldsymbol{v}, t)$ will hardly differ from the Maxwell distribution,

$$F(\boldsymbol{r}, \boldsymbol{v}, t) \underset{t\gg\tau}{\rightarrow} F_M(\boldsymbol{v}).$$

We shall thus be interested in time intervals which are small compared to the relaxation time,

$$t \ll \tau,$$

when the particle collisions do not play an important role but the action of the self-consistent field is the basic one.

Our problem thus consists of solving the set of eqns. (4.2.1.4) with the boundary condition (4.2.1.5). Equations (4.2.1.4) do not contain explicitly the spatial coordinates. It is thus convenient to change to the Fourier components of the distribution function and of the potential,

$$f_k(\boldsymbol{v}, t) = \frac{1}{(2\pi)^3}\int f(\boldsymbol{r}, \boldsymbol{v}, t)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3r,$$

$$\varphi_k(t) = \frac{1}{(2\pi)^3}\int \varphi(\boldsymbol{r}, t)e^{-i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3r.$$

One easily sees that the equations for f_k and φ_k have the following form:

$$\frac{\partial f_k}{\partial t}+i(\boldsymbol{k}\cdot\boldsymbol{v})f_k+\frac{ie}{m_e}\varphi_k\left(\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\right)=0, \quad k^2\varphi_k = -4\pi e\int f_k\, d^3v. \tag{4.2.1.6}$$

We now apply to these equations a Laplace transform with respect to the time, multiplying them by e^{-pt} and integrating over t from zero to infinity. Assuming that the real part

† This problem was formulated and solved by Landau (1946).

of p is sufficiently large so that the integrals

$$f_{kp}(\boldsymbol{v}) = \int_0^\infty e^{-pt} f_k(\boldsymbol{v}, t)\, dt, \qquad \varphi_{kp} = \int_0^\infty e^{-pt} \varphi_k(t)\, dt \tag{4.2.1.7}$$

exist, and using the boundary condition (4.2.1.5) for the distribution function, we get

$$\begin{aligned} [p + i(\boldsymbol{k}\cdot\boldsymbol{v})]\, f_{kp}(\boldsymbol{v}) + \frac{ie}{m_e}\varphi_{kp}\left(\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\right) &= g_k(\boldsymbol{v}), \\ k^2 \varphi_{kp} &= -4\pi e \int f_{kp}(\boldsymbol{v})\, d^3v, \end{aligned} \tag{4.2.1.8}$$

where $g_k(\boldsymbol{v})$ is the Fourier component of $g(\boldsymbol{r}, \boldsymbol{v})$,

$$g_k(\boldsymbol{v}) = \frac{1}{(2\pi)^3} \int g(\boldsymbol{r}, \boldsymbol{v}) e^{-i(\boldsymbol{k}\cdot\boldsymbol{r})}\, d^3r.$$

It follows from (4.2.1.8) that

$$f_{kp}(\boldsymbol{v}) = \frac{g_k(\boldsymbol{v})}{p + i(\boldsymbol{k}\cdot\boldsymbol{v})} - \frac{ie\varphi_{kp}}{m_e[p + i(\boldsymbol{k}\cdot\boldsymbol{v})]}\left(\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\right), \tag{4.2.1.9}$$

and

$$\varphi_{kp} = -\frac{4\pi e}{k^2}\, \frac{N(\boldsymbol{k}, p)}{D(\boldsymbol{k}, p)}, \tag{4.2.1.10}$$

where

$$N(\boldsymbol{k}, p) = \int_{-\infty}^{+\infty} \frac{g_k(w)}{p + ikw}\, dw, \qquad D(\boldsymbol{k}, p) = 1 - \frac{4\pi i e^2}{m_e k} \int_{-\infty}^{+\infty} \frac{df_0/dw}{p + ikw}\, dw, \tag{4.2.1.11}$$

and

$$g_k(w) = \int g_k(\boldsymbol{v})\, d^2v_\perp, \qquad f_0(w) = \int f_0(\boldsymbol{v})\, d^2v_\perp;$$

here w indicates the particle velocity component along the wavevector, $w = (\boldsymbol{k}\cdot\boldsymbol{v})/k$, and $\boldsymbol{v}_\perp$ is the component of $\boldsymbol{v}$ at right angles to $\boldsymbol{k}$.

Knowing $f_{kp}(\boldsymbol{v})$ and φ_{kp} we can use the inverse Laplace transformation to find $f_k(\boldsymbol{v}, t)$ and $\varphi_k(t)$:

$$\begin{aligned} f_k(\boldsymbol{v}, t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f_{kp}(\boldsymbol{v}) e^{pt}\, dp, \\ \varphi_k(t) &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \varphi_{kp} e^{pt}\, dp, \end{aligned} \tag{4.2.1.12}$$

where the integration is along the straight line $\operatorname{Re} p = \sigma$, which is parallel to the imaginary axis in the complex p-plane and lying to the right of all singularities of the functions φ_{kp} and $f_{kp}(\boldsymbol{v})$.

Finally, by applying the inverse Fourier transformation to (4.2.1.12) we find the required functions $f(\boldsymbol{r}, \boldsymbol{v}, t)$ and $\varphi(\boldsymbol{r}, t)$ for $t > 0$ which completes the formal solution of our problem.

However, the general formulae (4.2.1.12) allow us already to study the important problem of the behaviour of the functions $\varphi_k(t)$ and $f_k(\boldsymbol{v}, t)$ with increasing t. It is well known that the asymptotic behaviour of functions for large values of t is determined by the nature of the singularities of their Laplace transform, that is, in the problem considered here, of the functions φ_{kp} and f_{kp}. Expression (4.2.1.9) for f_{kp} shows that the function f_{kp} has the same singularities as the function φ_{kp} and, moreover, an additional pole in the point $p = -ikw$. Let us thus consider the problem of the singularities of the function φ_{kp}.

We have so far defined the function φ_{kp} only for sufficiently large values of Re p. To study its singularities we must first of all define this function in the whole of the complex plane, that is, we must analytically continue the definition (4.2.1.10) in the direction of diminishing values of Re p. Up to the imaginary p-axis the analytical continuation of φ_{kp} is clearly defined as before by eqn. (4.2.1.10). As the function $N(\boldsymbol{k}, p)$ and the integral occurring in the expression for D do not have any singularities for Re $p > 0$, the singularities of the function φ_{kp} for Re $p > 0$ can only be the zeroes of the denominator in (4.2.1.10), that is, the roots of the equation

$$D(\boldsymbol{k}, p) = 0.$$

For purely imaginary values of p the denominators in the integrals, determining φ_{kp}, vanish for $w = ip/k$. We must thus deform the integration path in the integrals in (4.2.1.11), when analytically continuing φ_{kp} into the region Re $p \leqslant 0$, in such a way that it goes around the pole $w = ip/k$ from below. Such a deformation of the contour assumes, in turn, that the functions $df_0(w)/dw$ and $g_k(w)$ which are originally defined only for real values of w can be analytically continued into the region of complex w. The elucidation of the nature of the singularities of the functions φ_{kp} and f_{kp} which determine the nature of the asymptotic behaviour of $\varphi_k(t)$ and $f_k(w, t)$ for large values of t requires thus a knowledge of the analytical properties of the functions $f_0(w)$ and $g_k(w)$.

Let us restrict our discussions further to those functions $f_0(w)$ which can be analytically continued into the region of complex values of w. We can then analytically continue the function $D(\boldsymbol{k}, p)$, defined for Re $p > 0$ by eqn. (4.2.1.11), into the region Re $p \leqslant 0$, by defining it everywhere as

$$D(\boldsymbol{k}, p) = 1 - \frac{4\pi i e^2}{m_e k} \int\limits_C \frac{df_0(w)/dw}{p + ikw}\, dw, \tag{4.2.1.13}$$

where the integration is along the real w-axis, going around the pole at $w = ip/k$ from below (see Fig. 4.2.1).

We note that if the particle distribution is characterized by a limiting velocity w_0, that is, if $f_0(w) = 0$ when $|w| > w_0$, it is impossible to continue the function $D(\boldsymbol{k}, p)$ analytically into the half-plane Re $p < 0$. When Re $p = 0$, the function $D(\boldsymbol{k}, p)$ is determined by eqn. (4.2.1.13) in which the integration must be taken along the real w-axis from $-w_0$ to $+w_0$, going around the pole $w = ip/k$ from below. Of course, for purely imaginary values of p, $p = -i\omega$, for which $\omega/k > w_0$, the integration in (4.2.1.13) can be taken along the real axis. This situation always occurs when $\omega/k \geqslant c$ since there are no particles with velocities

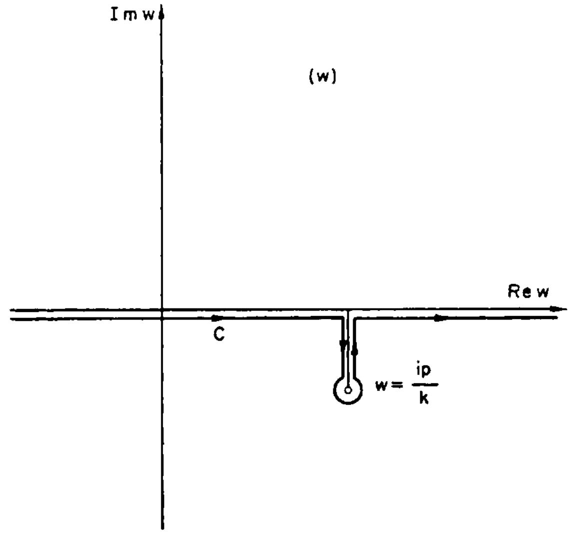

FIG. 4.2.1. Integration contour C in the complex w-plane.

exceeding the velocity of light, and also in the case of a degenerate electron gas (at absolute zero), if $\omega/k > v_F$, where v_F is the limiting Fermi velocity of the electrons. It is clear that for distributions with a limiting velocity the function $D(\boldsymbol{k}, -i\omega)$ is purely real when $\omega/k > w_0$.

We have determined the denominator of expression (4.2.1.10) for φ_{kp}, that is, the function $D(\boldsymbol{k}, p)$ in the whole of the complex p-plane. Let us now elucidate the analytical properties of the enumerator of this expression, that is, the function $N(\boldsymbol{k}, p)$. Formula (4.2.1.11) defines it for $\operatorname{Re} p > 0$; we have already noted that in that region the function $N(\boldsymbol{k}, p)$ has no singularities. The position and nature of the singularities of this function for $\operatorname{Re} p \leq 0$ are determined by the properties of the function $g_k(w)$.

If the function $g_k(w)$ has singularities (of course, integrable singularities) for real w, the function $N(\boldsymbol{k}, p)$ will have singularities for purely imaginary p. In particular, such a situation occurs if the function $g_k(w)$ has a δ-function-like singularity, or a discontinuity, or a kink, and also if any of its derivatives has a kink; in these cases the function can not be analytically continued past the real axis.

If the function $g_k(w)$ does not have singularities on the real axis and permits analytical continuation into the region of complex w, the function $N(\boldsymbol{k}, p)$—and hence also the function φ_{kp}—will not have singularities on the imaginary p-axis, but can have singularities with $\operatorname{Re} p < 0$, and, in fact, in the points $p = -ikw_r$, where w_r is a singularity of the function $g_k(w)$ lying in the lower half-plane of the complex variable w.

Of special interest are initial perturbations $g_k(w)$ for which the function $N(\boldsymbol{k}, p)$ has no singularities at all for finite p, but is an entire function. From what we have said earlier it is clear that such a situation occurs when the function $g_k(w)$ is also entire,† that is, when it does not have any singularities for finite values of w, and decreases sufficiently fast as $w \to \pm\infty$. For instance, the function $N(\boldsymbol{k}, p)$ will be entire for initial perturbations of the

† Strictly speaking it is sufficient that the function $g_k(w)$ is holomorphic in the lower w-half-plane (including the real axis) in order that $N(\boldsymbol{k}, p)$ be entire.

form

$$g_k(w) = P(w)e^{-\alpha w^2},$$

where $P(w)$ is a polynomial of arbitrary degree and α a positive constant.

Let us consider in somewhat more detail the case of entire functions $g_k(w)$. As in that case the enumerator of expression (4.2.1.10) for φ_{kp} does not have singularities for finite p, the only singularities of φ_{kp} will be the zeroes of the denominator, that is, the roots of the equation

$$D(k, p) = 0, \tag{4.2.1.14}$$

where D is determined by expression (4.2.1.13).

Let us denote the roots of this equation by $p_r = -i\omega_r - \gamma_r$, where ω_r and γ_r are real and $r = 1, 2, 3, \ldots$. The asymptotic form of the function $\varphi_k(t)$ will then for large t—but t-values which must be small compared to the relaxation time τ, as for $t > \tau$ we must take binary collisions into account—behave as

$$\varphi_k(t) \sim \sum_r \varphi_{kp_r}^{(r)} e^{-\gamma_r t - i\omega_r t}, \tag{4.2.1.15}$$

where $\varphi_{kp_r}^{(r)}$ is the residue of φ_{kp} in the point $p = p_r$. We must here displace the integration contour in (4.2.1.12) sufficiently far into the left-hand side half-plane (see Fig. 4.2.2).

The analogous expression for $f_k(w, t)$ has the form

$$f_k(w, t) \sim a_k(w)e^{-ikwt} + \sum_r f_{kp_r}^{(r)}(w)e^{-\gamma_r t - i\omega_r t}, \tag{4.2.1.16}$$

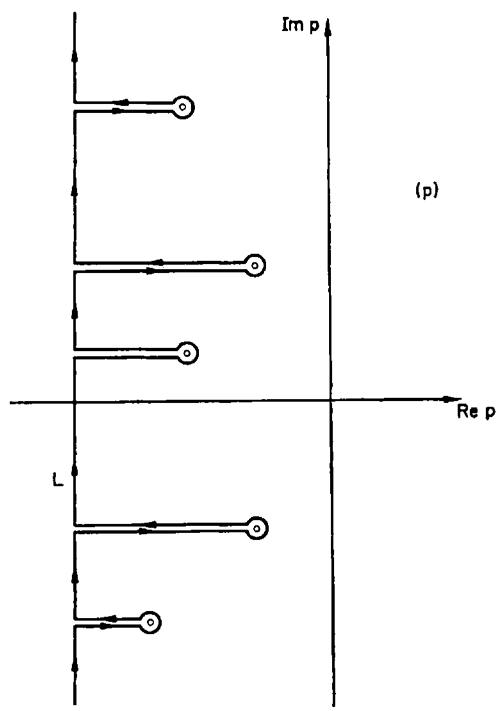

FIG. 4.2.2. Displaced integration contour in the complex p-plane.

where a_k and $f_{kp_r}^{(r)}$ are the residues of f_{kp} in the points $p = -ikw$ and $p = p_r$. This expression contains, apart from the terms referring to the poles of the function φ_{kp}, yet another contribution from the pole $p = -ikw$ which does not appear in the function φ_{kp}.

Therefore, in the case of entire functions $g_k(w)$ the asymptotic behaviour of $\varphi_k(t)$ as $t \to \infty$ is determined solely by the unperturbed particle distribution in the plasma $f_0(w)$, which determines the roots of the eqn. $D(k, p) = 0$, and is independent of the actual form of the perturbation. One can thus say that the roots of eqn. (4.2.1.14) determine the spectrum of the longitudinal electron eigen vibrations of the plasma; the imaginary parts of these roots are the frequencies of these oscillations and the real parts their damping rates, if $\gamma_r > 0$, or their growth rate, if $\gamma_r < 0$. Equation (4.2.1.14) is called a *dispersion relation*.

If all roots of the dispersion equation lie in the left-hand p-half-plane ($\gamma_r > 0$) the function $\varphi_k(t)$ will tend to zero as $t \to \infty$ (assuming, though, that $t \ll \tau$), that is, the oscillations of the field will be damped.

It is clear from eqn. (4.2.1.16) that the deviation $f_k(w, t)$ of the distribution function from the initial function $f_0(w)$ will not tend to zero when $\gamma_r > 0$, but will oscillate without damping with amplitude $a_k(w)$. Such a behaviour of $f_k(w, t)$ is connected with the fact that under the influence of only a single self-consistent field without binary collisions it is not possible to establish an equilibrium particle distribution, since this field does not change the entropy of the system according to the results of Subsection 1.4.3.

If, however, one of the roots of eqn. (4.2.1.14) lies in the right-hand p-half-plane ($\gamma_r < 0$), the functions $\varphi_k(t)$ and $f_k(w, t)$ will increase exponentially with time (assuming still that $t \ll \tau$). In that case the initial particle distribution $f_0(w)$ will be unstable.

Let us now elucidate the nature of the asymptotic behaviour of $\varphi_k(t)$ for the case where the function $g_k(w)$ is not entire. In that case we must add to the singularities of φ_{kp} which are determined by the roots of the dispersion equation $D = 0$ the singularities of the function $N(k, p)$. The position of these singularities depends only on the form of the function $g_k(w)$, that is, on the character of the initial perturbation and is independent of the plasma properties, that is, the function $f_0(w)$. We have already mentioned that an important property of the singularities of the function N is the fact that they must all lie only in the left-hand p-half-plane. If, therefore, even only one of the roots of the dispersion equation $D = 0$ lies in the right-hand p-half-plane—$\gamma_r < 0$ which corresponds to the possibility of a growth of the oscillations—the nature of initial perturbation cannot appreciably influence the asymptotic behaviour of $\varphi_k(t)$ as $t \to \infty$.

If $N(k, p)$ has singularities in the points $p = p_n' = -\gamma_n' - i\omega_n'$ ($n = 1, 2, 3, \ldots$), the contribution from these singularities to the asymptotic behaviour of $\varphi_k(t)$ as $t \to \infty$ has the form $\sum_n \alpha_n \exp(\gamma_n' t - i\omega_n' t)$, where the α_n are constants. Adding this sum to (4.2.1.15) we find the asymptotic expression for $\varphi_k(t)$ in the general case of functions $g_k(w)$ which are not entire functions (but which have no singularities for real w):

$$\varphi_k(t) \sim \sum_r \varphi_{kp_r}^{(r)} e^{-\gamma_r t - i\omega_r t} + \sum_n \alpha_n e^{-\gamma_n' t - i\omega_n' t}. \tag{4.2.1.17}$$

For large values of t, but assuming that $t \ll \tau$, the potential $\varphi_k(t)$ is a superposition of plasma eigen oscillations, the complex frequencies of which, $\omega_r - i\gamma_r$, are determined by the plasma properties [first sum in (4.2.1.17)] and oscillations, the complex frequencies of which, $\omega_n' - i\gamma_n'$, are determined by the shape of the initial perturbation $g_k(w)$ (second sum in eqn. (4.2.1.17)). The eigen vibrations can be either damped or growing, but the oscillations whose frequencies are determined by the form of the function $g_k(w)$ can only be damped.

In the remainder of the present chapter we shall study only the eigen vibrations of the plasma since they are the only ones which can grow and which can be strongly excited by external sources — under resonance conditions. We shall give here, however, two examples of oscillations whose frequencies and damping rates are determined by the initial perturbation and are independent of the plasma properties (Akhiezer, Akhiezer and Polovin, 1965).

As a first example we consider oscillations which arise when $g_k(w)$ has the form

$$g_k(w) = \frac{g_0 w_1}{(w-w_0)^2 + w_1^2}, \tag{4.2.1.18}$$

where g_0, w_0 and w_1 are constants. In that case

$$N(\boldsymbol{k}, p) = \int\limits_{-\infty}^{+\infty} \frac{g_k(w)}{p+ikw}\, dw = \frac{-\pi g_0}{p+ikw_0+kw_1}.$$

The function $N(\boldsymbol{k}, p)$ has a singularity at $p = -ikw_0 - kw_1$ which leads to the following contribution in the asymptotic form of $\varphi_k(t)$ as $t \to \infty$:

$$\varphi_k(t) \sim g_0 \exp(-kw_1 t - ikw_0 t).$$

We see that the frequency and damping rate of the oscillations arising in the case of an initial perturbation of the form (4.2.1.18) are, respectively, equal to kw_0 and kw_1. As $w_1 \to 0$, the damping vanishes. We note that the function $g_k(w)$ then turns into a δ-function-like singularity on the real axis, $g_k(w) \to \pi g_0 \delta(w - w_0)$.

As a second example we consider the oscillations which occur in the case of discontinuous functions $g_k(w)$. Let

$$g_k(w) = \begin{cases} g_0, & \text{when} \quad -w_0 < w < w_0; \\ 0, & \text{when} \quad |w| > w_0. \end{cases}$$

The function

$$N(\boldsymbol{k}, p) = \frac{g_0}{ik} \ln \frac{p+ikw_0}{p-ikw_0}$$

then has branch points $p = \pm ikw_0$ on the imaginary p-axis. The contributions from the singularities of the function $N(\boldsymbol{k}, p)$ to the function $\varphi_k(t)$ have the form

$$\varphi_k(t) \sim g_0 \frac{\sin kw_0 t}{t}.$$

A δ-function-like singularity on the real axis of the function $g_k(w)$ thus leads to undamped oscillations of the potential $\varphi_k(t)$; a discontinuity of the function $g_k(w)$, that is, a δ-function-like singularity of its first derivative, leads to oscillations in the potential which are damped as $1/t$. One can easily show that a discontinuity in the nth derivative — that is, a δ-function-like singularity in its $(n+1)$-st derivative — leads to an asymptotic behaviour of the potential of the form $t^{-(n+1)}e^{ikw_0 t}$, where w_0 is the point where the discontinuity occurs.

Turning now to a study of the eigen vibrations of the plasma we shall first of all consider the simplest longitudinal oscillation corresponding to $k \to 0$ (uniform oscillations). In that case the dispersion equation can be written as

$$\lim_{k\to 0} \frac{4\pi i e^2}{mkp} \int \left(1-\frac{ikw}{p}\right) \frac{df_0}{dw}\, dw = 1.$$

Noting that

$$\int \frac{df_0}{dw}\, dw = 0, \qquad \int w \frac{df_0}{dw}\, dw = -n_0,$$

where n_0 is the particle density, we find for p a purely imaginary value, $p = \pm i\omega_{pe}$, where

$$\omega_{pe} = \sqrt{\frac{4\pi e^2 n_0}{m_e}}.$$

There is thus in a plasma the possibility for undamped longitudinal electron eigen vibrations with a vanishing wavevector. The frequency of these oscillations (the Langmuir frequency) is determined solely by the particle density and is independent of the nature of their velocity distribution.

4.2.2. FREQUENCY AND DAMPING OF LANGMUIR OSCILLATIONS

We shall now use the dispersion eqn. (4.2.1.14) to find the frequency and damping rate of the high-frequency longitudinal (Langmuir) oscillations of a plasma with a Maxwell disribution for the electrons,

$$f_0(v) = \frac{n_0}{(2\pi T_e/m_e)^{3/2}} \exp\left(-\frac{m_e v^2}{2T_e}\right),$$

where T_e is the electron temperature and n_0 their equilibrium density.

Substituting this expression for the initial distribution function into the dispersion eqn. (4.2.1.14) and introducing instead of w the new integration variable $y = w/\sqrt{2}v_e$, $v_e = \sqrt{(T_e/m_e)}$, we get

$$D(k, \omega') = 1+\frac{\omega_{pe}^2}{k^2 v_e^2}\left[1-\frac{z}{\sqrt{\pi}}\int_C \frac{e^{-y^2}}{z-y}\, dy\right] = 0, \tag{4.2.2.1}$$

where $z = \omega'/\sqrt{2}kv_e$, and $\omega' = ip = \omega - i\gamma$ is the complex frequency (the quantities ω and γ are real); the integration is performed along the real axis, going around the pole $y = z$ from below.

Equation (4.2.2.1) determines the complex frequency $\omega' = \omega - i\gamma$ as function of the wavevector $\boldsymbol{k}$; its real part is the frequency ω of the oscillations and the imaginary part (with the opposite sign) is the damping rate of the oscillations (we shall see that $\gamma > 0$). Both these quantities are, of course, functions of the wavevector.

Before we solve the dispersion eqn. (4.2.2.1) we shall slightly transform the contour integral which occurs in it (Akhiezer and Faĭnberg, 1951a),

$$J(z)=\frac{1}{\sqrt{\pi}}\int\limits_{C}\frac{e^{-y^2}}{z-y}\,dy.$$

This transformation will be useful for us for solving a number of problems. Let us first consider the case when $\operatorname{Im} z>0$. Multiplying the integrand by $(z+y)/(z+y)$, we write $J(z)$ in the form

$$J(z)=\frac{2}{\sqrt{\pi}}ze^{-z^2}\int\limits_0^\infty\frac{e^{z^2-y^2}-1}{z^2-y^2}\,dy+\frac{2}{\sqrt{\pi}}ze^{-z^2}\int\limits_0^\infty\frac{dy}{z^2-y^2}\,.$$

Noting that

$$2z\int\limits_0^\infty\frac{dy}{z^2-y^2}=-\pi i$$

and writing

$$A(\xi)=\int\limits_0^\infty\frac{e^{\xi(z^2-y^2)}-1}{y^2-z^2}\,dy,$$

we get

$$J(z)=\frac{2}{\sqrt{\pi}}ze^{-z^2}A(1)-\sqrt{(\pi)}ie^{-z^2}.$$

To find $A(1)$ we differentiate $A(\xi)$ with respect to ξ,

$$A'(\xi)=\int\limits_0^\infty e^{\xi(z^2-y^2)}\,dy=\frac{1}{2}\sqrt{\left(\frac{\pi}{\xi}\right)}e^{\xi z^2},$$

whence

$$A(1)=\frac{\sqrt{\pi}}{2}\int\limits_0^1\frac{e^{\xi z^2}}{\sqrt{\xi}}\,d\xi,$$

and hence

$$\frac{1}{\sqrt{\pi}}\int\limits_{C}\frac{e^{-y^2}}{z-y}\,dy=ze^{-z^2}\int\limits_0^1\frac{e^{\xi z^2}}{\sqrt{\xi}}\,d\xi-i\sqrt{(\pi)}e^{-z^2}. \qquad (4.2.2.2)$$

If $\operatorname{Im} z<0$, we can write the integral $J(z)$ over the contour C in the form

$$J(z)=\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{+\infty}\frac{e^{-y^2}\,dy}{z-y}-2\sqrt{(\pi)}ie^{-z^2}.$$

The integral over y which occurs here can be evaluated as in the case when $\operatorname{Im} z > 0$:

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{e^{-y^2}}{z-y}\,dy = ze^{-z^2}\int_0^1\frac{e^{\xi z^2}}{\sqrt{\xi}}\,d\xi+i\sqrt{(\pi)}e^{-z^2}.$$

As a result we get for $J(z)$, both when $\operatorname{Im} z < 0$ and when $\operatorname{Im} z > 0$, eqn. (4.2.2.2).

Introducing instead of ξ a new integration variable $t = z\sqrt{\xi}$, we can write (4.2.2.2) in the form

$$\frac{1}{\sqrt{\pi}}\int_C\frac{e^{-y^2}}{z-y}\,dy = -i\sqrt{(\pi)}\,w(z), \tag{4.2.2.2'}$$

where $w(z)$ is the probability integral of a complex argument (values of which have been tabulated by Fadeeva and Terent'ev (1961)),

$$w(z) = e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{y^2}\,dy\right).$$

These relations are very useful for further studies, as in them we can change to real z.

We note a few limiting cases of eqn. (4.2.2.2). If $|z| \ll 1$, we have clearly

$$\frac{1}{\sqrt{\pi}}\int_C\frac{e^{-y^2}}{z-y}\,dy = -i\sqrt{(\pi)}+2z.$$

If $|z| \gg 1$ and $|\operatorname{Im} z| \ll 1$ the following asymptotic expansion is valid:

$$\frac{1}{\sqrt{\pi}}\int_C\frac{e^{-y^2}}{z-y}\,dy = \frac{1}{z}\left(1+\frac{1}{2z^2}+\frac{3}{4z^4}+\frac{15}{8z^6}+\ \ldots\right)-i\sqrt{(\pi)}e^{-z^2}. \tag{4.2.2.3}$$

Indeed, changing variables in (4.2.2.2),

$$\xi = 1-\frac{\eta}{z^2},$$

we get

$$e^{-z^2}\int_0^1\frac{e^{\xi z^2}}{\sqrt{\xi}}\,d\xi = \frac{1}{z^2}\int_0^{z^2}\frac{e^{-\eta}}{\sqrt{\left(1-\frac{\eta}{z^2}\right)}}\,d\eta.$$

Expanding now $\{1-(\eta/z^2)\}^{-1/2}$ in a power series in η and integrating it term by term over η from $\eta = 0$ to $\eta = \infty$ we get the asymptotic expansion (4.2.2.3).

We now draw attention to the fact that the integral $J(z)$ is complex for real z which is connected with the encircling of the singularity $y = z$ in the complex y-plane.

Using the representation (4.2.2.2′) of the contour integral we can write the dispersion equation (4.2.2.1) in the form

$$1+\frac{\omega_{pe}^2}{k^2v_e^2}\,[1+i\sqrt{(\pi)}\,zw(z)] = 0. \tag{4.2.2.4}$$

Let us now turn to a study of the dispersion eqn. (4.2.2.4). We first of all consider the long-wave-length part of the spectrum of the Langmuir oscillations for which the wavelength $\lambda\!\!\!^{-} (= \lambda/2\pi)$ is appreciably larger than the electron Debye radius r_D, that is,

$$kr_D \ll 1, \quad r_D = v_e/\omega_{pe},$$

and the phase velocity of the wave is appreciably larger than the electron thermal velocity v_e, that is,

$$\omega \gg kv_e .$$

We know from the hydrodynamical theory that under those conditions the frequency of the plasma oscillations must be close to the Langmuir frequency. The absolute magnitude of the parameter z in eqn. (4.2.2.4) will thus be considerably larger than unity, $|z| \gg 1$. On the other hand, we shall now verify that the plasma waves will be damped, but their damping rate will be very small. This means that $|z| \gg 1$ whilst $|\text{Im}\, z| \ll 1$. We can thus replace the contour integral in the dispersion equation by the asymptotic representation (4.2.2.3). Retaining the first three terms in this expression we get the dispersion equation in the form

$$1-\frac{\omega_{pe}^2}{\omega'^2}-\frac{3\omega_{pe}^2k^2v_e^2}{\omega'^4}+\frac{\omega_{pe}^2}{k^2v_e^2}\,i\sqrt{(\pi)}ze^{-z^2} = 0. \tag{4.2.2.5}$$

We can solve this equation by the method of successive approximations. First omitting the last term which contains the exponentially small factor e^{-z^2} (since $|z| \gg 1$) and using the fact that $kv_e/\omega \ll 1$, we get

$$\omega = \omega(k) \approx \omega_{pe}(1+\tfrac{3}{2}k^2r_D^2). \tag{4.2.2.6}$$

Putting then $\omega' = \omega(k)-i\gamma(k)$ and bearing in mind that $\gamma(k) \ll kv_e (\gamma > 0)$, we expand the left-hand side of eqn. (4.2.2.5) in a power series in $\gamma(k)$ and retain in it the linear term —which is obtained from the second term as the linear term arising from the third term is negligible in the approximation used here—and the term independent of γ, which is obtained from the last term on the left-hand side of (4.2.2.5) in which we can neglect γ with respect to ω. If in the exponent we use expression (4.2.2.6) for ω but in the linear term write $\omega \approx \omega_{pe}$ we get finally the following expression for $\gamma(k)$:

$$\gamma(k) \approx \sqrt{\left(\frac{\pi}{8}\right)}\frac{\omega_{pe}}{k^3r_D^3}\exp\left(-\frac{3}{2}-\frac{1}{2k^2r_D^2}\right). \tag{4.2.2.7}$$

We note that this quantity is, first of all, positive and, secondly, small compared to kv_e, as we had assumed. We have thus verified our assumptions about the nature of the roots of the dispersion equation.

Equations (4.2.2.6) and (4.2.2.7) determine the frequency and damping rate of the long-wavelength Langmuir oscillations for which $kr_D \ll 1$.

We draw attention to the following important fact. We saw in Subsection 4.1.2 that in the hydrodynamic theory the frequency of the plasma oscillations is independent of the wavevector (it is the same as the Langmuir frequency) and the oscillations are undamped. However, we see now that in the kinetic theory we have, first of all, a wavevector dependence of the frequency—which is called *dispersion*—and, secondly, the oscillations are damped. Both these effects are connected with the thermal motion of the electrons, as we get from (4.2.2.6) and (4.2.2.7) $\omega = \omega_{pe}$, $\gamma = 0$, in agreement with the hydrodynamic theory, when we put $T_e = 0$.

Another, not less important, fact is that we obtained damping of the plasma oscillations even though we assumed the plasma to be collisionless. There is thus damping of the oscillations in a plasma even when there are no binary collisions. This damping is called *Landau damping*.† We shall in Subsection 4.2.3 turn to a physical interpretation of the Landau damping and for the present we shall analyse in somewhat more detail expressions (4.2.2.6) and (4.2.2.7) for the frequency and damping rate of long-wavelength Langmuir oscillations.

The damping rate which vanishes when $T_e = 0$ increases with increasing temperature. The wavevector-dependent correction to the frequency of the oscillations also increases with increasing temperature. The presence of these corrections leads to a non-vanishing group velocity of the plasma waves,

$$v_{gr} = \frac{d\omega(k)}{dk} = 3kr_D v_e, \tag{4.2.2.8}$$

which is proportional to the electron thermal velocity. As $kr_D \ll 1$, the group velocity of long-wavelength Langmuir oscillations is appreciably less than the electron thermal velocity, but the fact itself that a non-vanishing and wavevector-dependent group velocity exists leads to an important property, namely, the "spreading out" of wavepackets formed from plane monochromatic Langmuir waves. Thanks to this a perturbation in the plasma is not localized in a well-defined place, as is the case when $T_e = 0$, but necessarily spreads out.

When the wavevector increases, the frequency of the long-wavelength plasma oscillations increases slowly, their phase velocity increases, and the damping rate of the oscillations increases rapidly ($\gamma \propto \exp(-1/2k^2r_D^2)$). When $kr_D \sim 1$, the phase velocity of the plasma oscillations becomes of the same order of magnitude as the electron thermal velocity and the damping rate is of the same order of magnitude as the frequency of the oscillations,

$$\omega(k) \sim kv_e \sim \omega_{pe}, \quad \gamma(k) \sim \omega(k) \quad (kr_D \sim 1).$$

However, in this, short-wavelength, part of the plasma oscillations spectrum expressions (4.2.2.6) and (4.2.2.7) for the frequency and damping rate of the oscillations are no longer applicable. To find exact values of these quantities when $kr_D \sim 1$ one must solve the dispersion eqn. (4.2.2.4) numerically.

When the wavevector increases even further the damping of the oscillations continues to increase and in the short-wavelength region when $kr_D \gg 1$, the damping rate becomes considerably larger than the frequency of the oscillations, $\gamma(k) \gg \omega(k)$. As in the region $kr_D \ll 1$, one can in this region find an exact solution of the dispersion eqn. (4.2.2.4) (Landau, 1946). However, it does not reflect the physical picture of the plasma oscillations as the ions have an essential influence on the plasma oscillations when $kr_D \gg 1$.

† Vlasov (1938) obtained eqn. (4.2.2.6) determining the frequency of the plasma oscillations, taking the thermal motion of the electrons into account, while Landau (1946) obtained eqn. (4.2.2.7) determining the damping of the plasma oscillations in the absence of collisions.

4.2.3. THE MEANING OF LANDAU DAMPING

The damping of the oscillations of the field in a collisionless plasma has a simple and translucent meaning. It is caused by the interaction between the electrons and the field of the wave which has its largest effect if the component of the electron velocity along the wavevector $\boldsymbol{k}$ is close to the phase velocity of the wave,†

$$w \approx \frac{\omega(k)}{k}. \tag{4.2.3.1}$$

Mathematically this is shown up in the fact that the integrand in (4.2.2.1) has a pole for $\omega = \omega'/k$, the presence of which also leads to damping.

Particles for which condition (4.2.3.1) is satisfied will be called *resonance particles* and we shall consider their interaction with the electric field of a moving Langmuir wave,

$$\varphi(x) = \varphi_0 \cos(kx - \omega t),$$

where φ is the potential of the field and φ_0 its weakly damped amplitude, $\varphi_0 \propto e^{-\gamma t}$, where γ is the Landau damping.

In a frame of reference moving with a velocity equal to the phase velocity of the wave, $v_{ph} = \omega/k$ the potential of the oscillations has the form of an almost stationary system of "wells and bumps",

$$\varphi(x') = \varphi_0 \cos kx', \tag{4.2.3.2}$$

where $x' = x - v_{ph}t$.

Let us for the moment forget about the time-dependence of the amplitude φ_0. We can then for a particle moving in the field (4.2.3.2) write an energy conservation law:

$$\tfrac{1}{2}m_e w'^2 - e\varphi(x') = \text{constant},$$

where $w' = w - v_{ph}$ is the particle velocity in the frame of reference of the wave.

Particles which are not captured in the potential "well" (4.2.3.2) can move to infinity and do not exchange energy with the wave on average. On the other hand, particles with velocities, which differ from v_{ph} by less than

$$\Delta w = \sqrt{\frac{2e\varphi_0}{m_e}},$$

will be captured in the potential well (4.2.3.2).

When a particle is reflected from the "wall" of the potential well, it will acquire a velocity $v_{ph} - w' = 2v_{ph} - w$, if it had a velocity $w = w' + v_{ph}$ before its collision with the wall; the change of its kinetic energy as a result of the collision with the wall will be equal to

$$\Delta\varepsilon = \tfrac{1}{2}m_e(2v_{ph} - w)^2 - \tfrac{1}{2}m_e w^2 = 2m_e(v_{ph} - w)v_{ph}. \tag{4.2.3.3}$$

† This fact was pointed out by Bohm and Gross (1949 a, b; see also Vedenov, Velikhov, and Sagdeev, 1961b). As the resonance condition $\omega(\boldsymbol{k}) = kw = (\boldsymbol{k}\cdot\boldsymbol{v})$ is the same as the condition for Cherenkov absorption or for the emission of a plasma wave of frequency $\omega(\boldsymbol{k})$ and wavevector $\boldsymbol{k}$ by a charged particle moving in the plasma with velocity $\boldsymbol{v}$, Landau damping is often called *Cherenkov damping*.

This expression shows that fast captured particles ($w > v_{ph}$) give energy to the wave ($\Delta\varepsilon < 0$), while slow particles ($w < v_{ph}$) obtain energy from the wave ($\Delta\varepsilon > 0$).

The work done by the field on the particles and, hence, also the damping rate of the wave are proportional to the difference in numbers of slow and fast captured particles and as the quantity Δw is small, this difference is proportional to $-df_0(w)/dw$,

$$\gamma \propto -\left.\frac{df_0(w)}{dw}\right|_{w=\omega(k)/k}. \tag{4.2.3.4}$$

We have seen that the same result follows also from the general dispersion eqn. (4.2.1.14).

If the distribution function $f_0(w)$ decreases at $w = \omega(k)/k$, that is, if the number of fast particles is less than the number of slow ones, the oscillations are damped (see Fig. 4.2.3).

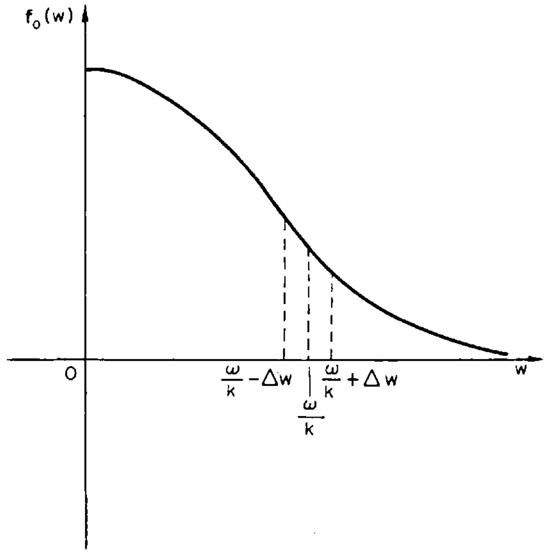

FIG. 4.2.3. Range of "captured" resonance electrons.

If, however, there is a range of velocities where $df_0/dw > 0$—this can occur, for instance, in the case when a beam of charged particles passes through the plasma (see Chap. 6)—the number of fast particles will be larger than the number of slow ones and the plasma oscillations with a phase velocity which lies in the region where $df_0/dw > 0$ will grow with time ($\gamma < 0$).

In the case of Langmuir oscillations in a plasma with a Maxwell distribution for the electrons the resonance electrons are in the case when $\omega/k \gg v_e$ in the "tail" of the Maxwell distribution where their number is exponentially small, so that the damping rate (4.2.2.7) is also exponentially small. We find in that case from eqn. (4.2.3.4) that γ is proportional to the characteristic factor

$$\gamma \propto \exp(-1/2k^2r_D^2).$$

When the wavevector increases the phase velocity of the Langmuir oscillations, $v_{ph} \approx \omega_{pe}/k$, decreases, the number of resonance electrons which effectively interact with the wave increases and the Landau damping therefore increases. When $kr_D \sim 1$, the phase velocity of the longitudinal Langmuir oscillations becomes comparable with the electron thermal velocity and the number of resonance electrons becomes so large that the damping rate becomes of the order of the frequency.

We emphasize that the picture of the absorption of oscillations by resonance electrons which we have considered here refers to the case of sufficiently weak fields. Indeed, this is necessary in order that over a free flight time of a particle "captured" by the potential well, which is equal to

$$\Delta t \sim \frac{1}{k\,\Delta w},$$

the amplitude of the field undergoes damping, that is,

$$\gamma\,\Delta t \sim \frac{\gamma}{k}\sqrt{\frac{m_e}{e\varphi_0}} \gg 1. \tag{4.2.3.5}$$

Indeed, in the opposite case the "captured" particle succeeds in performing several oscillations in the potential well before the amplitude of the field manages to change. If in one collision with the wall the particle acquired energy from the field, it would give off energy to the field in the next collision, and so on. As a result the exchange of energy between captured particles and field will be strongly inhibited and the damping of the field will thus be strongly diminished as compared to the damping given by the linear theory (when taking this effect into account one must use a non-linear theory).

One can also obtain inequality (4.2.3.5) from the condition for the applicability of the linear theory which clearly is of the form

$$\left|\frac{\partial f}{\partial w}\right| \ll \left|\frac{\partial f_0}{\partial w}\right|. \tag{4.2.3.6}$$

Taking into account that according to (4.2.1.9)

$$f \sim \frac{ek\varphi_0}{m_e(\omega' - kw)}\frac{df_0}{dw},$$

we get

$$\left|\frac{\partial f}{\partial w}\right|_{w=\omega(k)/k} \sim \frac{e\varphi_0}{m_e\gamma^2}\left|\frac{df_0}{dw}\right|_{w=\omega(k)/k}.$$

Substituting this expression into the inequality (4.2.3.6) we get condition (4.2.3.5).

Landau damping occurs thus only for very weak fields for which condition (4.2.3.5) is satisfied.

4.2.4. KINETIC THEORY OF ION-SOUND OSCILLATIONS

We now turn to a study of ion-sound oscillations based upon kinetic theory.† We shall assume that the electron- and ion-velocity distributions in the plasma are Maxwellian with temperatures T_e and T_i, respectively, when there are no oscillations. Linearizing the kinetic

† Gordeev (1954a) was the first to consider this problem.

equations for the electrons and the ions and solving them together with the Poisson equation, using Fourier and Laplace transformations, as we did for the case of purely electron oscillations, we get the following dispersion equation:

$$1+\frac{\omega_{pe}^2}{k^2v_e^2}[1+i\sqrt{(\pi)}\,z_e w(z_e)]+\frac{\omega_{pi}^2}{k^2v_i^2}[1+i\sqrt{(\pi)}\,z_i w(z_i)]=0, \qquad (4.2.4.1)$$

where

$$z_\alpha=\frac{\omega'}{\sqrt{(2)}kv_\alpha}, \quad v_\alpha=\sqrt{\frac{T_\alpha}{m_\alpha}}, \quad \alpha=\text{e, i}.$$

For ion-sound oscillations we can assume in accordance with inequalities (4.1.3.1) that $|z_e|\ll 1$ and $|z_i|\gg 1$. We can then put in (4.2.4.1)

$$w(z_e)\approx 1, \quad w(z_i)\approx\frac{i}{\sqrt{(\pi)}z_i}\left[1+\frac{1}{2z_i^2}+\frac{3}{4z_i^4}\right]+e^{-z_i^2},$$

after which the dispersion equation takes the form

$$1+\frac{\omega_{pe}^2}{k^2v_e^2}(1+i\sqrt{(\pi)}z_e)-\frac{\omega_{pi}^2}{\omega'^2}-\frac{3\omega_{pi}^2k^2v_i^2}{\omega'^4}+\frac{\omega_{pi}^2}{k^2v_i^2}i\sqrt{(\pi)}z_ie^{-z_i^2}=0. \qquad (4.2.4.2)$$

If we neglect here terms proportional to i and take into account that the quantity $k^2v_i^2/\omega^2$ is small, we get for the frequency of the ion-sound oscillations the following expression:

$$\omega(k)=\omega_s(k)\left(1+\frac{3}{2}\frac{k^2v_i^2}{\omega_s^2}\right), \qquad (4.2.4.3)$$

where

$$\omega_s(k)=\frac{kv_s}{\sqrt{(1+k^2r_D^2)}},$$

and where v_s is given by (4.1.3.5″). The small term proportional to $k^2v_i^2/\omega_s^2$ takes into account the influence of the thermal motion of the ions on the ion sound frequency.

In the short-wavelength region, $kr_D\gg 1$, the frequency $\omega(k)$ is close to ω_{pi}:

$$\omega(k)=\omega_{pi}\left(1-\frac{1}{2k^2r_D^2}+\frac{3}{2}k^2r_D'^2\right), \qquad (4.2.4.4)$$

where r_D' is the ion Debye radius, $r_D'=v_i/\omega_{pi}=\sqrt{(T_i/4\pi n_0e^2)}$. We note that when $T_i=0$, the frequency $\omega(k)=\omega_s(k)$ is always less than the frequency ω_{pi}, while for finite T_i the inequality $\omega(k)<\omega_{pi}$ holds only when $\sqrt{3}\,k^2r_Dr_D'<1$. In the region $\sqrt{3}\,k^2r_Dr_D'>1$ the frequency of the ion-sound oscillations is larger than ω_{pi}.

Taking in the next approximation the imaginary terms in (4.2.4.2) into account we find the damping rate of the ion-sound oscillations:

$$\gamma(k)=\gamma_e+\gamma_i, \qquad (4.2.4.5)$$

$$\gamma_e=\sqrt{\frac{\pi m_e}{8m_i}}\frac{kv_s}{(1+k^2r_D^2)^2}, \quad \gamma_i=\sqrt{\frac{\pi}{8}}\left(\frac{T_e}{T_i}\right)^{3/2}\frac{kv_s}{(1+k^2r_D^2)^2}\exp\left(-\frac{\omega^2(k)}{2k^2v_i^2}\right). \qquad (4.2.4.6)$$

The quantity γ_e determines the damping of ion sound caused by the absorption of ion-sound oscillations by resonance electrons and the quantity γ_i determines the damping caused by the absorption by resonance ions.

When $kr_D \lesssim 1$, we have, as to order of magnitude,

$$\frac{\gamma_e}{\omega(k)} \sim \sqrt{\frac{m_e}{m_i}} \ll 1.$$

Notwithstanding the fact that the number of resonance electrons is large when $v_{ph} \sim v_s \ll v_e$, the difference between the number of slow and fast electrons, which is proportional to $-df_0/dw$, is small when $w = v_{ph} \ll v_e$ so that the damping rate γ_e is also small. As $\omega/k \gg v_i$, the damping rate γ_i is exponentially small.

When the wavevector increases, the quantity γ_i increases and when $kr'_D \sim 1$, that is, when $\omega/k \sim v_i$, the oscillations are strongly absorbed by the ions, $\gamma_i \sim \omega \sim kv_i$. When $T_e \gg T_i(1+k^2r_D^2)$, the condition $v_{ph} \gg v_i$ is satisfied. When the difference between the ion and electron temperatures decreases, the damping of the ion-sound oscillations, caused by resonance ions, increases fast and when $T_e \sim T_i$ the ion-sound oscillations become strongly damped, $\gamma_i \sim \omega_s(k)$, when $kr_D \sim kr'_D \lesssim 1$.

4.3. Kinetic Theory of Electromagnetic Waves in a Plasma

4.3.1. THE DIELECTRIC PERMITTIVITY TENSOR AND THE DISPERSION EQUATION FOR ELECTROMAGNETIC WAVES IN A UNIFORM PLASMA

In the preceding section we developed a kinetic theory for longitudinal plasma oscillations and applied it to the actual case of a uniform plasma with a Maxwellian velocity distribution for the particles. We shall now turn to a further study of oscillations in a collisionless plasma without assuming that the electrical field is irrotational. We shall thus develop a kinetic theory for the propagation of electromagnetic waves in a uniform collisionless plasma. One of the results of this theory consists in that in the general case of a plasma with an anisotropic velocity distribution for the particles one can only approximately distinguish longitudinal oscillations.

We shall start from the well-known equation

$$\text{curl curl}\, \boldsymbol{E} + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}}{\partial t}, \tag{4.3.1.1}$$

which connects the electrical field $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r}, t)$ with the electrical current density $\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{r}, t)$; this equation follows from the Maxwell eqns. (4.1.1.3).

In the problem in which we are interested $\boldsymbol{E}$ is the electrical field of an electromagnetic wave propagating in the plasma and $\boldsymbol{j}$ the current density produced by the particles in the plasma,

$$\boldsymbol{j} = \sum_\alpha \boldsymbol{j}_\alpha(\boldsymbol{r}, t),$$

where j_α is the current density produced by particles of the α-th kind which is determined by the distribution function $F_\alpha(r, v, t)$ for those particles,

$$j_\alpha(r,t) = e_\alpha \int v F_\alpha(r, v, t)\, d^3v,$$

where e_α is the charge of a particle of the α-th kind.

In turn the distribution function $F_\alpha(r, v, t)$ is determined by the kinetic equation with a self-consistent field and without a collision integral (we consider the case of high frequencies),

$$\frac{\partial F_\alpha}{\partial t} + (v\cdot\nabla)F_\alpha + \frac{e_\alpha}{m_\alpha}\left(\left\{E + \frac{1}{c}[v\wedge B]\right\}\cdot\frac{\partial F_\alpha}{\partial v}\right) = 0,$$

where m_α is the mass of a particle of the α-th kind and B the magnetic field of the wave, which is connected with the electrical field through the relation

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\operatorname{curl} E. \tag{4.3.1.2}$$

In the present section we shall study only small plasma oscillations and we put therefore

$$F_\alpha(r, v, t) = f_{\alpha 0}(v) + f_\alpha(r, v, t),$$

where $f_{\alpha 0}(v)$ is the velocity distribution function of the particles when there are no oscillations and $f_\alpha(r, v, t)$ a small correction which is connected with the plasma oscillations, $|f_\alpha| \ll f_{\alpha 0}$. The kinetic equation can then be linearized in f_α:

$$\frac{\partial f_\alpha}{\partial t} + (v\cdot\nabla)f_\alpha + \frac{e_\alpha}{m_\alpha}\left(\left\{E + \frac{1}{c}[v\wedge B]\right\}\cdot\frac{\partial f_{\alpha 0}}{\partial v}\right) = 0. \tag{4.3.1.3}$$

We shall assume that the distribution functions in the absence of oscillations, taken together, produce neither a charge nor a current, that is, that we have the relations

$$\sum_\alpha e_\alpha \int f_{\alpha 0}(v)\, d^3v = 0, \quad \sum_\alpha e_\alpha \int v f_{\alpha 0}(v)\, d^3v = 0.$$

The current density occurring in (4.3.1.1) is thus determined by the formula

$$j(r, t) = \sum_\alpha e_\alpha \int v f_\alpha(r, v, t)\, d^3v. \tag{4.3.1.4}$$

Equations (4.3.1.1) to (4.3.1.4) are a closed self-consistent set of equations describing the propagation of small amplitude electromagnetic waves in a uniform collisionless plasma. We can find a solution of this set of equations together with the appropriate boundary conditions by proceeding as we did in Subsection 4.2.1 when studying longitudinal oscillations, that is, using Fourier transforms for the spatial coordinates and a Laplace transform for the time.

We shall be interested in the asymptotic behaviour of the field and of the particle distribution functions for large t (but with $t \ll \tau$!). We saw in Subsection 4.2.1 that in the case of longitudinal oscillations this asymptotic behaviour was determined by the roots of a transcendental equation—the dispersion equation—which could be established without solving the corresponding set of equations for the field and the particle distribution functions with well-defined boundary conditions. It was there important that the dispersion equation did not carry any traces of the boundary conditions and was solely determined by the particle distribution functions in the original state of the plasma, that is, when there were no oscillations.

The situation is similar also in the general case of non-longitudinal plasma oscillations and we must, indeed, when establishing the dispersion equation, assume that all variable quantities occurring in the set (4.3.1.1) to (4.3.1.4), that is, the quantities $\boldsymbol{E}$, $\boldsymbol{B}$, and f_α have the form of plane monochromatic waves,

$$\boldsymbol{E},\ \boldsymbol{B},\ f_\alpha \propto e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega' t},$$

where $\boldsymbol{k}$ is the wavevector and ω' the complex frequency of the oscillations, and we must find the condition that the linear set (4.3.1.1) to (4.3.1.4) can be solved.

For plane monochromatic waves eqns. (4.3.1.1) to (4.3.1.3) clearly become

$$\begin{aligned} &[\boldsymbol{k}\wedge[\boldsymbol{k}\wedge\boldsymbol{E}]]+\frac{\omega'^2}{c^2}\boldsymbol{E} = -\frac{4\pi i}{c^2}\omega'\boldsymbol{j}, \\ &\boldsymbol{B} = \frac{c}{\omega'}[\boldsymbol{k}\wedge\boldsymbol{E}], \\ &f_\alpha = -\frac{ie_\alpha}{m_\alpha[\omega'-(\boldsymbol{k}\cdot\boldsymbol{v})]}\left(\left\{\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial f_{\alpha 0}}{\partial \boldsymbol{v}}\right), \end{aligned} \tag{4.3.1.5}$$

where the current density, defined by eqn. (4.3.1.4), can be written in the form

$$\begin{aligned} &\boldsymbol{j} = \sum_\alpha \boldsymbol{j}_\alpha, \quad j_{\alpha i} = \sum_j \sigma_{ij}^{(\alpha)}E_j, \\ &\sigma_{ij}^{(\alpha)}(\boldsymbol{k},\omega') = -\frac{ie_\alpha^2}{m_\alpha}\int v_i\left\{\left[1-\frac{(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega'}\right]\frac{\partial f_{\alpha 0}}{\partial v_j}+\frac{v_j}{\omega'}\left(\boldsymbol{k}\cdot\frac{\partial f_{\alpha 0}}{\partial \boldsymbol{v}}\right)\right\}\frac{d^3v}{\omega'-(\boldsymbol{k}\cdot\boldsymbol{v})}, \end{aligned} \tag{4.3.1.6}$$

if we use the last of eqns. (4.3.1.5). The integration is performed here, in accordance with the results of Subsection 4.2.1, over $w = (\boldsymbol{k}\cdot\boldsymbol{v})/k$ from $-\infty$ to $+\infty$ along the real axis, encircling the singularity $w = \omega'/k$ from below. One can also use the equivalent formula

$$\frac{1}{\omega'-(\boldsymbol{k}\cdot\boldsymbol{v})} = \mathcal{P}\frac{1}{\omega'-(\boldsymbol{k}\cdot\boldsymbol{v})}-\pi i\delta\{\omega'-(\boldsymbol{k}\cdot\boldsymbol{v}\}, \tag{4.3.1.7}$$

where $\mathcal{P}$ is the principal value symbol (see also van Kampen, 1955).

We can put the set of eqns. (4.3.1.5) and (4.3.1.6) in a simpler and symmetrical form by introducing the electrical induction of the plasma,

$$\boldsymbol{D} = \boldsymbol{E}+\frac{4\pi i}{\omega'}\boldsymbol{j},$$

and the high-frequency dielectric permittivity tensor of the plasma, $\varepsilon_{ij} = \varepsilon_{ij}(\boldsymbol{k},\omega')$,

$$D_i = \sum_j \varepsilon_{ij}E_j, \tag{4.3.1.8}$$

where

$$\varepsilon_{ij} = \delta_{ij} + \sum_\alpha \pi_{ij}^{(\alpha)},$$

$$\pi_{ij}^{(\alpha)} = \frac{4\pi i}{\omega'}\sigma_{ij}^{(\alpha)} = \frac{4\pi e_\alpha^2}{m_\alpha \omega'^2}\int \sum_l \frac{\partial f_{\alpha 0}}{\partial v_l}\,\frac{v_i\{[\omega' - (\boldsymbol{k}\cdot\boldsymbol{v})]\delta_{lj} + k_l v_j\}}{\omega' - (\boldsymbol{k}\cdot\boldsymbol{v})}\,d^3v; \tag{4.3.1.9}$$

the quantity $\sigma_{ij}^{(\alpha)}$ is called the high-frequency conductivity of the α-th component of the plasma and $\pi_{ij}^{(\alpha)}$ the high-frequency polarizability of this component.

Introducing the dielectric permittivity tensor of the plasma we can write eqns. (4.3.1.5) and (4.3.1.6) in the form

$$\sum_j \Lambda_{ij}(\boldsymbol{k}, \omega')E_j = 0, \tag{4.3.1.10}$$

where

$$\Lambda_{ij}(\boldsymbol{k}, \omega') = n^2(\varkappa_i\varkappa_j - \delta_{ij}) + \varepsilon_{ij}(\boldsymbol{k}, \omega'), \tag{4.3.1.10'}$$

$\boldsymbol{\varkappa} = \boldsymbol{k}/k$ is a unit vector in the direction of the wave propagation and $n \equiv kc/\omega'$ is the refractive index.

The condition that the homogeneous set (4.3.1.10) of linear algebraic equations for the E_j can be solved gives a dispersion equation for the electromagnetic oscillations in a plasma:

$$\Lambda(\boldsymbol{k}, \omega') \equiv \text{Det}\,|\Lambda_{ij}(\boldsymbol{k}, \omega')| = 0. \tag{4.3.1.11}$$

This equation determines for given $\boldsymbol{k}$ a number of complex frequencies $\omega' = \omega_\nu(\boldsymbol{k}) - i\gamma_\nu(\boldsymbol{k})$, corresponding to different branches of plasma oscillations. The quantity $\omega_\nu(\boldsymbol{k})$ is here the frequency and $\gamma_\nu(\boldsymbol{k})$ the damping rate (when $\gamma_\nu > 0$) or the growth rate (when $\gamma_\nu < 0$) of the ν-th eigen oscillation.

Equations (4.3.1.9) and (4.3.1.11) completely determine the behaviour of plane electromagnetic waves in a uniform plasma when there are no external fields, but taking into account the thermal motion of the particles in the plasma.

A characteristic peculiarity of the expressions which we obtained for the conductivity and the dielectric constant of the plasma is that these quantities depend not only on the frequency but also on the wavevector. In that case one speaks of *temporal and spatial dispersion of the dielectric permittivity.*

The presence of such a dispersion means that the magnitude of the current density in some point in space $\boldsymbol{r}$ and at a given time t is determined by the field not only in that point of space and at that moment in time, but also in all other points of space and at all earlier times. Indeed, we see easily, if we use (4.3.1.9) that

$$j_i(\boldsymbol{r}, t) = \iint j_i(\boldsymbol{k}, \omega)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}) - i\omega t}\,d^3k\,d\omega = \iint K_{ij}(\boldsymbol{r}-\boldsymbol{r}', t-t')\,E_j(\boldsymbol{r}', t')\,d^3r'\,dt', \tag{4.3.1.12}$$

where

$$K_{ij}(\boldsymbol{r}, t) = \frac{1}{2\pi}\iint \sigma_{ij}(\boldsymbol{k}, \omega)e^{i(\boldsymbol{k}\cdot\boldsymbol{r}) - i\omega t}\,d^3k\,d\omega,$$

and

$$\sigma_{ij}(\boldsymbol{k}, \omega) = \sum_\alpha \sigma_{ij}^{(\alpha)}(\boldsymbol{k}, \omega).$$

By virtue of the causality principle $K_{ij}(r, t)$ must clearly vanish when $ct > r$,

$$K_{ij}(r, t) = 0, \quad \text{when} \quad ct > r.$$

One can, indeed, easily verify by using the actual expression (4.3.1.9) for $\sigma_{ij}^{(\alpha)}$ that this condition is satisfied.

In the general case the determinant Λ is a complex function of ω and $\boldsymbol{k}$, so that condition (4.3.1.11) reduces to requiring that the real and imaginary parts of Λ vanish separately:

$$\operatorname{Re} \Lambda(\boldsymbol{k}, \omega') = 0, \quad \operatorname{Im} \Lambda(\boldsymbol{k}, \omega') = 0. \tag{4.3.1.13}$$

In the region where the plasma is transparent—the region of $\boldsymbol{k}$ and ω'-values for which the anti-Hermitean part of the dielectric permittivity tensor is small compared to the Hermitean part—the imaginary part of Λ will be small compared to its real part. When neglecting the damping of the waves we can thus write the dispersion equation approximately in the form

$$\operatorname{Re} \Lambda(\boldsymbol{k}, \omega) = 0.$$

If we determine the eigenfrequency of the wave from this, $\omega = \omega_\nu(\boldsymbol{k})$, and if we assume the damping to be small, we find easily from eqn. (4.3.1.11) the damping rate:

$$\gamma_\nu(\boldsymbol{k}) = \left. \frac{\operatorname{Im} \Lambda(\omega, \boldsymbol{k})}{\partial \operatorname{Re} \Lambda(\omega, \boldsymbol{k})/\partial\omega} \right|_{\omega=\omega_\nu(\boldsymbol{k})}. \tag{4.3.1.14}$$

In concluding this subsection we want to discuss the problem of energy transfer in a dispersive medium with weak damping. Let us consider a wavepacket which we can write in the form

$$\boldsymbol{E}(\boldsymbol{r}, t) = \operatorname{Re} \boldsymbol{E}_0(\boldsymbol{r}, t) e^{i(\boldsymbol{k}\cdot\boldsymbol{r}) - i\omega t}, \tag{4.3.1.15}$$

where $\boldsymbol{E}_0(\boldsymbol{r}, t)$ is the amplitude of the wave which changes slowly in space and time, and $\omega = \omega(\boldsymbol{k})$ is the "central" frequency of the wavepacket, corresponding to the "central" wavevector $\boldsymbol{k}$. From the Maxwell equations we obtain the energy conservation law

$$\frac{\partial}{\partial t} \frac{E^2 + B^2}{8\pi} + \operatorname{div} \frac{c}{4\pi} [\boldsymbol{E} \wedge \boldsymbol{B}] + (\boldsymbol{j} \cdot \boldsymbol{E}) = 0. \tag{4.3.1.16}$$

Taking the time-average of eqn. (4.3.1.16) and using the fact that the anti-Hermitean parts of the tensor ε_{ij} are small compared to the Hermitean parts ε'_{ij}, we get the following equation for the energy transfer in a weakly absorbing medium with spatial dispersion:

$$\frac{\partial w}{\partial t} + \operatorname{div} \boldsymbol{S} + Q = 0, \tag{4.3.1.17}$$

where w and $\boldsymbol{S}$ are the average values of the energy density and the energy flux density, while Q is the magnitude of the energy loss. These quantities are defined by the equations

$$w = \frac{1}{\omega} \sum_{i,j} \frac{\partial \omega^2 \varepsilon'_{ij}(\boldsymbol{k}, \omega)}{\partial \omega} \frac{E_{0i}^* E_{0j}}{16\pi}, \tag{4.3.1.18}$$

$$S = \frac{c^2}{8\pi\omega}\left\{k|E_0|^2 - \frac{1}{2}(k\cdot E_0)E_0^* - \frac{1}{2}(k\cdot E_0^*)E_0\right\} - \omega\sum_{i,j}\frac{\partial\varepsilon'_{ij}(k,\omega)}{\partial k}\frac{E_{0i}^*E_{0j}}{16\pi}, \tag{4.3.1.19}$$

$$Q = -\frac{i\omega}{8\pi}\sum_{i,j}(\varepsilon_{ij} - \varepsilon_{ji}^*)E_{0i}^*E_{0j}. \tag{4.3.1.20}$$

The quantity w includes both the energy of the electromagnetic field and the kinetic energy of the particle oscillations in the field of the wave. The first term in (4.3.1.19) is the flux of the electromagnetic energy and the second one the energy flux density of the particles in the plasma. The quantities w and S are connected through the relation

$$S = v_{gr}w, \tag{4.3.1.21}$$

where v_{gr} is the group velocity

$$v_{gr} = \frac{\partial\omega(k)}{\partial k}. \tag{4.3.1.22}$$

It follows from (4.3.1.20) that the energy losses are proportional to the anti-Hermitean parts ε''_{ij} of the tensor ε_{ij}.

Equations (4.3.1.17) to (4.3.1.21) were introduced by Shafranov (1967).

4.3.2. POLARIZATION OF PLASMA WAVES

Each plane monochromatic electromagnetic wave possesses a well-defined polarization. The unit vector e which is directed along the electrical field of the wave will be called the polarization vector or, simply, the polarization. According to (4.3.1.10) the polarization vector $e^{(\nu)}$ of the ν-th branch of the oscillations satisfies the equation

$$\sum_j \Lambda_{ij}(k, \omega_\nu(k))e_j^{(\nu)} = 0; \tag{4.3.2.1}$$

its general solution has the form

$$e_i^{(\nu)} = C\sum_j \lambda_{ij}(k, \omega_\nu(k))a_j, \tag{4.3.2.2}$$

where a is an arbitrary vector, C a constant determined by the normalization condition

$$(e^{(\nu)}\cdot e^{(\nu)*}) = 1,$$

and the λ_{ij} are the cofactors of the Λ_{ij} in the determinant $\Lambda = \mathrm{Det}|\Lambda_{ij}|$,

$$\sum_j \Lambda_{ij}\lambda_{jk} = \Lambda\delta_{ik}, \quad \lambda_{ij} = \sum_{k,l,m,n}\tfrac{1}{2}\varepsilon_{ikl}\varepsilon_{jmn}\Lambda_{mk}\Lambda_{nl}, \tag{4.3.2.3}$$

where ε_{ikl} is the totally antisymmetric third rank unit tensor. To check the validity of (4.3.2.2) we must use the definitions (4.3.2.2) and the fact that the determinant Λ vanishes for the eigenfrequencies,

$$\Lambda(k, \omega_\nu(k)) = 0.$$

We shall show that eqn. (4.3.2.2) determines the polarization vector apart from a phase factor. We note first of all that the matrices λ_{ij} and Λ_{ij} are connected with one another through the relation

$$\lambda_{ij}\lambda_{kl} = \lambda_{il}\lambda_{kj} + \Lambda \sum_{n,m} \varepsilon_{ikm}\varepsilon_{jnl}\Lambda_{nm} . \tag{4.3.2.4}$$

To check the validity of this equation we multiply the left-hand and right-hand sides of the equation $\Lambda\varepsilon_{abc} = \sum_{m,n,p} \varepsilon_{mnp}\Lambda_{ma}\Lambda_{nb}\Lambda_{pc}$ by $\lambda_{aj}\lambda_{bl}\varepsilon_{ikc}$ and sum over a, b, and c.

For eigenfrequencies which satisfy the dispersion equation $\Lambda = 0$, eqn. (4.3.2.4) simplifies to

$$\lambda_{ij}\lambda_{kl} = \lambda_{il}\lambda_{kj} . \tag{4.3.2.5}$$

Neglecting the anti-Hermitean part of Λ in the transparency region one can easily deduce from (4.3.2.5) the equation

$$\frac{\sum_{l,k} \lambda_{il}a_l\lambda^*_{jk}a_k}{\sum_{m,n} \lambda_{mn}a_m a_n} = \frac{\sum_{l,k} \lambda_{il}a'_l\lambda^*_{jk}a'_k}{\sum_{m,n} \lambda_{mn}a'_m a'_n} (= \lambda_{ij}), \tag{4.3.2.6}$$

where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are arbitrary vectors. We note also that for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$ the scalar products $(\boldsymbol{a}\hat{\boldsymbol{\lambda}}\boldsymbol{a})$ and $(\boldsymbol{a}'\hat{\boldsymbol{\lambda}}\boldsymbol{a}')$ have the same sign:

$$(\boldsymbol{a}\hat{\boldsymbol{\lambda}}\boldsymbol{a})(\boldsymbol{a}'\hat{\boldsymbol{\lambda}}\boldsymbol{a}') = |(\boldsymbol{a}\hat{\boldsymbol{\lambda}}\boldsymbol{a}')|^2 . \tag{4.3.2.7}$$

In particular, the diagonal elements of the Hermitean part of the matrix λ_{ij} have the same sign.

Using now eqn. (4.3.2.5) we can write the normalization constant C in the form

$$C = \{(\boldsymbol{a}\hat{\boldsymbol{\lambda}}\boldsymbol{a}) \operatorname{Tr} \lambda\}^{-1/2} .$$

The normalized polarization vector $\boldsymbol{e}^{(\nu)}$ thus has the form

$$\boldsymbol{e}^{(\nu)} = \left. \frac{\hat{\boldsymbol{\lambda}}\boldsymbol{a}}{\sqrt{[(\boldsymbol{a}\hat{\boldsymbol{\lambda}}\boldsymbol{a}) \operatorname{Tr} \lambda]}} \right|_{\omega=\omega_\nu(\boldsymbol{k})} . \tag{4.3.2.8}$$

According to (4.3.2.6) the product $e_i e_j^*$ is invariant under a change in the vector $\boldsymbol{a}$ so that under an arbitrary change in the vector $\boldsymbol{a}$ only a phase factor can change in eqn. (4.3.2.8).

4.3.3. EXCITATION OF WAVES IN A PLASMA

Let us now consider the excitation of electromagnetic waves in a plasma by external currents. It follows from the Maxwell equations that the strength of the excitation of waves, that is, the increase in the electromagnetic energy of the plasma per unit time, is determined by the equation

$$I \equiv \frac{1}{4\pi}\int d^3\boldsymbol{r}\left[\left(\boldsymbol{E}\cdot\frac{\partial \boldsymbol{D}}{\partial t}\right)+\left(\boldsymbol{H}\cdot\frac{\partial \boldsymbol{B}}{\partial t}\right)\right] = \frac{c}{4\pi}\int (d^2\boldsymbol{S}\cdot[\boldsymbol{E}\wedge\boldsymbol{B}]) - \int d^3\boldsymbol{r}(\boldsymbol{j}_0\cdot\boldsymbol{E}), \tag{4.3.3.1}$$

where $\boldsymbol{j}_0 \equiv \boldsymbol{j}_0(\boldsymbol{r}, t)$ is the density of the external currents which we shall here assume to be a given function of the coordinates and the time.

In the case of an unbounded plasma the surface integral in (4.3.3.1) vanishes. Using a complex notation we can write the strength of the excitation in this case in the form

$$I = -\tfrac{1}{2}\,\mathrm{Re}\int (\boldsymbol{j}_0^*\cdot\boldsymbol{E})\,d^3r. \tag{4.3.3.2}$$

To find the total energy P transferred by the external currents to the plasma we must integrate this expression over time,

$$P = -\tfrac{1}{2}\,\mathrm{Re}\iint d^3r\,dt(\boldsymbol{j}_0^*\cdot\boldsymbol{E}). \tag{4.3.3.3}$$

Expanding the external current $\boldsymbol{j}_0$ and the electrical field of the excited waves $\boldsymbol{E}$ in Fourier integrals we can write this expression in the form

$$P = -\frac{1}{2(2\pi)^4}\,\mathrm{Re}\iint d^3k\,d\omega(\boldsymbol{j}_0^*(\boldsymbol{k},\omega)\cdot\boldsymbol{E}(\boldsymbol{k},\omega)). \tag{4.3.3.4}$$

The electrical field strength $\boldsymbol{E}(\boldsymbol{r}, t)$ when there are external currents present is determined from the equation

$$\mathrm{curl}\;\mathrm{curl}\;\boldsymbol{E}+\frac{1}{c^2}\frac{\partial^2\boldsymbol{D}}{\partial t^2} = -\frac{4\pi}{c^2}\frac{\partial\boldsymbol{j}_0}{\partial t},$$

or, in Fourier components

$$\sum_j \Lambda_{ij}(\boldsymbol{k},\omega)E_j(\boldsymbol{k},\omega) = \frac{4\pi i}{\omega}j_{0i}(\boldsymbol{k},\omega). \tag{4.3.3.5}$$

Hence we find that

$$\boldsymbol{E}(\boldsymbol{k},\omega) = -\frac{4\pi i}{\omega\Lambda(\boldsymbol{k},\omega)}\,\hat{\boldsymbol{\lambda}}\boldsymbol{j}_0(\boldsymbol{k},\omega) \tag{4.3.3.6}$$

As in the transparency region eigen oscillations are excited in the plasma for which the following relation holds:

$$\lambda_{ij} = e_i e_j^*\,\mathrm{Tr}\,\lambda, \tag{4.3.3.7}$$

we have

$$P = \frac{1}{(2\pi)^3}\,\mathrm{Re}\left(i\iint d^3k\,d\omega\,\frac{\mathrm{Tr}\,\lambda}{\omega\Lambda}\,|(\boldsymbol{e}\cdot\boldsymbol{j}_0^*(\boldsymbol{k},\omega))|^2\right). \tag{4.3.3.8}$$

Using the fact that Tr λ is real, we can write this equation in the form

$$P = \frac{1}{(2\pi)^3}\iint d^3k\,d\omega\,\frac{\mathrm{Tr}\,\lambda}{\omega}\,\frac{\mathrm{Im}\,\Lambda}{|\Lambda|^2}\,|(\boldsymbol{e}\cdot\boldsymbol{j}_0^*(\boldsymbol{k},\omega))|^2. \tag{4.3.3.9}$$

The quantity Im Λ can be connected with the magnitude of the electrical losses in the plasma,

$$Q = -\frac{i\omega}{8\pi}\sum_{i,j}(\varepsilon_{ij}-\varepsilon_{ji}^*)e_i^*e_j\,|\boldsymbol{E}|^2. \tag{4.3.3.10}$$

Indeed, one can easily verify directly that Im Λ in the transparency region can be expressed in terms of the Hermitean part of the matrix λ_{ij} and the anti-Hermitean part of the dielectric permittivity ε_{ij},

$$\operatorname{Im} \Lambda = \frac{1}{4i} \sum_{i,j} (\varepsilon_{ij} - \varepsilon_{ji}^*)(\lambda_{ji} + \lambda_{ij}^*) \tag{4.3.3.11}$$

Using now eqn. (4.3.3.7) we find

$$\operatorname{Im} \Lambda = \frac{\operatorname{Tr} \lambda}{2i} \sum_{i,j} (\varepsilon_{ij} - \varepsilon_{ji}^*) e_i^* e_j , \tag{4.3.3.12}$$

and we see the relation with Q.

As for a thermodynamically stable system we must, by virtue of the second law of thermodynamics, have $Q > 0$, then the following condition must hold:

$$\frac{\operatorname{Tr} \lambda}{\omega} \operatorname{Im} \Lambda > 0. \tag{4.3.3.13}$$

Using this condition and noting that in the transparency region

$$\frac{\operatorname{Im} \Lambda}{|\Lambda|^2} \to \pi \delta(\Lambda),$$

we get finally for the total energy transferred from the currents to the plasma the following expression:

$$P = \frac{1}{8\pi^2} \iint d^3k \, d\omega \left| \frac{\operatorname{Tr} \lambda}{\omega} \right| |(\boldsymbol{e} \cdot \boldsymbol{j}_0^*(\boldsymbol{k}, \omega))|^2 \, \delta\{\Lambda(\boldsymbol{k}, \omega)\}. \tag{4.3.3.14}$$

4.3.4. THE DIELECTRIC PERMITTIVITY TENSOR IN THE CASE OF AN ISOTROPIC PARTICLE DISTRIBUTION

If the velocity distribution of the particles in the plasma is isotropic, that is, if the distribution function depends only on the particle energy, $\varepsilon = \frac{1}{2} m_\alpha v^2$,

$$f_{\alpha 0} \equiv f_{\alpha 0}(\varepsilon),$$

the polarizability tensor $\pi_{ij}^{(\alpha)}(\boldsymbol{k}, \omega)$ takes the form ($f'_{\alpha 0} \equiv \partial f_{\alpha 0}/\partial \varepsilon$)

$$\pi_{ij}^{(\alpha)}(\boldsymbol{k}, \omega') = \frac{4\pi e_\alpha^2}{\omega'} \int \frac{v_i v_j}{\omega' - (\boldsymbol{k} \cdot \boldsymbol{v})} f'_{\alpha 0}(\varepsilon) \, d^3\boldsymbol{v}. \tag{4.3.4.1}$$

One can easily establish the general structure of this tensor and hence also of the tensor ε_{ij}. As in the case considered we have only one vector at our disposal, namely, the wavevector, we can construct only two independent second rank tensors: the unit tensor δ_{ij} and the tensor $k_i k_j$. The tensor ε_{ij} must thus have the following structure:

$$\varepsilon_{ij}(\boldsymbol{k}, \omega') = (\delta_{ij} - \varkappa_i \varkappa_j) \, \varepsilon_t(\boldsymbol{k}, \omega') + \varkappa_i \varkappa_j \varepsilon_l(\boldsymbol{k}, \omega'), \tag{4.3.4.2}$$

where $\boldsymbol{\varkappa} = \boldsymbol{k}/k$ and ε_t and ε_l are functions of $\boldsymbol{k}$ and ω'. They are called the *transverse and the longitudinal plasma permittivity*, respectively.

Comparing eqns. (4.3.4.1) and (4.3.4.2) we can easily find the longitudinal and transverse plasma permittivities:†

$$\begin{aligned} \varepsilon_l(\boldsymbol{k}, \omega') &= 1+\sum_\alpha \frac{4\pi e_\alpha^2}{k^2}\int \frac{(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega'-(\boldsymbol{k}\cdot\boldsymbol{v})} f'_{\alpha 0}(\varepsilon)\, d^3v, \\ \varepsilon_t(\boldsymbol{k}, \omega') &= 1+\sum_\alpha \frac{2\pi e_\alpha^2}{\omega'}\int \frac{v_\perp^2}{\omega'-(\boldsymbol{k}\cdot\boldsymbol{v})} f'_{\alpha 0}(\varepsilon)\, d^3v, \end{aligned} \tag{4.3.4.3}$$

where $\boldsymbol{v}_\perp$ is the component of $\boldsymbol{v}$ at right angles to $\boldsymbol{k}$, $\boldsymbol{v}_\perp = [\boldsymbol{\varkappa}\wedge[\boldsymbol{v}\wedge\boldsymbol{\varkappa}]]$.

We see thus that in the kinetic theory the dielectric properties of a plasma with an isotropic velocity distribution for the particles are determined by two scalar functions of the wavevector and the frequency, $\varepsilon_t(\boldsymbol{k}, \omega')$ and $\varepsilon_l(\boldsymbol{k}, \omega')$, whereas in the hydrodynamic theory theory these properties are described by only one function of the frequency, $\varepsilon(\omega')$, that is, in that case we have $\varepsilon_l(\boldsymbol{k}, \omega') = \varepsilon_t(\boldsymbol{k}, \omega') = \varepsilon(\omega')$. The dielectric constant tensor is in the hydrodynamic theory proportional to the unit tensor δ_{ij},

$$\varepsilon_{ij} = \varepsilon(\omega')\delta_{ij}, \quad T \to 0,$$

in agreement with the general structure (4.3.4.2) of this tensor.

The increase in the number of functions describing the dielectric properties of the plasma when we change from the hydrodynamic to the kinetic theory is connected—as can be seen from (4.3.4.2)—with taking into account the wavevector dependence of ε_{ij}—so that we then must introduce the second independent tensor k_ik_j as well as the unit tensor δ_{ij}—that is, taking into account spatial dispersion. However, the introduction of spatial dispersion is possible only when we take into account the thermal motion of the particles and this is done in kinetic theory.

Let us now turn to the dispersion eqn. (4.3.1.11) and substitute in it expression (4.3.4.2) for ε_{ij}. As a result we get the following dispersion equation for a plasma with an isotropic velocity distribution of the particles:

$$\Lambda = \varepsilon_l(\boldsymbol{k}, \omega')\left[\frac{k^2c^2}{\omega'^2} - \varepsilon_t(\boldsymbol{k}, \omega')\right]^2 = 0.$$

It splits into the equation

$$\varepsilon_l(\boldsymbol{k}, \omega') = 0, \tag{4.3.4.4}$$

determining the frequencies of the longitudinal oscillations (such an equation has already been studied by us in Section 4.2) and the equation

$$\frac{k^2c^2}{\omega'^2} = \varepsilon_t(\boldsymbol{k}, \omega'), \tag{4.3.4.5}$$

determining the frequencies of the transverse electromagnetic waves in the plasma. It is clear that each solution of this last equation corresponds to two waves with mutually perpendicular polarization vectors, which are also perpendicular to the wavevector.

We saw that the phase velocity of the transverse electromagnetic waves which propagate

† Gertsenshtein (1952) introduced the expressions for ε_t and ε_l.

in the plasma is larger than the velocity of light, $\omega/k > c$. Hence it follows that for such waves the denominator in the integrand in eqn. (4.3.4.3) for $\varepsilon_t(k, \omega')$ nowhere vanishes—the particle velocity is always less than the velocity of light. The integration in this formula can thus be taken along the real axis without any encircling of singularities. This means that the quantity ε_t will be real for $\omega/k > c$ and hence that there is for transverse electromagnetic waves propagating in a collisionless plasma no damping similar to the Landau damping for longitudinal plasma waves. The damping is in this case completely caused by the particle collisions.

We emphasize that this conclusion refers to waves propagating in an unbounded plasma for which $\omega/k > c$. If we consider a bounded plasma, for instance, a plasma occupying a half-space, near its boundary there may occur the propagation of waves which are damped in the bulk of the plasma and which have a phase velocity which will be less than the velocity of light (skin-effect). Damping occurs in this case also in a collisionless plasma.

Before discussing this effect in detail we turn to volume transverse waves in the case of a non-relativistic plasma, when $T \ll m_\alpha c^2$, where T is the temperature of the plasma. The average thermal velocity of the particles will then be considerably less than the velocity of light,

$$v_\alpha \ll c,$$

and we can in the denominator of the integrands occurring in $\varepsilon_t(\boldsymbol{k}, \omega')$ neglect $(\boldsymbol{k}\cdot\boldsymbol{v})$ as compared to ω'. Noting that

$$\int v_\perp^2 f'_{\alpha 0}(\varepsilon)\, d^3v = -\frac{2}{m_\alpha} n_{\alpha 0},$$

where $n_{\alpha 0}$ is the equilibrium density of particles of the α-th kind, we get for ε_t the hydrodynamic expression

$$\varepsilon_t = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}, \tag{4.3.4.6}$$

where

$$\omega_{p\alpha} = \sqrt{\left(\frac{4\pi e^2 n_{\alpha 0}}{m_\alpha}\right)}.$$

Taking the thermal motion of the particles into account leads only to small corrections, of order $T/m_\alpha c^2$, to this expression. The relativistic terms in the kinetic equations which we have neglected are also of the same order of magnitude.

The thermal motion of the electrons can only affect the behaviour of the electromagnetic waves appreciably when $\omega/k \ll c$. Such a situation occurs, as we noted earlier, under skin-effect conditions when $\omega \ll \omega_{pe}$. In that case the absolute magnitude of the wavevector $|\boldsymbol{k}| \sim \omega_{pe}/c$ can, in the case of a plasma with a large density, be large, and one can neglect the thermal motion of the electrons only if the inequality

$$\frac{v_e}{c}\frac{\omega_{pe}}{\omega} \ll 1$$

is satisfied. If, however, this condition is not satisfied, we must use instead of eqn. (4.3.4.6) the exact expression for $\varepsilon_t(\boldsymbol{k}, \omega)$.

If the electrons are characterized by a Maxwellian velocity distribution, the dispersion eqn. (4.3.4.5) for the transverse electromagnetic waves takes the form

$$n^2 = \varepsilon_t = 1 + i\sqrt{\left(\frac{\pi}{2}\right)}\frac{\omega_{pe}^2}{\omega k v_e}w(z_e), \tag{4.3.4.7}$$

where $z_e = \omega/\sqrt{2}kv_e$.

Let us consider in detail the limiting case of low frequencies when the inequality which is the opposite of the one we had a moment ago is valid:

$$\frac{v_e}{c}\frac{\omega_{pe}}{\omega} \gg 1. \tag{4.3.4.8}$$

Assuming that $|z_e| \ll 1$ and $|\varepsilon_t| \gg 1$ we can put $w(z_e) = 1$ in the right-hand side of eqn. (4.3.4.7) and neglect the 1 as compared to the term $\propto \omega_{pe}^2$. As a result we obtain for the wavevector as function of ω the expression:

$$k = \frac{\sqrt{(3)}+i}{2}\left[\sqrt{\left(\frac{\pi}{2}\right)}\frac{\omega_{pe}^2\omega}{c^2 v_e}\right]^{1/3}. \tag{4.3.4.9}$$

We see that if condition (4.3.4.8) is satisfied, the electromagnetic field penetrates into the plasma to a depth of the order of

$$l_a = \left[\frac{c^2 v_e}{\omega_{pe}^2\omega}\right]^{1/3}. \tag{4.3.4.10}$$

In deriving eqn. (4.3.4.9) we assumed that $|z_e| \ll 1$ and $|\varepsilon_t| \gg 1$. Using eqn. (4.3.4.9) we can easily check that the inequality $|z_e| \ll 1$ is satisfied if inequality (4.3.4.8) is valid and that the inequality $|\varepsilon_t| \gg 1$ is satisfied when

$$\frac{v_e}{c} \ll \frac{\omega_{pe}^2}{\omega^2}. \tag{4.3.4.11}$$

One sees easily that if together with the inequality (4.3.4.11) the condition

$$\frac{v_e}{c}\frac{\omega_{pe}}{\omega} \gtrsim 1$$

is satisfied, $\mathrm{Re}\,k \sim \mathrm{Im}\,k \sim 1/l_a$ and the penetration depth of the electromagnetic field into the plasma is as before of the order of l_a.

We note that the penetration depth l_a is larger than the skin depth

$$l_0 = \frac{c}{\omega_{pe}}$$

which corresponds to a neglect of the thermal motion of the electrons. One speaks therefore of the *anomalous skin-effect*† when $(v_e/c)(\omega_{pe}/\omega) \gg 1$ in contrast to the normal skin-effect which occurs when $(v_e/c)(\omega_{pe}/\omega) \ll 1$.

† Reuter and Sondheimer (1948) developed the theory of the anomalous skin-effect for metals. Silin (1955) considered this problem for a plasma with a Maxwellian velocity distribution for the electrons.

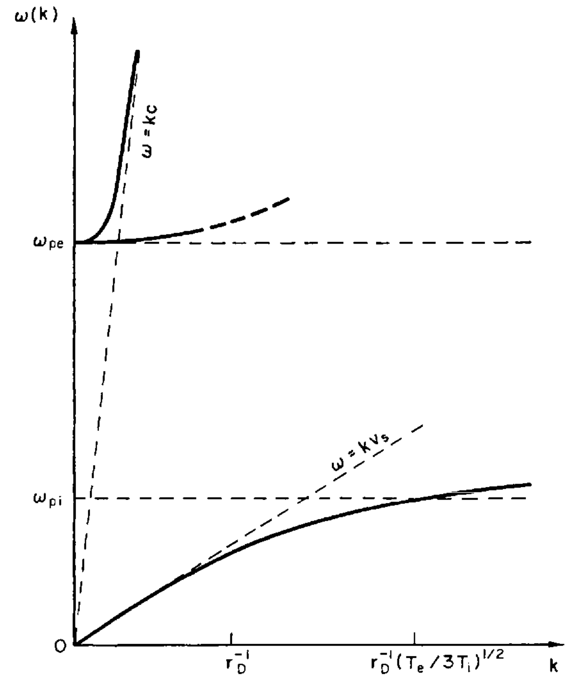

FIG. 4.3.1. Wavenumber dependence of the eigen frequencies of the oscillations of an unmagnetized isotropic plasma.

To conclude this section let us give a summary. The analysis of the dispersion equation given here shows that there are four kinds or branches of weakly damped oscillations in an unmagnetized plasma which differ in their dispersion laws and their polarization properties. They are the longitudinal electron (Langmuir) oscillations, the longitudinal ion-sound oscillations, and the transverse electromagnetic waves—of which there are two with different polarizations.

The frequency of the Langmuir oscillations lies close to the Langmuir frequency [eqn. (4.2.2.6)]. They are weakly damped when $kr_D \ll 1$ and the damping in that case is determined by the interaction of resonance particles with the field of the wave [Landau damping; eqn. (4.2.2.7)]. When $kr_D \gtrsim 1$, these oscillations are strongly damped ($\gamma \gtrsim \omega$).

The ion-sound oscillations are weakly damped only in a strongly non-isothermal plasma with $T_e \gg T_i$. If $kr_D \ll 1$, they are characterized by a linear dispersion law [eqn. (4.2.4.3)], but when $kr_D > 1$, their frequency lies close to the ionic Langmuir frequency [eqn. (4.2.4.4)]. If $T_e \sim T_i$, the ion-sound oscillations are strongly damped ($\gamma \sim \omega$).

Transverse electromagnetic waves propagating through a plasma have a phase velocity which is larger than the velocity of light and they are undamped in a collisionless plasma. One can use the hydrodynamic theory for their description in the case of a non-relativistic plasma. The anomalous skin-effect [eqn. (4.3.4.10)] occurs for frequencies considerably below the plasma frequency.

A sketch of the dispersion of the different branches of the oscillations is given in Fig. 4.3.1.

CHAPTER 5

Oscillations of a Plasma in a Magnetic Field

5.1. Hydrodynamical Theory of Oscillations of a Plasma in a Magnetic Field

5.1.1. DIELECTRIC PERMITTIVITY TENSOR OF A COLD PLASMA IN A MAGNETIC FIELD

We now turn to a study of the oscillations of a plasma which is in an external constant and uniform magnetic field.† As in the preceding chapter we shall consider oscillations with frequencies appreciably higher than the frequency of the binary collisions between particles.

We start with considering the case when we can neglect the effect of the thermal motion of the particles on the propagation of the waves. For this it is necessary that the phase velocity of the wave is considerably larger than the thermal velocity of the particles and, moreover, as we shall show below, that the frequency of the wave is not close to the cyclotron frequency of the electrons or of the ions.

In that case we can use for the description of the plasma oscillations the hydrodynamic equations for the electronic ($\alpha = \mathrm{e}$) and the ionic ($\alpha = \mathrm{i}$) components of the plasma:

$$\frac{d_\alpha \boldsymbol{u}_\alpha}{dt} = \frac{e_\alpha}{m_\alpha}\left\{\boldsymbol{E} + \frac{1}{c}\,[\boldsymbol{u}_\alpha \wedge (\boldsymbol{B}+\boldsymbol{B}_0)]\right\},$$

$$\frac{\partial n_\alpha}{\partial t} + \operatorname{div} n_\alpha \boldsymbol{u}_\alpha = 0, \qquad (5.1.1.1)$$

where

$$\frac{d_\alpha}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{u}_\alpha \cdot \nabla),$$

and $\boldsymbol{B}_0$ is the strength of the external magnetic field.

