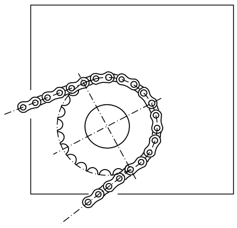
Structural mechanics



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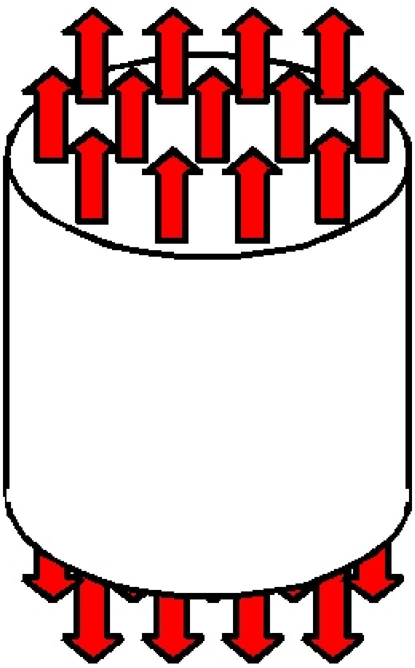

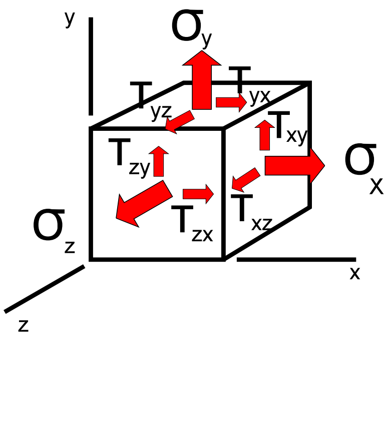

Dirk Pons

Structural Mechanics

Third Edition, 2010

Mechanical forces cause stress and deformation in parts. If the force is too high then the part breaks and the product can fail. Therefore Design Engineers in particular find it useful to know the equations describing these effects.

This volume lists the generally accepted equations of structural mechanics. The purpose is simply to list the equations in a way that makes them ready to apply in engineering design.

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Structural Mechanics

This volume describes various classical analyses for structural mechanics (also called strength of materials). This subject is taught extensively in engineering colleges, but with an analytical approach. This volume takes a different approach, which is to present only the final equations, generally without derivation. In this way it condenses several study years into one consistent summary. It is intended for use in mechanical engineering design, but it assumes some previous exposure to the subject. Derivations of the equations are not given, and it is left to the discretion of the reader to select appropriate equations for use. All equations assume quantities involved are nominally positive. Please note this work uses the comma version of the decimal point.

Important concepts found here

- ☐ *What is stress?*
- ☐ *What is strain, and how is it different to stress?*
- ☐ *Connecting stress and strain with modulus of elasticity.*
- ☐ *Special stress and strain conditions: plane stress, plane strain...*

1 STRESS AND STRAIN

This section describes the basic equations that are used for stress and strain, and the conversion between the two.

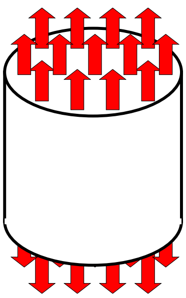
1.1 Stress

Normal (tension or compression) stress is
S

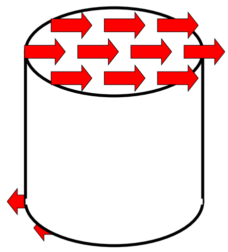
$$\sigma = \frac{F}{A_{normal}}$$

hear stress is:

$$\tau = \frac{F}{A_{parallel}}$$



Normal stress (tension illustrated)



Shear stress

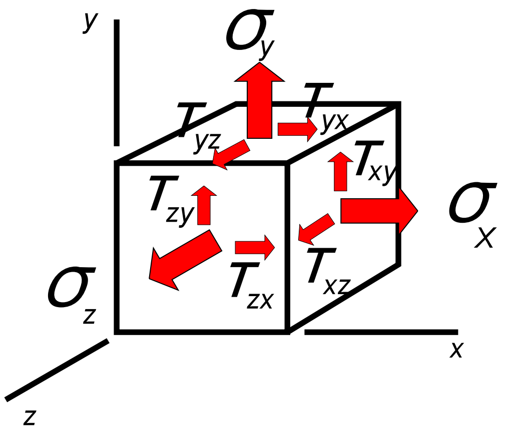
1.1.1 State of stress

Stresses on a small block are written with a double subscript, the first referring to the surface on which the stress acts, and the second to the direction. The full stress tensor is:

s_{xx}	s_{xy}	s_{xz}
s_{yx}	s_{yy}	s_{yz}
s_{zx}	s_{zy}	s_{zz}

The tensor is symmetrical about the principal diagonal as the stresses on opposite faces are numerically equal for a small block, eg $s_{xz} = s_{zx}$. The stresses s_{xx}, s_{yy}, and s_{zz} are all direct stresses. All the other stresses are shear stresses. Therefore it is often more convenient to rewrite the stresses as