The electrical field strength $\boldsymbol{E}$ and the magnetic field strength $\boldsymbol{B}$ of the wave are determined from the Maxwell eqns. (4.1.1.3) in which we must take for $\boldsymbol{j}$ and ϱ the expressions

$$\boldsymbol{j} = \sum_\alpha e_\alpha n_\alpha \boldsymbol{u}_\alpha \quad \text{and} \quad \varrho = \sum_\alpha e_\alpha n_\alpha .$$

† The results reported below were obtained—neglecting ions—in connection with a study of the propagation of electromagnetic waves in the ionosphere of the Earth by a number of authors (see, for example, Appleton and Barnett, 1925; Nichols and Schelling, 1925 a, b; Lassen, 1927; Appleton, 1927). Later, several authors took the motion of the ions into account (see, for example, Aström, 1950, 1951; Stepanov, 1959a; Körper, 1957; Shafranov, 1958b, 1967; Booker, 1935; Stix, 1957).

We shall restrict our considerations to small amplitude oscillations. Assuming that in the equilibrium state $n_e = n_i = n_0$, $\boldsymbol{u}_\alpha = \boldsymbol{E} = \boldsymbol{B} = 0$ we get by linearizing eqn. (5.1.1.1)

$$\frac{\partial \boldsymbol{u}_\alpha}{\partial t} = \frac{e_\alpha}{m_\alpha}\left\{\boldsymbol{E}+\frac{1}{c}[\boldsymbol{u}_\alpha \wedge \boldsymbol{B}_0]\right\}, \quad \frac{\partial n'_\alpha}{\partial t}+n_0 \operatorname{div} \boldsymbol{u}_\alpha = 0, \tag{5.1.1.2}$$

where $n'_\alpha = n_\alpha - n_0$ is the variable part of the density of particles of the α-th kind.

In the case of plane monochromatic waves, when all variable quantities are proportional to $e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega t}$, eqns. (5.1.1.2) take the form

$$-i\omega \boldsymbol{u}_\alpha - \frac{e_\alpha}{m_\alpha c}[\boldsymbol{u}_\alpha \wedge \boldsymbol{B}_0] = \frac{e_\alpha}{m_\alpha}\boldsymbol{E}, \quad n'_\alpha = n_0\frac{(\boldsymbol{k}\cdot\boldsymbol{u}_\alpha)}{\omega}. \tag{5.1.1.3}$$

Taking the z-axis along $\boldsymbol{B}_0$ we get from (5.1.1.3) the velocity components of the particles of the α-th kind:

$$u_{\alpha x} = \frac{e_\alpha(i\omega E_x - \omega_{B\alpha}E_y)}{m_\alpha(\omega^2-\omega_{B\alpha}^2)}, \quad u_{\alpha y} = \frac{e_\alpha(i\omega E_y + \omega_{B\alpha}E_x)}{m_\alpha(\omega^2-\omega_{B\alpha}^2)}, \quad u_{\alpha z} = \frac{ie_\alpha E_z}{m_\alpha \omega}, \tag{5.1.1.4}$$

where $\omega_{B\alpha} = e_\alpha B_0/m_\alpha c$ is the cyclotron frequency—gyro-frequency—of the particles of the α-th kind.

Using now eqns. (5.1.1.4) we can easily find the current density $\boldsymbol{j} = en_0(\boldsymbol{u}_i - \boldsymbol{u}_e)$, the conductivity tensor σ_{ij}, and the dielectric permittivity tensor ε_{ij}, defined by eqn. (4.3.1.8). The dielectric permittivity tensor of a cold plasma has the following structure:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \tag{5.1.1.5}$$

where the components ε_1, ε_2, and ε_3 are equal to

$$\varepsilon_1 = 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2-\omega_{B\alpha}^2}, \quad \varepsilon_2 = -\sum_\alpha \frac{\omega_{p\alpha}^2\omega_{B\alpha}}{\omega(\omega^2-\omega_{B\alpha}^2)}, \quad \varepsilon_3 = 1-\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}. \tag{5.1.1.6}$$

When $\boldsymbol{B}_0 = 0$, the plasma is isotropic and the tensor ε_{ij} becomes proportional to the unit tensor, $\varepsilon_{ij} = \varepsilon_3\delta_{ij}$.

We note that the tensor (5.1.1.5), though not a real, is a Hermitean tensor. This means that a cold plasma for which collisions are neglected is a loss-free medium.

The fact that there are off-diagonal elements, proportional to ε_2, in (5.1.1.5) when $B_0 \neq 0$ leads to the fact that electromagnetic waves in a plasma in a magnetic field cannot be linearly polarized—except for the particular case of the propagation of waves perpendicular to the magnetic field—and are in the general case elliptically polarized. This is connected with the fact that when $B_0 \neq 0$ the matrix Λ_{ij} given by (4.3.1.10′) is not real and cannot be reduced to diagonal form by means of a real transformation, that is, it is impossible to reduce the equation

$$\sum_i \Lambda_{ij}E_j = 0$$

to the form $\Lambda'_i E'_i = 0$ through a rotaion of the axes. A plasma in a magnetic field is thus not only anisotropic, but also an optically active, gyrotropic, medium. It is therefore often called a magneto-active plasma.

The quantities ε_1 and ε_2 have singularities when $|\omega| = |\omega_{B\alpha}|$, corresponding to cyclotron resonance. We note that, as we shall prove in the next subsection, eqns. (5.1.1.6) become inapplicable as $|\omega| \to |\omega_{B\alpha}|$ due to the occurrence of a strong cyclotron damping of the waves.

We showed in the preceding chapter that the dispersion equation for electromagnetic waves in an anisotropic medium with a dielectric permittivity tensor ε_{ij} has the form [see (4.3.1.11)]:

$$\Lambda = \text{Det}\,|\Lambda_{ij}| = 0,$$

where

$$\Lambda_{ij} = n^2\left(\frac{k_i k_j}{k^2} - \delta_{ij}\right) + \varepsilon_{ij},$$

while $n = kc/\omega$ is the refractive index.

Substituting here expression (5.1.1.5) for the tensor ε_{ij} we can write the dispersion equation for a cold magneto-active plasma in the form

$$\Lambda = An^4 + Bn^2 + C = 0, \tag{5.1.1.7}$$

where

$$\begin{aligned} A &= \varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta, \\ B &= -\varepsilon_1\varepsilon_3(1+\cos^2\theta) - (\varepsilon_1^2 - \varepsilon_2^2)\sin^2\theta, \\ C &= \varepsilon_3(\varepsilon_1^2 - \varepsilon_2^2), \end{aligned} \tag{5.1.1.8}$$

while θ is the angle between the wavevector $\boldsymbol{k}$ and the magnetic field $\boldsymbol{B}_0$. From (5.1.1.8) and (5.1.1.6) it follows that the coefficients A, B, and C, which appear in the dispersion equation, are known functions of the frequency.

It follows from (5.1.1.7) that it is possible that two waves can propagate in a plasma with a given frequency but with different values of the refractive index,

$$n^2 = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}. \tag{5.1.1.9}$$

If we take the wavevector $\boldsymbol{k}$ as being given, eqn. (5.1.1.7) determines the eigenfrequencies $\omega = \omega^{(\nu)}(k, \theta)$. As eqn. (5.1.1.7) is of the fifth degree in ω^2, it defines ten eigenfrequencies. As corresponding to each eigenfrequency $\omega^{(\nu)}$ there is an eigenfrequency $-\omega^{(\nu)}$, we shall assume for the sake of simplicity that all eigenfrequencies are positive and thus distinguish five branches of oscillations of a cold magneto-active plasma.

It is only possible in some limiting cases to find the form of $\omega = \omega^{(\nu)}(k, \theta)$, $\nu = 1, 2, 3, 4, 5$, explicitly. However, one can find in the general case the nature of the behaviour of the eigenfrequencies when the wavevector varies and also the nature of the frequency-dependence of the refractive index (5.1.1.9). To do this we must first of all find the region of transparency, where $n^2 > 0$, and the region of total internal reflection, where $n^2 < 0$, and after that determine the position of the zeroes and of the poles of n^2.

One can choose the polarization vector for electromagnetic waves with a given frequency in the form (φ is the azimuthal angle in $\boldsymbol{k}$-space)

$$\boldsymbol{e} = \left\{\cos\varphi - \frac{i\varepsilon_2}{n^2-\varepsilon_1}\sin\varphi,\ \sin\varphi + \frac{i\varepsilon_2}{n^2-\varepsilon_1}\cos\varphi,\ \frac{n^2\cos\theta\sin\theta}{n^2\sin^2\theta-\varepsilon_1}\right\}. \tag{5.1.1.10}$$

5.1.2. PLASMA (HYBRID) RESONANCES IN A COLD PLASMA

One of the refractive indexes (5.1.1.9) tends to infinity when the frequency of the wave approaches a resonance frequency determined by the condition

$$A = 0. \tag{5.1.2.1}$$

This refractive index then behaves as

$$n^2 = -\frac{B}{A}. \tag{5.1.2.2}$$

The refractive index of the other wave remains finite for $A = 0$:

$$n^2 = -\frac{C}{B}.$$

We shall show that as one approaches the resonance frequency the electromagnetic waves become longitudinal, that is, the component of the electrical field strength parallel to the wavevector, $\boldsymbol{E}_l = \boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})/k^2$ becomes appreciably larger than the components at right angles to $\boldsymbol{k}$. To see this we multiply the equation $\sum_j \Lambda_{ij}E_j = 0$ by k_i and sum over i from 1 to 3. As a result we get

$$\sum_{i,j} \varepsilon_{ij}k_iE_j = 0.$$

Putting $\boldsymbol{E} = \boldsymbol{E}_l + \boldsymbol{E}_t$, we find from this equation

$$E_l = -\frac{\sum_{i,j} \varepsilon_{ij}k_iE_{tj}}{kA}, \tag{5.1.2.3}$$

where

$$A = \frac{\sum_{i,j} \varepsilon_{ij}k_ik_j}{k^2} = \varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta,$$

so that, indeed, $|\boldsymbol{E}_l|/|\boldsymbol{E}_t| \to \infty$ as $A \to 0$.

As one can always put $\boldsymbol{E}_l = -\nabla\varphi = -i\boldsymbol{k}\varphi$ and $\boldsymbol{E}_t = \text{curl}\,\boldsymbol{\mathcal{A}} = i[\boldsymbol{k} \wedge \boldsymbol{\mathcal{A}}]$, where φ and $\boldsymbol{\mathcal{A}}$ are scalar and vector potentials, one can say that as $A \to 0$ the plasma oscillations become irrotational (quasi-electrostatic) oscillations.

When there is no magnetic field the longitudinal oscillations are plasma (Langmuir) oscillations. Therefore, for the case when $B_0 \neq 0$, one speaks of the oscillations determined from the condition $A = 0$ as longitudinal plasma oscillations in a magnetic field, and the frequencies are called hybrid or plasma resonance frequencies—as they are determined by both the Langmuir and the cyclotron frequencies. One also talks of plasma or hybrid resonances.

Substituting expression (5.1.1.6) for ε_1 and ε_2 into eqn. (5.1.2.1), we can write this equation in the form

$$1 - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pe}^2}{\omega^2 - \omega_{Be}^2}\sin^2\theta - \frac{\omega_{pi}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pi}^2}{\omega^2 - \omega_{Bi}^2}\sin^2\theta = 0. \tag{5.1.2.4}$$

This equation which is of third degree in ω^2 determines three resonance frequencies:

$$\omega = \omega_\infty^{(j)}(\theta), \qquad j = 1, 2, 3; \tag{5.1.2.5}$$

the index ∞ indicates that the frequencies (5.1.2.5) are the eigenfrequencies for infinitely large wavevectors, $\omega_\infty^{(j)}(\theta) = \lim_{k\to\infty} \omega^{(j+2)}(k, \theta)$.

Using the fact that the ratio of the electron mass to the ion mass is small, $m_e/m_i \ll 1$, we can simplify eqn. (5.1.2.4). Indeed, we can neglect in (5.1.2.4) the term proportional to $\omega_{pi}^2 \cos^2\theta$ as it is smaller by a factor m_i/m_e than the term proportional to $\omega_{pe}^2 \cos^2\theta$. The term proportional to $\omega_{pi}^2 \sin^2\theta$ is also small compared to the terms proportional to $\omega_{pe}^2\cos^2\theta$ or to $\omega_{pe}^2 \sin^2\theta$ except when ω is close to ω_{Bi} or θ close to $\pi/2$. If we neglect in (5.1.2.4) the contribution from the ions we get the following expressions for $\omega_\infty^{(1,2)}(\theta)$:

$$\begin{aligned} \omega_\infty^{(1)}(\theta) &= \left[\tfrac{1}{2}(\omega_{pe}^2+\omega_{Be}^2)+\tfrac{1}{2}\sqrt{\{(\omega_{pe}^2+\omega_{Be}^2)^2-4\omega_{pe}^2\omega_{Be}^2\cos^2\theta\}}\right]^{1/2}, \\ \omega_\infty^{(2)}(\theta) &= \left[\tfrac{1}{2}(\omega_{pe}^2+\omega_{Be}^2)-\tfrac{1}{2}\sqrt{\{(\omega_{pe}^2+\omega_{Be}^2)^2-4\omega_{pe}^2\omega_{Be}^2\cos^2\theta\}}\right]^{1/2}. \end{aligned} \tag{5.1.2.6}$$

As $\theta \to 0$ the resonance frequencies $\omega_\infty^{(1,2)}(\theta)$ approach the values

$$\omega_\infty^{(1)} = \mathrm{Max}\,(\omega_{pe}, |\omega_{Be}|); \qquad \omega_\infty^{(2)} = \mathrm{Min}\,(\omega_{pe}, |\omega_{Be}|). \tag{5.1.2.7}$$

When the angle θ increases the frequency $\omega_\infty^{(1)}(\theta)$ increases and the frequency $\omega_\infty^{(2)}(\theta$ decreases. When $\theta \to \pi/2$, the quantities $\omega_\infty^{(1,2)}(\theta)$ tend to the values

$$\omega_\infty^{(1)} = \sqrt{(\omega_{pe}^2+\omega_{Be}^2)}, \tag{5.1.2.8}$$

$$\omega_\infty^{(2)} = \frac{\omega_{pe}|\omega_{Be}|\cos\theta}{\sqrt{(\omega_{pe}^2+\omega_{Be}^2)}}. \tag{5.1.2.9}$$

In the case of a dense plasma, when $\omega_{pe} \gg |\omega_{Be}|$, expressions (5.1.2.6) can be considerably simplified:

$$\omega_\infty^{(1)}(\theta) \approx \omega_{pe}\left(1+\frac{1}{2}\frac{\omega_{Be}^2}{\omega_{pe}^2}\sin^2\theta\right), \quad \omega_\infty^{(2)}(\theta) \approx |\omega_{Be}|\cos\theta. \tag{5.1.2.10}$$

In the opposite, case when $\omega_{pe} \ll |\omega_{Be}|$, we get the formulae

$$\omega_\infty^{(1)}(\theta) \approx |\omega_{Be}|\left(1+\frac{1}{2}\frac{\omega_{pe}^2}{\omega_{Be}^2}\sin^2\theta\right), \quad \omega_\infty^{(2)}(\theta) \approx \omega_{pe}\cos\theta. \tag{5.1.2.11}$$

The frequency $\omega_\infty^{(1)}(\theta)$ is always appreciably larger than ω_{Bi}. In the case of a not too rarefied plasma ($\omega_{pe} \gg \omega_{Bi}$) the frequency $\omega_\infty^{(2)}(\theta)$ is also considerably larger than ω_{Bi} (this conclusion is not valid for angles θ close to $\pi/2$ for the case when $\omega_{pi} \lesssim \omega_{Bi}$).

We note that eqn. (5.1.2.9) for $\omega_\infty^{(2)}$ becomes inapplicable when $\theta \to \pi/2$. Indeed, this formula was obtained, assuming that the following inequalities held:

$$\frac{\omega_{pe}^2\sin^2\theta}{|\omega^2-\omega_{Be}^2|} \approx \frac{\omega_{pe}^2}{\omega_{Be}^2} \gg \frac{\omega_{pi}^2\sin^2\theta}{|\omega^2-\omega_{Bi}^2|} \approx \frac{\omega_{pi}^2}{\omega^2},$$

which are valid provided

$$\cos^2\theta \gg \frac{m_e}{m_i}. \tag{5.1.2.12}$$

We shall now find the frequency $\omega_\infty^{(3)}(\theta)$. Assuming that $\omega \sim \omega_{Bi} \ll |\omega_{Be}|$, ω_{pe}, and retaining in (5.1.2.4) the last term and the term proportional to $\omega_{pe}^2 \cos^2\theta$, we get

$$\omega_\infty^{(3)}(\theta) = \omega_{Bi}\left(1 - \frac{1}{2}\frac{m_e}{m_i}\tan^2\theta\right). \tag{5.1.2.13}$$

This formula is, like eqn. (5.1.2.9), applicable, provided the inequalities (5.1.2.12) hold.

Let us in concluding this subsection give expressions for $\omega_\infty^{(2)}(\theta)$ and $\omega_\infty^{(3)}(\theta)$ for the case when $\theta \approx \pi/2$ which are valid also when the inequality $\cos^2\theta \gg m_e/m_i$ is not satisfied. Bearing in mind that in that case $\omega \ll |\omega_{Be}|$, we get from eqn. (5.1.2.4) (Stepanov, 1959a)

$$\omega_\infty^{(2,3)}(\theta) = \left\{\frac{1}{2(1+\eta)}[\omega_{pe}^2\cos^2\theta + \omega_{pi}^2 + \omega_{Bi}^2] \pm \sqrt{[(\omega_{pe}^2\cos^2\theta+\omega_{pi}^2+\omega_{Bi}^2)^2 - 4(1+\eta)\omega_{Bi}^2\omega_{pe}^2\cos^2\theta]}\right\}^{1/2}, \tag{5.1.2.14}$$

where $\eta = \omega_{pe}^2/\omega_{Be}^2$.

The frequencies determined by these equations decrease when the angle θ increases. In the range of angles θ, for which the condition (5.1.2.12) is satisfied, expressions (5.1.2.14) for $\omega_\infty^{(2,3)}$ go over into expressions (5.1.2.9) and (5.1.2.13). If, however, we have the inequality $\cos^2\theta \ll m_e/m_i$, it follows from (5.1.2.14) that the frequency $\omega_\infty^{(2)}(\theta)$ remains finite as $\theta \to \pi/2$ (Körper, 1957),

$$\omega_\infty^{(2)}(\theta) \approx \sqrt{\frac{\omega_{Bi}^2+\omega_{pi}^2}{1+\eta}}, \tag{5.1.2.15}$$

while the frequency $\omega_\infty^{(3)}(\theta)$ tends to zero:

$$\omega_\infty^{(3)}(\theta) = \frac{\omega_{Bi}\omega_{pe}\cos\theta}{\sqrt{(\omega_{Bi}^2+\omega_{pi}^2)}}. \tag{5.1.2.15'}$$

In a not too rarefied plasma when $\omega_{pi} \gg \omega_{Bi}$ expressions (5.1.2.14) take the particularly simple form

$$\omega_\infty^{(2)}(\theta) = \sqrt{\left(\frac{\omega_{pe}^2\cos^2\theta+\omega_{pi}^2}{1+\eta}\right)}, \tag{5.1.2.16}$$

$$\omega_\infty^{(3)}(\theta) = \frac{\omega_{Bi}\omega_{pe}\cos\theta}{\sqrt{(\omega_{pe}^2\cos^2\theta+\omega_{pi}^2)}}. \tag{5.1.2.17}$$

We have sketched schematically in Fig. 5.1.1 the behaviour of the resonance frequencies $\omega_\infty^{(j)}(\theta)$ as functions of θ.

The oscillations with the frequency $\omega_\infty^{(1)}(\theta)$ are purely electronic oscillations. The oscillations with the frequency $\omega_\infty^{(2)}(\theta)$ are also purely electronic when $\cos^2\theta \gg m_e/m_i$. When $\cos^2\theta \lesssim m_e/m_i$, their dispersion is determined both by the electrons and by the ions. The dispersion of the oscillations with the frequency $\omega_\infty^{(3)}(\theta)$ is determined both by the electrons and by the ions for any θ.

The plasma resonances play an important role for the propagation of electromagnetic waves in a plasma. The damping of the waves and the level of thermal noise increases steeply close to them. The refractive index for electromagnetic waves is large in the vicinity of these

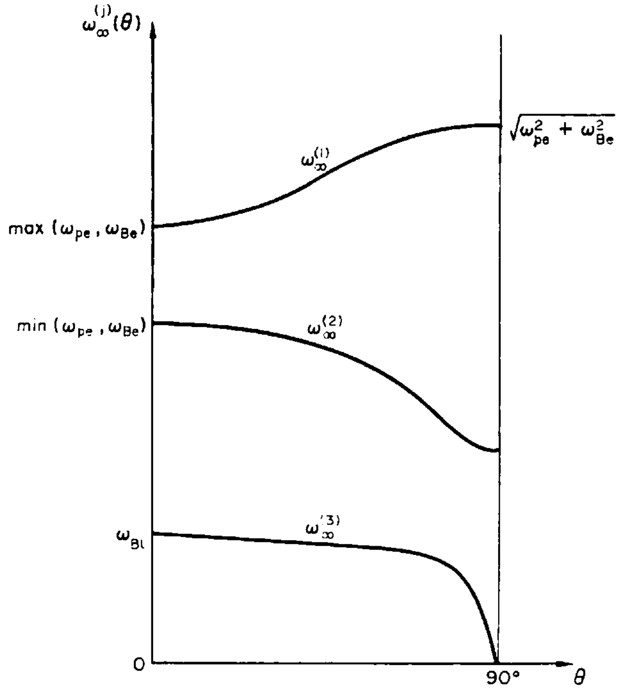

FIG. 5.1.1. The eigenfrequencies of longitudinal plasma oscillations in a magnetic field as functions of the angle θ.

resonances ($n \gg 1$), and the phase velocity is appreciably less than the velocity of light, that is, the waves become slow and hence the interaction of charged particles with the plasma occurs most effectively just close to the plasma resonances.

5.1.3. GENERAL PICTURE OF THE SPECTRA OF THE OSCILLATIONS OF A COLD MAGNETO-ACTIVE PLASMA

In the preceding subsection we found the poles $\omega = \omega_\infty^{(j)}(\theta)$ of the square of the refractive index $n(\omega)$. In order to elucidate the general picture of the frequency-dependence of the refractive index it is necessary to find also the zeroes of $n(\omega)$, and also to determine the values of n for $\omega = 0$ and $\omega = \infty$.

It is clear that $n^2 = 0$ when $C = \varepsilon_3(\varepsilon_1^2 - \varepsilon_2^2) = 0$. From this we find, neglecting the small ionic contributions to the quantities ε_1, ε_2, and ε_3, the zeroes of n^2:

$$\omega_0^{(1)} = \sqrt{(\omega_{pe}^2 + \tfrac{1}{4}\omega_{Be}^2)} + \tfrac{1}{2}\,|\omega_{Be}|, \quad \omega_0^{(2)} = \omega_{pe}, \quad \omega_0^{(3)} = \sqrt{(\omega_{pe}^2 + \tfrac{1}{4}\omega_{Be}^2)} - \tfrac{1}{2}\,|\omega_{Be}|; \quad (5.1.3.1)$$

the index 0 indicates that the quantities $\omega_\infty^{(j)}$ are the limiting values of the eigenfrequencies as $k \to 0$, $\omega_0^{(j)} = \lim_{k \to 0} \omega^{(j)}(k, \theta)$.

One easily checks that the following inequalities hold:

$$\omega_\infty^{(2)} < \omega_0^{(2)} < \omega_\infty^{(1)} < \omega_0^{(1)}; \qquad \omega_\infty^{(3)} < \omega_0^{(3)}. \qquad (5.1.3.2)$$

As to the frequency $\omega_0^{(3)}$, it can be either larger or smaller than $\omega_\infty^{(2)}(\theta)$, depending on the magnitude of the angle θ.

If $\omega \to \infty$, we have $\varepsilon_2 \to 0$, $\varepsilon_1 \to 1$, $\varepsilon_3 \to 1$ and, hence, $n \to 1$. If $\omega \to 0$, we have $\varepsilon_2 \to 0$ and $\varepsilon_3 \to -\infty$ and from (5.1.1.9) and (5.1.1.8) we find that n^2 is equal to either n_A^2 or $n_A^2/\cos^2\theta$, where

$$n_A^2 = 1 + \frac{\omega_{pi}^2}{\omega_{Bi}^2} + \frac{\omega_{pe}^2}{\omega_{Be}^2} = \varepsilon_1(\omega)\bigg|_{\omega=0}. \tag{5.1.3.3}$$

Once we know the position of the zeroes and poles of the function $n^2(\omega)$, as well as its values for $\omega = 0$ and $\omega = \infty$, we can easily schematically construct its shape. In Fig. 5.1.2 we show graphs of $n^2(\omega)$ for the case when θ is neither 0 nor $\pi/2$. The transparency regions where $n^2 > 0$ correspond to eigenfrequencies $\omega = \omega^{(j)}(k, \theta)$ which are schematically shown in Fig. 5.1.3.

There are always five branches of electromagnetic oscillations in a cold plasma; we shall call them the Alfvén branch (A), the fast magneto-sound branch (FMS), the slow extra-ordinary branch (SE), the ordinary branch (O), and the fast extra-ordinary branch (FE).† In Figs. 5.1.2 and 5.1.3 for $n^2(\omega)$ and $\omega^{(j)}(k)$ we have shown the SE branches by a full-drawn line for the case $\omega_0^{(3)} < \omega_\infty^{(2)}$ and by a dashed line for the case $\omega_0^{(3)} > \omega_\infty^{(2)}$.

It is clear from Fig. 5.1.2 that in the case when $\omega < \omega_\infty^{(3)}$ two waves (A and FMS) can propagate; only one wave (FMS) can propagate when $\omega_\infty^{(2)} < \omega < \omega_0^{(3)} < \omega_\infty^{(2)}$, and two waves (FMS and SE) can propagate when $\omega_0^{(3)} < \omega < \omega_\infty^{(2)}$. However, if $\omega_0^{(3)} > \omega_\infty^{(2)}$, only the FMS wave will propagate in the interval $\omega_\infty^{(3)} < \omega < \omega_\infty^{(2)}$, while no wave can propagate in the range $\omega_\infty^{(2)} < \omega < \omega_0^{(3)}$, while one wave (SE) can propagate in the range $\omega_0^{(3)} < \omega < \omega_0^{(2)}$; two waves (SE and 0) propagate in the interval $\omega_0^{(2)} < \omega < \omega_\infty^{(1)}$, one wave (O) when $\omega_0^{(1)} < \omega < \omega_0^{(1)}$, and two waves (O and FE) when $\omega > \omega_0^{(1)}$.

The frequencies of the propagating waves, $\omega^{(j)}(k, \theta)$ monotonically increase with increasing wavevector. This means that the dispersion is always normal in a cold plasma, and the angle between the direction of wave propagation and the group velocity $v_{gr} = \partial\omega/\partial k$ is always less than $\pi/2$.

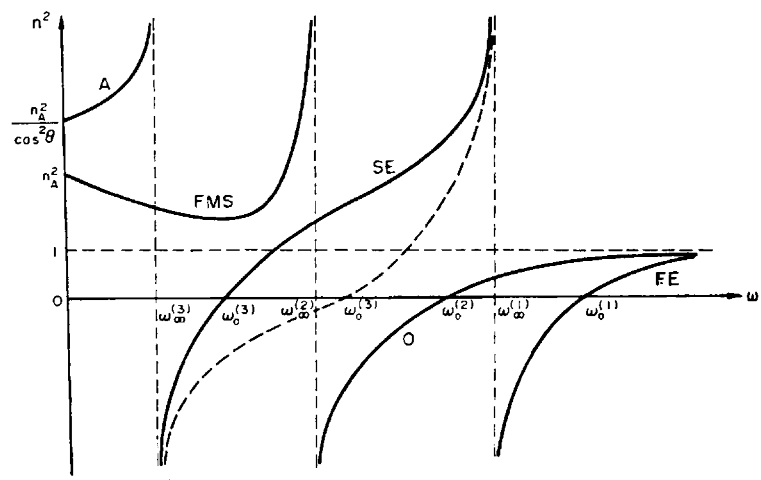

FIG. 5.1.2. Wavenumber-dependence of the square of the refractive index of electromagnetic waves in a "cold" magneto-active plasma for "oblique" propagation ($\theta \neq 0$ or $\pi/2$).

† This nomenclature is very convenient, and was introduced by Shafranov (1967).

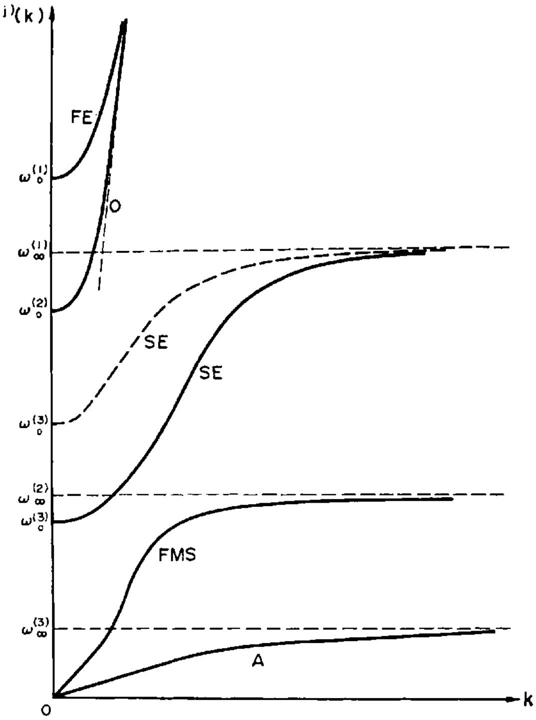

FIG. 5.1.3. Wavenumber-dependence of the eigenfrequencies of the oscillations of a "cold" magneto-active plasma in the case of "oblique" propagation ($\theta \neq 0,\ \pi/2$).

5.1.4. HIGH-FREQUENCY (ELECTRONIC) BRANCHES OF THE OSCILLATIONS IN A COLD MAGNETO-ACTIVE PLASMA

We can neglect the contribution from the ions in the high-frequency range, $\omega^2 \gg |\omega_{Be}|\,\omega_{Bi}$ and we can write the refractive index (5.1.1.9) in the following form (Lassen, 1927; Appleton, 1927):

$$n^2 = n_\pm^2 = 1 - \frac{2\omega_{pe}^2(\omega^2-\omega_{pe}^2)}{2\omega^2(\omega^2-\omega_{pe}^2)-\omega^2\omega_{Be}^2\sin^2\theta \pm [\omega^4\omega_{Be}^4\sin^4\theta+4\omega^2\omega_{Be}^2(\omega^2-\omega_{pe}^2)^2\cos^2\theta]^{1/2}}\,. \tag{5.1.4.1}$$

This expression determines the refractive index of the high-frequency (electronic) branches of the oscillations. The quantity $n_+(\omega)$ is the refractive index for the ordinary wave (when $\omega > \omega_0^{(2)} = \omega_{pe}$) and the refractive index for the high-frequency part of the FMS branch (when $\omega < \omega_\infty^{(2)}$), while the quantity $n_-(\omega)$ is the refractive index for the FE branch (when $\omega > \omega_0^{(1)}$) and the SE branch (when $\omega_0^{(3)} < \omega < \omega_\infty^{(1)}$).

The eqns. (5.1.4.1) are as yet rather complicated. We can simplify them in the following limiting cases:

(a) In the high-frequency region, $\omega \gg |\omega_{Be}|$, the plasma anisotropy is small and the refractive index (5.1.4.1) is close to that of an isotropic plasma, $n^2 = 1-(\omega_{pe}^2/\omega^2)$:

$$n^2 = 1-\frac{\omega_{pe}^2}{\omega^2}\left[1 \mp \frac{|\omega_{Be}|}{\omega}\cos\theta\right]. \tag{5.1.4.2}$$

If $\omega_{pe} \gg |\omega_{Be}|$, this expression determines the refractive index for the SE, O, and FE branches in the region $\omega > \omega_{pe}$, except for a narrow region around the first hybrid resonance $\omega \approx \omega_\infty^{(1)} \sim \omega_{pe}$ of the SE branch. If $\omega_{pe} \lesssim |\omega_{Be}|$ eqn. (5.1.4.2) applies only to the region of transparency of the O and FE branches where $n \approx 1$.

(b) In a low-density plasma, $\omega \gg \omega_{pe}$, $|\omega_{Be}| \gg \omega_{pe}$, the refractive index is close to unity, provided the frequency of the wave does not lie close to $\omega_\infty^{(1)}(\theta) \approx |\omega_{Be}|$:

$$n^2 = 1 - \frac{2\omega_{pe}^2}{2\omega^2 - \omega_{Be}^2 \sin^2\theta \pm [\omega_{Be}^4 \sin^4\theta + 4\omega^2\omega_{Be}^2 \cos^2\theta]^{1/2}}\,. \tag{5.1.4.3}$$

This formula describes the behaviour of the refractive index for the SE, O, and FE branches in the region $\omega \gg \omega_0^{(2)} = \omega_{pe}$.

(c) In a high-density plasma, in the low-frequency region, when $\omega_{pe} \gg \omega$, $\omega_{pe} \gg |\omega_{Be}|$, and θ not close to $\pi/2$ we have (Booker, 1935):

$$n^2 = \frac{\omega_{pe}^2}{\omega(|\omega_{Be}|\cos\theta - \omega)}, \quad n^2 = \frac{-\omega_{pe}^2}{\omega(|\omega_{Be}|\cos\theta + \omega)}\,. \tag{5.1.4.4}$$

We see that in that case only a single FMS wave can propagate (when $\omega < |\omega_{Be}|\cos\theta = \omega_\infty^{(2)}$).

As $\theta = 0$ for longitudinal propagation the refractive index of a FMS wave is determined by the equation

$$n^2 = \frac{\omega_{pe}^2}{\omega(|\omega_{Be}| - \omega)}\,,$$

which differs from (5.1.4.4) only in that in (5.1.4.4) we have $\omega_{Be}\cos\theta$ rather than ω_{Be}, this case is called the case of *quasi-longitudinal propagation*.

In the case of quasi-longitudinal propagation the frequency of the FMS wave as function of the wavevector is according to (5.1.4.4) given by the expression

$$\omega(k, \theta) = \frac{|\omega_{Be}|\, c^2k^2\cos\theta}{\omega_{pe}^2 + c^2k^2}\,. \tag{5.1.4.5}$$

In the frequency range $\omega \ll |\omega_{Be}|$ expressions (5.1.4.4) and (5.1.4.5) can be simplified

$$n^2 = \frac{\omega_{pe}^2}{\omega\,|\omega_{Be}|\cos\theta}\,, \tag{5.1.4.6}$$

$$\omega(k, \theta) = \frac{|\omega_{Be}|\, c^2k^2\cos\theta}{\omega_{pe}^2}\,. \tag{5.1.4.7}$$

In this frequency range the FMS wave with refractive index (5.1.4.6) and frequency (5.1.4.7), which is proportional to k^2, is called an *atmospheric whistler* or simply a *whistler*—and also, because of its circular polarization, a *spiral wave* or *helicon*.

We note that as expressions (5.1.4.6) and (5.1.4.7) are determined by only the component ε_2 of the tensor ε_{ij}, when $\omega_{pe} \gg \omega, |\omega_{Be}|$ (one can write eqn. (5.1.4.6) in the form $n^2 = \varepsilon_2/\cos\theta$) which is appreciably larger than ε_1, while the contribution of the ionic terms to ε_2 turns out to be negligible already when $\omega \gg \omega_{Bi}$, one can use eqns. (5.1.4.6) and (5.1.4.7) for the whistlers in the frequency range $\omega_{Bi} \ll \omega \ll |\omega_{Be}|$.

5.1.5. LOW-FREQUENCY BRANCHES OF THE OSCILLATIONS OF A COLD MAGNETO-ACTIVE PLASMA

In the low-frequency region, $\omega \ll |\omega_{Be}|$, $\omega \ll \omega_{pe}$, there are two branches of electromagnetic waves: Alfvén waves and magneto-sound waves. One can considerably simplify expression (5.1.1.9) for the refractive index of these waves if one takes into account that in the low-frequency region $|\varepsilon_3| \approx \omega_{pe}^2/\omega^2 \gg |\varepsilon_{1,2}|$ (Shafranov, 1958b; Stix, 1957):

$$n^2 = \frac{1}{2\cos^2\theta}[\varepsilon_1(1+\cos^2\theta) \pm \{\varepsilon_1^2(1+\cos^2\theta)^2 - 4(\varepsilon_1^2-\varepsilon_2^2)\cos^2\theta\}^{1/2}], \quad (5.1.5.1)$$

where

$$\varepsilon_1 \approx 1+\frac{\omega_{pe}^2}{\omega_{Be}^2}-\frac{\omega_{pi}^2}{\omega^2-\omega_{Bi}^2}, \quad \varepsilon_2 \approx -\frac{\omega_{pe}^2}{\omega\,|\omega_{Be}|}-\frac{\omega_{pi}^2\omega_{Bi}}{\omega(\omega^2-\omega_{Bi}^2)} \approx -\frac{\omega_{pi}^2\omega_{Bi}}{\omega(\omega^2-\omega_{Bi}^2)}. \quad (5.1.5.2)$$

Using eqn. (5.1.5.1) for n^2 one can also easily obtain explicit expressions for the frequencies of the fast magneto-sound and the Alfvén waves. If $\omega_{pi} \gg \omega_{Bi}$, we have

$$\omega^2(k,\theta) = \frac{1}{2}k^2v_A^2\left\{1+\cos^2\theta+\frac{k^2c^2\cos^2\theta}{\omega_{pi}^2} \pm \left[\left((1+\cos^2\theta+\frac{k^2c^2\cos^2\theta}{\omega_{pi}^2}\right)^2-4\cos^2\theta\right]^{1/2}\right\}, \quad (5.1.5.3)$$

where v_A is the Alfvén velocity,

$$v_A = c\frac{\omega_{Bi}}{\omega_{pi}} = \frac{B_0}{\sqrt{(4\pi n_0 m_i)}} \ll c.$$

In the low-frequency region, $\omega \ll \omega_{Bi}$ or $kc/\omega_{pi} \ll 1$, we get from (5.1.5.1) and (5.1.5.3) (Aström, 1950, 1951)

$$n^2 = \frac{c^2}{v_A^2}, \qquad \omega(k,\theta) = kv_A; \quad (5.1.5.4)$$

$$n^2 = \frac{c^2}{v_A^2\cos^2\theta}, \quad \omega(k,\theta) = kv_A\cos\theta. \quad (5.1.5.5)$$

Equations (5.1.5.4) and (5.1.5.5) determine the refractive index and the frequencies of the fast magneto-sound and the Alfvén waves, respectively.

It is of interest to note that in the case considered of a collisionless plasma the frequencies (5.1.5.4) and (5.1.5.5) are the same as the expressions (2.1.1.10) for the Alfvén and the fast magneto-sound wave obtained in the magneto-hydrodynamic approximation which is valid in the opposite case of a large collision frequency, when $\omega \ll \nu$. When comparing expressions (5.1.5.4) and (2.1.1.10) we must in (2.1.1.10) neglect the sound velocity $c_s \sim v_i = \sqrt{(T/m_i)}$ since in the case considered of a cold plasma $v_{ph} \sim v_A \gg v_i$.

When the frequency approaches the ion cyclotron frequency the refractive index of the Alfvén branch tends to infinity. In that frequency range the Alfvén branch is called an ion-cyclotron wave. Its refractive index and frequency are according to (5.1.5.1) and (5.1.5.3) determined by the equations (Stix, 1957)

$$n^2 = \frac{c^2}{v_A^2} \frac{1+\cos^2\theta}{2\cos^2\theta} \frac{\omega}{\omega_{Bi}-\omega}, \quad \omega \approx \omega_{Bi}, \tag{5.1.5.6}$$

$$\omega(k, \theta) = \omega_{Bi}\left(1 - \frac{1+\cos^2\theta}{2\cos^2\theta} \frac{\omega_{pi}^2}{k^2c^2}\right), \quad \frac{\omega_{pi}}{kc} \ll 1. \tag{5.1.5.7}$$

We note that the refractive index of the fast magneto-sound wave remains finite at $\omega = \omega_{Bi}$

$$n^2 = \frac{c^2}{v_A^2} \frac{1}{1+\cos^2\theta}, \quad \omega = \omega_{Bi}. \tag{5.1.5.8}$$

According to (5.1.5.6) $n^2 \to \infty$ as $\omega \to \omega_{Bi}$. In actual fact the refractive index does not have a singularity at $\omega = \omega_{Bi}$ because when deriving eqn. (5.1.5.6) we neglected terms of order m_e/m_i and the difference $\omega_{Bi}-\omega$ in the denominator of expression (5.1.5.6) must thus be large compared to $\omega_{Bi}(m_e/m_i)\tan^2\theta$. If these small terms are taken into account we must replace the difference $\omega_{Bi}-\omega$ in the denominator of (5.1.5.6) by $\omega_\infty^{(3)}(\theta)-\omega$, where the resonance frequency $\omega_\infty^{(3)}(\theta)$ is determined by eqn. (5.1.2.13). The resonance frequency is thus not the ion-cyclotron frequency but the third hybrid frequency $\omega_\infty^{(3)}(\theta)$ which is close to it. However, in practice this difference cannot turn out to be particularly important as dissipative effects connected either with collisions or with the collisionless cyclotron damping of the wave can set in before the difference $\omega_{Bi}-\omega$ becomes of the order of $\omega_{Bi}(m_e/m_i)\tan^2\theta$.

In the high-frequency region ($\omega \gg \omega_{Bi}$ or $kc/\omega_{pi} \gg 1$) eqns. (5.1.5.1) and (5.1.5.3) give expressions for the refractive index and frequency of whistlers:

$$n^2 = \frac{\omega_{pi}^2}{\omega\omega_{Bi}\cos\theta}, \quad \omega(k, \theta) = \frac{\omega_{Bi}c^2k^2\cos\theta}{\omega_{pi}^2}.$$

5.1.6. PROPAGATION OF ELECTROMAGNETIC WAVES IN A COLD MAGNETO-ACTIVE PLASMA PARALLEL TO THE MAGNETIC FIELD

The determinant in (5.1.1.7) splits into three factors when $\theta = 0$, and the dispersion eqn. (5.1.1.7) becomes

$$\Lambda = \varepsilon_3(n^2-\varepsilon_1-\varepsilon_2)(n^2-\varepsilon_1+\varepsilon_2) = 0. \tag{5.1.6.1}$$

The equation $\varepsilon_3 = 0$ has the solution

$$\omega(k) = \sqrt{(\omega_{pe}^2+\omega_{pi}^2)},$$

corresponding to Langmuir oscillations—it is clear that the magnetic field can not affect the longitudinal oscillations when $\theta = 0$.

The equations

$$n^2 = \varepsilon_1 \pm \varepsilon_2$$

determine the refractive index of two electromagnetic waves with circular polarization.

Recalling eqns. (5.1.1.6) for ε_1 and ε_2 we can write these equations in the form

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega - |\omega_{Be}|)} - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{Bi})}, \tag{5.1.6.2}$$

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega + |\omega_{Be}|)} - \frac{\omega_{pi}^2}{\omega(\omega - \omega_{Bi})}. \tag{5.1.6.3}$$

The refractive index of the wave whose electrical field vector rotates in the direction of the electron rotation in the magnetic field $\boldsymbol{B}_0$ has a singularity at $\omega = |\omega_{Be}|$, while the refractive index of the wave whose electrical field vector rotates in the direction of the ion rotation in the magnetic field $\boldsymbol{B}_0$ has a singularity at $\omega = \omega_{Bi}$.

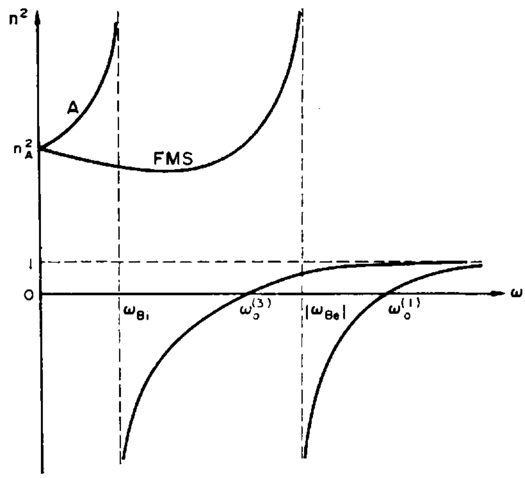

FIG. 5.1.4. Frequency dependence of the square of the refractive index of electromagnetic waves in a "cold" magneto-active plasma for the case of longitudinal propagation ($\theta = 0$).

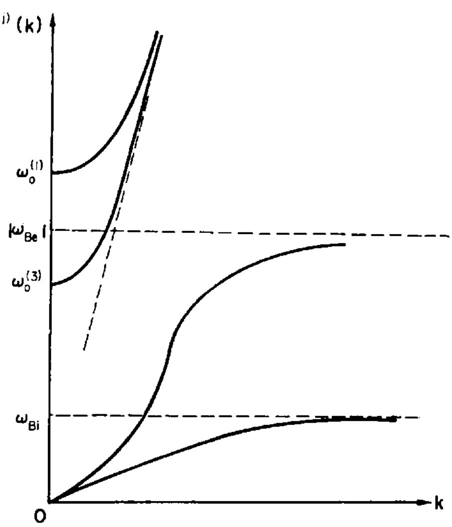

FIG. 5.1.5. Wavenumber dependence of the eigenfrequencies of the oscillations of a "cold" magneto-active plasma for the case of longitudinal propagation ($\theta = 0$).

The refractive indexes of the electromagnetic waves in a cold plasma therefore have singularities for $\omega = |\omega_{B\alpha}|$ only for longitudinal propagation—we showed earlier that when $\theta \neq 0$ these singularities coincide with the hybrid frequencies $\omega_{\infty}^{(j)}(\theta)$ ($\neq |\omega_{B\alpha}|$).

We show in Figs. 5.1.4 and 5.1.5 the frequency dependence of the refractive indexes (5.1.6.2) and (5.1.6.3) and the wavenumber dependence of the eigenfrequencies $\omega^{(j)}(k)$. In contrast to the general case, for the case when $\theta = 0$ eqns. (5.1.6.2) and (5.1.6.3) determine only four branches of oscillations—the branch which has "vanished" in fact is supplemented by the Langmuir oscillations with the frequency $\omega = \sqrt{(\omega_{pe}^2+\omega_{pi}^2)}$.

5.1.7. TRANSVERSE PROPAGATION OF ELECTROMAGNETIC WAVES IN A COLD MAGNETO-ACTIVE PLASMA

When $\theta = \pi/2$ the dispersion eqn. (5.1.1.7) splits into two equations:

$$n^2 = \varepsilon_3, \tag{5.1.7.1}$$

$$n^2 = \frac{\varepsilon_1^2-\varepsilon_2^2}{\varepsilon_1}. \tag{5.1.7.2}$$

Equation (5.1.7.1) determines the refractive index of the linearly polarized ordinary wave whose electrical field strength is parallel to the external magnetic field $\boldsymbol{B}_0$. The magnetic field does not affect the propagation of this wave.

We show in Figs. 5.1.6 and 5.1.7 the frequency dependence of the refractive index (5.1.7.2). and the wavenumber dependence of the corresponding three eigenfrequencies—the graphs of $n^2(\omega)$ and $\omega(k)$ for the ordinary wave are shown by the dashed curves.

There are four branches of oscillations when $\theta = \pi/2$, as for the case when $\theta = 0$. When $\theta \to \pi/2$, the Alfvén branch "disappears"; it has a frequency less than $\omega_{\infty}^{(3)}(\theta)$ and $\omega_{\infty}^{(3)}(\theta)$ tends to zero as $\theta \to \pi/2$ (see eqn. (5.1.2.15')) The resonance frequencies $\omega_{\infty}^{(1)}$ and $\omega_{\infty}^{(2)}$ are determined by eqns. (5.1.2.8) and (5.1.2.16) when $\theta = \pi/2$.

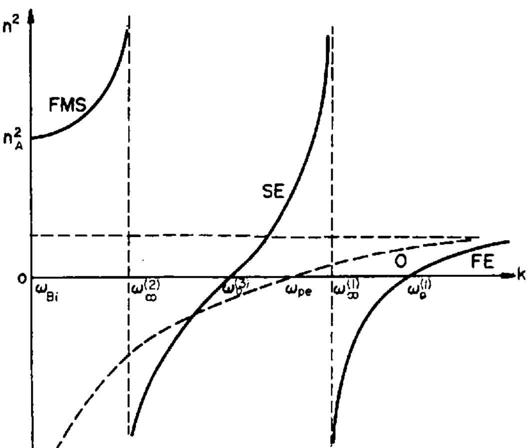

FIG. 5.1.6. Frequency dependence of the square of the refractive index of electromagnetic waves in a "cold" magneto-active plasma for the case of transverse propagation ($\theta = \pi/2$).

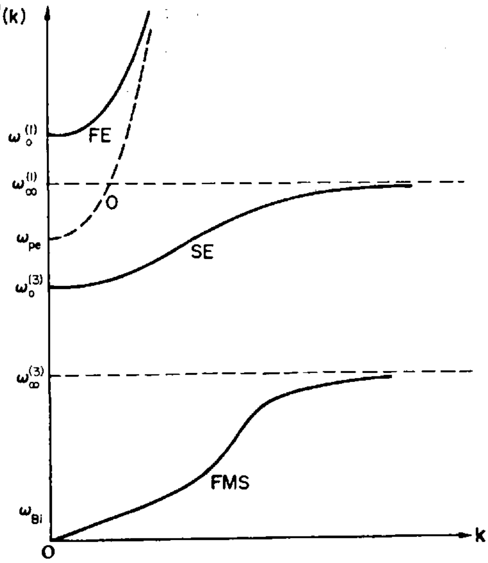

FIG. 5.1.7. Wavenumber dependence of the eigenfrequencies of the oscillations of a "cold" magneto-active plasma for the case of transverse propagation ($\theta = \pi/2$).

In the low-frequency region, $\omega \ll \sqrt{(|\omega_{Be}|\,\omega_{Bi})}$ the frequency of the FMS branch is for $\omega_{pi} \gg \omega_{Bi}$ determined by the relation

$$\omega(k) = kv_A ,$$

which, in contrast to (5.1.5.4) is valid not only when $\omega \ll \omega_{Bi}$ but also in the region where $\omega \sim \omega_{Bi}$.

Let us summarize. We have shown that a cold plasma in an external magnetic field is an anisotropic medium with temporal dispersion in which there are five branches of oscillations: the ordinary, the fast and the slow extra-ordinary, the Alfvén, and the fast magnetosound waves. The frequency dependence of the refractive index of these waves is determined by eqn. (5.1.1.9) (see Fig. 5.1.2). All waves have normal dispersion, that is, their frequencies increase with increasing wavenumber (see Fig. 5.1.3), and are elliptically polarized.

The dispersion of the high-frequency branches (O waves, FE waves, and also SE waves with θ not too close to $\pi/2$) is determined by the electrons alone, while the dispersion of the low-frequency branches (FMS waves, A waves, and also SE waves with $\theta \approx \pi/2$) is determined by both electrons and ions.

5.2. Kinetic Theory of Plasma Oscillations in a Magnetic Field

5.2.1. DIELECTRIC PERMITTIVITY TENSOR OF A MAGNETO-ACTIVE PLASMA IN THE KINETIC APPROXIMATION

The hydrodynamical theory of oscillations in a magneto-active plasma which we developed in the preceding section turns out to be insufficient in a number of cases as it neglects the thermal motion of the particles in the plasma. The presence of thermal motion of electrons and ions in the plasma leads to the appearance of qualitatively new features when electromagnetic waves propagate in a magneto-active plasma. First of all, there appear a number of new branches of oscillations in a magneto-active plasma due to the thermal motion of the particles in the plasma; these are not present in a cold plasma and they can be either weakly or strongly damped, either long-wavelength or short-wavelength oscillations. We remember that when there is no magnetic field taking into account the thermal motion of the electrons in a plasma with cold ions and hot electrons leads to the appearance of only one extra branch of oscillations, namely, the ion-sound waves. Secondly, the interaction of resonance particles with the electrical field of the wave leads, as we saw in Section 4.2, to Landau damping of the wave—we are dealing with a plasma whose initial state is an equilibrium state. Such an effect also occurs in a magneto-active plasma. However, besides this there arises in a magneto-active plasma yet another damping mechanism: cyclotron damping which is connected with the emission and absorption of electromagnetic waves by the charged particles in the plasma, which move along spirals in the magnetic field, at frequencies which are equal to the cyclotron frequency or multiples of it.

To study these effects we shall start from the kinetic equations for the particles in the plasma with self-consistent interactions. First of all, we shall use the kinetic theory to obtain the dielectric permittivity tensor for a plasma in a magnetic field, and then we shall use that tensor to study the propagation of waves in a magneto-active plasma, taking the thermal motion of electrons and ions into account.

We recall that when there are no oscillations, the distribution function $f_{\alpha 0}$ of the plasma particles of the α-th kind satisfies the equation

$$\frac{e_\alpha}{m_\alpha c}\left([\boldsymbol{v}\wedge\boldsymbol{B}_0]\cdot\frac{\partial f_{\alpha 0}}{\partial \boldsymbol{v}}\right) = \left[\frac{\partial f_{\alpha 0}}{\partial t}\right]_c, \tag{5.2.1.1}$$

from which it follows that $f_{\alpha 0}$ is a Maxwell distribution function,

$$f_{\alpha 0} = n_0\left(\frac{m_\alpha}{2\pi T}\right)^{3/2}\exp\left(-\frac{m_\alpha v^2}{2T}\right),$$

where T is the plasma temperature and n_0 the equilibrium particle density.

In the case of a high-temperature plasma the relaxation time is very large and we can consider a quasi-equilibrium state in which the distribution function varies slowly due to collisions and satisfies in zeroth approximation eqn. (5.2.1.1) without the collision integral:

$$\frac{e_\alpha}{m_\alpha c}\left([\boldsymbol{v}\wedge\boldsymbol{B}_0]\cdot\frac{\partial f_{\alpha 0}}{\partial \boldsymbol{v}}\right) = -\omega_{B\alpha}\frac{\partial f_{\alpha 0}}{\partial \varphi} = 0. \tag{5.2.1.2}$$

From this it follows that the initial distribution function $f_{\alpha 0}$ is an arbitrary function of two variables, $v_{\parallel}$ and $v_{\perp}$, which are the velocity components parallel and at right angles to the magnetic field, and is independent of the azimuthal angle in velocity space (see Fig. 5.2.1),

$$f_{\alpha 0} = f_{\alpha 0}(v_{\parallel}, v_{\perp}).$$

Let us now consider small oscillations of the plasma around an initial state described by the distribution function $f_{\alpha 0}$. If the oscillations are high-frequency oscillations, $\omega\tau \gg 1$,

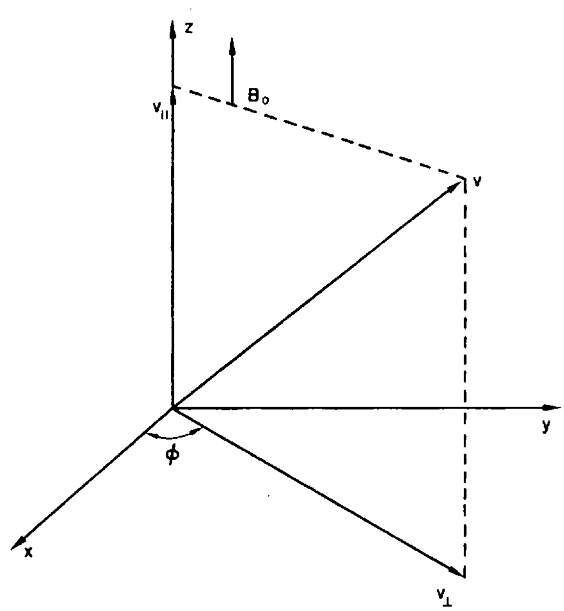

FIG. 5.2.1. The coordinate system in velocity space.

we can in the kinetic equation for the distribution function F_α,

$$\frac{\partial F_\alpha}{\partial t}+(\boldsymbol{v}\cdot\nabla)F_\alpha+\frac{e_\alpha}{m_\alpha}\left(\left\{\boldsymbol{E}+\frac{1}{c}\,[\boldsymbol{v}\wedge(\boldsymbol{B}+\boldsymbol{B}_0)]\right\}\cdot\frac{\partial F_\alpha}{\partial \boldsymbol{v}}\right)=\left[\frac{\partial F_\alpha}{\partial t}\right]_c,$$

neglect the collision integral and put

$$F_\alpha(\boldsymbol{r},\boldsymbol{v},t)=f_{\alpha 0}(v_{\|},v_{\perp})+f_\alpha(\boldsymbol{r},\boldsymbol{v},t),$$

where f_α is a small oscillating correction to $f_{\alpha 0}$. Linearizing the kinetic equation we get

$$\frac{\partial f_\alpha}{\partial t}+(\boldsymbol{v}\cdot\nabla)f_\alpha+\frac{e_\alpha}{m_\alpha}\left(\left\{\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial f_{\alpha 0}}{\partial \boldsymbol{v}}\right)-\omega_{B\alpha}\frac{\partial f_\alpha}{\partial\varphi}=0. \qquad (5.2.1.3)$$

This equation, together with the Maxwell equations for the variable fields $\boldsymbol{E}$ and $\boldsymbol{B}$, determines the linear oscillations of a magneto-active plasma. Using Fourier–Laplace transforms to solve this set of equations, as we did in Section 4.2 for the case of longitudinal oscillations, we can show that the asymptotic behaviour of the Fourier components of the fields as $t\to\infty$ is determined by solving a dispersion relation. As we noted already in Subsection 4.3.1, it is sufficient to restrict our considerations to the case of plane waves $\propto e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega t}$ if we want to find the dispersion equation, and to find by means of the kinetic equation the dielectric permittivity tensor ε_{ij}. The condition that the Maxwell eqns. (4.3.1.10) can be solved gives us the dispersion equation.

We shall show how one can find the tensor ε_{ij}.† In the case of plane monochromatic

† Gertsenshteĭn (1954) and Sitenko and Stepanov (1957; Galitskiĭ and Migdal, 1958; Stepanov, 1958a; Shafranov, 1958a; Trubnikov, 1958; Sagdeev and Shafranov, 1961; Stepanov and Kitsenko, 1961) have obtained expressions for the dielectric permittivity tensor of a plasma in an external magnetic field, taking into account the thermal motion of the electrons and ions.

waves eqn. (5.2.1.3) becomes

$$-i[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]f_\alpha-\omega_{B\alpha}\frac{\partial f_\alpha}{\partial\varphi}=-\frac{e_\alpha}{m_\alpha}\left(\left\{\boldsymbol{E}\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega}\right)+\frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})}{\omega}\right\}\cdot\frac{\partial f_{\alpha0}}{\partial\boldsymbol{v}}\right),\quad(5.2.1.4)$$

where we have used the fact that $\boldsymbol{B}=(c/\omega)\,[\boldsymbol{k}\wedge\boldsymbol{E}]$. Integrating this equation we find

$$f_\alpha=\frac{e_\alpha}{m_\alpha\omega_{B\alpha}}\exp\left\{\frac{i}{\omega_{B\alpha}}\int_0^{\varphi}[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]\,d\varphi'\right\}\left[\int_0^{\varphi}\exp\left\{-\frac{i}{\omega_{B\alpha}}\int_0^{\varphi'}[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]\,d\varphi''\right\}\right.$$

$$\left.\times\left(\left\{\boldsymbol{E}\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega}\right)+\frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})}{\omega}\right\}\cdot\frac{\partial f_{\alpha0}}{\partial\boldsymbol{v}}\right)d\varphi'+C\right],\quad(5.2.1.5)$$

where the constant $C=C(v_{\|},v_\perp)$, which is independent of φ and must be determined from the periodicity condition $f_\alpha(\varphi+2\pi)=f_\alpha(\varphi)$, is equal to

$$C=\exp\left\{\frac{i}{\omega_{B\alpha}}\int_0^{2\pi}[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]\,d\varphi\right\}$$

$$\times\int_0^{2\pi}d\varphi\left(\left\{\boldsymbol{E}\left(1-\frac{(\boldsymbol{k}\cdot\boldsymbol{v})}{\omega}\right)+\frac{\boldsymbol{k}(\boldsymbol{k}\cdot\boldsymbol{E})}{\omega}\right\}\cdot\frac{\partial f_{\alpha0}}{\partial\boldsymbol{v}}\right)\exp\left\{-\frac{i}{\omega_{B\alpha}}\int_0^{\varphi}[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]\,d\varphi'\right\}.\quad(5.2.1.6)$$

It is convenient to write eqn. (5.2.1.5) in a slightly different form. To do this we introduce a coordinate system in which $k_y=0$, so that $\boldsymbol{k}=(k_x,0,k_z)=(k\sin\theta,0,k\cos\theta)$; we then have

$$\int_0^{\varphi}[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]\,d\varphi'=(k_zv_{\|}-\omega)\varphi+k_xv_\perp\sin\varphi.\quad(5.2.1.7)$$

Substituting this expression into (5.2.1.6) and putting under the integral sign the following series expansion:

$$e^{-i\lambda\sin\varphi}=\sum_{l=-\infty}^{+\infty}J_l(\lambda)e^{-il\varphi},\quad(5.2.1.8)$$

where the $J_l(\lambda)$ are Bessel functions and $\lambda=k_xv_\perp/\omega_{B\alpha}$, we get the following expression for f_α:

$$f_\alpha=\frac{ie_\alpha}{m_\alpha}e^{i\lambda\sin\varphi}\sum_{l=-\infty}^{+\infty}\frac{(\boldsymbol{a}_l\cdot\boldsymbol{E})}{k_zv_{\|}+l\omega_{B\alpha}-\omega}e^{il\varphi},\quad(5.2.1.9)$$

where

$$a_{l1}=\left[\frac{k_zv_\perp}{\omega}\frac{\partial f_{\alpha0}}{\partial v_{\|}}+\left(1-\frac{k_zv_{\|}}{\omega}\right)\frac{\partial f_{\alpha0}}{\partial v_\perp}\right]\frac{l}{\lambda}J_l(\lambda),$$

$$a_{l2}=\left[\frac{k_zv_\perp}{\omega}\frac{\partial f_\alpha}{\partial v_{\|}}+\left(1-\frac{k_zv_{\|}}{\omega}\right)\frac{\partial f_{\alpha0}}{\partial v_\perp}\right]iJ_l'(\lambda),$$

$$a_{l3}=\frac{\partial f_{\alpha0}}{\partial v_{\|}}J_l(\lambda)+\left[\frac{k_xv_{\|}}{\omega}\frac{\partial f_{\alpha0}}{\partial v_\perp}-\frac{k_xv_{\|}}{\omega}\frac{\partial f_{\alpha0}}{\partial v_{\|}}\right]\frac{l}{\lambda}J_l(\lambda).$$

It can be seen from (5.2.1.9) that in the case of weakly damped oscillations the perturbation of the distribution function has a steep maximum when

$$v_{||} \approx v_{||\text{res}} \equiv \frac{\omega(k) - l|\omega_{B\alpha}|}{k_z}, \qquad l = 0, \pm 1, \pm 2, \ldots \tag{5.2.1.10}$$

We shall call particles with such a velocity resonance particles.

Putting $l = 0$ in (5.2.1.10) we get the relation

$$\omega(k) = k_z v_{||}, \tag{5.2.1.11}$$

which is the condition for Cherenkov resonance between a wave moving along the magnetic field $\boldsymbol{B}_0$ with a phase velocity $\omega(k)/k_z$ and a resonance particle—with a velocity component along the magnetic field $v_{||}$ being equal to the Cherenkov value. The absorption (excitation) of waves by these resonance particles is called Cherenkov absorption (excitation) of waves.

We can interpret condition (5.2.1.10) when $l \neq 0$ as follows. We can consider a charged particle moving in the plasma along a spiral with a velocity $v_{||}$ along the magnetic field $\boldsymbol{B}_0$ to be an oscillator which in its eigenframe of reference has the eigenfrequencies $|\omega_{B\alpha}|$, $2|\omega_{B\alpha}|$, $3|\omega_{B\alpha}|$, ... The resonance condition (5.2.1.10) for $l = 1, 2, \ldots$,

$$\omega(k) = l|\omega_{B\alpha}| + k_z v_{||}, \tag{5.2.1.12}$$

is nothing but the condition that the wave frequency and the frequency of the oscillator are equal, taking into account the Doppler shift $k_z v_{||}$. The phase velocity of the wave along $\boldsymbol{B}_0$, which is equal to $\omega(k)/k_z$, is larger than the particle velocity $v_{||}$ in this case and the Doppler effect is the normal one.

We shall call the absorption (excitation) of waves by these resonance particles cyclotron absorption (excitation) under normal Doppler effect conditions.

When $l = -1, -2, -3, \ldots$ the condition (5.2.1.10),

$$\omega(k) = k_z v_{||} - |l\omega_{B\alpha}|, \tag{5.2.1.13}$$

also means that the wave frequency and the eigenfrequency of the oscillator in the laboratory frame of reference are equal, but when $l < 0$, the phase velocity of the wave along $\boldsymbol{B}_0$ is less than the particle velocity $v_{||}$, that is, the Doppler effect is anomalous. Correspondingly, we shall call the absorption (excitation) of waves by these resonance particles cyclotron absorption (excitation) of waves under anomalous Doppler effect conditions.

Using eqn. (5.2.1.9) for f_α we can find the current density and the dielectric permittivity tensor ε_{ij}. In the frame of reference in which $k_y = 0$ this tensor has the following form:

$$\varepsilon_{ij} = \delta_{ij} + \sum_{\alpha=e,i} \frac{4\pi e_\alpha^2}{m_\alpha \omega^2} \left\{ \sum_{l=-\infty}^{+\infty} \int d^3v \left(\frac{\omega - k_z v_{||}}{v_\perp} \frac{\partial f_{\alpha 0}}{\partial v_\perp} + k_z \frac{\partial f_{\alpha 0}}{\partial v_{||}} \right) \frac{\Pi_{ij}^{(l)}}{\omega - k_z v_{||} - l\omega_{B\alpha}} \right.$$
$$\left. - \left(l + \int \frac{v_{||}^2}{v_\perp} \frac{\partial f_{\alpha 0}}{\partial v_\perp} d^3v \right) b_i b_j, \right. \tag{5.2.1.14}$$

where

$$\Pi_{ij}^{(l)}(\boldsymbol{v}) = \begin{pmatrix} \frac{l^2\omega_{B\alpha}^2}{k_x^2} J_l^2 & iv_\perp \frac{l\omega_{B\alpha}}{k_x} J_l J_l' & v_{\|} \frac{l\omega_{B\alpha}}{k_x} J_l^2 \\ -iv_\perp \frac{l\omega_{B\alpha}}{k_x} J_l J_l' & v_\perp J_l'^2 & -iv_{\|} v_\perp J_l J_l' \\ v_{\|} \frac{l\omega_{B\alpha}}{k_x} J_l^2 & iv_{\|} v_\perp J_l J_l' & v_{\|}^2 J_l^2 \end{pmatrix}, \quad (5.2.1.15)$$

$J_l = J_l(\lambda)$ and $J_l'(\lambda) = dJ_l/d\lambda$ are Bessel functions and their derivatives, $\lambda = k_x v_\perp/\omega_{B\alpha}$, and $\boldsymbol{b} = \boldsymbol{B}_0/B_0$ is a unit vector in the direction of the external magnetic field.

In accordance with the rule formulated in Section 4.2, the integration over $v_{\|} = v_z$ in (5.2.1.14) is from $-\infty$ to $+\infty$ encircling the singularities $v_{\|} = (\omega - l\omega_{B\alpha})/k_z$ from below when $k_z > 0$ and from above when $k_z < 0$.

5.2.2. THE DIELECTRIC PERMITTIVITY TENSOR OF A PLASMA WITH A MAXWELL DISTRIBUTION

If the initial state of the plasma is an equilibrium state, that is, is determined by a Maxwellian velocity distribution for the particles, one can greatly simplify expression (5.2.1.14) for the dielectric permittivity tensor. One can in that case perform the integration over $v_\perp$ using the well-known expression for the second exponential Weber integral (Watson, 1958)

$$\int_0^\infty e^{-p^2t^2} J_l(at) J_l(bt) t \, dt = \frac{1}{2p^2} \exp\left(-\frac{a^2+b^2}{4p^2}\right) I_l\left(\frac{ab}{2p^2}\right),$$

where $|\arg p| < \pi/4$ and where $I_l(x)$ is a modified Bessel function.

Secondly, the integrals over $v_{\|}$ are, according to (4.2.2.2′), reducible to the probability integral,

$$\int_C dv_{\|} \frac{k_z \exp(-v_{\|}^2/2v_\alpha^2)}{k_z v_{\|} - (\omega - l\omega_{B\alpha})} = i\pi w(z_l), \quad (5.2.2.1)$$

where

$$w(z_l) = \exp(-z_l^2)\left[\frac{k_z}{|k_z|} + \frac{2i}{\sqrt{\pi}} \int_0^{z_l} e^{-t^2} dt\right], \quad (5.2.2.2)$$

$$z_l = \frac{\omega - l\omega_{B\alpha}}{\sqrt{(2)} k_z v_\alpha}, \quad v_\alpha = \sqrt{\frac{T_\alpha}{m_\alpha}}. \quad (5.2.2.3)$$

As a result we obtain the following expressions for the components of the dielectric permittivity tensor of a magneto-active plasma with a Maxwell distribution:

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \delta_{ij} + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\left[\sum_{l=-\infty}^{+\infty} i\sqrt{(\pi)}\, z_0 w(z_l) \Pi_{ij}^{(\alpha)} + 2z_0^2 b_i b_j\right], \quad (5.2.2.4)$$

where

$$\Pi_{ij}^{(\alpha)} = \begin{pmatrix} \frac{l^2}{a_\alpha} A_l(a_\alpha) & ilA_l'(a_\alpha) & \sqrt{(2/a_\alpha)}\, lz_l A_l(a_\alpha) \\ -ilA'(a_\alpha) & \frac{l^2}{a_\alpha} A_l(a_\alpha) - 2a_\alpha A_l'(a_\alpha) & -i\sqrt{(2a_\alpha)}\, z_l A_l'(a_\alpha) \\ \sqrt{(2/a_\alpha)}\, lz_l A_l(a_\alpha) & i\sqrt{(2a_\alpha)}\, z_l A_l'(a_\alpha) & 2z_l^2 A_l(a_\alpha) \end{pmatrix},$$

$$A_l(x) = e^{-x} I_l(x), \quad A_l'(x) = \frac{dA_l}{dx}, \quad \sqrt{a_\alpha} = \frac{k_x v_\alpha}{\omega_{B\alpha}}, \quad \text{sgn}\,\sqrt{a_\alpha} = \frac{e_\alpha}{|e_\alpha|}.$$

We note that the tensor ε_{ij} is non-Hermitean. This is connected with the presence of absorption caused by the interaction with resonance particles. We note further that the following relations hold:

$$\varepsilon_{21} = -\varepsilon_{12}, \quad \varepsilon_{31} = \varepsilon_{13}, \quad \varepsilon_{32} = -\varepsilon_{23}.$$

Finally, we note that if we change the sign of the magnetic field, $\boldsymbol{B}_0 \to -\boldsymbol{B}_0$, the components ε_{11}, ε_{22}, ε_{33}, and ε_{13} remain unchanged, while the components ε_{12} and ε_{23} change sign.

Knowing the components of the dielectric permittivity tensor we can, as we did in Subsection 4.3.1, write down the dispersion equation for electromagnetic waves in a magneto-active plasma:

$$\Lambda = \text{Det}\,|\Lambda_{ij}| = An^4 + Bn^2 + C = 0, \tag{5.2.2.5}$$

where

$$\begin{aligned} A &= \varepsilon_{11}\sin^2\theta + 2\varepsilon_{13}\sin\theta\cos\theta + \varepsilon_{33}\cos^2\theta, \\ B &= -\varepsilon_{11}\varepsilon_{33} - (\varepsilon_{22}\varepsilon_{33} + \varepsilon_{23}^2)\cos^2\theta - (\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2)\sin^2\theta \\ &\quad + 2(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13})\sin\theta\cos\theta + \varepsilon_{13}^2, \\ C &= \text{Det}\,|\varepsilon_{ij}| = \varepsilon_{33}(\varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2) + \varepsilon_{11}\varepsilon_{23}^2 + 2\varepsilon_{12}\varepsilon_{13}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{13}^2. \end{aligned} \tag{5.2.2.6}$$

We shall here give only the expression for the coefficient A in the case of a plasma with a Maxwellian velocity distribution of the particles (Gordeev, 1952):

$$A(\boldsymbol{k}, \omega) = 1 + \sum_{\alpha = e, i} \frac{\omega_{p\alpha}^2}{k^2 v_\alpha^2}\left[1 + i\sqrt{(\pi)}\cdot z_0 \sum_{l=-\infty}^{\infty +} e^{-a_\alpha} I_l(a_\alpha)\, w(z_l)\right]. \tag{5.2.2.7}$$

We shall in what follows give a detailed analysis of the dispersion eqn. (5.2.2.5) for a plasma with a Maxwellian velocity distribution and we shall also study the damping of the wave caused by resonance particles and find the different branches of the oscillations.

5.2.3. KINETIC THEORY OF PLASMA RESONANCES

We showed in Subsection 5.1.2 that when the frequency of an electromagnetic wave approaches the frequency of a plasma resonance, determined by the condition $A(\omega) = 0$ the refractive index (and the wavevector) tends to infinity and the phase velocity to zero. However, we can use the equations of Subsection 5.1.2 which were obtained for a cold plasma only in the case when the refractive index is not too large.

We shall now show that taking the thermal motion of the electrons into account qualitatively changes the behaviour of the refractive index in the resonance region even for phase velocities which are appreciably larger than the thermal velocity of the electrons. In particular, the refractive index does not become infinite in the resonance region when the thermal motion is taken into account, although it will still be considerably larger than unity.

We shall first of all obtain an expression for the tensor ε_{ij}, taking into account the corrections caused by the thermal motion of the electrons and the ions. We shall assume, first of all, that the wavelength of the oscillations considered in the direction of the magnetic field is appreciably longer than the Larmor radius, $\varrho_\alpha = v_\alpha/|\omega_{B\alpha}|$, of particles with thermal velocity $v_\alpha \sim v_\perp$; secondly, that the phase velocity of the wave along $\boldsymbol{B}_0$ is appreciably larger than the thermal velocity of the particles and, finally, that the frequency of the wave does not lie close to $|\omega_{B\alpha}|$ or to $2|\omega_{B\alpha}|$ so that the inequalities

$$(k_\perp \varrho_\alpha)^2 = a_\alpha \ll 1, \quad |z_l| = \left|\frac{\omega - l|\omega_{B\alpha}|}{\sqrt{(2)}k_z v_\alpha}\right| \gg 1, \qquad l = 0, \pm 1, \pm 2, \tag{5.2.3.1}$$

are satisfied. In that case we can in eqns. (5.2.2.4) expand the quantity $\exp(-a_\alpha)$ as well as the Bessel functions $I_l(a_\alpha)$ and their derivatives $I_l'(a_\alpha)$ in power series in a_α and use the asymptotic expression (4.2.2.3) of the functions $w(z_l)$ for $|\mathrm{Re}\, z_l| \gg 1$ ($l = 0, \pm 1, \pm 2$). Retaining terms of order $(k_\perp \varrho_\alpha)^2$ and z_l^{-2} we get (Sitenko and Stepanov, 1957)

$$\begin{aligned}
\varepsilon_{11} &= \varepsilon_1 - \sum_\alpha \frac{\omega_{p\alpha}^2 k^2 v_\alpha^2}{\omega^2 - \omega_{B\alpha}^2}\left[\frac{\omega^2 + 3\omega_{B\alpha}^2}{(\omega^2 - \omega_{B\alpha}^2)^2}\cos^2\theta + \frac{3\sin^2\theta}{\omega^2 - 4\omega_{B\alpha}^2}\right],\\
\varepsilon_{22} &= \varepsilon_1 - \sum_\alpha \frac{\omega_{p\alpha}^2 k^2 v_\alpha^2}{\omega^2 - \omega_{B\alpha}^2}\left[\frac{\omega^2 + 3\omega_{B\alpha}^2}{(\omega^2 - \omega_{B\alpha}^2)^2}\cos^2\theta + \frac{\omega^2 + 8\omega_{B\alpha}^2}{\omega^2(\omega^2 - 4\omega_{B\alpha}^2)}\sin^2\theta\right],\\
\varepsilon_{33} &= \varepsilon_3 - \sum_\alpha \frac{\omega_{p\alpha}^2 k^2 v_\alpha^2}{\omega^4}\left(3\cos^2\theta + \frac{\omega^2 \sin^2\theta}{\omega^2 - \omega_{B\alpha}^2}\right),\\
\varepsilon_{12} &= i\varepsilon_2 - i\sum_\alpha \frac{\omega_{p\alpha}^2 \omega_{B\alpha} k^2 v_\alpha^2}{\omega(\omega^2 - \omega_{B\alpha}^2)}\left[\frac{3\omega^2 + \omega_{B\alpha}^2}{(\omega^2 - \omega_{B\alpha}^2)^2}\cos^2\theta + \frac{6\sin^2\theta}{\omega^2 - 4\omega_{B\alpha}^2}\right],\\
\varepsilon_{13} &= -\sum_\alpha \frac{2\omega_{p\alpha}^2 k^2 v_\alpha^2}{(\omega^2 - \omega_{B\alpha}^2)^2}\sin\theta\cos\theta, \qquad \varepsilon_{23} = i\sum_\alpha \frac{\omega_{p\alpha}^2 \omega_{B\alpha} k^2 v^2 (3\omega^2 - \omega_{B\alpha}^2)}{\omega^3(\omega^2 - \omega_{B\alpha}^2)^2}\cos\theta\sin\theta,
\end{aligned} \tag{5.2.3.2}$$

where the quantities ε_1, ε_2, and ε_3 are determined by eqns. (5.1.1.6).

We shall restrict ourselves to a study of the high-frequency hybrid resonances with frequencies $\omega \approx \omega_\infty^{(j)}(\theta)$ ($j = 1, 2$) determined by eqns. (5.1.2.6) in which case we can neglect the ion motion. Using (5.2.3.2) we can in this case write the quantities A, B, and C which occur in the dispersion eqn. (5.2.2.5) in the following form:

$$\begin{aligned}
A &= A_0 - \beta_e^2 n^2 A_1,\\
A_1 &= \frac{\omega_{pe}^2}{\omega^2}\left[3\cos^4\theta + \frac{6\omega^6 - 3\omega^4\omega_{Be}^2 + \omega^2\omega_{Be}^4}{(\omega^2 - \omega_{Be}^2)^3}\cos^2\theta\sin^2\theta + \frac{3\omega^4\sin^4\theta}{(\omega^2 - \omega_{Be}^2)(\omega^2 - 4\omega_{Be}^2)}\right],
\end{aligned} \tag{5.2.3.3}$$

$$B = B_0[1 + O(a_e^2, z_l^{-2})], \qquad l = 0, \pm 1, \pm 2,$$
$$C = C_0[1 + O(a_e^2, z_l^{-2})], \qquad l = 0, \pm 1, \pm 2.$$

Here $\beta_e = v_e/c = \sqrt{(T_e/m_e c^2)}$, $\beta_e \ll 1$, and A_0, B_0, and C_0 are the hydrodynamic values of the quantities A, B, and C, that is, their values for $T_\alpha = 0$. They are determined by eqns. (5.1.1.8) in which we can neglect the contribution from the ions for the high-frequency case considered.

When $\omega \approx \omega_\infty^{(j)}(\theta)$, $j = 1, 2$, we get

$$\begin{aligned} A_0 &= \frac{(\omega^2-\omega_\infty^{(1)2})(\omega^2-\omega_\infty^{(2)2})}{\omega^2(\omega^2-\omega_{Be}^2)}, \\ B_0 &= \frac{(\omega^2-\omega_{pe}^2)(2\omega_{pe}^2+\omega_{Be}^2-\omega^2)}{\omega^2(\omega^2-\omega_{Be}^2)}, \\ C_0 &= \left(1-\frac{\omega_{pe}^2}{\omega^2}\right)\left[1-\frac{\omega_{pe}^2(2\omega^2-\omega_{pe}^2)}{\omega^2(\omega^2-\omega_{Be}^2)}\right]. \end{aligned} \tag{5.2.3.4}$$

Neglecting small terms of order a_e^2 and z_l^- in the coefficients B and C in (5.2.3.3) and using the expression given for A by (5.2.3.3), we can write the dispersion eqn. (5.2.2.5) in the form[†]

$$(A_0-\beta_e^2 n^2 A_1)n^4+B_0 n^2+C_0 = 0. \tag{5.2.3.5}$$

This equation is of the third degree in n^2, in contrast to the dispersion equation corresponding to the hydrodynamic approximation ($\beta_e = 0$) which is a quadratic equation. Equation (5.2.3.5) thus determines three different refractive indexes.[‡]

We can study eqn. (5.2.3.5) using the fact that the parameter $\beta_e^2 n^2 = (v_e/v_{ph})^2 \ll 1$ is small (Sitenko and Stepanov, 1957). To fix the ideas we shall assume that $\omega \sim \omega_{pe} \sim |\omega_{Be}|$. We have then, as to order of magnitude, $A_1 \sim B_0 \sim C_0 \sim 1$; as far as A_0 is concerned, $A_0 \sim 1$ then ω is not too close to $\omega_\infty^{(j)}$ and $A_0 \ll 1$, if $\omega \approx \omega_\infty^{(j)}(\theta)$.

If $\omega \approx \omega_\infty^{(1)}$ or $\omega \approx \omega_\infty^{(2)}$ one of the roots of eqn. (5.2.3.5) is of order unity. To find this root we can neglect the small terms $(A_0-\beta_e^2 n^2 A_1)n^4$ when compared to $B_0 n^2$ and C_0 and we then get for n^2 the hydrodynamical expression

$$n^2 = -\frac{C_0}{B_0}.$$

The two other refractive indexes are, when $\omega \approx \omega_\infty^{(j)}$, considerably larger than unity and to obtain them we can neglect the term $C_0 \sim 1$ in comparison with $B_0 n^2$ and we find then

$$n^2 = \frac{A_0 \pm \sqrt{(A_0^2+4A_1B_0\beta_e^2)}}{2\beta_e^2 A_1}; \tag{5.2.3.6}$$

we can put $\omega = \omega_\infty^{(j)}(\theta)$ in the expressions for $A_0(\omega)$, $B_0(\omega)$, and $C_0(\omega)$.

When getting away from the resonance point, when $|A_0^2| \gg 4\beta_e^2\,|A_1|B_0$, one of the expressions (5.2.3.6) changes to the hydrodynamical expression (5.1.2.2),

$$n^2 = -\frac{B_0}{A_0}, \tag{5.2.3.7}$$

† This equation was obtained by Gershman (1953a).

‡ This fact was noted by Pargamanik (1948) who took the thermal motion into account in the hydrodynamic approximation by introducing in the electron equation of motion a pressure gradient, $\nabla p_e = T_e \nabla n_e$.

while the other one is determined by the equation (Gershman, 1953a)

$$n^2 = \frac{A_0}{\beta_e^2 A_1}. \tag{5.2.3.8}$$

We recall that when $\omega \approx \omega_\infty^{(1)}(\theta)$ the wave with the refractive index (5.2.3.7) corresponds to a slow extra-ordinary wave, and it corresponds to a fast magneto-sound wave when $\omega \approx \omega_\infty^{(2)}(\theta)$. As expression (5.2.3.8) is a solution of the dispersion equation for longitudinal oscillations,

$$A = A_0 - \beta_e^2 n^2 A_1 = 0,$$

the wave with the refractive index (5.2.3.8) is called a plasma wave.

It is convenient to write expression (5.2.3.6) in the form

$$n^2 = \sqrt{\left(\frac{B_0}{\beta_e^2 |A_1|}\right)}\, \varphi_\pm(x), \tag{5.2.3.9}$$

where

$$\varphi_\pm(x) = \alpha\{x \pm \sqrt{(x^2+\alpha)}\}, \quad x = \frac{A_0}{2\beta_e \sqrt{(B_0 |A_1|)}}, \quad \alpha = \operatorname{sgn} A_1. \tag{5.2.3.10}$$

As the coefficients A_1 and B_0 are frequency-independent, the functions $\varphi_\pm(x)$ characterize the frequency-dependence of the refractive index in the resonance region $(x \sim (\omega-\omega_\infty^{(j)})/\omega\beta_e)$. The behaviour of the functions $\varphi_+(x)$ and $\varphi_-(x)$ is shown in Fig. 5.2.2a for $\alpha = 1$ and in Fig. 5.2.2b for $\alpha = -1$.

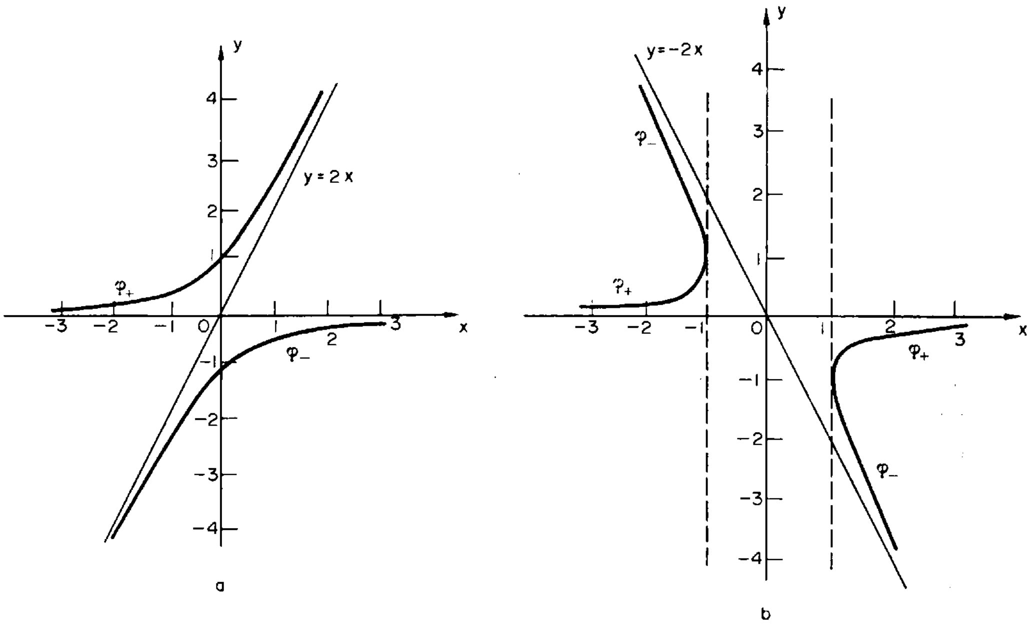

FIG. 5.2.2. The functions $\varphi_+(x)$ and $\varphi_-(x)$: (a) $\alpha = 1$; (b) $\alpha = -1$.

If $\alpha = 1$, only one of the refractive indexes (5.2.3.9), $n^2 \propto \varphi_+(x)$, corresponds to a propagating wave ($n^2 > 0$), while the other wave with $n^2 \propto \varphi_-(x)$ cannot propagate, as for it we have $n^2 < 0$.

Let us trace the behaviour of the frequency-dependence of the refractive index when $\alpha = 1$, that is, its behaviour as a function of x. When the frequency increases the refractive index of this wave increases from the hydrodynamical value (5.2.3.7), $n \sim \sqrt{(\omega/\{\omega_\infty^{(j)}-\omega\})}$, when $\omega < \omega_\infty^{(j)}$ ($x < 0$, $|x| \gg 1$), reaching a value of the order of $n \sim \beta_e^{-1/2}$ in the region $|\omega-\omega_\infty^{(j)}| \sim \beta_e\omega$ and changing in the region $\omega > \omega_\infty^{(j)}$ for $x \gg 1$ into an electrostatic plasma wave with refractive index (5.2.3.8).

The square of the refractive index of the non-propagating wave ($n^2 \propto \varphi_- < 0$) increases with increasing frequency from the hydrodynamical value (5.2.3.7) when $\omega > \omega_\infty^{(j)}$ and $x \gg 1$ to a value $n^2 \sim -\beta_e^{-1}$, and changing in the region $x \gg 1$ and $\omega < \omega_\infty^{(j)}$ into a non-propagating electrostatic wave with $n^2 \sim A_0/\beta_e < 0$.

The behaviour of n^2 for $\alpha = -1$ is different. This case is realized when $|\omega_{Be}| < \omega < 2|\omega_{Be}|$, and as $\omega_\infty^{(2)} < |\omega_{Be}|$ the frequency must lie close to $\omega_\infty^{(1)}$.

If $x < -1$, both refractive indexes (5.2.3.9) are real ($n^2 \propto \varphi_\pm > 0$), that is, two waves can propagate in the frequency range $\omega < \omega_\infty^{(1)}$ when $x < -1$. The wave with the smaller refractive index, $n^2 \propto \varphi_+$, must correspond to a slow extra-ordinary wave, as for $|x| \gg 1$ and $x < 0$ ($\omega < \omega_\infty^{(1)}$) expression (5.2.3.9) for n^2 changes to expression (5.2.3.7) for the square of the refractive index of the SE branch for $\omega \approx \omega_\infty^{(1)}$. The wave with the larger refractive index, $n^2 \propto \varphi_-$, can be called a plasma wave as it changes for $|x| \gg 1$, $x < 0$ into a longitudinal plasma wave with the refractive index determined by eqn. (5.2.3.8). We emphasize that such a division has a meaning only if the frequency of the wave is given; if, however, the wavevector is given, both waves mentioned here correspond to one branch of SE waves. When $x = -1$, the refractive indexes (5.2.3.9) are the same, and when $|x| \sim 1$ theyare of the order of $n \sim \beta_e^{-1/2}$.

When $x > 1$ both values of the square of the refractive index (5.2.3.9) are negative, as $\varphi_\pm < 0$, and waves can not propagate (region of complete internal reflection). When $x \gg 1$, the wave for which $n^2 \propto \varphi_+$ changes into a non-propagating longitudinal wave, and the wave for which $n^2 \propto \varphi_-$ changes into the wave which also exists in a cold plasma with the refractive index (5.2.3.7). When $x = +1$, the refractive indexes of the two waves are the same. As to order of magnitude $n^2 \sim -\beta_e^{-1}$ when $x \sim 1$, $x > 1$.

When $-1 < x < 1$ the propagation of waves is also impossible, as both expressions (5.2.3.9) are complex, while the real and imaginary parts of n are of the same order of magnitude.

We note that in the hydrodynamical approximation the quantity n^2 is always real when there are no collisions, and the region of non-transparency corresponds to values $n^2(\omega) < 0$. The occurrence of a region of non-transparency for which $n^2(\omega)$ is a complex quantity even when there are no dissipative effects is an effect characteristic for media with spatial dispersion.