σ_x	T_{xy}	T_{xz}
	σ_y	T_{yz}
		σ_z



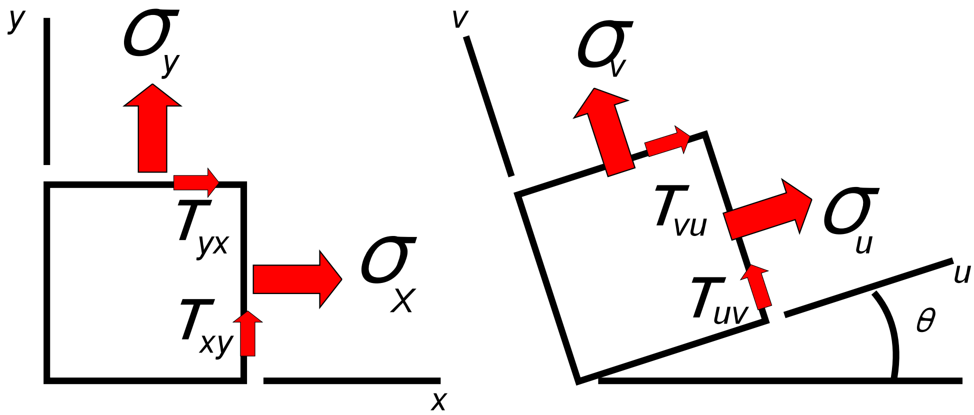
Stress tensor, showing normal and shear stresses on a small cube of material.

Note that the first subscript refers to the surface on which the stress acts, and the second to the direction. Therefore there are six possible stresses on a small cube of material, three direct stresses, and three shear stresses. Tensile stresses are taken as positive, anc compressive ones as negative.

1.1.2 Principal Stresses

Stresses on an inclined plane

At any point in a structure there may exist all six of the stresses. For the moment consider a simplification, where there are no stresses in the z direction, so that the problem is reduced to just two dimensions. Referring to the figure above for the stress tensor, imagine that all stresses in the z direction were zero. That would make τ_{xz} zero since it is on the x plane and in



Stresses on any inclined plane may be calculated, and are equivalent to the first orientation.

the z direction. However τ_{zx} is the same, so it will also be zero. Therefore the only stresses to exist would be σ_x, σ_y, and τ_{xy} .

In this 2D case, if we measure the stress at a point in some arbitrary direction, we would get a certain value. If we selected another direction instead (still at the same point though), then we would get a different value of stress. Therefore the value we get depends heavily on the direction which we chose. The common engineering practice is to take the first direction as the x direction, and then take the y direction as being perpendicular to that. Then we can determine stresses σ_x and σ_y. We can also determine the shear stress on the xy plane. Below are the equations that are used to determine the stresses on other planes. The direct stresses on some plane inclined to that on which the stresses are known, are:

$$\sigma_{u,v} = \frac{\sigma_x+\sigma_y}{2} \pm \frac{\sigma_x-\sigma_y}{2}.\cos(2.\theta) \pm \tau_{xy}.\sin(2.\theta)$$

where both are plus or both minus (no mixing). The shear stress on the inclined plane is

$$\tau_{uv} = \frac{\sigma_y - \sigma_x}{2} . \sin(2.\theta) + \tau_{xy} . \cos(2.\theta)$$

Principal stresses in biaxial loading

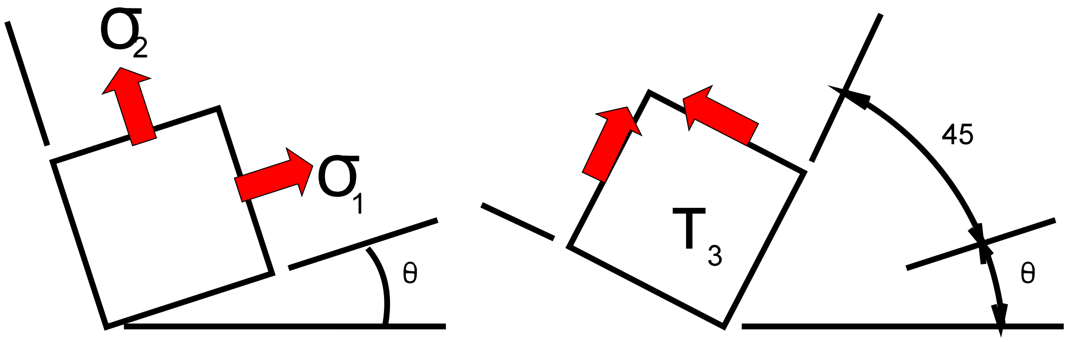
But we have no guarantee that the stresses that we found are the maximum ones for that point. What about the other directions? For engineering applications it is not so much the stress on some arbitrary plane that is of interest, but the maximum stress that can exist at the point in concern. The reason for this is that the material will tend to fail when this maximum stress gets too close to the yield strength of the material. This maximum stress is called the principal stress. In fact there are two principal stresses: one is the maximum and the other is the minimum. When the normal stresses reach their maximum and minimum values, the shear stresses become zero. The maximum and minimum normal stresses are called principal stresses, and are:

$$\sigma_{1,2} = \frac{1}{2}\left[(\sigma_x + \sigma_y) \pm \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \right]$$