We have studied the behaviour of the refractive indexes as functions of frequency in the region of the high-frequency hybrid resonances. Let us now consider the wavenumber dependence of the eigenfrequencies in the region of these resonances, taking the thermal motion of the electrons into account. We recall that the frequencies of the hybrid resonances, $\omega =$

$= \omega_\infty^{(j)}(\theta)$, are the limiting values of the frequencies, $\omega = \omega^{(j+2)}(k, \theta)$, of the oscillations of a cold plasma as $k \to \infty$. Putting

$$\omega^{(j+2)}(k, \theta) = \omega_\infty^{(j)}(\theta)\,[1+\Delta_j(k, \theta)], \qquad j = 1, 2, \tag{5.2.3.11}$$

where $|\Delta_j| \ll 1$, we get from the dispersion eqn. (5.2.3.5), dropping the small term C_0,

$$\Delta_j(k, \theta) = \frac{(\beta_e^2 A_1 n^4 - B_0)\,|\omega^2 - \omega_{Be}^2|}{2n^2[(\omega_\infty^{(1)})^2 - (\omega_\infty^{(2)})^2]}\Bigg|_{\omega=\omega_\infty^{(j)}}. \tag{5.2.3.12}$$

The behaviour of the eigenfrequencies $\omega^{(j+2)}(k, \theta)$ of the SE and FMS waves in the region of large wavenumber values ($kc \gg \omega$), as functions of k which is determined by eqns. (5.2.3.11) and (5.2.3.12) is shown schematically in Fig. 5.2.3a for $A_1 > 0$ and in Fig. 5.2.3b for $A_1 < 0$.

If $A_1 > 0$, the frequencies $\omega^{(j+2)}(k, \theta)$ monotonically increase with increasing wavenumber and $\omega^{(j+2)} < \omega_\infty^{(j)}$ for $k < k_0$ and $\omega^{(j+2)} > \omega_\infty^{(j)}$ for $k > k_0$, where

$$k_0 = \frac{\omega_\infty^{(j)}}{c}\left[\frac{B_0}{\beta_e^2|A_1|}\right]^{1/4}.$$

We recall that in a cold plasma the frequencies $\omega^{(j+2)}(k, \theta)$ monotonically increase and approach $\omega_\infty^{(j)}$ as $k \to \infty$, while all the time $\omega^{(j+2)} < \omega_\infty^{(j)}$.

If $A_1 < 0$, the frequency of the slow extra-ordinary wave, $\omega = \omega^{(3)}(k, \theta)$, monotonically increases with increasing wavenumber in the region $k < k_0$, reaches at $k = k_0$ a maximum and monotonically decreases for $k > k_0$. In the point $k = k_0$ the frequency $\omega^{(3)}(k, \theta)$ differs from $\omega_\infty^{(1)}(\theta)$ by an amount of the order of β_e:

$$\omega^{(3)}(k_1, \theta) = \omega_\infty^{(1)}(\theta)\left\{1+\beta_e\frac{\sqrt{(B_0|A_1|)}\,[\omega_{Be}^2-(\omega_\infty^{(1)})^2]}{[(\omega_\infty^{(1)})^2-(\omega_\infty^{(2)})^2]}\right\}.$$

The SE waves therefore have an anomalous dispersion when $k > k_1$ and $A_1 < 0$. We recall that in a cold plasma all branches of the oscillations have normal dispersion.

Let us, finally, consider the conditions for the applicability of the expressions we obtained for $n^2(\omega)$ and $\omega^{(j+2)}(k, \theta)$. When the frequencies get away from $\omega_\infty^{(j)}(\theta)$ the refractive index of a plasma wave increases and its phase velocity diminishes. In that region the collisionless damping of the waves increases. Equations (5.2.3.6) to (5.2.3.12), are obtained when inequalities (5.2.3.1) are satisfied when the effects of the collisionless Cherenkov damping and the cyclotron damping on the first and the second harmonics are exponentially small. In the region $|\omega-\omega_\infty^{(j)}| \sim \omega$ these inequalities are violated, the wavelength of the longitudinal oscillations becomes of the order of the electron Debye or Larmor radius, and the damping rate becomes of the order of the frequency (strong damping region, shown by dashed lines in Fig. 5.2.3).

Let us summarize. When the thermal motion of the plasma particles is taken into account the frequencies of the electromagnetic oscillations of the plasma in a magnetic field are determined by the dispersion eqn. (5.2.2.5) in which the dielectric permittivity tensor of the plasma, $\varepsilon_{ij}(k, \omega)$ depends both on the frequency and on the wavevector. It is determined by eqns. (5.2.1.14) or (5.2.2.4).

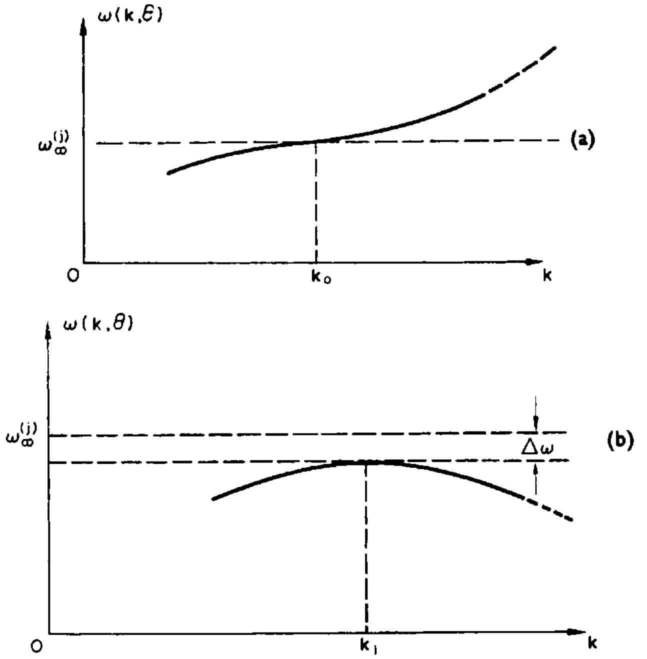

FIG. 5.2.3. The wavenumber dependence of the eigenfrequencies of electromagnetic waves in a magneto-active plasma in the region of a hybrid resonance, when the thermal motion of the electrons is taken into account: (a) $A_1 > 0$; (b) $A_1 < 0$.

The occurrence of singularities in the integrands in (5.2.1.14) for $v_{||} = v_{||\,\text{res}}$ is connected with the effects of Cherenkov and cyclotron resonance between the wave and particles moving along a spiral along the magnetic field with a velocity $v_{||} = v_{||\,\text{res}}$. This resonance interaction leads to Cherenkov and cyclotron damping (or excitation) of the waves.

A study of the dispersion equation for frequencies close to the hybrid resonance frequencies, $\omega = \omega_\infty^{(j)}(\theta)$ $(j = 1, 2)$, shows that when we take the thermal motion of the electrons into account the refractive indexes of the SE and FMS waves remain finite in the resonance region [see eqn. (5.2.3.6)] and there arises then the possibility of the appearance of a plasma wave. The dispersion of the oscillations for $\omega \approx \omega_\infty^{(2)}$ is normal, and for $\omega \approx \omega_\infty^{(1)}$ it can be either normal or anomalous (when $\omega_\infty^{(1)} < 2\,|\omega_{Be}|$).

5.3. Damping of High-frequency Electromagnetic Waves in a Magneto-active Plasma

5.3.1. ELECTRON-CYCLOTRON ABSORPTION OF THE EXTRA-ORDINARY WAVE IN A HOT LOW-DENSITY PLASMA

We shall now study the damping of electromagnetic waves caused by the interaction of these waves with resonance particles. We first of all consider the absorption of the extra-ordinary wave which has a frequency close to the electron-cyclotron frequency.† We shall

† The existence of strong absorption when there is cyclotron resonance was proved by Silin (1952, 1955).

assume that the plasma density is sufficiently low; in fact, we shall assume that the contribution $\delta\varepsilon_{ij}$ to the dielectric permittivity tensor due to electrons is considerably less than unity. The refractive index of electromagnetic waves is in that case close to unity and the presence of the plasma leads only to an unimportant change in the real part of the refractive index and to the occurrence of weak cyclotron damping.

As $\omega/k \approx c$, the quantities $a_e \sim (v_e/c)^2$ and $z_l^{-1} \sim v_e/c$ $(l \neq 1)$ in eqns. (5.2.2.4) for the components of the tensor ε_{ij} are considerably less than unity. We can thus expand the function $\exp(-a_e)\, I_l(a_e)$ in a power series in a_e and use the asymptotic expression of the functions $w(z_l)$ $(l \neq 1)$ for $|z_l| \gg 1$. As a result we get the following expressions for the $\delta\varepsilon_{ij}$:

$$\delta\varepsilon_{11} = \delta\varepsilon_{22} = i\,\delta\varepsilon_{12} = i\sqrt{\left(\frac{\pi}{8}\right)}\frac{\omega_{pe}^2}{\omega^2\beta_e}w(z),$$
$$\delta\varepsilon_{33} = -\frac{\omega_{pe}^2}{\omega^2}, \quad |\delta\varepsilon_{13}| \sim |\delta\varepsilon_{23}| \sim \beta_e|\delta\varepsilon_{11}|, \tag{5.3.1.1}$$

where

$$\beta_e = \frac{v_e}{c} = \sqrt{\frac{T_e}{m_e c^2}}, \quad z = \frac{\omega - |\omega_{Be}|}{\sqrt{(2)}\beta_e\omega\cos\theta}.$$

Substituting these expressions into the dispersion eqn. (5.2.2.5) we find the correction to the refractive index, δn, and the damping coefficient, $\varkappa = \operatorname{Im} n$, for the extra-ordinary wave,

$$n = 1 + \delta n + i\varkappa, \tag{5.3.1.2}$$

where

$$\delta n = -\frac{\omega_{pe}^2}{\sqrt{(8)}\beta_e\omega^2}\,\frac{1+\cos^2\theta}{\cos\theta}\,e^{-z^2}\int_0^z dt\,e^{t^2},$$
$$\varkappa = \frac{1}{4}\sqrt{\left(\frac{\pi}{2}\right)}\frac{\omega_{pe}^2}{\beta_e\omega^2}\,\frac{1+\cos^2\theta}{\cos\theta}\,e^{-z^2}. \tag{5.3.1.3}$$

We see that the quantities δn and $\varkappa$ are of order $\omega_{pe}^2/\beta_e\omega_{Be}^2$ when $z \sim 1$. The absorption line shape is symmetric with respect to the resonance frequency $\omega = |\omega_{Be}|$ and is determined by the function e^{-z^2}. Expressions (5.3.1.3) were obtained assuming that $|\delta\varepsilon_{ij}| \ll 1$ which is valid when $z \sim 1$ only for a very rarefied plasma, when

$$\frac{\omega_{pe}^2}{\beta_e\omega_{Be}^2} \ll 1. \tag{5.3.1.4}$$

The corrections to the refractive index of the ordinary wave are β_e times smaller than the quantities δn and $\varkappa$ for the extra-ordinary wave. This is connected with the fact that the ordinary wave, in contrast to the extra-ordinary one, interacts weakly with the electrons since the electric field strength vector of this wave rotates in the direction opposite to the rotation of the electrons in a magnetic field $\boldsymbol{B}_0$ while the electrical field strength vector of the extra-ordinary wave rotates in the same direction as the electrons in the field $\boldsymbol{B}_0$.

5.3.2. ELECTRON-CYCLOTRON ABSORPTION OF THE SLOW EXTRA-ORDINARY WAVE AND OF THE ORDINARY WAVE IN A HIGH-DENSITY PLASMA

Let us now consider electron cyclotron resonance in a high-density plasma, when the following inequality holds:

$$\frac{\omega_{pe}^2}{\beta_e \omega_{Be}^2} \gg 1, \tag{5.3.2.1}$$

which is the opposite of (5.3.1.4).

We shall assume that for the waves considered $k\varrho_e \ll 1$ and $|z_l| \gg 1$. In that case the refractive indexes of electromagnetic waves are determined by expressions (5.1.1.9) and (5.1.4.1), which were obtained for a "cold" plasma. It can be seen from Figs. 5.1.2 and 5.1.3 that in that case the curves $\omega = \omega^{(j)}(k, \theta)$ $(j = 2, 3)$ which determine the dispersion of the SE and O waves can intersect the straight line $\omega = |\omega_{Be}|$. However, when the frequency approaches $|\omega_{Be}|$ the inequality $|z| \equiv |z_1| = |(\omega - |\omega_{Be}|)/(\sqrt{2}\beta_e n \cos\theta)| \gg 1$ which was assumed to be valid when we derived equation (5.1.4.1) is violated and we need in the region $|z| \ll 1$ kinetic considerations.

The anti-Hermitean terms in the tensor ε_{ij} are of the same order as the Hermitean ones when $|z| \lesssim 1$. The SE and O waves must therefore, it seems, be strongly damped in that region. However, we shall show in what follows that the damping of these waves remains weak also when $|z| \lesssim 1$, except in those cases where the angle θ between the wavevector $\boldsymbol{k}$ and the magnetic field $\boldsymbol{B}_0$ lies close to zero or to $\pi/2$.

Let us study in more detail the behaviour of the eigenfrequencies $\omega = \omega^{(j)}(k, \theta)$ and the refractive index $n(\omega)$ of the SE and O waves in the cyclotron resonance region $\omega \approx |\omega_{Be}|$ in a dense plasma. Assuming as before that

$$a_e \ll 1, \quad |z_l| = \left|\frac{\omega - l|\omega_{Be}|}{\sqrt{(2)}\beta_e n\omega \cos\theta}\right| \gg 1, \qquad l \neq 1,$$

we get from (5.2.2.4) the following expressions for the components of the tensor ε_{ij}:

$$\begin{aligned}
\varepsilon_{11} &= i\sigma + 1 - \tfrac{1}{4}q,\\
\varepsilon_{22} &= i\sigma + 1 - \tfrac{1}{4}q - 2i\alpha,\\
\varepsilon_{12} &= \sigma - \tfrac{1}{4}iq - \alpha,\\
\varepsilon_{33} &= 1 - q + \varepsilon_{33}',\\
\varepsilon_{33}' &= \frac{q\sin^2\theta}{\sqrt{(2)}\cos\theta}\beta_e nz[1 + i\sqrt{(\pi)}\, zw(z)],\\
\varepsilon_{13} &= -i\varepsilon_{23} = \tfrac{1}{2}q\tan\theta[1 + i\sqrt{(\pi)}\, zw(z)],
\end{aligned} \tag{5.3.2.2}$$

where

$$\sigma = \sqrt{\left(\frac{\pi}{8}\right)}\frac{qw(z)}{\beta_e n\cos\theta}, \quad \alpha = (\beta_e n \sin\theta)^2\sigma, \quad q = \left(\frac{\omega_{pe}}{\omega_{Be}}\right)^2,$$

$$\text{and} \quad z = \frac{\omega - |\omega_{Be}|}{\sqrt{(2)}\beta_e n\omega\cos\theta};$$

z can take on any value.

Let us assume, in accordance with inequality (5.3.2.1), that $|\sigma| \gg 1$. If $q \lesssim 1$, we have then as to order of magnitude $\varepsilon_{11} \sim \varepsilon_{22} \sim \varepsilon_{12} \sim \sigma$ and $\varepsilon_{33} \sim \varepsilon_{13} \sim \varepsilon_{23} \sim 1$. Substituting expressions (5.3.2.2) into the dispersion eqn. (5.2.2.5) we can check that the highest order terms, that is, terms of order σ^2 in the coefficients B and C, cancel one another; retaining terms of order σ and of order unity in the coefficients A, B, and C we can write the dispersion eqn. (5.2.2.5) in the form

$$\Lambda = i\sigma\Lambda_0+\Lambda_1+i\sqrt{(\pi)}zw(z)\Lambda_2 = 0, \tag{5.3.2.3}$$

where

$$\Lambda_0 = \sin^2\theta\, n^4-(2+\sin^2\theta-2q)n^2+(1-q)(2-q), \tag{5.3.2.4}$$

$$\begin{aligned}\Lambda_1 = {}&[1-q(1+\tfrac{1}{4}\sin^2\theta)]n^4-[(1-q)(1-\tfrac{1}{4}q)(1+\cos^2\theta)+(1-\tfrac{1}{2}q)(1+q)\sin^2\theta\\ &-\tfrac{1}{4}q^2\tan^2\theta(1+\cos^2\theta)]n^2+(1-q)(1-\tfrac{1}{2}q)-\tfrac{1}{4}q^2(2-q)\tan^2\theta,\end{aligned} \tag{5.3.2.5}$$

$$\Lambda_2 = q\sin^2\theta\, n^4-[q(1-\tfrac{1}{2}q)\sin^2\theta+\tfrac{1}{4}q^2(1+\cos^2\theta)\tan^2\theta]n^2-\tfrac{1}{2}q^2(2-q)\tan^2\theta.$$

We can in eqn. (5.3.2.3) neglect in zeroth approximation ($\sigma \to \infty$) the terms Λ_1 and $i\sqrt{\pi}\,zw(z)\,\Lambda_2$, if we assume that $n^2 \ll |\sigma|$. As a result we obtain the following expressions for the refractive indexes of the ordinary waves, $n = n_-$, and of the slow extra-ordinary waves, $n = n_+$ (Sitenko and Stepanov, 1957):

$$n_\pm^2 = \frac{1}{\sin^2\theta}\left[1+\frac{1}{2}\sin^2\theta-q\pm\sqrt{\left\{\left(1+\frac{1}{2}\sin^2\theta-q\right)^2-(1-q)(2-q)\sin^2\theta\right\}}\right]. \tag{5.3.2.6}$$

These expressions for $n_\pm$ are the same as the expressions (5.1.4.1) for the refractive index of electromagnetic waves in a cold plasma, if we put in the latter $\omega = |\omega_{Be}|$. This is connected with the fact that if we use both in the kinetic and in the hydrodynamical approximations the inequalities $|\sigma| \gg 1$ and $|\sigma| \gg n^2$, the expressions for ε_{33}, ε_{13}, and ε_{23} do not differ in zeroth approximation.

It follows from eqn. (5.3.2.6) that $n_-^2 > 0$ only when $q < 1$ and $n_+^2 > 0$ only when $q < 2$, that is, the SE wave can propagate only when $q < 2$ and the O wave only when $q < 1$.

Retaining in the dispersion eqn. (5.3.2.3) terms of order unity we get

$$n = n_\pm+\delta n_\pm+i\varkappa_\pm, \tag{5.3.2.7}$$

where $\delta n_\pm$ and $\varkappa_\pm$ are appreciably smaller than $n_\pm$. One can show that†

$$\delta n_\pm = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\beta_e\cos\theta}{qn_\pm(2\sin^2\theta\, n_\pm^2-2+2q-\sin^2\theta)}\varphi(z), \tag{5.3.2.8}$$

$$\varkappa_\pm = \sqrt{\left(\frac{2}{\pi}\right)}\frac{\beta_e\cos\theta\Lambda_1}{qn_\pm(2\sin^2\theta\, n_\pm^2-2+2q-\sin^2\theta)}f(z), \tag{5.3.2.9}$$

where

$$\varphi(z) = \Lambda_1\frac{2}{\sqrt{\pi}}\frac{e^{-z^2}\int\limits_0^z dt e^{t^2}}{|w(z)|^2}-\Lambda_2\sqrt{(\pi)}z,\quad f(z) = \frac{e^{-z^2}}{|w(z)|^2},\quad z = \frac{\omega-|\omega_{Be}|}{\sqrt{(2)}\beta_e n_\pm\omega\cos\theta}. \tag{5.3.2.10}$$

† Stepanov (1959c, 1962a) and Gershman (1960) obtained eqn. (5.3.2.9), the second author for the case $|z| \ll 1$.

If $\theta \sim 1$, $q \sim 1$ we have when $|z| \leqslant 1$ as to order of magnitude $n_{\pm} \sim 1$, $\delta n_{\pm} \sim \beta_e$, and $\varkappa_{\pm} \sim \beta_e$, that is, in a dense plasma the cyclotron damping of SE and O waves is, indeed, small. The absorption line shape, determined by eqn. (5.3.2.10) is symmetric with respect to the cyclotron frequency $\omega = |\omega_{Be}|$. The behaviour of the function $f(z)$ is shown in Fig. 5.3.1. The width of the absorption line is of the order $\Delta\omega \sim \beta_e \omega \ll \omega$. The damping of the O wave usually turns out to be an order of magnitude smaller than the damping of the SE wave because of the numerical smallness of the quantity Δ_1, occuring in (5.3.2.9), for the O wave (Gershman, 1960).

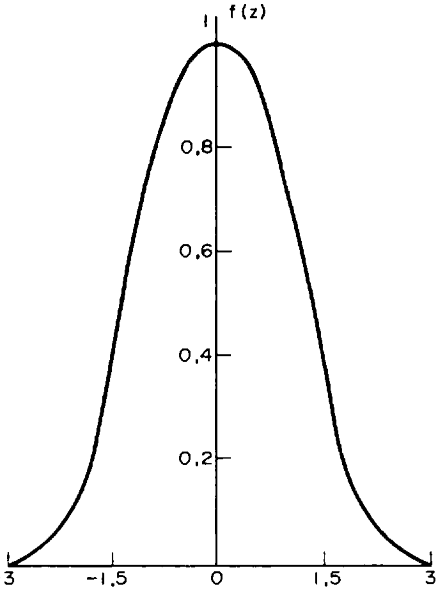

FIG. 5.3.1. The function $f(z)$.

When the angle θ approaches zero the refractive index of the O wave remains finite, $n_-^2 = 1 - \frac{1}{2}q$, and its damping tends to zero,

$$\varkappa_- = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\beta_e \theta^2}{1-q} n_-, \qquad \theta \ll 1.$$

The refractive index of the SE wave and its damping coefficient increase as $\theta \to 0$:

$$n_+^2 \approx \frac{2(1-q)}{\theta^2}, \qquad \theta \ll 1, \tag{5.3.2.11}$$

$$\varkappa_+ \approx \sqrt{\left(\frac{2}{\pi}\right)} \frac{\beta_e n_+^4}{q} \propto \frac{1}{\theta^4}, \qquad \theta \ll 1. \tag{5.3.2.12}$$

However, these expressions can only be used if the angle θ is not too small, namely, provided the inequalities $\varkappa_+ \ll n_+$ or $n_+^2 \ll |\sigma|$ are satisfied, that is,

$$\theta^3 \gg \beta_e \quad (\omega_{pe} \sim |\omega_{Be}|).$$

If the angle θ is so close to zero that $\theta^3 \lesssim \beta_e$, it is clear from the dispersion eqn. (5.3.2.3) that $n^2 \sim i\sigma$, that is, when $|z| \lesssim 1$, we have

$$\mathrm{Re}\, n \approx \varkappa \approx \beta_e^{-1/3} \gg 1. \tag{5.3.2.13}$$

When $\theta = 0$, the dispersion equation for the SE wave has the form $n^2 = i\sigma$. Hence we get when $|z| \ll 1$ (Silin, 1952, 1955):

$$n = \frac{\sqrt{(3)}+i}{2}\left[\sqrt{\left(\frac{\pi}{2}\right)}\frac{v_e}{c}\frac{\omega_{pe}^2}{\omega_{Be}^2}\right]^{1/3}. \tag{5.3.2.14}$$

The slow extra-ordinary wave is thus for almost longitudinal propagation, $\theta^3 \lesssim \beta_e$, strongly damped when $|z| \lesssim 1$. The electromagnetic wave penetrates in that case into the plasma to a depth

$$l = \frac{1}{\mathrm{Im}\,\varkappa} \sim l_a \equiv \left[\frac{c^2 v_e}{\omega_{pe}^2 |\omega_{Be}|}\right]^{1/3}, \tag{5.3.2.15}$$

that is, the region of frequencies close to the cyclotron resonance is in the case when $\theta^3 \lesssim \beta_e$ a region of an anomalous skin effect. The condition $|z| \lesssim 1$ is satisfied when

$$\left|\frac{\omega - |\omega_{Be}|}{\omega}\right| \lesssim \left[\frac{v_e}{c}\frac{\omega_{pe}}{|\omega_{Be}|}\right]^{2/3} \ll 1.$$

In the frequency region where $|z| \gg 1$, the plasma is transparent to the SE wave which has a frequency less than $|\omega_{Be}|$, while the cyclotron damping coefficient is exponentially small for these waves,

$$\varkappa = \sqrt{\left(\frac{\pi}{32}\right)}\frac{\omega_{pe}^2}{\omega^2 \beta_e n^2} e^{-z^2}, \quad z = \frac{\omega - |\omega_{Be}|}{\sqrt{(2)}\beta_e n\omega}, \tag{5.3.2.16}$$

where n is the refractive index of the SE wave,

$$n^2 = \frac{2\omega_{pe}^2}{\omega_{Be}^2 - \omega^2} \gg 1.$$

In the intermediate case when $\sigma \sim \omega_{pe}^2/kv_e|\omega_{Be}| \sim 1$, electromagnetic waves with a frequency close to the cyclotron frequency are strongly damped when $|z| \sim |(\omega - |\omega_{Be}|)/\beta_e\omega| \lesssim 1$:

$$\mathrm{Im}\, n \sim \mathrm{Re}\, n \sim \frac{\omega_{pe}^2 c}{\omega_{Be}^2 v_e} \sim 1. \tag{5.3.2.17}$$

5.3.3. ELECTRON-CYCLOTRON RESONANCE AT HIGHER HARMONICS AND ELECTRON CHERENKOV DAMPING OF HIGH-FREQUENCY WAVES

If the frequency of the slow extra-ordinary wave lies close to $l|\omega_{Be}|$ ($l = 2, 3, \ldots$) collisionless cyclotron damping caused by the absorption of these waves by resonance electrons under normal Doppler effect conditions becomes important.

The fast magneto-sound and the slow extra-ordinary waves can have phase velocities which are much smaller than the velocity of light; these waves can therefore effectively interact with fast electrons in Cherenkov resonance and in cyclotron resonance under anomalous Doppler effect conditions so that Cherenkov absorption and cyclotron absorption under anomalous Doppler effect conditions can turn out to be important for these waves.

We shall now obtain expressions for the Cherenkov and cyclotron damping coefficients of high-frequency waves (O, SE, FE and FMS branches) in the case when the conditions for the applicability of the hydrodynamic approximation are satisfied, that is, when $a_e \sim \sim k^2\varrho_e^2 \ll 1$ and $|z_l| \gg 1$ ($l = 0, \pm 1$). The components of the tensor ε_{ij} in that case have the form

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_1 + \sum_{l \neq 0} 2i\sigma_l, \quad \varepsilon_{12} = i\varepsilon_2 - \sum_{l \neq 0} 2i\sigma_l, \quad \varepsilon_{33} = \varepsilon_3 + 2i\sigma_0, \quad \varepsilon_{13} \approx \varepsilon_{23} \approx 0, \tag{5.3.3.1}$$

where

$$\sigma_l = \sqrt{\left(\frac{\pi}{8}\right)} \frac{\omega_{pe}^2}{\omega k_z v_e} \frac{l^2 a_e^{|l|-1}}{2^{|l|}|l|!} \exp(-z_l^2),$$

$$\sigma_0 = \sqrt{(\pi)} \frac{\omega_{pe}^2}{\omega^2} z_0^3 \exp(-z_0^2), \tag{5.3.3.2}$$

$$z_l = \frac{\omega - l|\omega_{Be}|}{\sqrt{(2)}\beta_e n\omega \cos\theta}, \quad l = 0, \pm 1, \ldots; \quad a_e = \left(\frac{kv_e \sin\theta}{\omega_{Be}}\right)^2.$$

The quantities ε_1, ε_2 and ε_3 are determined by the hydrodynamical eqns. (5.1.1.6) in which we must retain only the electron terms.

Taking into account that the quantities σ_l in (5.3.3.1) are small compared to the ε_i we find from the dispersion equation that in zeroth approximation ($\sigma_l \to 0$) the refractive indexes are determined by eqns. (5.1.4.1) which are valid for a cold plasma. In the next approximation which takes into account terms $\propto \sigma_l$ we find the damping coefficient $\varkappa = \operatorname{Im} n$ (Stepanov, 1959b, 1960)

$$\varkappa = \sum_l \varkappa_l, \tag{5.3.3.3}$$

where

$$\varkappa_l = \sigma_l \frac{\sin^2\theta\, n^4 - \varepsilon_3(1+\cos^2\theta)n^2 + 2(\varepsilon_1 - \varepsilon_2)(\varepsilon_3 - \sin^2\theta\, n^2)}{2C + Bn^2}, \tag{5.3.3.4}$$

$$\varkappa_0 = \sigma_0 \frac{\cos^2\theta\, n^4 - \varepsilon_1(1+\cos^2\theta)n^2 + \varepsilon_1^2 - \varepsilon_2^2}{2C + Bn^2}, \tag{5.3.3.5}$$

$$B = -\varepsilon_1\varepsilon_3(1+\cos^2\theta) - (\varepsilon_1^2 - \varepsilon_2^2)\sin^2\theta, \quad C = \varepsilon_3(\varepsilon_1^2 - \varepsilon_2^2),$$

$$\varepsilon_1 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{Be}^2}, \quad \varepsilon_2 = \frac{\omega_{pe}^2|\omega_{Be}|}{\omega(\omega^2 - \omega_{Be}^2)}, \quad \varepsilon_3 = 1 - \frac{\omega_{pe}^2}{\omega^2}.$$

If $\omega \approx l|\omega_{Be}|$ ($l = 2, 3, \ldots$) and $n \sim 1$, we have as to order of magnitude

$$\varkappa_l \sim \alpha_l \beta_e^{2l-3} \exp(-z_l^2),$$

where the coefficients α_l decrease very fast with increasing l — as $(l!)^{-1}$. The absorption line shape is in that case determined by the function $\exp(-z_l^2)$.

If $\omega \approx 2|\omega_{Be}|$ and $n \sim 1$, we have $\varkappa_2 \sim \beta_e$, that is, the damping coefficient is in that case of the order of the damping coefficient (5.3.2.9) of the SE and O waves with frequency $\omega \approx |\omega_{Be}|$ in a dense plasma.

Notwithstanding the fast decrease of the damping coefficient (5.3.3.4) with increasing l, the cyclotron damping can in a high-temperature plasma appreciably exceed the damping due to Coulomb collisions, $\varkappa_C \sim \nu/\omega$. For instance, when $n_0 \sim 10^{13}$ cm^{-3}, $T_e \sim 10^7$ °K and $\omega \sim 10^{11}$ s^{-1}, we get $\beta_e^2 \approx 2\times 10^{-3}$, $\varkappa_C \sim 2\times 10^{-7}$, $\varkappa_1 \sim \varkappa_2 \sim \beta_e \sim 0.05$, $\varkappa_3 \sim 3\times 10^{-5}$, $\varkappa_4 \sim 4\times 10^{-7}$, that is, the cyclotron damping is for the first three harmonics appreciably larger than the collisional damping, while damping at the fourth harmonic is comparable with the collisional damping.

The damping coefficients $\varkappa_l$ caused by cyclotron resonance for electrons under anomalous Doppler effect conditions ($l = -1, -2, -3, \ldots$) and the Cherenkov absorption coefficient (5.3.3.5) are exponentially small as $|z_l| \gg 1$ when $l = 0, -1, -2, \ldots$.

5.3.4. DAMPING OF WAVES NEAR PLASMA RESONANCES

When the frequency of a wave approaches a plasma resonance frequency $\omega_\infty^{(j)}(\theta)$, $j = 1, 2$, the refractive indexes of the FMS and SE waves and their damping coefficients increase fast:

$$\varkappa_l = \sigma_l \frac{\sin^2\theta\, n^2}{B}, \quad l \neq 0; \qquad \varkappa_0 = \sigma_0 \frac{\cos^2\theta\, n^2}{B}, \tag{5.3.4.1}$$

where $n^2 = -B_0/A_0$ and the coefficients $B_0 = B(\omega)\big|_{\omega=\omega_\infty^{(j)}}$ and A_0 are determined by eqns. (5.2.3.4).

We can use these expressions only when $n^2 \ll \beta_e^{-2}$. If that inequality is not satisfied, the refractive indexes of electromagnetic waves in the vicinity of the frequencies $\omega_\infty^{(j)}$ are determined by eqns. (5.2.3.7).

One can easily obtain the damping coefficients of waves with refractive indexes (5.2.3.7) by taking into account that as $n^2 \gg 1$ the anti-Hermitean terms in the dispersion eqn. (5.2.2.5) need only be taken into account in the coefficient A. Retaining in the expression for A, determined by eqn. (5.2.2.7), terms of order a_e^2 and z_l^{-2} ($l = 0, \pm 1, \pm 2$) we get

$$A = A_0 - \beta_e^2 n^2 A_1 + 2i\sigma_0 \cos^2\theta + \sum_{l\neq 0} 2i\sigma_l \sin^2\theta, \tag{5.3.4.2}$$

where A_1 is determined by eqn. (5.2.3.3). The coefficients B and C in (5.2.2.5) are then determined by their hydrodynamical expressions B_0 and C_0. We then get for the damping coefficients the expressions (Stepanov, 1959c, 1960, 1962a)

$$\varkappa_l = \sigma_l \frac{\sin^2\theta}{2\beta_e^2 A_1 n^2 - A_0}, \quad l \neq 0; \qquad \varkappa_0 = \sigma_0 \frac{\cos^2\theta}{2\beta_e^2 A_1 n^2 - A_0}. \tag{5.3.4.3}$$

When one gets away from the resonance point $\omega = \omega_\infty^{(j)}(\theta)$ in the direction of diminishing n^2, that is, if we go over to the hydrodynamical sections of the FMS and SE branches, we can in the region where $\beta_e^2 n^2 \ll |A_0|$ neglect the term $\propto \beta_e^2$ in the denominators of expressions (5.3.4.3); in that case eqns. (5.3.4.3) change to eqns. (5.3.4.1).

In the region $|A_0| \lesssim \beta_e^2$ we have as to order of magnitude $n^2 \sim \beta_e^{-1}$ and, if $|z_l| \lesssim 1$ ($l = 3, 4, \ldots$), the coefficients of the cyclotron damping (5.3.4.3) are as to order of magnitude equal to

$$\varkappa_l \sim \beta_e^{l-5/2}, \qquad l = 3, 4, \ldots . \tag{5.3.4.4}$$

Comparing this expression with the damping coefficient of waves outside the resonance region where $v_{ph} \sim c$ and $\varkappa_l \sim \beta_e^{2l-3}$, we see that for $\omega = \omega_\infty^{(1)}(\theta) = l|\omega_{Be}|$ the damping is increased by a factor $\beta_e^{1/2-l}$ ($l \geqslant 3$). In particular, for $\omega = \omega_\infty^{(1)} = 3|\omega_{Be}|$, the damping coefficient of the SE wave, $\varkappa_3 \sim \sqrt{\beta_e}$ is $\beta_e^{-1/2}$ times larger than the damping coefficient of waves with $v_{ph} \sim c$ at $\omega = |\omega_{Be}|$ and $\omega = 2|\omega_{Be}|$.

For a plasma wave $n^2 \sim A_0/\beta_e^2 A_1$ and expressions (5.3.4.3) become (Sitenko and Stepanov, 1957)

$$\varkappa_l = \sigma_l \frac{\sin^2\theta}{A_0}, \quad l \neq 0; \qquad \varkappa_0 = \sigma_0 \frac{\cos^2\theta}{A_0}. \tag{5.3.4.4'}$$

The damping coefficients of an SE wave in the region where they change to electrostatic plasma oscillations (see Fig. 5.1.7), which are determined by eqns. (5.3.4.4′), are appreciably larger than the damping coefficients of this wave when $\omega = \omega_\infty^{(l)}$. When we go further away from the resonance point the Cherenkov damping coefficient becomes of the order of the refractive index: when $|\omega - \omega_\infty^{(l)}| \sim \omega$, we have $A_0 \sim 1$, $n \sim c/v_e$, $z_0 \sim 1$, and $\varkappa_0 \sim c/v_e$, that is, when $|\omega - \omega_\infty^{(l)}| \sim \omega$ the plasma wave is damped over a distance of the order $r_D = v_e/\omega_{pe}$.

Equations (5.3.4.3) for $\varkappa_l$ and expression (5.2.3.7) for n were obtained assuming that $|z_2| = |(\omega - 2|\omega_{Be}|)/\sqrt{(2)}\beta_e n\omega \cos\theta| \gg 1$; in that case the damping coefficient $\varkappa_2$ is exponentially small, $\varkappa_2 \propto \exp(-z_2^2)$. When the frequency of the wave approaches $2|\omega_{Be}|$ the cyclotron damping will be very strong, if simultaneously the frequency ω lies close to the plasma resonance frequency $\omega_\infty^{(1)}(\theta)$. This case needs a special study. We shall therefore consider the double resonance $\omega \approx \omega_\infty^{(1)} \approx 2|\omega_{Be}|$ in more detail.

We shall, first of all, assume that $a_e \ll 1$ and $z_l^2 \gg 1$ ($l \neq 2$) and we then find that the coefficient A in the dispersion equation is equal to

$$A = A_0 + i\sqrt{(2\pi)}\beta_e n \frac{\sin^4\theta}{\cos\theta}\frac{\omega_{pe}^2}{\omega^2} w(z),$$

where

$$z = \frac{\omega - 2|\omega_{Be}|}{\sqrt{(2)}\beta_e n \cos\theta\, \omega}.$$

Using this expression for A and neglecting the small term $C \approx C_0$ in the dispersion equation, we find that in the region of the double resonance the dispersion equation has the form (Stepanov, 1959c, 1962)

$$\left[A_0 + i\sqrt{(2\pi)}\beta_e n \frac{\sin^4\theta}{\cos\theta}\frac{\omega_{pe}^2}{\omega^2} w(z)\right] n^2 + B_0 = 0. \tag{5.3.4.5}$$

Hence we find that when $\omega = \omega_\infty^{(1)} = 2|\omega_{Be}|$, or more precise, when $|z| \ll 1$ and $|A_0| \ll \beta_e n$, the refractive index is determined by the equation

$$n = \frac{i \pm \sqrt{3}}{2} \left[\frac{B_0}{\beta_e} \frac{\cos\theta}{\sqrt{(2\pi)} \sin^4\theta} \frac{\omega^2}{\omega_{pe}^2} \right]^{1/3}. \tag{5.3.4.6}$$

If $|A_0| \lesssim \beta_e n$ and $|z| \lesssim 1$, Re n and Im n will also be determined by eqn. (5.3.4.6), at least as to order of magnitude, that is,

$$\mathrm{Re}\, n \sim \mathrm{Im}\, n \sim \beta_e^{-1/3}.$$

Therefore, if the frequency of the wave lies close to $2|\omega_{Be}|$ and to the plasma resonance frequency $\omega_\infty^{(1)}(\theta)$, the wave cannot propagate because of strong damping, and it will penetrate into the plasma to a depth

$$l_a = \frac{(c^2 v_e)^{1/3}}{\omega_{pe}}. \tag{5.3.4.7}$$

Let us give a brief summary of the study of the damping of high-frequency (electron) electromagnetic waves in a collisionless magneto-active plasma. This damping is caused by the interaction of the waves with resonance electrons. Both in a rarefied and in a dense plasma the damping of electromagnetic waves is weak under electroncyclotron resonance conditions, $\omega \sim l|\omega_{Be}|$, $l = 1, 2, \ldots$. An exception occurs only in the case of cyclotron resonance at the fundamental frequency, $\omega = |\omega_{Be}|$, in a dense plasma for longitudinal (or almost longitudinal, $\theta \ll 1$) propagation and in a plasma with an intermediate value of the density when $\omega_{pe}^2/\omega_{Be}^2 \sim \beta_e$, and also in the case of double resonance, $\omega = 2|\omega_{Be}| = \omega_\infty^{(1)}(\theta)$.

The Cherenkov damping of the waves considered is exponentially small except in the short-wavelength region ($kr_D \gtrsim 1$) for the plasma (electrostatic) branches of the oscillations.

5.4. Absorption of Alfvén and Fast Magneto-sound Waves

5.4.1. CHERENKOV ABSORPTION OF ALFVÉN AND FAST MAGNETO-SOUND WAVES IN A LOW-PRESSURE PLASMA

The phase velocities of low-frequency A and FMS waves in a sufficiently dense plasma are appreciably lower than the velocity of light and these waves can thus effectively interact with the plasma electrons under Cherenkov resonance conditions. We obtained in Subsection 5.1.5 expressions for the refractive index and frequencies of A and FMS waves [see eqns. (5.1.5.1) and (5.1.5.3)] in the case of a cold plasma for which the phase velocities of these waves are considerably larger than the electron thermal velocity. In the frequency region $\omega \lesssim \omega_{Bi}$ the phase velocities of both waves are of the order of v_A and the condition $v_A \gg v_e$ is satisfied, provided

$$\xi_e = \frac{n_0 T_e}{B_0^2/8\pi} \ll \frac{m_e}{m_i}. \tag{5.4.1.1}$$

In actual fact weakly damped A and FMS waves with refractive index (5.1.5.1) and frequency (5.1.5.3) exist in a plasma for considerably weaker limiting conditions on the plasma pressure, namely, provided the conditions $v_A \gg v_i$, $v_A \gg v_s$ $(= \sqrt{(T_e/m_i)})$, are satisfied, that is, provided

$$\xi = \frac{n_0(T_e+T_i)}{B_0^2/8\pi} \ll 1. \tag{5.4.1.2}$$

For a plasma with a gas-kinetic pressure appreciably lower than the magnetic pressure $B_0^2/8\pi$ the phase velocity of low-frequency waves can be of the order of or even appreciably lower than the electron thermal velocity. In that case the number of resonance electrons is large, but all the same the damping of both waves remains weak.

We shall now find the frequencies and damping coefficients of A and FMS waves in a low-pressure plasma. We shall assume that the wavelength in the direction at right angles to the magnetic field is considerably longer than the Larmor radius of ions (and electrons) with thermal velocities, that is, $k_x\varrho_\alpha \ll 1$, while the phase velocity of the wave is appreciably larger than the ion thermal velocity and the sound velocity. In that case we have

$$\begin{aligned}
\varepsilon_{11} &= \varepsilon_1 = -\frac{\omega_{pi}^2}{\omega^2-\omega_{Bi}^2},\\
\varepsilon_{22} &= \varepsilon_1+\varepsilon_{22}' = \varepsilon_1+i\sqrt{\pi}\frac{m_e}{m_i}\left(\frac{\omega_{pi}}{\omega_{Bi}}\tan\theta\right)^2\frac{w(z)}{z},\\
\varepsilon_{33} &= \frac{\omega_{pi}^2}{k^2v_s^2\cos^2\theta}[1+i\sqrt{(\pi)}zw(z)],\\
\varepsilon_{12} &= i\varepsilon_2 = -\frac{i\omega_{pi}^2\omega}{\omega_{Bi}(\omega^2-\omega_{Bi}^2)},\\
\varepsilon_{23} &= -i\frac{\omega_{pi}^2}{\omega\omega_{Bi}}\cot\theta\,[1+i\sqrt{(\pi)}zw(z)],
\end{aligned} \tag{5.4.1.3}$$

where

$$z = \frac{\omega}{\sqrt{(2)}kv_e\cos\theta} = \frac{1}{\sqrt{(2)}\beta_e n\cos\theta}.$$

We note that the quantities ε_1 and ε_2 are the same as the corresponding expressions in the hydrodynamical approximation. Substituting these expressions into the dispersion eqn. (5.2.2.5) we can write it in the form

$$\begin{aligned}
&\varepsilon_{33}[\cos^2\theta n^4-\varepsilon_1(1+\cos^2\theta)n^2+\varepsilon_1^2-\varepsilon_2^2]+\varepsilon_1\sin^2\theta\,n^4\\
&\quad+[2\varepsilon_{12}\varepsilon_{23}\cos\theta\sin\theta-(\varepsilon_{22}'\varepsilon_{33}+\varepsilon_{23}^2)\cos^2\theta-(\varepsilon_1^2-\varepsilon_2^2)\sin^2\theta]n^2+\varepsilon_{22}'\varepsilon_{33}\varepsilon_1+\varepsilon_1\varepsilon_{23}^2 = 0.
\end{aligned} \tag{5.4.1.4}$$

Retaining in this equation in zeroth approximation — that is, when $\varepsilon_{33}\to\infty$ — the largest terms, of order $\varepsilon_{33}\,\varepsilon_1^2$ and $\varepsilon_{33}\,\varepsilon_2^2$, we get

$$\cos^2\theta\,n^4-\varepsilon_1(1+\cos^2\theta)n^2+\varepsilon_1^2-\varepsilon_2^2 = 0.$$

Hence expressions (5.1.5.1) and (5.1.5.3) follow for n^2 and $\omega(k, \theta)$, which were obtained in Subsection 5.1.5 under the condition (5.4.1.1) which is more restricted than (5.4.1.2).

In the next approximation, retaining in the dispersion eqn. (5.4.1.4) terms of order $\varepsilon_1 n^4$, $\varepsilon_1 n^2$ and $\varepsilon_2 n^2$, we obtain the damping coefficient $\varkappa = \operatorname{Im} n = \varkappa_e$ (Stepanov, 1960):

$$\varkappa_e = \frac{1}{2}\operatorname{Im}\left\{\frac{1}{\varepsilon_{33}}[\varepsilon_1 \sin^2\theta\, n^4 + \varepsilon_{33}\varepsilon'_{22}(\varepsilon_1 - \cos^2\theta\, n^2) + (2\varepsilon_{12}\varepsilon_{23}\cos\theta\sin\theta - \varepsilon_{23}^2\cos^2\theta - (\varepsilon_1^2 - \varepsilon_2^2)\sin^2\theta)\, n^2]\,[(1+\cos^2\theta)\varepsilon_1 n - 2\cos^2\theta\, n^3]^{-1}\right\},$$

or

$$\varkappa_e = \frac{\sqrt{\pi}}{4}\frac{m_e}{m_i}\frac{e^{-z^2}}{z}\left[\frac{\omega_{pi}^2}{\omega_{Bi}^2}\tan^2\theta(\varepsilon_1 - \cos^2\theta\, n^2) + \frac{\omega^2\sin^2\theta\, n^2(\varepsilon_1^2 - \varepsilon_2^2 - \varepsilon_1 n^2)}{\omega_{pi}^2\,|1 + i\sqrt{(\pi)}\, z w(z)|^2}\right][\varepsilon_1 n(1+\cos^2\theta) - 2n^3\cos^2\theta]^{-1}. \qquad (5.4.1.5)$$

We shall study eqn. (5.4.1.5) for the damping coefficient of A and FMS waves in some special cases.

(a) In the high-frequency region, $\omega_{Bi} \ll \omega \ll |\omega_{Be}|$ (we recall that the FMS waves in that region are called atmospheric whistlers) we get from (5.4.1.5)

$$\varkappa_e = \frac{\sqrt{\pi}}{4}\frac{\sin^2\theta}{\cos\theta}\frac{\omega}{|\omega_{Be}|}\, n\varphi_1(z), \qquad (5.4.1.6)$$

where $n = \omega_{pe}/\sqrt{(\omega\,|\omega_{Be}|\cos\theta)}$ is the refractive index of the atmospheric whistler, and

$$\varphi_1(z) = \frac{1}{2z}\left[1 + \frac{1}{|1 + i\sqrt{(\pi)} z w(z)|^2}\right]e^{-z^2}. \qquad (5.4.1.7)$$

This function decreases fast with increasing z (see Fig. 5.4.1).

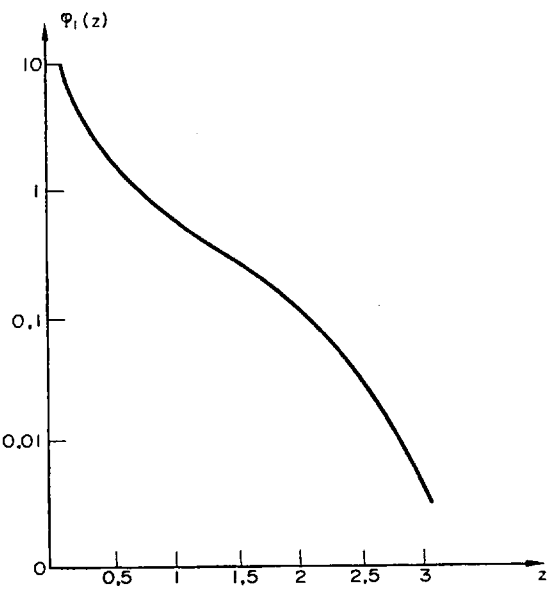

FIG. 5.4.1. The function $\varphi_1(z)$.

In the region of small phase velocities ($z \ll 1$) we have $\varphi_1(z) \approx 1/z$ and the damping coefficient of the whistlers is equal to

$$\varkappa_e = \sqrt{(\pi/8)} \sin^2\theta \, k\varrho_e n, \tag{5.4.1.8}$$

where $\varrho_e = v_e/|\omega_{Be}|$ is the electron Larmor radius.

In the region of large phase velocities ($z \gg 1$) the Cherenkov damping of the whistlers is exponentially small (Gershman, 1958b; Shafranov, 1958c):

$$\varkappa_e = \frac{\sqrt{\pi}}{2} \frac{\sin^2\theta}{\cos\theta} \frac{\omega}{|\omega_{Be}|} n z^3 e^{-z^2}. \tag{5.4.1.9}$$

(b) In the frequency region $\omega \lesssim \omega_{Bi}$ the phase velocity of both waves is of order v_A (except for the cyclotron resonance region for the A branch) and when $v_A \lesssim v_e$, that is, when $1 \gg \xi_e \gtrsim m_e/m_i$, we have as to order of magnitude

$$\frac{\varkappa_e}{n} \sim \frac{m_e}{m_i} \frac{v_e}{v_A} = \sqrt{\left(\frac{m_e}{m_i}\right)} \frac{v_s}{v_A} \ll 1. \tag{5.4.1.10}$$

(c) For the Alfvén branch in the region of ion cyclotron resonance the phase velocity and refractive index increase rapidly,

$$n^2 \approx \frac{c^2}{v_A^2} \frac{1+\cos^2\theta}{2\cos^2\theta} \frac{\omega}{\omega_{Bi}-\omega}, \quad \omega \approx \omega_{Bi},$$

the Cherenkov damping also increases,

$$\frac{\varkappa_e}{n} \approx \frac{\sqrt{\pi}}{8} \frac{m_e}{m_i} \tan^2\theta \frac{\omega}{\omega_{Bi}-\omega} \varphi_2(z), \tag{5.4.1.11}$$

where

$$\varphi_2(z) = \frac{1}{z\,|1+i\sqrt{(\pi)} z w(z)|^2} e^{-z^2}. \tag{5.4.1.12}$$

We note that for $z \ll 1$ and $z \gg 1$ we can use the expressions

$$\varphi_2(z) \approx \frac{1}{z}, \quad z \ll 1; \qquad \varphi_2(z) \approx 4z^3 e^{-z^2}, \quad z \gg 1.$$

A comparison of eqns. (5.4.1.10) and (5.4.1.11) shows that when the frequency of the wave approaches ω_{Bi} the damping rate of the A wave increases by a factor $[\omega/(\omega_{Bi}-\omega)]^{3/2}$ as compared with the non-resonance case.

(d) Equation (5.4.1.5) can be greatly simplified for magneto-hydrodynamic waves ($\omega \ll \omega_{Bi}$). For the magneto-hydrodynamic Alfvén wave, $\omega = kv_A|\cos\theta|$, we have

$$\frac{\varkappa_e}{n} = \sqrt{\left(\frac{\pi}{8}\right)} \frac{m_e}{m_i} \frac{v_e}{v_A} \frac{\omega^2}{\omega_{Bi}^2} \left[\cot^2\theta + \frac{\tan^2\theta}{|1+i\sqrt{(\pi)} z w(z)|^2}\right] e^{-z^2}. \tag{5.4.1.13}$$

For the fast magneto-sound wave, $\omega = kv_A$, the damping coefficient is equal to†

$$\frac{\varkappa_e}{n} = \sqrt{\left(\frac{\pi}{8}\right)} \frac{m_e}{m_i} \frac{v_e}{v_A} \frac{\sin^2\theta}{\cos\theta} e^{-z^2}. \tag{5.4.1.14}$$

From this it follows that the magneto-hydrodynamic Alfvén wave is damped appreciably more weakly (by a factor ω^2/ω_{Bi}^2) than the fast magneto-sound wave.

We note that the magneto-hydrodynamic Alfvén wave is weakly damped even in a high-pressure plasma ($\xi \sim 1$) when its phase velocity is of the order of the ion thermal velocity; in that case the propagation of a FMS wave, and also of A waves and FMS waves with frequencies of the order of ω_{Bi}, becomes completely impossible due to the strong Cherenkov damping by ions (see Stepanov, 1958a).

Equations (5.4.1.13) and (5.4.1.14) are inapplicable in the small-angle region, $\theta^2 \lesssim \omega/\omega_{Bi}$. In that case the damping of both magneto-hydrodynamic waves is the same as to order of magnitude,

$$\frac{\varkappa_e}{n} \sim \frac{m_e}{m_i} \frac{v_e}{v_A} \theta^2,$$

and tends to zero as $\theta \to 0$. The exact expressions for $\varkappa_e$ have the form (Stepanov, 1958a)

$$\frac{\varkappa_e}{n} = \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{m_e}{m_i} \frac{v_e}{v_A} \theta^2 \left[1 \pm \frac{\theta^2}{\sqrt{[\theta^4 + 4(\omega/\omega_{Bi})^2]}}\right] e^{-z^2}. \tag{5.4.1.15}$$

When $\theta^2 \gg 2\omega/\omega_{Bi}$ these expressions change to expressions (5.4.1.13) and (5.4.1.14) if we let θ tend to zero in them.

5.4.2. ION-CYCLOTRON RESONANCE

If the frequency of the FMS wave lies close to the ion-cyclotron frequency or to a multiple of it, $\omega \approx l\omega_{Bi}$ ($l = 1, 2, \ldots$) this wave will suffer cyclotron absorption.‡

The frequency of the A branch is less than $\omega_\infty^{(3)}(\theta) \approx \omega_{Bi}$ so that cyclotron absorption of the A branch becomes important only at the edge of the transparency band when $\omega \approx \omega_{Bi}$. We shall, as before, assume that the plasma pressure is low, that is, $\xi \ll 1$. The tensor ε_{ij} is then for $k\varrho_i \ll 1$ and $|z_l| = |(\omega - l\omega_{Bi})/(\sqrt{2}\,\beta_i l\omega \cos\theta)| \gg 1$ ($l = 0, \pm 1$) determined by eqns. (5.4.1.3), where we must add to ε_{11} and ε_{22} for $\omega \approx l\omega_{Bi}$ the small term $2i\sigma_l$ and to ε_{12} the term $-2\sigma_l$, where

$$\sigma_l = \sqrt{\left(\frac{\pi}{8}\right)} \frac{l^2}{2^l l!} \frac{\omega_{pi}^2}{\omega k v_i \cos\theta} a_i^{l-1} \exp(-z_l^2), \quad z_l = \frac{\omega - l\omega_{Bi}}{\sqrt{(2)} k v_i \cos\theta}, \quad a_i = \left[\frac{k v_i \sin\theta}{\omega_{Bi}}\right]^2. \tag{5.4.2.1}$$

† Stepanov (1958a) obtained eqn. (5.4.1.14); this expression for the case when $z \ll 1$ was obtained also by Braginskiĭ and Kazantsev (1958) and by Gershman (1958a; the latter result must be multiplied by T_e/T_i) and for the case when $z \gg 1$ by Gershman (1958a).

‡ Gershman (1953b) was the first to consider ion cyclotron damping for magneto-hydrodynamic waves ($\omega < \omega_{Bi}$) for $\theta = 0$, when the damping rate is exponentially small; Stix (1958) was the first to consider it in the region of strong damping ($\omega \approx \omega_{Bi}$) for the A branch.

Using these expressions for the tensor ε_{ij} we easily obtain the following expression for the damping coefficient of a FMS wave under multiple ion cyclotron resonance conditions, $\omega \approx l\omega_{Bi}$ (Stepanov, 1960)

$$\varkappa_l = \sigma_l \frac{(1+\cos^2\theta)n^2 - 2(\varepsilon_1 - \varepsilon_2)}{2n^3\cos^2\theta - \varepsilon_1 n(1+\cos^2\theta)}. \tag{5.4.2.2}$$

The absorption line shape of the FMS wave in the ion cyclotron resonance region, $\omega \approx l\omega_{Bi}$, is determined by the function $\exp(-z_l^2)$, and the absorption line width is equal to $\Delta\omega \sim (v_i/v_A)\,\omega\cos\theta$. When $|z_l| \lesssim 1$, we have as to order of magnitude

$$\frac{\varkappa_l}{n} \sim \left(\frac{v_i}{v_A}\right)^{2l-3} \sim \xi_i^{l-3/2}, \tag{5.4.2.3}$$

where $\xi_i = n_0T_i/(B_0^2/8\pi)$. In particular, for $\omega = 2\omega_{Bi}$ we have

$$\frac{\varkappa_l}{n} \sim \frac{v_i}{v_A} \sim \sqrt{\xi_i}\,.$$

The damping (5.4.2.2) decreases fast with increasing harmonic number l and becomes less than the electron Cherenkov damping, if $v_A < v_e$, already for relatively small values of l so that in practice cyclotron damping of FMS waves is important only for the first harmonics.

Let us now turn to a study of the ion-cyclotron resonance at the principal frequency, $\omega \approx \omega_{Bi}$. As before, the component ε_{33} is in this case determined by eqn. (5.4.1.3) while the other components of the ε_{ij} tensor are equal to

$$\begin{aligned}
\varepsilon_{11} &= \varepsilon_{22} = i\sigma_1 - \frac{1}{4}\left(\frac{\omega_{pi}}{\omega_{Bi}}\right)^2, \quad \varepsilon_{12} = -\sigma_1 - i\frac{3}{4}\left(\frac{\omega_{pi}}{\omega_{Bi}}\right)^2,\\
\varepsilon_{23} &= -i\left(\frac{\omega_{pi}}{\omega_{Bi}}\right)^2 \tan\theta\left[\frac{3}{2} + i\sqrt{(\pi)}z_2w(z_2) + \tfrac{1}{2}i\sqrt{(\pi)}z_1w(z_1)\right],\\
\varepsilon_{13} &= -\frac{1}{2}\left(\frac{\omega_{pi}}{\omega_{Bi}}\right)^2 \tan\theta[1 + i\sqrt{(\pi)}z_1w(z_1)],
\end{aligned} \tag{5.4.2.4}$$

where $z_1 = (\omega - \omega_{Bi})/(\sqrt{(2)}kv_i\cos\theta)$, $z_2 = \omega/(\sqrt{(2)}kv_i\cos\theta)$, while the quantity σ_1 is determined by expression (5.4.2.1) for $l = 1$ multiplied by $2w(z_1)$.

Bearing in mind that $|\varepsilon_{33}| \gg |\varepsilon_{11}| \approx |\varepsilon_{12}| \approx |\sigma_1| \gg |\varepsilon_{23}| \gtrsim |\varepsilon_{13}|$, we can retain in the dispersion eqn. (5.2.2.5) solely the terms proportional to ε_{33} and write it in the form

$$\cos^2\theta\, n^4 - \varepsilon_{11}(1+\cos^2\theta)n^2 + \varepsilon_{11}^2 + \varepsilon_{12}^2 = 0. \tag{5.4.2.5}$$

For an FMS wave $n^2 \sim c^2/v_A^2 \ll |\varepsilon_{11}|$; using this we can drop in zeroth approximation ($|\varepsilon_{11}| \to \infty$) the term $\cos^2\theta\, n^4$ term in (5.4.2.5). As a result we get for n^2 the hydrodynamic expression (5.1.5.8),

$$n^2 = \frac{c^2}{v_A^2}\frac{1}{1+\cos^2\theta}.$$

In the next approximation, retaining in the dispersion equation terms of order $(c/v_A)^4$, we find the cyclotron damping coefficient for an FMS wave with $\omega = \omega_{Bi}$ (Stepanov, 1960):

$$\frac{\varkappa_1}{n} = \frac{\cos\theta \sin^4\theta}{\sqrt{(2\pi)}\,(1+\cos^2\theta)^{5/2}} \frac{v_i}{v_A} f(z_1), \tag{5.4.2.6}$$

where the function $f(z_1)$ is determined by eqn. (5.3.2.10),

$$f(z_1) = \frac{1}{|w(z_1)|^2} \exp(-z_1^2).$$

The behaviour of the function $f(z)$ was shown in Fig. 5.3.1.

When θ lies not too close to 0 or $\pi/2$ we have as to order of magnitude $\varkappa_1/n \sim v_i/v_A \sim \sim \sqrt{\xi_i} \ll 1$. Comparing this estimate with (5.4.2.3) we find that the damping of an FMS wave is of the same order of magnitude for $\omega \approx \omega_{Bi}$ as for $\omega \approx 2\omega_{Bi}$.

Let us now find the damping of the A branch in the region $\omega \approx \omega_{Bi}$; we recall that in the region $\omega \approx \omega_{Bi}$ this wave is called the ion-cyclotron wave. The refractive index of the Alfvén wave is large when $\omega \approx \omega_{Bi}$: $n^2 \sim \varepsilon_{11} \sim \sigma$. Using this fact we can simplify the dispersion eqn. (5.4.2.5), dropping in it the last two terms $\varepsilon_{11}^2 + \varepsilon_{12}^2 \sim \varepsilon_{11}(c^2/v_A^2)$ when compared with $\cos^2\theta\, n \sim \varepsilon_{11}^2$ and $\varepsilon_{11} n^2 \sim \varepsilon_{11}^2$:

$$\cos^2\theta\, n^2 - i\sigma_1(1+\cos^2\theta) = 0. \tag{5.4.2.7}$$

From this we find that when $|z_1| \gg 1$ the refractive index of the A wave is determined by the hydrodynamic expression (5.1.5.6)

$$n^2 = \frac{c^2}{v_A^2} \frac{\omega}{\omega_{Bi}-\omega} \frac{1+\cos^2\theta}{2\cos^2\theta},$$

while the cyclotron damping coefficient is exponentially small (Shafranov, 1958b; Stix, 1958)

$$\frac{\varkappa}{n} = \frac{\sqrt{\pi}}{2} |z_1| \exp(-z_1^2), \tag{5.4.2.8}$$

where

$$z_1 = \left[\frac{\omega_{Bi}}{\omega} - 1\right]^{3/2} \frac{v_A}{v_i} \frac{1}{(1+\cos^2\theta)^{1/2}} \gg 1.$$

When ω approaches ω_{Bi} the quantity $|z_1|$ decreases and the damping coefficient (5.4.2.8) increases. In the region $|z_1| \lesssim 1$ the damping of the ion-cyclotron wave becomes so strong that it becomes impossible for this wave to propagate:

$$\operatorname{Re} n \sim \operatorname{Im} n \sim \left[\frac{c^3(1+\cos^2\theta)}{v_A^2 v_i \cos^3\theta}\right]^{1/3}. \tag{5.4.2.9}$$

If $|z_1| \ll 1$, by putting in (5.4.2.7) $w(z_1) = 1$ we can obtain an exact expression for n (Stepanov, 1960; Doyle and Neufeld, 1958):

$$n = \frac{\sqrt{3}+i}{2}\left[\sqrt{\left(\frac{\pi}{8}\right)} \frac{c^3(1+\cos^2\theta)}{v_A^2 v_i \cos^3\theta}\right]^{1/3}. \tag{5.4.2.10}$$

When $|z_1| \lesssim 1$, the electromagnetic field penetrates into the plasma to a depth of the order of

$$l_i = \frac{(v_A^2 v_i)^{1/3}}{\omega_{Bi}}. \tag{5.4.2.11}$$

In strong magnetic fields and for high plasma densities the skin depth (5.4.2.11) becomes very small. For instance, for $n_0 \sim 10^{15}$ cm^{-3}, $B_0 \sim 10^5$ G, $T_i \sim 10^6$ °K and $m_i \sim 10^{-24}$ g, we get $v_A \sim 10^9$ cm/s, $v_i \sim 10^7$ cm/s, $\omega_{Bi} \sim 10^9$ s^{-1} and $l_i \sim 0.2$ cm.

Let us summarize. Weakly damped low-frequency waves (the FMS branch for $\omega \ll \omega_{Bi}$ and the A branch) exist not only in a very-low-pressure plasma ($v_A \gg v_e$) when the Cherenkov damping of these waves by the electrons is exponentially small, but also in a plasma with an appreciably higher pressure (v_s, $v_i \ll v_A \lesssim v_e$). The Cherenkov damping coefficient of both waves due to electrons is determined in this case by eqn. (5.4.1.5).

The ion cyclotron damping of an FMS wave, determined by eqn. (5.4.2.2) for resonance at a harmonic and by eqn. (5.4.2.6) for $\omega \approx \omega_{Bi}$, is always small. The ion-cyclotron damping of the A branch is large when $\omega \approx \omega_{Bi}$ and this wave is damped over a distance of the order of the wavelength [see (5.4.2.11)].

5.5. Low-frequency Oscillations of a Hot Plasma in a Magnetic Field

5.5.1. LONGITUDINAL OSCILLATIONS OF A PLASMA WITH HOT ELECTRONS AND COLD IONS

In the preceding two sections we have studied the influence of the thermal motion of the electrons and ions on the damping of those branches of electromagnetic oscillations of the plasma which exist also in a cold plasma, and we have also studied the influence of the thermal motion of the electrons on the behaviour of the refractive index near high-frequency plasma resonances.

In the present and later sections of the present chapter we shall study new branches of oscillations of a magneto-active plasma which arise solely thanks to the finite electron and ion temperatures. We shall start with a consideration of the longitudinal oscillations of a plasma with hot electrons and cold ions ($T_e \gg T_i$) in a magnetic field (Stepanov, 1959a). We showed in Subsections 4.1.3 and 4.2.4 that if there is no external magnetic field weakly damped ion-sound oscillations can propagate in such a plasma with a dispersion law

$$\omega = \omega_s(k) = \frac{kv_s}{\sqrt{(1+k^2r_D^2)}}. \tag{5.5.1.1}$$

It is clear that if the frequency (5.5.1.1) is considerably higher than the ion-cyclotron frequency ω_{Bi} and the wavelength considerably shorter than the ion Larmor radius ϱ_i, the influence of the magnetic field on such oscillations will be weak. The question therefore arises how the ion-sound oscillations are changed when $\omega_s(k) \lesssim \omega_{Bi}$ and $k\varrho_i \lesssim 1$.

To solve that problem we turn to the dispersion equation $A = 0$ which describes the longitudinal oscillations of a plasma in a magnetic field:

$$A = 1+\delta\varepsilon_e+\delta\varepsilon_i = 0, \tag{5.5.1.2}$$

where

$$\delta\varepsilon_\alpha = \frac{\omega_{p\alpha}^2}{k^2 v_\alpha^2}\left[1 + i\sqrt{(\pi)} z_0 \sum_{l=-\infty}^{+\infty} A_l(a_\alpha) w(z_l)\right], \quad z_l = \frac{\omega - l|\omega_{B\alpha}|}{\sqrt{(2)} k v_\alpha \cos\theta}. \quad (5.5.1.3)$$

We shall assume that the wavelength is appreciably longer than the ion and electron Larmor radius, the phase velocity of the wave appreciably larger than the ion thermal velocity, but appreciably smaller than the electron thermal velocity, and the frequency of the oscillations not very close to the ion-cyclotron frequency and considerably smaller than the electron-cyclotron frequency, so that the following inequalities hold:

$$a_\alpha = (k_x \varrho_\alpha)^2 \ll 1, \quad v_i \ll \frac{\omega}{k_z} \ll v_e, \quad \frac{|\omega - \omega_{Bi}|}{k_z v_i} \gg 1, \quad |\omega_{Be}| \gg \omega,\ k_z v_e. \quad (5.5.1.4)$$

The quantities $\delta\varepsilon_e$ and $\delta\varepsilon_i$ then have the form

$$\begin{aligned} \delta\varepsilon_e &= \frac{\omega_{pe}^2}{k^2 v_e^2}(1 + i\sqrt{(\pi)} z_e) + \frac{\omega_{pe}^2}{\omega_{Be}^2}, \\ \delta\varepsilon_i &= -\frac{\omega_{pi}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pi}^2 \sin^2\theta}{\omega^2 - \omega_{Bi}^2} + i\sqrt{\left(\frac{\pi}{2}\right)} \frac{\omega_{pi}^2 \omega}{k^3 v_i^3 \cos\theta} \sum_{l=0}^{\infty} \frac{a_i^l}{2^l l!} \exp(-z_l^2), \end{aligned} \quad (5.5.1.5)$$

$$z_e = \frac{\omega}{\sqrt{(2)} k_z v_e}, \quad z_l = \frac{\omega - l\omega_{Bi}}{\sqrt{(2)} k_z v_i}. \quad (5.5.1.6)$$

In the expression for $\delta\varepsilon_e$ we took into account only one anti-Hermitean term, corresponding to the Cherenkov absorption of the oscillations by electrons, since in accordance with the condition $|\omega_{Be}| \gg k_z v_e$ the electron-cyclotron damping of the oscillations considered is exponentially small. In the expression for $\delta\varepsilon_i$ we did not take into account the exponentially small terms corresponding to the cyclotron absorption of the oscillations under anomalous Doppler effect conditions.

Substituting expressions (5.5.1.5) and (5.5.1.6) into the dispersion eqn. (5.5.1.2) we get

$$\begin{aligned} 1 + \frac{\omega_{pe}^2}{\omega_{Be}^2} + \frac{\omega_{pi}^2}{k^2 v_s^2} - \frac{\omega_{pi}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pi}^2}{\omega^2 - \omega_{Bi}^2}\sin^2\theta + i\sqrt{\pi}\frac{\omega_{pe}^2}{k^2 v_e^2} z_e \\ + i z_0 \sqrt{\pi} \frac{\omega_{pi}^2}{k^2 v_i^2} \sum_{l=0}^{\infty} \frac{a_i^l}{2^l l!} \exp(-z_l^2) = 0. \quad (5.5.1.7) \end{aligned}$$

Bearing in mind that according to (5.5.1.4) $\omega_{pe}^2/\omega_{Be}^2 \ll \omega_{pi}^2/k^2 v_s^2$ and neglecting the small anti-Hermitean terms, we get from the dispersion eqn. (5.5.1.7) the following expressions for the frequencies of the longitudinal oscillations of a strongly non-isothermic plasma with hot electrons and cold ions:

$$\omega^2(k, \theta) = \tfrac{1}{2}(\omega_s^2 + \omega_{Bi}^2) \pm \tfrac{1}{2}\sqrt{[(\omega_s^2 + \omega_{Bi}^2)^2 - 4\omega_s^2 \omega_{Bi}^2 \cos^2\theta]}, \quad (5.5.1.8)$$

where $\omega_s(k)$ is determined by eqn. (5.5.1.1). There are thus in a plasma in a magnetic field two branches of longitudinal low-frequency oscillations instead of the single branch of ion-sound oscillations when there is no magnetic field.

It follows from (5.5.1.8) that as $k \to 0$ the lower of the frequencies tends to the value

$$\omega(k, \theta) = kv_s \cos \theta, \tag{5.5.1.9}$$

and the higher to the value

$$\omega(k, \theta) = \omega_{Bi}\left(1+\frac{k^2 v_s^2 \sin^2 \theta}{2\omega_{Bi}^2}\right), \tag{5.5.1.10}$$

that is, it tends to the ion-cyclotron frequency.

As $k \to \infty$ ($kr_D \gg 1$) the frequencies (5.5.1.8) tend to the limiting values

$$\omega_{\pm}^2(\theta) = \tfrac{1}{2}(\omega_{pi}^2+\omega_{Bi}^2) \pm \tfrac{1}{2}\sqrt{[(\omega_{pi}^2+\omega_{Bi}^2)^2-4\omega_{pi}^2\omega_{Bi}^2 \cos^2 \theta]}. \tag{5.5.1.11}$$

The wavenumber dependence of the frequencies (5.5.1.8) is shown in Fig. 5.5.1.

As expressions (5.5.1.11) were obtained assuming that $k^2\varrho_i^2 \ll 1$, they can be used provided $(T_i/T_e)(\omega_{pi}^2/\omega_{Bi}^2) \ll 1$. If $\omega_{pi} \gg \omega_{Bi}$ this condition cannot be satisfied and eqns. (5.5.1.11) become inapplicable. We can in that case use expressions (5.5.1.8) only in the long-wave-

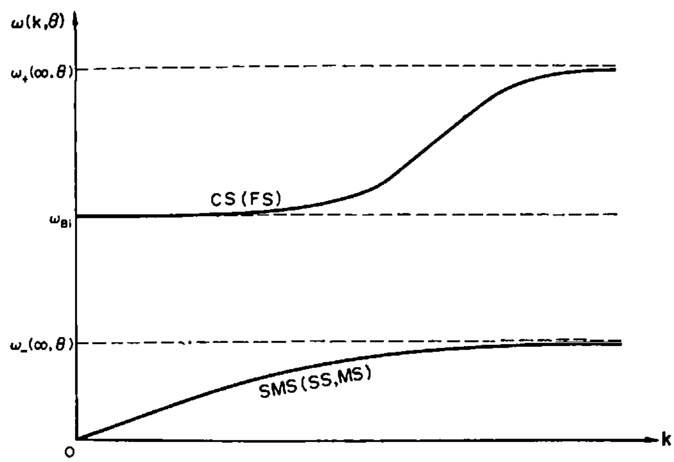

FIG. 5.5.1. Wavenumber dependence of fast and slow sound oscillations (cyclotron sound and slow magneto sound waves) of a non-isothermal plasma ($T_e \gg T_i$).

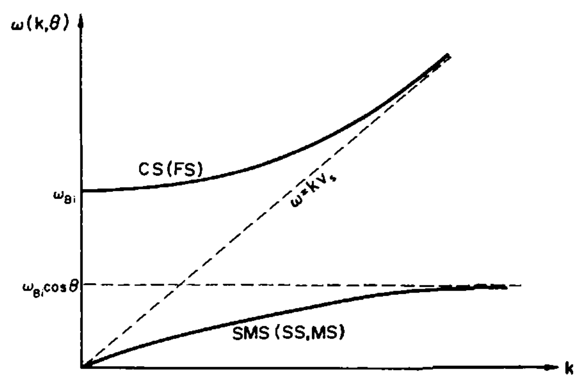

FIG. 5.5.2. Wavenumber dependence of the fast and slow sound waves in the long-wavelength region ($kr_D \ll 1$).

length region ($kr_D \ll 1$). The wavenumber dependence of the frequencies (5.5.1.8) in that case is shown in Fig. 5.5.2. The larger of the frequencies (5.5.1.8) is the same as that of the sound oscillations when there is no magnetic field when $kv_s \gg \omega_{Bi}$ (but $kr_D \ll 1$),

$$\omega(k) = kv_s\,, \tag{5.5.1.12}$$

while the smaller of the frequencies (5.5.1.8) is determined by the equation

$$\omega(k, \theta) = \omega_{Bi} \cos \theta. \tag{5.5.1.13}$$

As the higher of the frequencies (5.5.1.8) tends to ω_{Bi} as $k \to 0$ ($kv_s \ll \omega_{Bi}$), while in a dense plasma ($\omega_{pi} \gg \omega_{Bi}$) in the regions $kr_D \ll 1$ and $kv_s \gg \omega_{Bi}$ the frequency of the oscillations is the same as the sound frequency in a plasma when there is no magnetic field, this branch of the oscillations can be called the cyclotron-sound (CS) branch.

As expression (5.5.1.9) differs from the expression $\omega = kv_s$ for the frequency of the ion-sound oscillations when there is no magnetic field by the replacement of $k_z = k \cos \theta$ for k, the oscillations with frequency (5.5.1.9) are called magnetized sound (MS waves). The oscillations with the lower of the frequencies (5.5.1.8) are thus called magnetized sound oscillations. These oscillations are also called slow magneto-sound (SMS) oscillations as expression (5.5.1.9) is the same as the expression for the frequency of the slow magneto-sound magneto-hydrodynamic waves in the magneto-hydrodynamics approximation, if we assume the oscillations to be isothermal and neglect v_s^2 compared to v_A^2.

As the phase velocity of the oscillations with the higher of the frequencies (5.5.1.8) is larger than the phase velocity of the waves with the lower of these frequencies, it is natural to call these branches the fast and slow sound oscillations (FS and SS waves).

Let us now find the damping of the CS and SMS oscillations. Assuming the anti-Hermitean terms in the dispersion equation (5.5.1.7) to be small we get the following expression for the damping rate of these oscillations:

$$\gamma = \gamma_e + \sum_{l=0}^{\infty} \gamma_l\,, \tag{5.5.1.14}$$

where

$$\gamma_e = \frac{\sqrt{\pi}}{2} \frac{(\omega^2 - \omega_{Bi}^2)^2}{\omega k^2 v_s^2} z_e\,, \tag{5.5.1.15}$$

$$\gamma_l = \frac{\sqrt{\pi}}{2} \frac{(\omega^2 - \omega_{Bi}^2)^2}{\omega k^2 v_i^2} \frac{a_i^l}{2^l l!} z_0 \exp(-z_l^2). \tag{5.5.1.16}$$

The damping rate γ_e caused by the Cherenkov absorption of the sound oscillations by the electrons is, when $\omega \sim kv_s \sim \omega_{Bi}$, as to order of magnitude equal to

$$\frac{\gamma_e}{\omega} \sim \sqrt{\frac{m_e}{m_i}}\,.$$

The damping rate γ_0 caused by the Cherenkov absorption of sound oscillations by ions and the damping rate γ_1 caused by the cyclotron absorption of the oscillations by the ions at the principal frequency are exponentially small; however, as for $\omega \sim kv_s \sim \omega_{Bi}$ the quantities $z_l^2 \sim (T_e/T_i)\{1-l(\omega_{Bi}/\omega)\}^2$ ($l = 0, 1$) are not very large compared to unity, these terms can be comparable with γ_e. The damping rate of the cyclotron-sound wave increases particularly when $\theta \to 0$ or when $\omega_{Bi} \gg \omega_s$, when the frequency (5.5.1.10) of this wave lies close to ω_{Bi}. However, eqns. (5.5.1.10) and (5.5.1.16) for $\omega(k, \theta)$ and γ_1 can be used only provided $|z_1| = (\omega_s/\sqrt{8}\omega_{Bi})(T_e/T_i)^{1/2}(\sin^2\theta/\cos\theta)(1+k^2r_D^2)^{-1/2} \gg 1$, and this condition is satisfied when the angle θ is not too close to zero and when the ratio ω_s/ω_{Bi} is not too small, that is, when the frequency ω lies not too close to ω_{Bi}.

The damping rates γ_l ($l \geq 2$) are caused by the ion-cyclotron resonance at frequencies which are multiples of ω_{Bi} under normal Doppler effect conditions. As the expressions for γ_l are proportional to the small parameter $a_i^l = (k_x\varrho_i)^{2l}$, the quantities γ_l turn out to be comparable with γ_e only when $\omega \sim l\omega_{Bi}$ in the region $|z_l| \lesssim 1$.

If the frequency of the wave is given, the wavevector (or the refractive index) of the longitudinal sound oscillations is determined by the equation

$$n^2 = n_s^2 \frac{\omega^2-\omega_{Bi}^2}{\omega^2-\omega_{Bi}^2\cos^2\theta}, \qquad (5.5.1.17)$$

where $n_s = c/v_s$. The regions of transparency for these waves are determined by the inequalities $\omega < \omega_{Bi}\cos\theta$ or $\omega > \omega_{Bi}$. The behaviour of the function (5.5.1.17) is shown in Fig. 5.5.3.

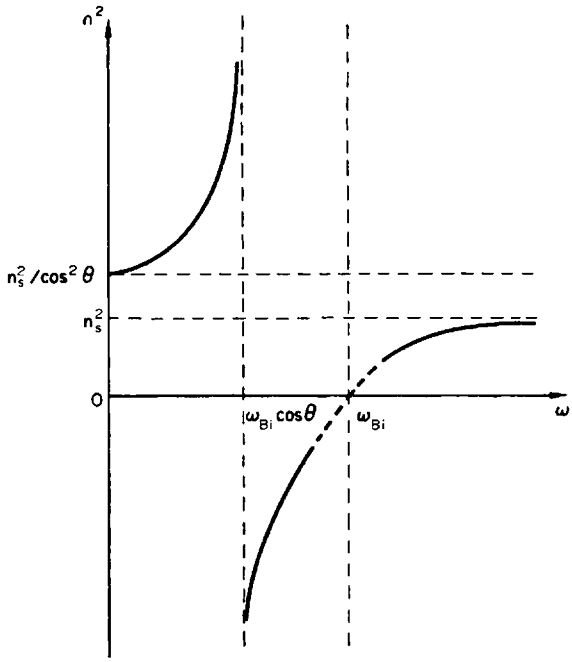

FIG. 5.5.3. Frequency dependence of the refractive index of the fast and slow sound waves in the long-wavelength region ($kr_D \ll 1$).

To conclude this subsection we shall elucidate under what conditions we can use the approximate equation for the longitudinal oscillations, $A = 0$, instead of the exact equiton

$$A + \frac{B}{n^2} + \frac{C}{n^4} = 0.$$

To do this it is necessary that the expressions $|B|/n^2$ and $|C|/n^4$ are appreciably smaller than the main terms occurring in A. Putting $\omega(k) \sim kv_s \sim \omega_{Bi}$, $\cos\theta \sim \sin\theta \sim 1$ and using (5.5.1.7) we find that this is the case if the following inequality holds:

$$\xi_e = \frac{n_0 T_e}{B_0^2/8\pi} \sim \frac{v_s^2}{v_A^2} \sim \frac{\omega_{pi}^2}{k^2c^2} \ll 1, \tag{5.5.1.18}$$

that is, the branches considered of the CS and MS oscillations are purely longitudinal only in the case of a low-pressure plasma.

5.5.2. LOW-FREQUENCY ELECTROMAGNETIC WAVES IN A PLASMA WITH FINITE PRESSURE AND $T_e \gg T_i$

If the pressure of the electron gas in a plasma with hot electrons and cold ions is comparable with or larger than the magnetic pressure, so that inequality (5.5.1.18) is not satisfied, one can no longer consider the cyclotron-sound oscillations and the magnetized sound to be purely longitudinal. It is in that case necessary to study the complete dispersion eqn. (5.2.2.5).

When conditions (5.5.1.4) are satisfied one can write the dispersion eqn. (5.2.2.5) in the form (Pakhomov, 1965)

$$\cos^2\theta(\cos^2\theta-\eta)n^6-\cos^2\theta[2n_A^2+(1-\eta)n_s^2]n^4+[n_A^2+(1+\cos^2\theta)n_s^2]n_A^2n^2-n_A^4n_s^2$$
$$= iA_e+i\sum_{l=0}^{\infty} A_l, \tag{5.5.2.1}$$

where $\eta = (\omega/\omega_{Bi})^2$ and

$$A_e = \sqrt{(\pi)}z_e\left\{2\cos^2\theta\sin^2\theta(\cos^2\theta-\eta)\frac{n_A^2}{n_s^2}n^6+\cos^2\theta\left[2\sin^2\theta\left(1-\frac{n_A^2}{n_s^2}\right)n_A^2+(1-\eta)n_s^2\right]n^4\right.$$
$$\left.-(1+\cos^2\theta)n_A^2n_s^2n^2+n_A^4n_s^2\right\}, \tag{5.5.2.2}$$

$$A_l = 2\sigma_l\frac{\eta(1-\eta)\cos^2\theta\, n^2}{n_A^2}\left\{\cos^2\theta\sin^2\theta\,\eta n^6+\cos^2\theta\left(2\cos^2\theta+\sin^2\theta\frac{2\eta+2\sqrt{(\eta)}+1}{\eta+\sqrt{\eta}}\right)\right.$$
$$\left.\times n_A^2n^4-[\sin^2\theta\, n_A^2+(1+\cos^2\theta)n_s^2]n_A^2n^2-\frac{2n_A^4n_s^2}{1+\sqrt{\eta}}\right\}. \tag{5.5.2.3}$$

The quantity σ_l occurring in eqn. (5.5.2.3) is determined by eqn. (5.4.2.1), z_e by eqn. (5.5.1.6), $n_s = c/v_s$, and n_A is defined by eqn. (5.1.3.3).

If we neglect the right-hand side of eqn. (5.5.2.1), that is, the Cherenkov damping by electrons and ions and the cyclotron damping by ions under normal Doppler effect conditions, eqn. (5.5.2.1) will determine the refractive index (or the eigenfrequencies) of three waves: the Alfvén wave, the fast and the slow magneto-sound waves (the equation is cubic in n^2 for given ω and cubic in ω^2 for given $\boldsymbol{k}$). The frequencies of these waves increase monotonically with increasing wavenumber k.

Once we know n it is easy to find the damping coefficients,

$$\varkappa = \varkappa_e + \sum_{l=0}^{\infty} \varkappa_l , \tag{5.5.2.4}$$

where

$$\varkappa_e = \frac{A_e}{D} N , \tag{5.5.2.5}$$

$$\varkappa_l = \frac{A_l}{D} N , \tag{5.5.2.6}$$

and

$$D = 6\cos^2\theta(\cos^2\theta-\eta)n^6 - 4\cos^2\theta[2n_A^2+(1-\eta)n_s^2]n^4 + 2[n_A^2+(1+\cos^2\theta)n_s^2]\, n_A^2 n^2 . \tag{5.5.2.7}$$

Let us study the dispersion eqn. (5.5.2.1) for some limiting cases.

(a) In the low-frequency region, $\omega \ll \omega_{Bi}$, or, what amounts to the same, in the long-wavelength region, $k \to 0$, we find from (5.5.2.1) the following expressions for the frequencies of the Alfvén, the fast, and the slow magneto-sound waves (Stepanov, 1959d)

$$\omega = k_z v_A , \quad \omega = k v_\pm , \tag{5.5.2.8}$$

where

$$v_\pm^2 = \tfrac{1}{2}(v_A^2 + v_s^2) \pm \tfrac{1}{2}\sqrt{\{(v_A^2+v_s^2)^2 - 4v_A^2 v_s^2 \cos^2\theta\}}. \tag{5.5.2.9}$$

We note that these expressions for the phase velocities of the fast and the slow magneto-sound waves are the same as the expressions for $v_\pm$ in the magneto-hydrodynamical approximation; we must here understand by v_s the velocity of propagation of isothermal oscillations, $v_s \equiv c_s = \sqrt{(\partial p/\partial \varrho)_T} = \sqrt{(p/\varrho)}$.

The electron Cherenkov damping coefficient of the magneto-sound waves is determined by the equation (Stepanov, 1959d)

$$\frac{\varkappa_\pm}{n} = \frac{\gamma_\pm}{\omega} = \sqrt{\left(\frac{\pi}{8}\frac{m_e}{m_i}\right)}\, \frac{v_\pm^4 + 2\cos^2\theta\, v_s^2(\cos^2\theta\, v_s^2 - v_\pm^2)}{(v_\pm^2 - \cos^2\theta\, v_s^2)(2v_\pm^2 - v_A^2 - v_s^2)} . \tag{5.5.2.10}$$

When $v_A \sim v_s$, we have the order of magnitude estimate $\gamma/\omega \sim \sqrt{(m_e/m_i)}$. The damping of the Alfvén wave turns out to be considerably smaller in this case: $\gamma/\omega \sim (\omega/\omega_{Bi})^2 \sqrt{(m_e/m_i)}$.

(b) In the short-wavelength region ($k \to \infty$) the lowest of the three solutions, corresponding to the SMS branch of eqn. (5.5.2.1), tends to the limiting value $\omega_{Bi}\cos\theta$. In that region the refractive index and the frequency are determined by the equations

$$n^2 = \frac{(2n_A^2 + \sin^2\theta\, n_s^2)\omega_{Bi}^2}{\cos^2\theta\, \omega_{Bi}^2 - \omega^2} , \tag{5.5.2.11}$$

$$\omega(k,\theta) = \omega_{Bi}\cos\theta\left[1 - \frac{(2v_s^2 + \sin^2\theta\, v_A^2)\omega_{Bi}^2}{2k^2 v_A^2 v_s^2}\right]. \tag{5.5.2.12}$$

For a low-pressure plasma ($v_A \gg v_s$) eqns. (5.5.2.11) and (5.5.2.12) go over into expressions (5.5.1.17) and (5.5.1.3) for the refractive index and frequency of longitudinal oscillations.

One can simplify the damping coefficient (5.5.2.5) and (5.5.2.6) for an SMS wave in the short-wavelength region:

$$\frac{\varkappa_e}{n} = \frac{\sqrt{\pi}}{2} z_e \frac{\sin^2\theta[2n_A^4 + 2n_A^2 n_s^2(1+\sin^2\theta) + n_s^4]}{n_s^2(2n_A^2 + \sin^2\theta\, n_s^2)}, \tag{5.5.2.13}$$

$$\frac{\varkappa_i}{n} = \sigma_i \cos^6\theta \sin^4\theta \frac{n^4}{n_A^2(2n_A^2 + \sin^2\theta\, n_s^2)}. \tag{5.5.2.14}$$

In the case of a low-pressure plasma ($n_s \gg n_A$) expressions (5.5.2.13) and (5.5.2.14) correspond to expressions (5.5.1.15) and (5.5.1.16) for the damping rate of longitudinal SMS oscillations.

The frequencies of the two other branches tend to infinity as $k \to \infty$ ($\omega \gg \omega_{Bi}$). The frequency of the FMS wave approaches the whistler frequency (5.1.4.7) as $k \to \infty$,

$$\omega(k, \theta) = \frac{\omega_{Bi} c^2 k^2 \cos\theta}{\omega_{pi}^2},$$

and the frequency of the A branch the sound frequency,

$$\omega(k) = kv_s.$$

Expression (5.5.2.13) for $\varkappa_e$ in the case of an FMS wave goes over into expression (5.4.1.8) for the damping coefficient of a whistler, and in the case of oscillations with a frequency $\omega = kv_s$ it corresponds to expression (5.5.1.15) for the damping rate of longitudinal oscillations.

In the region of large wavenumber values the A branch thus goes over into the branch of the CS oscillations, studied in the preceding subsection.

(c) The frequencies of the FMS and A branches can be the same as the ion cyclotron frequency (see Fig. 5.5.4). The corresponding values of the wavenumber $k = k_{1,2}$ are determined from the equation

$$\cos^2\theta \sin^2\theta\, n^6 + \cos^2\theta\, 2n_A^2 n^4[n_A^2 + (1+\cos^2\theta)n_s^2]n_A^2 n^2 + n_A^4 n_s^2 = 0,$$

where $n = n_{1,2} = ck_{1,2}/\omega_{Bi}$. One must, however, note that the ion-cyclotron damping increases steeply for the A branch as $\omega \to \omega_{Bi}$ and the dispersion eqn. (5.5.2.1) becomes itself inapplicable in the region where the condition $|z_1| = |\omega(k, \theta) - \omega_{Bi}|/\sqrt{(2)}k_z v_i \gg 1$ is violated. It is therefore, strictly speaking, impossible to state that the sections of the A branch which correspond to $\omega > \omega_{Bi}$ and to $\omega < \omega_{Bi}$, indeed, refer to the same branch of oscillations. Such a conclusion could only be reached after solving the dispersion equation in the strong damping region, $|\omega - \omega_{Bi}| \lesssim k_z v_i$.

(d) In the case of a low-pressure plasma ($\xi_e \ll 1$, $n_A \ll n_s$) one can easily find a solution of the dispersion equation and elucidate the general behaviour of the eigenfrequencies of the low-frequency oscillations of a strongly non-isothermal plasma and the frequency dependence of the refractive index of these waves (see Figs. 5.5.4 and 5.5.5).

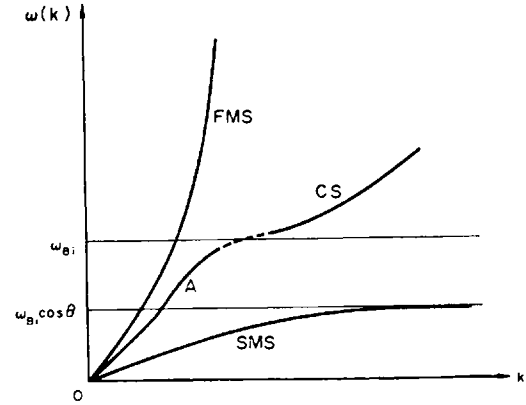

FIG. 5.5.4. The wavenumber dependence of the frequencies of the Alfvén, the fast and the slow magneto-sound waves in a low-pressure non-isothermal plasma.

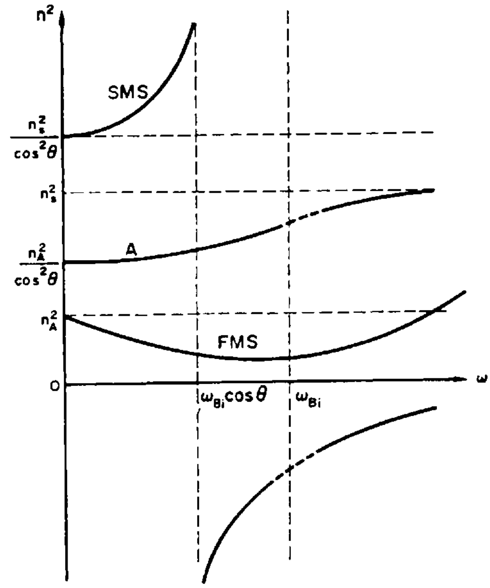

FIG. 5.5.5. The frequency-dependence of the refractive indexes of the Alfvén, the fast and the slow magneto-sound waves in a low-pressure non-isothermal plasma.

The frequency of the FMS wave is determined by eqn. (5.1.5.3) and its refractive index by eqn. (5.1.5.1) which are valid both for a cold plasma and for the case considered here.

The frequency of the A wave is determined for $k < k_2$ by eqn. (5.1.5.3) and its refractive index by eqn. (5.1.5.1) which are obtained for a cold plasma but which are applicable also in the present case. This wave is strongly damped in the region $|\omega - \omega_{Bi}| \lesssim k_z v_i$ when $k \approx k_2$ (see Subsection 5.4.2). When $k > k_2$ the frequency $\omega(k, \theta)$ for this branch is given by formula (5.5.1.8) for the frequency of the longitudinal CS oscillations which in the region $kv_s \gg \omega_{Bi}$ go over into sound oscillations with a frequency $\omega = kv_s$. The refractive index of the CS oscillations is given by formula (5.5.1.17).

The FMS oscillations are longitudinal in a low-pressure plasma and their frequency is given by eqn. (5.5.1.8) and their refractive index by formula (5.5.1.17).

The values $k = k_{1,2}$, for which $\omega(k, \theta) = \omega_{Bi}$, are when $\xi \ll 1$ determined by the equations

$$k_1 = \frac{\omega_{Bi}}{v_A \sqrt{(1+\cos^2\theta)}}, \quad k_2^2 = \frac{\omega_{Bi}^2 \sqrt{(1+\cos^2\theta)}}{v_A v_s \cos\theta \sin\theta}.$$

(e) In a high-pressure plasma ($\xi_e \gg 1$, $v_A \ll v_s$) one can also easily obtain expressions for the eigenfrequencies and refractive indexes of the oscillations considered. Using the inequality $n_A \gg n_s$ we find from (5.5.2.1) (Kovner, 1961a)

$$n^2 = n_s^2, \quad n^2 = \frac{n_A^2}{\cos\theta} \frac{\omega_{Bi}}{\cos\theta\, \omega_{Bi} \pm \omega}. \tag{5.5.2.15}$$

These values of n^2 correspond to the frequencies

$$\omega(k) = kv_s, \tag{5.5.2.16}$$

$$\omega(k, \theta) = \frac{1}{2} k_z v_A \left[\sqrt{\left(4 + \frac{k^2 v_A^2}{\omega_{Bi}^2}\right)} + \frac{k v_A}{\omega_{Bi}} \right], \tag{5.5.2.17}$$

$$\omega(k, \theta) = \frac{1}{2} k_z v_A \left[\sqrt{\left(4 + \frac{k^2 v_A^2}{\omega_{Bi}^2}\right)} - \frac{k v_A}{\omega_{Bi}} \right]. \tag{5.5.2.18}$$

The sound frequency (5.5.2.16) and the whistler frequency (5.5.2.17) intersect in the frequency range $\omega \gg \omega_{Bi}$ when $k = k_0$, where

$$k_0 = \frac{v_s}{c^2} \frac{\omega_{pi}^2}{\omega_{Bi}}.$$

When $k \approx k_0$ we have instead of the above expressions the following formula for $\omega(k, \theta)$ (Kondratenko and Stepanov, 1968)

$$\omega(k, \theta) = kv_s(1+\Delta), \tag{5.5.2.19}$$

where

$$\Delta = \Delta_\pm = \frac{1}{2}\left(1 - \frac{k_0}{k}\right)[1 \pm \sqrt{(1+\eta)}], \quad \eta = \frac{\sin^2\theta}{\cos\theta} \frac{\omega_{Bi}}{k_0 v_s} \frac{k^2}{(k-k_0)^2}.$$

The damping rate for $k \approx k_0$ is determined by the expression

$$\gamma = \frac{1}{2} \gamma_{wh} \left(1 \pm \frac{1}{\sqrt{(1+\eta)}}\right), \tag{5.5.2.20}$$

where γ_{wh} is the whistler damping rate, determined by the formula

$$\gamma_{wh} = \sqrt{\left(\frac{\pi}{2}\right)} k \varrho_e \omega \sin^2\theta, \tag{5.5.2.21}$$

corresponding to eqn. (5.4.1.8) for $\varkappa$.

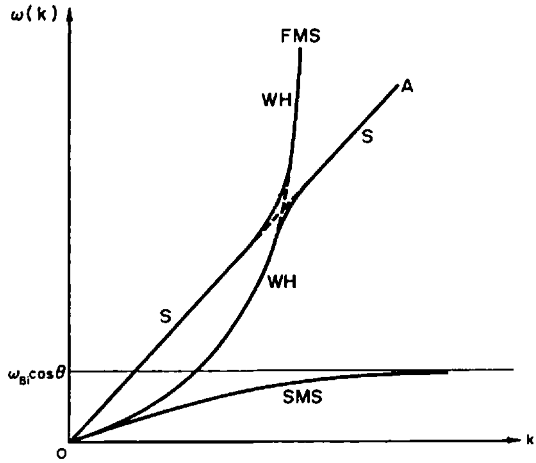

FIG. 5.5.6. Wavenumber dependence of the frequencies of the Alfvén, the fast and the slow magneto-sound waves in a high-pressure non-isothermal plasma.

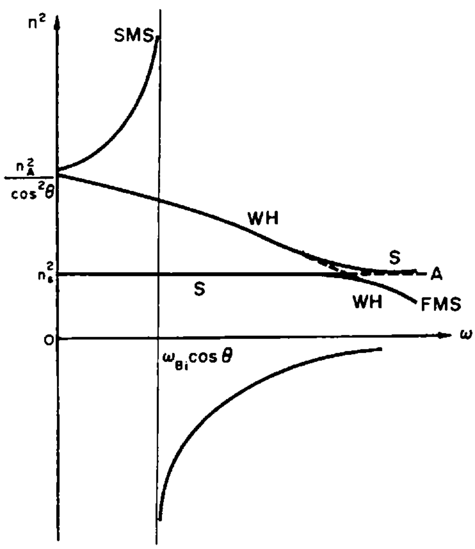

FIG. 5.5.7. Frequency dependence of the refractive indexes of the Alfvén, the fast and the slow magneto-sound waves in a high-pressure non-isothermal plasma

We show in Figs. 5.5.6 and 5.5.7 the behaviour of the functions $\omega(k, \theta)$ and n^2 for the case of a high-pressure plasma, corresponding to eqns. (5.5.2.15) to (5.5.2.20). The frequency of the SMS branch is given by eqn. (5.5.2.18). The frequency of the A branch is for $k < k_0$ given by eqn. (5.5.2.17), in the region $\omega \gg \omega_{Bi}$ this wave is an atmospheric whistler; when $k \gg k_0$ these waves are sound oscillations with frequency (5.5.2.16), while in the region $k \approx k_0$ the frequency of this branch is given by formula (5.5.2.19) with $\Delta = \Delta_-$. The FMS branch is for $k < k_0$ a sound wave with frequency (5.5.2.16) and for $k > k_0$ a whistler with frequency (5.5.2.17), while for $k \approx k_0$ the frequency of this branch is given by eqn. (5.5.2.19) with $\Delta = \Delta_+$.

5.5.3. HIGH-FREQUENCY ELECTRON SOUND

Let us now consider the longitudinal oscillations of a non-isothermal plasma with hot ions and cold electrons ($T_i \gg T_e$) (Mikhaĭlovskiĭ, 1965). We shall assume that the frequency of the oscillations is appreciably higher than the ion cyclotron frequency, that the wavelength is considerably smaller than the Larmor radius of ions with thermal velocity, and that the phase velocity is appreciably less than the ion thermal velocity, that is, $\omega \gg \omega_{Bi}$, $k\varrho_i \gg 1$, and $\omega \ll kv_i$. We can in that case assume the motion of the ions to be unmagnetized, that is, we can neglect the Lorentz force in the kinetic equation for the ions. We can then use for the contribution $\delta\varepsilon_i$ from the ions to the dispersion eqn. (5.5.1.2) for longitudinal oscillations,

$$A = 1+\delta\varepsilon_e+\delta\varepsilon_i = 0,$$

the expression

$$\delta\varepsilon_i = \frac{\omega_{pi}^2}{k^2v_i^2}[1+i\sqrt{(\pi)}z_iw(z_i)], \tag{5.5.3.1}$$

where $z_i = \omega/\sqrt{(2)}kv_i \ll 1$, which is valid when $\boldsymbol{B}_0 = 0$.

We shall further assume that the frequency of the wave is appreciably lower than the electron-cyclotron frequency, that the wavelength is considerably larger than the electron Larmor radius, and that the phase velocity of the oscillations along the magnetic field is appreciably larger than the electron thermal velocity, that is, $\omega \ll |\omega_{Be}|$, $k\varrho_e \ll 1$, $\omega \gg\gg kv_e\,|\cos\theta|$. Expression (5.5.1.3) for $\delta\varepsilon_e$ then becomes

$$\delta\varepsilon_e = \frac{\omega_{pe}^2}{\omega_{Be}^2}-\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta+i\sqrt{\pi}\frac{\omega_{pe}^2}{k^2v_e^2}z_e\exp(-z_e^2), \tag{5.5.3.2}$$

where $z_e = \omega/\sqrt{(2)}kv_e\cos\theta \gg 1$. It is clear that the conditions $kv_e\cos\theta \ll \omega \ll kv_i$ can be satisfied only when $\theta \sim \frac{1}{2}\pi$, as usually $v_e \gg v_i$, even if $T_i \gg T_e$.

Substituting expressions (5.5.3.1) and (5.5.3.2) into the dispersion equation and using the fact that the terms corresponding to Cherenkov absorption of the oscillations considered by the electrons and ions are small, we get for the frequency and damping rate the expressions

$$\omega(k,\theta) = \frac{kv_{se}\cos\theta}{\sqrt{(1+k^2\varrho^2+k^2r_{Di}^2)}}, \tag{5.5.3.3}$$

$$\gamma = \gamma_e+\gamma_i, \tag{5.5.3.4}$$

$$\gamma_e = \sqrt{(\pi)}z_e^3\omega\exp(-z_e^2), \tag{5.5.3.5}$$

$$\gamma_i = \sqrt{\left(\frac{\pi}{8}\frac{m_i}{m_e}\right)}\frac{kv_{se}\cos^2\theta}{(1+k^2\varrho^2+k^2r_{Di}^2)^2}, \tag{5.5.3.6}$$

where $v_{se} = \sqrt{(T_i/m_e)}$, $\varrho = v_{se}/|\omega_{Be}|$, $r_{Di} = v_i/\omega_{pi}$.

In the case of long-wavelength oscillations, $k\varrho \ll 1$, $kr_{Di} \ll 1$, eqn. (5.5.3.3) for the frequency becomes

$$\omega(k, \theta) = k_z v_{se}, \tag{5.5.3.7}$$

which is similar to formula (5.5.1.12) for magnetized ion sound in a low-pressure plasma. The oscillations considered here are called *high-frequency electron sound*, and v_{se} the velocity of electron sound, as the quantity v_{se} is determined by the ion temperature and the electron mass. We show in Fig. 5.5.8 the k-dependence of $\omega(k, \theta)$.

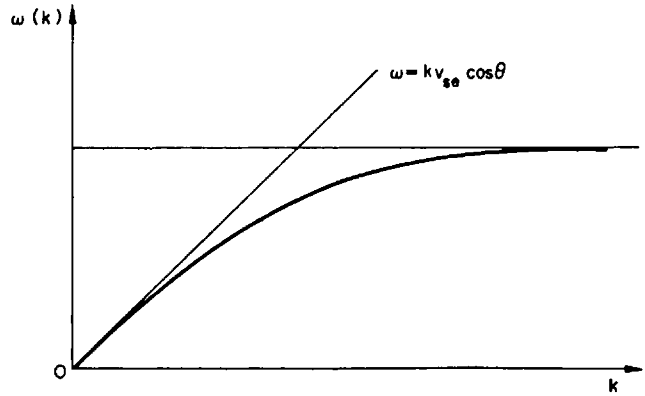

FIG. 5.5.8. The wavenumber dependence of the frequency of high-frequency electron-sound oscillations in a non-isothermal plasma ($T_i \gg T_e$).

The Cherenkov damping (5.5.3.5) of electron sound by electrons is exponentially small as the ion temperature is appreciably higher than the electron temperature,

$$z_e^2 = \frac{T_i}{T_e}\frac{1}{2(1+k^2\varrho^2+k^2r_{Di}^2)} \gg 1.$$

The Cherenkov damping (5.5.3.6) of electron sound by ions is of the order of

$$\frac{\gamma_i}{\omega} \sim \sqrt{\left(\frac{m_i}{m_e}\right)}\cos\theta, \quad k\varrho \lesssim 1, \quad kr_{Di} \lesssim 1.$$

It is small when $\cos\theta \ll \sqrt{(m_e/m_i)}$.

On the other hand, the sound frequency (5.5.3.3) must be considerably higher than ω_{Bi}; this leads to the condition $kv_{se}\cos\theta/\sqrt{(1+k^2\varrho^2+k^2r_{Di}^2)} \gg \omega_{Bi}$. The simultaneous validity of this inequality and the inequality $\cos^2\theta \ll m_e/m_i$ is possible only in a narrow range of angles $\theta \approx \pi/2$. For instance, when $k\varrho \sim 1$ and $\varrho \gtrsim r_{Di}$, these inequalities are satisfied when

$$\frac{m_e}{m_i} \ll \cos\theta \ll \sqrt{\frac{m_e}{m_i}}.$$

5.5.4. LOW-FREQUENCY ELECTRON SOUND

The third hybrid resonance frequency is appreciably lower than the ion-cyclotron frequency when $|\frac{1}{2}\pi-\theta| \ll \sqrt{(m_e/m_i)}$,

$$\omega(k, \theta) = \omega_\infty^{(3)}(\theta) = \frac{\omega_{Bi}\omega_{pe}\cos\theta}{\sqrt{(\omega_{pi}^2+\omega_{Bi}^2)}}. \tag{5.5.4.1}$$

This expression was obtained in subsection 5.1.2 for a cold plasma, in particular, for the case when $k\varrho_i \ll 1$.

Let us first of all elucidate how expression (5.6.4.1) changes for finite $k\varrho_i$ (Mikhaĭlovskiĭ, 1968). Assuming that $k\varrho_e \ll 1, \omega/k_z v_e \gg 1$, and $\omega/k_z v_i \gg 1$, we can write the dispersion eqn. (5.5.1.2) for longitudinal oscillations in the form

$$1-\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta+\frac{\omega_{pi}^2}{k^2 v_i^2}[1-A_0(k\varrho_i)] = 0, \quad A_0(x) \equiv e^{-x^2} I_0(x^2). \tag{5.5.4.2}$$

Hence we find that

$$\omega(k, \theta) = \frac{k v_{se} \cos\theta}{\sqrt{(1-A_0(k\varrho_i)+k^2 r_{Di}^2)}}. \tag{5.5.4.3}$$

In the region $k\varrho_i \gg 1$ and $kr_{Di} \ll 1$ the frequency (5.5.4.3) is the same as the frequency of electron sound,

$$\omega(k, \theta) = k_z v_{se}.$$

The oscillations with frequency (5.5.4.3) which form the short-wavelength part of the slow magneto-sound wave branch are therefore called low-frequency electron sound. The wavenumber dependence of the frequency (5.5.4.3) is shown in Fig. 5.5.9.

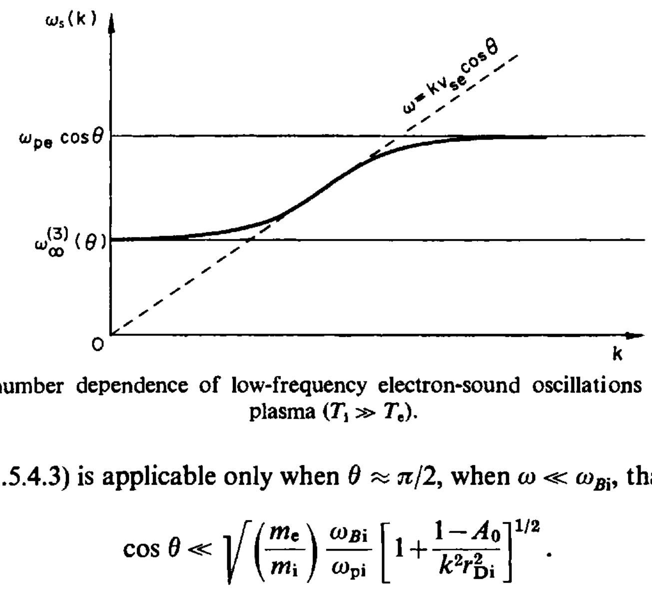

FIG. 5.5.9. Wavenumber dependence of low-frequency electron-sound oscillations in a non-isotherma plasma ($T_i \gg T_e$).

Expression (5.5.4.3) is applicable only when $\theta \approx \pi/2$, when $\omega \ll \omega_{Bi}$, that is,

$$\cos\theta \ll \sqrt{\left(\frac{m_e}{m_i}\right)}\frac{\omega_{Bi}}{\omega_{pi}}\left[1+\frac{1-A_0}{k^2 r_{Di}^2}\right]^{1/2}.$$

In particular, when $kr_{Di} \lesssim 1$ and $k\varrho_i \gtrsim 1$ this inequality is satisfied, if

$$\cos\theta \ll \frac{1}{k\varrho_i}\sqrt{\frac{m_e}{m_i}},$$

that is, low-frequency electron sound exist in a narrow interval of angles close to $\theta \approx \pi/2$.

Let us summarize. In a strongly non-isothermal plasma with hot electrons and cold ions ($T_e \gg T_i$) in the low-frequency region ($\omega \ll |\omega_{Be}|$) three weakly damped long-wavelength

($k\varrho_i \ll 1$) branches of oscillations (the SMS, A, and FMS branches) exist. In the long-wavelength region these branches correspond to the magneto-hydrodynamical SMS, A, and FMS waves.

In the short-wavelength region the frequency of the SMS branch lies close to the frequency $\omega_{Bi} \cos\theta$ of the longitudinal oscillations, the A branch goes over into the unmagnetized ion-sound oscillations, and the FMS branch into atmospheric whistlers.

In a strongly non-isothermal plasma with hot ions and cold electrons ($T_i \gg T_e$) there exists, when $\theta \approx \pi/2$, in the region $\omega \gg \omega_{Bi}$ a branch of short-wavelength ($k\varrho \gg 1$) high-frequency electron-sound oscillations; and in the region $\omega \ll \omega_{Bi}$ a branch of low-frequency electron-sound oscillations which are a continuation into the short-wavelength ($k\varrho_i \gtrsim 1$) region of the A branch.

5.6. Cyclotron Waves in a Plasma for the Case of Quasi-transverse Propagation

5.6.1. LONGITUDINAL ION-CYCLOTRON OSCILLATIONS IN A PLASMA FOR QUASI-TRANSVERSE PROPAGATION

We showed earlier that if the frequency of a wave is close to a multiple of the electron or ion-cyclotron frequency, collisionless damping of the wave occurs which is caused by the resonance interaction between plasma particles and the wave. The cyclotron damping at higher harmonics ($\omega \approx l|\omega_{B\alpha}|$, $l \geqslant 2$) is an effect which is connected with the fact that the Larmor radius of the particles is finite, namely, the damping rate of long-wavelength oscillations ($k\varrho_\alpha \ll 1$, where ϱ_α is the Larmor radius of the α-th kind of particle) is for $\omega \approx l|\omega_{B\alpha}|$ proportional to $(k\varrho_\alpha)^{2n}$ and vanishes as $\varrho_\alpha \to 0$.

Another effect caused by the finite size of the Larmor radius of the particles is the occurrence when $\theta \approx \pi/2$ of a number of new branches of plasma eigenoscillations with frequencies which as $k \to 0$ and as $k \to \infty$ approach the frequencies $l|\omega_{B\alpha}|$ ($l = 1, 2, 3, \ldots$). Such oscillations are called *cyclotron* oscillations.

The behaviour of the ion-cyclotron oscillations depends in an essential manner on how close θ is to $\pi/2$. If the phase velocity of the wave along the magnetic field is appreciably smaller than the electron thermal velocity, but considerably larger than the ion thermal velocity,

$$v_i \ll \frac{\omega}{k_z} \ll v_e\,, \tag{5.6.1.1}$$

we shall say that we have a *quasi-transverse* propagation of the cyclotron waves. If $\omega/k_z \gg v_e$, we shall call it transverse propagation of these waves.

We shall start with a study of the longitudinal ion-cyclotron oscillations for the case of quasi-transverse propagation in a low-pressure plasma (Drummond and Rosenbluth, 1962; Lominadze and Stepanov, 1964a; Drummond and Rosenbluth were the first to prove the existence of these oscillations). We shall assume that both inequality (5.6.1.1) and the inequality

$$|z_l| = \frac{|\omega - l\omega_{Bi}|}{\sqrt{(2)}k_z v_i} \gg 1, \qquad l = 0, \pm 1, \ldots \tag{5.6.1.2}$$

are satisfied. We then find that the quantities $\delta\varepsilon_e$ and $\delta\varepsilon_i$ in the dispersion equation for longitudinal oscillations,

$$A = 1+\delta\varepsilon_e+\delta\varepsilon_i = 0,$$

will be given by the formulae

$$\delta\varepsilon_e = \frac{\omega_{pe}^2}{k^2 v_e^2}(1+i\sqrt{(\pi)}z_e), \tag{5.6.1.3}$$

$$\delta\varepsilon_i = \frac{\omega_{pi}^2}{k^2 v_i^2}\left[1-\sum_{l=-\infty}^{+\infty}\frac{\omega}{\omega-l\omega_{Bi}}A_l(k\varrho_i)\right]+i\sqrt{\pi}\frac{\omega_{pi}^2}{k^2 v_i^2}z_0\sum_{l=-\infty}^{+\infty}A_l(k\varrho_i)\exp(-z_l^2),$$

$$A_l(x) \equiv e^{-x^2}I_l(x^2). \tag{5.6.1.4}$$

Assuming for the sake of simplicity that $k^2 \ll \omega_{p\alpha}^2/v_\alpha^2$ we can in (5.5.1.2) neglect unity when compared with $\delta\varepsilon_e$ and $\delta\varepsilon_i$. Dropping also the small dissipative terms in the $\delta\varepsilon_\alpha$ we can write the dispersion equation $A = \delta\varepsilon_e+\delta\varepsilon_i = 0$ in the form

$$1+\frac{T_i}{T_e} = f(\omega), \tag{5.6.1.5}$$

where

$$f(\omega) = A_0(k\varrho_i)+2\omega^2\sum_{l=1}^{\infty}\frac{A_l(k\varrho_i)}{\omega^2-l^2\omega_{Bi}^2}. \tag{5.6.1.6}$$

It is convenient to solve eqn. (5.6.1.5) graphically. We have sketched the function $f(\omega)$ in Fig. 5.6.1. The intersections 1′, 2′, ... of the various parts of the curve $y = f(\omega)$ with the horizontal line $y = 1+(T_i/T_e)$ correspond to the solutions of the dispersion eqn. (5.6.1.5)

$$\omega = \omega^{(l)}(k), \qquad l = 1, 2, \ldots . \tag{5.6.1.7}$$

It is clear from Fig. 5.6.1 that the larger T_i/T_e, the closer $\omega^{(l)}(k)$ lies to $l\omega_{Bi}$. We have sketched in Fig. 5.6.2 the k-dependence of the first three $\omega^{(l)}(k)$.

In the limiting cases of short wavelengths ($k\varrho_i \gg 1$) and long wavelengths ($k\varrho_i \ll 1$) the frequencies $\omega^{(l)}(k)$ are close to $l\omega_{Bi}$. Using this we put

$$\omega^{(l)}(k) = l\omega_{Bi}[1+\psi_l(k)], \tag{5.6.1.8}$$

where $|\psi_l| \ll 1$. One can easily show that

$$\psi_l(k) = \frac{A_l(k\varrho_i)}{1+\dfrac{T_i}{T_e}+\sum\limits_{m=-\infty}^{+\infty}\dfrac{l}{m-l}A_m(k\varrho_i)}. \tag{5.6.1.9}$$

The frequency $\omega^{(l)}(k)$ is close to $l\omega_{Bi}$ for arbitrary values of $k\varrho_i$ in the case of a non-isothermal plasma with hot ions and cold electrons ($T_i \gg T_e$). In that case

$$\omega^{(l)}(k) = l\omega_{Bi}\left[1+\frac{T_e}{T_i}A_l(k\varrho_i)\right], \qquad \frac{T_e}{T_i}A_l \ll 1. \tag{5.6.1.10}$$

When $T_e > T_i$ the frequencies do not lie close to $l\omega_{Bi}$ when $k\varrho_i \gtrsim 1$, and when $T_e \gg T_i$ they can lie close to $(l+1)\omega_{Bi}$ (see Fig. 5.6.3 in which we show the frequencies $\omega^{(1)}(k)$ and $\omega^{(2)}(k)$ as functions of $(k\varrho_i)^2$ for different values of the parameter T_i/T_e; the frequencies were obtained by numerically solving eqn. (5.6.1.5)).

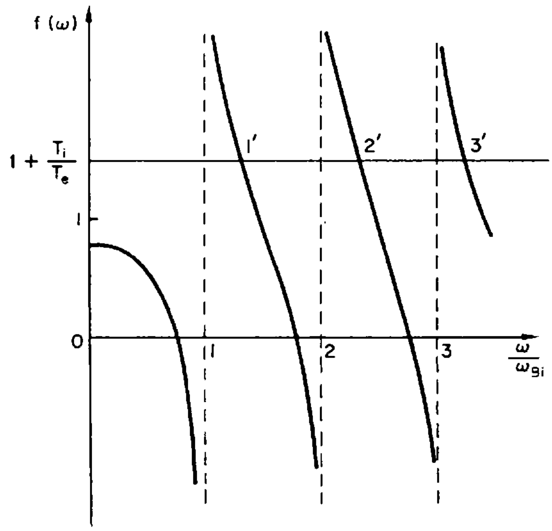

FIG. 5.6.1. Graphical solution of the dispersion eqn. (5.6.1.5).

Taking the small anti-Hermitean terms in eqns. (5.6.1.3) and (5.6.1.4) into account we easily find the damping rates of the oscillations considered:

$$\gamma^{(l)} = \gamma_e^{(l)} + \sum_{m=-\infty}^{+\infty} \gamma_m^{(l)}, \tag{5.6.1.11}$$

where

$$\gamma_e = \sqrt{\left[\frac{\pi}{32}\frac{m_e}{m_i}\frac{T_i^3}{T_e^3}\right]}\frac{\omega_{Bi}}{k\varrho_i\cos\theta\, F(\omega^{(l)})},$$

$$\gamma_m^{(l)} = \sqrt{\left(\frac{\pi}{32}\right)}\frac{\omega_{Bi}}{k\varrho_i\cos\theta\, F(\omega^{(l)})} A_m(k\varrho_i)\exp(-z_m^2), \tag{5.6.1.12}$$

$$F(\omega^{(l)}) = \sum_{m'=1}^{\infty}\frac{m^2\omega_{Bi}^4}{[\omega^{(l)2}-m'^2\omega_{Bi}^2]^2}A_{m'}(k\varrho_i), \qquad z_m = \frac{\omega^{(l)}(k)-m\omega_{Bi}}{\sqrt{(2)}kv_i\cos\theta}.$$

The quantities $\gamma_e^{(l)}$ and $\gamma_0^{(l)}$ are the damping rates caused by the Cherenkov absorption of the oscillations by electrons and ions, while the $\gamma_m^{(l)}$ are the ion-cyclotron damping rates. As $l\omega_{Bi} < \omega^{(l)}(k) < (l+1)\omega_{Bi}$ and $\omega/kv_i\cos\theta \gg 1$, we can drop in the sum (5.6.1.11) all exponentially small terms $\gamma_m^{(l)} \propto \exp(-z_m^2)$, except the terms with the smallest z_m^2, that is,

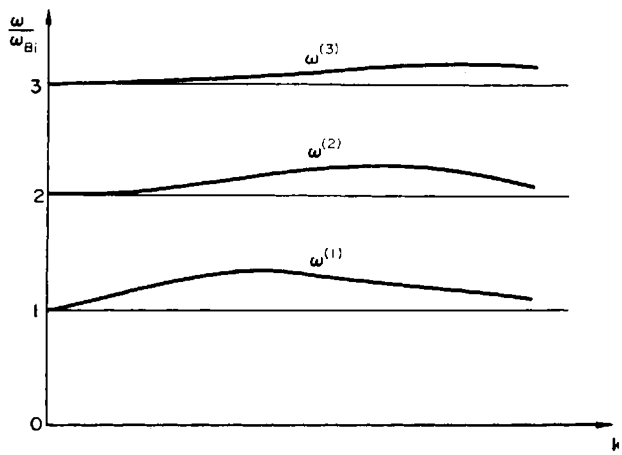

FIG. 5.6.2. Sketch of the wavenumber dependence of the frequencies of the longitudinal ion-cyclotron oscillations.

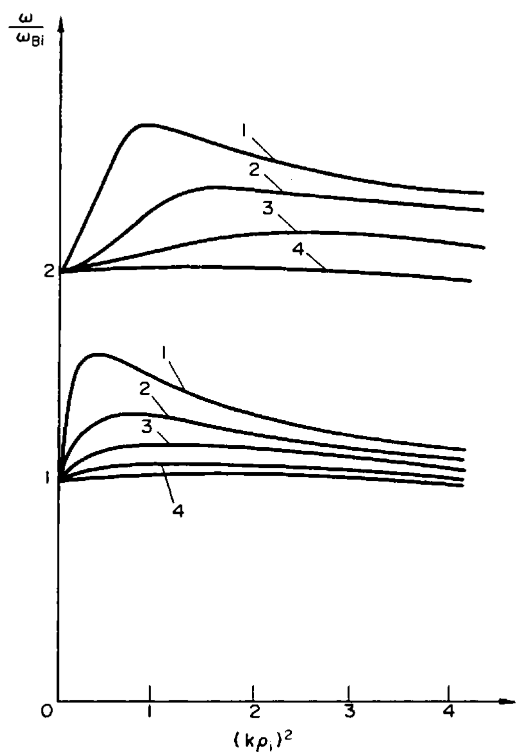

FIG. 5.6.3. The frequencies of the longitudinal ion-cyclotron oscillations as functions of $(k\varrho_i)^2$ for different values of the parameter T_i/T_e. The curves 1–4 correspond to T_i/T_e = 0.1, 0.3, 1, and 3, respectively

$\gamma_l^{(l)}$ and $\gamma_{l+1}^{(l)}$. If $\omega^{(l)}$ lies close to $l\omega_{Bi}$, we have $\gamma^{(l)} \approx \gamma_e^{(l)} + \gamma_l^{(l)}$, where

$$\gamma_e^{(l)} = \sqrt{\left(\frac{\pi m_e T_i^3}{2m_i T_e^3}\right)} \frac{l^2 \omega_{Bi} \psi_l^2}{k\varrho_i \cos\theta A_l(k\varrho_i)},$$

$$\gamma_l^{(l)} = \sqrt{\frac{\pi}{2}} \frac{l^2 \psi_l^2 \omega_{Bi}}{k\varrho_i \cos\theta} \exp(-z_l^2) \qquad (5.6.1.13)$$

$$z_l^2 = \frac{l^2 \psi_l^2}{2k^2 \varrho_i^2 \cos^2\theta} \gg 1,$$

while the quantity ψ_l is given by eqn. (5.6.1.9).

Weakly damped ion-cyclotron oscillations exist in a limited range of wavenumbers k and a narrow range of angles θ (close to $\pi/2$).

The wavenumber interval is limited by the condition $\gamma_l^{(l)} \ll l\omega_{Bi}$: when the wavenumber decreases or increases the frequency $\omega^{(l)}(k)$ gets so close to $l\omega_{Bi}$ that the quantity z_l becomes of the order of unity and the cyclotron damping rate $\gamma_l^{(l)}$ of the order of $k_z v_i$.

When the angle θ approaches $\pi/2$ the Cherenkov damping by electrons ($\gamma_e^{(l)}$) increases, and when the angle θ moves away from $\pi/2$ the cyclotron damping rate $\gamma_m^{(l)}$ increases; the range of angles θ for which the damping is weak is limited by the conditions

$$z_e \approx \frac{l\omega_{Bi}}{kv_e \cos\theta} \ll 1, \quad |z_m| = \frac{|\omega^{(l)} - m\omega_{Bi}|}{\sqrt{2}\cdot kv_i \cos\theta} \gg 1.$$

The frequencies $\omega^{(l)}(k)$ get much more closely to $l\omega_{Bi}$ when the number l of the branch increases—as the quantities $A_l(k\varrho_i)$ are appreciably less than unity when $l \gg 1$, we can for sufficiently large values of l use eqn. (5.6.1.8) for $\omega^{(l)}(k)$. The damping $\gamma_l^{(l)}$ therefore becomes comparable to the frequency for sufficiently large l, so that there are no weakly damped ion-cyclotron oscillations with large branch numbers.

We note in conclusion that we can use the approximate dispersion equation for longitudinal ion-cyclotron oscillations, $A = 0$, if the quantities $\delta\varepsilon_e$ and $\delta\varepsilon_i$ are considerably larger than $|B|/n^2$ and $|C|/n^4$. These conditions are satisfied in a low-pressure plasma, if

$$\xi_e = \frac{n_0 T_e}{B_0^2/8\pi} \ll \cos^2\theta \ll 1. \qquad (5.6.1.14)$$

5.6.2. NON-POTENTIAL ION-CYCLOTRON WAVES IN AN ISOTHERMAL LOW-PRESSURE PLASMA FOR THE CASE OF QUASI-TRANSVERSE PROPAGATION

If $\xi_e \gtrsim \cos^2\theta$, the ion-cyclotron oscillations considered in the previous subsection can no longer be purely longitudinal. We shall now consider ion-cyclotron oscillations in this case, assuming as before that the plasma pressure is low compared to the magnetic pressure, that is, $\xi_e \ll 1$. (Lominadze and Stepanov, 1964 b; Mikhailovskiĭ aud Pashitskiĭ, 1965). Moreover, we shall assume that $T_e \sim T_i$.

Assuming, as in the previous subsection, that the following inequalities hold:

$$\cos^2\theta \ll 1, \quad z_e = \frac{\omega}{\sqrt{2}\cdot k_z v_e} \ll 1, \quad |z_l| = \frac{|\omega - l\omega_{Bi}|}{\sqrt{2}\cdot k_z v_i} \gg 1,$$

we get for the components of the tensor ε_{ij} the expressions

$$\varepsilon_{11} = 1-\frac{\omega_{pi}^2}{\omega^2}\sum_{l=-\infty}^{+\infty}\frac{l^2A_l}{a_i}\left[\frac{\omega}{\omega-l\omega_{Bi}}-i\sqrt{\pi}\cdot z_0\exp(-z_l^2)\right],$$

$$\varepsilon_{22} = 1-\frac{\omega_{pi}^2}{\omega^2}\sum_{l=-\infty}^{+\infty}\left(\frac{l^2A_l}{a_i}-2a_iA_l'\right)\left(\frac{\omega}{\omega-l\omega_{Bi}}-i\sqrt{\pi}\cdot z_0\exp(-z_l^2)\right)+2\frac{\omega_{pi}^2}{\omega^2}a_i i\sqrt{\pi}\cdot z_e\frac{T_e}{T_i},$$

$$\varepsilon_{33} = 1-\frac{\omega_{pi}^2}{\omega^2}\sum_{l=-\infty}^{+\infty}A_l\left(\frac{\omega}{\omega-l\omega_{Bi}}-i\sqrt{\pi}\cdot z_0\exp(-z_l^2)\right)+\frac{\omega_{pe}^2}{k_z^2v_e^2}(1+i\sqrt{\pi}\cdot z_e),$$

$$\varepsilon_{12} = -i\frac{\omega_{pi}^2}{\omega^2}\sum_{l=-\infty}^{+\infty}A_l'\left(\frac{\omega}{\omega-l\omega_{Bi}}-i\,z_0\sqrt{\pi}\cdot\exp(-z_l^2)\right)-i\frac{\omega_{pi}^2}{\omega\omega_{Bi}}, \tag{5.6.2.1}$$

$$\varepsilon_{13} = -\frac{\omega_{pi}^2}{\omega^2}\sum_{l=-\infty}^{+\infty}lA_l\left[\frac{\omega\omega_{Bi}\cos\theta}{(\omega-l\omega_{Bi})^2}-i\sqrt{\frac{2\pi}{a_i}}\,z_0z_l\exp(-z_l^2)\right],$$

$$\varepsilon_{23} = i\frac{\omega_{pi}^2}{\omega^2}\sum_{l=-\infty}^{+\infty}\sqrt{(2a_i)}A_l'\left[\frac{z_0}{2z_l^2}-i\sqrt{\pi}\cdot z_0z_l\exp(-z_l^2)\right]-i\frac{\omega_{pi}^2}{\omega\omega_{Bi}}\tan\theta\,(1+i\sqrt{\pi}\cdot z_e),$$

$$A_l = \exp(-a_i)I_l(a_i), \qquad A_l' = \frac{dA_l}{da_i}.$$

Using relations (5.6.2.1) we write the dispersion equation $A+(B/n^2)+(C/n^4) = 0$ in the form

$$\delta\varepsilon_e+\delta\varepsilon_i-\frac{\varepsilon_{11}\varepsilon_{33}}{n^2} = 0, \tag{5.6.2.2}$$

where $\delta\varepsilon_e$ and $\delta\varepsilon_i$ are given by eqns. (5.6.1.3) and (5.6.1.4). This equation contains, in contrast to the dispersion equation $\delta\varepsilon_e+\delta\varepsilon_i = 0$ for longitudinal cyclotron oscillations, an extra term $-\varepsilon_{11}\,\varepsilon_{33}/n^2$ and describes non-potential cyclotron waves.

Neglecting in (5.6.2.2) the small anti-Hermitean terms, we can write this equation in the form

$$1+\frac{T_i}{T_e}\,\frac{1}{1-\zeta(\omega)} = f(\omega) \tag{5.6.2.3}$$

where
$$\zeta(\omega) = \left(\frac{T_i}{T_e}\right)^2\frac{\omega^2\xi_e}{2\omega_{Bi}^2a_i^2\cos^2\theta}, \tag{5.6.2.4}$$

while the function $f(\omega)$ is given by formula (5.6.1.6).

If $|\zeta| \ll 1$, which is equivalent to inequality (5.6.1.14) when $k\varrho_i \sim 1$ and $\omega \sim \omega_{Bi}$, the dispersion eqn. (5.6.2.3) goes over into the dispersion eqn. (5.6.1.5) for longitudinal ion-cyclotron oscillations.

We shall use graphical methods to study eqn. (5.6.2.3) in more detail. The functions $y_1 = f(\omega)$ and $y_2 = (T_i/T_e)(1-\zeta(\omega))^{-1}$ have been sketched in Figs. 5.6.4 and 5.6.5. The point $\omega = \omega^*$ for which $\zeta(\omega^*) = 1$ lies in the interval $l\omega_{Bi} < \omega^* < (l+1)\omega_{Bi}$ $(l = 1, 2, \ldots)$ in Fig. 5.6.4, while the point $\omega = \omega^*$ lies to the left of the point $\omega = \omega_{Bi}$ in Fig. 5.6.5. The solutions $\omega = \omega^{(l)}(k, \theta)$ $(l = 1, 2, \ldots)$ of eqn. (5.6.2.3) correspond to the intersections 1, 2, 3, ... of the curves $y = y_1$ and $y = y_2$.

It is clear from Fig. 5.6.4 that eqn. (5.6.2.3) for the frequencies of non-potential waves has, in contrast to the dispersion eqn. (5.6.1.5) for longitudinal waves, an additional root

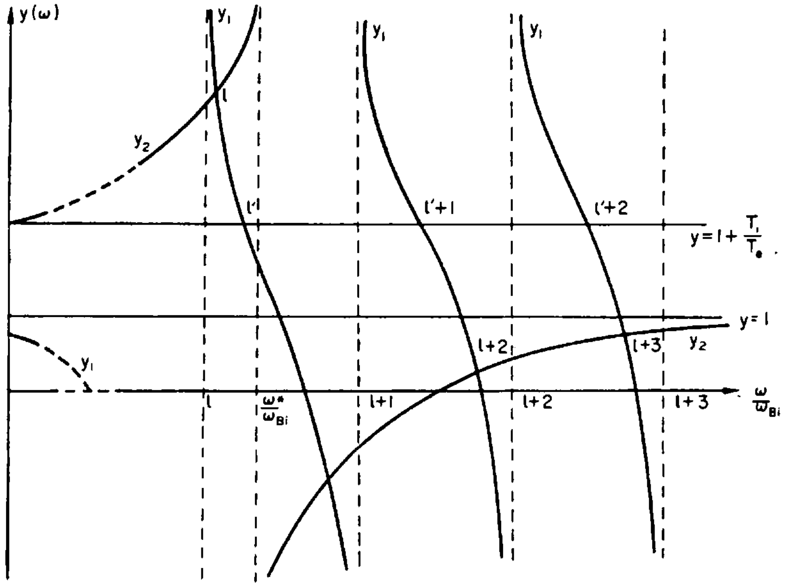

FIG. 5.6.4. Graphical solution of the equation $y_1(\omega) = y_2(\omega)$ when $l\omega_{Bi} < \omega^* < (l+1)\omega_{Bi}$.

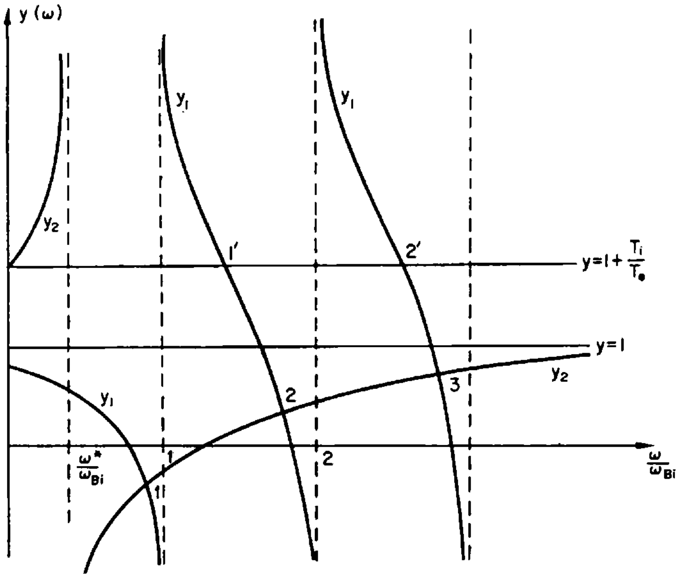

FIG. 5.6.5. Graphical solution of the equation $y_1(\omega) = y_2(\omega)$ when $\omega^* < \omega_{Bi}$.

$\omega = \omega^{(1)}(k, \theta)$ which is less than ω_{Bi}. In the interval $l\omega_{Bi} < \omega < (l+1)\omega_{Bi}$ eqn. (5.6.2.3) has two solutions when $\zeta \neq 0$, in contrast to the case of longitudinal oscillations ($\zeta = 0$) when there is only one solution.

The frequency $\omega^{(l)}(k, \theta)$ is equal to $nl\omega_{Bi}$ when $k = k_l$ ($\zeta = 1$), where

$$k_l(\theta) = \frac{\omega_{Bi}}{v_i} \sqrt[4]{\left(\frac{T_i^2 \xi_e}{2T_e^2 \cos^2\theta}\right)}.$$

We have sketched in Fig. 5.6.6 the k-dependence of the eigenfrequencies $\omega^{(l)}(k, \theta)$.

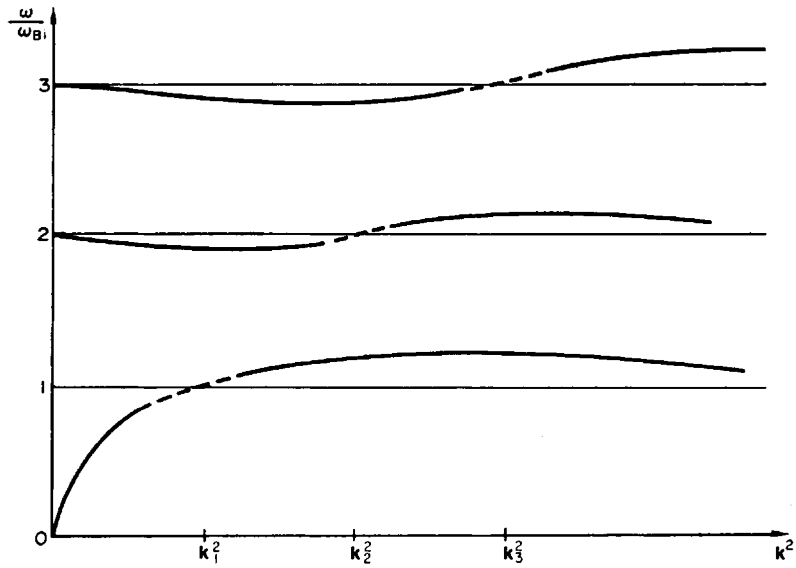

FIG. 5.6.6. Wavenumber dependence of the frequencies of the non-potential ion-cyclotron oscillations.

The longitudinal cyclotron oscillations can be split off when the parameter ξ_e tends to zero. In that case $k_l \to 0$ (see Fig. 5.6.6) and the general picture of the spectrum of the cyclotron oscillations is the same as that of the spectrum shown in Fig. 5.6.2.

The frequency $\omega^{(l)}(k, \theta)$ is close to $l\omega_{Bi}$ in three cases when $T_e \gtrsim T_i$: when $k\varrho_i \ll 1$, when $k\varrho_i \gg 1$, and when $k \approx k_l$; while for $T_i \gg T_e$ this is the case for any value of $k\varrho_i$. In those cases one finds easily that

$$\omega^{(l)}(k, \theta) = l\omega_{Bi}(1+\Delta_l), \tag{5.6.2.5}$$

where the quantity Δ_l is equal to

$$\Delta_l = \frac{A_l}{\frac{T_i}{T_e}\left(1-\frac{l^2\xi_e T_i^2}{2a_i^2\cos^2\theta\, T_e^2}\right)^{-1}+1+\sum_{m(\neq l)}\frac{l}{m-l}A_m}. \tag{5.6.2.6}$$

One can use eqn. (5.6.2.6) only when k lies not too close to k_l, that is, provided

$$\left|\frac{k-k_l}{k}\right| \gg \frac{kv_i\cos\theta}{l\omega_{Bi}A_l}\,\frac{T_i}{T_e}. \tag{5.6.2.7}$$

If this inequality is not satisfied, there occurs strong cyclotron damping.

In the long-wavelength region, $k\varrho_i \ll 1$, the quantity $\delta\varepsilon_e \approx \varepsilon_{33}\cos^2\theta$ in formula (5.6.2.2) is appreciably larger than $\delta\varepsilon_i$, if ω is not close to ω_{Bi}. Using this fact, we can write eqn. (5.6.2.2) in the form

$$n^2 \approx \frac{\varepsilon_{11}}{\cos^2\theta} \approx \frac{\omega_{pi}^2}{\cos^2\theta(\omega_{Bi}^2-\omega^2)}. \tag{5.6.2.8}$$

This equation is the same as the dispersion equation for the Alfvén branch (the A branch) of oscillations in a cold plasma when $\theta \approx \pi/2$. The frequency of these oscillations is equal to

$$\omega(k, \theta) = \frac{k_z v_A}{\sqrt{[1+(k_z c/\omega_{pi})^2]}} \cdot \tag{5.6.2.9}$$

When $k_z c \ll \omega_{pi}$ this frequency is the same as the frequency of the magneto-hydrodynamic Alfvén wave, $\omega = k_z v_A$. The additional branch of oscillations which for $k < k_1$ has a frequency less than ω_{Bi} is thus an Alfvén branch. When $k > k_1$ it goes over into cyclotron oscillations. We note that when $k \approx k_1$ these oscillations are strongly damped.

We shall now find the damping rate of the oscillations considered, assuming that the frequencies $\omega^{(l)}(k, \theta)$ are known. Taking the small anti-Hermitean terms in (5.6.2.2) into account, we find

$$\gamma = \gamma_e^{(l)} + \sum_{m=-\infty}^{+\infty} \gamma_m^{(l)},$$

where

$$\gamma_e^{(l)} = \frac{\sqrt{\pi}}{\alpha}\frac{T_i}{T_e} z_e\left[1+\frac{\xi_e(f-1)}{2a_i \cos^2\theta}\frac{\omega^2}{\omega_{Bi}^2}\right]\omega, \tag{5.6.2.10}$$

$$\gamma_m^{(l)} = \frac{\sqrt{\pi}}{\alpha} A_m z_0 \exp(-z_m^2)\left[1+\frac{\xi_e(f-1)}{4a_i\cos^2\theta}\left(\frac{\omega}{\omega_{Bi}}-l\right)^2 - \frac{\xi_e l^2}{2a_i\cos^2\theta}\frac{T_i}{T_e}\right]\omega, \tag{5.6.2.11}$$

with

$$\alpha = (\zeta-1)\omega\frac{\partial f}{\partial\omega}+2\zeta(f-1).$$

In the expressions $\alpha = \alpha(\omega)$ and $f = f(\omega)$ we must here put $\omega = \omega^{(l)}(k, \theta)$.

When $|\zeta| \ll 1$, these expressions are the same as the formulae for $\gamma_e^{(l)}$ and $\gamma_m^{(l)}$ obtained in the preceding subsection for longitudinal cyclotron waves. As the quantities z_m^2 are large, we must clearly take into account the largest of the exponentially small quantities $\gamma_m^{(l)} \propto \exp(-z_m^2)$, that is, $\gamma_l^{(l)}$ and $\gamma_{l\pm1}^{(l)}$.

5.7. Cyclotron Waves in the Case of Transverse Propagation

5.7.1. ORDINARY CYCLOTRON WAVES

We now turn to a study of cyclotron waves in the case of transverse propagation.† When $\theta = \pi/2$ the components of the dielectric permittivity tensor ε_{13} and vanish and the

† Gross (1951) observed longitudinal electron cyclotron waves. Silin (1959a) proved the existence of cyclotron waves in Fermi gases and Fermi liquids. Subsequently these oscillations have been studied by many authors (see, for instance, Gershman, 1953a; Sitenko and Stepanov, 1957; Dnestrovskiĭ and Kostomarov, 1959, 1961, 1962; Kitsenko and Stepanov, 1964; Drummond, 1958; Stepanov, 1959c, 1962b; Bernstein, 1958).

dispersion equation splits into two equations. One of them,

$$n^2 - \varepsilon_{33} = 0, \tag{5.7.1.1}$$

determines in the high-frequency region ($\omega > \omega_{pe}$) the refractive index of the linearly polarized ordinary wave (see Subsection 5.1.7). Furthermore, this equation has the solution $\omega = \omega(k)$ which tends to $l|\omega_{Be}|$ ($l = 1, 2, \ldots$) as $k \to 0$ or $k \to \infty$. These branches of the oscillations are called *ordinary* electron and ion cyclotron waves.

We begin with a study of the ordinary electron-cyclotron waves. We can then neglect the ion contribution to ε_{33} and start from the following dispersion equation:

$$n^2 = \varepsilon_{33} = 1 - \frac{\omega_{pe}^2}{\omega} \sum_{l=-\infty}^{+\infty} \frac{A_l(a_e)}{\omega - l|\omega_{Be}|}. \tag{5.7.1.2}$$

Before embarking upon an analysis of the dispersion eqn. (5.7.1.2) we note that, as we have neglected relativistic effects of order v_e^2/c^2, and, in particular, the transverse (relativistic) Doppler effect, we must in expression (5.7.1.2) for ε_{33} and also in the other components of the tensor ε_{ij} assume the difference $\omega - l|\omega_{Be}|$ to be sufficiently large so that the inequality

$$\left| \frac{\omega - l|\omega_{Be}|}{\omega} \right| \gg \frac{v_e^2}{c^2} \tag{5.7.1.3}$$

is satisfied; only when this condition is fulfilled can we neglect relativistic effects.

If the wavelength is of the order of the electron Larmor radius ($k\varrho_e \sim 1$), the square of the refractive index of the ordinary cyclotron wave will for $\omega \sim |\omega_{Be}|$ be of the order of $n^2 \sim c^2/v_e^2 \gg 1$. We can thus in (5.7.1.2) in the expression for ε_{33} neglect unity in comparison with n^2 and the terms $\propto \omega_{pe}^2$. It is clear that the terms in $\varepsilon_{33} \sim \omega_{pe}^2 Al/\omega(\omega - l|\omega_{Be}|)$ can always compensate, for sufficiently small $|\omega - l|\omega_{Be}||$, the quantity $n^2 \sim c^2/v_e^2$. However, if $\omega_{pe} \lesssim |\omega_{Be}|$, this compensation is possible only if $|(\omega - l|\omega_{Be}|)/\omega| \lesssim v_e^2/c^2$, since $A_l < 1$, and this violates condition (5.7.1.3). We must thus restrict our study to the case where $\omega_{pe} \gg |\omega_{Be}|$.

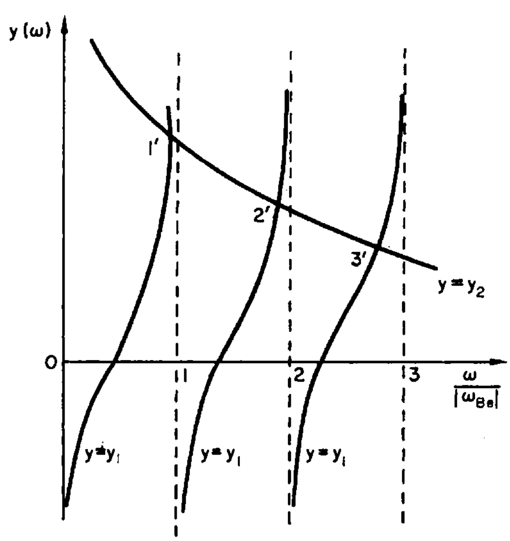

FIG. 5.7.1. Graphical solution of the equation $y_1(\omega) = y_2(\omega)$.

We shall study eqn. (5.7.1.1) graphically. The solutions of eqn.(5.7.1.1), $\omega = \omega^{(l)}(k)$ ($l = 1, 2, 3, \ldots$) correspond to the points of intersection of the curves $y_1(\omega) = \varepsilon_{33}(k, \omega)$ and $y_2(\omega) = k^2c^2/\omega^2$ (see Fig. 5.7.1). It is clear from Fig. 5.7.1 that the frequency $\omega^{(l)}(k)$ lies in the interval

$$(l-1)\,|\omega_{Be}| < \omega^{(l)}(k) < l\,|\omega_{Be}|, \qquad l = 1, 2, \ldots .$$

One can obtain analytical expressions for the eigenfrequencies $\omega^{(l)}(k)$ in the case of a low-pressure plasma, $\xi_e = n_0T_e/(B_0^2/8\pi) \ll 1$. Taking only the resonance term, proportional to $A_l/(\omega - l\,|\omega_{Be}|)$, into account in the expression for ε_{33} when $a_e = k^2\varrho_e^2 \gg \xi_e$, we find that

$$\omega^{(l)}(k) = l\,|\omega_{Be}|\left[1 - \frac{\xi_e A_l(a_e)}{2a_e}\right], \qquad l = 1, 2, \ldots . \tag{5.7.1.4}$$

In the short-wavelength region, $k\varrho_e \gg 1$, this formula becomes

$$\omega^{(l)}(k) = l\,|\omega_{Be}|\left(1 - \frac{1}{\sqrt{(8\pi)}}\frac{\xi_e}{a_e^{3/2}}\right). \tag{5.7.1.5}$$

If $a_e \lesssim \xi_e$ we must take in ε_{33} apart from the resonance term also the term with $l = 0$ into account. As a result we get

$$\omega^{(l)}(k) = l\,|\omega_{Be}|\left[1 - \frac{\xi_e A_l(a_e)}{2a_e + \xi_e}\right], \qquad l = 1, 2, \ldots . \tag{5.7.1.6}$$

This formula determines the frequency $\omega^{(l)}(k)$ when $\xi_e \ll 1$ for arbitrary values of $k\varrho_e$. We have sketched in Fig. 5.7.2 the wavenumber dependence of the frequencies $\omega^{(l)}(k)$ given by eqns. (5.7.1.6).

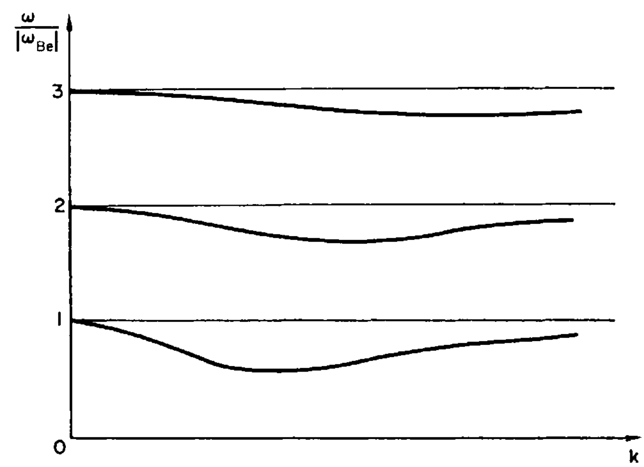

FIG. 5.7.2. Wavenumber dependence of the frequencies of the ordinary electron-cyclotron waves.

The frequencies $\omega^{(l)}(k)$ decrease in the region $k < k^{(l)}_{\min}$, where $k^{(l)}_{\min}\varrho_e \sim 1$, reach a minimum for $k = k^{(l)}_{\min}$, where $\Delta\omega = l\,|\omega_{Be}| - \omega^{(l)} \sim \xi_e\omega^{(l)}$, and then increase, approaching $l\,|\omega_{Be}|$.

For a plasma with a finite pressure $\xi_e \gtrsim 1$, the behaviour of the frequencies $\omega^{(l)}(k)$ is qualitatively the same as when $\xi_e \ll 1$. However, whereas the frequencies $\omega^{(l)}$ for the case when $\xi_e \ll 1$ are close to $l\,|\omega_{Be}|$ for any values of $k\varrho_e$, when $\xi_e \gtrsim 1$ and $k\varrho_e \sim 1$ these frequencies are quite different from $l\,|\omega_{Be}| : \Delta\omega \sim |\omega_{Be}|$.

When $k\varrho_e \ll 1$, eqn. (5.7.1.6) for $\omega^{(l)}$ is also valid for a plasma with a finite pressure; in that case

$$\omega^{(l)}(k) = l\,|\omega_{Be}|\,[1 - A_l(a_e)], \quad \xi_e \gtrsim 1, \quad k\varrho_e \ll 1, \tag{5.7.1.7}$$

where $A_l \approx a_e^l/2^l l!$.

Let us now study the ordinary ion-cyclotron waves. Assuming that $a_e \ll 1$, $\omega \sim \omega_{Bi} \ll |\omega_{Be}|$ we retain in ε_{33} only the electron term with $l = 0$,

$$\varepsilon_{33} = -\frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega} \sum_{l=-\infty}^{+\infty} \frac{A_l(a_i)}{\omega - l\omega_{Bi}}. \tag{5.7.1.8}$$

The solution of the dispersion eqn. (5.7.1.1) is found in this case in the same way as in the case of electron cyclotron waves. A sketch of the k-dependence of the eigenfrequencies of the ion-cyclotron waves is shown in Fig. 5.7.2, in which the index e must be replaced by i.

We note that thanks to the presence in ε_{33} of the large electron term $-\omega_{pe}^2/\omega^2$ the dispersion equation can be satisfied only when $\omega \approx l\omega_{Bi}$. Retaining in ε_{33} apart from the electron term the resonance ion term which is proportional to $(\omega - l\omega_{Bi})^{-1}$, we find that

$$\omega^{(l)}(k) = l\omega_{Bi}\left[1 - \frac{\xi_i A_l(a_i)}{2a_i + \xi_i(m_i/m_e)}\right], \qquad l = 1, 2, \ldots . \tag{5.7.1.9}$$

It follows from (5.7.1.9) that the frequencies $\omega^{(l)}(k)$ are extremely close to $l\omega_{Bi}$ both when $\xi_i \lesssim 1$ and when $\xi_i > 1$. If $\xi_i \gtrsim (m_e/m_i)a_i$, we have $\Delta\omega/\omega_{Bi} \sim (m_e/m_i)\,A_l(a_i)$.

5.7.2. LONGITUDINAL ELECTRON-CYCLOTRON OSCILLATIONS

When $\theta = \pi/2$, the dispersion equation for the limiting oscillations, $A = \varepsilon_{11} = 0$ has, in the high-frequency region, $\omega \gg \omega_{Bi}$, when we can neglect the motion of the ions, the form

$$\varepsilon_{11} = 1 - \frac{\omega_{pe}^2}{\omega} \sum_{l=-\infty}^{+\infty} \frac{l^2 A_l(a_e)}{a_e} \frac{1}{\omega - l\,|\omega_{Be}|} = 0. \tag{5.7.2.1}$$

It is easy to solve this equation graphically (see Fig. 5.7.3). The points of intersection 1′, 2′, ... of the curve $y = y_1(\omega) = (\omega_{Be}^2/\omega)\Sigma l^2 A_l/[a_e(\omega - l\,|\omega_{Be}|)]$ with the straight line $y = y_2(\omega) = \omega_{Be}^2/\omega_{pe}^2$ correspond to the solutions $\omega = \omega^{(l)}(k)$ of eqn. (5.7.2.1) lying in the interval $l\,|\omega_{Be}| < \omega^{(l)}(k) < (l+1)\,|\omega_{Be}|$.

In the long-wavelength region, $k\varrho_e \ll 1$, we find, by expanding the quantities $A(a_e)$ in power series in $a_e = k^2\varrho_e^2$, that when ω is not too close to $l\,|\omega_{Be}|$ the frequency of the

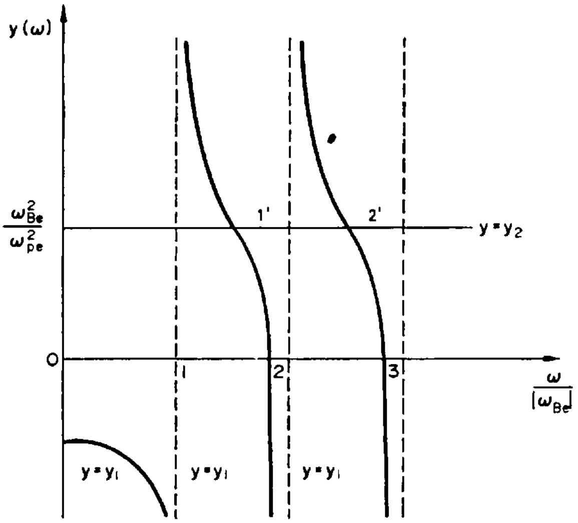

FIG. 5.7.3. Graphical solution of the dispersion eqn. (5.7.2.1).

longitudinal oscillations is close to the hybrid frequency $\omega_\infty^{(1)}$:

$$\omega(k) = \omega_\infty^{(1)}(1+\xi), \tag{5.7.2.2}$$

where

$$\omega_\infty^{(1)} = \sqrt{(\omega_{pe}^2+\omega_{Be}^2)}, \quad \xi(k) = \left.\frac{3k^2\varrho_e^2\omega_{pe}^2\omega_{Be}^2}{2\omega^2(\omega^2-4\omega_{Be}^2)}\right|_{\omega=\omega_\infty^{(1)}}.$$

On the other hand, the quantity A_n is small when $k\varrho_e \ll 1$ and one can solve the dispersion equation assuming that the frequency $\omega(k)$ lies close to $l\,|\omega_{Be}|$:

$$\omega(k) = l\,|\omega_{Be}|\,(1+\xi'), \qquad l = 2, 3, \ldots, \tag{5.7.2.3}$$

where

$$\xi'(k) = \frac{(l^2-1)\,\omega_{pe}^2 A_l(a_e)}{a_e[(l^2-1)\omega_{Be}^2-\omega_{pe}^2]}.$$

In the short-wavelength region, $k\varrho_e \gg 1$, the quantity A_l/a_e is also small, and in this region there is a solution of eqn. (5.7.2.1) close to $l\,|\omega_{Be}|$:

$$\omega(k) = l\,|\omega_{Be}|\left(1+\frac{1}{\sqrt{(2\pi a_e^3)}}\,\frac{\omega_{pe}^2}{\omega_{Be}^2}\right). \tag{5.7.2.4}$$

We have sketched the wavenumber dependence of the frequencies $\omega^{(l)}(k)$ in Fig. 5.7.4 for the case where $\omega_{pe}^2 < 3\omega_{Be}^2$ and in Fig. 5.7.5 for the case when $m^2 < 1+(\omega_{pe}/\omega_{Be})^2 < (m+1)^2$, $m \geqslant 2$.

It is clear from Fig. 5.7.5 that the frequencies $\omega^{(l)}(k)$ monotonically decrease with increasing k when $l < m$, tending to $l\,|\omega_{Be}|$ as $k\varrho_e \to \infty$, the frequency of this wave is determined by formula (5.7.2.4) when $k\varrho_e \gg 1$. The frequency $\omega^{(l)}(k)$ $(m\,|\omega_{Be}| < \omega_\infty^{(1)} < (m+1)\,|\omega_{Be}|)$ is for small values of $k\varrho_e$ given by eqn. (5.7.2.3), increases monotonically with increasing k when

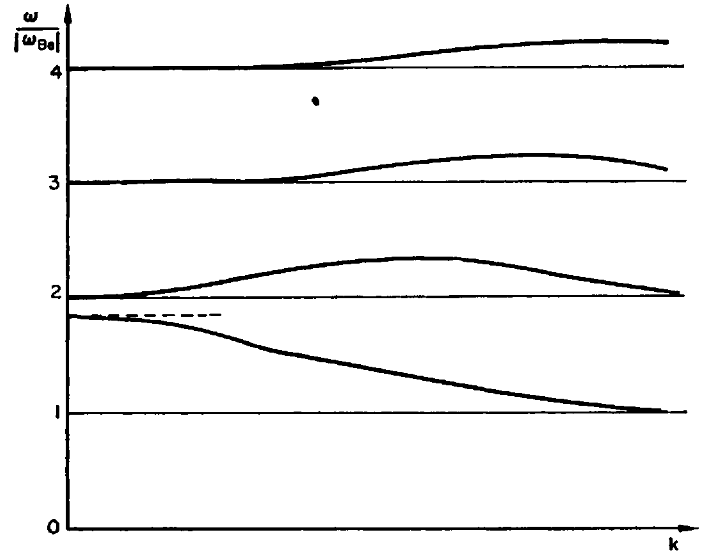

FIG. 5.7.4. Wavenumber dependence of the frequencies of the longitudinal electron-cyclotron oscillations for the case where $\omega^2_{pe} < 3\,\omega^2_{Be}$.

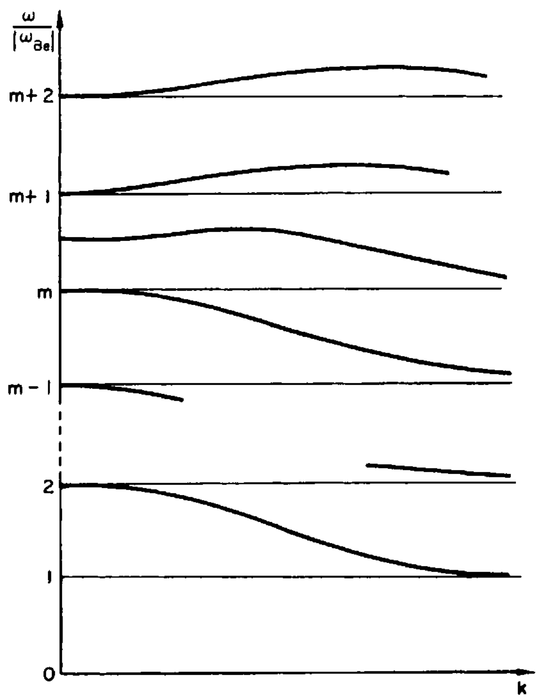

FIG. 5.7.5. Wavenumber dependence of the frequencies of longitudinal electroncyclotron oscillations for the case when $m^2 < 1+(\omega^2_{pe}/\omega^2_{Be}) < (m+1)^2$, $m \geqq 2$.

$m \geqslant 2$, reaches a maximum when $k\varrho_e \sim 1$, and then decreases, tending to $m\,|\omega_{Be}|$ as $k\varrho_e \to \infty$. If $m = 1$, this frequency decreases monotonically with increasing k from a value $\omega_\infty^{(1)}$ at $k = 0$ to $|\omega_{Be}|$ as $k \to \infty$. A frequency $\omega^{(l)}(k)$ which is larger than $m\,|\omega_{Be}|$, that is, $l > m$, increases with increasing k from a value $l\,|\omega_{Be}|$ at $k = 0$, reaches a maximum when $k\varrho_e \sim 1$, and then decreases, approaching $l\,|\omega_{Be}|$ as $k \to \infty$.

When $k\varrho_e \ll 1$ the eigenfrequencies are given by eqns. (5.7.2.2) and (5.7.2.3), while for $k\varrho_e \gg 1$ they are given by eqn. (5.7.2.4). When $k\varrho_e \sim 1$ it is necessary to solve eqn. (5.7.2.1) numerically.

We note that formulae (5.7.2.2) and (5.7.2.3) are inapplicable when the hybrid frequency $\omega_\infty^{(1)}$ lies close to $l\,|\omega_{Be}|$, where $l = m$ or $l = m+1$. In this case we get by retaining in the sum in eqn. (5.7.2.1) both the terms with $l = \pm 1$ and the resonance term with $l = m$ (or $l = m+1$):

$$\omega(k) = \omega_\infty^{(1)}(1+\xi_\pm), \tag{5.7.2.5}$$

where

$$\xi_\pm = \frac{1}{2}\left(\frac{l\,|\omega_{Be}|}{\omega_\infty^{(1)}} - 1\right) \pm \frac{1}{2}\sqrt{\left[\left(\frac{l\,|\omega_{Be}|}{\omega_\infty^{(1)}} - 1\right)^2 + \frac{2(l^2-1)^2}{l^2 a_e} A_l\right]}. \tag{5.7.2.6}$$

If the difference $\omega_\infty^{(1)} - l\,|\omega_{Be}|$ is sufficiently large so that $[(l\,|\omega_{Be}|/\omega_\infty^{(1)}) - 1]^2 \gg A_l/a_e$, eqn. (5.7.2.5) goes over into formula (5.7.2.2) or (5.7.2.3).

It follows from these formulae that the eigenfrequencies are always different from $l\,|\omega_{Be}|$. In particular, when $\sqrt{(\omega_{pe}^2+\omega_{Be}^2)} = l\,|\omega_{Be}|$ the eigenfrequencies are separated by a finite interval (the Gross "gap") from one another. For instance, when $\omega_\infty^{(1)} = 2\,|\omega_{Be}|$ we find from (5.7.2.5) and (5.7.2.6) that

$$\omega(k) = 2\,|\omega_{Be}|\,(1 \pm \tfrac{3}{8}k\varrho_e),$$

that is, there is a finite distance between the two neighbouring branches, $\Delta\omega = \frac{3}{2}k\varrho_e\,|\omega_{Be}| = \frac{3}{2}kv_e$.

5.7.3. LONGITUDINAL ION-CYCLOTRON OSCILLATIONS

In the low-frequency region, $\omega \ll |\omega_{Be}|$, the dispersion equation, $\varepsilon_{11} = 0$, for longitudinal oscillations has for $k\varrho_e \ll 1$ the form

$$\frac{\omega_{Bi}^2}{\omega_{pi}^2}\left(1+\frac{\omega_{pe}^2}{\omega_{Be}^2}\right) = \frac{\omega_{Bi}^2}{\omega}\sum_{l=-\infty}^{+\infty}\frac{l^2 A_l(a_i)}{a_i}\,\frac{1}{\omega - l\omega_{Bi}}. \tag{5.7.3.1}$$

As in the case of the high-frequency oscillations, one can easily verify that eqn. (5.7.3.1) has a solution $\omega = \omega^{(l)}(k)$ which lies in the interval

$$l\omega_{Bi} < \omega^{(l)}(k) < (l+1)\omega_{Bi}, \qquad l = 1, 2, \ldots .$$

In the long-wavelength region, $k\varrho_i \ll 1$, one of the solutions of eqn. (5.7.3.1) lies close to the hybrid frequency $\omega_\infty^{(2)}$:

$$\omega(k) = \omega_\infty^{(2)}(1+\xi), \tag{5.7.3.2}$$

where

$$\omega_\infty^{(2)} = \sqrt{\frac{(\omega_{pi}^2+\omega_{Bi}^2)\omega_{Be}^2}{\omega_{pe}^2+\omega_{Be}^2}},$$

$$\xi(k) = \frac{3k^2\varrho_i^2\omega_{pi}^2\omega_{Bi}^2\omega_{Be}^2}{2(\omega_\infty^{(2)})^2(\omega_{pe}^2+\omega_{Be}^2)}\left[\frac{1}{(\omega_\infty^{(2)})^2-4\omega_{Bi}^2}+\frac{1}{4}\frac{T_e}{T_i}\frac{(\omega_\infty^{(2)})^2-\omega_{Bi}^2}{\omega_{Be}^2\omega_{Bi}^2}\right].$$

Other solutions of eqn. (5.7.3.1) lie for $k\varrho_i \ll 1$ close to $l\omega_{Bi}$:

$$\omega(k) = l\omega_{Bi}(1+\xi'), \qquad l = 2, 3, \ldots, \tag{5.7.3.3}$$

where

$$\xi'(k) = \frac{(l^2-1)A_l(a_i)\omega_{pi}^2}{a_i[(l^2-1)\omega_{Bi}^2(1+\{\omega_{pe}^2/\omega_{Be}^2\})-\omega_{pi}^2]}.$$

In the short-wavelength region, $k\varrho_i \gg 1$, the solution of eqn. (5.7.3.1) is also close to $l\omega_{Bi}$:

$$\omega^{(l)}(k) = l\omega_{Bi}\left[1+\frac{\omega_{pi}^2\omega_{Be}^2}{\omega_{Bi}^2(\omega_{pe}^2+\omega_{Be}^2)\sqrt{(2\pi a_i^3)}}\right]. \tag{5.7.3.4}$$

To find the eigenfrequencies when $k\varrho_i \sim 1$ one must solve eqn. (5.7.3.1) numerically.

The qualitative behaviour of the frequencies $\omega^{(l)}(k)$ of the longitudinal ion-cyclotron oscillations for transverse propagation as functions of the wavenumber k can be seen from Figs. 5.7.4 and 5.7.5 in which we need only change the index e to i.

If the hybrid frequency $\omega_\infty^{(2)}$ lies close to $l\omega_{Bi}$ ($l = m$ or $l = m+1$), eqn. (5.7.3.2) for the frequency $\omega^{(m)} \approx \omega_\infty^{(2)}$ and eqn. (5.7.3.3) for the neighbouring frequency, close to it, may no longer be applicable. Instead of them we must in this case use the equations

$$\omega(k) = \omega_\infty^{(2)}(1+\xi_\pm), \tag{5.7.3.5}$$

where

$$\xi_\pm(k) = -\frac{1}{2}\left(1-\frac{\omega_\infty^{(2)}}{l\omega_{Bi}}\right)\pm\frac{1}{2}\sqrt{\left[\left(1-\frac{\omega_\infty^{(2)}}{l\omega_{Bi}}\right)^2+2\left(1-\frac{1}{l^2}\right)^2\frac{l^2A_l(a_i)}{a_i}\right]}.$$

In the region $[1-(\omega_\infty^{(2)}/l\omega_{Bi})]^2 \gg l^2A_l/a_i$ eqn. (5.7.3.5) approaches eqn. (5.7.3.2) and (5.7.3.3).

The minimum distance between neighbouring frequencies is, when $\omega_\infty^{(2)} = l\omega_{Bi}$, equal to

$$\Delta\omega = 2\xi_+\omega_\infty^{(2)} = \sqrt{\frac{(l^2-1)^2}{2^l\, l!}}\,(k\varrho_i)^{l-1}\,\omega_{Bi}.$$

5.7.4. EXTRA-ORDINARY ELECTRON-CYCLOTRON WAVES

When $\theta = \pi/2$, the second of the dispersion equations into which the complete dispersion eqn. (5.2.2.5) split up has the form

$$\varepsilon_{11}n^2-\varepsilon_{11}\varepsilon_{22}-\varepsilon_{12}^2 = 0. \tag{5.7.4.1}$$

In the case of high-frequency (electron) oscillations the contribution from the ions to

ε_{ij} is negligibly small, and the quantities ε_{ij} have the form

$$\begin{aligned}\varepsilon_{11} &= 1-\frac{\omega_{pe}^2}{\omega}\sum_{l=-\infty}^{+\infty}\frac{l^2A_l(a_e)}{a_e}\,\frac{1}{\omega-l\,|\omega_{Be}|},\\ \varepsilon_{22} &= 1-\frac{\omega_{pe}^2}{\omega}\sum_{l=-\infty}^{+\infty}\left[\frac{l^2A_l(a_e)}{a_e}-2a_eA_l'(a_e)\right]\frac{1}{\omega-l\,|\omega_{Be}|},\\ \varepsilon_{12} &= i\frac{\omega_{pe}^2}{\omega}\sum_{l=-\infty}^{+\infty}\frac{lA_l'(a_e)}{\omega-l\,|\omega_{Be}|}.\end{aligned} \tag{5.7.4.2}$$

When $T_e = 0$, eqn. (5.7.4.1) determines in the high-frequency region the refractive index of the high-frequency extra-ordinary waves. We shall therefore call the cyclotron waves whose frequencies are given by eqn. (5.7.4.1) *extra-ordinary* cyclotron waves.

In the preceding subsections we studied approximate solutions of eqn. (5.7.4.1), corresponding to the equation $\varepsilon_{11} = 0$ for longitudinal cyclotron oscillations. If $\omega \sim \omega_{pe} \sim |\omega_{Be}|$ such oscillations can be distinguished, if $n^2 \gg 1$, as we considered waves with $k\varrho_e \sim 1$ and $n \sim c/v_e$ so that corrections to the equation $\varepsilon_{11} = 0$ will be of order v_e^2/c^2 and they can be neglected. However, we shall show in a moment that, apart from the longitudinal cyclotron waves, eqn (5.7.4.1) has additional roots close to $l\,|\omega_{Be}|$.

We now turn to a study of the dispersion eqn. (5.7.4.1) assuming, as in Subsection 5.7.1, that $\omega_{pe} \gg |\omega_{Be}|$ and that the pressure in the plasma is low, $\xi_e \ll 1$. If $a_e \gg \xi_e$, we can retain in eqns. (5.7.4.2) for the ε_{ij} merely the resonance terms; we then get

$$\omega(k) = l\,|\omega_{Be}|\,[1-\xi_e\varphi(a_e)], \qquad l = 1, 2, \ldots, \tag{5.7.4.3}$$

where

$$\varphi(a_e) = \frac{1}{2}\,A_l(a_e)\left[\frac{l^2}{a_e}+1-\left(\frac{I_l'(a_e)}{I_l(a_e)}\right)^2\right]. \tag{5.7.4.4}$$

The function $\varphi(a_e)$ increases monotonically in the range of small a_e values, reaches a maximum when $a_e \sim 1$ and then tends to zero.

In the region $a_e \lesssim \xi_e$ we must when $\omega = l\,|\omega_{Be}|$ $(l = 2, 3, \ldots)$ take into account in the tensor ε_{ij} not only the resonance terms which are proportional to $(\omega - l\,|\omega_{Be}|)^{-1}$, but also the terms proportional to $(\omega \mp |\omega_{Be}|)^{-1}$. In that case we find instead of (5.7.4.3)

$$\omega(k) = l\,|\omega_{Be}|\left[1-\frac{\xi_ea_e\varphi(a_e)}{a_i+2l(l+1)^{-1}\xi_e}\right], \qquad l = 2, 3, \ldots . \tag{5.7.4.5}$$

When $\xi_e \ll a_e \ll 1$ eqns. (5.7.4.3) and (5.7.4.5) "merge" into another.

Apart from the electron-cyclotron wave branches whose frequencies are given by eqns. (5.7.4.3) and (5.7.4.5) there are also branches of oscillations with frequencies given, when $k\varrho_e \ll 1$, by the expression

$$\omega(k) = l\,|\omega_{Be}|\left[1-2I_l(a_e)\,\frac{(l^2-1)a_e+l(l-1)\xi_e}{a_e(\xi_e+2a_e)}\right]. \tag{5.7.4.6}$$

The frequencies of these branches lie for given k much further from $l\,|\omega_{Be}|$ than the frequencies (5.7.4.3) or (5.7.4.5).

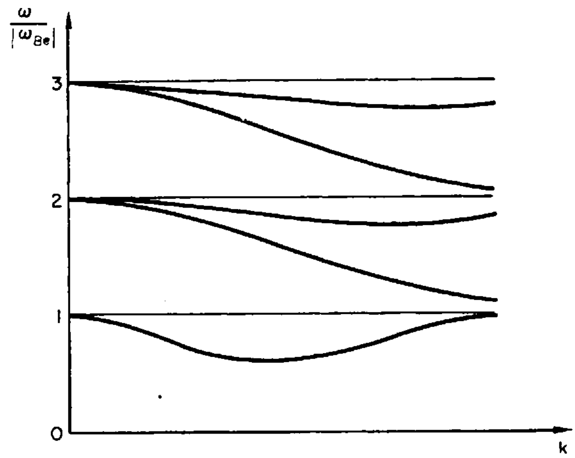

FIG. 5.7.6. Wavenumber-dependence of the frequencies of the extra-ordinary electron-cyclotron waves for the case when $\omega_{pe} > \omega_{Be}$.

In the region $\xi_e \ll a_e \ll 1$ eqn. (5.7.4.6) goes over into formula (5.7.2.3) for the frequency of longitudinal cyclotron oscillations. The waves with the frequencies (5.7.4.6) are thus the long-wavelength part of the longitudinal cyclotron oscillations. In the region $k\varrho_e \gtrsim 1$ the frequencies of these waves follow from the equation $\varepsilon_{11} = 0$ and when $k\varrho_e \gg 1$ these frequencies tend to $(l-1)|\omega_{Be}|$. Figure 5.7.6 shows qualitatively the wavenumber dependence of the extra-ordinary cyclotron waves.

5.7.5. EXTRA-ORDINARY ION-CYCLOTRON WAVES

In the low-frequency region, $\omega \sim \omega_{Bi}(\omega \ll \sqrt{(\omega_{Bi}|\omega_{Bi}|)})$ the components of the tensor ε_{ij} are for $k\varrho_e \ll 1$ and a dense plasma $(\omega_{pi} \gg \omega_{Be})$ given by the equations

$$\begin{aligned}
\varepsilon_{11} &= -\frac{\omega_{pi}^2}{\omega}\sum_{l=-\infty}^{+\infty}\frac{l^2A_l(a_i)}{a_i}\,\frac{1}{\omega-l\omega_{Bi}},\\
\varepsilon_{22} &= -\frac{\omega_{pi}^2}{\omega}\sum_{l=-\infty}^{+\infty}\left[\frac{l^2}{a_i}A_l(a_i)-a_iA_l'(a_i)\right]\frac{1}{\omega-l\omega_{Bi}},\\
\varepsilon_{12} &= -i\frac{\omega_{pi}^2}{\omega}\sum_{l=-\infty}^{+\infty}\frac{lA_l'(a_i)}{\omega-l\omega_{Bi}}-i\frac{\omega_{pi}^2}{\omega\omega_{Bi}}.
\end{aligned}\tag{5.7.5.1}$$

We shall now study the dispersion eqn. (5.7.4.1) in the low-frequency region using these expressions for the components of the dielectric permittivity tensor.

First of all we recall that if $k\varrho_i \ll 1$ when we can neglect the thermal motion of the ions eqn. (5.7.4.1) has a solution corresponding to a fast magneto-sound wave

$$\omega(k) = kv_A. \tag{5.7.5.2}$$

Apart from this eqn. (5.7.4.1) has additional roots corresponding to cyclotron waves. We obtain the frequencies of these waves, retaining in eqns. (5.7.5.1) only the resonance terms

$$\omega(k) = l\omega_{Bi}\left[1-\frac{\xi_i a_i \varphi(a_i)}{a_i-a_0}\right], \qquad l = 1, 2, \ldots, \tag{5.7.5.3}$$

and

$$\omega(k) = l\omega_{Bi}\left[1-\frac{2(l^2-1)\, I_l(a_i)\,(a_i-a_0)}{a_i(2a_i-l^2\xi_i)}\right], \qquad l = 2, 3, \ldots, \tag{5.7.5.4}$$

where the quantity $\varphi(a_i)$ is given by formula (5.7.4.4) and

$$a_0 = \xi_i \frac{l^2}{l^2+1}.$$

Equation (5.7.5.4) was obtained for the case when $a_i \ll 1$. We note that this equation is for $a_i \gg \xi_i$ the same as eqn. (5.7.3.3) for the frequency of the longitudinal oscillations. The frequency (5.7.5.4) has an extremum in the points

$$a_i = a_\pm = \xi_i \frac{l^2}{4(l+1)}[l+4\pm\sqrt{(l^2+8)}], \qquad l = 2, 3, \ldots . \tag{5.7.5.4'}$$

Equations (5.7.5.2) and (5.7.5.4), as well as eqn. (5.7.5.3) for $l = 1$, are inapplicable in the region where these branches intersect, that is, when $a_i \approx \frac{1}{2}\xi_i l^2$. In that case we must use instead of these equations the formulae

$$\omega(k) = \omega_\pm = \frac{1}{2} l\omega_{Bi}\left[\sqrt{\frac{2a_i}{\xi_i}}+l \pm \sqrt{\left[\left(\frac{2a_i}{\xi_i}-1\right)^2+2l^2(l-1)^2\frac{I_l}{a_i}\right]}\right] \tag{5.7.5.5}$$

for $l = 2, 3, \ldots$ and the formulae

$$\omega(k) = \omega_\pm = \frac{1}{2}\omega_{Bi}\left[\sqrt{\frac{2a_i}{\xi_i}}+1 \pm \sqrt{\left[\left(\sqrt{\frac{2a_i}{\xi_i}}-1\right)^2+\frac{1}{2}a_i^2\right]}\right] \tag{5.7.5.6}$$

for $l = 1$.

These formulae go for $[\sqrt{(2a_i/\xi_i)}-1]^2 \gg I_l/a_i$ $(l \neq 1)$ and $[\sqrt{(2a_i/\xi_i)}-1]^2 \gg a_i^2$ over into eqns. (5.7.5.2) and (5.7.5.4) and into (5.7.5.2) and (5.7.5.3), respectively.

Equations (5.7.5.3) and (5.7.5.4) are also inapplicable when $a_i \approx a_0$. In that case we have instead of (5.7.5.3) and (5.7.5.6)

$$\omega(k) = \omega_\pm^* = l\omega_{Bi}\left[1+\frac{\sqrt{(l+1)}}{l} I_l(a_i)\, \psi_\pm(x)\right], \qquad l = 2, 3, \ldots \tag{5.7.5.7}$$

where $\psi_\pm(x) = x\pm\sqrt{(x^2+1)}$, $x = (l\sqrt{(l+1)}/a_0^2)(a_i-a_0)$. In the region $|x| \gg 1$ eqn. (5.7.5.7) gives for $\omega(k)$ expressions (5.7.5.3) and (5.7.5.4).

We show in Fig. 5.7.7 the wavenumber dependence of the frequencies $\omega^{(l)}(k)$ determined by eqns. (5.7.5.2) to (5.7.5.7).

Let us now consider in detail various parts of the curves 1, 2, 3, ... shown in Fig. 5.7.7. In the region $2a_i < \xi_i$ curve 1 corresponds to a magneto-sound wave with frequency (5.7.5.2).

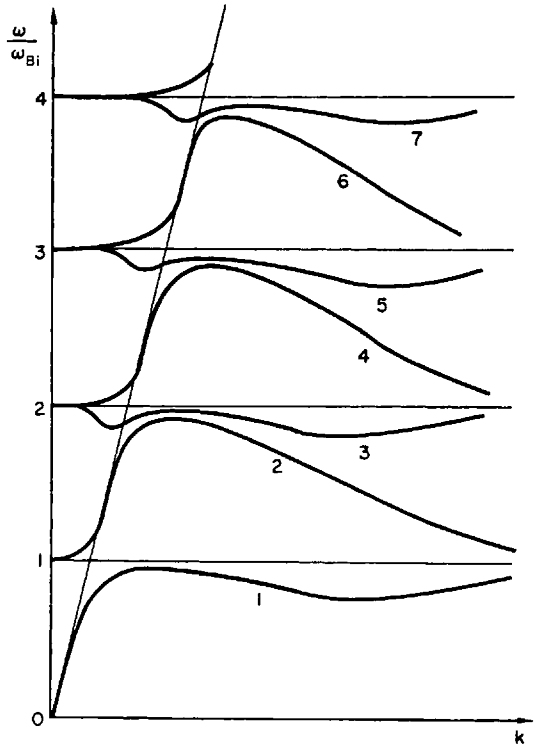

FIG. 5.7.7. The wavenumber dependence of the extra-ordinary ion-cyclotron waves and of the fast magnetosound wave.

In the region $2a_i \approx \xi_i$ the frequency of that wave is given by formula (5.7.5.6) for ω_-, and for $2a_i > \xi_i$ by formula (5.7.5.3). Curve 1 reaches its maximum in that region for $a_i = \xi_i$, then decreases with increasing k, reaches a minimum when $a_i \sim 1$ and again increases, approaching ω_{Bi}.

The frequency of the branch 2 of the oscillations is given by eqn. (5.7.5.3) for $2a_i < \xi_i$, by (5.7.5.6) for ω_+ when $2a_i \approx \xi_i$, by (5.7.5.2) when $\xi_i < 2a_i < 4\xi_i$, by (5.7.5.5) for ω_- when $l = 2$ and by (5.7.5.4) when $l = 2$ for the range $2\xi_i < a_i \ll 1$; in this region the frequency $\omega(k)$ decreases, approaching ω_{Bi} as $k \to \infty$—when $a_i \gg \xi_i$, these oscillations are longitudinal.

The curves $m = 2l-1 = 3, 5, 7, \ldots$, whose shape for $a_i < a_0$ is determined by eqn. (5.7.5.4), decrease with increasing k in that region from a value $\omega^{(l)} = l\omega_{Bi}$ at $k = 0$, reach a minimum at $a_i = a_0$ and then increase. When $a_i \approx a_0$ the shape of these curves is determined by formula (5.7.5.7) for ω_+^* and for $|a_i - a_0| \gg \xi_i^2$ by eqn. (5.7.5.3). In that region the mth curve reaches a maximum at $a_i = a_+$, then decreases, reaches a minimum at $a_i = a_l \sim 1$, and then approaches $l\omega_{Bi}$. The point a_l moves to the right with increasing l.

The curves $m = 2l = 4, 6, \ldots$ are determined by eqn. (5.7.5.3) for $a_i < a_0$, by (5.7.5.7) for ω_+^* when $a_i \approx a_0$, by (5.7.5.4) when $a_0 < a_i < \frac{1}{2}l^2\xi_i$, by (5.7.5.5) for ω_+ when $a_i \approx \frac{1}{2}l^2\xi_i$, by (5.7.5.2) when $\frac{1}{2}l^2\xi_i < a_i < \frac{1}{2}(l+1)^2\xi_i$, by (5.7.5.5), in which l must be replaced by $l+1$, for ω_- when $a_i \approx \frac{1}{2}(l+1)^2\xi_i$, and by (5.7.5.4), in which also l must be replaced by

$l+1$, when $\frac{1}{2}(l+1)^2 \xi_i < a_i \ll 1$. When $a_i \gg \xi_i$, the frequency of these oscillations, which are purely longitudinal, is determined from the equation $\varepsilon_{11} = 0$; in this region the frequency of the branch $m = 2l$ decreases monotonically, approaching $l\omega_{Bi}$. The curve $m = 2l$ has a maximum at $a_i = a_+$, which lies above the minimum of the curve $m = 2l+1$ in the point $a_i = a_-$, where a_- is determined by eqn. (5.7.5.4′) in which we must replace l by $l+1$.

Let us summarize. When $\theta = \pi/2$ several groups of electron and ion cyclotron wave branches exist: ordinary cyclotron waves, longitudinal cyclotron oscillations, and extra-ordinary cyclotron waves. The wavenumber dependence of the frequencies of these frequencies is shown in Figs. 5.7.2 and 5.7.4 to 5.7.7. In the case of a low-pressure plasma one can obtain explicit expressions for the eigenfrequencies $\omega(k)$ of the cyclotron waves for arbitrary values of $k\varrho_\alpha$.

CHAPTER 6

Interaction between Charged Particle Beams and a Plasma
Stable and Unstable Particle Distributions in a Plasma

6.1. Interaction of Charged Particle Beams with the Oscillations of an Unmagnetized Plasma

6.1.1. DISPERSION EQUATION FOR A BEAM-PLASMA SYSTEM

In the preceding chapters we have found the spectra of the oscillations of an equilibrium and a quasi-equilibrium (two-temperature) plasma and we have shown that these oscillations are damped even when we neglect binary collisions between particles. This important peculiarity of the oscillations of an equilibrium or quasi-equilibrium plasma is explained by the fact that in a collisionless plasma the damping of the oscillations is caused by the resonance interaction between particles and waves, in which particles which move in phase with the wave obtain more energy from the field than they give off to it. This fact, in turn, is connected with the nature of the particle distribution function which in the equilibrium and the quasi-equilibrium case is isotropic and decreases with increasing particle energy.

Let us now consider the case where the distribution function of the particles in the plasma is not an equilibrium one. We have already indicated that the mean free path increases with increasing temperature and the time to establish equilibrium also increases. For a sufficiently hot and not too dense plasma the particle distribution can thus for long periods differ appreciably from the equilibrium one. Such distributions may, naturally, be anisotropic and may be functions of the particle velocities which are not monotonically decreasing. In that case, the oscillations of the plasma, caused by the self-consistent field, may not necessarily be damped—as in the case of an equilibrium distribution—as the energy which the waves give off to the particles is not necessarily larger than the energy obtained by the waves from the particles in the case of non-equilibrium distributions. If the particle distribution is such that the waves obtain more energy from the particles than the particles from the waves, the plasma oscillations will grow in time and the particle distribution will be unstable.

Such a situation occurs, for instance, in a plasma in which the particle velocity distribution has the form of a sum of a Maxwell distribution and a narrow peak around some velocity $\mathbf{v} = \mathbf{u}$. Such a distribution corresponds to the passage of a beam of charged particles moving with velocity $\mathbf{u}$ through an equilibrium plasma.

In this case the fluctuations in the density and in the velocities of the particles in the beam propagate thanks to the collective interaction of the beam particles and the plasma in the form of waves with increasing amplitudes and the electrical field which then arises is of the same nature. As a result of the growth of the waves the initial particle distribution becomes unstable, that is, the beam-plasma system is unstable.†

The general problem thus crops up to establish criteria for the conditions under which plasma oscillations will be damped or growing. The present chapter is devoted to the study of the nature of the oscillations in a non-equilibrium (but spatially uniform) plasma and to the elucidation of criteria for the stability or instability of various particle distributions.

We shall start with the study of the simplest and and the same time the most important case of a non-equilibrium plasma, namely, the beam-plasma system, that is, an equilibrium unmagnetized plasma through which passes an overall neutral beam of charged particles. We shall assume that the particles of the plasma and of the beam are characterized by Maxwell distributions $f_{\alpha 0}$ and $f'_{\alpha 0}$ with different temperatures T_α and T'_α:

$$f_{\alpha 0} = n_0\left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} \exp\left(-\frac{m_\alpha v^2}{2T_\alpha}\right), \quad f'_{\alpha 0} = n'_0\left(\frac{m_\alpha}{2\pi T'_\alpha}\right)^{3/2} \exp\left(-\frac{m_\alpha(\boldsymbol{v}-\boldsymbol{u})^2}{2T'_\alpha}\right),$$

where n_0 and n'_0 are the density of the particles (of one kind) in the plasma and in the beam, while $\boldsymbol{u}$ is the average directed velocity of the beam, which we assume to be small compared to the velocity of light.

We showed in Subsection 4.3.1 that the oscillatory properties of a plasma are determined by the dielectric permittivity tensor. Using the general formula (4.3.1.9) and substituting in it for the distribution function of particles of kind α the sum $f_{\alpha 0}+f'_{\alpha 0}$ we find the dielectric permittivity tensor of the beam-plasma system:

$$\varepsilon_{ij} = \varepsilon_{ij}^{(p)} + \varepsilon_{ij}^{(b)}. \tag{6.1.1.1}$$

Here $\varepsilon_{ij}^{(p)}$ is the dielectric permittivity tensor of a plasma with a Maxwellian particle velocity distribution, which is given by formula (4.3.4.2),

$$\varepsilon_{ij}^{(p)} = (\delta_{ij}-\varkappa_i\varkappa_j)\,\varepsilon_t(\boldsymbol{k}, \omega)+\varkappa_i\varkappa_j\varepsilon_l(\boldsymbol{k}, \omega),$$

$$\varepsilon_t = 1+\sum_\alpha i\sqrt{\frac{\pi}{2}}\,\frac{\omega_{p\alpha}^2}{\omega k v_\alpha}\,w(z_\alpha),$$

$$\varepsilon_l = 1+\sum_\alpha \frac{\omega_{p\alpha}^2}{k^2 v_\alpha^2}\,[1+i\sqrt{(\pi)}z_\alpha w(z_\alpha)],$$

where $\boldsymbol{\varkappa} = \boldsymbol{k}/k$, $z_\alpha = \omega/\sqrt{(2)}kv_\alpha$, $\omega_{p\alpha} = \sqrt{(4\pi e^2 n_0/m_\alpha)}$, while $\varepsilon_{ij}^{(b)}$ is the correction to the dielectric permittivity tensor caused by the particles in the beam which is given by the expression

† Akhiezer and Faĭnberg (1949, 1951 a, b) and Bohm and Gross (1949 a, b) discovered the instability of the beam-plasma system (beam instability) caused by the excitation of high-frequency oscillations. Pierce (1947, 1948, 1949) had established the possibility of exciting low-frequency oscillations when electrons are moving relative to the ions. Haeff (1948, 1949) had developed the theory of amplifying oscillations in a two-stream tube with two "cold" electron beams.

$$\varepsilon_{11}^{(b)} = \sum_{\alpha}\left\{\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \cos^2\theta\, i\sqrt{(\pi)}z'_\alpha w(z'_\alpha) + \left(\frac{\omega'_{p\alpha}\omega' \sin\theta}{\omega k v'_\alpha}\right)^2 [1+i\sqrt{(\pi)}z'_\alpha w(z'_\alpha)]\right\},$$

$$\varepsilon_{22}^{(b)} = \sum_{\alpha}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 i\sqrt{(\pi)}z'_\alpha w(z'_\alpha),$$

$$\varepsilon_{33}^{(b)} = \sum_{\alpha}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \sin^2\theta\, i\sqrt{(\pi)}z'_\alpha w(z'_\alpha) + \left(\frac{\omega'_{p\alpha}(\omega\cos\theta + ku\sin^2\theta)}{\omega k v'_\alpha}\right)^2 [1+i\sqrt{(\pi)}z'_\alpha w(z'_\alpha)],$$

$$\varepsilon_{13}^{(b)} = \varepsilon_{31}^{(b)} = \sum_{\alpha}\left\{-\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \sin\theta\cos\theta\, i\sqrt{(\pi)}z'_\alpha w(z'_\alpha)\right.$$

$$\left. + \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{\omega'\sin\theta}{k^2 v'^2_\alpha}(\omega\cos\theta + ku\sin^2\theta)[1+i\sqrt{(\pi)}z'_\alpha w(z'_\alpha)]\right\}, \qquad (6.1.1.2)$$

$$\varepsilon_{12}^{(b)} = \varepsilon_{21}^{(b)} = \varepsilon_{23}^{(b)} = \varepsilon_{32}^{(b)} = 0,$$

where θ is the angle between the wavevector $\boldsymbol{k}$ and the beam velocity $\boldsymbol{u}$,

$$\omega' = \omega - k_z u, \quad z'_\alpha = \frac{\omega'}{\sqrt{(2)}kv'_\alpha}, \quad v'_\alpha = \sqrt{\frac{T'_\alpha}{m_\alpha}}, \quad \omega'_{p\alpha} = \sqrt{\frac{4\pi e^2 n'_0}{m_\alpha}}.$$

The tensor $\varepsilon_{ij}^{(b)}$ is given in a coordinate system in which the z-axis is parallel to the beam velocity, while the vector $\boldsymbol{k}$ lies in the xz-plane, so that $k_x = k\sin\theta$, $k_y = 0$, $k_z = k\cos\theta$.

The general dispersion equation (4.3.1.11) for electromagnetic waves splits in the case of a beam–plasma system into two equations:

$$n^2 = \varepsilon_{22}, \qquad (6.1.1.3)$$

$$An^2 - \varepsilon_{11}\varepsilon_{33} + \varepsilon_{13}^2 = 0, \qquad (6.1.1.4)$$

where

$$A = \varepsilon_{11}\sin^2\theta + 2\varepsilon_{13}\sin\theta\cos\theta + \varepsilon_{33}\cos^2\theta.$$

Equation (6.1.1.3) determines the frequency of the transverse electromagnetic waves (for which $(\boldsymbol{k}\cdot\boldsymbol{E}) = 0$), and eqn. (6.1.1.4) the frequencies of the longitudinal-transverse waves (for which $(\boldsymbol{k}\cdot\boldsymbol{E}) \neq 0$ and $[\boldsymbol{k}\wedge\boldsymbol{E}] \neq 0$, when $\theta \neq 0$).

It turns out that the transverse waves are not growing waves and we shall therefore not consider them any further. As far as the longitudinal-transverse waves are concerned, they can be growing waves and can, thus, lead to instabilities of the beam–plasma system. Their study is therefore of special great interest.

If the separate terms occurring in the coefficient A are appreciably larger than $\varepsilon_{11}\varepsilon_{33}/n^2$ and ε_{13}^2/n^2, the dispersion equation (6.1.1.4) becomes

$$A = 1 + \sum_{\alpha}\left\{\left(\frac{\omega_{p\alpha}}{kv_\alpha}\right)^2 [1+i\sqrt{(\pi)}\,z_\alpha w(z_\alpha)] + \left(\frac{\omega'_{p\alpha}}{kv'_\alpha}\right)^2 [1+i\sqrt{(\pi)}\,z'_\alpha w(z'_\alpha)]\right\} = 0. \quad (6.1.1.5)$$

This equation which describes the longitudinal oscillations of the beam–plasma system has clearly the following structure:

$$A = \varepsilon_l(\mathbf{k}, \omega) = 1 + 4\pi \sum_\alpha \varkappa_\alpha(\mathbf{k}, \omega - (\mathbf{k}\cdot\mathbf{u}_\alpha)) = 0, \tag{6.1.1.6}$$

where $\varepsilon_l(\mathbf{k}, \omega)$ is the longitudinal dielectric permittivity of the beam–plasma system, $\varkappa_\alpha(\mathbf{k}, \omega)$ the polarizability of the particles of a given kind, and the summation is over all kinds of particles in the plasma and in the beam.

6.1.2. EXCITATION OF LONGITUDINAL PLASMA OSCILLATIONS BY RESONANCE BEAM PARTICLES

We have shown in Chapter 4 that there are in an isotropic plasma two branches of weakly damped oscillations, namely, the high-frequency branch of the Langmuir oscillations and the low-frequency branch of the ion-sound oscillations (in a strongly non-isothermal plasma). When a beam of electron passes through the plasma, the electrons which are in phase with the wave,

$$\omega(\mathbf{k}) \approx (\mathbf{k}\cdot\mathbf{v}),$$

and which we shall call resonance electrons, can excite both these branches of oscillations. We shall find the growth rates of these oscillations in the case of a low-density electron beam with a large thermal velocity spread (the case of a "hot" beam).

We shall begin with a study of the Langmuir oscillations (Bohm and Gross, 1949a, b). For the Langmuir oscillations $\omega/k \gg v_e$ and we can neglect the effect of the ions. The dispersion equation (6.1.1.5) then becomes:

$$1 - \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}^2}{k^2 v_e^2}\, i\sqrt{(\pi)} z e^{-z^2} + \left(\frac{\omega_{pe}'}{k v_e'}\right)^2 [1 + i\sqrt{(\pi)}\, z_e' w(z_e')] = 0. \tag{6.1.2.1}$$

Assuming the effect of the beam to be small ($\omega_{pe}' \ll k v_e'$) we can from this equation find the frequency $\omega(k) = \operatorname{Re}\omega$ and the growth rate $\gamma(k) = \operatorname{Im}\omega$ of the Langmuir oscillations† excited by a "hot" beam:

$$\begin{aligned} \omega(k) &= \omega_{pe}\left(1 - \frac{\omega_{pe}'^2}{2k^2 v_e'^2}\left[1 - 2\exp(-z_e'^2)\int_0^{z_e'} dt\, e^{t^2}\right]\right), \\ \gamma(k) &= -\frac{\sqrt{\pi}}{2}\omega_{pe}\left[\frac{\omega_{pe}^2}{k^2 v_e^2} z_e \exp(-z_e^2) + \frac{\omega_{pe}'^2}{k^2 v_e'^2} z_e' \exp(-z_e'^2)\right], \end{aligned} \tag{6.1.2.2}$$

where

$$z_e = \frac{\omega(k)}{\sqrt{(2)} k v_e}, \quad z_e' = \frac{\omega_{pe} - k_z u}{\sqrt{(2)} k v_e'}.$$

We see that the frequency of the oscillations differs only by small terms which are proportional to the beam density from the Langmuir plasma frequency. As regards the growth rate of the oscillations, notwithstanding the fact that the contribution of the beam electrons is

† In contrast to previous chapters we shall in the present chapter assume that positive values of γ correspond to a growth in time of the oscillations.

proportional to the small parameter $(\omega'_{pe}/kv'_e)^2$, it is nevertheless important because of the exponentially small value of the Landau damping by the resonance particles in the plasma ($z_e \gg 1$).

If the damping of the Langmuir oscillations by the electrons in the plasma is small, the oscillations will be excited when $\omega_{pe}/k_z < u$ (in that region $\partial f'_{e0}/\partial v_z > 0$), while the interaction of the Langmuir oscillations with the resonance particles in the beam will lead to their damping when $\omega_{pe}/k_z > u$ (in that region $\partial f'_{e0}/\partial v_z < 0$). The growth rate reaches its maximum value when $z'_e = -1/\sqrt{2}$, that is, when $\omega(k) = \omega_{pe} = k_z u - kv'_e$:

$$\gamma_{\max} = \sqrt{\frac{\pi}{8}} \left(\frac{\omega'_{pe}}{kv'_e}\right)^2 \omega_{pe}. \tag{6.1.2.3}$$

When the magnitude of the directed beam velocity u decreases the region of phase velocities of growing oscillations diminishes and for some critical value $u = u_c$ of the beam velocity the oscillations become damped. As the phase velocity for Langmuir oscillations is larger than the thermal velocity of the electrons in the plasma, we have clearly $u_c > v_e$.

A beam with a large thermal velocity spread and a small average velocity ($u < v_e$) can not excite oscillations in the plasma when $T_e \lesssim T_i$, as such oscillations are strongly absorbed by the electrons in the plasma.

If $T_e \gg T_i$, it is possible to excite the ion-sound oscillations which are weakly damped when there is no beam (Gordeev, 1954b). In that case the dispersion equation (6.1.1.5) becomes

$$1+\left(\frac{\omega_{pe}}{kv_e}\right)^2 (1+i\sqrt{(\pi)}z_e)-\frac{\omega_{pi}^2}{\omega^2}+\left(\frac{\omega'_{pe}}{kv'_e}\right)^2 [1+i\sqrt{(\pi)}\, z'_e w(z'_e)] = 0.$$

Hence we easily find the frequency and growth rate the oscillations:

$$\begin{aligned} \omega(k) &= \frac{\omega_{pi}}{\sqrt{\left[1+\left(\frac{\omega_{pe}}{kv_e}\right)^2+\left(\frac{\omega'_{pe}}{kv'_e}\right)^2\left(1-2\exp(-z_e'^2)\int\limits_0^{z'_e} dt\, e^{t^2}\right)\right]}}, \\ \gamma(k) &= -\frac{\sqrt{\pi}}{2}\frac{\omega^3(k)}{\omega_{pi}^2}\left[\frac{\omega(k)\omega_{pe}^2}{\sqrt{(2)}k^3v_e^3}+\left(\frac{\omega'_{pe}}{kv'_e}\right)^2 z'_e \exp(-z_e'^2)\right], \end{aligned} \tag{6.1.2.4}$$

where $z'_e = (\omega(k)-k_z u)/\sqrt{(2)}kv'_e$.

For the excitations of ion-sound oscillations it is necessary that the beam velocity is larger than their phase velocity, $\omega(k)/k \sim v_s$. However, in order that the growth rate caused by the beam electrons exceeds the damping of the oscillations caused by the resonance electrons in the plasma, that is, in order that $\gamma > 0$, it is necessary that a more rigorous inequality holds:

$$u > v_s \frac{n_0}{n'_0}\left(\frac{T'_e}{T_e}\right)^{3/2}\frac{1}{(1+k^2r_D^2)^{1/2}}, \quad u \lesssim v'_e.$$

Resonance beam ions can also excite Langmuir and ion-sound oscillations. The growth rates and corrections to the frequencies of these oscillations, caused by the beam ions, are given by eqns. (6.1.2.2) and (6.1.2.4) in which the index e must be replaced by i for the quantities ω'_{pe}, v'_e, and z'_e. The maximum growth rate of Langmuir oscillations, caused by the beam ions,

$$\gamma_{max} = \sqrt{\left(\frac{\pi}{8}\right)}\left(\frac{\omega'_{pi}}{kv'_i}\right)^2 \omega_{pe}, \tag{6.1.2.5}$$

is for $T'_i \sim T'_e$ of the same order of magnitude as (6.1.2.3). However, the range of phase velocities of unstable Langmuir waves which can be excited by the beam ions, $\Delta(\omega/k_z) \sim v'_i$, is appreciably narrower than the range of phase velocities excited by beam electrons, $\Delta(\omega/k_z) \sim v'_e$. The interaction of an ion beam with Langmuir oscillations is thus considerably weaker than that of an electron beam with the same density. For the same reason the interaction of ion-sound oscillations with an ion beam is less effective than with an electron beam.

In the case considered of beams of charged particles with large thermal velocity spreads the oscillations are excited by resonance particles. One can say that each resonance particle excites the oscillations independently. The growth rate of the oscillations is in that case proportional to the beam density n'_0. In what follows we shall show that if the velocity spread of the beam particles is sufficiently small, the situation may occur in which the beam particles excite the oscillations coherently; the growth rate of the oscillations is then proportional to $(n'_0)^{1/2}$ or to $(n'_0)^{1/3}$.

6.1.3. EXCITATION OF LONGITUDINAL OSCILLATIONS BY A MONOENERGETICAL BEAM

The expressions obtained a moment ago for the growth rates of longitudinal oscillations are applicable, provided

$$\frac{\gamma_{max}}{kv'_e} \ll 1.$$

As $\gamma_{max} \sim n'_0/T'_e$, this inequality is satisfied, provided the thermal spread of the beam particles is sufficiently large and the density small.

If the opposite inequality,

$$\frac{\gamma_{max}}{kv'_e} \gg 1,$$

(corresponding to the case of a "cold" or monoenergetical beam) holds, the spread in the velocity of the beam particles is unimportant.

We shall first consider the excitation of Langmuir oscillations by a monoenergetical beam (Bohm and Gross, 1949a, b; Akhiezer and Fainberg, 1949, 1951a, b). Assuming in that case that $|z_e| \gg 1$ and $|z'_e| \gg 1$, and neglecting the contribution of the ions to the dispersion equation, we get

$$1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega'^2_{pe}}{(\omega - k_z u)^2} = 0. \tag{6.1.3.1}$$

This equation is of fourth degree in ω. The presence of the beam thus changes the dispersive properties of the plasma in an essential manner. We shall find the solutions of the dispersion equation (6.1.3.1) for the case of a low-density beam ($n'_0 \ll n_0$). As $\omega'_{pe} \ll \omega_{pe}$, two of the roots of this equation lie close to $\pm\omega_{pe}$:

$$\omega^{(1,2)} = \pm\omega_{pe}\left[1+\frac{\omega_{pe}'^2}{2(k_z u \mp \omega_{pe})^2}\right]. \tag{6.1.3.2}$$

The other two roots lie close to $\omega = k_z u$. Putting

$$\omega^{(3,4)} = k_z u + \eta^{(3,4)}, \tag{6.1.3.3}$$

where $|\eta^{(3,4)}| \ll |k_z u|$, we find

$$\eta^{(3,4)} = \pm\frac{\omega_{pe}'}{\sqrt{\left[1-\left(\frac{\omega_{pe}}{k_z u}\right)^2\right]}}. \tag{6.1.3.4}$$

From this it follows that the long-wavelength oscillations ($k_z u < \omega_{pe}$) are unstable and the growth rate of these oscillations is equal to

$$\gamma = \operatorname{Im}\eta = \frac{\omega_{pe}'}{\sqrt{\left[\left(\frac{\omega_{pe}}{k_z u}\right)^2-1\right]}}.$$

As to order of magnitude we have

$$\gamma \sim \sqrt{\left(\frac{n_0'}{n_0}\right)}k_z u \sim \sqrt{\left(\frac{n_0'}{n_0}\right)}\omega, \quad k_z u < \omega_{pe}, \tag{6.1.3.5}$$

When $k_z u \sim \omega_{pe}$, we see from this that $\gamma \sim \omega_{pe}'$.

Equations (6.1.3.2) and (6.1.3.4) for $\omega = \omega^{(1)}$ and $\omega = \omega^{(3,4)}$ are inapplicable near the resonance point $k_z u = \omega_{pe}$ where (in the limit as $n_0' \to 0$) the three frequencies $\omega^{(1)} = \omega_{pe}$ and $\omega^{(3,4)} = k_z u$ intersect. Putting as before $\omega^{(j)} = k_z u + \eta^{(j)}$ ($j = 1, 3, 4$) we get in this case for the correction $\eta = \eta^{(j)}$ to the frequency the cubic equation

$$\eta^3 + \left(1-\frac{\omega_{pe}}{k_z u}\right)\eta^2\omega_{pe} - \frac{1}{2}\omega_{pe}\omega_{pe}'^2 = 0. \tag{6.1.3.6}$$

When $|\eta| \gg |k_z u - \omega_{pe}|$ we find from this that

$$\eta = \xi^{(j)}(\tfrac{1}{2}\omega_{pe}\omega_{pe}'^2)^{1/3}, \tag{6.1.3.7}$$

where

$$\xi^{(j)} = \sqrt[3]{1} = \left(1, \frac{-1+i\sqrt{3}}{2}, -\frac{1+i\sqrt{3}}{2}\right).$$

The growth rate for the unstable branch is equal to

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{n_0'}{n_0}\right)^{1/3}\omega_{pe}. \tag{6.1.3.8}$$

The oscillations of the low-density-beam-plasma system have thus a maximum growth rate which under the resonance conditions $\omega_{pe} = k_z u$ is proportional to $n_0'^{1/3}$.

We note that the expressions obtained can be used when the inequality $|z'_e| \sim |\eta|/k_z v'_e \gg 1$ is satisfied. This condition is satisfied for oscillations with a growth rate (6.1.3.5) when

$$u \gg \sqrt{\left(\frac{n_0}{n'_0}\right)} v'_e ,$$

that is, the velocity spread in the electron beam must be very small. For oscillations with the maximum growth rate the condition $|z'_e| \gg 1$ is somewhat weaker:

$$u \gg \left(\frac{n_0}{n'_0}\right)^{1/3} v'_e .$$

If these conditions are not satisfied, it is impossibly coherently to excite Langmuir oscillations. However, it is in that case possible to excite ion-sound oscillations (Gordeev, 1954b). The dispersion equation for the ion-sound oscillation in the presence of a mono-energetic electron beam has the form

$$1+\frac{1}{k^2 r_D^2}-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}'^2}{(\omega-k_z u)^2} = 0. \tag{6.1.3.9}$$

This equation is like eqn. (6.1.3.1) of the fourth degree in ω. If $n'_0 \ll n_0(m_e/m_i)$, the roots of this equation are determined by eqns. (6.1.3.2) to (6.1.3.4) in which we must replace ω_{pe} by $\omega_s = kv_s/\sqrt{(1+k^2r_D^2)}$, where $v_s = \sqrt{(T_e/m_i)}$ is the sound velocity, and ω'_{pe} by $\omega'_{pe}kr_D/\sqrt{(1+k^2r_D^2)}$. The oscillations are unstable when $k_z u < \omega_s(k)$. The growth rate is in that case of the order of

$$\gamma \sim \sqrt{\left(\frac{n'_0}{n_0}\frac{m_i}{m_e}\right)} k_z u, \quad kr_D \sim 1, \quad ku < \omega_{pe} .$$

The growth rate reaches a maximum under the resonance conditions $\omega_s(k) = k_z u$:

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{\omega'_{pe}}{\omega_{pi}}\right)^{2/3} \omega_s . \tag{6.1.3.10}$$

Let us consider the conditions for the applicability of these expressions. They are obtained when the condition $|z'_e| \gg 1$ is satisfied, which in the non-resonance case occurs when

$$u \gg \left(\frac{n_0}{n'_0}\frac{m_e}{m_i}\right)^{1/2} v'_e ,$$

and when $\omega_s = k_z u$ this condition holds, provided

$$u \gg \left(\frac{n_0}{n'_0}\frac{m_e}{m_i}\right)^{1/3} v'_e ,$$

that is, only provided $u \gg v'_e$.

Moreover, the following inequalities must hold:

$$|\omega| \gg kv_i \quad \text{and} \quad |\omega| \ll kv_e .$$

As $\omega \sim ku$, it is necessary that the following inequality holds:

$$v_i \ll u \ll v_e.$$

It is clear that the inequalities obtained will be compatible only if the thermal velocity of the beam particles is very small.

If the beam density is such that $n_0' \gg n_0(m_e/m_i)$, we have, taking the small term $\propto \omega_{pi}^2$ into account,

$$\omega = k_z u \pm \omega_s'(k) \pm \frac{\omega_s'^3}{(k_z u \pm \omega_s')^2} \frac{\omega_{pi}^2}{\omega_{pe}'^2}, \tag{6.1.3.11}$$

where

$$\omega_s'(k) = \frac{\omega_{pe}' k r_D}{\sqrt{(1+k^2 r_D^2)}}.$$

The two other roots of this equation are considerably smaller than $k_z u$:

$$\omega = \pm \frac{\omega_{pi}}{\sqrt{\left[1+\frac{1}{k^2 r_D^2} - \left(\frac{\omega_{pe}'}{k_z u}\right)^2\right]}}. \tag{6.1.3.12}$$

Oscillations for which $k_z u < \omega_s'(k)$ are unstable. The growth rate of the oscillations is in that case as to order of magnitude equal to

$$\gamma \sim \left(\frac{n_0}{2n_0'} \frac{m_e}{m_i}\right)^{1/2} k_z u, \quad k r_D \sim 1.$$

Under resonance conditions, when $\omega_s'(k) = k_z u$,

$$\omega = \xi^{(i)} \left(\frac{n_0}{n_0'} \frac{m_e}{m_i}\right)^{1/3} \omega_s'. \tag{6.1.3.13}$$

The maximum growth rate is according to (6.1.3.13) equal to

$$\gamma = \frac{\sqrt{3}}{2} \left(\frac{n_0 m_e}{2 n_0' m_i}\right)^{1/3} \omega_s'.$$

The conditions for the applicability of the expressions obtained, $|z_e'| \gg 1$, $|z_e| \ll 1$, and $|z_i| \gg 1$, are satisfied for $\omega_s' \sim k_z u$ when $v_i, v_e' \ll u \ll v_e$.

We have studied the excitation by a monoenergetical beam of Langmuir and ion-sound waves which are damped when there is no beam. We shall now show that a monoenergetic beam can excite plasma oscillations with a phase velocity of the order of the thermal velocity of the electrons and ions, which are strongly damped when there is no beam. Assuming as before that $|z_e'| \gg 1$, we can write the dispersion equation for longitudinal oscillations, $A = 0$, in the form

$$\varepsilon_l^{(p)}(\mathbf{k}, \omega) - \left(\frac{\omega'_{pe}}{\omega - k_z u}\right)^2 = 0, \quad |z'_e| \gg 1, \tag{6.1.3.14}$$

where $\varepsilon_l^{(p)} = A(\mathbf{k}, \omega)$, when $n'_0 = 0$. As the quantity ω'_{pe} is small, we can put $\omega = k_z u + \eta$, and we find that

$$\eta = \pm \frac{\omega'_{pe}}{\sqrt{\varepsilon_l^{(p)}(\mathbf{k}, k_z u)}}. \tag{6.1.3.15}$$

As the quantity $\varepsilon_l^{(p)}$ has an imaginary part (caused by the presence of resonance plasma particles) one of the roots (6.1.3.15) always has a positive imaginary part. For instance, when $u \sim v_e$ and $kr_D \sim 1$, we find $\text{Re}\,\varepsilon_l^{(p)} \sim \text{Im}\,\varepsilon_l^{(p)} \sim 1$, so that

$$\gamma \sim \omega'_{pe}.$$

When $u \sim v_i$ and $kr_D \sim 1$ we also have $\gamma \sim \omega'_{pe}$. In the case considered the excitation of the oscillations by a monoenergetic beam is caused by the presence of strong absorption of the oscillations by the resonance electrons and ions in the plasma. This mechanism is effective only in the velocity region $u \sim v_{e,i}$. If $u \gg v_e$, the number of resonance electrons is exponentially small. In that case eqn. (6.1.3.15) becomes

$$\eta = \pm \frac{\omega'_{pe}}{\sqrt{\left[1 - \left(\frac{\omega_{pe}}{k_z u}\right)^2 + i\sqrt{\frac{\pi}{2}}\,\frac{\omega_{pe}^2 u}{k^2 v_e^3} \exp\left(-\frac{u^2}{2v_e^2}\right)\right]}}.$$

When $\omega < \omega_{pe}$ there occurs a coherent excitation of the oscillations and we can neglect the small imaginary term under the radical in this expression. If $\omega > \omega_{pe}$, the growth rate is exponentially small

$$\gamma = \text{Im}\,\eta = \frac{\omega'_{pe}}{\left[1 - \left(\frac{\omega_{pe}}{k_z u}\right)^2\right]^{3/2}} \sqrt{\frac{\pi}{8}}\,\frac{\omega_{pe}^2 u}{k^2 v_e^3} \exp\left(-\frac{u^2}{2v_e^2}\right).$$

It follows from this expression that when we approach the resonance region, $k_z u = \omega_{pe}$, the excitation of the oscillations due to Cherenkov absorption is amplified.

Let us now consider the problem of the excitation of plasma oscillations by a low-density monoenergetic ion beam. The expressions obtained earlier for the frequencies and growth rates of the oscillations remain applicable also in this case, provided we replace ω'_{pe} by ω'_{pi}. As the growth rates are under non-resonance conditions proportional to $\omega'_{p\alpha}$, coherent excitation of plasma oscillations by a monoenergetic low-density ion beam proceeds less effectively than the excitation of oscillations by an electron beam.

In concluding this subsection we shall consider the excitation of ion-sound oscillations in interpenetrating currents with comparable densities (Kadomtsev, 1961; Vedenov, Velikhov, and Sagdeev, 1961b). Assuming that $|z_e| \ll 1$, $|z'_e| \ll 1$, $|z_i| \gg 1$, and $|z'_i| \gg 1$, we can write the dispersion equation (6.1.1.5) in the form

$$1 + \frac{1}{k^2 r_D^2} + \frac{1}{k^2 r_D'^2} - \frac{\omega_{pi}^2}{\omega^2} - \left(\frac{\omega'_{pi}}{\omega - k_z u}\right)^2 = 0,$$

where $r'_D = v'_e/\omega'_{pe}$. In the case of long-wavelength oscillations ($kr_D \ll 1$, $kr'_D \ll 1$) we can in this equation neglect unity as compared to $(kr_D)^{-2}$ and $(kr'_D)^{-2}$. As a result we easily obtain the following criterion for instability:

$$\cos^2\theta\, u^2 < \frac{(v_s v'_s)^2\left[\left(\frac{n_0}{m_i}\right)^{1/3} + \left(\frac{n'_0}{m'_i}\right)^{1/3}\right]^3}{\frac{n_0 v_s'^2}{m_i} + \frac{n'_0 v_s^2}{m'_i}},$$

where θ is the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{u}$. When this inequality is satisfied there occurs in the interpenetrating currents a strong instability with a growth rate of the oscillations of the order of

$$\gamma \sim ku\cos\theta, \quad n_0 \sim n'_0, \quad T_e \sim T'_e.$$

We must note that the quantity $u\cos\theta$ must be appreciably larger than the thermal velocity of the ions in both beams, as in the opposite case the excitation of oscillations becomes impossible due to the appearance of strong damping by resonance ions.

6.1.4. INSTABILITY OF A PLASMA IN WHICH THE ELECTRONS MOVE RELATIVE TO THE IONS

In the preceding subsection we considered the excitation of longitudinal plasma oscillations by an overall neutral electron beam. We shall now consider the excitation of longitudinal oscillations in a plasma in which the electrons move relative to the ions. Such a motion may be produced by a constant electrical field imposed on the plasma. We denote this field by $\boldsymbol{E}_0$. Electrons moving in this field will gain on a mean free path length l ($l = 1/\sigma n_0$, where $\sigma = \pi e^4 L/T_e^2$ is the scattering cross-section for Coulomb collisions with L the Coulomb logarithm) an energy $\delta\varepsilon = eE_0 l$. If $\delta\varepsilon \ll T_e$, the electron will give off its energy to other particles in the plasma and the result will be Joule heating of the plasma while the electron gas moves relative to the ions with a velocity

$$u = \frac{eE_0}{m_e \nu_C},$$

which is considerably smaller than the thermal velocity of the electrons ($\nu_C \sim v_e/l$ is the Coulomb collision frequency).

The situation is changed in an essential manner in strong electrical fields, when $\delta\varepsilon > T_e$. As the electron mean free path increases steeply with increasing energy, Coulomb collisions by themselves can not prevent the electrons to change to a regime of continuous acceleration in which the velocity of the electron gas linearly increases with time,

$$u = \frac{e}{m_e} E_0 t.$$

This transition cannot be prevented by the Coulomb scattering of electrons by ions, but it can be prevented by another mechanism, namely, the scattering of electrons by unstable longitudinal plasma oscillations which occur when electrons move relatively to the ions.

We shall consider the occurrence of this instability in some detail (Pierce, 1948a; Budker, 1956; Buneman, 1958, 1959; Kovrizhnykh and Rukhadze, 1960). We shall assume that the electrical field is stronger than the critical field, $E_0 > E_c$, which is determined by the condition

$$eE_c l = T_e,$$

and that the velocity of the directed motion of the electrons, u, is considerably larger than the thermal velocity. On the other hand, we shall assume that the field E_0 is not too large so that in a time $1/\gamma$, where γ is the growth rate of the oscillations considered, the increase in electron velocity, $\Delta u = eE_0/m_e\gamma$, is small, $\Delta u \ll u$, and that we may assume that during the time that the oscillations develop the electron beam travels uniformly and rectilinearly (adiabatic approximation). Because of their large mass the ions acquire in the field E_0 a velocity which is small compared to the electron velocity and we can assume them to be at rest when there are no oscillations.

Under those conditions the dispersion equation for the longitudinal plasma oscillations will be of the form

$$1+4\pi\varkappa^{(e)}(\boldsymbol{k}, \omega-k_z u)+4\pi\varkappa^{(i)}(\boldsymbol{k}, \omega) = 0.$$

Neglecting the thermal motion of the electrons and the ions we can write this equation in the form

$$1-\frac{\omega_{pe}^2}{(\omega-k_z u)^2}-\frac{\omega_{pi}^2}{\omega^2} = 0, \qquad (6.1.4.1)$$

where $\omega_{pi} = \sqrt{(4\pi e^2 n_0/m_i)}$ is the ion Langmuir frequency.

If $\omega \gtrsim k_z u$, the ion term in (6.1.4.1) is clearly negligibly small, and we find from (6.1.4.1) for the expression for the frequencies of the usual Langmuir oscillations which are shifted by the amount $\Delta\omega = k_z u$ becaûse of the Doppler effect

$$\omega = k_z u \pm \omega_{pe}.$$

We shall be interested in the low-frequency branch of the electron oscillations, $\omega \ll k_z u$. In that case we must retain in eqn. (6.1.4.1) the ion term:

$$1-\frac{\omega_{pe}^2}{k_z^2 u^2}\left(1+2\frac{\omega}{k_z u}\right)-\frac{\omega_{pi}^2}{\omega^2} = 0. \qquad (6.1.4.2)$$

For the case when $k_z u$ lies not too close to ω_{pe} we can neglect the small term $\propto \omega/k_z u$, and we get

$$\omega = \pm i\frac{\omega_{pi}}{\sqrt{\left[\left(\frac{\omega_{pe}}{k_z u}\right)^2-1\right]}}, \qquad (6.1.4.3)$$

that is, the oscillations considered are unstable when $k_z u < \omega_{pe}$.

As to order of magnitude the growth rate of the oscillations considered, (6.1.4.3), is equal to

$$\gamma \sim \sqrt{\left(\frac{m_e}{m_i}\right)} k_z u, \quad \omega_{pe} > k_z u.$$

This growth rate becomes particularly large as $k_z u \to \omega_{pe}$. In that case, however, eqn. (6.1.4.3) ceases to be applicable and we must determine the quantity ω from the cubic eqn. (6.1.4.2).

When $\omega_{pe} = k_z u$ we find from (6.1.4.2) that

$$\omega = \zeta^{(i)}(\tfrac{1}{2}\omega_{pe}\omega_{pi}^2)^{1/3},$$

where

$$\zeta^{(i)} = \sqrt[3]{(-1)} = \left(-1,\ \frac{1-i\sqrt{3}}{2},\ \frac{1+i\sqrt{3}}{2}\right).$$

The maximum growth rate is thus equal to

$$\gamma_{\max} = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{m_e}{m_i}\right)^{1/3}\omega_{pe} = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{m_i}{m_e}\right)^{1/6}\omega_{pi}. \qquad (6.1.4.4)$$

Concluding this subsection let us consider the conditions for the applicability of the adiabatic approximation, used by us (Lovetskiĭ and Rukhadze, 1966). It is clear that we can use eqn. (6.1.4.4) if during a time $\sim 1/\gamma_{\max}$ the resonance condition $|\omega_{pe}-k_z u| \lesssim \gamma_{\max}$ does not become violated due to acceleration, that is, when $k\Delta u \lesssim \gamma_{\max}$, where $\Delta u = eE_0/m_e\gamma_{\max}$ and $k = \omega_{pe}/u$.

In sufficiently strong fields and short systems the plasma may turn out to be stable as it is impossible to excite either the resonance oscillations, $\omega_{pe} \approx ku$, or the non-resonance oscillations with growth rate $\gamma \sim ku\sqrt{(m_e/m_i)}$. Indeed, when the acceleration of the electrons is taken into account the oscillations in the ion density are described by the equation

$$\frac{\partial^2 n_i'}{\partial t^2} + \frac{\omega_{pi}^2}{\left(\dfrac{\omega_{pe}}{k_z u}\right)^2 - 1} n_i' = 0,$$

where $u = (eE_0/m_e)t$. The solutions of this equation for $\omega_{pe} > k_z u$ have the form of growing solutions,

$$n_i' = \sqrt{t}\left[C_1 I_{1/4}\left(\frac{t^2}{t_1^2}\right) + C_2 I_{-1/4}\left(\frac{t^2}{t_1^2}\right)\right],$$

where $t_1 = (4m_e m_i/k^2e^2E_0^2)^{1/4}$. In the region where $k_z u > \omega_{pe}$ the solutions of this equation oscillate.

When $t < t_1$ the oscillations increase very slowly

$$n_i' = C_1 + C_2 t, \quad t \ll t_1,$$

and in that stage it is impossible to develop the initial perturbation. When $t \gg t_1$, we find an exponential growth, corresponding to the adiabatic approximation:

$$n_i' \propto \frac{1}{\sqrt{t}} \exp\left(\frac{t^2}{t_1^2}\right), \quad t \gg t_1.$$

In order that it be impossible to develop the oscillations it is necessary that within a time $t \lesssim t_1$ the condition $k_z u < \omega_{pe}$ is violated for the longest-wavelength oscillations with

$k_{\min} \sim 2\pi/L_{||}$, where $L_{||}$ is the length of the system. From this we get the criterion for stability,

$$k_{\min}\frac{eE_0}{m_1}t_1 > \omega_{pe},$$

that is,

$$E_0 > E_{\min},$$

where

$$E_{\min} = en_0L_{||}\sqrt{\frac{m_e}{m_i}}.$$

In fields $E_0 > E_{\min}$ the electron–ion beam instability is not developed.

If the plasma is in an electrical field E_0 considerably smaller than E_c, the average electron velocity will be considerably smaller than the thermal electron velocity. If in such a plasma the electron temperature is appreciably larger than the ion temperature, the relative motion of the electrons with respect to the ions may lead to the excitation of ion-sound oscillations (Gordeev, 1954a).

We shall assume that the electron and ion distribution functions are Maxwellian when there are no oscillations and that the average ion velocity is zero while the electrons have an average velocity $\boldsymbol{u}$. The dispersion equation for the ion-sound oscillations then has the following form:

$$1+\frac{1}{k^2r_D^2}\left(1+i\sqrt{\left(\frac{\pi}{2}\right)}\frac{\omega-k_zu}{kv_e}\right)-\frac{\omega_{pi}^2}{\omega^2}+i\sqrt{\left(\frac{\pi}{2}\right)}\frac{\omega_{pi}^2\omega}{k^3v_i^3}\exp\left(-\frac{\omega^2}{2k^2v_i^2}\right) = 0.$$

From this we easily find the frequency $\omega(k)$ and the growth rate $\gamma(k)$ of the oscillations,

$$\omega = \omega_s(k), \quad \gamma(k) = \gamma_e - \gamma_i,$$

where

$$\omega_s(k) = \frac{kv_s}{\sqrt{(1+k^2r_D^2)}},$$

$$\gamma_e(k) = \sqrt{\left(\frac{\pi}{8}\frac{m_e}{m_i}\right)}\frac{kv_s}{(1+k^2r_D^2)^2}\left(\frac{k_zu}{\omega_s}-1\right), \quad \gamma_i(k) = \sqrt{\left(\frac{\pi}{8}\right)}\frac{\omega_s^4}{k^3v_i^3}\exp\left(-\frac{\omega_s^2}{2k^2v_i^2}\right). \tag{6.1.4.5}$$

It is clear that the ion-sound oscillations can become unstable only when $u > \omega_s/k$. Neglecting the damping by the ions we get for the case when $kr_D \sim 1$ and $u \gtrsim v_s$ as to order of magnitude:

$$\omega(k) \sim \omega_{pi}, \quad \gamma(k) \sim \omega_{pi}\frac{u}{v_e}. \tag{6.1.4.6}$$

When the drift velocity of the electrons decreases, the growth rate also decreases. Instability sets in only when $u > u_c$ where the magnitude of the critical velocity is determined from the condition $\gamma_e = \gamma_i (\gamma < 0$ when $u < u_c$ for any value of $k)$. As to order of magnitude $u_c \sim 3$ to $4v_i$.

6.1.5. EXCITATION OF ELECTROMAGNETIC WAVES IN A PLASMA BY CHARGED PARTICLE CURRENTS

We have considered the excitation of unstable potential oscillations in a plasma in the case when currents of charged particles pass through the plasma. We shall now show that in that case it is possible that electromagnetic (non-potential) waves are excited in that case (Fried, 1959; Neufeld and Doyle, 1961; Makhan'kov and Rukhadze, 1962).

To make clear the mechanism through which these waves are excited and the plasma instabilities connected with them, we shall consider the following simple example (Fried, 1959). Let the plasma consist of two electron beams moving along the z-axis in opposite directions with a relative velocity $2u$ so that

$$f_{e0} = \tfrac{1}{2}n_0\,\delta(v_x)\,\delta(v_y)\,[\delta(v_z-u)+\delta(v_z+u)].$$

If there occurs in the plasma a magnetic field along the y-axis,

$$B_y \propto e^{ikx}$$

it will curve the electron trajectories. After a time δt the electrons with a velocity $v_z = u$ obtain an increase in their velocity along the x-axis,

$$\delta v_x = -\frac{e}{m_e c}\,[\boldsymbol{v}\wedge\boldsymbol{B}]_x\,\delta t = \frac{eu}{m_e c}\,B_y\,\delta t.$$

There is thus along the x-axis a flux of the z-component of the electron momentum

$$m_e\frac{1}{2}\,n_0 v_z\,\delta v_x = \frac{n_0 u^2}{2c}\,B_y\,\delta t.$$

Electrons moving with a velocity $v_z = -u$ obtain after a time δt an increase in their velocity along the x-axis which is equal to $-\delta v_x$. The flux of z-component of the momentum of these electrons through unit area perpendicular to the x-axis will also be equal to

$$\frac{n_0 u^2}{2c}\,B_y\,\delta t.$$

The total flux of z-component of momentum of both groups of electrons per unit time through unit area perpendicular to the x-axis will be equal to

$$\frac{\partial p_{zx}}{\partial t} = \frac{n_0 u^2}{c}\,B_y. \tag{6.1.5.1}$$

The transfer of momentum leads to the appearance of a variable component of the electron velocity along the z-axis:

$$m_e n_0\frac{\partial v_z'}{\partial t} = -\frac{\partial p_{zx}}{\partial x} = -ikp_{zx}. \tag{6.1.5.2}$$

Because of this there appears an electrical current density in the z-direction:

$$j_z = -en_0 v_z'. \tag{6.1.5.3}$$

On the other hand, the current density j_z is equal to

$$j_z = \frac{c}{4\pi} \operatorname{curl}_z \boldsymbol{B} = \frac{ick}{4\pi} B_y. \tag{6.1.5.4}$$

Taking the time-derivative of eqn. (6.1.5.2) and using eqns. (6.1.5.1), (6.1.5.3), and (6.1.5.4) we get the following equation for B_y:

$$\frac{\partial^2 B_y}{\partial t^2} = \frac{u^2}{c^2} \frac{4\pi e^2 n_0}{m_e} B_y. \tag{6.1.5.5}$$

We see that the field B_y will grow exponentially in time with a growth rate

$$\gamma = \frac{u}{c} \omega_{pe}. \tag{6.1.5.6}$$

The mechanism of the instability considered consists thus in the fact that an initial magnetic field B_y curves the electron trajectories and leads to the appearance of a flux of z-component of electron momentum in the x-direction. A change in this flux leads to the appearance of a variable component v_z' of the electron velocity, that is, to a current density j_z, which causes the appearance of a magnetic field, thus amplifying the initially occurring perturbation of the magnetic field.

The analysis given here is, however, incomplete, as it does not take into account the appearance of a rotational electrical field:

$$E_z = -\frac{i}{ck} \frac{\partial B_y}{\partial t}.$$

We can neglect its effect in the equation of motion,

$$\frac{\partial v_z'}{\partial t} = -\frac{e}{m_e} E_z - \frac{1}{mn_0} \frac{\partial p_{zx}}{\partial x},$$

only when

$$\frac{\partial v_z'}{\partial t} \sim \gamma v_z' \sim \frac{ck\gamma}{4\pi e n_0} B_y \gg \frac{e}{m_e} E_z \sim \frac{e\gamma}{m_e ck} B_y,$$

that is, when the inequality

$$k^2 c^2 \gg \omega_{pe}^2$$

is satisfied.

Let us now consider the same instability, taking into account both the rotational electrical field which occurs and the thermal motion of the particles in the plasma and in the beam (Makhan'kov and Rukhadze, 1962). To do this we use the general dispersion equation (6.1.1.4) for electromagnetic waves in a beam–plasma system. In the case of waves propagating at right angles to the z-axis in the x-direction this equation has the form

$$\varepsilon_{11}(n^2 - \varepsilon_{33}) + \varepsilon_{13}^2 = 0. \tag{6.1.5.7}$$

The components of the dielectric permittivity tensor for a plasma and a beam with Maxwellian velocity distributions are in this case equal to

$$\varepsilon_{11} = 1+\sum_\alpha \left\{\frac{\omega_{p\alpha}^2}{k^2v_\alpha^2}[1+i\sqrt{(\pi)}\,z_\alpha w(z_\alpha)]+\frac{\omega_{p\alpha}'^2}{k^2v_\alpha'^2}[1+i\sqrt{(\pi)}\,z_\alpha' w(z_\alpha')]\right\},$$

$$\varepsilon_{33} = 1+\sum_\alpha \left\{\frac{\omega_{p\alpha}^2}{\omega^2}i\sqrt{(\pi)}\,z_\alpha w(z_\alpha)+\left(\frac{\omega_{p\alpha}'}{\omega}\right)^2 i\sqrt{(\pi)}\,z_\alpha' w(z_\alpha') + \left(\frac{\omega_{p\alpha}'}{\omega}\frac{u}{v_\alpha'}\right)^2[1+i\sqrt{(\pi)}\,z_\alpha' w(z_\alpha')]\right\}, \quad (6.1.5.8)$$

$$\varepsilon_{13} = \sum_\alpha \frac{\omega_{p\alpha}'^2 u}{\omega k v_\alpha'^2}[1+i\sqrt{(\pi)}\,z_\alpha' w(z_\alpha')],$$

where

$$z_\alpha = \frac{\omega}{\sqrt{(2)}kv_\alpha}, \quad z_\alpha' = \frac{\omega}{\sqrt{(2)}kv_\alpha'}.$$

We shall study eqn. (6.1.5.7) in a number of limiting cases.

(a) In the region of large phase velocities when $|\omega/k| \gg v_e, v_e'$, the tensor has the form

$$\varepsilon_{11} = 1-\sum_\alpha \frac{\omega_{p\alpha}^2+\omega_{p\alpha}'^2}{\omega^2},$$

$$\varepsilon_{33} = 1-\sum_\alpha \left\{\frac{\omega_{p\alpha}^2}{\omega^2}+\frac{\omega_{p\alpha}'^2}{\omega^2}\left(1+\frac{k^2u^2}{\omega^2}\right)\right\}, \quad (6.1.5.9)$$

$$\varepsilon_{13} = -\sum_\alpha \frac{\omega_{p\alpha}'^2 ku}{\omega^3}.$$

Substituting this into (6.1.5.7) we get

$$\omega^2 = -k^2\frac{(\Sigma\omega_{pe}^2)(\Sigma\omega_{pe}^2u^2)-(\Sigma\omega_{pe}^2u)^2}{(k^2c^2+\Sigma\omega_{pe}^2)(\Sigma\omega_{pe}^2)}, \quad (6.1.5.10)$$

where the summations are over the particles in the plasma and in the beam. As $\omega^2 < 0$, the oscillations are unstable. In a frame of reference in which

$$\Sigma\omega_{pe}^2 u = 0$$

expression (6.1.5.10) simplifies:

$$\omega^2 = -k^2\frac{\Sigma\omega_{pe}^2u^2}{k^2c^2+\Sigma\omega_{pe}^2} < 0. \quad (6.1.5.11)$$

In the particular case of two identical beams, moving against one another, eqn. (6.1.5.11) is the same as eqn. (6.1.5.6) for γ when $k^2c^2 \gg \omega_{pe}^2$.

Let us now consider cases when the thermal spread of the particles is important. For the sake of simplicity we shall consider a system consisting of a plasma at rest with density n_0 and an overall neutral beam of density n_0' moving with velocity u; we denote the temperatures of the electrons and the ions in the plasma and in the beam by T_e, T_i, T_e', and T_i', respectively.

(b) If the beam is cold and the electrons in the plasma hot, we have

$$v_e', v_i, v_i' \ll \left|\frac{\omega}{k}\right| \ll v_e,$$

and the tensor ε_{ij} becomes

$$\begin{aligned}\varepsilon_{11} &= 1+\frac{\omega_{pe}^2}{k^2v_e^2}-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}'^2}{\omega^2},\\ \varepsilon_{33} &= 1+\frac{\omega_{pe}^2}{k^2v_e^2}\,i\sqrt{(\pi)}z_e-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}'^2(\omega^2+k^2u^2)}{\omega^4},\\ \varepsilon_{13} &= -\frac{\omega_{pe}'^2ku}{\omega^3}.\end{aligned} \tag{6.1.5.12}$$

Substituting expressions (6.1.5.12) into eqn. (6.1.5.7) and neglecting in the expression for ε_{33} the term $\propto iz_e$ we get for ω^2 the equation

$$A\omega^4+B\omega^2+C = 0, \tag{6.1.5.13}$$

where

$$\begin{aligned}A &= \frac{\omega_{pe}^2}{k^2v_e^2}(k^2c^2+\omega_{pe}'^2+\omega_{pi}^2),\\ B &= -(k^2c^2+\omega_{pe}'^2+\omega_{pi}^2)(\omega_{pe}'^2+\omega_{pi}^2)+\frac{u^2}{v_e^2}\omega_{pe}^2\omega_{pe}'^2,\\ C &= -\omega_{pi}^2\omega_{pi}'^2k^2u^2.\end{aligned}$$

Hence we have

$$\omega^2 = \frac{-B\pm\sqrt{(B^2-4AC)}}{2A}. \tag{6.1.5.14}$$

As $AC < 0$, one of the roots (6.1.5.14) is always negative, that is, the oscillations considered with phase velocities considerably less than v_e are always unstable. One should, however note that the conditions under which expressions (6.1.5.14) were obtained are satisfied under rather rigid limitations.

One can easily check that the oscillations considered are also excited when $u \lesssim v_e$.

(c) If the beam is cold and the electrons and ions in the plasma hot, we have

$$v_e', v_i' \ll \left|\frac{\omega}{k}\right| \ll v_e, v_i,$$

and the components of the tensor ε_{ij} are equal to

$$\begin{aligned}\varepsilon_{11} &= 1+\frac{\omega_{pe}^2}{k^2v_e^2}+\frac{\omega_{pi}^2}{k^2v_i^2}-\frac{\omega_{pe}'^2}{\omega^2},\\ \varepsilon_{33} &= 1+\frac{\omega_{pe}^2}{\omega^2}\,i\sqrt{(\pi)}z_e+\frac{\omega_{pi}^2}{\omega^2}\,i\sqrt{(\pi)}z_i-\frac{\omega_{pe}'^2(k^2u^2+\omega^2)}{\omega^4},\\ \varepsilon_{13} &= -\frac{\omega_{pe}'^2ku}{\omega^3}.\end{aligned}$$

Substituting these expressions into (6.1.5.7) (and neglecting small terms) we find

$$\omega^2 = -\omega_{pe}'^2\left[\frac{k^2u^2}{k^2c^2+\omega_{pe}'^2}-\frac{k^2v_e^2v_i^2}{\omega_{pe}^2v_i^2+\omega_{pi}^2v_e^2}\right]. \tag{6.1.5.15}$$

The oscillations will be unstable when the following inequality holds:

$$\frac{u^2}{k^2c^2+\omega_{pe}'^2} > \frac{v_e^2v_i^2}{\omega_{pe}^2v_i^2+\omega_{pi}^2v_e^2}.$$

(d) In the region of phase velocities appreciably lower than the thermal velocities of the electrons in the plasma and in the beam and considerably larger than the thermal velocities of the ions in the plasma and in the beam,

$$v_i,\ v_i' \ll \left|\frac{\omega}{k}\right| \ll v_e,\ v_e',$$

the tensor ε_{ij} has the form

$$\varepsilon_{11} = \frac{\omega_{pe}^2}{k^2v_e^2}+\frac{\omega_{pe}'^2}{k^2v_e'^2}-\frac{\omega_{pi}^2+\omega_{pi}'^2}{\omega^2},$$

$$\varepsilon_{33} = \frac{\omega_{pe}^2}{\omega^2}i\sqrt{(\pi)}z_e+\frac{\omega_{pe}'^2}{\omega^2}i\sqrt{(\pi)}z_e'+\frac{\omega_{pe}'^2u^2}{\omega^2v_e'^2}-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pi}'^2}{\omega^2}\left(1+\frac{k^2u^2}{\omega^2}\right),$$

$$\varepsilon_{13} = \frac{\omega_{pe}'^2u}{\omega kv_e'^2}-\frac{\omega_{pi}'^2ku}{\omega^3}.$$

Substituting these expressions into eqn. (6.1.5.7), and neglecting in ε_{33} the terms $\propto iz_e$ and $\propto iz_e'$, we get for ω eqn. (6.1.5.13) in which the coefficients A, B, and C are equal to

$$\begin{aligned} A &= \left(\frac{\omega_{pe}^2}{k^2v_e^2}+\frac{\omega_{pe}'^2}{k^2v_e'^2}\right)\left(k^2c^2+\omega_{pi}^2+\omega_{pi}'^2-\frac{u^2}{v_e'^2}\omega_{pe}'^2\right)+\frac{\omega_{pe}'^4u^2}{k^2v_e'^4},\\ B &= -(\omega_{pi}^2+\omega_{pi}'^2)\left(k^2c^2+\omega_{pi}^2+\omega_{pi}'^2-\frac{u^2}{v_e'^2}\omega_{pe}'^2\right)\\ &\quad -\left(\frac{\omega_{pe}^2}{k^2v_e^2}+\frac{\omega_{pe}'^2}{k^2v_e'^2}\right)\omega_{pi}'^2k^2u^2-2\frac{u^2}{v_e'^2}\omega_{pe}'^2\omega_{pi}'^2,\\ C &= -\omega_{pi}^2\omega_{pe}^2k^2u^2. \end{aligned} \tag{6.1.5.16}$$

The square of the frequency is then given by formula (6.1.5.14). The oscillations will be unstable, if $A > 0$ or if $A < 0$ and $B < 0$. If, for instance, $n_0' \sim n_0$ and $u \sim v_e \sim v_e'$, the growth rate of the oscillations will be of the order of,

$$\gamma \sim kv_e\sqrt{\frac{m_e}{m_i}}.$$

The condition $|\omega| \gg kv_i$ is only satisfied when $T_e \gg T_i$.

(e) Finally, if the electrons in the beam are hot and the plasma cold, in the phase velocity region

$$v_e \ll \left|\frac{\omega}{k}\right| \ll v_e'$$

the components of the tensor ε_{ij} are of the form

$$\begin{aligned}
\varepsilon_{11} &= \frac{\omega_{pe}'^2}{k^2 v_e'^2} - \frac{\omega_{pe}^2}{\omega^2}, \\
\varepsilon_{33} &= \frac{\omega_{pe}'^2 u^2}{\omega^2 v_e'^2} - \frac{\omega_{pe}^2}{\omega^2} + \frac{\omega_{pe}'^2}{\omega^2} i\sqrt{(\pi)} z_e' \left(1 + \frac{u^2}{v_e'^2}\right), \\
\varepsilon_{13} &= \frac{\omega_{pe}'^2 u}{\omega k v_e'^2}.
\end{aligned} \tag{6.1.5.17}$$

If $|\varepsilon_{13}^2| \ll |\varepsilon_{11}\varepsilon_{33}|$ we can neglect the quantity ε_{13}^2 in equation (6.1.5.7) and that equation becomes

$$n^2 - \varepsilon_{33} = 0.$$

Substituting here expression (6.1.5.17) for ε_{33}, we get

$$\omega = i\sqrt{\left(\frac{2}{\pi}\right)} \frac{k v_e'^3}{\omega_{pe}'^2 (v_e'^2 + u^2)} \left[\frac{u^2}{v_e'^2} \omega_{pe}'^2 - k^2 c^2 - \omega_{pe}^2\right], \tag{6.1.5.18}$$

where

$$k^2 c^2 + \omega_{pe}^2 \approx \frac{u^2}{v_e'^2} \omega_{pe}'^2.$$

Instability occurs in this case when the following inequality holds:

$$k^2 c^2 + \omega_{pe}^2 < \left(\frac{u}{v_e'} \omega_{pe}'\right)^2.$$

We have dwelt in so much detail on the instabilities connected with the excitation of electromagnetic waves as they occur when the velocities of the directed motion of beams are either larger or smaller than the thermal velocities of the electrons and ions, in cases when the beam instabilities connected with the excitation of longitudinal oscillations, which are, in general, characterized by much larger growth rates of the oscillations, cannot develop.

6.1.6. INSTABILITY OF A PLASMA WITH AN ANISOTROPIC VELOCITY DISTRIBUTION

Earlier we have studied the instability of a system consisting of two overall neutral plasma beams. We shall show that when there are no beams in the plasma, that is, when the distribution function of the particles in the plasma decreases in all directions with increasing velocity, there still can occur instabilities, provided the particle distribution is anisotropic. We shall consider the instability of a plasma with the following electron distribution function (Weibel, 1959; Stefanovich, 1962; Makhan'kov and Shevchenko, 1965):

$$f_{e0}(\mathbf{v}) = f_{e0}(v_z^2, v_\perp^2),$$

where $v_\perp = \sqrt{(v_x^2+v_y^2)}$ is the velocity component perpendicular to the distinguished anisotropy direction (the z-axis).

We shall as before consider waves propagating along the x-axis. Assuming that $|\varepsilon_{13}^2| \ll |\varepsilon_{11}\varepsilon_{33}|$, we then find from (6.1.1.4) the dispersion equation

$$n^2 = \varepsilon_{33},$$

or

$$k^2c^2+\omega_{pe}^2(1-\eta) = -2\omega_{pe}^2\omega \int d^3v v_z^2 \frac{\partial f_{e0}}{\partial v_x^2} \frac{1}{kv_x-\omega}, \tag{6.1.6.1}$$

where

$$\eta \equiv -2\int d^3v v_z^2 \frac{\partial f_{e0}}{\partial v_x^2}. \tag{6.1.6.2}$$

Assuming that $|\omega| \ll k\,|v_x|$, we get from this

$$\omega = \frac{ik}{2\pi} \frac{k^2c^2+\omega_{pe}^2(1-\eta)}{\omega_{pe}^2\left(\frac{\partial}{\partial v_x^2}\iint dv_y\, dv_z\, v_z^2 f_{e0}\right)_{v_x^2=0}}. \tag{6.1.6.3}$$

If, in particular, the electron velocity distribution is an anisotropic Maxwell distribution,

$$f_{e0} = \frac{n_0 m_e^{3/2}}{(2\pi)^{3/2}T_{\parallel}^{1/2}T_\perp} \exp\left[-\frac{m_e v_\perp^2}{2T_\perp} - \frac{m_e v_z^2}{2T_{\parallel}}\right],$$

where $T_\perp$ and $T_\parallel$ are the "transverse" and "longitudinal" electron temperatures, which determine the average values of $v_\perp^2$ and v_z^2, eqn. (6.1.6.3) becomes

$$\omega = -i\sqrt{\left(\frac{2T_\perp}{\pi m_e}\right)}\frac{k^2c^2+\omega_{pe}^2(1-\eta)}{\omega_{pe}^2\eta}k, \quad \eta = \frac{T_\parallel}{T_\perp}. \tag{6.1.6.4}$$

We see that instability occurs when

$$\eta > 1,$$

that is, when

$$T_\parallel > T_\perp.$$

We note that the condition $|\omega| \ll k\,|v_x|$ which was used to derive eqns. (6.1.6.3) and (6.1.6.4) is satisfied when

$$k^2 \approx k_0^2 = \frac{\omega_{pe}^2}{c^2}\left(\frac{T_\parallel}{T_\perp}-1\right).$$

If $T_\parallel < T_\perp$, there also occurs an instability, but it arises thanks to the growth of waves propagating along the anisotropy direction (along the z-axis). In that case the dispersion equation is of the form

$$n^2 = \varepsilon_{11},$$

where

$$\varepsilon_{11} = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2}\frac{1}{n_0}\int \frac{k\,dv_z}{kv_z - \omega}\frac{\partial}{\partial v_z}\int_0^\infty dv_\perp v_\perp^3 f_{e0}(v_z^2, v_\perp^2).$$

Assuming that $|\omega| \ll k\,|v_z|$ we find from this that

$$\omega = \frac{i}{\pi}\frac{k^2c^2 + \omega_{pe}^2(1-\zeta)}{\omega^2_e\varphi'(0)}k, \tag{6.1.6.5}$$

where

$$\varphi(v_z^2) = \frac{2\pi}{n_0}\int_0^\infty d^2v_\perp v_\perp^3 f_{e0}(v_z^2, v_\perp^2),$$

and

$$\zeta = -\frac{1}{n_0}\int d^3v v_\perp^2 \frac{\partial f_{e0}}{\partial v_z^2}.$$

For an anisotropic Maxwell distribution the parameter ζ is equal to

$$\zeta = \frac{T_\perp}{T_\parallel}.$$

It is clear that Im $\omega > 0$, if $\zeta > 1$ ($T_\perp > T_\parallel$) and

$$k^2 < k_0^2 \equiv \frac{\omega_{pe}^2}{c^2}(\zeta - 1).$$

The condition $|\omega| \ll k\,|v_z|$ is now satisfied only when $k \approx k_0$.

6.2. Interaction of a Charged Particle Beam with Plasma Oscillations in a Magnetic Field

6.2.1. THE DIELECTRIC PERMITTIVITY TENSOR OF A BEAM–PLASMA SYSTEM IN A MAGNETIC FIELD

We now turn to a study of the interaction between a charged particle beam and a plasma when there is present an external constant and uniform magnetic field $\boldsymbol{B}_0$. We shall assume that the average velocity of the particles in the beam, $\boldsymbol{u}$, is parallel to the field $\boldsymbol{B}_0$ and that the velocity distributions of the particles in the plasma and in the beam are Maxwellian. For a study of the dispersive properties of such a system we need to know its dielectric permittivity tensor. This tensor can clearly be written in the form

$$\varepsilon_{ij} = \varepsilon_{ij}^{(p)} + \varepsilon_{ij}^{(b)}, \tag{6.2.1.1}$$

where $\varepsilon_{ij}^{(p)}$ is the dielectric permittivity tensor of a magneto-active plasma given by eqn. (5.2.2.4), and $\varepsilon_{ij}^{(b)}$ the contribution to the tensor ε_{ij} coming from the particles in the beam. Using the general formula (5.2.1.14) for the dielectric permittivity tensor of a magneto-active plasma for the case of a Maxwellian particle distribution shifted by an amount $\boldsymbol{u}$, we find (Kitsenko and Stepanov, 1961a)

$$
\begin{aligned}
\varepsilon_{11}^{(b)} &= \sum_\alpha \sum_{l=-\infty}^{+\infty} \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{l^2 A_l(a'_\alpha)}{a'_\alpha}\, i\sqrt{(\pi)}\, z'_0 w(z'_l), \\
\varepsilon_{22}^{(b)} &= \sum_\alpha \sum_{l=-\infty}^{+\infty} \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \left[\frac{l^2}{a'_\alpha} A_l(a'_\alpha) - 2a'_\alpha A'_l(a'_\alpha)\right] i\sqrt{(\pi)}\, z'_0 w(z'_l), \\
\varepsilon_{33}^{(b)} &= \sum_\alpha \sum_{l=-\infty}^{+\infty} 2\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 A_l(a'_\alpha)\,[y_0^2 + y_l^2 i\sqrt{(\pi)}\, z'_0 w(z'_l)], \\
\varepsilon_{12}^{(b)} &= -\sum_\alpha \sum_{l=-\infty}^{+\infty} \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 l A'_l(a'_\alpha)\sqrt{(\pi)}\, z'_0 w(z'_l), \\
\varepsilon_{13}^{(b)} &= \sum_\alpha \sum_{l=-\infty}^{+\infty} \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{l A_l(a'_\alpha)}{\sqrt{a'_\alpha}}\, y_l i\sqrt{(2\pi)}\, z'_0 w(z'_l), \\
\varepsilon_{23}^{(b)} &= \sum_\alpha \sum_{l=-\infty}^{+\infty} \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \sqrt{(a'_\alpha)}\, A'_l(a'_\alpha)\, y_l \sqrt{(2\pi)}\, z'_0 w(z'_l),
\end{aligned}
\tag{6.2.1.2}
$$

where

$$
A_l(x) = e^{-x} I_l(x); \qquad A'_l(x) = \frac{dA_l}{dx}; \qquad a'_\alpha = \left(\frac{k_x v'_\alpha}{\omega_{B\alpha}}\right)^2; \qquad z'_l = \frac{\omega - l\omega_{B\alpha} - k_z u}{\sqrt{(2)} k_z v'_\alpha};
$$

$$
y_l = \frac{\omega - l\omega_{B\alpha}}{\sqrt{(2)} k_z v'_\alpha}.
$$

In the case of a "cold" (monoenergetic) beam of charged particles, when

$$
a'_\alpha \ll 1, \quad |z'_l| \gg 1,
$$

we get from (6.2.1.12)

$$
\begin{aligned}
\varepsilon_{11}^{(b)} = \varepsilon_{22}^{(b)} &= -\sum_\alpha \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{(\omega - k_z u)^2}{(\omega - k_z u)^2 - \omega_{B\alpha}^2}, \\
\varepsilon_{33}^{(b)} &= -\sum_\alpha \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \left[\frac{\omega^2}{(\omega - k_z u)^2} + \frac{(k_x u)^2}{(\omega - k_z u)^2 - \omega_{B\alpha}^2}\right], \\
\varepsilon_{12}^{(b)} &= -i\sum_\alpha \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{\omega_{B\alpha}(\omega - k_z u)}{(\omega - k_z u)^2 - \omega_{B\alpha}^2}, \\
\varepsilon_{13}^{(b)} &= -\sum_\alpha \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{(\omega - k_z u) k_x u}{(\omega - k_z u)^2 - \omega_{B\alpha}^2}, \\
\varepsilon_{23}^{(b)} &= i\sum_\alpha \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 \frac{\omega_{B\alpha} k_x u}{(\omega - k_z u)^2 - \omega_{B\alpha}^2}.
\end{aligned}
\tag{6.2.1.3}
$$

We note first of all that the anti-Hermitean parts of the tensor $\varepsilon_{ij}^{(b)}$ caused by the presence of resonance particles are proportional to the quantity z'_0, that is, to $\omega - k_z u$. When there is no beam, resonance particles in the case of a Maxwellian velocity distribution always cause damping of the electromagnetic waves. When, however, there is a beam, slow waves will be excited by the resonance particles in the beam—when the condition $u > \omega(\mathbf{k})/k_z$ is satisfied—and if the damping of these waves by the resonance particles in the plasma is sufficiently weak, the amplitude of the wave will grow exponentially in time.

In the case of a monoenergetic beam, the components of the tensor $\varepsilon_{ij}^{(b)}$ have singularities when

$$\omega = k_z u, \quad \omega = k_z u \pm |\omega_{B\alpha}|. \tag{6.2.1.4}$$

This means that when a low density beam passes through a magneto-active plasma there are, apart from the oscillation branches $\omega^{(j)}(\mathbf{k})$, which exist in the plasma when there is no beam, new branches of oscillations which in the limit as $n_0' \to 0$ are given by eqn. (6.2.1.4).

A cold beam therefore significantly alters the dispersion properties of a magneto-active plasma. Let us recall in this connection that when a monoenergetic beam passes through an unmagnetized plasma there also arise new branches of oscillations with frequencies $k_z u$ —as $n_0' \to 0$ the frequency $\omega = k_z u$ was two-fold degenerate.

The beam oscillations with $\omega = k_z u$ and $\omega = k_z u - |\omega_{B\alpha}|$ may be unstable. This instability is particularly large in the resonance region where the frequency of the plasma eigenoscillations $\omega^{(j)}(\mathbf{k})$ gets close to the frequencies $\omega = k_z u$ or $\omega = k_z u - |\omega_{B\alpha}|$. In that case one speaks of the *Cherenkov resonance* between the beam particles and the wave, when $\omega^{(j)} = k_z u$, and of the *cyclotron resonance* under anomalous Doppler effect conditions, when $\omega^{(j)} = k_z u - |\omega_{B\alpha}|$.†

The beam oscillations with frequency $\omega = k_z u + |\omega_{B\alpha}|$ are stable. Instability can not even occur when there is cyclotron resonance under normal Doppler effect conditions, when $\omega^{(j)} = k_z u + |\omega_{B\alpha}|$.

In what follows we shall study the excitation of various branches of oscillations in a magneto-active plasma, both by resonance beam particles and by a monoenergetic beam.

6.2.2. EXCITATION OF LONGITUDINAL OSCILLATIONS OF A PLASMA IN A MAGNETIC FIELD BY AN ELECTRON BEAM

We start with a study of the excitation of high-frequency (electron) longitudinal oscillations in a beam-plasma system by the electrons in a beam (Kitsenko and Stepanov, 1961a; Stepanov, 1958; Rappoport, 1960; Kovner, 1960 a, b). To find the complex frequencies of the longitudinal oscillations—or, to be more precise, the almost longitudinal oscillations with $E_l \gg E_t$—we must put the quantity A in the dispersion eqn. (5.2.2.5) equal to zero. As we are considering high-frequency oscillations, we can in our evaluation of A neglect the contribution to the tensor ε_{ij} made by the ions in the plasma and the beam.

Using expressions (6.2.1.2) we can write the dispersion equation $A = 0$ in the form

$$A = 1 + \delta\varepsilon^{(p)} + \delta\varepsilon^{(b)} = 0, \tag{6.2.2.1}$$

† Ginzburg (1960) and Zheleznyakov (1959) have discussed the instability of the oscillations of a magneto-active plasma caused by Cherenkov and cyclotron emission of resonance particles in the beams from a quantal point of view.

where

$$\delta\varepsilon^{(p)} = \frac{\omega_{pe}^2}{k^2 v_e^2}\left[1+i\sqrt{(\pi)}z_0 \sum_{l=-\infty}^{+\infty} A_l(a_e)\, w(z_l)\right],$$
$$\delta\varepsilon^{(b)} = \frac{\omega_{pe}'^2}{k^2 v_e'^2}\left[1+i\sqrt{(\pi)}z_0' \sum_{l=-\infty}^{+\infty} A_l(a_e')\, w(z_l')\right], \tag{6.2.2.2}$$
$$z_l = \frac{\omega - l\,|\omega_{Be}|}{\sqrt{(2)}k_z v_e}, \quad z_l' = \frac{\omega - l\,|\omega_{Be}| - k_z u}{\sqrt{(2)}k_z v_e'}.$$

(a) To begin with let us consider the excitation of oscillations of a cold plasma by a low-density beam with a large thermal velocity spread. In that case

$$\delta\varepsilon^{(p)} = -\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pe}^2}{\omega^2-\omega_{Be}^2}\sin^2\theta, \tag{6.2.2.3}$$

and when there is no beam, the dispersion equation $A = 1+\delta\varepsilon^{(p)} = 0$ gives two frequencies $\omega = \omega_\infty^{(1,2)}(\theta) = \omega_\pm$ given by eqns. (5.1.2.6). Putting

$$\omega = \omega_\pm + i\gamma,$$

and neglecting the real part of $\delta\varepsilon^{(b)}$, which leads merely to a small change in the frequency of the oscillations, we find

$$\gamma = -\frac{\sqrt{(\pi)}\omega_{pe}'^2\omega}{2k^2 v_e'^2} z_0' \sum_{l=-\infty}^{+\infty} A_l(a_e') \exp(-z_l'^2)\left[\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta + \frac{\omega_{pe}^2\omega^2\sin^2\theta}{(\omega^2-\omega_{Be}^2)^2}\right]^{-1}, \tag{6.2.2.4}$$

where

$$z_l' = \frac{\omega_\pm - l\,|\omega_{Be}| - k_z u}{\sqrt{(2)}k_z v_e'}.$$

The quantity γ will thus be positive, that is, the oscillations will grow, if the beam velocity exceeds the phase velocity of the oscillations along the magnetic field,

$$u > \frac{\omega_\pm}{k_z}.$$

If $u < \omega_\pm/k_z$, the presence of the beam leads to a damping of the oscillations.

Equation (6.2.2.4) was obtained under the assumption that $\gamma \ll k_z v_e$, that is, that we are dealing with low-density beams with a large thermal velocity spread. In that case the growth rate is proportional to n_0'—as in the case when there is no magnetic field.

If $u \gg v_e'$, the quantity $|z_l'|$ is much larger than unity, provided the frequency $\omega_\pm$ is not close to $k_z u + l\,|\omega_{Be}|$. We can thus in the sum over l in (6.2.2.4) for the case when $u \gg v_e'$ retain only the single term corresponding to the minimum $|z_l'|$. When the condition

$$\omega_\pm \approx k_z u + l\,|\omega_{Be}|, \qquad l = 0, \pm 1, \ldots, \tag{6.2.2.5}$$

is satisfied, the growth rate is thus given by the expression

$$\gamma = -\frac{\sqrt{(\pi)}\omega_{pe}'^2\omega_\pm}{2k^2 v_e'^2}\,\frac{a_e^{|l|}}{2^{|l|}\,|l|!}\left[\frac{\omega_{pe}^2}{\omega_\pm^2}\cos^2\theta + \frac{\omega_{pe}^2\omega_\pm^2\sin^2\theta}{(\omega_\pm^2-\omega_{Be}^2)^2}\right]^{-1} z_0' \exp(-z_l'^2). \tag{6.2.2.6}$$

In deriving this formula we used the fact that $a_e = (k_x v_e'/\omega_{Be})^2 \ll 1$ and $A_l(a_e) \approx a_e^{|l|}/2^{|l|}|l|!$.

If condition (6.2.2.5) is satisfied for $l = 0$, we are dealing with Cherenkov excitation of oscillations; when $l = -1, -2, -3, \ldots$ the excitation is called cyclotron excitation of the oscillations.

It is clear from expression (6.2.2.6) that the growth rate of the long-wavelength oscillations ($k_x \varrho_e' = |\sqrt{a_e'}| \ll 1$) increases very rapidly with increasing number of the excited harmonic.

(b) Equations (6.2.2.4) and (6.2.2.6) determine the growth rate in the hot beam–cold plasma system. Let us now consider the excitations in the cold beam–cold plasma system. In that case the dispersion eqn. (6.2.2.1) becomes

$$1-\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta-\frac{\omega_{pe}^2\sin^2\theta}{\omega^2-\omega_{Be}^2}-\frac{\omega_{pe}'^2\cos^2\theta}{(\omega-k_z u)^2}-\frac{\omega_{pe}'^2\sin^2\theta}{(\omega-k_z u)^2-\omega_{Be}^2}=0. \tag{6.2.2.7}$$

When there is no beam this equation determines four eigenfrequencies: $\omega = \pm\omega_+$ and $\omega = \pm\omega_-$; when there is a beam, the number of eigenfrequencies of longitudinal plasma oscillations becomes equal to eight.

If the beam density is low ($n_0' \ll n_0$), we can put

$$\omega = k_z u + \eta, \tag{6.2.2.8}$$

and assuming that η is small, $|\eta| \ll 1$, we get for η the equation

$$\eta = \pm\omega_{pe}' \sqrt{\left[\frac{\omega^2(\omega^2-\omega_{Be}^2)}{(\omega^2-\omega_+^2)(\omega^2-\omega_-^2)}\right]_{\omega=k_z u}}. \tag{6.2.2.8'}$$

It is clear that the quantity η will be purely imaginary—corresponding to an instability of the oscillations—if $k_z u < \omega_-$, or if $|\omega_{Be}| < k_z u < \omega_+$.

As to order of magnitude, the growth rate is equal to

$$\gamma \sim \sqrt{\left(\frac{n_0'}{n_0}\right)}\, k_z u, \qquad \omega_{pe} \sim |\omega_{Be}| \gtrsim k_z u.$$

It is clear from eqn. (6.2.2.8') that the growth rate becomes particularly large when the frequency $k_z u$ approaches one of the eigenfrequencies of the longitudinal plasma oscillations ω_+ or ω_-. However, when $\omega \approx k_z u \approx \omega_\pm$, eqn. (6.2.2.8') may become inapplicable. In that case the quantity η is determined from the equation

$$\eta^3+(k_z u-\omega_\pm)\eta^2-\frac{\omega_{pe}'^2\cos^2\theta(\omega_+^2-\omega_-^2)\,\omega_\pm}{2\,|\omega_\pm^2-\omega_{Be}^2|}=0.$$

If $|k_z u-\omega_\pm| \gg |\eta|$, we have

$$\eta = \xi^{(i)}\left[\frac{\omega_{pe}'^2\cos^2\theta(\omega_+^2-\omega_-^2)\,\omega_\pm}{2\,|\omega_\pm^2-\omega_{Be}^2|}\right]^{1/3}, \tag{6.2.2.9}$$

where

$$\xi^{(i)} = \sqrt[3]{1} = \left(1,\ \frac{-1+i\sqrt{3}}{2},\ \frac{-1-i\sqrt{3}}{2}\right).$$

From this we get for the maximum growth rate the expression

$$\gamma = \frac{\sqrt{3}}{2}\left[\frac{\omega_{pe}'^2 \cos^2\theta(\omega_+^2-\omega_-^2)\,\omega_\pm}{2\,|\omega_\pm^2-\omega_{Be}^2|}\right]^{1/3}. \tag{6.2.2.10}$$

When $\omega_{pe} \sim |\omega_{Be}|$, we have as to order of magnitude

$$\gamma \sim \omega_{pe}\sqrt[3]{\frac{n_0'}{n_0}}.$$

Under Cherenkov resonance conditions, $\omega_\pm = k_z u$, the growth rate is thus proportional to $(n_0')^{1/3}$, as in the case of Cherenkov resonance in a plasma without a magnetic field, when $\omega_{pe} = k_z u$.

Let us now consider the oscillations in a low-density cold beam-cold plasma system with frequencies close to $\omega = k_z u - |\omega_{Be}|$. Putting

$$\omega = k_z u - |\omega_{Be}| + \eta,$$

where $|\eta| \ll \big|k_z u - |\omega_{Be}|\big|$, we find that for ω not too close to $\omega_\pm$ the correction η to the frequency is purely real:

$$\eta = -\frac{\omega_{pe}'^2 \sin^2\theta(\omega^2-\omega_{Be}^2)\,\omega^2}{2\,|\omega_{Be}|\,(\omega^2-\omega_+^2)(\omega^2-\omega_-^2)}\bigg|_{\omega = k_z u - |\omega_{Be}|}.$$

When $k_z u - |\omega_{Be}| \approx \omega_\pm$, we have

$$\eta = \frac{1}{2}(\omega_\pm - k_z u + |\omega_{Be}|) \pm \frac{1}{2}\sqrt{\left\{(\omega_\pm - k_z u + |\omega_{Be}|)^2 - \frac{\omega_{pe}'^2 \sin^2\theta\,|\omega_\pm^2-\omega_{Be}^2|\,\omega_\pm}{|\omega_{Be}|\,(\omega_+^2-\omega_-^2)}\right\}}. \tag{6.2.2.11}$$

From this it follows that when the frequency $\omega = k_z u - |\omega_{Be}|$ approaches the eigenfrequencies of the plasma oscillations the oscillations become unstable, and the growth rate,

$$\gamma = \frac{1}{2}\sqrt{\left\{\frac{\omega_{pe}'^2 \sin^2\theta\,|\omega_\pm^2-\omega_{Be}^2|\,\omega_\pm}{|\omega_{Be}|\,(\omega_+^2-\omega_-^2)} - (\omega_\pm - k_z u + |\omega_{Be}|)^2\right\}},$$

reaches its maximum value when $\omega_\pm = k_z u - |\omega_{Be}|$,

$$\gamma_{max} = \frac{1}{2}\sqrt{\left[\frac{\omega_{pe}'^2 \sin^2\theta\,|\omega_\pm^2-\omega_{Be}^2|\,\omega_\pm}{|\omega_{Be}|\,(\omega_+^2-\omega_-^2)}\right]}. \tag{6.2.2.12}$$

When $\omega_{pe} \sim |\omega_{Be}|$ we have as to order of magnitude

$$\gamma_{max} \sim \omega_{pe}\sqrt{\frac{n_0'}{n_0}},$$

that is, the maximum growth rate of the oscillations with frequencies $\omega_\pm$ in cyclotron resonance under anomalous Doppler effect conditions is less than the growth rate of these oscillations under Cherenkov resonance conditions, when $\gamma_{max} \sim \omega_{pe}\sqrt[3]{(n_0'/n_0)}$.

Oscillations with frequencies $\omega = k_z u + |\omega_{Be}|$ are always stable. Indeed, putting

$$\omega = k_z u + |\omega_{Be}| + \eta,$$

where $|\eta| \ll \big|k_z u + |\omega_{Be}|\big|$, we get for η a purely real expression,

$$\eta = \frac{\omega_{pe}'^2 \sin^2\theta\, \omega^2(\omega^2 - \omega_{Be}^2)}{2|\omega_{Be}|(\omega^2-\omega_+^2)(\omega^2-\omega_-^2)}\bigg|_{\omega = k_z u + |\omega_{Be}|},$$

if $k_z u + |\omega_{Be}|$ is not too close to $\omega_\pm$, and

$$\eta = \frac{1}{2}(\omega_\pm - k_z u - |\omega_{Be}|) \pm \frac{1}{2}\sqrt{\left\{(\omega_\pm - k_z u - |\omega_{Be}|)^2 + \frac{\omega_{pe}'^2 \sin^2\theta\, |\omega_\pm^2 - \omega_{Be}^2|\,\omega_\pm}{|\omega_{Be}|(\omega_+^2 - \omega_-^2)}\right\}},$$

when $k_z u + |\omega_{Be}| \approx \omega_\pm$.

(c) Let us now consider the interaction of a low-density cold beam with a hot plasma. In that case the dispersion eqn. (6.2.2.1) has the form

$$1 + \delta\varepsilon^{(p)} - \frac{\omega_{pe}'^2 \cos^2\theta}{(\omega - k_z u)^2} - \frac{\omega_{pe}'^2 \sin^2\theta}{(\omega - k_z u)^2 - \omega_{Be}^2} = 0, \tag{6.2.2.13}$$

where $\delta\varepsilon^{(p)}(\boldsymbol{k}, \omega)$ is given by eqn. (6.2.2.2).

If the beam velocity u is of the order of or less than the thermal velocity, v_e, of the electrons in the plasma, it is no longer possible to neglect the thermal motion of the electrons in the plasma for oscillations with frequencies $\omega = k_z u$ or $\omega = k_z u \pm |\omega_{Be}|$.

If in that case we put $\omega = k_z u + \eta$ we find

$$\eta = \pm \frac{\omega_{pe}' \cos\theta}{\sqrt{\{1 + \delta\varepsilon^{(p)}(\boldsymbol{k}, k_z u)\}}}. \tag{6.2.2.14}$$

As due to the resonance Cherenkov and cyclotron absorption of the oscillations by the electrons $\mathrm{Im}\,\delta\varepsilon^{(p)}(\boldsymbol{k}, k_z u) > 0$, one of the roots of eqn. (6.2.2.14) always has a positive imaginary part, that is, because of the presence of the strong damping of the plasma eigenoscillations the oscillations with frequency $\omega = k_z u$, excited by a monoenergetic beam, are always unstable. When $u \sim v_e$ and $kr_D \lesssim 1$ we have as to order of magnitude

$$\gamma \sim k v_e \sqrt{\frac{n_0'}{n_0}}. \tag{6.2.2.14'}$$

Oscillations with $\omega = k_z u - |\omega_{Be}|$ are also unstable, because of the presence of damping of the plasma eigenoscillations. Putting $\omega = k_z u - |\omega_{Be}| + \eta$, we find that

$$\eta = -\frac{\omega_{pe}'^2 \sin^2\theta}{2|\omega_{Be}|[1 + \delta\varepsilon^{(p)}(\boldsymbol{k}, k_z u - |\omega_{Be}|)]}. \tag{6.2.2.15}$$

Hence we get

$$\gamma = \mathrm{Im}\,\eta = \frac{\omega_{pe}'^2 \sin^2\theta\, \mathrm{Im}\,\delta\varepsilon^{(p)}(\boldsymbol{k}, k_z u - |\omega_{Be}|)}{2|\omega_{Be}|\,|1 + \delta\varepsilon^{(p)}(\boldsymbol{k}, k_z u - |\omega_{Be}|)|^2}.$$

As $\mathrm{Im}\,\delta\varepsilon^{(p)} > 0$, we have $\gamma > 0$, that is, oscillations with $\omega = k_z u - |\omega_{Be}|$ are, indeed, unstable.

If $k\varrho_e \sim 1$, where ϱ_e is the electron Larmor radius, $kr_D \lesssim 1$, and $\omega_{pe} \sim |\omega_{Be}|$, we have as to order of magnitude

$$\gamma \sim kv_e \frac{n_0'}{n_0}.$$

Comparing this expression with eqn. (6.2.2.14′) we see that when absorption is taken into account the excitation of oscillations is much more effective for $\omega = k_z u$ than for $\omega = k_z u - |\omega_{Be}|$.

We note that oscillations with frequency $\omega = k_z u + |\omega_{Be}|$ are stable. Indeed, in this case we find for $\eta = \omega - k_z u - |\omega_{Be}|$ the expression

$$\eta = \frac{\omega_{pe}'^2 \sin^2\theta}{2|\omega_{Be}|\,[1+\delta\varepsilon^{(p)}(\boldsymbol{k}, k_z u + |\omega_{Be}|)]}.$$

Hence it follows that $\gamma = \operatorname{Im} \eta < 0$.

Concluding this subsection we note that the excitation of low-frequency longitudinal oscillations with frequencies $\omega = \omega_\infty^{(2)}(\theta)$ $(\theta \approx \pi/2)$ and $\omega = \omega_\infty^{(3)}(\theta)$ by a low-density beam can be considered in exactly the same way as we did for the high-frequency oscillations (Lominadze and Stepanov, 1964a).

When the electrons move relative to the ions under the influence of an external electrical field with a velocity u, which is appreciably larger than the thermal electron velocity, it is also possible in a magnetic field to excite unstable longitudinal oscillations with a growth rate $\gamma \sim \sqrt[3]{(m_e/m_i)}\omega_{pe}$ (Stepanov, 1963a; Buneman, 1962).

If the drift velocity u of the electrons caused by the action of an electrical field along the magnetic field $\boldsymbol{B}_0$ lies in the interval $v_s \lesssim u < v_e$, it is possible to excite in a low-pressure plasma with hot electrons and cold ions ($T_e \gg T_i$) the slow magneto-sound waves and the cyclotron-sound waves which were studied in Subsection 5.5.1, while in a low-pressure plasma in the case of $T_i \gtrsim T_e$ longitudinal ion-cyclotron oscillations, which were studied in Subsection 5.6.1, are excited. The damping (growth) rates of SMS and CS oscillations and of ion-cyclotron oscillations are, as before, given by eqns. (5.5.1.14) to (5.5.1.16) and (5.6.1.11) and (5.6.1.12), in which it is necessary to multiply the quantity γ_e by $1-(k_z u/\omega(\boldsymbol{k}))$ (Drummond and Rosenbluth, 1962; Lominadze and Stepanov, 1965; Mikhaĭlovskiĭ and Pashitskiĭ, 1965).

6.2.3. INSTABILITY OF A MAGNETO-ACTIVE PLASMA IN THE FIELD OF A LOW-FREQUENCY ELECTROMAGNETIC WAVE

If a plasma is subject to a variable electrical field $\boldsymbol{E}$, for instance, the field of an ion-cyclotron or fast magneto-sound wave, which has a component at right angles to an external magnetic field $\boldsymbol{B}_0$, the electrons and ions will acquire different velocities $\boldsymbol{u}_e$ and $\boldsymbol{u}_i$ under the action of that field. If the frequency ω_0 of the field is considerably lower than the electron cyclotron frequency, the electron and ion velocities will be given by the equations

$$\boldsymbol{u}_e = c\frac{[\boldsymbol{E}\wedge\boldsymbol{B}_0]}{B_0^2} - i\frac{e(\boldsymbol{E}\cdot\boldsymbol{b})\boldsymbol{b}}{m_e\omega_0},$$

$$\boldsymbol{u}_i = \frac{e}{m_i(\omega_{Bi}^2-\omega_0^2)}\{\omega_{Bi}[\boldsymbol{E}\wedge\boldsymbol{b}] + i\omega_0[[\boldsymbol{E}\wedge\boldsymbol{b}]\wedge\boldsymbol{b}]\},$$

where $\boldsymbol{b} = \boldsymbol{B}_0/B_0$ is a unit vector in the direction of the external magnetic field. In a low-frequency field the component, $\boldsymbol{E}_\parallel = \boldsymbol{b}(\boldsymbol{E}\cdot\boldsymbol{b})$, of the electrical field parallel to the magnetic field $\boldsymbol{B}_0$ will be small, and the drift velocities of the electrons and ions in the plasma in directions perpendicular to the magnetic field $\boldsymbol{B}_0$ may be comparable with the electron velocity along the magnetic field or even larger than this velocity. We shall restrict ourselves to this case.

The appearance of a relative velocity of the electrons with respect to the ions, $\boldsymbol{u} = \boldsymbol{u}_i - \boldsymbol{u}_e$, may lead to the occurrence of a beam instability (Sagdeev, 1962; Kurilko and Miroshnichenko, 1963; Babykin, Zavoĭskiĭ, Rudakov and Skoryupin, 1962; Babykin, Gavrin, Zavoĭskiĭ, Rudakov, Skoryupin and Sholin, 1964; Zavoĭskiĭ, 1963; Stepanov, 1965; Sizonenko and Stepanov, 1967).

We shall assume that the frequency and the growth rate of the oscillations which appear are much larger than the ion-cyclotron frequency and that the wavelength is appreciably longer than the ion Larmor radius. In that case we may consider the motion of the ions in the field of the unstable high-frequency longitudinal oscillations, which appear, to be unmagnetized, that is, we can neglect the Lorentz force in the kinetic equation for the ions. We can, furthermore, assume that during the period when the instability of the oscillations is established the ion gas moves uniformly with respect to the electron gas with a constant velocity $\boldsymbol{u} = \boldsymbol{u}_i - \boldsymbol{u}_e$.

In the frame of reference which moves with the electrons the dispersion equation for the longitudinal oscillations has the form

$$A = 1 + \delta\varepsilon^{(e)} + \delta\varepsilon^{(i)} = 0, \tag{6.2.3.1}$$

where

$$\delta\varepsilon^{(e)} = \frac{\omega_{pe}^2}{k^2 v_e^2}\left[1 + i\sqrt{(\pi)}z_0 \sum_{l=-\infty}^{+\infty} A_l(a_e)\, w(z_l)\right],$$

$$\delta\varepsilon^{(i)} = \frac{\omega_{pi}^2}{k^2 v_i^2}\,[1 + i\sqrt{(\pi)}\, z_i w(z_i)],$$

$$z_l = \frac{\omega - l\,|\omega_{Be}|}{\sqrt{(2)}k_z v_e}, \quad z_i = \frac{\omega - (\boldsymbol{k}\cdot\boldsymbol{u})}{\sqrt{(2)}k v_i}.$$

We shall study this equation in a number of limiting cases.

(a) If the relative velocity is also much larger than the electron thermal velocity, the ion beam excites high-frequency, long-wavelength ($k_x \varrho_e \ll 1$) plasma oscillations with a large phase velocity ($|\omega/k_z| \gg v_e$) (Buneman, 1962; Stepanov, 1965). One can consider in that case both electrons and ions in the plasma to be "cold" and the expressions for $\delta\varepsilon^{(\alpha)}$ become

$$\delta\varepsilon^{(e)} = -\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta - \frac{\omega_{pe}^2 \sin^2\theta}{\omega^2 - \omega_{Be}^2}, \quad \delta\varepsilon^{(i)} = \frac{-\omega_{pi}^2}{[\omega - (\boldsymbol{k}\cdot\boldsymbol{u})]^2}.$$

Using these expressions we get from (6.2.3.1)

$$\omega = (\boldsymbol{k}\cdot\boldsymbol{u}) + \eta,$$

where

$$\eta = \pm\omega_{pi}\left[\frac{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)}{\omega^2(\omega^2 - \omega_{Be}^2)}\right]^{-1/2}_{\omega = (\boldsymbol{k}\cdot\boldsymbol{u})}. \tag{6.2.3.2}$$

These oscillations will be unstable, if $(\boldsymbol{k}\cdot\boldsymbol{u}) < \omega_-$ or if $|\omega_{Be}| < (\boldsymbol{k}\cdot\boldsymbol{u}) < \omega_+$. When $\omega_{pe} \sim |\omega_{Be}|$ we have as to order of magnitude

$$\gamma \sim (\boldsymbol{k}\cdot\boldsymbol{u}) \sqrt{\frac{m_e}{m_i}}.$$

In the resonance region $(\boldsymbol{k}\cdot\boldsymbol{u}) \approx \omega_\pm$ where the growth rate is a maximum we get instead of (6.2.3.2)

$$\eta = \xi^{(i)}\left[\frac{\omega_{pi}^2\,|\omega_\pm^2-\omega_{Be}^2|\,\omega_\pm}{2(\omega_+^2-\omega_-^2)}\right]^{1/3}. \tag{6.2.3.3}$$

We note that for the oscillations considered the condition $|z_0| \gg 1$ is satisfied when $\sin\theta \sim \cos\theta \sim 1$, if $u \gg v_e$. However, also when $u \lesssim v_e$—and even when $u \ll v_e$—the excitation of hydrodynamic oscillations at the second hybrid frequency is possible when the angle θ is close to $\pi/2$. In that case $\omega \ll |\omega_{Be}|, \omega_{pe}$, and eqns. (6.2.3.2) and (6.2.3.3) can be simplified:

$$\eta = \pm\omega_{pi}\left[\frac{\omega_{Be}^2(\boldsymbol{k}\cdot\boldsymbol{u})^2}{(\omega_{pe}^2+\omega_{Be}^2)\,[(\boldsymbol{k}\cdot\boldsymbol{u})^2-\omega_-^2]}\right]^{1/2}, \quad \eta = \xi^{(i)}\left[\frac{\omega_{pe}^2\omega_{Be}^2\omega_-}{2(\omega_{pe}^2+\omega_{Be}^2)}\right]^{1/3}, \tag{6.2.3.4}$$

where

$$\omega_- = \frac{\omega_{pe}\cos\theta}{\sqrt{[1+(\omega_{pe}^2/\omega_{Be}^2)]}}.$$

We can use these expressions when $\cos^2\theta \gg m_e/m_i$. If $\cos^2\theta \lesssim m_e/m_i$ the dispersion equation for the oscillations considered with frequency $|\omega| \ll |\omega_{Be}|$ is of the form

$$1+\frac{\omega_{pe}^2}{\omega_{Be}^2}-\frac{\omega_{pe}^2}{\omega^2}\cos^2\theta-\frac{\omega_{pi}^2}{[\omega-(\boldsymbol{k}\cdot\boldsymbol{u})]^2} = 0. \tag{6.2.3.5}$$

If $\cos^2\theta \sim m_e/m_i$ we get from this equation as to order of magnitude

$$\mathrm{Re}\,\omega \sim \mathrm{Im}\,\omega \sim (\boldsymbol{k}\cdot\boldsymbol{u}) \sim \frac{\omega_{pi}}{\sqrt{[1+(\omega_{pe}^2/\omega_{Be}^2)]}}. \tag{6.2.3.6}$$

In particular, in a dense plasma, when $\omega_{pe} > |\omega_{Be}|$,

$$\mathrm{Re}\,\omega \sim \mathrm{Im}\,\omega \sim (\boldsymbol{k}\cdot\boldsymbol{u}) \sim \sqrt{(|\omega_{Be}|\,\omega_{Bi})}.$$

(b) When $u \lesssim v_e$ it is also possible to excite oscillations due to their absorption (or emission) by resonance electrons (Stepanov, 1965). Assuming, as before, that the ion beam is cold, that is, $|z_i| \gg 1$, we get

$$\omega = (\boldsymbol{k}\cdot\boldsymbol{u})+\eta,$$

where

$$\eta = \pm\frac{\omega_{pi}}{\sqrt{[1+\delta\varepsilon^{(e)}(\boldsymbol{k}, (\boldsymbol{k}\cdot\boldsymbol{u}))]}}. \tag{6.2.3.7}$$

When $u \sim v_e$, $\omega_{pe} \sim |\omega_{Be}|$, and $\cos\theta \sim \sin\theta \sim 1$, we get from this

$$\mathrm{Re}\,\omega \sim \mathrm{Im}\,\omega \sim kv_s, \quad kr_D \lesssim 1.$$

The condition $|z_i| \gg 1$ is satisfied only if $T_e \gg T_i$. In particular, if $u \lesssim v_s \ll v_e$, we get the following expressions for the frequency and growth rate of ion-sound oscillations:

$$\omega = (\boldsymbol{k}\cdot\boldsymbol{u}) \pm \omega_s(k)$$

$$\gamma = \mathrm{Im}\,\eta = \frac{\sqrt{(\pi)}\omega\omega_s^2}{2k^2v_i^2} z_i \exp(-z_i^2) \mp \sqrt{\left(\frac{\pi m_e}{8m_i}\right)} \frac{A_0(k_x^2\varrho_e^2)}{|\cos\theta|}, \qquad (6.2.3.8)$$

where

$$\omega_s(k) = \frac{kv_s}{\sqrt{(1+k^2r_D^2)}}, \quad v_s = \sqrt{\frac{T_e}{m_e}}, \quad z_i = \pm\frac{\omega_s(k)}{\sqrt{(2)}kv_i}, \quad |z_i| \gg 1.$$

Ion-sound instability occurs if $u_\perp > u_c \sim 3v_i$. The growth rate reaches a maximum for $k\varrho_e \gg 1$, when $\omega \approx \pm k_z v_e(\cos\theta - u/v_e)$,

$$\gamma = \gamma_{\max} \sim \sqrt{(|\omega_{Be}|\,\omega_{Bi})}.$$

(c) In a plasma with hot ions and cold electrons the relative motion of ions with respect to electrons across the magnetic field can lead to the excitation of electron sound (Sizonenko and Stepanov, 1967; see Subsection 5.5.3). For those oscillations $k\varrho_e \ll 1$, $k_z v_e \ll |\omega| \ll kv_i$ so that the dispersion equation has the form

$$1 + \frac{\omega_{pe}^2}{\omega_{Be}^2} - \frac{\omega_{pe}^2}{\omega^2}\cos^2\theta + \frac{\omega_{pi}^2}{k^2v_i^2}(1 + i\sqrt{(\pi)}z_i) = 0.$$

When $\omega_{pe} \gg |\omega_{Be}|$ we get from this equation

$$\omega(k) = \frac{k_z v_{se}}{\sqrt{(1+k^2\varrho^2)}}, \quad \gamma(k) = \sqrt{\left(\frac{\pi m_i}{8m_e}\right)} \frac{\cos\theta[(\boldsymbol{k}\cdot\boldsymbol{u}_\perp) - \omega(k)]}{(1+k^2\varrho^2)^{3/2}}, \qquad (6.2.3.9)$$

where

$$v_{se} = \sqrt{\frac{T_i}{m_e}}, \quad \varrho = \frac{v_{se}}{|\omega_{Be}|}.$$

The condition $|\omega| \gg k_z v_e$ is satisfied when

$$1 + k^2\varrho^2 \ll \frac{T_i}{T_e},$$

and the condition $\omega_{Bi} \ll |\omega| \ll |\omega_{Be}|$, when

$$k_z\varrho_e \sqrt{\left[\frac{T_i}{T_e}(1+k^2\varrho^2)\right]} \ll 1, \quad k_z\varrho_i \sqrt{\left[\frac{m_i}{m_e}(1+k^2\varrho^2)\right]} \gg 1.$$

The electron-sound instability occurs when the following inequality holds:

$$k_\perp u_\perp \cos\varphi > \omega = \frac{k_z v_{se}}{\sqrt{(1+k^2\varrho^2)}},$$

where φ is the angle between the vectors $\boldsymbol{k}_\perp$ and $\boldsymbol{u}_\perp$. These oscillations grow most rapidly when $\varphi = 0$, $k\varrho = 1$, and $\cos\theta = u_\perp/\sqrt{(2)}v_{se}$. In that case the frequency and growth rate of the oscillations are equal to

$$\omega = \omega_m = \frac{u_\perp}{2v_i}\sqrt{(|\omega_{Be}|\,\omega_{Bi})}, \quad \gamma = \gamma_m = \frac{\sqrt{\pi}}{16\sqrt{2}}\left(\frac{u_\perp}{v_i}\right)^2\sqrt{(|\omega_{Be}|\,\omega_{Bi})}. \quad (6.2.3.10)$$

The ration $\gamma_m/\omega_m = 0.15u_\perp/v_i$ remains small when $u_\perp \sim v_i$ only because of the small numerical factor. We note that the electron-sound instability is also possible when $u_\perp \ll v_i$.

Equations (6.2.3.9) and (6.2.3.10) which we have obtained for the frequency and growth rate of electron-sound oscillations are applicable when $\gamma(\boldsymbol{k}) \ll \omega_0$. If the frequency of the low-frequency wave which causes the relative motion of the electrons with respect to the ions is of the order of the ion cyclotron frequency, this condition takes for $\gamma \sim \gamma_m$ the form

$$\left(\frac{u_\perp}{v_i}\right)^2 \gg 10\sqrt{\frac{m_e}{m_i}},$$

that is, the ratio $u_\perp/v_i$ cannot be very small.

(d) If the plasma is isothermal ($T_e \sim T_i$) the instability of the longitudinal oscillations also occurs when $u_\perp \sim v_i$ (Sizonenko and Stepanov, 1967[a]) and in that case the dispersion eqn. (6.2.3.1) becomes, when we use the inequality $|\omega| \ll |\omega_{Be}|$,

$$1+\frac{\omega_{pe}^2}{k^2v_e^2}\left[1+i\sqrt{(\pi)}\,z_0w(z_0)\,A_0(k^2\varrho_e^2)\right]+\frac{\omega_{pi}^2}{k^2v_i^2}\left[1+i\sqrt{(\pi)}\,z_iw(z_i)\right] = 0. \quad (6.2.3.11)$$

In the case of long-wavelength oscillations ($kr_D \ll 1$ and $k\varrho_e \ll 1$) we can in that equation neglect unity in comparison with $(\omega_{p\alpha}/kv_\alpha)^2$ and put $A_0 = 1$. Putting the imaginary and real parts of this equation equal to zero for $\varphi = 0$ and $\gamma = 0$ we get two equations determining the limits of the stability region $u_\perp = u_c$ (the plasma is unstable when $u_\perp > u_c$):

$$z_0\exp(-z_0^2)+\frac{T_e}{T_i}z_i\exp(-z_i^2) = 0,$$

$$1-2z_0\int_0^{z_0}dt\exp(t^2-z_0^2)+\frac{T_e}{T_i}\left(1-2z_i\int_0^{z_i}dt\exp(t^2-z_i^2)\right) = 0. \quad (6.2.3.12)$$

When $T_e = T_i$ we find from the first of these equations that $z_0 = -z_i$, after which we get from the second eqn. (6.2.3.12)

$$1-z_0\int_0^{z_0}dt\exp(t^2-z_0^2) = 0,$$

whence: $z_0 \approx 0.9$. The limiting value of the velocity is thus equal to $u_c = \sqrt{(2)}z_0v_i \approx 1.3v_i$ when $T_e = T_i$.

If $T_e \sim T_i$ and $u_\perp > u_c$, so that $|u_\perp - u_c| \sim u_c$, we have as to order of magnitude

$$\operatorname{Re}\omega \sim \gamma \sim (\boldsymbol{k}_\perp\cdot\boldsymbol{u}_\perp) \sim \sqrt{(|\omega_{Be}|\,\omega_{Bi})}, \quad \omega_{pe} \gtrsim |\omega_{Be}|.$$

6.2.4. THE EXCITATION OF FAST MAGNETO-SOUND AND ALFVÉN WAVES BY ELECTRON AND ION BEAMS

Electron and ion beams may excite not only slow longitudinal plasma oscillations but also any other slow wave. We recall that in a plasma consisting of electrons and one kind of ions the slow waves are Alfvén and fast magneto-sound (FMS) waves.

(a) In the high-frequency region, $\omega \gg \omega_{Bi}$, the FMS branch is a purely electron branch in a plasma with a high density and its refractive index and frequency are given by the equations

$$n^2 = \frac{\omega_{pe}^2}{\omega(|\omega_{Be}|\cos\theta - \omega)}, \quad \omega(k, \theta) = \frac{|\omega_{Be}|\cos\theta||}{1+r}, \tag{6.2.4.1}$$

where $r = (\omega_{pe}/kc)^2$ (the angle θ is assumed to be not too close to $\pi/2$). We recall that in the frequency interval $\omega_{Bi} \ll |\omega| \ll |\omega_{Be}|$ $(r \gg 1)$ this branch is called that of the atmospheric whistlers.

We study the excitation of oscillations with frequency (6.2.4.1) by a low-density electron beam (Kitsenko and Stepanov, 1961a). Assuming both the plasma and the beam to be cold and using eqn. (5.1.1.5) for the dielectric permittivity tensor of the plasma and eqn. (6.2.1.3) for the tensor $\varepsilon_{ij}^{(b)}$ we get from the dispersion eqn. (5.2.2.5) under Cherenkov resonance conditions, $\omega(k, \theta) = k_z u$,

$$\gamma = \frac{\sqrt{3}}{2^{4/3}} \left(\frac{n_0'}{n_0} \frac{\sin^2\theta}{1+r}\right)^{1/3} \omega(k, \theta). \tag{6.2.4.2}$$

For cyclotron resonance, $\omega(k, \theta) = k_z u - |\omega_{Be}|$, the growth rate is equal to

$$\gamma = \sqrt{\left(\frac{n_0'}{n_0}\right)} \frac{(1+r+\cos\theta)(1-\cos\theta)|\cos\theta - r|}{2(1+r)(\cos\theta)^{3/2}} \omega(k, \theta). \tag{6.2.4.3}$$

We note that the growth rates (6.2.4.2) and (6.2.4.3) are of the same order of magnitude when $r \sim 1$ as the growth rates (6.2.2.10) and (6.2.2.12) of the longitudinal plasma oscillations at the second hybrid frequency $\omega = \omega_- \approx |\omega_{Be}| \cos\theta$. This fact is not surprising as for $r \ll 1$ the FMS wave becomes longitudinal and its frequency (6.2.4.2) tends to the second hybrid resonance frequency as $r \to 0$.

When an FMS wave is excited by an electron beam with a large thermal velocity spread the growth rates are given by the following formulae:

for Cherenkov resonance, $\omega(k, \theta) \approx k_z u$,

$$\frac{\gamma}{\omega} = -\frac{\sqrt{(\pi)}\sin^2\theta\, \omega^2}{2k_z^2 v_e'^2(1+r)} \frac{n_0'}{n_0} z_0 \exp(-z_0^2), \tag{6.2.4.4}$$

where

$$z_0 = \frac{\omega(k, \theta) - k_z u}{\sqrt{(2)} k_z v_e'};$$

for cyclotron resonance under anomalous Doppler effect conditions, $\omega(k, \theta) \approx k_z u - l|\omega_{Be}|$,

$$\frac{\gamma}{\omega} = \frac{\sqrt{(\pi)}\,(a_e')^{l-1}\, l\,|\omega_{Be}|}{2^{(l+3)/2} l!\, k v_e'}\;\frac{(1-\cos\theta)^2\,[lr(1+r)-\cos\theta-\cos^2\theta]^2}{\cos^3\theta(1+r)^3}\;\frac{n_0'}{n_0}\exp(-z_l^2), \quad (6.2.4.5)$$

where

$$z_l = \frac{\omega(k, \theta)-k_z u+l\,|\omega_{Be}|}{\sqrt{(2)}k_z v_e'}.$$

These expressions were obtained when $k\varrho_e' \ll 1$. If $k\varrho_e' \gtrsim 1$, one can easily obtain a general expression for the growth rate which is analogous to eqn. (6.2.2.4) for the growth rate of longitudinal oscillations.

We note that when the frequency is lowered the growth rates (6.2.4.2) to (6.2.4.4) decrease, that is, the excitation of an FMS wave by an electron beam is most effective in the high-frequency region when $\omega \sim |\omega_{Be}|$; whistlers, $\omega \ll |\omega_{Be}|$, are weakly excited (Kitsenko and Stepanov, 1961a; Kovner, 1961a, c).

(b) We now consider the excitation of plasma oscillations by beams of fast ions with a large thermal velocity spread (Krasovitskiĭ and Stepanov, 1963). It follows from the resonance condition $\omega \approx k_z u - l|\omega_{Bi}|$, $l = 0, 1, \ldots$, that the frequency $\omega(k, \theta)$ of the excited wave can be much higher than the ion-cyclotron frequency when the beam velocity is large. The resonance condition can thus be satisfied for large numbers of the harmonics ($l \gg 1$). The cyclotron emission of the beam ions, leading to instability, has in this case a quasi-continuous spectrum.

We shall consider the case when the spectrum of the emission by the ions lies in the frequency region $\omega_{Bi} \ll \omega \ll |\omega_{Be}|$. In this case, in a high-density plasma electromagnetic waves (whistlers) with frequency

$$\omega(k, \theta) = \frac{|\omega_{Be}|\cos\theta}{r}, \qquad r = \frac{\omega_{pe}^2}{k^2c^2} \gg 1,$$

will be excited. We shall further assume that the wavelength of the excited waves is much shorter than the Larmor radius of the beam ions ($k\varrho_i' = k_i'/\omega_{Bi} \gg 1$). We can then neglect the effect of the magnetic field on the beam ions and use eqns. (6.1.1.2) for the components of the beam dielectric permittivity tensor which are valid when $\boldsymbol{B}_0 = 0$. The anti-Hermitean parts of the components of the tensor $\varepsilon_{ij}^{(b)}$ responsible for the excitation of atmospheric whistlers have the form

$$\varepsilon_{11}^{(b)\prime\prime} = \varepsilon_{22}^{(b)\prime\prime}(1+2z^2\tan^2\theta)\cos^2\theta = \left(\frac{\omega_{pi}'}{\omega}\right)^2\sqrt{(\pi)}z_i\exp(-z_i^2)(\cos^2\theta+2z^2\sin^2\theta), \quad (6.2.4.6)$$

where $z_i = (\omega(k, \theta)-k_z u)/\sqrt{(2)}kv_i'$. The other components $\varepsilon_{ij}^{(b)\prime\prime}$ do not exceed (6.2.4.6), but we can neglect them in the dispersion equation as they are multiplied by the small factor $\omega/|\omega_{Be}|$.

Substituting these expressions into the dispersion eqn. (5.2.2.5) and using the fact that $|\varepsilon_{33}| = |\varepsilon_3| \gg |\varepsilon_2| \gg |\varepsilon_1|$, where the ε_{ij} are given by eqn. (5.1.1.5), we get for the growth rate of the oscillations the expression

$$\frac{\gamma}{\omega} = -\frac{\omega^2}{2k_z^2c^2}(\varepsilon_{11}^{(b)\prime\prime}+\varepsilon_{22}^{(b)\prime\prime}\cos^2\theta), \tag{6.2.4.7}$$

or

$$\frac{\gamma}{\omega} = -\sqrt{(\pi)}\frac{n_0'}{n_0}\frac{\omega_{Bi}\cos\theta}{\omega}(1+z_i^2\tan^2\theta)z_i\exp(-z_i^2). \tag{6.2.4.7'}$$

The growth rate (6.2.4.7') is proportional to the small parameter n_0'/n_0—as in all other cases of the excitation of slow waves by a hot beam—and also to a second small parameter, ω_{Bi}/ω.

If $u \gg v_i'$, the growth rate (6.2.4.7') is exponentially small, if $k_z u$ is not close to $\omega(k, \theta)$. Resonance occurs for $k = k_0$, where $k_0 = u\omega_{pe}^2/c^2\omega_{Be}^2$. The growth rate reaches a maximum for a given θ, if $z_i = -z_m$, where

$$z_m^2 = \tfrac{1}{2}\cot^2\theta[\sqrt{\{(2-3\tan^2\theta)^2+8\tan^2\theta\}}-2+3\tan^2\theta].$$

The maximum growth rate is equal to

$$\gamma_m = \sqrt{(\pi)}\omega_{Bi}\frac{n_0'}{n_0}f(\theta), \tag{6.2.4.8}$$

where

$$f(\theta) = \cos\theta(1+z_m^2\tan^2\theta)z_m\exp(-z_m^2).$$

When $\theta = 0$ we have $z_m = \frac{1}{2}$ and $f(0) = 0.43$. If $\theta = \pi/2$, $z_m = \sqrt{3}$ and $f(\pi/2) = 0.851/\cos\theta \gg 1$. The maximum growth rate thus increases as θ approaches $\pi/2$. However, one can only use the expressions given here for the growth rates, like the expression $\omega(k, \theta) = [|\omega_{Be}|\cos\theta]/r$ for the frequency of the atmospheric whistlers, when θ is not too close to $\pi/2$.

Let us therefore consider separately the excitation of waves with frequencies $\omega_{Bi} \ll \omega \ll |\omega_{Be}|$ by an ion beam for the case when $\theta \approx \pi/2$. As before we assume that $\omega_{pe} \gg |\omega_{Be}|$ and we get the following expression for the frequency of an FMS wave:

$$\omega^2(k, \theta) = \frac{\omega_{pe}^2\left[\frac{m_e}{m_i}(1+r)+\cos^2\theta\right]}{(1+r)\left[1+(1+r)\frac{\omega_{pe}^2}{\omega_{Be}^2}\right]}; \tag{6.2.4.9}$$

from this equation we get for $r \gg 1$ and $\cos^2\theta \gg (m_e/m_i)r$ the expression for the atmospheric whistler frequency.

The growth rate of the oscillations with frequency (6.2.4.9) is equal to

$$\frac{\gamma}{\omega} = -\frac{\omega^2}{2k^2c^2}\frac{(1+r)\varepsilon_{11}^{(b)\prime\prime}}{r\left[\frac{m_e}{m_i}(1+r)+\cos^2\theta\right]}, \tag{6.2.4.10}$$

where $\varepsilon_{11}^{(b)\prime\prime}$ is given by formula (6.2.4.6).

If $(m_e/m_i)(1+r) \ll \cos^2\theta$, this formula is the same as formula (6.2.4.7′), if we neglect in (6.2.4.7) $\varepsilon_{22}^{(b)''}$ in comparison with $\varepsilon_{11}^{(b)''}/\cos^2\theta$.

If $\cos^2\theta \sim m_e/m_i$, $r \sim 1$, and $k \sim k_0$, the maximum growth rate is, as to order of magnitude, equal to

$$\gamma_m \sim \frac{n_0'}{n_0}\sqrt{\left(\frac{m_i}{m_e}\right)}\omega_{Bi} = \frac{n_0'}{n_0}\sqrt{(|\omega_{Be}|\,\omega_{Bi})}.$$

When $\cos^2\theta \lesssim (m_e/m_i)(1+r)$ and $\omega_{pe} \gtrsim |\omega_{Be}|$ we have as to order of magnitude

$$\omega(k,\theta) \sim \sqrt{(|\omega_{Be}|\,\omega_{Bi})}, \quad \gamma \sim \frac{n_0'}{n_0}\omega z^3 e^{-z^2}. \tag{6.2.4.11}$$

The growth rate reaches a maximum when $k \sim k_0 \sim \sqrt{(|\omega_{Be}|\,\omega_{Bi})}/u\cos\theta$,

$$\gamma_m \sim \frac{n_0'}{n_0}\omega \sim \frac{n_0'}{n_0}\sqrt{(|\omega_{Be}|\,\omega_{Bi})}. \tag{6.2.4.12}$$

The growth rate of the high-frequency part of the FMS branch thus reaches a maximum for $\cos^2\theta \lesssim m_e/m_i$. In that case γ_m is larger by a factor $\sqrt{(m_e/m_i)}$ than the growth rate of the oscillations for the case when $\cos\theta \sim 1$.

(c) Let us now study the excitation of low-frequency ($\omega \sim \omega_{Bi}$) Alfvén and fast magnetosound waves by electron and ion beams, moving along the magnetic field through a low-pressure plasma with a velocity of the order of the Alfvén velocity $u \sim v_A$ (Kitsenko and Stepanov, 1961a; Kovner, 1961c). The dispersion equation for these waves in a low-pressure plasma ($v_A \gg v_i$, v_s) when there is a beam present has the form

$$\cos^2\theta\, n^4 - (\varepsilon_{11} + \varepsilon_{22}\cos^2\theta)\, n^2 + \varepsilon_{11}\varepsilon_{22} + \varepsilon_{12}^2 = 0; \tag{6.2.4.13}$$

in obtaining this equation from the general dispersion eqn. (5.2.2.5) we used the fact that the component ε_{33} is much larger than the other components of the tensor ε_{ij}.

From this equation we get when there is no beam

$$\omega^2(k,\theta) = \tfrac{1}{2}k^2v_A^2[1+\cos^2\theta+q\cos^2\theta \pm \sqrt{\{(1+\cos^2\theta+q\cos^2\theta)^2 - 4\cos^2\theta\}}], \tag{6.2.4.14}$$

where $q = (kc/\omega_{pi})^2$.

If the particle beam is monoenergetic, we can use expression (6.2.1.3) for the components of the beam dielectric permittivity tensor, and we get under cyclotron resonance conditions, $\omega(k,\theta) = k_z u - \omega_{Bi}$

$$\gamma = \left\{\frac{\omega_{p\alpha}'^2\,|\omega_{B\alpha}|}{4\omega Q}\left[(1+\cos^2\theta)\frac{k^2c^2}{\omega^2} - \frac{2\omega_{pi}^2}{\omega_{Bi}(\omega_{Bi}\mp\omega)}\right]\right\}^{1/2}, \tag{6.2.4.15}$$

where

$$Q = \frac{\omega_{pi}^4}{(\omega^2-\omega_{Bi}^2)^2}\left[(1+\cos^2\theta)\frac{k^2v_A^2}{\omega^2} + \frac{\omega^2}{\omega_{Bi}^2} - 2\right].$$

The upper (lower) sign is taken in eqn. (6.2.4.15) if the resonance occurs for an ion (electron) beam.

If the velocity spread in the beam is large, the growth rate is under Cherenkov resonance conditions equal to

$$\gamma = -\frac{\sqrt{(\pi)}\omega_{p\alpha}'^2 a_\alpha'}{\omega Q}\left(\frac{k_z^2c^2}{\omega^2}+\frac{\omega_{pi}^2}{\omega^2-\omega_{Bi}^2}\right) z_0 \exp(-z_0^2). \qquad (6.2.4.16)$$

For cyclotron resonance for the beam particles, $\omega(k, \theta) \approx k_z u - l|\omega_{B\alpha}|$, $l = 1, 2, \ldots$, we have

$$\gamma = -\frac{\sqrt{(\pi)}\omega_{p\alpha}'^2 l^2 a_\alpha'^{l-1}}{2^{l+1}l!\,\omega Q}\left[(1+\cos^2\theta)\frac{k^2c^2}{\omega^2}-\frac{2\omega_{pi}^2}{\omega(\omega_{Bi}\mp\omega)}\right] z_0 \exp(-z_l^2),$$

where

$$z_l = \frac{\omega(k, \theta)-k_z u+l\,|\omega_{B\alpha}|}{\sqrt{(2)}k_z v_\alpha'}.$$

As $a_\alpha' = (k\varrho_\alpha')^2 \ll 1$, cyclotron excitation by resonance beam particles occurs more strongly under one-fold resonance conditions, $\omega = k_z u - |\omega_{B\alpha}|$. We note that when longitudinal long-wavelength oscillations ($k\varrho_\alpha' \ll 1$) are excited the growth rate is a maximum in the case of Cherenkov resonance, $\omega \approx k_z u$.

6.3. Excitation of Electromagnetic Waves in a Plasma by Oscillator Beams

6.3.1. DIELECTRIC PERMITTIVITY TENSOR OF AN OSCILLATOR BEAM-PLASMA SYSTEM

In the preceding two sections we have studied the stability of the beam–plasma system assuming that the velocity distributions in the plasma and in the beam were Maxwellian. We shall now assume that the particle distribution in the plasma is Maxwellian, as before, but that the particle distribution in the beam is appreciably different from a Maxwellian one.

We shall assume that when there are no oscillations the beam particles have the same Larmor radius — such particles will be called oscillators† — and have a Maxwell velocity distribution along the magnetic field:

$$f_{\alpha 0}' = \frac{n_0}{(2\pi)^{3/2}v_\alpha' v_\perp}\,\delta(v_\perp - v_{\perp 0})\exp\left(-\frac{(v_\parallel-u)^2}{2v_\alpha'^2}\right), \qquad (6.3.1.1)$$

where $v_\alpha' = \sqrt{(T_\alpha'/m_\alpha)}$, and T_α' is the "longitudinal beam temperature".

† Malmfors (1950) indicated the possibility that there could occur an instability in a plasma in which the electrons have the same Larmor radius; Sen (1952) obtained a criterion for the instability for such a plasma. A number of authors (see, for instance, Gaponov, 1961; Zheleznyakov, 1960 a, b, 1961 a, b; Kitsenko and Stepanov, 1961 a, b, 1963; Shevchenko, 1963; Petelin, 1961; Harris, 1961; Pistunovich, 1963; Krasovitskiĭ and Stepanov, 1964; Timofeev and Pistunovich, 1970) subsequently studied the stability of a plasma in which there were – electron and ion – oscillators.

The dielectric permittivity tensor of the oscillator beam–plasma system can be put in the form

$$\varepsilon_{ij} = \varepsilon_{ij}^{(p)} + \varepsilon_{ij}^{(b)},$$

where $\varepsilon_{ij}^{(p)}$ is the dielectric permittivity tensor of the plasma when there is no beam and $\varepsilon_{ij}^{(b)}$ is the dielectric permittivity tensor of the oscillator beam. For the case of a distribution function of the form (6.3.1.1) the components of the tensor $\varepsilon_{ij}^{(b)}$ have the form (Kitsenko and Stepanov, 1961b, 1963)

$$\varepsilon_{11}^{(b)} = -\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left(1-\frac{\lambda^2}{2a'_\alpha}\right) + \sum_{l=-\infty}^{+\infty}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left[\frac{z'_l}{a'_\alpha}\, l^2 J_l^2 + \frac{\omega_{B\alpha} y'_0}{\omega\lambda}\, 2l^3 J_l J'_l\right] i\sqrt{(\pi)}w(z'_l),$$

$$\varepsilon_{22}^{(b)} = -\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left(1-\frac{\lambda^2}{2a'_\alpha}\right) + \sum_{l=-\infty}^{+\infty}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left[\frac{\lambda^2 z'_l}{a'_\alpha}\, J_l'^2 + \frac{l\omega_{B\alpha} y'_0}{\omega\lambda}\,(\lambda^2 J_l'^2)'\right] i\sqrt{(\pi)}w(z'_l),$$

$$\varepsilon_{33}^{(b)} = \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left(2y_0'^2 + \frac{\lambda^2}{2a'_\alpha}\tan^2\theta - \tan^2\theta\right)$$
$$+ \sum_{l=-\infty}^{+\infty}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2 y_l'^2\left[z'_l J_l^2 + \frac{2a'_\alpha y'_0\omega_{B\alpha}}{\lambda\omega}\, l J_l J'_l\right] 2i\sqrt{(\pi)}w(z'_l), \qquad (6.3.1.2)$$

$$\varepsilon_{12}^{(b)} = -\sum_{l=-\infty}^{+\infty}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left[\frac{l^2\omega_{B\alpha} y'_0}{\omega\lambda}\,(\lambda J_l J'_l)_l^2 + \frac{l\lambda z'_l}{a'_\alpha}\, J_l J'_l\right]\sqrt{(\pi)}w(z'_l),$$

$$\varepsilon_{13}^{(b)} = \left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\left(1-\frac{\lambda^2}{2a'_\alpha}\right)\tan\theta + \sum_{l=-\infty}^{+\infty}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\tan\theta\,\frac{\omega}{\omega_{B\alpha}}\, y'_l\left[\frac{2l^2\omega_{B\alpha}}{\omega\lambda}\, J_l J'_l + \frac{l z'_l}{y'_0 a'_\alpha}\, J_l^2\right] i\sqrt{(\pi)}w(z'_l),$$

$$\varepsilon_{23}^{(b)} = \sum_{l=-\infty}^{\infty}\left(\frac{\omega'_{p\alpha}}{\omega}\right)^2\tan\theta\, y_l\left[\frac{l}{\lambda}\,(\lambda J_l J'_l)' + \frac{\lambda z'_l\omega}{\omega_{B\alpha} a'_l y'_0}\, J_l J'_l\right[\sqrt{(\pi)}w(z'_l),$$

where $J_l \equiv J_l(\lambda)$ is a Bessel function, $J'_l = dJ_l/d\lambda$, and

$$\lambda = \frac{k_x v_{\perp 0}}{\omega_{B\alpha}}, \quad a'_\alpha = \left(\frac{k_x v'_\alpha}{\omega_{B\alpha}}\right)^2, \quad y'_l = \frac{\omega - l\omega_{B\alpha}}{\sqrt{(2)}k_z v'_\alpha}, \quad z'_l = \frac{\omega - l\omega_{B\alpha} - k_z u}{\sqrt{(2)}k_z v'_\alpha}.$$

In the case of a cold beam ($T_\alpha \to 0$) eqns. (6.3.1.2) can be simplified:

$$\varepsilon_{11}^{(b)} = -\frac{\omega_{p\alpha}'^2}{\omega^2} - \sum_{l=-\infty}^{+\infty}\frac{\omega_{p\alpha}'^2}{\omega^2}\left[\frac{\omega_{B\alpha}^2\cot^2\theta\, l^2 J_l^2}{(\omega - l\omega_{B\alpha} - k_z u)^2} + \frac{2\omega_{B\alpha} l^3 J_l J'_l}{\lambda(\omega - l\omega_{B\alpha} - k_z u)}\right],$$

$$\varepsilon_{22}^{(b)} = -\frac{\omega_{p\alpha}'^2}{\omega^2} - \sum_{l=-\infty}^{+\infty} \frac{\omega_{p\alpha}'^2}{\omega^2}\left[\frac{\omega_{B\alpha}^2 \cot^2\theta\, \lambda^2 J_l'^2}{(\omega - l\omega_{B\alpha} - k_z u)^2} + \frac{\omega_{B\alpha} l(\lambda^2 J_l'^2)'}{\lambda(\omega - l\omega_{B\alpha} - k_z u)}\right],$$

$$\varepsilon_{33}^{(b)} = -\frac{\omega_{p\alpha}'^2}{\omega^2}\tan^2\theta - \sum_{l=-\infty}^{+\infty} \frac{\omega_{p\alpha}'^2}{\omega^2}\left[\frac{(\omega - l\omega_{B\alpha})^2 J_l^2}{(\omega - l\omega_{B\alpha} - k_z u)^2} + \frac{2\tan^2\theta(\omega - l\omega_{B\alpha}) l J_l J_l'}{\lambda\omega_{B\alpha}(\omega - l\omega_{B\alpha} - k_z u)}\right],$$

$$\varepsilon_{12}^{(b)} = -i\sum_{l=-\infty}^{+\infty} \frac{\omega_{p\alpha}'^2}{\omega^2}\left[\frac{\omega_{B\alpha}^2 \cot^2\theta\, l\lambda J_l J_l'}{(\omega - l\omega_{B\alpha} - k_z u)^2} + \frac{\omega_{B\alpha} l^2(\lambda J_l J_l')'}{\lambda(\omega - l\omega_{B\alpha} - k_z u)}\right],$$

$$\varepsilon_{13}^{(b)} = \frac{\omega_{p\alpha}'^2}{\omega^2}\tan^2\theta - \sum_{l=-\infty}^{+\infty} \frac{\omega_{p\alpha}'^2}{\omega^2}\left[\frac{\omega_{B\alpha}(\omega - l\omega_{B\alpha})\cot\theta\, l J_l^2}{(\omega - l\omega_{B\alpha} - k_z u)^2} + \frac{2(\omega - l\omega_{B\alpha})\tan\theta\, l^2 J_l J_l'}{\lambda(\omega - l\omega_{B\alpha} - k_z u)}\right],$$

$$\varepsilon_{23}^{(b)} = i\sum_{l=-\infty}^{+\infty} \frac{\omega_{p\alpha}'^2}{\omega^2}\left[\frac{\omega_{B\alpha}(\omega - l\omega_{B\alpha})\cot\theta\lambda J_l J_l'}{(\omega - l\omega_{B\alpha} - k_z u)^2} + \frac{(\omega - l\omega_{B\alpha})\tan\theta\, l(\lambda J_l J_l')'}{\lambda(\omega - l\omega_{B\alpha} - k_z u)}\right].$$

The excitation of plasma oscillations by oscillators possesses a few important characteristic features:

Firstly, while for a hot beam with an isotropic distribution function the excitation of oscillations by resonance particles took place under anomalous Doppler effect conditions ($\omega/k_z < u$) for the Cherenkov and cyclotron resonance, while the cyclotron resonance for beam particles under normal Doppler effect conditions led to the damping of the oscillations, for a hot oscillator beam the excitation of oscillations by resonance particles can occur also for cyclotron resonance under normal Doppler effect conditions, and damping of oscillations for Cherenkov resonance and cyclotron resonance under anomalous Doppler effect conditions. The anti-Hermitean parts of the tensor $\varepsilon_{ij}^{(b)}$ for an oscillator beam — in contrast to a particle beam with a Maxwell distribution function — is not proportional to the quantity $z_0' = (\omega - k_z u)/\sqrt{(2)}k_z v_\alpha'$ and can change sign when the quantity $\lambda = k_x \varrho_\alpha'$ is changed: the sign of the anti-Hermitean terms in $\varepsilon_{ij}^{(b)}$ responsible for the damping of excitation of the oscillations is determined both by the sign of the quantity z_l' and by the sign of the quantities $J_l J_l'$ and $(\lambda J_l J_l')'$.

Secondly, it is clear from eqns. (6.3.1.3) that the quantities $\varepsilon_{ij}^{(b)}$ contain resonance denominators proportional to $(\omega - l|\omega_{B\alpha}| - k_z u)^2$, where $l = 0, \pm 1, \pm 2, \ldots$, while in the case of a beam with a Maxwell distribution the resonance denominators only contain the term with $l = 0$. The growth rate of the plasma eigenoscillations with frequency $\omega(\boldsymbol{k})$ under resonance conditions $\omega(\boldsymbol{k}) \approx k_z u + l|\omega_{B\alpha}|$, $l = 0, \pm 1, \ldots$, for an oscillator beam is thus proportional to $(n_0')^{1/3}$ as in the case of Cherenkov resonance for an isotropic beam; we recall that for cyclotron resonance, $\omega(\boldsymbol{k}) \sim k_z' u - |\omega_{B\alpha}|$ the growth rate of the oscillations for an isotropic beam is proportional to $\sqrt{n_0'}$.

The cyclotron excitation of plasma oscillations by an oscillator beam is thus stronger than the excitation by a beam of particles with a Maxwell velocity distribution.

Finally, an oscillator beam is unstable by itself (when there is no plasma) while a beam of particles with a Maxwellian velocity distribution is, of course, stable when there is no plasma.

6.3.2. EXCITATION OF LONGITUDINAL HIGH-FREQUENCY OSCILLATIONS BY AN ELECTRON OSCILLATOR BEAM

The dispersion equation for high-frequency longitudinal oscillations has the form

$$A = 1+\delta\varepsilon^{(p)}(\boldsymbol{k}, \omega)+\delta\varepsilon^{(b)}(\boldsymbol{k}, \omega), \tag{6.3.2.1}$$

where the quantity $\delta\varepsilon^{(p)}$ is given by eqn. (6.2.2.2) while the quantity $\delta\varepsilon^{(b)}$ is equal to

$$\delta\varepsilon^{(b)} = \left(\frac{\omega'_{p\alpha}}{kv'_\alpha}\right)^2 \left[1+i\sqrt{\pi} \sum_{l=-\infty}^{+\infty} \left(z'_l J_l^2 + \frac{2l\,|\omega_{B\alpha}|}{\omega} a'_\alpha y'_0 J_l J'_l\right) w(z'_l)\right]. \tag{6.3.2.2}$$

If the thermal spread in longitudinal velocities in the oscillator beam is small,

$$\delta\varepsilon^{(b)} = -\sum_{l=-\infty}^{+\infty} \omega'^2_{p\alpha} \left[\frac{\cos^2\theta J_l^2}{(\omega-l\,|\omega_{B\alpha}|-k_z u)^2} + \frac{2\sin^2\theta J_l J'_l}{\lambda\,|\omega_{B\alpha}|\,(\omega-l\,|\omega_{B\alpha}|-k_z u)}\right]. \tag{6.3.2.3}$$

In the limiting case of a small Larmor radius ($\lambda \to 0$) this expression goes over into the expression for $\delta\varepsilon^{(b)}$ for a cold monoenergetic beam.

In the case of a low-density oscillator beam one can easily study the dispersion eqn. (6.3.2.1) in the same way as was done in the preceding subsection. We shall consider a few limiting cases (Kitsenko and Stepanov, 1961b, 1963; Shevchenko, 1963).

(a) Let us first of all consider the excitation of high-frequency longitudinal oscillations of a cold plasma ($\omega = \omega_\infty^{(1,2)}(\theta) = \omega_\pm$) by an oscillator beam with a large spread of velocities along the magnetic field. The growth rate of the oscillations is in that case given by the expression

$$\gamma = \sum_{l=-\infty}^{+\infty} \gamma_l, \tag{6.3.2.4}$$

where

$$\gamma_l = -\frac{\sqrt{\pi}}{2}\left(\frac{\omega'_{p\alpha}}{kv'_\alpha}\right)^2 \left[z'_l J_l^2 + \frac{2l\,|\omega_{B\alpha}|\,a'_\alpha y'_0}{\omega\lambda} J_l J'_l\right] \frac{|\omega_\pm^2-\omega_{Be}^2|\,\omega_\pm}{\omega_+^2-\omega_-^2} \exp(-z_l'^2)$$

$$z_l = \frac{\omega_\pm - l\,|\omega_{B\alpha}|-k_z u}{\sqrt{(2)}k_z v'_\alpha}. \tag{6.3.2.5}$$

If the quantity v'_e is sufficiently small the main contribution to the sum (6.3.2.4) is given by the term by the smallest $|z'_l|$, so that $\gamma \approx \gamma_l$. As the quantity γ_l is proportional to the expression

$$z'_l J_l^2 + \frac{2l\,|\omega_{Be}|\,a'_e y'_0}{\omega\lambda} J_l J'_l,$$

which, depending on the parameters occurring in it, can have different signs, both cyclotron excitation and cyclotron damping, both under normal and under anomalous Doppler effect conditions can occur for $\omega \approx l\,|\omega_{Be}|+k_z u$ and $u \gg v'_e$. For instance, when $z'_l = 0$

excitation of the oscillations under normal Doppler effect conditions ($l > 0$) occurs when $J_l J_l' < 0$, and damping when $J_l J_l' > 0$, while under anomalous Doppler effect conditions ($l < 0$) the oscillations are excited when $J_l J_l' > 0$ and damped when $J_l J_l' < 0$.

(b) When oscillations in a cold plasma are excited by a monoenergetic oscillator beam we must retain in eqn. (6.3.2.3) only the term proportional to $(\omega - l|\omega_{B\alpha}| - k_z u)^{-2}$. Putting in that case

$$\omega = k_z u + l|\omega_{Be}| + \eta, \qquad l = 0, \pm 1, \pm 2, \ldots,$$

where $|\eta| \ll |\omega|$, we find that

$$\eta = \pm \omega_{pe}' \cos\theta\, |J_l(\lambda)| \left[\frac{\omega^2(\omega^2 - \omega_{Be}^2)}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)}\right]^{1/2}_{\omega = k_z u + l|\omega_{Be}|} \tag{6.3.2.6}$$

Instability occurs when $\omega < \omega_-$ for $|\omega_{Be}| < \omega < \omega_+$.

Under resonance conditions, $\omega = k_z u + l|\omega_{Be}| = \omega_\pm$ we get instead of (6.3.2.6)

$$\eta = \xi^{(i)} \left[\frac{\omega_{pe}' \cos^2\theta J_l^2 |\omega_\pm^2 - \omega_{Be}^2| \omega_\pm}{2(\omega_+^2 - \omega_-^2)}\right]^{1/3}. \tag{6.3.2.7}$$

These expressions are applicable only provided the angle θ lies not too close to $\pi/2$ so tha the conditions

$$\frac{\cos^2\theta J_l^2}{|\eta|} \gg \frac{\sin^2\theta\, |J_l J_l'|}{|\lambda\omega_{Be}|}$$

are satisfied. This inequality is also violated as $\lambda \to 0$ for $l = \pm 1$ and $l = \pm 2$. In that case expressions (6.3.2.7) are applicable only when $\lambda^2 \gg |\gamma/\omega_{Be}|$. If, however, $\lambda^2 \ll |\gamma/\omega_{Be}|$, the oscillations can grow only for resonance with $l = -1$; in this case the growth rate of the oscillations is given by eqn. (6.2.2.12).

The maximum growth rate of the plasma oscillations under cyclotron resonance conditions for an oscillator beam is thus, indeed, proportional to $(n_0')^{1/3}$, as in the case of Cherenkov resonance.

(c) In the case of a monoenergetic oscillator beam passing through a plasma with a large thermal velocity spread, oscillations can be excited through their absorption by resonance electrons in the plasma. In that case the quantity η equals

$$\eta = \pm \frac{\omega_{p\alpha}' \cos\theta J_l}{\sqrt{[1 + \delta\varepsilon^{(p)}(k, k_z u + l|\omega_{B\alpha}|)]}}. \tag{6.3.2.8}$$

(d) In concluding this subsection we shall consider the problem of the stability of a plasma consisting of electrons with a Maxwellian velocity distribution and monoenergetic ion-oscillators (Kitsenko and Stepanov, 1961a; Harris, 1961; Pistunovich, 1963; Krasovitskiĭ and Stepanov, 1964). Such an ion distribution may occur in adiabatic traps when molecular ions are injected into the trap at right angles to the magnetic field. The trap is filled by a plasma because of the dissociation of the molecular ions on the residual gas. Such a velocity distribution of ions occurs in traps when the capture of the ions in the trap is realized because of Lorentz ionization, that is, the ionization of atoms moving at right angles to the magnetic field.

We shall assume that the plasma density is so small that $|\omega_{Be}| \gg \omega_{pe}$ and we shall consider plasma oscillations with frequencies $\omega \sim \omega_{pe} \sim \omega_{Bi}$; such conditions can be realized in large adiabatic traps. The dispersion equation for the longitudinal oscillations of such a plasma is of the form

$$A = 1+\delta\varepsilon^{(e)}+\delta\varepsilon^{(i)}, \tag{6.3.2.9}$$

where $\delta\varepsilon^{(e)}$ is given by eqn. (6.2.2.2) and $\delta\varepsilon^{(i)}$ by eqn. (6.3.2.3) in which we must replace the index e by the index i and put $n_0' = n_0$.

Using the fact that in the case considered oscillations are excited with a low frequency $|\omega| \ll |\omega_{Be}|$ and a long wavelength $k_x\varrho_e \ll 1$, we write (6.3.2.9) in the form

$$1+\frac{\omega_{pe}^2}{k^2v_e^2}[1+i\sqrt{(\pi)}\,z_e w(z_e)]-\sum_{l=-\infty}^{+\infty}\omega_{pi}^2\left[\frac{\cos^2\theta J_l^2}{(\omega-l\omega_{Bi})^2}+\frac{2\sin^2\theta l J_l J_l'}{\lambda\omega_{Bi}(\omega-l\omega_{Bi})}\right]=0, \tag{6.3.2.10}$$

where $z_e = \omega/\sqrt{(2)}k_z v_e$.

If the phase velocity of the excited oscillations is much larger than the thermal velocity of the electrons, $|z_e| \gg 1$. Retaining in that case in (6.3.2.10) only the single resonance term proportional to $(\omega-l\omega_{Bi})^{-2}$, we get

$$\omega = l\omega_{Bi}+i\gamma,$$

where

$$\gamma = \frac{\omega_{pi}\,|\cos\theta J_l|}{\sqrt{[(\omega_{pe}\cos\theta/l\omega_{Bi})^2-1]}}. \tag{6.3.2.11}$$

Instability occurs when $\cos\theta > l\omega_{Bi}/\omega_{pe}$. To satisfy that inequality it is necessary that the condition $\omega_{pe} > \omega_{Bi}$ holds. The growth rate of the oscillations is clearly a maximum when the frequency $\omega = l\omega_{Bi}$ lies close to the second hybrid frequency $\omega = \omega_\infty^{(2)}(\theta) = |\omega_{Be}|\cos\theta$; in that case

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{m_e}{m_i}J_l^2(\lambda)\right)^{1/3} l\omega_{Bi}. \tag{6.3.2.12}$$

As to order of magnitude we have under non-resonance conditions when $\lambda \sim 1$ and small values of the number l

$$\gamma \sim \left(\frac{m_e}{m_i}\right)^{1/2}\omega_{Bi},$$

and under resonance conditions, $\omega_{pe}\cos\theta = l\omega_{Bi}$,

$$\gamma \sim \left(\frac{m_e}{m_i}\right)^{1/3}\omega_{Bi}.$$

Equations (6.3.2.11) and (6.3.2.12) are valid when the angle θ is not too close to $\pi/2$ so that

$$\frac{J_l\cos^2\theta}{|\gamma|} \gg \frac{|J_l'|}{\lambda\omega_{Bi}}.$$

If that inequality is not satisfied, we have instead of (6.3.2.11)

$$\omega = l\omega_{Bi}+\eta,$$

where

$$\eta = \frac{1}{1-(\omega_{pe}\cos\theta/l\omega_{Bi})^2}\left\{\frac{\omega_{pi}^2 lJ_l J_l'}{\lambda\omega_{Bi}} \pm\left[\left(\frac{\omega_{pi}^2 lJ_l J_l'}{\lambda\omega_{Bi}}\right)^2+\left(1-\frac{\omega_{pe}^2\cos^2\theta}{l^2\omega_{Bi}^2}\right)\omega_{pi}^2\cos^2\theta\right]^{1/2}\right\}. \tag{6.3.2.13}$$

From this it follows that for $\theta \approx \pi/2$ the oscillations are unstable provided

$$\cos\theta > \frac{l^2\,|J_l J_l'|\,\omega_{pi}}{\lambda\sqrt{[\omega_{pe}^2\cos^2\theta-l^2\omega_{Bi}^2]}},$$

and

$$\omega_{pe}\cos\theta > l\omega_{Bi}.$$

When $\omega \lesssim k_z v_e$ a *monoenergetic ion* gas excites oscillations due to Landau damping by resonance electrons. Indeed, putting as before $\omega = l\omega_{Bi}+\eta$, we get

$$\eta = \pm\frac{\omega_{pi}\cos\theta J_l}{\sqrt{\left\{1+\frac{\omega_{pe}^2}{k^2v_e^2}[1+i\sqrt{(\pi)}\,z_e w(z_e)]\right\}}}, \tag{6.3.2.14}$$

where $z_e = l\omega_{Bi}/\sqrt{(2)}k_z v_e$. Due to the presence of Landau damping, instability can thus occur also when $\omega_{pe} < \omega_{Bi}$.

In a low-density plasma with a strongly anisotropic ion distribution function,

$$f_{i0} \propto \delta(v_\perp - v_{\perp 0}),$$

there occur unstable longitudinal oscillations with a frequency equal to the ion cyclotron frequency or to a multiple of it. This form of instability is called *cyclotron instability*. Cyclotron instability can develop not only in a plasma consisting of ion oscillators, but also in a plasma with an arbitrary anisotropic ion distribution function $f_{i0}(v_{||}, v_\perp)$, if the anisotropy is sufficiently large, in particular, in a plasma with an anisotropic ion Maxwell distribution ($T_\perp \neq T_{||}$) (Harris, 1961; Pistunovich, 1963; Timofeev, 1961; Dnestrovskiĭ, Kostomarov and Pistunovich, 1963). When the density increases (with $\omega_{pe} \gg |\omega_{Be}|$) the excitation of harmonics with a high number becomes possible and the growth rate becomes larger than the ion cyclotron frequency. The cyclotron instability of a plasma with a non-monotonic behaviour of the ion velocity distribution in directions at right angles to the magnetic field, $\partial f_{i0}(v_x)/\partial v_x > 0$, goes in that case over into the so-called loss-cone instability (Dnestrovskiĭ, 1963; Krasovitskiĭ and Stepanov, 1964; Mikhaĭlovskiĭ, 1965; Post and Rosenbluth, 1965, 1966)—we have written here $f_{i0}(v_x) = \iint f_{i0}(v_{||}, v_\perp)\,dv_y\,dv_{||}$).

6.3.3. EXCITATION OF FAST MAGNETO-SOUND WAVES AND ALFVÉN WAVES BY OSCILLATOR BEAMS

We shall now consider the excitation of the high-frequency part of the branch of FMS waves, $\omega \gg \omega_{Bi}$ in a dense plasma ($\omega_{pe} \gg |\omega_{Be}|$) by an electron oscillator beam (Kitsenko and Stepanov, 1961a).

Using expressions (6.3.1.3) for $\varepsilon_{ij}^{(b)}$ we get the following expressions for the growth rate of these waves in the case of a hot oscillator beam:

$$\gamma = \frac{\sqrt{\pi}}{2}\frac{n_0'}{n_0}\frac{R}{1+r}\exp(-z_l^2)\,\omega(k,\theta), \tag{6.3.3.1}$$

where $\omega(k, \theta) = |\omega_{Be}| \cos\theta/(1+r)$, $r = \omega_{pe}^2/k^2c^2$, is the frequency of the wave which is close to $k_z u + l|\omega_{Be}|$, and

$$R = \left(z_l' J_l^2 + \frac{2l\omega_{Be}y_0'}{\omega\lambda} a_e' J_l J_l'\right)\left[\left(2y_0'^2\cos^2\theta + r\frac{l^2}{a_e'}\right)\left(1+r-\frac{\omega_{Be}^2}{\omega^2}\right) + 2y_l'^2 r(1+r)\right]$$
$$-2r\frac{\omega_{Be}}{\omega}\left(\sin^2\theta + r\frac{l\omega_{Be}}{\omega}\right)\left[\frac{ly_0'}{\lambda}(\lambda J_l J_l')' + \frac{\lambda z_l'\omega}{a_e'\omega_{Be}} J_l J_l'\right]$$
$$-r^2(1+r)\left[\frac{\lambda^2 z_l'}{a_e'} J_l'^2 + \frac{ly_0'\omega_{Be}}{\omega\lambda}(\lambda^2 J_l'^2)\right],$$

$a_e' = (k_x v_e'/\omega_{Be})^2 \ll 1$, $v_e' \ll u$, and we have put in the expressions for $R: \omega = \omega(k, \theta)$.

It follows from eqn. (6.3.3.1) that the oscillations can grow both under anomalous and under normal Doppler effect conditions.

In the case of a cold oscillator beam the growth rate of the oscillations for resonance, $\omega(k, \theta) \approx k_z u + l|\omega_{Be}|$, is given by the equation

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left[\frac{n_0'}{n_0}\frac{\cot^2\theta\,|R|}{(1+r)^2}\right]^{1/3}\omega, \tag{6.3.3.2}$$

where

$$R = -J_l^2\left[\left(1-\frac{\omega_{Be}^2}{\omega^2}\right)\left(\sin^2\theta + l^2 r\frac{\omega_{Be}^2}{\omega^2}\right) + r\sin^2\theta + r(1+r)\tan^2\theta\left(1-\frac{l\omega_{Be}}{\omega}\right)^2\right.$$
$$\left. + l^2 r^2\frac{\omega_{Be}^2}{\omega^2}\right] + 2r\frac{\omega_{Be}^2}{\omega^2}\left(\sin^2\theta + lr\frac{\omega_{Be}}{\omega}\right)\lambda J_l J_l' + r^2(1+r)\frac{\omega_{Be}^2}{\omega^2}\lambda^2 J_l'^2.$$

In the right-hand side of this equation we put $\omega = \omega(k, \theta)$. When $l \sim 1$ and $\lambda \sim 1$, we have as to order of magnitude

$$\frac{\gamma}{\omega} \sim \left(\frac{n_0'}{n_0}\right)^{1/3}.$$

Equation (6.3.3.2) was obtained under conditions where the resonance terms in $\varepsilon_{ij}^{(b)}$, proportional to $(\omega - l|\omega_{Be}| - k_z u)^{-2}$ are much larger than the terms proportional to $(\omega - l|\omega_{Be}| - k_z u)^{-1}$. In the long-wavelength region ($\lambda \ll 1$) this condition may not be satisfied for $l = \pm 1$ and $l = \pm 2$, if the condition $|\gamma| \ll \lambda^2\omega$ is violated. If $\gamma \gg \lambda^2\omega$, cyclotron excitation is possible at the first harmonic only under anomalous Doppler effect conditions. In that case the growth rate has the same form, (6.2.4.3), as in the case of a cold monoenergetic beam.

Therefore, as in the case of longitudinal oscillations the excitation of an FMS wave by an oscillator beam occurs for cyclotron resonance between the wave and beam particles both under anomalous and under normal Doppler effect conditions.

Let us now consider the excitation of the A and FMS branches in a low-pressure plasma by an oscillator beam in the range of frequencies of the order of ω_{Bi}. If the thermal motion of the beam particles is important ($\gamma \ll k_z v'_e$), the growth rate of these waves is given by the equation

$$\gamma = -\frac{\sqrt{(\pi)}\omega'^2_{p\alpha}P_1}{2\omega Q} \exp(-z_l'^2), \qquad (6.3.3.3)$$

where

$$P_1 = (n^2-\varepsilon_1)\left[\frac{z_l'}{a_\alpha'} l^2 J_l^2 + 2l^3 J_l J_l' \frac{\omega_{Bi} y_0'}{\omega\lambda}\right] + (n^2\cos^2\theta - \varepsilon_1)\left[\frac{\lambda^2 z_l'}{a_\alpha'} J_l'^2 + \frac{l\omega_{Bi}y_0'}{\omega\lambda}(\lambda^2 J_l'^2)'\right]$$
$$+2\varepsilon_2\left[\frac{\omega_{Bi}y_0' l^2}{\omega\lambda}(\lambda J_l J_l')' + \frac{l\lambda z_l'}{a_\alpha'} J_l J_l'\right],$$
$$Q = \varepsilon_1^2\left[(1+\cos^2\theta)\frac{k^2 v_A^2}{\omega^2} - 2 + \frac{\omega^2}{\omega_{Bi}^2}\right].$$

In these formulae we must put $\omega = \omega(k, \theta)$, where $\omega(k, \theta)$ is given by eqn. (6.2.4.14).

A monoenergetic oscillator beam excites A and FMS oscillations with a frequency $\omega = k_z u + l\omega_{B\alpha}$ where the growth rate is maximum at resonance, $\omega(k, \theta) = k_z u + l\omega_{B\alpha}$,

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{\omega'^2_{p\alpha}\omega^2_{B\alpha}\cot^2\theta\, P_2}{\omega^4 Q}\right)^{1/3}\omega, \qquad (6.3.3.4)$$

where

$$P_2 = (n^2-\varepsilon_1)\,l^2 J_l^2 + (n^2\cos^2\theta - \varepsilon_1)\,\lambda^2 J_l'^2 + 2\varepsilon_2 \lambda l J_l J_l',$$

$$\varepsilon_1 = \frac{\omega_{pi}^2}{\omega_{Bi}^2 - \omega^2}, \qquad \varepsilon_2 = \frac{\omega_{pi}^2\omega}{\omega_{Bi}(\omega_{Bi}^2 - \omega^2)}.$$

6.4. Excitation of Electromagnetic Waves in a Plasma by Relativistic Charged Particle Beams

6.4.1. THE DIELECTRIC PERMITTIVITY TENSOR OF A RELATIVISTIC PLASMA BEAM

In the preceding sections of this chapter we have studied the interaction between non-relativistic charged particle beams and plasma oscillations and we have shown that it leads to the excitation in the plasma of slow electromagnetic waves with phase velocities much smaller than the velocity of light ($v_{ph} \ll c$). We now turn to a study of the interaction between plasma oscillations and relativistic charged particle beams, and we shall show that in this case both slow and fast ($v_{ph} \gtrsim c$) waves can be excited.†

The dispersion properties of the relativistic beam–plasma system are determined by its dielectric permittivity tensor $\varepsilon_{ij}(k, \omega)$ which as in the case when a non-relativistic beam

† Gaponov (1961) and Zheleznyakov (1960a) suggested that an oscillator beam might excite fast electromagnetic waves.

passes through the plasma is in the form of a sum

$$\varepsilon_{ij} = \varepsilon_{ij}^{(p)} + \varepsilon_{ij}^{(b)},$$

where $\varepsilon_{ij}^{(p)}$ and $\varepsilon_{ij}^{(b)}$ are the dielectric permittivity tensors of the plasma and of the beam.

The tensor $\varepsilon_{ij}^{(p)}$ is given by eqns. (5.1.1.5) and (5.1.1.6). As to the tensor $\varepsilon_{ij}^{(b)}$, we must obtain new formulae for it which take relativistic effects into account. As before we shall now start from the relation

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \delta_{ij} + \frac{4\pi i}{\omega} \sigma_{ij}(\boldsymbol{k}, \omega),$$

where $\sigma_{ij}(\boldsymbol{k}, \omega)$ is the conductivity tensor which connects the current density $\boldsymbol{j}(\boldsymbol{k}, \omega)$ with the electrical field $\boldsymbol{E}(\boldsymbol{k}, \omega)$:

$$j_i(\boldsymbol{k}, \omega) = \sum_j \sigma_{ij}(\boldsymbol{k}, \omega) E_j(\boldsymbol{k}, \omega), \quad \boldsymbol{j} = \sum_\alpha e_\alpha \int d^3p \boldsymbol{v} F_\alpha(\boldsymbol{r}, \boldsymbol{p}, t), \tag{6.4.1.1}$$

where $F_\alpha(\boldsymbol{r}, \boldsymbol{p}, t)$ is the distribution function of the αth kind of particles, in coordinate ($\boldsymbol{r}$) and momentum ($\boldsymbol{p}$) space. This function satisfies in the case of a collisionless plasma the following kinetic equation:

$$\frac{\partial F_\alpha}{\partial t} + (\boldsymbol{v} \cdot \nabla F_\alpha) + e_\alpha \left(\left\{ \boldsymbol{E} + \frac{1}{c} [\boldsymbol{v} \wedge \{\boldsymbol{B} + \boldsymbol{B}_0\}] \right\} \cdot \frac{\partial F_\alpha}{\partial \boldsymbol{p}} \right) = 0, \tag{6.4.1.2}$$

where the momentum $\boldsymbol{p}$ and the velocity $\boldsymbol{v}$ of a particle of kind α are connected through the relativistic relation

$$\boldsymbol{p} = \frac{m_\alpha \boldsymbol{v}}{\sqrt{\left[1 - \frac{v^2}{c^2}\right]}},$$

$\boldsymbol{B}_0$ is the external constant and uniform magnetic field, and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the self-consistent variable electrical and magnetic fields. Linearizing the kinetic eqn. (6.4.1.2) in the deviation of the distribution function from the initial distribution function $f'_{\alpha 0}(\boldsymbol{p})$ and expanding the perturbed quantities in a series of plane monochromatic waves we get from (6.4.1.1) the following expressions for the components of the dielectric permittivity tensor of the beam, $\varepsilon_{ij}^{(b)} \equiv \varepsilon_{ij}^{(b)}(\boldsymbol{k}, \omega)$ (Zayed and Kitsenko, 1968):

$$\varepsilon_{11}^{(b)} = -\sum_\alpha \frac{2\pi\omega'^2_{p\alpha}}{\omega\omega^{(0)}_{B\alpha}} \sum_{l=-\infty}^{+\infty} \int_0^\infty p_\perp^2 \, dp_\perp \int dp_z \frac{l^2 J_l^2}{\lambda^2(l+b)} R,$$

$$\varepsilon_{22}^{(b)} = -\sum_\alpha \frac{2\pi\omega'^2_{p\alpha}}{\omega\omega^{(0)}_{B\alpha}} \sum_{l=-\infty}^{+\infty} \int_0^\infty p_\perp^2 \, dp_\perp \int dp_z \frac{J_l'^2}{l+b} R,$$

$$\varepsilon_{33}^{(b)} = \sum_\alpha \frac{2\pi\omega'^2_{p\alpha}}{\omega\omega^{(0)}_{B\alpha}} \int_0^\infty dp_\perp \int p_z \, dp_z \frac{\omega_{B\alpha}}{\omega} \left(p_\perp \frac{\partial f'_{\alpha 0}}{\partial p_z} - p_z \frac{\partial f'_{\alpha 0}}{\partial p_\perp} \right)$$

$$- \sum_\alpha \frac{2\pi\omega'^2_{p\alpha}}{\omega\omega^{(0)}_{B\alpha}} \sum_{l=-\infty}^{+\infty} \int_0^\infty dp_\perp \int p_z^2 \, dp_z \frac{J_l^2}{l+b} R, \tag{6.4.1.3}$$

$$\varepsilon_{12}^{(b)} = -i \sum_{\alpha} \frac{2\pi\omega_{p\alpha}'^2}{\omega\omega_{B\alpha}^{(0)}} \sum_{l=-\infty}^{+\infty} \int_0^\infty p_\perp^2 \, dp_\perp \int dp_z \frac{lJ_lJ_l'}{\lambda^2(l+b)} R,$$

$$\varepsilon_{13}^{(b)} = -\sum_{\alpha} \frac{2\pi\omega_{p\alpha}'^2}{\omega\omega_{B\alpha}^{(0)}} \sum_{l=-\infty}^{+\infty} \int_0^\infty p_\perp \, dp_\perp \int p_z \, dp_z \frac{lJ_l^2}{\lambda(l+b)} R,$$

$$\varepsilon_{23}^{(b)} = i \sum_{\alpha} \frac{2\pi\omega_{p\alpha}'^2}{\omega\omega_{B\alpha}^{(0)}} \sum_{l=-\infty}^{+\infty} \int_0^\infty p_\perp \, dp_\perp \int p_z \, dp_z \frac{J_lJ_l'}{l+b} R,$$

where

$$J_l \equiv J_l(\lambda), \quad J_l' = dJ_l/d\lambda,$$

$$R = \frac{1}{n_{\alpha 0}'} \left[\left(1 - \frac{k_z v_z}{\omega}\right) \frac{\partial f_{\alpha 0}'}{\partial p_\perp} + \frac{k_z v_\perp}{\omega} \frac{\partial f_{\alpha 0}'}{\partial p_z} \right],$$

$$\lambda = \frac{k_x v_\perp}{\omega_{B\alpha}}, \quad b = \frac{k_z v_z - \omega}{\omega_{B\alpha}}, \quad \omega_{p\alpha}' = \sqrt{\frac{4\pi e^2 n_{\alpha 0}'}{m_\alpha}},$$

$$\omega_{B\alpha}^{(0)} = \frac{e_\alpha B_0}{m_\alpha c}, \quad \omega_{B\alpha} = \omega_{B\alpha}^{(0)} \sqrt{(1-\beta^2)}, \quad \beta^2 = \frac{v_\perp^2 + v_z^2}{c^2},$$

and $n_{\alpha 0}'$ is the density in the beam of particles of kind α in the laboratory frame of reference.

In particular, for an oscillator beam with an initial distribution function for the beam particles of the form

$$f_{\alpha 0}' = \frac{n_{\alpha 0}'}{2\pi p_\perp} \delta(p_\perp - p_{\perp 0}) \, \delta(p_z - p_{z0}) \tag{6.4.1.4}$$

the tensor $\varepsilon_{ij}^{(b)}$ is given by the equations

$$\begin{aligned}
\varepsilon_{11}^{(b)} &= \sum_{\alpha, l=-\infty}^{+\infty} \left[\frac{2lJ_lJ_l'}{\lambda} P + l^2 J_l^2 Q \right]. \\
\varepsilon_{22}^{(b)} &= \sum_{\alpha, l=-\infty}^{+\infty} \left[\frac{1}{\lambda} (\lambda^2 J_l'^2)' P + \lambda^2 J_l'^2 Q \right], \\
\varepsilon_{33}^{(b)} &= -\sum_{\alpha} \frac{\omega_{p\alpha}'^2}{\omega^2} (1-\beta_0^2)^{1/2} + \sum_{\alpha, l=-\infty}^{+\infty} \left[\left(\frac{2k_z u}{\omega - k_z u} J_l^2 + \frac{u^2}{v_{\perp 0}^2} 2\lambda J_l J_l' \right) P + \frac{u^2}{v_{\perp 0}^2} \lambda^2 J_l^2 Q \right], \\
\varepsilon_{12}^{(b)} &= -i \sum_{\alpha, l=-\infty}^{+\infty} \left[\frac{l}{\lambda} (\lambda J_l J_l')' P + l\lambda J_l J_l' Q \right], \\
\varepsilon_{13}^{(b)} &= \sum_{\alpha, l=-\infty}^{+\infty} \left[\left(\cot\theta \frac{l\omega_{B\alpha}}{\omega - k_z u} J_l^2 + \frac{u}{v_{\perp 0}} 2lJ_lJ_l' \right) P + \frac{u}{v_{\perp 0}} \lambda l J_l^2 Q \right], \\
\varepsilon_{23}^{(b)} &= i \sum_{\alpha, l=-\infty}^{+\infty} \left[\left\{ \cot\theta \frac{\omega_{B\alpha}}{\omega - k_z u} \lambda J_l J_l' + \frac{u}{v_{\perp 0}} (\lambda J_l J_l')' \right\} P + \frac{u}{v_{\perp 0}} \lambda^2 J_l J_l' Q \right],
\end{aligned} \tag{6.4.1.5}$$

where

$$P = \frac{\omega_{p\alpha}'^2 (\omega - k_z u)(1-\beta_0^2)^{1/2}}{\omega^2(\omega - k_z u - l\omega_{B\alpha})}, \quad Q = \frac{\omega_{p\alpha}'^2 \omega_{B\alpha}^2 (h - \cot^2\theta)(1-\beta_0^2)^{1/2}}{\omega^2(\omega - k_z u - l\omega_{B\alpha})^2},$$

$$\omega_{B\alpha} = \omega_{B\alpha}^{(0)} (1-\beta_0^2)^{1/2}, \quad \beta_0^2 = \frac{v_{\perp 0}^2 + u^2}{c^2}, \quad h = \frac{\omega^2}{c^2 k_x^2}.$$

These formulae are the relativistic generalizations of the formulae for $\varepsilon_{ij}^{(b)}$ obtained in Subsection 6.3.1 for a slow ($u \ll c$) beam.

If $p_{\perp 0} = 0$, that is, if the beam moves parallel to the magnetic field, we have (Wright, Wiginton and Neufeld, 1965)

$$\begin{aligned}
\varepsilon_{11}^{(b)} = \varepsilon_{22}^{(b)} &= -\sum_\alpha \frac{\omega_{p\alpha}'^2(\omega-k_z u)^2(1-\beta^2)^{1/2}}{\omega^2[(\omega-k_z u)^2-\omega_{B\alpha}^2]}, \\
\varepsilon_{33}^{(b)} &= -\sum_\alpha \left\{ \frac{\omega_{p\alpha}'^2(1-\beta^2)^{3/2}}{(\omega-k_z u)^2} + \frac{\omega_{p\alpha}'^2 k_x^2 u^2(1-\beta^2)^{1/2}}{\omega^2[(\omega-k_z u)^2-\omega_{B\alpha}^2]} \right\}, \\
\varepsilon_{12}^{(b)} &= -i\sum_\alpha \frac{\omega_{p\alpha}'^2(\omega-k_z u)\,\omega_{B\alpha}(1-\beta^2)^{1/2}}{\omega^2[(\omega-k_z u)^2-\omega_{B\alpha}^2]}, \qquad (6.4.1.6)\\
\varepsilon_{13}^{(b)} &= -\sum_\alpha \frac{\omega_{p\alpha}'^2(\omega-k_z u)\,k_x u(1-\beta^2)^{1/2}}{\omega^2[(\omega-k_z u)^2-\omega_{B\alpha}^2]}, \\
\varepsilon_{23}^{(b)} &= i\sum_\alpha \frac{\omega_{p\alpha}'^2\omega_{B\alpha} k_x u(1-\beta^2)^{1/2}}{\omega^2[(\omega-k_z u)^2-\omega_{B\alpha}^2]},
\end{aligned}$$

where $\beta = (1-u^2/c^2)^{1/2}$. These formulae are the relativistic generalizations of eqns. (6.2.1.3) for the components of the tensor $\varepsilon_{ij}^{(b)}$ obtained in the non-relativistic case for a mono-energetic beam moving along the external magnetic field.

To conclude this subsection we shall obtain the equations for the transformation of the dielectric permittivity tensor when changing from one system of reference to another Rukhadze, 1962; Zayed and Kitsenko, 1968). In a frame of reference in which the beam is at rest the components of the current density are given by the equations

$$j_j' = \sum_l \frac{\omega'}{4\pi i}(\varepsilon_{jl}' - \delta_{jl})E_l, \qquad j, l = 1, 2, 3, \qquad (6.4.1.7)$$

where the primes indicate variables in the frame of reference fixed in the beam, where the components of the tensor $\varepsilon_{jl}' \equiv \varepsilon_{jl}(\boldsymbol{k}', \omega')$ are given by the non-relativistic formulae (for $u = 0$), and where the frequency ω' and wavevector $\boldsymbol{k}'$ in the rest frame of the beam are connected with the frequency ω and wavevector $\boldsymbol{k}$ in the laboratory frame by the relations

$$\omega' = \frac{\omega-(\boldsymbol{k}\cdot\boldsymbol{u})}{\sqrt{(1-\beta^2)}}, \quad \boldsymbol{k}' = \boldsymbol{k} - \frac{\boldsymbol{u}(\boldsymbol{k}\cdot\boldsymbol{u})}{u^2}\left(1-\frac{1}{\sqrt{(1-\beta^2)}}\right) - \frac{\boldsymbol{u}\omega}{c^2\sqrt{(1-\beta^2)}}, \qquad (6.4.1.8)$$

where $\beta = u/c$.

The current density in the laboratory frame is connected with the current density and the volume charge density ϱ' in the beam rest frame by the relation

$$\boldsymbol{j} = \boldsymbol{j}' - \frac{\boldsymbol{u}}{u^2}(\boldsymbol{u}\cdot\boldsymbol{j}')\left(1-\frac{1}{\sqrt{(1-\beta^2)}}\right) + \frac{\boldsymbol{u}\varrho'}{\sqrt{(1-\beta^2)}}.$$

Substituting here for ϱ' the expression

$$\varrho' = \frac{(\boldsymbol{k}'\cdot\boldsymbol{j}')}{\omega'},$$

and using eqn. (6.4.1.8), we get

$$j_j = \sum_m \frac{\omega}{\omega'} \beta_{jm} j'_m, \tag{6.4.1.9}$$

where

$$\beta_{lm} = \frac{\omega'}{\omega} \delta_{lm} + \left(1 - \frac{1}{\sqrt{(1-\beta^2)}}\right) \frac{u_l u_m}{u^2} + \frac{k_m u_l}{\omega \sqrt{(1-\beta^2)}}. \tag{6.4.1.10}$$

The electrical field strength in the beam rest frame is connected with the electrical and magnetic fields in the laboratory frame by the relation (Fock, 1964)

$$\boldsymbol{E}' = (\boldsymbol{E} \cdot \boldsymbol{u}) \frac{\boldsymbol{u}}{u^2} + \frac{1}{\sqrt{(1-\beta^2)}} \left\{ \boldsymbol{E} - (\boldsymbol{E} \cdot \boldsymbol{u}) \frac{\boldsymbol{u}}{u^2} + \frac{1}{c} [\boldsymbol{u} \wedge \boldsymbol{B}] \right\}.$$

As we are considering plane monochromatic waves, we have

$$\boldsymbol{B} = \frac{c}{\omega} [\boldsymbol{k} \wedge \boldsymbol{E}],$$

and therefore

$$E'_l = \sum_n \beta_{nl} E_n, \tag{6.4.1.11}$$

where the quantities β_{nl} are given by eqn. (6.4.1.10). Using eqns. (6.4.1.7) and (6.4.1.11) we get from (6.4.1.9)

$$j_j = \sum_{m,l} \frac{\omega}{\omega'} \beta_{jm} \frac{\omega'}{4\pi i} (\varepsilon'_{ml} - \delta_{ml}) E'_l = \sum_{m,l,n} \frac{\omega}{4\pi i} \beta_{jm} (\varepsilon'_{ml} - \delta_{ml}) \beta_{nl} E_n .$$

On the other hand,

$$j_j = \sum_n \frac{\omega}{4\pi i} (\varepsilon_{jn} - \delta_{jn}) E_n.$$

The final formulae for the transformation of the tensor ε_{ij} thus have the form

$$\varepsilon_{jn} - \delta_{jn} = \sum_{m,l} (\varepsilon'_{ml} - \delta_{ml}) \beta_{jm} \beta_{nl} . \tag{6.4.1.12}$$

If the z-axis is parallel to $\boldsymbol{B}_0$ and the x-axis lies in the plane through the vectors $\boldsymbol{k}$ and $\boldsymbol{B}_0$, these formulae take the form

$$\begin{aligned}
&\varepsilon_{11} - 1 = \frac{\omega'^2}{\omega^2} (\varepsilon'_{11} - 1), \quad \varepsilon_{22} - 1 = \frac{\omega'^2}{\omega^2} (\varepsilon'_{22} - 1), \\
&\varepsilon_{33} = \varepsilon'_{33} + \frac{k_x^2 u^2 (\varepsilon'_{11} - 1)}{\omega^2 (1-\beta^2)} + \frac{2 k_x u \varepsilon'_{13}}{\omega (1-\beta^2)^{1/2}}, \\
&\varepsilon_{12} = \frac{\omega'^2}{\omega^2} \varepsilon'_{12}, \quad \varepsilon_{21} = -\varepsilon_{12}, \\
&\varepsilon_{13} = \frac{\omega' k_x u}{\omega^2 \sqrt{(1-\beta^2)}} (\varepsilon'_{11} - 1) + \frac{\omega'}{\omega} \varepsilon'_{13}, \quad \varepsilon_{31} = \varepsilon_{13}, \\
&\varepsilon_{32} = \frac{\omega' k_x u}{\omega^2 \sqrt{(1-\beta^2)}} \varepsilon'_{12} + \frac{\omega'}{\omega} \varepsilon'_{32}, \quad \varepsilon_{23} = -\varepsilon_{32} .
\end{aligned} \tag{6.4.1.13}$$

We note that when using these relations we must remember that the beam density in the laboratory frame, $n'_{\alpha 0}$, and the beam density in the beam rest frame, $n^*_{\alpha 0}$, are connected through the relation

$$n'_{\alpha 0} = \frac{n^*_{\alpha 0}}{\sqrt{(1-\beta^2)}}.$$

Equations (6.4.1.13) enable us, once we know the tensor ε_{ij} for a plasma at rest, to find the tensor ε_{ij} for a moving plasma. In particular, one can easily obtain eqns. (6.4.1.6) for the dielectric permittivity tensor of a monoenergetic relativistic beam from eqns. (6.2.1.3) by using these equations.

6.4.2. EXCITATION OF ELECTROMAGNETIC WAVES IN AN UNMAGNETIZED PLASMA BY A RELATIVISTIC ELECTRON BEAM

We shall first of all consider the interaction of a relativistic beam with an unmagnetized plasma (see also Bludman, Watson and Rosenbluth, 1960 a, b; Rukhadze, 1962).

Putting $\boldsymbol{B}_0 = 0$, we find

$$\varepsilon_{12} = \varepsilon_{23} = 0.$$

One sees easily that the equation

$$\Lambda = 0$$

then splits into two equations. One of them,

$$n^2 - \varepsilon_{11} = 0$$

determines the frequency of a transverse electromagnetic wave,

$$\omega(\boldsymbol{k}) = \pm\sqrt{\{k^2c^2 + \omega_{pe}^2 + \omega_{pe}'^2\sqrt{(1-\beta^2)}\}},$$

which is stable.

The second equation has the form

$$\varepsilon_l n^2 - \varepsilon_{11}\varepsilon_{33} + \varepsilon_{13}^2 = 0, \tag{6.4.2.1}$$

where

$$\begin{aligned}
\varepsilon_l &= \varepsilon - \frac{\omega_{pe}'^2(1-\beta^2\cos^2\theta)(1-\beta^2)^{1/2}}{(\omega - k_z u)^2},\\
\varepsilon_{11} &= \varepsilon - \frac{\omega_{pe}'^2}{\omega^2}(1-\beta^2)^{1/2},\\
\varepsilon_{33} &= \varepsilon - \frac{\omega_{pe}'^2(1-\beta^2)^{1/2}}{(\omega-k_z u)^2}\left(1-\beta^2+\frac{k^2u^2}{\omega^2}\sin^2\theta\right),\\
\varepsilon_{13} &= -\frac{\omega_{pe}'^2(1-\beta^2)^{1/2}ku\sin\theta}{\omega^2(\omega-k_z u)},\\
\varepsilon &= 1-\frac{\omega_{pe}^2}{\omega^2},
\end{aligned} \tag{6.4.2.2}$$

where θ is the angle between the wavevector and the beam velocity.

When $\theta = 0$ this dispersion equation splits into two equations. One of them, $n^2 = \varepsilon_{11}$ also gives the frequency of a transverse electromagnetic wave, while the other one determines the frequencies of the longitudinal oscillations:

$$1-\frac{\omega_{pe}^2}{\omega^2}-\frac{\omega_{ep}'^2}{(\omega-k_z u)^2}(1-\beta^2)^{3/2} = 0. \tag{6.4.2.3}$$

This dispersion equation is the same as the dispersion equation for longitudinal oscillations when a non-relativistic electron beam passes through a plasma—one needs only replace the beam density n_0' by $n_0'(1-\beta^2)^{3/2}$.

These oscillations are unstable, as in the non-relativistic case. In the case of a low-density beam the growth rate of oscillations with frequency $\omega = (\boldsymbol{k}\cdot\boldsymbol{u})$ equals

$$\gamma = \frac{\omega_{pe}'(1-\beta^2)^{3/4}}{\sqrt{\left[\left(\frac{\omega_{pe}}{(\boldsymbol{k}\cdot\boldsymbol{u})}\right)^2-1\right]}}. \tag{6.4.2.4}$$

We see that as compared with the non-relativistic case the growth rate is diminished by a factor $(1-\beta^2)^{3/4}$.

Under resonance conditions $\omega = (\boldsymbol{k}\cdot\boldsymbol{u}) = \omega_{pe}$ the growth rate is given by the equation

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{n_0'}{n_0}\right)^{1/3}\omega_{pe}(1-\beta^2)^{1/2}, \tag{6.4.2.5}$$

and is diminished as compared with the non-relativistic case by a factor $(1-\beta^2)^{1/2}$.

The growth rate of the longitudinal oscillations thus decreases considerably as $u \to c$. This is connected with the relativistic increase in the "longitudinal" electron mass,

$$m_{\parallel} = \frac{m_0}{(1-\beta^2)^{3/2}}.$$

In the case of "oblique" propagation ($\theta \neq 0, \pi/2$) and low-density beams ($n_0' \ll n_0$) we can in the dispersion eqn. (6.4.2.1) neglect the small terms containing ε_{13}^2:

$$\varepsilon(n^2-\varepsilon)-\frac{\omega_{pe}'^2(1-\beta^2)^{1/2}}{(\omega-k_z u)^2}\left[\left(1-\beta^2\cos^2\theta\right)n^2-\varepsilon\left(1-\beta^2+\frac{k_x^2u^2}{\omega^2}\right)\right] = 0. \tag{6.4.2.6}$$

This equation has, like the original eqn. (6.4.2.1), six roots corresponding in the limit as $n_0' \to 0$ to the Langmuir oscillations, $\omega = \pm\omega_{pe}$, transverse electromagnetic waves, $\omega = \pm\sqrt{(k^2c^2+\omega_{pe}^2)}$, and beam oscillations, $\omega = k_z u$.

We shall find the solution of eqn. (6.4.2.6) corresponding to the unperturbed frequency $\omega = k_z u$. Putting, as usual,

$$\omega = k_z u+\eta,$$

where $|\eta| \ll k_z u$, we find

$$\eta = \pm i\omega_{pe}'\left[\frac{(1-\beta^2\cos^2\theta)(1-\beta^2\varepsilon)(1-\beta^2)^{1/2}}{\beta^2\cos^2\theta(\varepsilon-n^2)\varepsilon}\right]^{1/2}_{\omega=k_z u}. \tag{6.4.2.7}$$

As in the non-relativistic case these oscillations will be unstable if $k_z u < \omega_{pe}$. In contrast to the case $\theta = 0$ growth rate diminishes as compared to the non-relativistic case not by a factor $(1-\beta^2)^{3/4}$ but only by a factor $(1-\beta^2)^{1/4}$.

As the frequency $\omega = k_z u$ approaches the Langmuir frequency $\omega = \omega_{pe}$, the growth rate (6.4.2.7) increases and the oscillations themselves become almost longitudinal when $k_z u \approx \omega_{pe}$. In that case ($k_z u \approx \omega_{pe}$) the quantity η follows from the equation

$$\varepsilon_l = 2\frac{\eta}{\omega_{pe}} + 2\left(1-\frac{\omega_{pe}}{k_z u}\right) - \frac{\omega_{pe}'^2}{\eta^2}(1-\beta^2)^{1/2}(1-\beta^2\cos^2\theta) = 0.$$

When $|k_z u - \omega_{pe}| \ll |\eta|$, we get from this equation

$$\eta = \xi^{(i)}[\tfrac{1}{2}\omega_{pe}'^2\omega_{pe}(1-\beta^2)^{1/2}(1-\beta^2\cos^2\theta)]^{1/3}.$$

The maximum growth rate is equal to

$$\gamma_m = \frac{\sqrt{3}}{2^{4/3}}\left(\frac{n_0'}{n_0}\right)^{1/3}\omega_{pe}(1-\beta^2)^{1/6}(1-\beta^2\cos^2\theta)^{1/3}. \tag{6.4.2.8}$$

The maximum growth rate is thus, when $\theta \neq 0$, diminished as compared with the non-relativistic case only by a factor $(1-\beta^2)^{1/6}$, in contrast to the case $\theta = 0$, when it is diminished by a factor $(1-\beta^2)^{1/2}$.

We note that although the growth rates of the oscillations excited by a relativistic beam are diminished as compared to the growth rates of the oscillations excited by a non-relativistic beam of the same density, a relativistic beam is more efficient in giving off its energy to the plasma oscillations. This is connected with the fact that the resonance condition $\omega_{pe} \approx k_z u$, which leads to the fastest growth of the oscillations, is violated for a non-relativistic beam due to the reaction of the oscillations—changing the beam velocity—at lower values of the amplitude of the excited high-frequency fields than for a relativistic beam.

It is also important that the condition for coherent excitation,

$$k v_e' \ll \gamma, \tag{6.4.2.9}$$

where v_e' is the thermal velocity spread of the beam particles, is for a relativistic beam, for $u \approx c$, much more easily satisfied than for a non-relativistic beam—with the same thermal energy spread, that is, with the same temperature.

6.4.3. DISPERSION EQUATION FOR "OBLIQUE" WAVE PROPAGATION

Substituting in the general dispersion eqn. (5.2.2.5) for the dielectric permittivity tensor the sum of the dielectric permittivity tensor (5.1.1.5) of the plasma and that of the beam, (6.4.1.3), $\varepsilon_{ij} = \varepsilon_{ij}^{(p)} + \varepsilon_{ij}^{(b)}$ we get, as we have already mentioned, the dispersion equation for the relativistic beam-plasma system in an external magnetic field $\boldsymbol{B}_0$. We shall study this equation in the case of a low-density beam (Zayed and Kitsenko, 1968).

Considering in eqn. (5.2.2.5) only the terms linear in $\varepsilon_{ij}^{(b)}$ we write the dispersion equation in the form

$$\Lambda = \Lambda_0 + \Lambda' = 0, \tag{6.4.3.1}$$

where

$$\Lambda_0 = \Lambda\Big|_{\varepsilon_{ij}^{(b)}=0} = (\varepsilon_1 \sin^2\theta + \varepsilon_3 \cos^2\theta)n^4 - [\varepsilon_1\varepsilon_3(1+\cos^2\theta) + (\varepsilon_1^2-\varepsilon_2^2)\sin^2\theta]n^2 + \varepsilon_3(\varepsilon_1^2-\varepsilon_2^2),$$

$$\Lambda' = [\varepsilon_{11}^{(b)}\sin^2\theta + 2\varepsilon_{13}^{(b)}\sin\theta\cos\theta + \varepsilon_{33}^{(b)}\cos^2\theta]n^4 + [2\cos\theta\sin\theta(i\varepsilon_2\varepsilon_{23}^{(b)} - \varepsilon_1\varepsilon_{13}^{(b)}) - (\varepsilon_1\varepsilon_{33}^{(b)} + \varepsilon_3\varepsilon_{11}^{(b)}) - (\varepsilon_1\varepsilon_{33}^{(b)} + \varepsilon_3\varepsilon_{22}^{(b)})\cos^2\theta - (\varepsilon_1\varepsilon_{22}^{(b)} + \varepsilon_1\varepsilon_{11}^{(b)} + 2i\varepsilon_2\varepsilon_{12}^{(b)})\sin^2\theta]n^2 + (\varepsilon_1^2-\varepsilon_2^2)\varepsilon_{33}^{(b)} + \varepsilon_3(\varepsilon_{11}^{(b)}\varepsilon_1 + \varepsilon_{22}^{(b)}\varepsilon_1 + 2i\varepsilon_{12}^{(b)}\varepsilon_2). \tag{6.4.3.2}$$

We shall restrict our considerations to the excitation of electromagnetic waves by a relativistic monoenergetic oscillator beam. In that case the quantities $\varepsilon_{ij}^{(b)}$ in (6.4.3.2) are given by eqns. (6.4.1.5).

In zeroth approximation ($n_0' \to 0$) the roots of the dispersion eqn. (6.4.3.1) are

$$\omega = \omega^{(p)}(\boldsymbol{k}), \quad \omega = \omega^{(b)} = k_z u + l|\omega_{Be}|, \quad l = 0, \pm 1, \ldots, \tag{6.4.3.3}$$

where the $\omega^{(p)}(\boldsymbol{k})$ are the eigenfrequencies of the plasma oscillations when there is no beam, which are the roots of the equation

$$\Lambda_0 = 0.$$

Of most interest is the case when the unperturbed frequencies (6.4.3.3) coincide, as then the growth rate of the oscillations will be especially large. We shall in what follows only consider that case; we now put in the dispersion equation

$$\omega = \omega^{(b)} + \eta, \quad \omega^{(b)} = \omega^{(p)},$$

where $|\eta| \ll |\omega^{(b)}|$. The quantity Λ' occurring in the dispersion equation then has the form

$$\Lambda' = \frac{\omega^2 R}{(\omega - k_z u - l|\omega_{Be}|)^2} + \frac{\omega S}{\omega - k_z u - l|\omega_{Be}|}, \tag{6.4.3.4}$$

where

$$R = \frac{\omega_{pe}'^2\omega_{Be}^2(h-\cot\theta)(1-\beta^2)^{1/2}}{\omega^4}\left\{\frac{\omega^2}{\omega_{Be}^2}\sin^2\theta J_l^2 n^4 - \left[2\sin\theta\cos\theta\frac{u}{v_{\perp 0}}(\varepsilon_1\lambda l J_l^2 + \varepsilon_2\lambda^2 J_l J_l') + \frac{u^2}{v_{\perp 0}^2}(1+\cos^2\theta)\varepsilon_1\lambda^2 J_l^2 + \varepsilon_3(l^2 J_l^2 + \cos^2\theta\lambda^2 J_l'^2) + \varepsilon_1(l^2 J_l^2 + \lambda^2 J_l'^2)\sin^2\theta + 2\varepsilon_2\sin^2\theta\lambda l J_l J_l'\right]n^2 + \varepsilon_1\varepsilon_3(l^2 J_l^2 + \lambda^2 J_l'^2) + 2\varepsilon_2\varepsilon_3\lambda l J_l J_l' + \frac{u^2}{v_{\perp 0}^2}\lambda^2 J_l^2(\varepsilon_1^2-\varepsilon_2^2)\right\}\Bigg|_{\omega=\omega^{(b)}}. \tag{6.4.3.5}$$

We need only take the term in (6.4.3.4) which is proportional to S into account in the case when $\lambda = k_x v_{\perp 0}/|\omega_{Be}| \ll 1$, as this term is much smaller than the first term, which is proportional to $R(\omega - l|\omega_{Be}| - k_z u)^{-2}$, when $\lambda \gtrsim 1$. We shall therefore give here the expression for S only for the case $\lambda \ll 1$:

$$S = -\frac{\omega_{pe}'^2(1-\beta_0^2)^{1/2}(\omega - k_z u)}{\omega^3}\,\frac{2|l^3|J_l^2}{\lambda^2}W, \tag{6.4.3.6}$$

where

$$W = \frac{\omega^2}{l^2\omega_{Be}^2}\sin^2\theta n^4 - \left[2\cos\theta\sin^2\theta\frac{ku}{l\omega_{Be}}(\varepsilon_1-\varepsilon_2)+\frac{k_x^2u^2}{l^2\omega_{Be}^2}(1+\cos^2\theta)\varepsilon_1+(1+\cos^2\theta)\varepsilon_3\right.$$
$$\left.+2\sin^2\theta(\varepsilon_1-\varepsilon_2)\right]n^2+\frac{k_x^2u^2}{l^2\omega_{Be}^2}(\varepsilon_1^2-\varepsilon_2^2)+2\varepsilon_3(\varepsilon_1-\varepsilon_2).$$

If $\lambda \gtrsim 1$ we neglect in the dispersion eqn. (6.4.3.1) the term proportional to S and we get for η a cubic equation:

$$\frac{\partial \Lambda_0}{\partial \omega}(\eta+\omega^{(b)}-\omega^{(p)})+\frac{\omega^2}{\eta^2}R = 0. \tag{6.4.3.7}$$

If $|\omega^{(b)}-\omega^{(p)}| \ll |\eta|$, we get from this equation

$$\gamma = \operatorname{Im}\eta = \frac{\sqrt{3}}{2}\left|\frac{R\omega^2}{\partial\Lambda_0/\partial\omega}\right|^{1/3}_{\omega=\omega^{(p)}}. \tag{6.4.3.8}$$

As in the cases studied in the preceding section of excitation of slow waves by a non-relativistic oscillator beam the growth rate of oscillations with frequency $\omega = \omega^{(b)} = \omega^{(p)}$, which are excited by a relativistic oscillator beam, is proportional to $(n_0')^{1/3}$.

For fast waves ($v_{ph} \gtrsim c$, $n \sim 1$) which are excited by a relativistic beam, $v_{\perp 0} \sim u \sim c$, the growth rate (6.4.3.8) of the oscillations is of the order of

$$\gamma \sim \left[\frac{n_0'}{n_0}\sqrt{(1-\beta_0^2)}\right]^{1/3}\omega^{(p)}. \tag{6.4.3.9}$$

We note that a non-relativistic beam can under resonance conditions, $\omega^{(b)} = \omega^{(p)}$, also excite fast waves, but, as in that case we have $\lambda \sim v_{\perp 0}/c \ll 1$, the growth rate of the oscillations will be much smaller than (6.4.3.9)

$$\gamma \sim \left(\frac{n_0'}{n_0}\right)^{1/3}\left(\frac{v_{\perp 0}}{c}\right)^{\frac{2}{3}l}\omega^{(p)}. \tag{6.4.3.10}$$

Let us now elucidate the conditions for the applicability of the expressions (6.4.3.8) to (6.4.3.10) which we have obtained for $\lambda \ll 1$. If $\lambda \ll 1$, we have as to order of magnitude

$$R \sim \frac{n_0'}{n_0}J_l^2, \quad S \sim \frac{n_0'}{n_0}\frac{J_l^2}{\lambda^2}.$$

The condition for the applicability of eqns. (6.4.3.8) to (6.4.3.10),

$$\left|\frac{\omega^2}{\gamma^2}R\right| \gg \left|\frac{\omega}{\gamma}S\right|,$$

is satisfied if

$$\left|\frac{\gamma}{\omega}\right| \ll \lambda^2.$$

As in the case when $n \gtrsim 1$ and $\omega \sim |\omega_{Be}| \sim \omega_{pe}$ we have as to order of magnitude

$$\gamma \sim \left[\frac{n_0'}{n_0}\sqrt{(1-\beta_0^2)}\right]^{1/3}\lambda^{\frac{2}{3}l}\omega,$$

the condition $\gamma \ll \lambda^2\omega$ can be violated only when $l = \pm 1$ and $l = \pm 2$.

If $\omega^{(p)} \approx \omega^{(b)}$, $l = \pm 1, \pm 2$, and $\lambda \ll 1$, we can neglect in the equation for Λ' the quantity $\omega^2 R/\eta^2$ in comparison with $\omega S/\eta$ and the dispersion equation (6.4.3.1) becomes

$$\frac{\partial \Lambda_0}{\partial \omega}(\eta + \omega^{(p)} - \omega^{(b)}) + \frac{\omega}{\eta} S = 0.$$

Hence it follows that

$$\frac{\eta}{\omega} = \frac{1}{2\omega\, \partial\Lambda_0/\partial\omega}\left[1 - \frac{\omega^{(b)}}{\omega^{(p)}} \pm \sqrt{\left\{\left(1 - \frac{\omega^{(b)}}{\omega^{(p)}}\right)^2 - 4S\omega \frac{\partial \Lambda_0}{\partial \omega}\right\}}\right]. \tag{6.4.3.11}$$

If $S\ \partial\Lambda_0/\partial\omega > 0$ the growth rate will, when $\omega^{(p)} = \omega^{(b)}$, equal

$$\gamma = \sqrt{\frac{S\omega}{\partial\Lambda_0/\partial\omega}}. \tag{6.4.3.12}$$

As to order of magnitude we get from this

$$\gamma \sim \left[\frac{n_0'}{n_0}\sqrt{(1-\beta_0^2)}\right]^{1/2} \lambda^{|l|-1}\omega.$$

We see thus that when a relativistic oscillator beam passes through a magneto-active plasma an efficient excitation of fast electromagnetic waves becomes possible, even though the relativistic increase in the mass somewhat decreases the growth rate of the oscillations ($\gamma \propto (1-\beta^2)^{1/6}$). We note that fast waves may be excited also by a non-relativistic oscillator beam, but the growth rate is in that case smaller by a factor $(c/v_{\perp 0})^{-2l/3}$ than the growth rate of the oscillations in the case of a relativistic beam.

6.4.4. TRANSVERSE PROPAGATION

The expressions we obtained earlier for the growth rates of the oscillations can be greatly simplified in the case of transverse propagation of the waves ($\theta = \pi/2$). A study of that case is especially interesting because for $\theta = \pi/2$ the usual Doppler term $k_z u$ in the tensor $\varepsilon_{ij}^{(b)}$ vanishes and the normal non-relativistic "beam" mechanism for the excitation of waves does not work. Let us consider this case in some detail (Zayed and Kitsenko, 1968).

When $\theta = \pi/2$ the general dispersion equation splits into two equations for the ordinary and for the extra-ordinary waves. The dispersion equation for the ordinary wave has the form

$$n^2 - \varepsilon_{33} = 0,$$

or

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pe}'^2}{\omega^2}(1-\beta_0^2)^{1/2}$$
$$- \sum_{l=-\infty}^{+\infty} \frac{u^2}{v_\perp^2}\left[\frac{\omega_{pe}'^2(1-\beta_0^2)^{1/2}\, 2\lambda J_l J_l'}{\omega(\omega - l\omega_{Be})} + \frac{\omega_{pe}'^2(1-\beta_0^2)^{1/2}\, \omega_{Be}^2 \lambda^2 J_l^2}{n^2\omega^2(\omega - l\omega_{Be})^2}\right]. \tag{6.4.4.1}$$

In zeroth approximation ($n_0' \to 0$) we find from this equation the frequency of the ordinary wave,

$$\omega(\boldsymbol{k}) = \sqrt{(k^2c^2 + \omega_{pe}^2)}.$$

The growth rate of the oscillations is maximum when $\omega(\boldsymbol{k}) = l|\omega_{Be}|$ and in that case has the form

$$\gamma = \frac{\sqrt{3}}{2^{4/3}}\left[\frac{\omega_{pe}'^2\sqrt{(1-\beta_0^2)}}{\omega^2}\frac{u^2}{c^2}J_l^2\right]^{1/3}\omega. \tag{6.4.4.2}$$

The fast transverse ordinary electromagnetic wave is thus excited by an oscillator beam if the beam moves along the magnetic field. The growth rate is proportional to $(n_0'u^2/n_0c^2)^{1/3}$ and is non-vanishing only due to relativistic effects.

The dispersion equation for extra-ordinary (fast and slow) waves in the oscillator beam–plasma system has for the case when $\theta = \pi/2$ the form

$$\varepsilon_1 n^2 - \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_{11}^{(b)} n^2 - \varepsilon_1(\varepsilon_{11}^{(b)} + \varepsilon_{22}^{(b)}) - 2i\varepsilon_2\varepsilon_{12}^{(b)} = 0. \tag{6.4.4.3}$$

From this equation we find in zeroth approximation ($n_0' \to 0$) the frequency of the extra-ordinary waves:

$$\omega^{(1,2)}(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left[2\omega_{pe}^2 + \omega_{Be}^{(0)2} + k^2c^2 \pm \sqrt{\{4\omega_{pe}^2\omega_{Be}^{(0)2} + (\omega_{Be}^{(0)2} - k^2c^2)^2\}}\right]^{1/2}. \tag{6.4.4.4}$$

If we now take into account in (6.4.4.3) terms proportional to n_0' we can find the growth rates for the extra-ordinary waves. They will be particularly large under resonance condition,

$$\omega^{(1,2)}(\boldsymbol{k}) = l|\omega_{Be}|;$$

in that case

$$\frac{\gamma}{\omega^{(1,2)}} = \frac{\sqrt{3}}{2^{4/3}}\left|\frac{\omega_{pe}'^2(1-\beta^2)^{1/2}\,\omega_{Be}^2(\omega^2-\omega_{Be}^{(0)2})\,(\varepsilon_1\lambda J_l' + \varepsilon_2 lJ_l^2)}{n^2\omega^2(\omega^{(1)2}-\omega^{(2)2})\,(\omega^{(1,2)}-\omega_{pe}^2-\omega_{Be}^{(0)2})}\right|^{1/3}. \tag{6.4.4.5}$$

Comparing this expression with expression (6.4.4.2) we see that when $\omega \sim |\omega_{Be}^{(0)}| \sim \omega_{pe} \sim kc$ and $n \sim 1$ the growth rates for the ordinary and for the extra-ordinary waves are of the same order of magnitude.

In the non-relativistic case ($v_{\perp 0}/c \ll 1$, $\lambda \ll 1$) expression (6.4.4.5) becomes

$$\frac{\gamma}{\omega^{(1,2)}} = \frac{\sqrt{3}}{2^{4/3}}\left|\frac{\omega_{pe}'^2}{\omega^4}\sqrt{(1-\beta_0^2)}J_l^2\frac{(\omega+\omega_0^{(1)})^2(\omega-\omega_0^{(2)})^2(\omega-\omega_{Be}^{(0)})^2}{(\omega^{(1)2}-\omega^{(2)2})\,(\omega^2-\omega_{pe}^2-\omega_{Be}^{(0)2})}\right|^{1/3}, \tag{6.4.4.6}$$

where

$$\omega_0^{(1,2)} = \omega^{(1,2)}(\boldsymbol{k})\Big|_{k=0}.$$

Using the fact that when $|\omega_{Be}| \sim \omega_{pe} \sim kc$ and $n \sim 1$ this expression is as to order of magnitude equal to

$$\frac{\gamma}{\omega} \sim \left(\frac{n_0'}{n_0}J_l^2\right)^{1/3}, \tag{6.4.4.7}$$

and comparing (6.4.4.7) and (6.4.4.2) we see that the growth rate of fast extra-ordinary waves, excited by a non-relativistic beam is larger by a factor $(c/u)^{2/3}$ than the growth rate of the ordinary wave.

6.5. General Criteria for the Stability of Particle Distributions in a Plasma

6.5.1. CRITERIA FOR THE STABILITY OR INSTABILITY OF PARTICLE DISTRIBUTIONS IN AN UNMAGNETIZED PLASMA

In the preceding four sections we have considered the interaction of charged particle beams with a plasma and shown that this interaction may lead to a growth of the oscillations, that is, to an instability of the beam–plasma system. We shall now turn to a study of the general problem of criteria for the stability or instability of particle distributions in a plasma, that is, to an elucidation of the conditions which must be satisfied by the particle distribution functions in order that the plasma oscillations are damped or growing.

We shall first of all consider an unmagnetized plasma in which there are longitudinal oscillations. According to the results of Section 4.2, the initial electron distribution function $F_0(w)$, where w is the component of the electron velocity along the wave vector $\boldsymbol{k}$, will be stable provided all roots of the equation

$$\frac{i\omega_p^2}{k}\int_C \frac{F_0'(w)\,dw}{p+ikw}=1 \tag{6.5.1.1}$$

lie in the left-hand p-half-plane. The integration is here along the real w-axis going around the possible pole $w = ip/k$ from below; the function $F_0(w)$ is assumed to be normalized by the condition $\int F_0(w)\,dw = 1$. In this case the spatial Fourier component of the potential $\varphi_k(t)$ will tend to zero for large t (see (4.2.1.15); we recall that t must be small compared to the relaxation time τ). The deviation $f_k(w, t)$ of the distribution function from the original function will undergo undamped oscillations with a constant amplitude and a frequency kw which depends on the particle velocity (see eqn. (4.2.1.16)).

On the other hand, if one of the roots of eqn. (6.5.1.1) lies in right-hand p-half-plane, the functions $\varphi_k(t)$ and $f_k(w, t)$ will grow without bound with time, and the original distribution will be unstable.

If several kinds of particles take part in the oscillations, we must in formula (6.5.1.1) make the substitution

$$\omega_p^2F_0(w) \to \psi(w) \equiv \sum_\alpha \omega_{p\alpha}^2 F_\alpha(w),$$

where $F_\alpha(w)$ is the initial distribution function for particles of the αth kind, normalized to unity, and

$$\omega_{p\alpha}^2 = \frac{4\pi n_\alpha e_\alpha^2}{m_\alpha}.$$

A necessary and sufficient condition for the stability of particle distributions in an unmagnetized plasma with respect to longitudinal oscillations consists thus in the absence of roots of the equation

$$y = G(s) \equiv \int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-s} = k^2 \tag{6.5.1.2}$$

in the upper s-half-plane ($s = ip/k$) for any value of $k (k > 0)$.

One sees easily that the presence of roots of eqn. (6.5.1.2) in the upper s-half-plane is equivalent to the situation where the curve $\mathcal{K}$, which describes the point $y = G(s)$, when s traces the real axis, intersects the positive y-axis. Indeed, if the curve $\mathcal{K}$ intersects the positive y-axis, for instance, in the point A (see Fig. 6.5.1b), there will always be near that point another point B corresponding to $\text{Im}\, s > 0$, for which $\text{Re}\, y > 0$, $\text{Im}\, y = 0$. The quantity s_B corresponding to the point B is a root of eqn. (6.5.1.2) with $k^2 = \text{Re}\, y_B$.

If the curve $\mathcal{K}$ does not intersect the positive y-axis, eqn. (6.5.1.2) does not have roots in the upper half-plane. Indeed, the integral, defining the function $y = G(s)$, exists for any value of s in the upper-half-plane. The whole region $\mathcal{D}$, which is the mapping of the upper s-half-plane, will thus lie in a finite part of the y-plane. As the curve $\mathcal{K}$ is the boundary of the region $\mathcal{D}$ it must be closed. The region $\mathcal{D}$ must thus lie inside a curve $\mathcal{K}$ which does not intersect the positive y-axis. It follows from this that the function $y = G(s)$ can for no value of s in the upper half-plane be equal to a positive value k^2, that is, the eqn. (6.5.1.2) has no roots in the upper s-half-plane. We show in Fig. 6.5.1 a, b, c three typical curves (Gertsenshteĭn, 1952; Penrose, 1960; Akhiezer, Lyubarskiĭ, and Polovin, 1961). Fig. 6.5.1a corresponds to the case where there are no roots of eqn. (6.5.1.2) in the upper half-plane, that is, a stable case. Figs. 6.5.1b and c correspond to cases where there are such roots, that is, to unstable cases.

Let us determine under what conditions the curve $\mathcal{K}$ intersects the positive y-axis. Such an intersection means that for some real value of s the quantity

$$y = \mathcal{P}\int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-s} + \pi i \psi'(s) \tag{6.5.1.3}$$

($\mathcal{P}$ indicates the principal value integral) is positive, that is,

$$\psi'(s) = 0, \quad \mathcal{P}\int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-s} > 0.$$

The first of these conditions is satisfied in the extremum points $w_1, w_2, \ldots, w_n$ of the function $\psi(w)$. (Therefore the principal value sign before the integral can be omitted.) For the instability of the distribution function it is thus necessary and sufficient that one of the inequalities

$$\int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-w_j} > 0, \quad j = 1, 2, \ldots, n, \tag{6.5.1.4}$$

is satisfied. If in all extremum points of the function $\psi(w)$ the opposite inequality

$$\int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-w_j} < 0 \tag{6.5.1.5}$$

is satisfied for all values $j = 1, 2, \ldots, n$, the distribution $\psi(w)$ will be stable.

One can easily prove that if condition (6.5.1.5) is satisfied only in the points of the minima, it is satisfied in all extremum points, that is, the distribution $\psi(w)$ will be stable in that case.

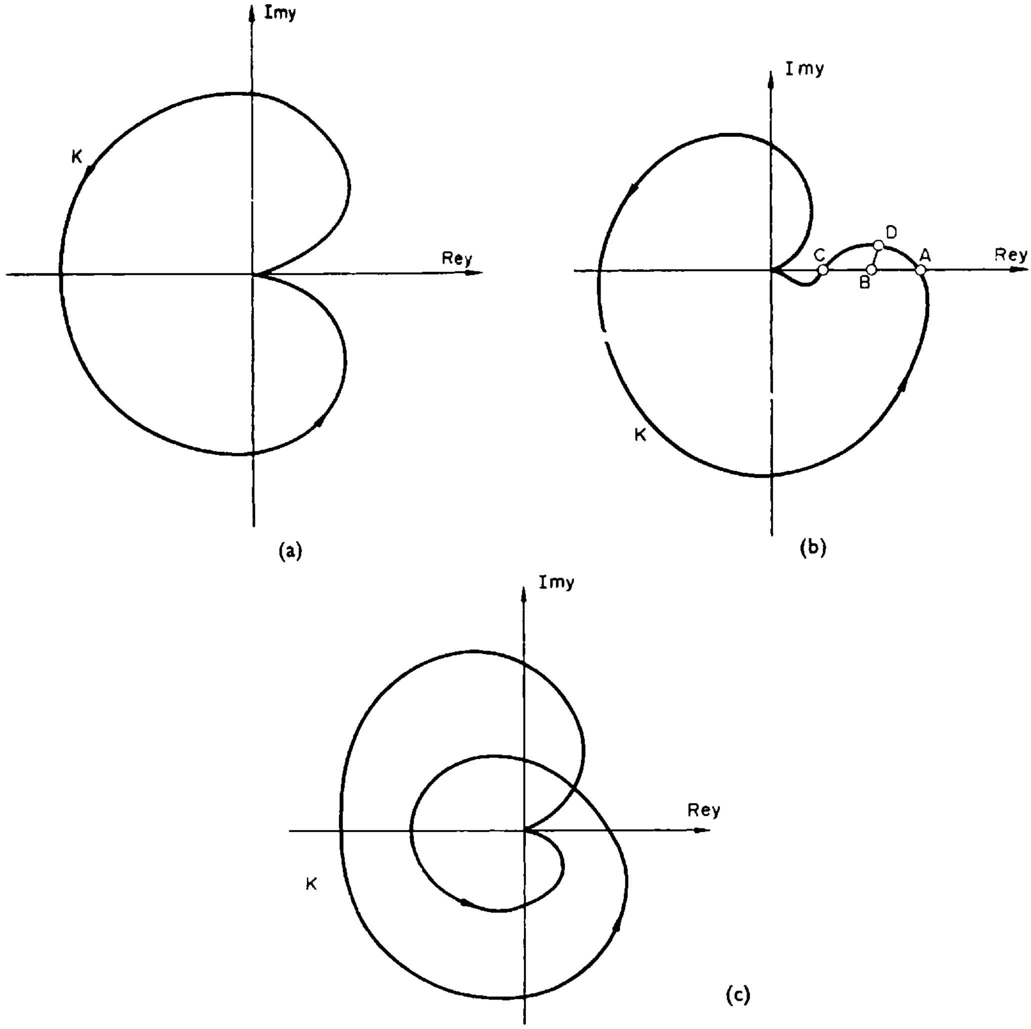

FIG. 6.5.1. The contour $\mathcal{K}$. (a) Stable distribution; (b) and (c) different kinds of unstable distributions.

Indeed, when the point s moves from $-\infty$ to $+\infty$ along the real axis, the upper s-half-plane remains to the left. When the point y moves along the curve $\mathcal{K}$, the region $\mathcal{D}$ lying inside the curve $\mathcal{K}$ also remains to the left. As the points $s = \pm\infty$ correspond to the origin of the y-plane, the curve $\mathcal{K}$ either does not intersect the positive y-axis at all or it intersects it at least once, coming from the lower into the upper half-plane (see Figs. 6.5.1 b, c). According to (6.5.1.3) the derivative $\psi'(w)$ then changes from negative to positive values, which is possible only in the vicinity of a minimum of $\psi(w)$.

The criterion for the stability of a distribution function therefore has the form†

$$\int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-w_j} < 0, \quad \psi'(w_j) = 0, \quad \psi''(w_j) > 0. \tag{6.5.1.6}$$

† This criterion was established independently by Gertsenshteĭn (1952), Penrose (1960), Akhiezer, Lyubarskiĭ, and Polovin (1961), and Noerdlinger (1960).

The stability condition (6.5.1.6) can be put in different form by integrating by parts (Penrose, 1960; Noerdlinger, 1960):

$$\int_{-\infty}^{+\infty} \frac{\psi(w)-\psi(w_j)}{(w-w_j)^2} dw < 0, \quad \psi'(w_j) = 0, \quad \psi''(w_j) > 0. \tag{6.5.1.7}$$

From this one can conclude that the distribution will be unstable if the function $\psi(w)$ vanishes in an isolated point—as by definition $\psi(w) \geqslant 0$, $\psi(w)$ has a minimum in that point.

We shall show that if the distribution $\psi(w)$ is unstable, the following inequality holds:

$$\psi'(v_{\text{ph}}) > 0, \tag{6.5.1.8}$$

where $v_{\text{ph}} = \text{Re}\, s$ and s is a root of eqn. (6.5.1.2).

The proof follows directly from Fig. 6.5.1b. Let A be a point where $\psi(w)$ is a minimum, B a point where s is a root of eqn. (6.5.1.2), and C a point where $\psi(w)$ is a maximum. We draw through the point B the line $\text{Re}\, s$ = constant which intersects the curve $\mathcal{K}$ in the point D which lies between the minimum A and the maximum C. As the values of s and $\psi(s)$ increase when we move along the arc AC, the derivative $\psi'(s)$ will be positive in all points on that arc, including the point D.

The instability condition (6.5.1.8) has a simple physical meaning: v_{ph} is clearly the phase velocity of the wave. Particles with velocities close to the wave velocity v_{ph} will interact efficiently with the wave. Particles which move faster than the wave ($w > v_{\text{ph}}$) will then give energy to the wave, while particles moving more slowly will receive energy from the wave. Condition (6.5.1.8) means that the number of particles giving off energy is larger than the number of particles receiving energy (Pierce, 1948). We recall that in an equilibrium plasma $F_0'(v_{\text{ph}}) < 0$, that is, the number of particles receiving energy from the wave is larger than the number of particles giving energy to the wave; as we mentioned earlier, this leads to the damping of oscillations in an equilibrium plasma.

It follows directly from the stability criterion (6.5.1.6) that distribution functions which have only one maximum are stable (Gertsenshteĭn, 1952; Walker, 1955; Auer, 1958; Akhiezer and Lyubarskiĭ (1955) and Berz (1956) proved this statement for the case of an even distribution function). In particular, one can easily check that if the initial distribution function $F_\alpha(v)$ is spherically symmetric—with an arbitrary number of maxima and minima—and nowhere vanishes, the function $\psi(w)$ has a single maximum, in the point $w = 0$. A spherically symmetric particle distribution is thus stable (Kovrizhnykh and Rukhadze, 1960; Ginzburg, 1960).

In deriving the instability criterion (6.5.1.4) for distribution functions we neglected the collisions between particles. These collisions clearly inhibit the development of instabilities. The growth rate of oscillations in the case of an unstable distribution function must thus exceed some minimum value determined by the effective collision frequency, in order that the instability in fact develops (Singhaus, 1964).

If the distribution function is unstable, there will occur together with a growth of the oscillations also a change in the particle distribution leading to a decrease, and ultimately a cessation of the growth of the oscillations. A study of this effect requires taking non-linear effects into account, and we shall turn to this in Section 9.1 in Volume II.

6.5.2. TWO-STREAM INSTABILITY

The conditions (6.5.1.6) and (6.5.1.7) can be applied to a study of the problem of the stability of a system of two infinite interpenetrating charged particle beams moving with parallel or antiparallel velocities. We shall denote the distribution functions in these beams by $F_1(w)$ and $F_2(w)$ and define the function

$$\psi(w) = \omega_{p1}^2 F_1(w) + \omega_{p2}^2 F_2(w).$$

We further denote by v_1 and v_2 the thermal velocities and by u_1 and u_2 the systematic velocities of the particles in the two beams.

It is clear that if the difference between the systematic velocities u_1 and u_2 is so large that the inequality

$$|u_1 - u_2| \gg v_1 + v_2, \tag{6.5.2.1}$$

holds, the minimum value of ψ will lie close to zero. We can conclude from this that, according to condition (6.5.1.7) the distribution function will be unstable. The two-beam system will thus be unstable, when inequality (6.5.2.1) holds (Vlasov, 1945; Lampert, 1956; Jackson, 1960; Faĭnberg, 1961). We emphasize that this criterion is valid, independent of the beam densities.

The instability condition (6.5.2.1) applies equally well to an overall neutral electron beam, moving through a plasma (if we neglect the ion oscillations) as to a plasma in which the electrons move relative to the ions—in which the ion oscillations are taken into account—and in contrast to the results of the preceding sections we have not made any assumptions here that the density of the beam must be low.

Let us now consider some cases when condition (6.5.2.1) is not satisfied. If

$$|u_1 - u_2| \ll v_1 + v_2, \quad v_1 \sim v_2, \tag{6.5.2.2}$$

the two-beam system will be stable, as the function $\psi(w)$ does not have a minimum.

If

$$|u_1 - u_2| \ll v_1, \quad v_1 \gg v_2, \quad r_{D1} \lesssim r_{D2}, \tag{6.5.2.3}$$

where r_{D1} and r_{D2} are the Debye radii of the particles in the beams, the contribution of the second beam to the integral (6.5.1.7) will be infinitesimally small. The value of that integral will therefore differ little from its value at the maximum of the function $F_1(w)$, where it is negative. It follows from this that the two-beam system will be stable when condition (6.5.2.3) is satisfied.

Finally, if

$$|u_1 - u_2| \ll v_1, \quad v_1 \gg v_2, \quad r_{D1} \gg r_{D2}, \tag{6.5.2.4}$$

we find from the necessary condition for instability—the existence of a minimum of the function $\psi(w)$—that we have the following condition for instability (Pierce, 1949):

$$|u_1 - u_2| \gg v_2. \tag{6.5.2.5}$$

We showed in Subsection 6.1.4 that in the particular case of a plasma with hot electrons which move relative to cold ions, the instability is connected with the excitation of ion

Langmuir oscillations. One can show that this instability is connected with the growth of oscillations with large values of k ($1/r_{D1} \ll k \ll 1/r_{D2}$) (Jackson, 1960; Imshennik and Morozov 1961).

In Table 6.5.1 we have given the conditions for the stability and instability of a two-beam system in various situations.

When $v_1 \gg v_2$ while the quantities $\lvert u_1 - u_2 \rvert$ and v_1 are of the same order of magnitude, the stability condition has a complicated form (Pierce, 1948; Jackson, 1960; Imshennik and Morozov, 1961). A qualitative sketch of the instability region is given in Fig. 6.5.2a (Baksht, 1964; Polovin, 1963 b, c).

TABLE 6.5.1. *Stability and instability conditions of a two-beam system*

$\lvert u_1-u_2 \rvert \gg v_1+v_2$	$\lvert u_1-u_2 \rvert \ll v_1+v_2$			
	$v_1 \sim v_2$	$v_1 \gg v_2$	$\lvert u_1-u_2 \rvert \ll v_1$	
		$r_{D1} \lesssim r_{D2}$	$r_{D1} \gg r_{D2}$	
			$\lvert u_1-u_2 \rvert \lesssim v_2$	$\lvert u_1-u_2 \rvert \gg v_2$
unstable	stable	stable	stable	unstable

So far we have considered a two-component system, but we can consider in exactly the same way the stability of a four-component system, for instance, the stability of two plasmas moving in opposite directions. To fix the ideas we consider the case where the plasmas have the same density, the same electron temperature T_e, and the same ion temperature T_i—the magnitudes of T_e and T_i can be different. One sees easily that if

$$v_i \ll u_0 \ll v_e ,$$

where u_0 is the relative velocity of the plasmas and v_e and v_i the thermal velocities of the electrons and ions, the instability condition has the form

$$u_0 > \sqrt{\frac{T_i}{m_e}} . \tag{6.5.2.6}$$

If $u_0 \ll v_i$, $u_0 \ll v_e$, opposite moving plasmas will be stable, while they are unstable when $u_0 \gg v_i$, $u_0 \gg v_e$. We have given in Fig. 6.5.2b the region of instability for arbitrary relations between u_0, v_e, and v_i (Nexsen, Cummins, Coengsen and Sherman, 1960). Kellogg and Liemohn (1960) have considered the collision of two plasmas with different densities and temperatures.

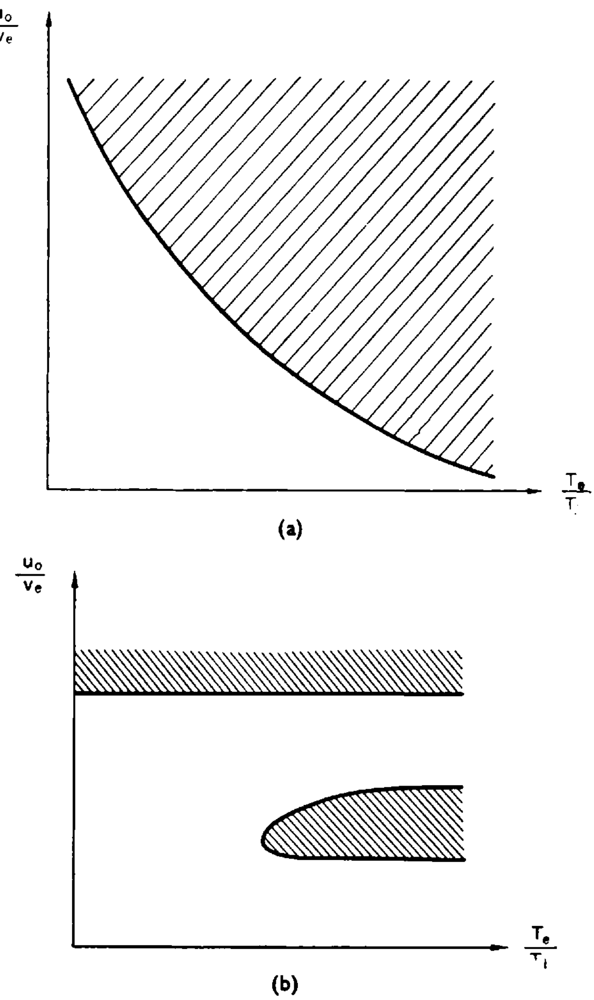

FIG. 6.5.2. Structure of the instability regions. (a) Two beams; (b) opposite moving identical plasmas. The instability regions are shaded.

6.5.3. STABILITY CRITERIA FOR ELECTRON DISTRIBUTIONS IN A PLASMA IN A MAGNETIC FIELD

One can generalize the discussion of the preceding subsections, which refer to an unmagnetized plasma, to the case of a plasma in an external magnetic field. We shall, for instance, make clear the stability conditions for electron distribution functions of a plasma in a magnetic field against excitation of longitudinal electron oscillations. We shall assume the wavelength of the plasma oscillations to be so large that the condition

$$\frac{kv_e}{\omega_{pe}} \ll 1$$

is satisfied. The dispersion equation for the longitudinal oscillations can, according to

(6.2.2.7), be written in the form

$$1-\frac{\omega_{pe}^2 \cos^2\theta}{k_z}\int\limits_{-\infty}^{+\infty}\frac{F_0'(w)\,dw}{k_z w-\omega}$$
$$+\frac{\omega_{pe}^2\sin^2\theta}{2\omega_{Be}}\int\limits_{-\infty}^{+\infty}\left(\frac{1}{k_z w-\omega+\omega_{Be}}-\frac{1}{k_z w-\omega-\omega_{Be}}\right)F_0(w)\,dw=0, \quad (6.5.3.1)$$

where $F_0(w)$ is the electron distribution function with respect to the longitudinal component (relative to the external magnetic field $\boldsymbol{B}_0$) of the velocity w; $k_z = k\cos\theta$, where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$.

Introducing the notation

$$\psi_B(w) = F_0(w)\cos^2\theta+\sin^2\theta\int\limits_0^w \frac{F_0(w+s_B)-F_0(w-s_B)}{2s_B}\,dw, \quad (6.5.3.2)$$

where $s_B = \omega_{Be}/k_z$, we can write eqn. (6.5.3.1) in the form

$$G_B(s)\equiv\int\limits_{-\infty}^{+\infty}\frac{\psi_B'(w)\,dw}{w-s}=\frac{k_z^2}{\omega_{pe}^2}, \quad (6.5.3.3)$$

where $s = \omega/k_z$.

Comparing this equation with eqn. (6.5.1.2) for the case when there is no magnetic field, we conclude easily that a necessary and sufficient condition for the stability of the distribution function $F_0(w)$ consists in that the roots of eqn. (6.5.3.3) do not lie in the upper s-half-plane.

We showed earlier that in the case of instability the function $y = G_B(s)$ maps the real s-axis on a curve $\mathcal{K}$ which intersects the positive real y-axis (we recall that in that case the curve $\mathcal{K}$ when s increases intersects the positive real axis at least once going from the lower to the upper half-plane).

The electron distribution will be stable if in all points of a minimum $s = w_j$, the function $\psi_B(w)$ satisfies the condition (Akhiezer, Lyubarskiĭ, and Polovin, 1961)

$$\int\limits_{-\infty}^{+\infty}\frac{\psi_B'(w)\,dw}{w-w_j}<0. \quad (6.5.3.4)$$

We note that the function $\psi_B(w)$ changes into $F(w)$ when $\theta = 0$, and also when $s_B = 0$ or when $s_B = \infty$. Therefore, if $\theta = 0$, or $s_B = 0$, or $s_B = \infty$, the stability condition (6.5.3.4) is the same as the stability condition when there is no magnetic field. Hence it follows that, generally speaking, a magnetic field narrows down the class of stable distribution functions (Simon, 1965). One can show (Polovin, 1964) that the stability condition (6.5.3.4) is satisfied solely by an even distribution function which has a single maximum.

In particular, the anisotropic distribution $F_0(v) = f(v_{\|}^2, v_{\perp}^2)$ is stable with respect to long-wavelength longitudinal electron oscillations. However, this function may turn out to be unstable with respect to longitudinal oscillations, if we take into account higher-order terms in the expansion of the dispersion equation in powers of kv_e/ω_{pe} (see Sen, 1952; Harris, 1959; Burt and Harris, 1961; Ozawa, Kaij, and Kito, 1962; McCune, 1965) and may also turn out to be unstable with respect to transverse oscillations (Sudan, 1963).

So far, when studying the stability condition of distribution functions we have not fixed the quantity k; when studying the stability of distribution functions in a magnetic field we have also left the quantities ω_{pe} and θ unfixed. Another way to state the stability problem is, however, possible (Ozawa, Kaij, and Kito, 1962), where the instability criterion is established for fixed values of k, θ, and ω_{pe}. One can find the stability criterion for such a way of stating the problem through somewhat altering the discussion of the preceding section.

In the case when the value of k^2 which occurs on the right-hand side of eqn. (6.5.1.2) is not fixed, instability occurs if the curve $\mathcal{K}$ which is the map of the real s-axis onto the plane $y = G(s)$ intersects the positive y-axis in some point. One can then always choose such a value of k that the region $\mathcal{D}$ bounded by the curve $\mathcal{K}$ contains the point $\operatorname{Re} y = k^2$, and this indicates instability.

If, however, the value of k is fixed, it is insufficient for instability that the curve $\mathcal{K}$ intersects the positive y-axis. It is also necessary that the point k^2 lies in the domain $\mathcal{D}$. In other words, the stability criterion for a given value of k consists in that the function

$$W(s) \equiv \int_{-\infty}^{+\infty} \frac{\psi'(w)\,dw}{w-s} - k^2$$

must map the upper half-plane $\operatorname{Im} s > 0$ onto a region $\mathcal{D}$ which does not contain the origin $W = 0$.

The stability criterion can be formulated in exactly the same way when there is a magnetic field present, if the values of the quantities k, θ, and ω_{pe} are given. One only needs replace the function $W(s)$ by the function

$$W_B(s) = \int_{-\infty}^{+\infty} \frac{\psi_B'(w)\,dw}{w-s} - \frac{k_z^2}{\omega_{pe}^2}.$$

To find out whether the point $W = 0$ falls into the region $\mathcal{D}$ it is sufficient to know the signs of the quantities $W(w_1)$, $W(w_2)$, ..., $W(w_n)$, where the z_i are the points of the extrema of the function $\psi_B(w)$ (Ozawa, Kaij, and Kito, 1962). If the signs of the quantities $W(w)$ are known, the problem whether the distribution is stable or unstable can be clarified as follows.

Let us arrange all quantities $w_1, w_2, \ldots, w_n$ in increasing order: $w_1 < w_2 < \ldots < w_n$. We arrange in the same order the (plus or minus) signs of the quantities $W(w_1)$, $W(w_2)$, ..., $W(w_n)$. We shall successively cross out two identical signs, if they stand next to one another. If after this crossing out only a single sign remains—which can only be a minus sign—the distribution is stable. If, however, after the crossing out there remains an alternating sign sequence—which can only be $-+-$, $-+-+-$, $-+-+-+-$, ... —the distribution is unstable (Akhiezer, Lyubarskiĭ, and Polovin, 1963). For instance, the sequence of signs $++-+-$ indicates instability, and the sequence of signs $+--+-$ indicates stability.

6.6. Absolute and Convective Instabilities

6.6.1. CRITERIA FOR ABSOLUTE AND CONVECTIVE INSTABILITY

In the preceding sections we have started in our study of the stability of the particle distribution functions in a plasma from the linearized kinetic equations and the Maxwell equations and we obtained as condition for their solubility the dispersion equation connecting the frequency ω and the wavevector $\boldsymbol{k}$ of the oscillations. If then for real $\boldsymbol{k}$ we find a complex ω with $\text{Im}\omega > 0$, a perturbation in the form of a plane monochromatic wave, $e^{i(\boldsymbol{k}\cdot\boldsymbol{r})-i\omega t}$, will grow in time without bounds and the distribution function will be unstable.

In reality, however, small perturbations do not have the form of separate plane monochromatic waves but are wavepackets, that is, a superposition of plane monochromatic waves. The asymptotic behaviour of a wavepacket may differ significantly from the behaviour of separate plane monochromatic waves. Indeed, even if the separate components in the wavepacket would grow in time without bounds, nevertheless the wavepacket as a whole may remain bounded in a given point of space as the perturbation may "be carried away" downstream.

If in the wavepacket $u(x, t)$, where u is some quantity characterizing the state of the system (for the sake of simplicity we consider a one-dimensional packet), the perturbation remains bounded for x = constant and $t \to \infty$, notwithstanding the presence of components with $\text{Im}\omega > 0$ (in this case the perturbation generally approaches zero),

$$\lim_{\substack{t\to\infty\\ x=\text{constant}}} u(x, t) = 0, \tag{6.6.1.1}$$

one speaks about a *convective* instability.

If, however, the perturbation $u(x, t)$ growth without bounds for given x as $t \to \infty$,

$$\lim_{\substack{t\to\infty\\ x=\text{constant}}} u(x, t) = \infty, \tag{6.6.1.2}$$

the instability is called an *absolute* instability.†

When studying instabilities it is thus insufficient to verify the existence of complex frequencies of the dispersion equation $D(\boldsymbol{k}, \omega) = 0$, but one must also find out how a wavepacket behaves in a given point of space as $t \to \infty$. It is clear that such a way of stating the problem is characteristic not only for a plasma but for any oscillatory system, provided it is sufficiently extended.

To solve the problem of the stability of a system it is necessary to determine the development of an initial perturbation in a way similar to the one used in Subsection 4.2.1 when studying longitudinal oscillations in a plasma. We shall assume that the state vector $u = (u_1, u_2, \ldots, u_n)$ satisfies a set of partial differential equations with constant coefficients:

$$\sum_{j=1}^{n} P_{ij}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u_j(x, t) = 0, \quad i = 1, 2, \ldots, n, \tag{6.6.1.3}$$

where the P_{ij} are polynomials in $\partial/\partial x$ and $\partial/\partial t$ with constant coefficients.

† The concepts of absolute and convective instabilities were introduced by Twiss (1951a) and Landau and Lifshitz (1959).

We must find a solution of this homogeneous system satisfying well-defined boundary conditions. It is, however, more convenient to solve the equivalent problem of the following set of inhomogeneous equations:

$$\sum_{j=1}^{n} P_{ij}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u_j(x, t) = f_i(x, t), \tag{6.6.1.4}$$

where the $f_i(x, t)$ are given functions ("external forces") which are continuous and non-vanishing in the bounded intervals (x_1, x_2) and (t_1, t_2).

One can express the solution $u_i(x, t)$ of the set (6.6.1.4) in terms of a Green matrix

$$u_i(x, t) = \sum_j \iint g_{ij}(x', t') f_j(x-x', t-t')\, dx'\, dt', \tag{6.6.1.4'}$$

where the $g_{ij}(x, t)$ are the solutions of the simpler set of equations

$$\sum_j P_{ij} g_{jk}(x, t) = \delta_{ik}\delta(x)\,\delta(t). \tag{6.6.1.5}$$

Let us introduce Fourier transforms with respect to the coordinate and Laplace transforms with respect to the time for the Green matrix $g_{ij}(x, t)$:

$$g_{ij}(x, t) = \frac{1}{2\pi i} \int_{\mathcal{L}} e^{pt}\, dp \int_{-\infty}^{+\infty} e^{ikx} g_{ij}(k, p)\, dk, \tag{6.6.1.6}$$

$$g_{ij}(k, p) = \int_{0}^{\infty} e^{-pt}\, dt \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} g_{ij}(x, t)\, dx, \tag{6.6.1.7}$$

where the integration contour $\mathcal{L}$ is a straight line parallel to the imaginary p-axis which lies to the right of all singularities of the function

$$\varphi_{ij}(k, p) = \int_{-\infty}^{+\infty} e^{ikx} g_{ij}(k, p)\, dk.$$

From (6.6.1.5) we find

$$g_{ij}(k, p) = \frac{1}{2\pi} P_{ij}^{-1}(ik, p),$$

where the P_{ij}^{-1} are elements of the matrix which is the inverse of the matrix P_{ij}. From this expression and (6.6.1.6) it follows that

$$g_{ij}(x, t) = \frac{1}{4\pi^2} \int_{\Omega} e^{-i\omega t}\, d\omega \int_{-\infty}^{+\infty} e^{ikx} P_{ij}^{-1}(ik, -i\omega)\, dk, \tag{6.6.1.8}$$

where $\omega = ip$ and the integration contour Ω is a straight line parallel to the real ω-axis and lying above all singularities of the functions $\varphi_{ij}(x, -i\omega)$.

Noting that the determinant $\mathrm{Det}|P_{ij}(ik, -i\omega)|$ is the same as the left-hand side of the dispersion equation

$$D(k, \omega) = 0,$$

we can write eqn. (6.6.1.8) in the form

$$g_{ij}(x, t) = \frac{1}{4\pi^2} \int_{\Omega} e^{-i\omega t} d\omega \int_{-\infty}^{+\infty} e^{ikx} \frac{A_{ji}(k, \omega)}{D(k, \omega)} dk, \tag{6.6.1.9}$$

where the $A_{ij}(k, \omega)$ are the cofactors of the elements of the matrix $P_{ij}(ik, -i\omega)$.

We first of all perform in eqn. (6.6.1.9) the integration over ω. To do this we complete the contour Ω by a semicircle of infinitely large radius below it. Using the fact that A_{ji}/D tends to zero as $|\omega| \to \infty$ and applying the residue theorem we find

$$g_{ij}(x, t) = \frac{-i}{2\pi} \int_{-\infty}^{+\infty} \sum_{m=1}^{n} \frac{A_{ji}[k, \omega_m(k)]}{D_{\omega}[k, \omega_m(k)]} e^{ikx - i\omega_m(k)t} dk, \tag{6.6.1.10}$$

where $D_{\omega} \equiv \partial D/\partial\omega$ and where the summation is over all roots $\omega = \omega_m(k)$ of the dispersion equation. We note that in the branch points of the function $\omega = \omega_m(k)$, in which

$$\omega_r(k) = \omega_s(k),$$

the denominator of the corresponding terms in the integrand in eqn. (6.6.1.10) vanishes, $D_{\omega} = 0$. The two terms corresponding to $m = r$ and $m = s$ then tend to infinity. However, the sum of those two terms vanishes, as they mutually cancel one another.

We shall now assume that to some real values of k there correspond complex values $\omega_m(k)$ and we shall find out the form of the instability—whether it is absolute or convective. To do this, we put $x = 0$ in eqn. (6.6.1.10) and replace the integration variable k by ω_m in the integrand:

$$g_{ij}(0, t) = \frac{-i}{2\pi} \sum_{m} \int_{\Omega_m} \frac{A_{ji}[k(\omega_m), \omega_m]}{D_{\omega}[k(\omega_m), \omega_m]} \frac{\exp(-i\omega_m t)}{V_m} d\omega_m,$$

where $V_m = d\omega_m/dk$ and Ω_m is a contour in the ω_m-plane corresponding to the real k-axis.

Using the fact that

$$D_{\omega}\frac{d\omega}{dk} + D_k = 0, \quad D_k \equiv \frac{\partial D}{\partial k},$$

we can write the expression for $g_{ij}(0, t)$ in the form

$$g_{ij}(0, t) = \frac{i}{2\pi} \sum_{m} \int_{\Omega_m} \frac{A_{ji}[k(\omega_m), \omega_m]}{D_k[k(\omega_m), \omega_m]} \exp(-i\omega_m t)\, d\omega_m. \tag{6.6.1.11}$$

If there are no singularities of the integrand of (6.6.1.11) between each contour Ω_m and the real ω_m-axis, we can deform each integration contour Ω_m into the real axis, and we find that

$$g_{ij}(0,t) = \frac{i}{2\pi}\sum_m \int_{-\infty}^{+\infty} \frac{A_{ji}}{D_k} \exp(-i\omega_m t)\, d\omega_m .$$

This expression is the Fourier transform of the function A_{ji}/D_k and tends to zero as $t \to \infty$, according to the Riemann–Lebesgue theorem (Titchmarsh, 1937), and this indicates the presence of convective instability.

Convective instability occurs therefore in the case when there are no singularities of the integrand of (6.6.1.11) between each contour Ω_m and the real ω_m-axis. However, if between one of the contours Ω_m and the real ω_m-axis there is at least one singularity $\omega' + i\omega''(\omega'' > 0)$ of the integrand of (6.6.1.11), we get a contribution from the singularity in the form $e^{\omega'' t}e^{-i\omega' t}$ when we deform the contour Ω_m into the real axis, and this indicates the presence of absolute instability.

In that case the function $u(0, t)$ has thus the following form as $t \to \infty$

$$u(0,t) = \sum_r a_r \exp(\omega_r'' t) \exp(-i\omega_r' t),$$

where $\omega_r' + i\omega_r''(\omega_r'' > 0)$ are the singularities of the function A_{ji}/D_k which are situated between one of the contours Ω_m and the real ω_m-axis, and the a_r are constants.

Let us now elucidate the nature of the singularities $\omega_r' + i\omega_r''$ of the function A_{ji}/D_k. These singularities may be of two kinds: (1) zeroes of the function D_k, (2) branch points of the function $k = k(\omega_m)$. To determine the branch points of the function $k(\omega)$ we investigate the behaviour of the left-hand side $D(k, \omega)$ of the dispersion equation close to the point (k_0, ω_0), where k_0 and ω_0 are connected through the dispersion equation

$$D(k_0, \omega_0) = 0.$$

Expanding $D(k, \omega)$ in a power series in $k - k_0$ and $\omega - \omega_0$, we get

$$D_\omega(\omega-\omega_0) + D_k(k-k_0) + \tfrac{1}{2}D_{kk}(k-k_0)^2 + \ldots = 0,$$

where D_ω, D_k, D_{kk}, ... are partial derivatives of the function $D(k, \omega)$ in the point k_0, ω_0.

If $D_\omega \neq 0$ and $D_k \neq 0$, $k(\omega)$ is a single-valued analytical function in the point $\omega = \omega_0$. If, however, $D_k = 0$, $D_\omega \neq 0$, $D_{kk} \neq 0$, the function $k(\omega)$ near the point $\omega = \omega_0$ behaves as

$$k - k_0 = \pm \sqrt{\left(\frac{2D_\omega}{D_{kk}}\right)} \sqrt{(\omega-\omega_0)} + \ldots,$$

that is, the point $\omega = \omega_0$ will be a second-order branch point of the function $k(\omega)$. If $D_\omega \neq 0$, $D_k = 0$, $D_{kk} = 0$, $D_{kkk} \neq 0$, the point $\omega = \omega_0$ will be a third-order branch point.

We see that to find the second-order branch points we must solve the set of equations

$$D(k,\omega) = 0, \quad D_k(k,\omega) = 0,$$

and to find the third order branch points we must solve the set of equations

$$D(k,\omega) = 0, \quad D_k(k,\omega) = 0, \quad D_{kk}(k,\omega) = 0.$$

It is clear that, as a rule the first set will have a solution, but the second set will not have a solution. We shall therefore limit ourselves to considering only second-order branch points.

We see that the zeroes of the function D_k are branch points of the function $k(\omega)$, that is, the nature of the instability is determined by the position of the branch points of the function $k(\omega)$. Indeed, if for no m there are branch points of the function $k = k(\omega_m)$ between the contour Ω_m and the real axis, the instability will be convective; in the opposite case the instability will be absolute (Faĭnberg, Kurilko, and Shapiro, 1961).

As an example we shall study the dispersion equation

$$(\omega - kv)^2 = k^2c^2 - \nu^2, \quad v, c, \nu > 0.$$

Solving this equation we find

$$\omega_{1,2}(k) = kv \pm \sqrt{(k^2c^2 - \nu^2)}. \tag{6.6.1.12}$$

When $-\nu/c < k < \nu/c$ the frequency ω is complex, that is, we have instability. To find the nature of this instability we find the inverse function

$$k(\omega) = \frac{v\omega \pm \sqrt{[c^2\omega^2 - (v^2 - c^2)\nu^2]}}{v^2 - c^2}.$$

We see that the function $k(\omega)$ has branch points when

$$\omega = \pm \frac{\nu}{c} \sqrt{(v^2 - c^2)}.$$

If $v > c$, the branch points lie on the real axis. In that case the contour Ω_m can be deformed into the real axis, that is, the instability is convective.

We shall show that, if $v < c$, one of the branch points,

$$\omega_0 = \frac{i\nu}{c} \sqrt{(c^2 - v^2)},$$

of the function $k(\omega)$ lies between the contour Ω_m and the real ω-axis, that is, if $v < c$ the instability is absolute. To do this, we note that the contour Ω_m is described by the point $\omega_m(k)$, when k moves along the real axis from $-\infty$ to $+\infty$, while $\omega_m(k)$ is given by eqn. (6.6.1.12). The two signs in front of the radical in eqn. (6.6.1.12) correspond to two contours Ω_1 and Ω_2. To identify the absolute instability it is sufficient to consider only the contour Ω_1 which lies in the upper ω-half-plane.

One can see from (6.6.1.12) that the contour Ω_1 intersects the imaginary axis for $k = 0$ in the point

$$\omega_* = i\nu.$$

As $\text{Im}\,\omega_0 < \text{Im}\,\omega_*$, the branch point ω_0 lies between the contour Ω_1 and the real axis, that is, the instability is absolute when $v < c$.

6.6.2. THE TRAVELLING WAVES METHOD

When we obtained the criterion for absolute and convective instability we first integrated over ω in eqn. (6.6.1.9). We obtain another form of this criterion, if we first integrate over k. To do this we complete the contour by adding to the real k-axis the semicircle of infinitely large radius which lies in the upper half-plane when $x > 0$ and in the lower half-plane when $k < 0$. Using the fact that the integrand tends to zero as $|k| \to \infty$, and using the residue theorem, we get

$$g_{ij}(x, t) = \begin{cases} \dfrac{i}{2\pi} \displaystyle\int_{\Omega} d\omega \sum_{\alpha} \frac{A_{ji}[k_\alpha(\omega), \omega]}{D_k[k_\alpha(\omega), \omega]} \exp(ik_\alpha(\omega)x - i\omega t), & \text{when} \quad x > 0, \\ -\dfrac{i}{2\pi} \displaystyle\int_{\Omega} d\omega \sum_{\beta} \frac{A_{ji}[k_\beta(\omega), \omega]}{D_k[k_\beta(\omega), \omega]} \exp(ik_\beta(\omega)x - i\omega t), & \text{when} \quad x < 0, \end{cases} \tag{6.6.2.1}$$

where $D_k = \partial D/\partial k$ and $k_\alpha(\omega)$ and $k_\beta(\omega)$ are roots of the dispersion equation which lie, respectively, in the upper and lower half-plane when $\text{Im}\,\omega > M$—the straight line $\text{Im}\,\omega = M$ lies above all singularities of the integrands of (6.6.2.1)—

$$\text{Im}\,k_\alpha(\omega) > 0, \quad \text{Im}\,k_\beta(\omega) < 0.$$

We shall say that the terms occurring in the first sum of (6.6.2.1) describe waves which propagate to the right, and the terms which occur in the second sum waves which propagate to the left.

We are interested in the asymptotic behaviour of $g_{ij}(x, t)$ as $t \to \infty$. To establish that behaviour we shall shift the contour Ω downwards. It is clear that if the first singularity of the integrand $\varphi(x, -i\omega)$ which we shall then meet with lies at the point ω_0, $g_{ij}(x, t)$ will behave as $\exp(-i\omega_0 t)$ as $t \to \infty$, that is, it will tend to infinity if $\text{Im}\,\omega_0 > 0$ and to zero if $\text{Im}\,\omega_0 \leq 0$. Convective instability thus occurs when there are no singularities of the integrand $\varphi(x, -i\omega)$ in the upper half-plane $\text{Im}\,\omega > 0$.

Repeating the discussion of the preceding subsection we find that the singularities of the integrand of (6.6.2.1) can only be the branch points of the function $k(\omega)$. However, not all branch points of the function $k(\omega)$ contribute to the integral (6.6.2.1). If different branches $k_r(\omega_0)$ and $k_s(\omega_0)$ become equal in the branch point ω_0, and if these two branches correspond to two waves moving in the same direction the point ω_0 will in actual fact not be a branch point of the integrand as we must sum over α or β in eqns. (6.6.2.1). The criterion for absolute instability thus consists of the requirement that there must be a branch point of the function $k(\omega)$ in the upper half-plane $\text{Im}\,\omega > 0$ in which two branches of the function $k(\omega)$ coincide which correspond to waves moving in opposite directions (Briggs, 1964).

We shall study by this method the nature of the instability of the system characterized by the dispersion equation

$$(\omega - kv)^2 = k^2c^2 - \nu^2, \quad v, c, \nu > 0,$$

which we studied in the preceding subsection.

If $v > c$ the imaginary part of the function

$$k(\omega) = \frac{v\omega \pm \sqrt{[c^2\omega^2 - (v^2 - c^2)\nu^2]}}{v^2 - c^2}$$

will be positive as $\text{Im}\,\omega \to +\infty$. Therefore, if $v > c$ both waves travel to the right and, hence, the instability is convective.

If, however, $v < c$ one of the waves with $k(\omega)$ propagates to the right, and the other one to the left. In that case one of the branch points, namely,

$$\omega = \frac{iv}{c}\sqrt{(c^2-v^2)}$$

lies in the upper half-plane. We therefore are dealing with absolute instability when $v < c$.

6.6.3. CRITERION FOR AMPLIFICATION AND BLOCKING OF OSCILLATIONS

Oscillatory systems with a dispersion equation which has complex solutions can, in principle, be used for the generation and amplification of oscillations. If the instability of the system is absolute, the system can be a generator of oscillations with frequencies lying in the range where there is absolute instability.

In the case of convective instability the perturbation "drifts with the stream"; this means that this form of instability corresponds to an amplification, but not to a generation of oscillations, in other words, systems with convective instabilities can be amplifiers of oscillations.†

Clearly, oscillations can also be amplified for which, for real ω, $\text{Im}\,k < 0$; we assume that the system is semi-infinite, $x > 0$, and the x-axis is chosen such that the amplified waves move in the direction of increasing x.

However, the condition $\text{Im}\,k < 0$ for real ω is by itself insufficient for amplification of the oscillations. For instance, in a waveguide oscillations with frequencies below the critical frequency cannot propagate, even though they correspond to imaginary values of k. A similar situation occurs in a plasma for electromagnetic waves with frequencies below the Langmuir frequency in the absence of a magnetic field.

The existence of complex k-values for real ω can thus indicate either amplification or blocking of the oscillations.‡ To establish criteria for the amplification and blocking of oscillations we choose the external forces $f_i(x, t)$ occurring on the right-hand side of eqns. (6.6.1.4) such that all waves propagate in the positive x-direction, that is, that we have the following condition:

$$u_i(x, t) = 0 \quad \text{when} \quad x < 0. \tag{6.6.3.1}$$

According to (6.6.2.1) the imaginary part of $k(\omega)$ for waves moving to the right must be positive when $\text{Im}\,\omega > M$. On the other hand, the condition for spatial growth has, clearly, the form

$$\text{Im}\,k(\omega) < 0, \quad \omega : \text{real}.$$

It is thus necessary for wave amplification that the imaginary part of $k(\omega)$ has different

† Nevertheless systems with convective instabilities can also be used to generate oscillations if the input and output in them are connected; due to this feedback is realized and the perturbation which is "carried away" turns back, that is, the instability in the system takes on the character of an absolute instability.

‡ Twiss (1951b) introduced the concepts of amplification and blocking of oscillations.

signs for $\operatorname{Im} \omega \to +\infty$ and for $\operatorname{Im} \omega = 0$. If, on the other hand, the quantity $\operatorname{Im} k(\omega)$ has the same sign for $\operatorname{Im} \omega \to +\infty$ and $\operatorname{Im} \omega = 0$, the oscillations are blocked (Briggs, 1964).

As an example we consider a system with the dispersion equation

$$\omega^2 = k^2c^2 + \nu^2 . \tag{6.6.3.2}$$

Clearly, complex values of k are possible only when $-\nu < \omega < \nu$, and then each value of ω corresponds to two solutions of the dispersion equation

$$k_1(\omega) = \frac{i}{c}\sqrt{(\nu^2-\omega^2)}, \quad k_2(\omega) = -\frac{i}{c}\sqrt{(\nu^2-\omega^2)}. \tag{6.6.3.3}$$

One sees easily that for $\operatorname{Im} \omega = 0$ and for $\operatorname{Im} \omega \to \infty$, we have

$$\operatorname{Im} k_1(\omega) > 0, \quad \operatorname{Im} k_2(\omega) < 0,$$

that is, in the frequency range $-\nu < \omega < \nu$ the system does not transmit oscillations. We note that this result is physically translucent, as eqn. (6.6.3.2) is the dispersion equation of a waveguide with a critical frequency ν, and it is well known that waveguides do not transmit oscillations with frequencies below the critical one.

As a second example we consider a system with the dispersion equation

$$(\omega - kv)^2 = k^2c^2 - \nu^2 .$$

Solving this equation for k we get

$$k_1(\omega) = \frac{v\omega + \sqrt{[c^2\omega^2 - \nu^2(v^2-c^2)]}}{v^2-c^2}, \quad k_2(\omega) = \frac{v\omega - \sqrt{[c^2\omega^2 - \nu^2(v^2-c^2)]}}{v^2-c^2}.$$

If $v < c$, all real values of ω correspond to real values of k. In that case the oscillations are transmitted. In fact, we showed in Subsections 6.6.1 and 6.6.2 that the system is absolutely unstable when $v < c$.

If, however, $v > c$ we find that when

$$-\frac{\nu}{c}\sqrt{(v^2-c^2)} < \omega < \frac{\nu}{c}\sqrt{(v^2-c^2)},$$

the quantity k becomes complex. In that case both waves $k_1(\omega)$ and $k_2(\omega)$ have a positive imaginary part as $\operatorname{Im} \omega \to +\infty$. When $\operatorname{Im} \omega = 0$ we have

$$\operatorname{Im} k_1(\omega) > 0, \quad \operatorname{Im} k_2(\omega) < 0.$$

The solution $k_1(\omega)$ therefore corresponds to blocking and the solution $k_2(\omega)$ to amplification of the oscillations.

6.6.4. STURROCK'S RULES

Generally speaking, it is very difficult to apply practically the criteria for the instability and amplification of waves, which were formulated in the preceding subsections, as the finding of branch points of the functions in regions which are bounded by the real axes of the ω_m-planes and the contours Ω_m—which themselves must be found—or the determination of the direction of propagation of the waves is a complex and laborious problem.

The situation is considerably simplified when the dispersion equation $D(k, \omega) = 0$ is an algebraic equation with real coefficients which in the large $|k|$-values region (or, what amounts to the same, in the large $|\omega|$-values region) splits into a product of factors of the form $\omega - sk$, where s is a non-vanishing constant. In this case which corresponds to replacing the initial integro-differential equations which describe the system by partial differential equations, neglecting dissipative effects and assuming the signal propagation velocity to be finite, we can establish the nature of the instability from the general shape of the curve which describes the dispersion equation in the k, ω-plane.

We shall to begin with assume that $D(k, \omega)$ is a second-degree polynomial in k and ω. Four kinds of dispersion curves, which are shown in Fig. 6.6.1, are then possible.

We shall first elucidate the problem of the instability of the systems corresponding to these dispersion curves. To do that we consider the intersection of straight lines, parallel to the ω-axis, with these curves. The straight line k = constant intersects the curves shown in Figs. 6.6.1a and 6.6.1b in two points for any value of the constant. These dispersion curves thus correspond to stable systems.

On the other hand, in the case of the dispersion curves shown in Figs. 6.6.1c and 6.6.1d, the straight line k = constant does not intersect the dispersion curves for all values of the constant. In fact, there is a range of real k-values, (k_A, k_B), which corresponds to complex ω-values. As we have assumed that the coefficients of the dispersion equation were real, the complex ω-values will be each other's complex conjugate, and one of them will have a positive imaginary part. These dispersion curves thus correspond to unstable systems.

We note that the direction of propagation of the waves can also be directly established from the shape of the dispersion curve. Indeed, in the region of large $|\omega|$ each branch of the dispersion curve is described by an equation of the form

$$k = \frac{\omega}{s},$$

where s is real. However, according to subsection 6.6.2, for waves propagating to the right $\operatorname{Im} k > 0$ as $\operatorname{Im} \omega \to +\infty$, and for waves propagating to the left, $\operatorname{Im} k < 0$ as $\operatorname{Im} \omega \to +\infty$ Waves propagating to the right therefore correspond to positive s and waves propagating to the left to negative s. On the other hand, positive s correspond to asymptotes of the dispersion curve inclined to the right and negative s to asymptotes inclined to the left. The slope of the asymptotes thus determines the direction of propagation of the waves.

One can easily explain the physical meaning of this result. To do this we recall that to find the slope of the asymptotes we must write the dispersion equation in the form

$$D(k, \omega) \equiv \prod_{j=1}^{n} (\omega - s_j k) + D_1(k, \omega) = 0, \tag{6.6.4.1}$$

where $D_1(k, \omega)$ is a polynomial of degree n_1, $n_1 < n$, and the s_j are constants. Neglecting in this eqation the term D_1, we find the slopes of the asymptotes:

$$\frac{\omega}{k} = s_j .$$

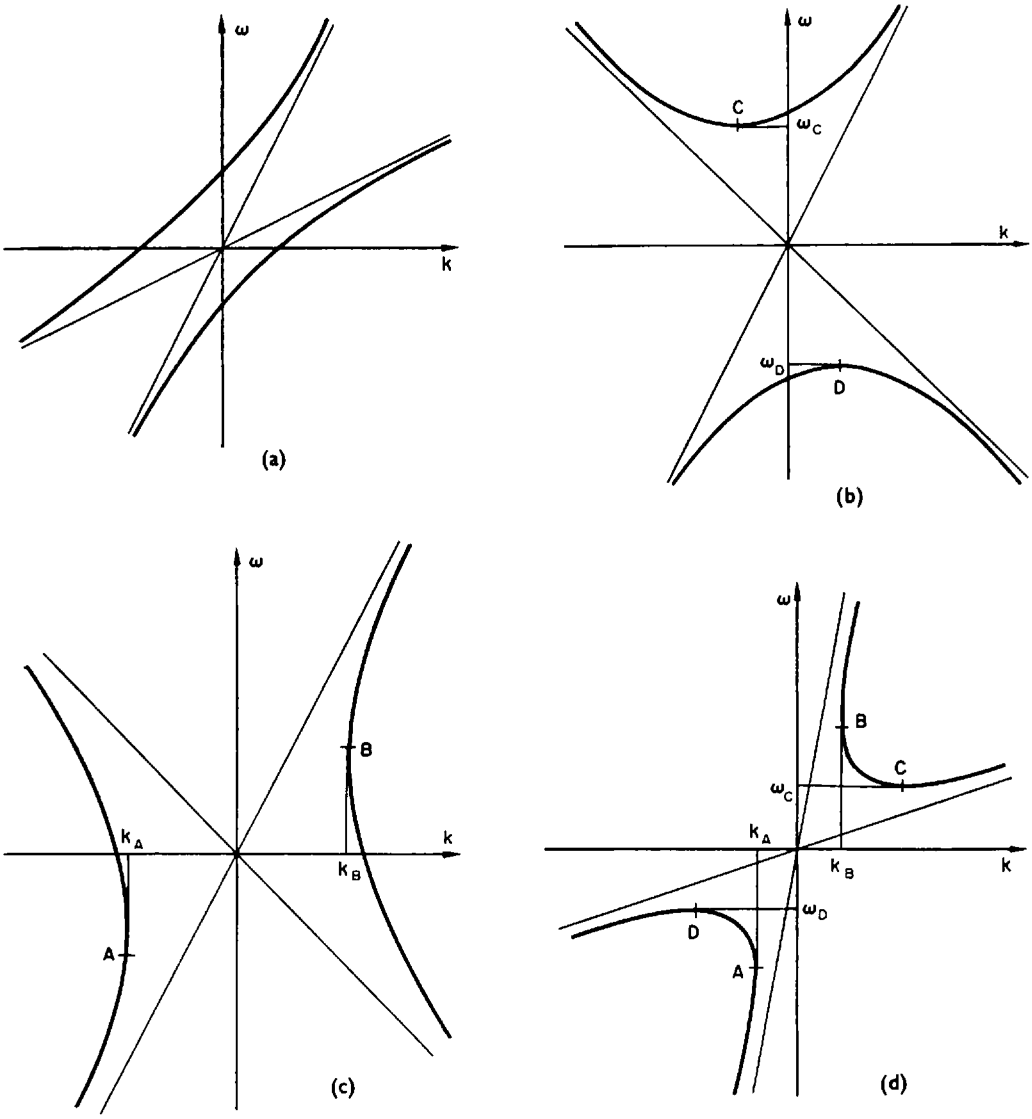

FIG. 6.6.1. Dispersion curves in the case when the dispersion equation is a second-degree polynomial. (a) Stability, transmission; (b) stability, blocking; (c) absolute instability, transmission; (d) convective instability, amplification.

The quantities s_j are clearly the phase velocities of the waves which propagate in the system when we neglect the term D_1 in the dispersion equation (6.6.4.1). We shall call these waves eigenwaves.

When the term D_1 is taken into account the properties of the waves propagating in the system will be different from the properties of the eigenwaves. One can say that the term D_1 describes the interaction between the eigenwaves. As the quantities s_j are real, while the quantities ω must be complex for instability, one can say that instability arises due to interactions between the waves.

Let us now turn to the problem of the nature of the instability. We have already seen that Figs. 6.6.1c and 6.6.1d correspond to unstable systems. The difference between the two curves consists in the fact that in the first case the asymptotes are inclined in different directions, while they are inclined in the same direction in the second case. As we have just shown this corresponds to waves moving in opposite directions in the case of Fig. 6.6.1c, and to waves moving in the same direction in the case of Fig. 6.6.1d. Therefore in the case of Fig. 6.6.1c in the branch points of the function $k(\omega)$ two branches of this function coincide corresponding to waves propagating in opposite directions, while in the case of Fig. 6.6.1d two branches coincide corresponding to waves moving in the same direction.

Moreover, in the case of Fig. 6.6.1c in the branch point of the function $k(\omega)$ the imaginary part of ω is positive, $\text{Im}\ \omega > 0$. (This follows from the fact that in branch points of the function $k(\omega)$ we have $d\omega/dk = 0$, that is, for real ω and k the tangent to the dispersion curve is horizontal in these points.) As there are no horizontal tangents in Fig. 6.6.1c, at least one of the quantities ω or k must be complex in the branch point. It is clear from Fig. 6.6.1c that real values of ω cannot correspond to complex values of k. Therefore, the quantity ω must be complex in the branch points of the function $k(\omega)$. As the polynomial $D(k, \omega)$ has real coefficients, the ω-values in the branch points of the function $k(\omega)$ are each other's conjugate complex and one of them must have a positive imaginary part.

According to the criteria for absolute and convective instability, established in Subsection 6.6.2, Fig. 6.6.1c corresponds to absolute instability, and Fig. 6.6.1d to convective instability. The instability will thus be absolute or convective depending on whether the asymptotes are inclined in different directions or in the same direction (*first Sturrock rule*; Sturrock, 1959).

Let us now explain how we can distinguish through the dispersion curve transmission, blocking, and amplification of oscillations. In the case of transmission real values of ω correspond to real values of k. The straight line ω = constant then intersects the dispersion curve for all values of the constant. This case is met with in Figs. 6.6.1a and 6.6.1c. On the other hand, in the case of Figs. 6.6.1b and 6.6.1d the straight line ω = constant does not intersect the dispersion curve for all values of the constant. When it does not intersect the dispersion curve the values of ω correspond to complex values of k, that is, we have either amplification or blocking of the oscillations.

The difference between Figs. 6.6.1b and 6.6.1d consists in the fact that in the case of Fig. 6.6.1b all real values of k correspond to real values of ω, while in the case of Fig. 6.6.1d the values of k in the interval (k_A, k_B) correspond to complex values of ω. From this it follows that in the case of Fig. 6.6.1b the sign of $\text{Im}\ k$ in the upper ω-half-plane cannot change, and according to the criterion established in Subsection 6.6.3 we then have blocking of the oscillations (Rolland, 1965). In the case of Fig. 6.6.1d some real values of k correspond to complex values of ω in which the sign of $\text{Im}\ k$ changes. As by assumption the coefficients of the dispersion equation are real, the values of ω for which $\text{Im}\ k$ changes sign are each other's conjugate complex, and one of them must lie in the upper half-plane. According to the criterion established in Subsection 6.6.3 we then have amplification of the oscillations.

The difference between Figs. 6.6.1b and 6.6.1d lies in the fact that the asymptotes of the dispersion curve in Fig. 6.6.1b are inclined in different directions while in Fig. 6.6.1d the asymptotes are inclined in the same direction. If the asymptotes of the dispersion curve are inclined in different directions we have thus blocking of oscillations, while we have amplification of oscillations if the asymptotes are inclined in the same direction (*second Sturrock rule*).†

We see that the same dispersion curve corresponds to the problem of instability and the problem of amplification of waves and that if we have absolute instability, we also have transmission of oscillations. If, however, we have convective instability, we have also amplification of oscillations. Which of these possibilities is realized depends, of course, on the actual physical circumstances.

So far we have assumed that the function $D(k, \omega)$ is a second-degree polynomial in k and ω. We shall now show that the general case when the function $D(k, \omega)$ is a polynomial of arbitrary degree n can be reduced to the earlier considered case of $n = 2$. To do this we assume that the coefficients of the polynomial $D(k, \omega)$ depend on a parameter ξ which we can vary in such a way that the polynomial $D(k, \omega)$ splits into a product of linear factors of the form $\omega - s_j k - a_j$. To fix the ideas we shall assume that the value of the parameter $\xi = 1$ corresponds to the original equation and the value $\xi = 0$ to the division of the polynomial into factors.

It is clear then when $\xi = 0$ the dispersion curves will be a set of straight lines. We shall assume that no more than two lines intersect in any point and that when ξ changes within the half-open interval $0 < \xi \leqslant 1$ the topological character of the dispersion curves does not change.

When ξ is small the bands of instability (or amplification) will be situated near the points where the straight lines $\omega - s_j k - a_j = 0$, into which the dispersion curves split up for $\xi = 0$, intersect. On the other hand, as we have assumed that the lines only intersect in pairs, for small ξ the dispersion curves must clearly be similar to the curves shown in Fig. 6.6.1. In other words, for small ξ the study of the general case of a polynomial of arbitrary degree n reduces to the earlier considered case of $n = 2$.

One can show that when the parameter ξ increases continuously the character of the instability cannot change (Polovin, 1962). This enables us to determine the character of the instability of the original dispersion equation corresponding to the value of the parameter $\xi = 1$.

Let us as an example consider the dispersion equation

$$\frac{\omega_b^2}{(\omega - k u_b)^2} + \frac{\omega_p^2}{\omega^2 - k^2 v_p^2} = 1, \qquad (6.6.4.2)$$

which in the case

$$\omega_b < \omega_p, \quad v_p < u_b < v_p\,[\sqrt{1 + (\omega_b^2/\omega_p^2)}] \qquad (6.6.4.3)$$

corresponds to the dispersion curve shown in Fig. 6.6.2c. We shall consider the quantity ω_p^2 to be variable and replace ω_p^2 by $\omega_p^2 \xi$:

$$\frac{\omega_b^2}{(\omega - k u_b)^2} + \frac{\omega_p^2 \xi}{\omega^2 - k^2 v_p^2} = 1. \qquad (6.6.4.4)$$

† This rule was formulated by Sturrock (1959), strictly proven by Polovin (1962, 1963d), and heuristically derived by Akhiezer, Akhiezer, Polovin, Sitenko, and Stepanov (1967).

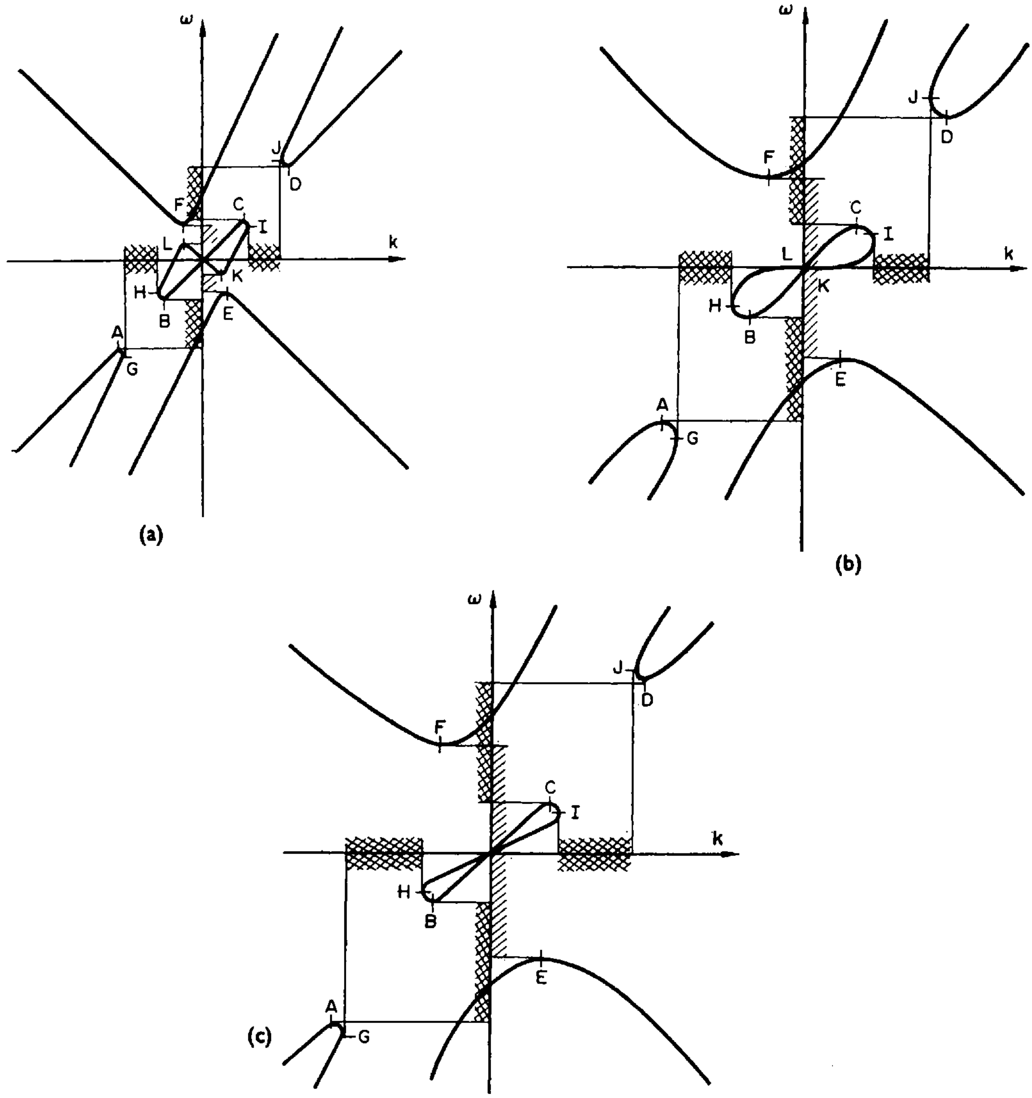

FIG. 6.6.2. Reduction of the dispersion equation to a second-degree polynomial. (a) Small ξ; (b) $\xi = \xi_0$; (c) $\xi = 1$.

When $\xi = 0$, eqn. (6.6.4.4) splits into four linear equations

$$\omega - ku_b = \pm\omega_b, \quad \omega = \pm kv_p,$$

and the dispersion curve degenerates into four straight lines.

For small ξ the dispersion curve has the form shown in Fig. 6.6.2a. It is clear from this figure that the system described by the dispersion equation (6.6.4.4) for small ξ shows convective instability. The bands of convective instability correspond to the wavenumber ranges (k_G, k_H) and (k_I, k_J), where k_M denotes the wavenumber corresponding to the point M.

If we solve the problem of the amplification of oscillations in the system described by the dispersion eqn. (6.6.4.4) for small ξ, it is immediately clear from Fig. 6.6.2a that there are two amplification bands in the frequency intervals (ω_A, ω_B) and (ω_C, ω_D) and also two blocking bands in the frequency intervals (ω_E, ω_K) and (ω_L, ω_F).

When ξ increases the dispersion curve (6.6.4.4) is deformed but its topological character is not changed. When the parameter ξ reaches the value

$$\xi_0 = \frac{\omega_b^2}{\omega_p^2} \frac{v_p^2}{u_b^2},$$

the tangent to one of the branches of the dispersion curve, passing through the origin, becomes horizontal (see Fig. 6.6.2b). The two blocking bands (ω_E, ω_K) and (ω_L, ω_F) then merge into one blocking band (ω_E, ω_F). It follows from inequalities (6.6.4.3) that this merging of the blocking bands occurs when $\xi < 1$.

When ξ further increases the dispersion curve takes the shape shown in Fig. 6.6.2c. As the curves shown in Figs. 6.6.2 a, b, and c are topologically equivalent, the conclusions about the character of the instability for small ξ remains valid also when $\xi = 1$, that is, for the original dispersion eqn. (6.6.4.2).

Comparing Figs. 6.6.2 a, b, and c we can conclude that the original system, described by the dispersion eqn. (6.6.4.2) is convectively unstable in the wavenumber intervals (k_G, k_H) and (k_I, k_J). Moreover, the system considered has two amplification bands in the frequency intervals (ω_A, ω_B) and (ω_C, ω_D) and also a blocking band (ω_E, ω_F).

6.6.5 GLOBAL INSTABILITY

So far we have assumed when studying the instability of dynamic systems that they are infinitely extended and we have therefore neglected the presence of boundaries. All the same, the presence of boundaries may be very important as waves may be reflected from them. Because of this there may be realized a feedback between the "input" and the "output" of the system as a result of which a system which is convectively unstable will behave as if it were absolutely unstable. An essential fact is here that this kind of effective absolute instability — which is called *global instabliity*† — will occur also in the limiting case of infinitely extended systems. It is also important that this conclusion is independent of the actual form of the boundary conditions.

It is well known that eigenoscillations in bounded systems arise as a result of the superposition of waves travelling in different directions. The frequencies of these oscillations are discrete and the system will be unstable if at least one of these frequencies has a positive imaginary part.

To distinguish waves travelling to the right from those travelling to the left we must, according to Subsection 6.6.2, determine the sign of the imaginary part of the function $k \equiv k(\omega)$ as $\mathrm{Im}\,\omega \to +\infty$: if in that case $\mathrm{Im}\,k > 0$, the wave travels to the right, and if $\mathrm{Im}\,k < 0$, the wave travels to the left. We shall denote the wavevectors of the waves which travel to the right by $k_1(\omega), k_2(\omega), \ldots, k_s(\omega)$, and the wavevectors of the waves travelling to the left by $k_{s+1}(\omega), k_{s+2}(\omega), \ldots, k_n(\omega)$. We note that these functions are solutions of the dispersion equation,

$$D(k, \omega) = 0,$$

of the unbounded system.

† Kulikovskiĭ (1966) introduced the concept of global instability.

Let now ω be an eigenfrequency of the bounded system. We arrange the wavevectors of the wave moving in a particular direction in order of decreasing Im $k(\omega)$:

$$\begin{gathered}\operatorname{Im} k_1(\omega) > \operatorname{Im} k_2(\omega) > \ldots > \operatorname{Im} k_s(\omega),\\ \operatorname{Im} k_{s+1}(\omega) > \operatorname{Im} k_{s+2}(\omega) > \ldots > \operatorname{Im} k_n(\omega).\end{gathered} \tag{6.6.5.1}$$

We note that for finite ω the quantities Im $k_\alpha(\omega)$ $(1 \leqslant \alpha \leqslant s)$ are not necessarily positive, or the quantities Im $k_\beta(\omega)$ $(s+1 \leqslant \beta \leqslant n)$ necessarily negative.

To obtain the equation for the eigenfrequencies ω we assume that at the left-hand boundary of the system $(x = -L)$ all waves $1, 2, \ldots, n$ with wavevectors $k_1(\omega), k_2(\omega), \ldots, k_n(\omega)$ are excited. Only the first s waves will then move to the right. By virtue of inequalities (6.6.5.1) when the right-hand boundary of the system $(x = L)$ is reached, the largest amplitude will belong to the sth wave, when L is large. If the amplitude of this wave were unity at $x = -L$, its amplitude at $x = L$ will equal

$$\exp(-2L \operatorname{Im} k_s(\omega)).$$

When the sth wave is reflected at the right-hand boundary, the $s+1$st, $s+2$nd, $\ldots$, nth waves with wavevectors $k_{s+1}, k_{s+2}, \ldots, k_n$, which will move to the left, will be excited. When they reach the left-hand boundary of the system, the $s+1$st wave will have the largest amplitude, and its amplitude at $x = -L$ will be equal to

$$T_+ \exp(-2L \operatorname{Im} k_s + 2L \operatorname{Im} k_{s+1}),$$

where T_+ is the coefficient for the transformation of the sth wave into the $s+1$st one at the right-hand boundary of the system. When the $s+1$thwave is reflected from the left-hand boundary of the system, there arises again the sth wave, with amplitude

$$T_+T_- \exp(-2L \operatorname{Im} k_s + 2L \operatorname{Im} k_{s+1}),$$

where T_- is the coefficient for the transformation of the $s+1$st wave into the sth one at the left-hand boundary of the system. Equating this expression to unity and letting L tend to infinity, we get an equation for the eigenfrequency ω:

$$\operatorname{Im} k_s(\omega) = \operatorname{Im} k_{s+1}(\omega). \tag{6.6.5.2}$$

This equation together with the dispersion equation,

$$D(k, \omega) = 0,$$

determines a line in the complex ω-plane. If this line has points situated in the upper half-plane, Im $\omega > 0$, the system will be globally unstable.†

We note that instead of a discrete spectrum of eigenfrequencies, we obtained the continuous line (6.6.5.2), because we took the limit as $L \to \infty$: each point of this line is a limiting point of the discrete eigenfrequencies as $L \to \infty$.

† We have given here a heuristic derivation for the criterion for global instability; a rigorous proof was given by Kulikovskiĭ (1966).

We remark here that a system with absolute instability will always be globally unstable. Indeed, absolute instability means that there are in the upper half-plane $\text{Im}\,\omega > 0$ points satisfying the dispersion equation $D(k, \omega) = 0$, for which

$$k_\alpha(\omega) = k_\beta(\omega), \quad 1 \leqslant \alpha \leqslant s, \quad s+1 \leqslant \beta \leqslant n.$$

In that case, the condition

$$\text{Im}\, k_\alpha(\omega) = \text{Im}\, k_\beta(\omega), \quad \text{Im}\,\omega > 0,$$

is clearly satisfied, and this condition is weaker than (6.6.5.2). By virtue of inequalities (6.6.5.1) from this last equation we find

$$\text{Im}\, k_s(\omega) \leqslant \text{Im}\, k_{s+1}(\omega), \quad \text{Im}\,\omega > 0.$$

On the other hand, from the definition of waves propagating to the right and to the left it follows that as $\text{Im}\,\omega \to +\infty$, we have the inequality

$$\text{Im}\, k_s(\omega) > \text{Im}\, k_{s+1}(\omega).$$

As the function $\text{Im}\,\{k_s(\omega) - k_{s+1}(\omega)\}$ is continuous it follows from this that there are solutions of eqn. (6.6.5.2) in the upper ω-half-plane, that is, that we have global instability.

Let us as an example consider a system with the following dispersion equation:

$$3\omega^2 - 4\omega k + k^2 + 1 = 0.$$

This dispersion equation corresponds to two waves, 1 and 2, with wavevectors

$$k_{1,2} = 2\omega \pm \sqrt{(\omega^2 - 1)}.$$

One sees easily that the system considered is convectively unstable. Indeed, as $\text{Im}\,\omega \to +\infty$ we get $\text{Im}\, k_1 > 0$, $\text{Im}\, k_2 > 0$. Both waves 1 and 2 therefore move to the right, that is, there are no branch points in which wavevectors of waves moving in different directions become equal.

Let us now consider the problem of the global instability of this system. As both waves move to the right, we do not get an equation such as (6.6.5.2) in this case; the system is thus globally stable.

Let us now consider another example where the dispersion equation has the form

$$(3\omega^2 - 4\omega k + k^2 + 1)(\omega + k) = 0.$$

In that case there occur three waves 1, 2, 3 with wavevectors

$$k_{1,2} = 2\omega \pm \sqrt{(\omega^2 - 1)}, \quad k_3 = -\omega,$$

and waves 1 and 2 move to the right, and wave 3 to the left.

It is clear that the system is convectively unstable as the wavevectors of the first two waves are the same as the wavevectors k_1 and k_2 of the previous example—the third wave does not lead to instability. However, in contrast to the previous example, the system will now be globally unstable, and the instability arises because of the existence of the third wave.

To check this, we determine the eigenfrequencies using eqn. (6.6.5.2):

$$\text{Im } k_{1,2}(\omega) = \text{Im } k_3(\omega). \tag{6.6.5.3}$$

Putting

$$\omega = \alpha + \beta i,$$

we get

$$\text{Im } k_{1,2} = 2\beta \pm \sqrt{\{\tfrac{1}{2}[\sqrt{\{(\alpha^2-\beta^2-1)^2+4\alpha^2\beta^2\}}-(\alpha^2-\beta^2-1)]\}},$$

and from eqn. (6.6.5.3) it follows that

$$\alpha^2+17\beta^2-1 = \sqrt{\{(\alpha^2-\beta^2-1)^2+4\alpha^2\beta^2\}}. \tag{6.6.5.4}$$

Squaring both sides of that equation, we find the equation of the curve (6.6.5.3):

$$8\alpha^2+72\beta^2 = 9. \tag{6.6.5.5}$$

We note that when solving the irrational eqn. (6.6.5.4) we might obtain redundant roots. The correct roots of this equation corresponds to the positive square root, that is, they are determined by the condition

$$\alpha^2+17\beta^2 \geqslant 1. \tag{6.6.5.6}$$

One sees easily that the ellipse (6.6.5.5) lies completely in the region (6.6.5.6). This means that all points of the curve (6.6.5.5) satisfy eqn. (6.6.5.3). As part of the ellipse (6.6.5.5) is situated in the upper half-plane ($\beta > 0$), the system considered is globally unstable.

6.6.6. NON-INVARIANT NATURE OF THE CONCEPTS OF ABSOLUTE AND CONVECTIVE INSTABILITIES

One sees easily that the concepts of absolute and convective instabilities are not invariant with respect to the choice of moving coordinate systems. On the other hand, the concepts of amplification and blocking have an invariant character. This can be illustrated by means of the Sturrock rules. Changing from a frame of reference $\mathcal{K}$ to a frame $\mathcal{K}'$ corresponds, clearly, to a rotation of the asymptotes of the dispersion curves by an angle: if φ and φ' denote the angles of inclination of the asymptotes to the k-axis in the frames $\mathcal{K}$ and $\mathcal{K}'$, we have

$$\tan\varphi' = \tan\varphi - v,$$

where v is the velocity of the frame $\mathcal{K}$ relative to the frame $\mathcal{K}'$.

When the magnitude of v changes the asymptotes rotate, going through a horizontal position, but not through a vertical position. The dispersion curve shown in Fig. 6.6.1c can thus change to the dispersion curve, shown in Fig. 6.6.1d. Hence it follows that absolute instability can change to convective instability, and vice versa.

On the other hand, changing from a frame of reference $\mathcal{K}$ to a frame $\mathcal{K}'$ cannot change the dispersion curves shown in Figs. 6.6.1b and 6.6.1d into one another, as this would mean that the asymptotes went through the vertical position, and that is impossible. Therefore, when changing from the frame $\mathcal{K}$ to the frame $\mathcal{K}'$ the amplification and non-transmission bands can vanish, but they cannot change into one another.

6.6.7. NATURE OF THE BEAM INSTABILITY

We shall illustrate the theory given here by the example of a two-stream tube (Sturrock, 1960), which has a dispersion equation of the form

$$\frac{\omega_1^2}{(\omega-ku_1)^2}+\frac{\omega_2^2}{(\omega-ku_2)^2} = 1,$$

where $\omega_{1,2}^2 = 4\pi e^2 n_{1,2}/m_{1,2}$, $n_{1,2}$ and $u_{1,2}$ are the densities and velocities of the two kinds of particles, and $m_{1,2}$ are the particle masses. This dispersion equation corresponds to the dispersion curve shown in Fig. 6.6.3a, if the velocities u_1 and u_2 are in the same direction, and to the dispersion curve shown in Fig. 6.6.3b, if the velocities u_1 and u_2 are in opposite directions. It is immediately clear from these figures that when u_1 and u_2 have the same sign there is an amplification band and a convective instability band, while there is a blocking band and an absolute instability band, when u_1 and u_2 have different signs.

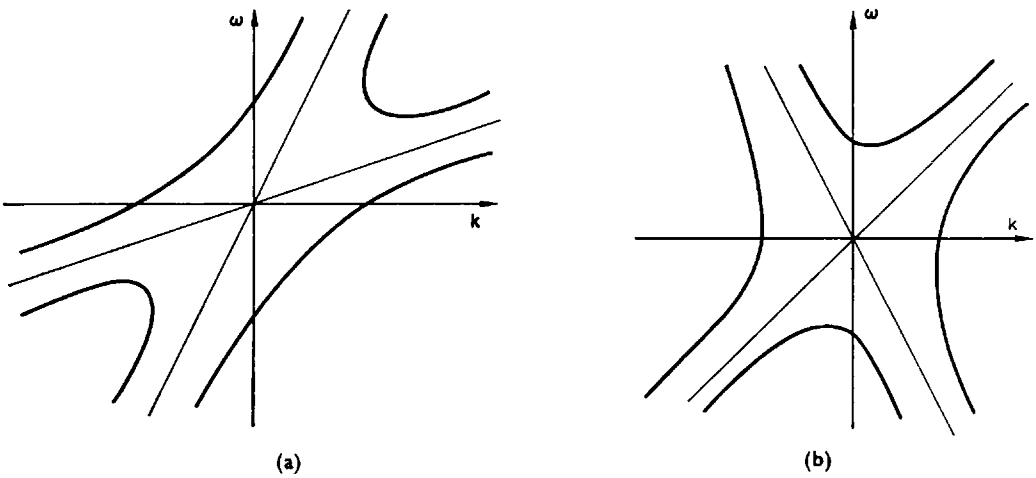

FIG. 6.6.3. Dispersion curve of a two-stream tube. (a) Parallel beam velocities; (b) antiparallel beam velocities.

Let us now elucidate the nature of the instability occurring when a charged particle beam interacts with a plasma. We saw in Subsection 6.1.1 that if we describe the plasma and the beam by means of kinetic equations the dispersion equation of the beam-plasma system will be transcendental. We shall therefore simplify the problem and describe the plasma hydrodynamically—the plasma pressure is assumed to be equal to n_0T; we shall assume the beam to be cold. Under those assumptions the dispersion equation of the beam-plasma system has the form (Akhiezer and Faĭnberg, 1951a)

$$\frac{\omega_b^2}{(\omega-ku_b)^2}+\frac{\omega_p^2}{\omega^2-k^2v_p^2} = 1, \tag{6.6.7.1}$$

where ω_p and ω_b are the Langmuir frequencies of the plasma and of the beam; v_p is the thermal velocity of the electrons in the plasma; and u_b is the beam velocity. This equation is an algebraic equation with real coefficients and we can thus apply the Sturrock rules.

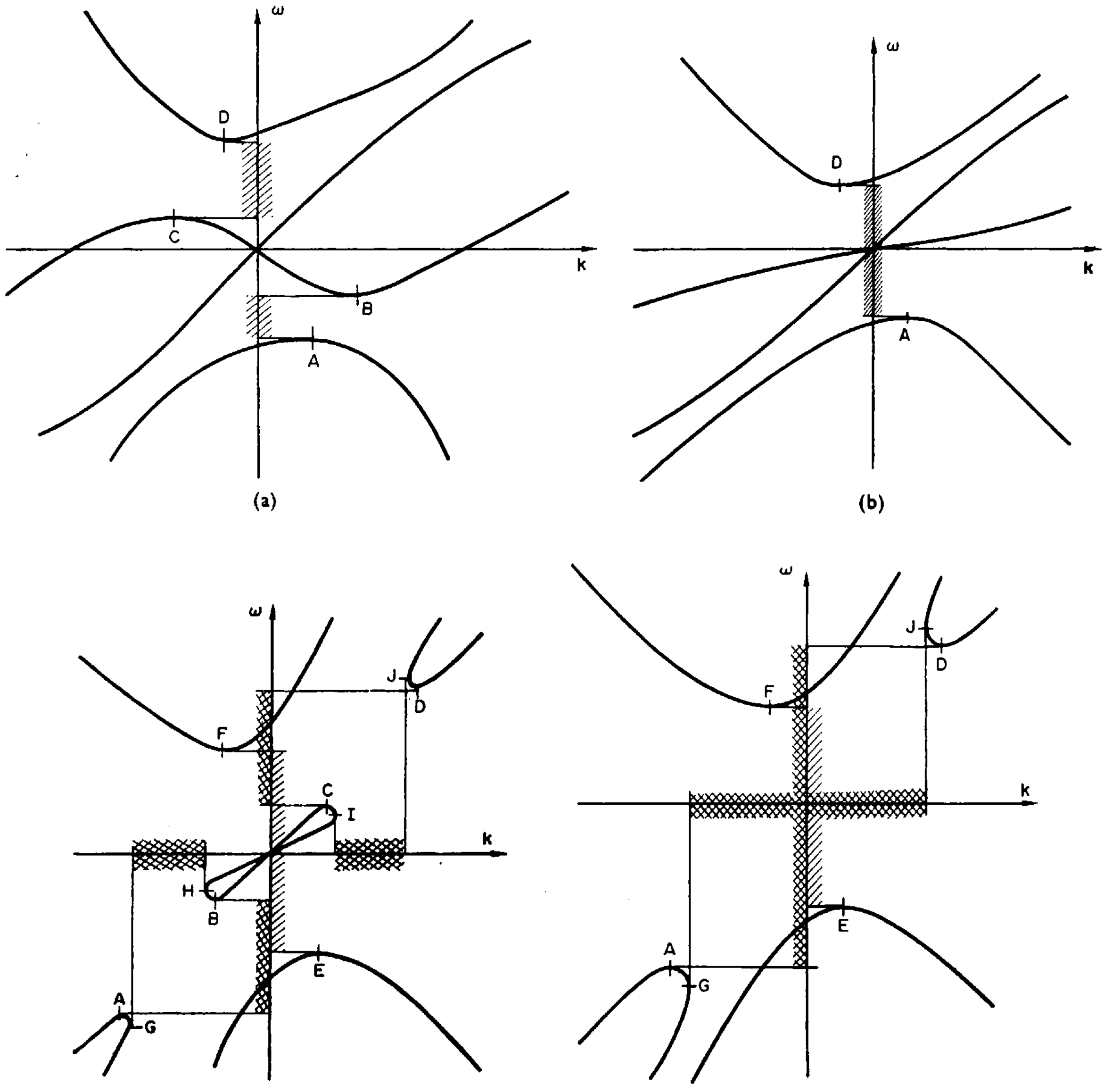

FIG. 6.6.4. Dispersion curves corresponding to the interaction between a cold beam and a hot plasma: (a) $u_b < v_p(\omega_b/\omega_p)$; (b) $v_p(\omega_b/\omega_p) < u_b < v_p$; (c) $v_p < u_b < v_p\sqrt{\{1+(\omega_b^2/\omega_p^2)\}}$; (d) $v_p\sqrt{\{1+(\omega_b^2/\omega_p^2)\}} < u_b$.

The dispersion curves corresponding to eqn. (6.6.7.1) are shown in Fig. 6.6.4, where to fix the ideas we have put $\omega_b < \omega_p$. Figures 6.6.4a and 6.6.4b refer to the case where $u_b < v_p$, and Figs. 6.6.4c and 6.6.4d to the case when $u_b > v_p$; Fig. 6.6.4a corresponds to beam velocities $u_b < v_p(\omega_b/\omega_p)$, Fig. 6.6.4b corresponds to $v_p(\omega_b/\omega_p) < u_b < v_p$, Fig. 6.6.4c corresponds to $v_p < u_b < v_p\sqrt{\{1+(\omega_b^2/\omega_p^2)\}}$, and, finally, Fig. 6.6.4d corresponds to $v_p\sqrt{\{1+(\omega_b^2/\omega_p^2)\}} < u_b$.

We see that in the case shown in Fig. 6.6.4a there are two blocking bands between the points (A, B) and (C, D)—the blocking bands are indicated by shading along the ω-axis. In Fig. 6.6.4b these two non-transmission bands have merged into a single one (A, D). It is clear from Figs. 6.6.4 a, b that the beam–plasma system is stable when $u_b < v_p$ and that amplification of waves is impossible in this system.†

If $u_b > v_p$ there is a blocking band in the interval (E, F) (see Fig. 6.6.4c). Moreover, there are either two amplification bands in the intervals (A, B) and (C, D) (see Fig. 6.6.4c; the amplification bands are indicated by the double hatching along the ω-axis) or a single amplification band (A, D) (see Fig. 6.6.4d). We see in Fig. 6.6.4c also two convective instability bands (G, H) and (I, J) (indicated by double hatching along the k-axis), while in Fig. 6.6.4 these two convective instability bands have merged into a single band (G, J).‡ There is no region of absolute instability.

We remark that an equation of the kind (6.6.7.1) describes the oscillations of a beam passing through a waveguide. One must then replace ω_p by the limiting frequency of the waveguide and v_p by the velocity of the waves in the waveguide when there is no beam. If $u_b > v_p$, we have convective instability or amplification of oscillations—depending on the conditions of the excitation. If $u_b < v_p$, the motion of the beam is stable and amplification of oscillations is impossible, although in that case the dispersion equation leads to complex values of k for real values of ω.

† This statement is valid only in the hydrodynamical approximation. If kinetic effects are taken into account the beam–plasma system can be unstable also when $u_b < v_p$ (see Subsection 6.1.3).

‡ We note that this conclusion is not connected with any assumption about a low density in the beam.

CHAPTER 7

Oscillations of a Partially Ionized Plasma

7.1. Electron Distribution Function and High-frequency Electron Oscillations in an External Electrical Field

7.1.1. THE KINETIC EQUATION

Having studied the oscillations of a completely ionized plasma, we shall now turn to a study of the oscillations of a partially ionized plasma. If there are in the plasma neutral particles as well as electrons and ions, the influence of the neutral particles on the plasma oscillations manifests itself, first of all, in an additional damping of the oscillations, caused by the collisions of the electrons and ions with the neutral particles. As the density of the neutral particles increases, so does the damping rate of the oscillations, and in a plasma with a sufficiently high neutral particle density, when the damping rate of the oscillations becomes of the same order of magnitude as their frequency, the propagation of weakly-damped oscillations—such as Langmuir or ion-sound oscillations—becomes impossible.

There may, however, exist in a partially ionized plasma specified oscillations which are qualitatively different from the oscillations of a completely ionized plasma. Their characteristic peculiarity is the non-conservation of the number of electrons, ions, and neutral particles, connected with ionization and recombination processes. The frequencies of these oscillations therefore are of the same order of magnitude as the characteristic reciprocal times of the ionization and recombination processes.

The presence of neutral particles in a plasma can have an important effect on the nature of the plasma oscillations also in the region of frequencies much higher than the reciprocal time of all particle scattering processes—both elastic and inelastic. Such a situation occurs in the case of a plasma in an external constant electrical field. In such a plasma the collisions between neutral particles and electrons lead to the appearance of an essentially non-Maxwellian electron distribution function, characterized by a directed velocity and a very high average energy of the random electron motion (effective temperature). Such a velocity distribution of the electrons—and not so much the collisions with the neutral particles themselves—leads in turn to peculiarities of ion-sound oscillations, such as the possibility for the growth of these oscillations.

We shall first study oscillations arising in a partially ionized plasma with an electron distribution function, modified by an external electrical field, and then we shall use a very simple example to consider specific plasma oscillations which are accompanied by a change in the number of particles of the various plasma components.

We shall describe the electron component of the plasma by a kinetic equation:

$$\frac{\partial F}{\partial t}+(\boldsymbol{v}\cdot\nabla)F-\frac{e}{m_e}\left(\left\{\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial F}{\partial \boldsymbol{v}}\right)+\mathcal{I}\{F\} = 0, \tag{7.1.1.1}$$

where $\mathcal{I}\{F\}$ is the collision integral. We shall assume that amongst the processes contributing to the collision integral $\mathcal{I}\{F\}$ the main role is played by the collisions between electrons and neutral particles. Because of the large mass difference of the colliding particles, the change in energy $\Delta\varepsilon$ of the electron in the collision is small, $\Delta\varepsilon \sim (m_e/m_0)$ (ε is the electron energy and m_0 the mass of the neutral particle). As to the characteristic change Δv in the electron velocity v, as to order of magnitude we have $|\Delta v| \sim v$. The neutral-electron collisions will therefore "mix" the electrons fast with respect to the velocity direction, but slowly with respect to the absolute magnitude of the velocity. Under the influence of such collisions there will thus be established an electron distribution which is close to being isotropic. In other words, if we write the electron distribution function in the form of a series in spherical harmonics $Y_{lm}(\boldsymbol{v}/v)$,

$$F(\boldsymbol{v}) = F_0(v)+\left(\frac{\boldsymbol{v}}{v}\cdot\boldsymbol{F}_1(v)\right)+\Delta F(\boldsymbol{v}), \tag{7.1.1.2}$$

where

$$\Delta F(\boldsymbol{v}) = \sum_{l,m;l\geq 2} F_{lm}(v)\,Y_{lm}\left(\frac{\boldsymbol{v}}{v}\right),$$

while $\boldsymbol{F}_1(v)$ and $F_{lm}(v)$ are functions of the absolute magnitude of the velocity, we must expect that the largest term in that expansion will be F_0. We shall in a moment make clear that the expansion which we have just written down is in fact an expansion in powers of the small parameter $(m_e/m_0)^{1/2}$. We retain in this expansion the first two terms and express the electron density n_e and the electrical current density $\boldsymbol{j}_e$ in terms of them:

$$n_e = \int F(\boldsymbol{v})\,d^3v = \int F_0(v)\,d^3v, \qquad \boldsymbol{j}_e = -e\int \boldsymbol{v}F(\boldsymbol{v})\,d^3v = -\frac{e}{3}\int v\boldsymbol{F}_1(v)\,d^3v. \tag{7.1.1.3}$$

Substituting (7.1.1.2) into the kinetic eqn. (7.1.1.1) and introducing the zeroth and first moments of the collision integral,

$$\mathcal{I}_0 = \int \mathcal{I}\{F\}\frac{d^2\omega}{4\pi}, \qquad \boldsymbol{\mathcal{I}}_1 = \frac{1}{3}\int \frac{\boldsymbol{v}}{v}\mathcal{I}\{F\}\frac{d^2\omega}{4\pi},$$

we get

$$\begin{aligned}\frac{\partial F_0}{\partial t}+\frac{v}{3}\operatorname{div}\boldsymbol{F}_1-\frac{e}{3m_e v^2}\frac{\partial}{\partial v}[v^2(\boldsymbol{E}\cdot\boldsymbol{F}_1)]+\mathcal{I}_0 &= 0,\\ \frac{\partial \boldsymbol{F}_1}{\partial t}+v\operatorname{grad}F_0-\frac{e}{m_e}\boldsymbol{E}\frac{\partial F_0}{\partial v}-\frac{e}{m_e c}[\boldsymbol{B}\wedge\boldsymbol{F}_1]+\boldsymbol{\mathcal{I}}_1 &= 0.\end{aligned} \tag{7.1.1.4}$$

The fact that the absolute magnitude of the electron velocity changes little when the electron collides with a neutral particle enables us to obtain simple closed expressions for the quantities

$\mathcal{J}_0$ and $\mathcal{J}_1$. In this connection we note that the number of electrons in the range $(\boldsymbol{v}, \boldsymbol{v}+d\boldsymbol{v})$ changes due to collisions in a time dt by

$$\mathcal{J}\{F(\boldsymbol{v})\}v^2\,dv\,d^2\omega\,dt = \{F(\boldsymbol{v})v^2\,2\pi\int\limits_0^\pi w(\vartheta, v)\sin\vartheta\,d\vartheta$$
$$-\int F(\boldsymbol{v}')\,W(\vartheta, v', \Delta v)v'^2 d\Delta v\,d^2\omega'\}\,dv\,d^2\omega\,dt, \qquad (7.1.1.5)$$

where $W(\vartheta, v, \Delta v)$ is the probability that per unit time an electron changes from a state with velocity $\boldsymbol{v}$ to a state with velocity $\boldsymbol{v}'$,

$$w(\vartheta, v) = \int\limits_{-v}^{\infty} W(\vartheta, v, \Delta v)d\,\Delta v, \qquad (7.1.1.6)$$

ϑ is the scattering angle, that is, the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{v}'$, and Δv is the change in the absolute magnitude of the electron velocity in the scattering, $\Delta v = |\boldsymbol{v}'|-|\boldsymbol{v}|$. Substituting the expansion (7.1.1.2) into (7.1.1.5) and assuming that $\Delta F \ll F_0$, $|\boldsymbol{F}_1|$, we find that

$$\mathcal{J}_0 = F_0(v)2\pi\int\limits_0^\pi w(\vartheta, v)\sin\vartheta\,d\vartheta - \int F_0(v')\,W(\vartheta, v', \Delta v)\frac{v'^2}{v^2}d\,\Delta v\,d^2\omega',$$

$$\left(\frac{\boldsymbol{v}}{v}\cdot\boldsymbol{\mathcal{J}}_1\right) = \left(\frac{\boldsymbol{v}}{v}\cdot\boldsymbol{F}_1(v)\right)2\pi\int\limits_0^\pi w(\vartheta, v)\sin\vartheta\,d\vartheta - \int\left(\frac{\boldsymbol{v}'}{v'}\cdot\boldsymbol{F}_1(v')\right)W(\vartheta, v', \Delta v)\frac{v'^2}{v^2}d\,\Delta v\,d^2\omega'.$$
$$(7.1.1.7)$$

Let us first evaluate the quantity $\mathcal{J}_0$. Assuming that $\Delta v \ll v$, we can expand the integrand on the right-hand side of eqn. (7.1.1.7) in a power series in Δv:

$$F_0(v')\,W(\vartheta, v', \Delta v)\frac{v'^2}{v^2} = F_0(v)\,W(\vartheta, v, \Delta v) + \frac{\Delta v}{v^2}\frac{\partial}{\partial v}\{v^2F_0(v)\,W(\vartheta, v, \Delta v)\}$$
$$+\frac{(\Delta v)^2}{2v^2}\frac{\partial^2}{\partial v^2}\{v^2F_0(v)\,W(\vartheta, v, \Delta v)\} + \ldots$$

Using then (7.1.1.6) and using the notation

$$\left\langle\frac{\Delta v}{\Delta t}\right\rangle = \int \Delta v W(\vartheta, v, \Delta v)d\Delta v\,d^2\omega, \quad \left\langle\frac{\Delta v^2}{\Delta t}\right\rangle = \int (\Delta v)^2\,W(\vartheta, v, \Delta v)d\Delta v\,d^2\omega, \quad (7.1.1.8)$$

we get

$$\mathcal{J}_0 = -\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2F_0(v)\left\langle\frac{\Delta v}{\Delta t}\right\rangle + \frac{1}{2}\frac{\partial}{\partial v}\left[v^2F_0(v)\left\langle\frac{\Delta v^2}{\Delta t}\right\rangle\right]\right\}. \qquad (7.1.1.9)$$

The quantities $\langle\Delta v/\Delta t\rangle$ and $\langle\Delta v^2/\Delta t\rangle$ depend, of course, on the form of the neutral particles distribution function and on the explicit form of the neutral-electron interaction potential. We shall now prove that in the case of a Maxwellian velocity distribution of the neutral particles both these quantities can be expressed in terms of a single parameter—the electron mean free path $l(v)$ which is connected with the probability $w(\vartheta, v)$ through the relation

$$l(v) = v\left\{\int w(\vartheta, v)\,(1-\cos\vartheta)\,d^2\omega\right\}^{-1}. \tag{7.1.1.10}$$

We note that the quantity $l(v)$ characterizes the deceleration rate of the electron current, in fact, the decrease of the average value of the electron momentum component in the direction of its initial motion—but not the rate of change in the average electron energy.

We shall first of all evaluate the quantity $\langle\Delta v^2/\Delta t\rangle$. To do this we note that the change in the absolute magnitude of the electron velocity in collisions with neutral particles is connected with the velocity of the neutral particles $\boldsymbol{u}$ through the relation

$$\Delta v = 2(\boldsymbol{u}\cdot\boldsymbol{\beta})\sin\tfrac{1}{2}\vartheta, \tag{7.1.1.11}$$

where $\boldsymbol{\beta} = (\boldsymbol{v}-\boldsymbol{v}')/|\boldsymbol{v}-\boldsymbol{v}'|$, with $\boldsymbol{v}$ and $\boldsymbol{v}'$ the electron velocities before and after the scattering, and ϑ the scattering angle. Indeed, the velocities $\boldsymbol{v}$ and $\boldsymbol{v}'$ are interconnected through the energy and momentum conservation laws by the obvious relation

$$\boldsymbol{v}' = \boldsymbol{v} - 2\beta\frac{m_0}{m_e+m_0}(\boldsymbol{\beta}\cdot\boldsymbol{v}_0),$$

where $\boldsymbol{v}_0 = \boldsymbol{v}-\boldsymbol{u}$ is the relative velocity of the colliding particles. Using also the fact that the scattering angle ϑ differs from the scattering angle in the centre of mass system, $\vartheta_0 = 2\arcsin\{(\boldsymbol{\beta}\cdot\boldsymbol{v}_0)/v_0\}$ only by terms which are proportional to the small parameter m_e/m_0, and neglecting terms quadratic in that parameter, we get eqn. (7.1.1.11).

Substituting (7.1.1.11) into the second of eqns. (7.1.1.8), integrating over Δv, and averaging over the velocities $\boldsymbol{u}$ of the neutral particles, we get

$$\left\langle\frac{\Delta v^2}{\Delta t}\right\rangle = \frac{2}{3}\overline{u^2}\int w(\vartheta, v)\,(1-\cos\vartheta)\,d^2\omega,$$

where $\overline{u^2}$ is the mean square velocity of the neutral particles. Noting that in the case of a Maxwellian velocity distribution of the neutral particles, $\overline{u^2} = 3T_0/m_0$, and using (7.1.1.10) we can express the quantity $\langle\Delta v^2/\Delta t\rangle$ in terms of the electron mean free path $l(v)$ and the temperature T_0 of the neutral particles:

$$\left\langle\frac{\Delta v^2}{\Delta t}\right\rangle = 2\frac{T_0}{m_0}\frac{v}{l(v)}. \tag{7.1.1.12}$$

We shall now determine the quantity $\langle\Delta v/\Delta t\rangle$. To do this, we note that the collision integral (7.1.1.9) must vanish if in it we substitute for the electron distribution function a Maxwell distribution with temperature T_0. Hence

$$\left\langle\frac{\Delta v}{\Delta t}\right\rangle = -\frac{1}{2}\exp\left(\frac{m_e v^2}{2T_0}\right)\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\exp\left(-\frac{m_e v^2}{2T_0}\right)\left\langle\frac{\Delta v^2}{\Delta t}\right\rangle\right\}.$$

Substituting here (7.1.1.12) we get

$$\left\langle\frac{\Delta v}{\Delta t}\right\rangle = \frac{m_e v^2}{m_0 l(v)} - \frac{T_0}{m_0}\frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{v^3}{l(v)}\right). \tag{7.1.1.13}$$

Substituting (7.1.1.12) and (7.1.1.13) into (7.1.1.9) we get the final expression for the quantity $\mathcal{J}_0$:

$$\mathcal{J}_0 = -\frac{1}{v^2}\frac{\partial}{\partial v}\left\{\frac{m_e v^4}{m_0 l(v)} F_0(v) + \frac{v^3}{m_0 l(v)} T_0 \frac{\partial F_0(v)}{\partial v}\right\}. \tag{7.1.1.14}$$

We finally evaluate the quantity $\boldsymbol{\mathcal{J}}_1$. Neglecting corrections proportional to $\Delta v/v$ and using (7.1.1.6), we can write the second of eqns. (7.1.1.7) in the form

$$\left(\frac{\boldsymbol{v}}{v}\cdot\boldsymbol{\mathcal{J}}_1\right) = \left(\frac{\boldsymbol{v}}{v}\cdot\boldsymbol{F}_1(v)\right)2\pi\int_0^{\pi} w(\vartheta, v)\sin\vartheta\, d\vartheta - \left(\boldsymbol{F}_1(v)\cdot\int\frac{\boldsymbol{v}'}{v'}\, w(\vartheta, v')\, d^2\boldsymbol{\omega}'\right).$$

Multiplying this equation by $\frac{1}{3}\boldsymbol{v}d^2\boldsymbol{\omega}/4\pi$ and integrating over the direction of the vector $\boldsymbol{v}$, we get

$$\boldsymbol{\mathcal{J}}_1\{F\} = \frac{v}{l(v)}\boldsymbol{F}_1(v), \tag{7.1.1.15}$$

where $l(v)$ is the electron mean free path, given by eqn. (7.1.1.10).

The kinetic eqn. (7.1.1.4) together with the collision integrals (7.1.1.14) and (7.1.1.15) completely describe the electron component of a partially ionized plasma for the case of an electron velocity distribution which is nearly isotropic.†

7.1.2. STATIONARY ELECTRON DISTRIBUTION FUNCTION

We shall now determine a stationary electron distribution function in a constant and uniform electrical field $\boldsymbol{E}_0$. In that case the second of eqns. (7.1.1.4) gives

$$\boldsymbol{F}_1 = \frac{e\boldsymbol{E}_0 l}{m_e v}\frac{\partial F_0}{\partial v}. \tag{7.1.2.1}$$

Substituting this relation into the first of eqns. (7.1.1.4) we get

$$\begin{aligned} F_0(v) &= C\exp\left\{-\int^{v}\frac{3(m_e v')^3\, dv'}{3(m_e v')^2 T_0 + m_0(eE_0 l)^2}\right\}, \\ \boldsymbol{F}_1(v) &= -e\boldsymbol{E}_0 l\left\{T_0 + \frac{m_0}{3m_e^2 v^2}(eE_0 l)^2\right\}^{-1} F_0, \end{aligned} \tag{7.1.2.2}$$

where C is a normalization constant. One sees easily that in the case of a weak electrical field ($eE_0\bar{l} \ll \sqrt{(m_e/m_0)}\, T_0$, where $\bar{l}$ is the average value of the electron mean free path) the distribution function (7.1.2.2) is nearly a Maxwell distribution with temperature T_0. However, in the case of a strong field ($e_0\bar{l}E \gg \sqrt{(m_e/m_0)}T_0$) in which we shall be primarily

† Davydov (1936, 1937) was the first to establish these equations. They are equally suitable for describing transport processes in a weakly ionized plasma as for describing them in semiconductors.

interested, we get from (7.1.2.2)

$$F_0(v) = C\exp\left\{-\frac{3m_e^3}{m_0(eE_0)^2}\int^v \frac{v'^3\,dv'}{l^2(v')}\right\},\quad F_1(v) = -\frac{3E_0(m_e v)^2}{eE_0^2 l m_0}F_0. \qquad (7.1.2.3)$$

If the neutral particles behave in collisions with the electrons as rigid elastic spheres, the mean free path l will be independent of the electron speed v. In that case the distribution (7.1.2.3) becomes (Druyvesteyn, 1930, 1934)

$$F_0(v) = C\exp\left\{-\left(\frac{m_e v^2}{2T_e}\right)^2\right\},\quad F_1(v) = \frac{\boldsymbol{E}_0}{E_0}\sqrt{\left(\frac{3m_e}{m_0}\right)}\frac{m_e v^2}{T_e}F_0(v),\quad C = \frac{n_{e0}}{\pi\Gamma(\frac{3}{4})}\left(\frac{m_e}{2T_e}\right)^{3/2}, \qquad (7.1.2.4)$$

where n_{e0} is the equilibrium electron density and

$$T_e = \sqrt{\left(\frac{m_0}{3m_e}\right)}eE_0 l. \qquad (7.1.2.5)$$

In this case the electron current density equals

$$\boldsymbol{j} = -\frac{4en_{e0}}{3\sqrt{\pi}}\Gamma\left(\frac{5}{4}\right)\frac{e\boldsymbol{E}_0 l}{\sqrt{(m_e T_e)}}. \qquad (7.1.2.6)$$

We see that in a partially ionized plasma in a strong external electrical field the electron velocity distribution is characterized by a very large random motion energy (effective temperature) T_e and a relatively small directed velocity $|j/en_{e0}| \sim \sqrt{(T_e/m_0)}$. A characteristic peculiarity of this distribution is the relatively small (when compared with the Maxwell distribution) number of electrons with energies above the average energy.

If the plasma is not only in an electrical, but also in a constant uniform magnetic field, we have instead of (7.1.2.1) the equation

$$\boldsymbol{F}_1 = \frac{el}{m_e v}\frac{\partial F_0}{\partial v}\,\frac{\boldsymbol{E}_0 - \dfrac{el}{m_e vc}[\boldsymbol{E}_0\wedge\boldsymbol{B}_0] + \left(\dfrac{el}{m_e vc}\right)^2\boldsymbol{B}_0(\boldsymbol{E}_0\cdot\boldsymbol{B}_0)}{1+\left(\dfrac{el}{m_e vc}\right)^2 B_0^2}, \qquad (7.1.2.7)$$

where $\boldsymbol{B}_0$ is the magnetic induction vector. Substituting this equation into the first of eqns. (7.1.1.4) we get

$$F_0(v) = C\exp\left\{-\int^v \frac{-3(m_e v')^3\,dv'}{3(m_e v')^2 T_0 + m_0(eE_0 l)^2\zeta(v')}\right\}, \qquad (7.1.2.8)$$

where C is a normalizing constant and

$$\zeta(v) = \frac{1+\left(\dfrac{eB_0 l}{m_e vc}\right)\cos^2\beta}{1+\left(\dfrac{eB_0 l}{m_e vc}\right)^2},$$

with β the angle between the vectors $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$.

Expressions (7.1.2.7) and (7.1.2.8) for the electron distribution function are valid for any uniform fields $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$, whatever their strengths or directions. In the case of a weak magnetic field ($eB_0\bar{l} \ll m_e\bar{v}c$, where $\bar{v}$ is the mean random electron velocity) these formulae go over into (7.1.2.2).

The case of a plasma in strong electrical and magnetic fields ($eE_0l \gg \sqrt{(m_e/m_0)}T_0$, $eB_0\,\bar{l} \gg m_e\bar{v}c$) is of particular interest. In that case we have, when l = constant

$$F_0(v) = C \exp\left\{-\left(\frac{m_e v^2}{2T_B}\right)^2\right\},$$
$$\boldsymbol{F}_1(v) = \frac{\boldsymbol{B}_0}{B_0}\sqrt{\left(\frac{3m_e}{m_0}\right)}\frac{m_e v^2}{T_B}\cos^2\beta\, F_0(v), \tag{7.1.2.9}$$

where

$$C = \frac{n_{e0}}{\pi\Gamma(\frac{3}{4})}\left(\frac{m_e}{2T_B}\right)^{3/2}, \quad T_B = \sqrt{\left(\frac{m_0}{3m_e}\right)}eE_0 l\,|\cos\beta|; \tag{7.1.2.10}$$

we have assumed that the fields $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are not at right angles to one another, $\cos\beta \gg m_e\bar{v}c/e\bar{l}B_0$. The electron current density is in this case equal to

$$\boldsymbol{j} = -\frac{4en_{e0}}{\sqrt{(3\pi)}}\Gamma\left(\frac{5}{4}\right)\frac{\boldsymbol{B}_0}{B_0}\sqrt{\left(\frac{T_B}{m_0\,|\cos\beta|}\right)}. \tag{7.1.2.11}$$

Comparing eqns. (7.1.2.9) with eqns. (7.1.2.4) and (7.1.2.5) we see that applying a strong magnetic field as well as a strong electrical field does not change the general structure of the electron distribution function. In fact, the expressions for the effective temperature T_e and the function $\boldsymbol{F}_1$ in the case of a strong magnetic field can be obtained from the corresponding expressions for the case when $\boldsymbol{B}_0 = 0$ through the substitution

$$T_e \to T_B; \qquad \boldsymbol{E}_0 \to \frac{\boldsymbol{B}_0}{B_0}E_0\cos^2\beta.$$

In concluding this subsection we note that according to (7.1.2.1) and (7.1.2.7) $|\boldsymbol{F}_1| \sim \sqrt{(m_e/m_0)}F_0$, as was stated above. Using the kinetic equation (7.1.1.1) and the explicit form (7.1.1.5) of the collision integral one checks easily that the expansion (7.1.1.2) of the electron distribution function in terms of spherical harmonics is—in the case of stationary and spatially uniform distributions—in fact a power series in the small parameter $\sqrt{(m_e/m_0)}$.

We emphasize once more the two-stage nature of the electron relaxation in a partially ionized plasma. In such a plasma first of all in a time of the order of $\bar{l}/\bar{v}$ an electron velocity distribution which is nearly isotropic is established; this stage of the plasma relaxation is described by the very general collision integral (7.1.1.5). After a time of the order $(\bar{l}/\bar{v})\sqrt{(m_0/m_e)}$ there follows a "mixing" of the electrons with respect to energy, and as a result of this the final stationary electron distribution is set up; this stage of the relaxation is described by the collision integrals (7.1.1.14) and (7.1.1.15).

7.1.3. HIGH-FREQUENCY ELECTRON OSCILLATIONS

We now turn to a study of the oscillations of a partially ionized plasma in external fields.†
We start from the linearized kinetic equation (7.1.1.1),

$$\frac{\partial f}{\partial t}+(v\cdot\nabla)f-\frac{e}{m_e}\left(\left\{E_0+\frac{1}{c}[v\wedge B_0]\right\}\cdot\frac{\partial f}{\partial v}\right)-\frac{e}{m_e}\left(\left\{E'+\frac{1}{c}[v\wedge B']\right\}\cdot\frac{\partial F^{(0)}}{\partial v}\right)+\mathcal{I}\{f\}=0, \tag{7.1.3.1}$$

where $E' = E-E_0$ and $B' = B-B_0$ are the variable electrical field and magnetic induction, $f = F-F^{(0)}$ is the deviation of the electron distribution function from its stationary value, and $\mathcal{I}$ is the collision integral which describes the collisions between the electrons and the neutral particles—we shall here and henceforth denote the stationary electron distribution function, determined in the preceding subsection, by $F^{(0)}$.

We note, first of all that, as to order of magnitude, $(\{eE_0/m_e\}\cdot\{\partial f/\partial v\}) \sim (eE_0\bar{l}/\bar{\varepsilon})\nu_e f$, where ν_e is the average electron-neutral collision frequency and $\bar{\varepsilon}$ the average energy of the random electron motion. Using the fact that $\bar{\varepsilon} \sim \text{Max}\,\{T_e, T_0\}$, where $T_e \sim eE_0\bar{l}\sqrt{(m_0/m_e)}$, we see that the term in the kinetic equation for the deviation f of the distribution function from its stationary value, which contains the constant electrical field, is small compared with the collision integral,

$$\left(\frac{eE_0}{m_e}\cdot\frac{\partial f}{\partial v}\right) \sim \sqrt{\left(\frac{m_e}{m_0}\right)}\,\text{Min}\left\{1;\ \frac{T_e}{T_0}\right\}\mathcal{I}\{f\} \ll \mathcal{I}\{f\},$$

and we shall therefore neglect it in what follows.

We draw attention that—in contrast to the kinetic equation for the function f—in the kinetic equation for the stationary distribution function $F^{(0)}$ the relative contribution of the term containing the constant electrical field was not small and that that term exactly cancelled the collision integral. This difference is connected with the two-stage nature of the electron relaxation in a partially ionized plasma: the effective collision integral occurring in the kinetic equation for the almost isotropic function F is smaller by a factor $\sqrt{(m_0/m_e)}$ than the collision integral $\mathcal{I}\{f\}$ which describes the initial stage of the damping of the correction f to the stationary distribution function.

We shall first consider the high-frequency oscillations of a partially ionized plasma in the case when there is no magnetic field. Using the fact that in the high-frequency region the electrical susceptibility of the ion and neutral components are small, we can start from the following expression for the dielectric permittivity of the plasma (see Chapter 4)

$$\varepsilon_{ij}(k,\omega) = \delta_{ij}+\frac{4\pi e^2}{m_e\omega^2}\sum_l\int v_i[\omega-(k\cdot v)+i\hat{\mathcal{I}}]^{-1}\{[\omega-(k\cdot v)]\delta_{lj}+k_l v_j\}\frac{\partial F^{(0)}}{\partial v_l}\,d^3v, \tag{7.1.3.2}$$

where $[\omega-(k\cdot v)+i\hat{\mathcal{I}}]^{-1}$ is an operator which is the inverse of the operator $[\omega-(k\cdot v)+i\hat{\mathcal{I}}]$ and $\hat{\mathcal{I}}$ is the neutral-electron collision operator, $\hat{\mathcal{I}}f \equiv \mathcal{I}\{f\}$.

To determine the frequencies and damping rates of the plasma eigenoscillations, one must, according to the general rule of Chapter 4, put the determinant of the tensor $\hat{\Lambda}$,

† Akhiezer and Sitenko (1959) considered high-frequency plasma oscillations in an electrical field.

which is given by eqn. (4.3.1.10′) equal to zero. Substituting (7.1.3.2) into the tensor $\hat{\Lambda}$ we see that in the high-frequency region ($\omega \gg k\bar{v}$, $\omega \gg \nu_e$) two kinds of weakly damped oscillations can propagate: transverse electromagnetic waves with frequencies $\pm\omega_t(k)$ ($\omega_t(k) = \sqrt{(\omega_{pe}^2+c^2k^2)}$) and longitudinal electron (Langmuir) oscillations with frequencies $\pm\omega_l(\boldsymbol{k})$, where

$$\omega_l(\boldsymbol{k}) = \omega_{pe}+\frac{k^2\bar{v}^2}{2\omega_{pe}}+(\boldsymbol{k}\cdot\boldsymbol{u}), \tag{7.1.3.3}$$

with ω_{pe} the electron plasma frequency, $\omega_{pe} = \sqrt{(4\pi e^2 n_{e0}/m_e)}$ (n_{e0} is the equilibrium electron density), $\bar{v}$ is the root mean square random electron velocity, and $\boldsymbol{u}$ is the average directed electron velocity,

$$\bar{v}^2 = \frac{1}{n_{e0}}\int F_0^{(0)}v^2\,d^3v, \quad \boldsymbol{u} = \frac{1}{3n_{e0}}\int F_1^{(0)}v\,d^3v. \tag{7.1.3.4}$$

The damping rate γ_t of the transverse oscillations is of the order of magnitude of collision frequency, $\gamma_t \sim \nu_e$. As far as the damping rate of the longitudinal oscillations γ_l is concerned, in the case of a not too low collision frequency ($\gamma_l^{(0)} \lesssim \nu_e$) we have, as to order of magnitude $\gamma_l \sim \nu_e$; in the case of very rare collisions ($\gamma_l^{(0)} \gg \nu_e$)$\gamma_l = \gamma_l^{(0)}$, where $\gamma_l^{(0)}$ is the collisionless Landau damping rate,

$$\gamma_l^{(0)}(\boldsymbol{k}) = \frac{\Gamma(\frac{1}{4})}{\sqrt{2}}\,\frac{\omega_{pe}^4 m_e^{3/2}}{k^3(2T_e)^{3/2}}\left\{1+\frac{m_e\omega_{pe}^2}{k^2T_e}\sqrt{\left(\frac{3m_e}{m_0}\right)}\cos\chi\right\}, \tag{7.1.3.5}$$

with χ the angle between the vectors $\partial\omega/\partial\boldsymbol{k}$ and $\boldsymbol{u}$; to fix the ideas we assume a strong external electrical field, $T_e \gg T_0$.

When an external magnetic field is present in a partially ionized plasma, two branches of longitudinal electron oscillations and two branches of transverse electromagnetic waves can propagate. The frequencies of all four oscillation branches are given by the general formulae of Chapter 5, in which we must substitute for the electron distribution function the function $F^{(0)}$ given in Subsection 7.1.2. For the damping rates of the oscillations we have, as to order of magnitude, $\gamma_t \sim \nu_e$, $\gamma_l \sim \text{Max}\{\nu_e; \gamma_l^{(0)}\}$, where $\gamma_l^{(0)}$ is given by eqn. (7.1.3.5).

We give here the expressions for the frequencies of the fast and slow longitudinal ($\omega_{l\pm}$) and transverse ($\omega_{t\pm}$) waves only for the case of not too large magnetic fields, $|\omega_{Be}| \ll \omega_{pe}$:

$$\omega_{l+}(\boldsymbol{k}) = \omega_{pe}+\frac{\omega_{Be}^2}{2\omega_{pe}}\sin^2\theta+\frac{k^2\bar{v}^2}{2\omega_{pe}}+(\boldsymbol{k}\cdot\boldsymbol{u}), \quad \omega_{l-}(\boldsymbol{k}) = \omega_{Be}|\cos\theta|+\frac{k^2\bar{v}^2}{2\omega_{pe}}+(\boldsymbol{k}\cdot\boldsymbol{u}),$$
$$\omega_{t\pm}(\boldsymbol{k}) = \sqrt{[\omega_{pe}^2+c^2k^2]}\pm\frac{1}{2}|\omega_{Be}|\,\frac{|(\boldsymbol{k}\cdot\boldsymbol{u})|}{ku}, \tag{7.1.3.6}$$

where $\omega_{Be} = eB_0/m_ec$ is the electron gyro-frequency and θ the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{B}_0$. Waves moving in the opposite direction have clearly frequencies $-\omega_\lambda(-\boldsymbol{k})$, where $\lambda = $ l+, l−, t+, or t−.

7.2. Ion-sound Oscillations in a Strong Electrical Field

7.2.1. ION-SOUND OSCILLATIONS IN THE ABSENCE OF AN EXTERNAL MAGNETIC FIELD

We now turn to a study of the oscillations of a partially ionized plasma in an external electrical field in the region of not too high frequencies ($kv_e \gg \omega \gg \nu_e$; v_e is the average random electron velocity). In this frequency range we must take into account not only the motion of the electrons, but also that of the ions in the plasma. Therefore, when we study such oscillations, we must add to the kinetic equation for the electron distribution function (7.1.3.1) and to the electrostatics equations,

$$\operatorname{div} \boldsymbol{D} = -4\pi e(n_e - Zn_i), \quad \operatorname{curl} \boldsymbol{E} = 0, \quad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E}, \tag{7.2.1.1}$$

where ε_0 is the dielectric permittivity of the neutral component, also a kinetic equation for the ion distribution function F_i,

$$\frac{\partial F_i}{\partial t} + (\boldsymbol{v}\cdot\nabla)F_i + \frac{Ze}{m_i}\left(\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{B}]\right\}\cdot\frac{\partial F_i}{\partial \boldsymbol{v}}\right) + \mathcal{J}_i\{F_i\} = 0, \tag{7.2.1.2}$$

where m_i is the mass and Ze the charge of an ion, while $\mathcal{J}_i\{F_i\}$ is the collision integral for the ions and n_e and n_i are the electron and ion densities.

We note first of all that by virtue of the large ion mass their stationary velocity distribution in an external electrical field will differ little from a Maxwell distribution and their temperature T_i will be close to the temperature T_0 of the neutral particles. In the case of a strong external electrical field—in which we shall be interested in what follows—the average energy of the random electron motion, $T_e = \sqrt{(m_0/3m_e)}eE_0\bar{l}$, can thus appreciably exceed the ion temperature. We noted in Chapter 4 that under such condition ion-sound oscillations can propagate in the plasma with frequencies in the range $kv_e \gg \omega \gg kv_i$, where v_i is the average thermal ion velocity.

Using the kinetic equations (7.1.3.1) and (7.2.1.2) we can easily write down the longitudinal dielectric permittivity of the plasma in the frequency range $kv_e \gg \omega \gg kv_i$, $\omega \gg \nu_e$ in the following form

$$\varepsilon_l(\boldsymbol{k}, \omega) = \varepsilon_0 + \frac{1}{k^2 r_D^2} - \frac{\omega_{pi}^2}{\omega^2} + 4\pi i\{\varkappa_e''(\boldsymbol{k}, \omega) + \varkappa_i''(\boldsymbol{k}, \omega)\}, \tag{7.2.1.3}$$

where $\omega_{pi} = \sqrt{(4\pi(Ze)^2 n_{i0}/m_i)}$ is the ion plasma frequency and r_D the electron Debye radius,

$$r_D = \sqrt{\left(\frac{T_e \Gamma(\frac{3}{4})}{2\pi e^2 n_{e0} \Gamma(\frac{1}{4})}\right)} = 0.815\sqrt{\left(\frac{T_e}{4\pi e^2 n_{e0}}\right)}, \tag{7.2.1.4}$$

$n_{\alpha 0}$ the equilibrium value of the density of the αth kind of particles (α = e, i),

$$\varkappa_e''(\boldsymbol{k}, \omega) = \frac{\pi}{4k^2 r_D^2}\sqrt{\left(\frac{6m_e}{m_0}\right)}\frac{1}{\Gamma^2(\frac{1}{4})}\left\{\frac{2^{3/4}}{\sqrt{(3\pi)}\sqrt{[1+k^2 r_D^2]}}\sqrt{\left(\frac{Zm_0}{m_i}\right)}\operatorname{sgn}\omega - \frac{(\boldsymbol{k}\cdot\boldsymbol{E}_0)}{kE_0}\right\}, \tag{7.2.1.5}$$

$$\varkappa_i''(\boldsymbol{k}, \omega) = \frac{2}{\pi}\sqrt{\left(\frac{2T_i}{\pi m_i}\right)}\frac{\omega_{pi}^2}{k^2}\operatorname{Im}\int[(\boldsymbol{k}\cdot\boldsymbol{v}) - \omega - i\hat{\mathcal{J}}_i]^{-1}(\boldsymbol{k}\cdot\boldsymbol{v})\exp\left(-\frac{m_i v^2}{2T_i}\right)d^3v, \tag{7.2.1.6}$$

and $\hat{\mathcal{J}}_i$ the collision operator, $\hat{\mathcal{J}}_i F_i = \mathcal{J}_i\{F_i\}$—for the sake of simplicity we assume here and henceforth that the electron mean free path l is velocity-independent.

Putting ε_l equal to zero we get an equation to determine the plasma eigenfrequencies. We see that if $\omega \gg \nu_{e,i}$, where ν_i^{-1} is the average relaxation time of the ion component of the plasma, weakly damped ion-sound oscillations can propagate in the plasma with a frequency (Angeleĭko and Kitsenko, 1965; Angeleĭko and Akhiezer, 1968)

$$\omega_s(\boldsymbol{k}) = \frac{kv_s}{\sqrt{(1+k^2r_D^2)}}, \quad v_s = \sqrt{\left(\frac{2ZT_e\Gamma(\frac{3}{4})}{m_i\Gamma(\frac{1}{4})}\right)} = 0.662\sqrt{\left(\frac{ZT_e}{m_i}\right)}, \tag{7.2.1.7}$$

and a damping rate

$$\gamma_s(\boldsymbol{k}) = 2\pi\omega_{pi}\frac{(kr_D)^3}{(1+k^2r_D^2)^{3/2}}\{\varkappa_e''(\boldsymbol{k},\omega)+\varkappa_i''(\boldsymbol{k},\omega)\}; \tag{7.2.1.8}$$

for the sake of simplicity we have neglected the electrical susceptibility of the neutral component of the plasma, putting $\varepsilon_0 = 1$.

The second term in eqn. (7.2.1.8) for the function $\gamma_s(\boldsymbol{k})$ is always positive and describes the damping of the ion-sound oscillations caused by the ions in the plasma. This term is of the order of magnitude of Max $\{\nu_i, \gamma_i^{(0)}\}$, where $\gamma_i^{(0)}$ is the collisionless Landau damping rate,

$$\gamma_i^{(0)}(\boldsymbol{k}) = \left(\frac{2\pi m_i}{T_i}\right)^{3/2}\frac{k}{2}\frac{v_s^4}{(1+k^2r_D^2)^2}\exp\left\{-\frac{m_iv_s^2}{2T_i(1+k^2r_D^2)}\right\}. \tag{7.2.1.9}$$

As to the first term in (7.2.1.8) describing the interaction of the ion-sound oscillations with the electrons in the plasma, it can be either positive (corresponding to absorption of the oscillations) or negative (corresponding to excitation of the oscillations by the electrons) depending on the angle χ between the vectors $\partial\omega/\partial\boldsymbol{k}$ and $\boldsymbol{E}_0$. Because of this the function $\gamma_s(\boldsymbol{k})$ is not necessarily positive, but can also become negative; in the latter case the quantity $|\gamma_s(\boldsymbol{k})|$ is, of course, the growth rate of the oscillations.

To fix the ideas we restrict ourselves to the case when $\gamma_i^{(0)} \gg \nu_i$, and we can write eqn. (7.2.1.8) in the form

$$\gamma_s(\boldsymbol{k}) = \omega_s(\boldsymbol{k})\frac{\pi^2}{1+k^2r_D^2}\sqrt{\left(\frac{3m_e}{2m_0}\right)}\frac{1}{\Gamma^2(\frac{1}{4})}(R-\cos\chi), \tag{7.2.1.10}$$

where χ is the angle between the vectors $\partial\omega/\partial\boldsymbol{k}$ and $\boldsymbol{E}_0$ and

$$R = \frac{2^{3/4}}{\sqrt{(3\pi)}(1+k^2r_D^2)^{1/2}}\sqrt{\left(\frac{Zm_0}{m_i}\right)}\left[1+\frac{\sqrt{(2\pi)}}{\Gamma(\frac{1}{4})}\right]Z\sqrt{\left(\frac{m_i}{m_e}\right)}\left(\frac{T_e}{T_i}\right)^{3/2}\exp\left\{-\frac{m_iv_s^2}{2T_i(1+k^2r_D^2)}\right\}. \tag{7.2.1.11}$$

One sees easily that when $R > 1$, we have $\gamma_s(\boldsymbol{k}) > 0$, and the plasma oscillations are damped. If $R < 1$, oscillations propagating at an angle χ to the direction of the electrical field which is larger than $\chi_c = \arccos R$ will be damped. As $\chi \to \chi_c$, the damping rate tends to zero; when $\chi < \chi_c$, the plasma oscillations grow with a growth rate $|\gamma_s(\boldsymbol{k})|$.

The presence of growing oscillations in the plasma leads to a plasma instability when $R \leqslant 1$. The nature of the instability of a partially ionized plasma in an external electrical field is the same as that of the instability of a two-temperature plasma with a directed electron motion, which we considered in Chapter 6.

Let us now investigate how the nature of the ion-sound oscillations depends on the magnitude of the external electrical field. When the field is not too strong, $R > 1$ so that all oscillations are damped. When E_0 increases the long-wavelength oscillations ($r_D k \ll 1$) which propagate along the field are the first to start growing. Using (7.2.1.11) we get for the corresponding critical value of the field

$$E_c = \frac{\alpha T_i}{Zel} \sqrt{\left(\frac{m_e}{m_i}\right)} \ln \frac{m_i}{m_e}; \qquad \alpha = \frac{\sqrt{(3)}\Gamma(\frac{3}{4})}{2\Gamma(\frac{3}{4})} = 2.62. \tag{7.2.1.12}$$

When the field increases further, oscillations with larger k and larger values of the angle χ between the vectors $\partial\omega/\partial \boldsymbol{k}$ and $\boldsymbol{E}_0$ will start to grow. The opening angle of the cone inside which the oscillations grow increases with increasing E_0, tending as $E_0 \to \infty$ to a maximum value determined by the ratio of the ion mass to the mass of the neutral particles in the plasma,

$$\chi_{max} = \arccos \left\{ \frac{2^{3/4}}{\sqrt{(3\pi)}} \sqrt{\left(\frac{Zm_0}{m_i}\right)} \right\}. \tag{7.2.1.13}$$

In particular, if $Zm_0/m_i > 3\pi \cdot 2^{-3/2} = 3.34$ the ion-sound oscillations do not become growing oscillations for any values of the external electrical field, so that the plasma does not become unstable.

7.2.2. ION-SOUND OSCILLATIONS IN EXTERNAL ELECTRICAL AND MAGNETIC FIELDS

Let us now consider the ion-sound oscillations of a plasma in both a strong external electrical field as well as in a strong external magnetic field ($|\omega_{Be}| \gg \nu_e$) (Akhiezer and Angeleĭko, 1969 a, b). The stationary electron distribution function $F_e^{(0)}$ is in this case given by eqns. (7.1.2.9); as before we can assume the stationary ion distribution function $F_i^{(0)}$ to be Maxwellian. Using the kinetic eqns. (7.1.3.1) and (7.2.1.2) and taking to fix the ideas $\gamma_i^{(0)} \gg \nu_i$, we get the following expression for the longitudinal (with respect to the wavevector $\boldsymbol{k}$) dielectric permittivity of the plasma (see Chapter 5)

$$\varepsilon_l(\boldsymbol{k}, \omega) = 1 + 4\pi \sum_{\alpha=e,\, i} \varkappa_\alpha(\boldsymbol{k}, \omega),$$

$$\varkappa_\alpha(\boldsymbol{k}, \omega) = \frac{2\pi e_\alpha^2}{k^2 m_\alpha} \sum_{l=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_{\parallel} \int_0^{\infty} dv_\perp\, v_\perp \frac{J_l(k_\perp v_\perp/\omega_{B\alpha})}{k_\parallel v_\parallel - \omega + l\omega_{B\alpha}} \left[k_\parallel \frac{\partial F_\alpha^{(0)}}{\partial v_\parallel} + \frac{l\omega_{B\alpha}}{k_\perp v_\perp} k_\perp \frac{\partial F_\alpha^{(0)}}{\partial v_\perp} \right], \tag{7.2.2.1}$$

where e_α and m_α are the charge and mass of the αth kind of particles (α = e, i), $\omega_{B\alpha} = e_\alpha B_0/m_\alpha c$; $\boldsymbol{k}_\perp$, $k_\parallel$ and $\boldsymbol{v}_\perp$, $v_\parallel$ are the components of the vectors $\boldsymbol{k}$ and $\boldsymbol{v}$ which are perpendicular and parallel to the magnetic field, and J_l is a Bessel function.

Assuming the electron component of the plasma to be strongly magnetized ($|\omega_{Be}| \gg \omega$) we can restrict our considerations to the term with $l = 0$ in eqn. (7.2.2.1) for the electrical susceptibility of the electrons, $\varkappa_e$. As a result we get

$$\varkappa_e(k, \omega) = \frac{1}{4\pi(kr_D)^2} + \frac{\pi i e^2}{k^2 m_e} \operatorname{sgn} \omega \int \delta\left(v_{\parallel} - \frac{\omega}{k_{\parallel}}\right) \frac{\partial F_e^{(0)}}{\partial v_{\parallel}} d^3 v, \tag{7.2.2.2}$$

where r_D is the electron Debye radius which is related to the effective electron temperature $T_B = \sqrt{(m_0/3m_e)} eE_0 l\,|\cos\beta|$ by the equation

$$r_D = \sqrt{\left(\frac{T_B \Gamma(\frac{3}{4})}{2\pi e^2 n_{e0} \Gamma(\frac{1}{4})}\right)} = 0.815 \sqrt{\left(\frac{T_B}{4\pi e^2 n_{e0}}\right)}. \tag{7.2.2.3}$$

As far as the ion component of the plasma is concerned, we shall consider two limiting cases: the case of weakly magnetized ions ($\omega_{Bi} \ll \omega$) and the case of strongly magnetized ions ($\omega_{Bi} \gg \omega$). In the first case we can use for the ion electrical susceptibility $\varkappa_i$ the equations of the preceding subsection. In the second case we can restrict ourselves in eqn. (7.2.2.1) for the function $\varkappa_i$ to the contribution from the term with $l = 0$, and we have

$$\varkappa_i(k, \omega) = -\frac{\omega_{pi}^2}{4\pi\omega^2} + i\frac{(Ze)^2 n_{i0}}{k^2 T_i} \sqrt{\left(\frac{\pi m_i}{2T_i}\right)} \frac{\omega}{k_{\parallel}} \exp\left\{-\frac{m_e \omega^2}{2T_i k_{\parallel}^2}\right\}. \tag{7.2.2.4}$$

Let us first consider the case of weakly magnetized ions, $\omega_{Bi} \ll \omega$. Putting the function ε_l equal to zero we find the frequency and damping rate of the ion-sound oscillations:

$$\omega_s(k) = \frac{kv_s}{\sqrt{(1+k^2 r_D^2)}}, \quad v_s = \sqrt{\left(\frac{2ZT_B\Gamma(\frac{3}{4})}{m_i\Gamma(\frac{1}{4})}\right)} = 0.662 \sqrt{\left(\frac{ZT_B}{m_i}\right)}, \tag{7.2.2.5}$$

$$\gamma_s(k) = \omega_s(k) \frac{\pi^2 \mu^{\pm}}{\Gamma^2(\frac{1}{4})(1+k^2 r_D^2)\,|\cos\chi|} \sqrt{\left(\frac{3m_e}{2m_0}\right)} (R^{\pm} - \cos\chi), \tag{7.2.2.6}$$

where χ is the angle between the group velocity $\partial\omega/\partial \boldsymbol{k}$ and the external magnetic field, and

$$\begin{aligned} R^{\pm} &= 2^{3/4} \sqrt{\left(\frac{Zm_0}{m_e}\right)} \frac{1}{\mu^{\pm}\sqrt{[3\pi(1+k^2 r_D^2)]}}, \\ \mu^{\pm} &= 1 \mp \frac{2^{3/4} Z^{3/2} \Gamma(\frac{3}{4})}{\pi\sqrt{3}\sqrt{(1+k^2 r_D^2)}} \sqrt{\left(\frac{m_0}{m_e}\right)} \left(\frac{T_B}{T_i}\right)^{3/2} \exp\left\{-\frac{m_i v_s^2}{2T_i(1+k^2 r_D^2)}\right\}, \end{aligned} \tag{7.2.2.7}$$

and the upper (lower) sign refers to the case $\chi < \pi/2$ ($\chi > \pi/2$)—eqn. (7.2.2.6) for the damping rate is applicable for angles which are not too close to $\pi/2$, $\cos\chi \gg \operatorname{Max}\{\sqrt{(Zm_e/m_i)};\ \nu_{e,i}/\omega_s\}$.

We see that—as in the case when there is no magnetic field—if $R^+ > 1$, we have $\gamma_s > 0$, and the plasma oscillations are damped. If $R^+ < 1$; oscillations propagating at an angle χ to the direction of the external magnetic field exceeding $\chi_c^+ = \arccos R^+$ are damped. As $\chi \to \chi_c^+$ the damping rate tends to zero; in the range $\chi < \chi_c^+$ the plasma oscillations grow with a growth rate $|\gamma_s(\boldsymbol{k})|$. The presence of growing oscillations in the plasma leads to the instability of the plasma when $R^+ \leqslant 1$.

Let us investigate how the nature of the ion-sound oscillations depends on the magnitude and direction of the external electrical field. When the field is not too strong, we have $R^+ > 1$, and all oscillations are therefore damped. When E_0 increases the long-wavelength oscillations ($r_D k \ll 1$) propagating along the magnetic field will be the first to start growing. We find for the corresponding critical value of the field, using eqns. (7.2.2.6) and (7.2.2.7)

$$E_c' = \frac{E_0}{|\cos\beta|}, \tag{7.2.2.8}$$

where the quantity E_c is given by eqn. (7.2.1.12). When E_0 increases further, oscillations with larger k and larger values of the angle χ between the vectors $\partial\omega/\partial \boldsymbol{k}$ and $\boldsymbol{B}_0$ will begin to grow. The opening angle of the cone inside which the oscillations grow increases with increasing E_0, tending as $E_0 \cos\beta \to \infty$ to the maximum value given by eqn. (7.2.1.13).

If the ion component of the plasma is strongly magnetized ($\omega_{Bi} \gg \omega$), we can use (7.2.2.2) and (7.2.2.4) to find for the frequency and damping rate of the oscillations, which in this case are called magneto-sound oscillations,

$$\begin{aligned} \omega_m(\boldsymbol{k}) &= \frac{k v_s |\cos\chi|}{\sqrt{(1+k^2 r_D^2)}}, \\ \gamma_m(\boldsymbol{k}) &= \omega_m(\boldsymbol{k}) \frac{\pi^2}{\Gamma^2(\frac{1}{4})} \sqrt{\left(\frac{3m_e}{2m_0}\right)} \frac{1}{1+k^2 r_D^2} \{\xi(\boldsymbol{E}_0) - \operatorname{sgn}\cos\chi\}, \end{aligned} \tag{7.2.2.9}$$

where the velocity v_s as before is given by eqn. (7.2.2.5), and

$$\xi(\boldsymbol{E}_0) = \frac{2^{3/4}}{\sqrt{(3\pi)}} \sqrt{\left(\frac{Zm_0}{m_i}\right)} \left\{1 + \frac{Z\Gamma(\frac{3}{4})}{\sqrt{\pi}} \sqrt{\left(\frac{m_0}{m_e}\right)} \left(\frac{T_B}{T_i}\right)^{3/2} \exp\left(-\frac{m_i v_s^2}{2T_i(1+k^2 r_D^2)}\right)\right\}. \tag{7.2.2.10}$$

If the external electrical field is not too strong, we have $\xi(\boldsymbol{E}_0) > 1$; the plasma oscillations are damped in that case. When the external electrical field increases the function $\xi(\boldsymbol{E}_0)$ decreases, becoming equal to unity at the critical value of the field E_c', given by eqn. (7.2.2.8). When the electrical field increases further oscillations with wavevectors in the half-space $\chi < \pi/2$ start to grow. This leads to the fact that when $E_0 \geqslant E_c'$ the plasma is unstable.

We draw attention to the fact that in a very strong magnetic field ($\omega_{Bi} \gg \omega$) all oscillations with wavevectors in the half-space $\chi < \pi/2$ will grow when the critical value of the electrical field is reached. In contrast to this, when $\omega_{Bi} \ll \omega$—as in the case when there is no external magnetic field—only those oscillations grow which are propagating inside a cone with an angle χ_c^+ at the vertex.

Concluding this subsection we emphasize that if $Zm_0/m_i > 3\pi \cdot 2^{-3/2} = 3.34$, both when there is a magnetic field present and when there is no magnetic field, the ion-sound (or magnetosound) oscillations of a partially ionized plasma become non-growing however large the value of the external electrical field.

7.3. Low-frequency Oscillations of a Partially Ionized Plasma

7.3.1. OSCILLATIONS INVOLVING NON-CONSERVATION OF THE NUMBERS OF PARTICLES OF THE DIFFERENT COMPONENTS OF THE PLASMA

Oscillations which are accompanied by a change in the total number of charged particles play an important role in a partially ionized plasma, just as the kinds of oscillations which we have considered in which the total number of electrons N_e and of ions N_i in the whole volume of the plasma does not change. In particular, it is this kind of oscillations which leads apparently to the occurrence of striations in the positive column of a gas discharge.

To study plasma oscillations accompanied by a change in the total number of particles —such oscillations are sometimes called ionization waves—we must, generally speaking, start from the complete set of kinetic equations for electrons, ions, and neutral particles —as well as the equations of electrostatics. As well as the self-consistent field and the usual collision integrals—which take into account elastic scattering processes of the particles—we must introduce in these equations collision integrals which describe inelastic processes, such as ionization and recombination processes or transitions of atoms and ions into excited states. If this set of equations is linearized and if we look for its solution in the form of plane monochromatic waves, we can, in principle, determine the frequencies and damping (or growth) rates of the ionization waves.

The problem of a theoretical study of the ionization wave spectra turns out, however, to be extra-ordinarily complex. The fact is that the collision integrals—describing both elastic processes as well as processes involving a change in the number of particles—are very complex functions of the velocities of the colliding particles. These functions are usually determined from experiments and in many cases they have so far not been found. At the same time the dispersion laws of the ionization waves are very sensitive to the explicit form of the collision integrals.

The equations describing ionization waves can be somewhat simplified if we take into account that in those waves—in contrast to the normal plasma waves—usually only the particle density changes, but not the momentum per unit volume of the plasma. Instead of the kinetic equations we can therefore for the description of these waves use the continuity equations for each kind of particle taking into account inelastic processes—and for the charged particles also the quasi-neutrality condition for the plasma or the equations of motion including the transport coefficients. However, even if we use this approach the equations remain very complex. Therefore a theoretical study has been possible so far only of a few particular cases and there does not exist a consistent study of ionization waves for a wide range of plasma parameters, such as densities, temperatures, or external magnetic field strengths.

We shall not aim at expounding with any degree of completeness the theoretical results relating to ionization waves.† We note merely that the frequencies of the ionization waves are of the order of magnitude of the reciprocals of the times characteristic for the processes responsible for the occurrence of these waves, such as ionization and recombination processes and are of the order of 10^3 to 10^5 s^{-1}. The oscillations of a partially ionized plasma which are accompanied by a change in the total numbers of particles are therefore sometimes called the low-frequency oscillations of such a plasma, in contrast to the usual plasma oscillations.

In order to elucidate the mechanism of the oscillations of a partially ionized plasma which are accompanied by a change in the total numbers of particles we shall consider the simplest case of uniform oscillations (Roth, 1967). Assuming that the dimensions of the plasma are large compared to the Debye radius, we can take it that $N_i = N_e$ (we assume the ions to be

† Nedospasov (1968) and Pekarek (1968) have given detailed reviews of experiments and theories relating to these waves.

singly charged). We shall start from the simplest set of equations for the time-derivatives of the total number of electrons N_e and the total number of neutral particles N:

$$\begin{aligned} \dot{N}_e &= A_0 + A_1 N + A_2 N_e + A_3 N N_e + A_4 N^2 + A_5 N_e^2, \\ \dot{N} &= B_0 + B_1 N + B_2 N_e + B_3 N N_e + B_4 N^2 + B_5 N_e^2. \end{aligned} \tag{7.3.1.1}$$

Here A_0, B_0 are the number of particles injected into or leaving the volume considered of the plasma per unit time, A_1, A_2 and B_1, B_2 are coefficients characterizing those processes which are responsible for the change in the numbers of particles which are not connected with collisions between particles (for instance, ionization processes due to a high temperature in a spark discharge); finally, A_3, A_4, A_5 and B_3, B_4, B_5 are coefficients characterizing the rate of ionization and recombination processes which are the result of collisions between particles.

Equations (7.3.1.1) are analogous to the equations in the classical Volterra problem about the numbers of two kinds of fishes which eat one another (see, for instance, Andronov, Vitt, and Khaĭkin, 1959). It is well known that such equations can have bounded, periodic solutions.

To fix our ideas, let is consider a volume V of a plasma into which B_0 neutral particles are injected per unit time. Ions may leave this volume through a surface S (for instance, a cross-section perpendicular to the magnetic field lines); the escape of ions per unit time is clearly

$$A_2 N_e = -N_e \frac{v_i S}{4V},$$

where v_i is the average thermal velocity of the ions.

We shall assume that the ionization of neutral particles when they collide with electrons is the main source of ions. Introducing the cross-section σ of this process we can write an expression for the changes in the numbers of neutral and of charged particles per unit time, caused by these proceses, in the form

$$A_3 N N_e = -B_3 N N_e = N N_e \frac{\langle \sigma v \rangle}{V},$$

where v is the relative velocity of the colliding particles and the pointed brackets indicate an average. Neglecting other processes which may possibly change the numbers of charged and neutral particles we get the following equation for the functions N and N_e:

$$\dot{N} = B_0 - A_3 N N_e, \quad \dot{N}_e = N_e(A_2 + A_3 N). \tag{7.3.1.2}$$

Putting the right-hand sides of eqns. (7.3.1.2) equal to zero, we get the equilibrium particle numbers N_0 and N_{e0}:

$$N_0 = -\frac{A_2}{A_3}, \quad N_{e0} = -\frac{B_0}{A_2}; \tag{7.3.1.3}$$

we recall that the coefficients B_0 and A_3 are positive and that the coefficient A_2 is negative. Introducing the deviations of the particle numbers from their equilibrium values, $\Delta N =$

$N-N_0$, $\Delta N_e = N_e - N_{e0}$, and linearizing eqns. (7.3.1.2), we find

$$\Delta\dot{N} = -\omega_0 \sqrt{\left(\frac{N_0}{N_{e0}}\right)} \Delta N_e - \omega_0 \sqrt{\left(\frac{N_{e0}}{N_0}\right)} \Delta N, \quad \Delta\dot{N}_e = \omega_0 \sqrt{\left(\frac{N_{e0}}{N_0}\right)} \Delta N, \quad (7.3.1.4)$$

where

$$\omega_0 = A_3 \sqrt{(N_0 N_{e0})} = \langle \sigma v \rangle \frac{\sqrt{(N_0 N_{e0})}}{V}. \quad (7.3.1.5)$$

We can determine the frequency ω and the damping rate γ of the oscillations from the condition that eqns. (7.3.1.4) are compatible, and we find

$$\omega = \omega_0 \sqrt{\left(1 - \frac{N_{e0}}{4N_0}\right)}, \quad \gamma = \frac{\omega_0}{2} \sqrt{\left(\frac{N_{e0}}{N_0}\right)}. \quad (7.3.1.6)$$

We see that the oscillations described by eqns. (7.3.1.2) are, in general, damped. However, in the case of a weakly ionized plasma, when $N_{e0} \ll N_0$, the damping rate of these oscillations is small compared to the frequency.

Equations such as (7.3.1.1) can also have undamped—or even growing—solutions. Let us briefly consider the simplest example of undamped oscillations. We shall assume that the increase in the number of neutral particles per unit time is proportional to the number of neutral particles (Roth (1969) has described an experimental situation when this occurs). In that case we must start not from eqns. (7.3.1.2), but from the equations

$$\dot{N} = N(B_1 - A_3 N_e), \quad \dot{N}_e = N_e(A_2 + A_3 N). \quad (7.3.1.7)$$

In the present case the equilibrium numbers of particles are related to the A and B coefficients through the equations

$$N_0 = -\frac{A_2}{A_3}, \quad N_{e0} = \frac{B_1}{A_3}. \quad (7.3.1.8)$$

Linearizing eqns. (7.3.1.7) with respect to small deviations of the particle numbers from their equilibrium values, we get

$$\Delta\dot{N} = -\omega_0 \sqrt{\left(\frac{N_0}{N_{e0}}\right)} \Delta N_e, \quad \Delta\dot{N}_e = \omega_0 \sqrt{\left(\frac{N_{e0}}{N_0}\right)} \Delta N, \quad (7.3.1.9)$$

where the frequency ω_0 is given by eqn. (7.3.1.5). These equations describe clearly undamped uniform plasma oscillations with frequency ω_0.

Without wanting to discuss in detail the case of non-uniform plasma oscillations which are accompanied by a change in the total particle numbers we note solely that such oscillations can be characterized by a great variety of dispersion relations and a great variety in the wavevector-dependence of the damping (or growth) rates. The frequencies and damping rates of these oscillations vary strongly, even for small changes in the plasma pressure or temperature. The velocities of the ionization waves usually lie in the range from 10^3 to 10^5 cm/s, and the group and phase velocities may differ a great deal one from another, and can even be in opposite directions.

7.3.2. LARGE AMPLITUDE UNIFORM LOW-FREQUENCY OSCILLATIONS

Let us now consider uniform low-frequency oscillations of a partially ionized plasma without assuming the amplitude of the oscillations to be small (Roth, 1967.) A study of such non-linear oscillations enables us, firstly, to find the way the frequencies of the oscillations depend on their amplitude and, secondly, to find the characteristic shape of the pulsations—which turns out to be sinusoidal only in the small amplitude limit.

We shall restrict ourselves to the case of a weakly ionized plasma ($N_e \ll N$); in that case we can simplify eqns. (7.3.1.2) considerably. Eliminating for this purpose the quantity $\dot{N}$ from these equations, we get

$$\ddot{N}_e - A_3BN_e + A_3^2N_0N_e^2 + \Delta NA_3N_e(A_3N_e - 2A_3N_0 - 2A_2) - (\Delta N)^2 A_3^2N_e = 0, \quad (7.3.2.1)$$

where N_0 is the time-averaged value of the function $N(t)$; $\Delta N(t) = N(t) - N_0$, and

$$B = B_0 + A_3^{-1}(A_2 + A_3N_0)^2 .$$

We shall show in what follows that in the case of a weakly ionized plasma the relative amplitude of the oscillations of the number of neutral particles is small, $\Delta N/N_0 \ll 1$, while the relative amplitude of the oscillations of the number of electrons is, in general, not small, $\Delta N_e/N_e \sim 1$. We can therefore neglect in eqn. (7.3.2.1) terms containing ΔN and use the equation

$$\ddot{N}_e - A_3BN_e + A_3^2N_0N_e^2 = 0. \quad (7.3.2.2)$$

Equations (7.3.1.7) reduce in the case when $N_e \ll N$ to a similar equation. Indeed, eliminating $\dot{N}$ from these equations and neglecting terms containing ΔN we get the same eqn. (7.3.2.2) in which we must substitute for B the expression $B = B_1N_0$.

As the time origin we take the moment when the derivative $\dot{N}_e(t)$ vanishes, $\dot{N}_e(0) = 0$, and we introduce the notation

$$N_e = N_{e0}(1+Q), \quad \eta = \frac{B}{A_3N_0N_{e0}}, \quad (7.3.2.3)$$

where $N_{e0} \equiv N_e(0)$ is the extremal value of the function $N_e(t)$; we can then write eqn. (7.3.2.2) in the form

$$\omega_0^{-2}\ddot{Q} + Q^2 + (2-\eta)Q + (1-\eta) = 0, \quad (7.3.2.4)$$

where the frequency ω_0 is given by eqn. (7.3.1.5). We can express the solution of this equation in terms of elliptical Jacobi functions. To do this, we multiply eqn. (7.3.2.4) by $\dot{Q}$ and integrate once,

$$\left(\frac{\dot{Q}}{\omega_0}\right)^2 + \frac{2}{3}Q(Q-Q_1)(Q-Q_2) = 0, \quad (7.3.2.5)$$

where

$$Q_{1,2} = -\tfrac{3}{4}\{(2-\eta) \pm \sqrt{[(\eta - \tfrac{2}{3})(\eta+2)]}\}.$$

Let us first consider the case when $\frac{2}{3} < \eta < 1$. Integrating eqn. (7.3.2.5) and using the fact that in this case both Q_1 and Q_2 are negative, we get

$$Q = Q_2 \,\mathrm{sn}^2([\tfrac{1}{6}|Q_1|]^{1/2}\omega_0 t;\ [Q_2/Q_1]^{1/2}), \tag{7.3.2.6}$$

where $\mathrm{sn}(x; k)$ is a Jacobian elliptical function which satisfies the equation (see, for instance, Gradshteĭn and Ryzhik, 1966)

$$\frac{d}{dx}\,\mathrm{sn}\,(x;k) = [1-\mathrm{sn}^2(x;k)]^{1/2}\,[1-k^2\,\mathrm{sn}^2(x;k)]^{1/2}.$$

In the case considered the frequency of the oscillations is equal to

$$\omega = \tfrac{1}{2}\omega_0[\tfrac{1}{6}\,|Q_1|]^{1/2}K^{-1}([Q_2/Q_1]^{1/2}), \tag{7.3.2.7}$$

where $K(k)$ is a complete elliptic integral of the first kind; the extremum N_{e0}/V is for this range of η-values the maximum value of the electron density.

Let us now consider the case $\eta > 1$. We can in this case as before use eqn. (7.3.2.6) for the quantity Q. However, bearing in mind that $Q_2 > 0 > Q_1$, and using the well-known relation (see Gradshteĭn and Ryzhik, 1966)

$$\mathrm{sn}\,(x;ik) =$$

$$\frac{1}{\sqrt{(1+k^2)}}\,\mathrm{sn}\left(x\sqrt{(1+k^2)};\ \frac{k}{\sqrt{(1+k^2)}}\right)\left\{1-\frac{k^2}{1+k^2}\,\mathrm{sn}^2\left(x\sqrt{(1+k^2)};\ \frac{k}{\sqrt{(1+k^2)}}\right)\right\}^{-1/2},$$

it is advantageous to write the equation for Q in the form

$$Q = \frac{Q_1Q_2\,\mathrm{sn}^2\left(\frac{1}{2}\omega_0 t[(\eta-\frac{2}{3})(\eta+2)]^{1/4};\ \sqrt{[Q_2/(Q_2-Q_1]}\right)}{Q_1-Q_2+Q_2\,\mathrm{sn}^2\left(\frac{1}{2}\omega_0 t[(\eta-\frac{2}{3})(\eta+2)]^{1/4};\ \sqrt{[Q_2/(Q_2-Q_1)]}\right)}. \tag{7.3.2.8}$$

The frequency of the oscillations is in that case equal to

$$\omega = \tfrac{1}{4}\omega_0[(\eta-\tfrac{2}{3})(\eta+2)]^{1/4}K^{-1}([Q_2/(Q_2-Q_1)]^{1/2}); \tag{7.3.2.9}$$

now the quantity N_{e0}/V is the minimum value of the electron density.

When $\eta \approx 1$, we have $Q \ll 1$. In that case eqn. (7.3.2.2) describes the small amplitude harmonic oscillations with frequency ω_0 which we considered in the preceding subsection. The case $\eta = 1$ corresponds to the stationary regime, $N_e(t) \equiv N_{e0}$.

Let us estimate the quantity ΔN which we neglected when changing from eqns. (7.3.1.2) or (7.3.1.7) to eqn. (7.3.2.2). Using the fact that according to the first of eqns. (7.3.1.2) or (7.3.1.7) $\Delta N \sim \omega_0^{-1}N_0N_{e0}A_3$, we see that

$$\frac{\Delta N}{N_0} \sim \sqrt{\left(\frac{N_{e0}}{N_0}\right)}.$$

Therefore, eqn. (7.3.2.2) correctly describes plasma oscillations, provided the degree of ionization of the plasma is small, $N_{e0} \ll N_0$.

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† As far as possible the English translations of Russian papers and the originals of other papers are given (*Translator*). This list is common to Volumes 1. and 2. and thus contains papers not referred to in Volume 1.

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GLOSSARY

A-wave: Alfvén wave
$a_\alpha = (kv_\alpha/\omega_{B\alpha})^2 = (k\rho_\alpha)^2$ (cf. p. 231)

$\boldsymbol{B}$: magnetic induction
$\boldsymbol{B}^{(e)}$: external magnetic induction
$\boldsymbol{B}^{(p)}$: internal (self-consistent) magnetic field
$\boldsymbol{b} = \boldsymbol{B}/B$

C_s: s-particle correlation function
CS-wave: cyclotron-sound wave
c_p: specific heat at constant pressure
c_s: sound velocity
c_v: specific heat at constant volume

$D(k, p)$: function defined by eqn. (4.2.1.11)
$\mathcal{D}$: probability density

$\boldsymbol{E}$: electrical field
$\boldsymbol{E}^{(e)}$: external electrical field
$\boldsymbol{E}^{(p)}$: internal (self-consistent) electrical field
e: elementary charge ($-e$ is the electron charge)
$\boldsymbol{e}$: polarization vector
e_α: charge of particle of type α

F: single-particle distribution function
$F^{(M)}$: Maxwell distribution function
FE-wave: fast extra-ordinary wave
FMS-wave: fast magneto-sound wave
FS: fast sound wave
f_s: s-particle distribution function

G: pair correlation function
g: plasma parameter given by eqn. (1.2.1.3)

$\mathcal{H}$: Hamiltonian

I-wave: ionization wave
$\mathcal{I}$: collision integral

j: current density
$j^{(e)}$: external current density
$j^{(p)}$: current density in plasma

L: Coulomb integral
$\mathcal{L}$: differential operator given by eqn. (1.1.3.7)
l: mean free path
l_s: penetration depth

M: Mach number
MS-wave: magnetized sound wave
m: mass of plasma particle
m_0: mass of neutral particle
m_e: electron mass
m_i: ion mass

N_i: total number of particles of kind i
n: refractive index
n_α: density of particles of kind α
$n'_{\alpha\bullet}$: beam density

O-wave: ordinary wave

$\mathcal{P}$: principal value symbol
p: pressure
p^*: effective pressure

Q, q: energy flux density
q: dissipated energy density

R_i: Riemann invariant
R_ν: Reynolds number
R_σ: magnetic Reynolds number, Lundquist number
r_0 electron radius
r_D: Debye radius; usually electron Debye radius
r_{Di}: ion Debye radius

S: entropy
SE-wave: slow extra-ordinary wave
SMS-wave: slow magneto-sound wave
SS-wave: slow sound wave
s: entropy density

T: temperature (in energy units)
T_0: temperature of neutral particles
T_e: electron temperature
T_i: ion temperature
T^{ik}: energy-momentum tensor

U: group velocity
$\boldsymbol{u}$: hydrodynamical velocity
$u_{\alpha\beta}$: velocity deformation tensor

V: volume
V: phase velocity
$\bar{v}$: mean thermal velocity
v_+: velocity of fast magneto-sound waves
v_-: velocity of slow magneto-sound waves
v_A: Alfvén velocity
v_e: electron thermal velocity
v_{gr}: group velocity
v_i: ion thermal velocity
v_s: ion sound velocity
v_{se}: electron sound velocity
v_{ph}: phase velocity

w: enthalpy
$w(z)$: function given by eqn. (5.2.2.2)

Ze: ion charge
z_i: quantity given by eqn. (5.2.2.3)
z_i': quantity given by eqn. (6.2.1.2)

$\beta = \sqrt{(1-u^2/c^2)}$
$\beta_e = v_e/c$

γ: adiabatic index; ratio of specific heats ($= c_p/c_v$)
$\gamma(\boldsymbol{k})$: damping or growth rate

ε: dielectric constant
ε: internal energy per unit mass

ζ: viscosity coefficient

η: viscosity coefficient

θ: temperature
$\theta[x]$: step function given by eqn. (2.1.6.15')

$\varkappa$: thermal conductivity
$\varkappa$: damping coefficient
$\boldsymbol{\varkappa} = \boldsymbol{k}/k$

Λ_{ik}: matrix elements defined by eqn. (4.3.1.10')
$\lambda = k_x v_\perp/\omega_{B\alpha}$(see p. 228)

ν, ν_1, ν_2: viscosity coefficients
ν_e: Coulomb collision frequency
ν_m : magnetic viscosity

$\Pi_{\alpha\beta}$: momentum current density tensor
$\boldsymbol{\pi}$: electromagnetic energy flux density
$\pi_{ij}^{(\alpha)}$: polarizability tensor
$\pi_{\alpha\beta}$: viscous stress tensor

ξ_e: ratio of electron pressure to magnetic pressure (see eqn. (5.6.1.14))

ϱ, ϱ_e: charge density
ϱ : external charge density
ϱ_m: mass density
ϱ^p: charge density in plasma
ϱ_α: Larmor radius of particle of type α

σ: electrical conductivity
σ: scattering cross-section
$\sigma_{\alpha\beta}$: stress tensor

τ: relaxation time

φ: electrostatic potential
φ_k: Fourier transform of φ
φ_{ij}: two-particle electrostatic potential

ω_{Be}: electron gyro-frequency
$\omega_{B\alpha}(= e_\alpha B/m_\alpha c)$: gyro-frequency of particle of type α
ω_p: plasma frequency, Langmuir frequency

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