(I.e σ_1 is determined with the + sign, and σ_2 with the - sign)
The angle at which these occur, relative to the original stress system, is:

$$\tan(2.\theta) = \frac{2.\tau_{xy}}{\sigma_x - \sigma_y}$$

Actually, in practical engineering applications we are often not very interested in the direction of the principal stress, since its magnitude is what has the



Principal stresses are the maximum normal stresses at a point, and occur for some special orientation θ. The shear stresses are zero.

Maximum shear stress is an alternative to principal stresses, and occurs for 45 deg off the principal direction. The principal stresses are zero.

greater effect on the resistance of the part to failure.

Conversely, maximum shear stress occurs at planes 45° from the planes of max or min normal stress, and is given by:

$$\tau_3 = \pm\frac{\sigma_1 - \sigma_2}{2} = -\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$

and similarly for the other axes.

A common mode of failure for a material is to fail when the maximum shear stress reaches a certain limit. For loading in one axis (eg tensile test specimen), the maximum shear stress will occur on planes at 45 deg to the axis of loading. This is precisely the form of the slip lines ("Luder's lines") seen on such specimens.

Principal stresses for three dimensional loading
It is always possible to determine the three principal stresses for a full complement of three dimensional loading. However the equations are significantly more difficult to handle than for the two dimensional case. In general, it is probably better to use distortion energy (see later) to combine 3D loading into one equivalent stress, at least for the ductile materials.

For a given stress tensor of:

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ & & \sigma_z \end{matrix}$$

The principal stresses are the three roots of β in the following equation

$$\beta^3 - I_1.\beta^2 + I_2.\beta - I_3 = 0$$

where the three invariants are

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$

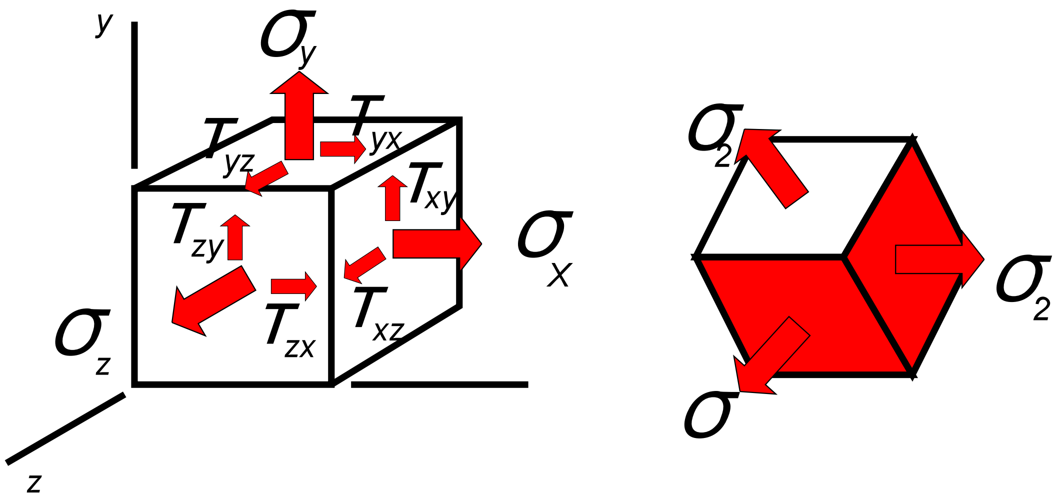
$$I_2 = \det\begin{bmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{bmatrix} + \det\begin{bmatrix} \sigma_z & \tau_{zx} \\ \tau_{xz} & \sigma_x \end{bmatrix} + \det\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

$$= \sigma_y.\sigma_z - \tau_{yz}^2 + \sigma_z.\sigma_x - \tau_{xz}^2 + \sigma_x.\sigma_y - \tau_{xy}^2$$

$$I_3 = \det \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$$= \sigma_x.\sigma_y.\sigma_z + 2\tau_{xy}.\tau_{yz}.\tau_{zx} - \sigma_x.\tau_{yz}^2 - \sigma_z.\tau_{xy}^2 - \sigma_y.\tau_{xz}^2$$

There are three solutions of β, corresponding to the three principal stresses. In the general case it will be necessary to solve the equation numerically.



Principal stresses for three axis loading. Left shows full loading, and right shows equivalent principal stresses.

1.1.3 Special Stress Systems

The most general stress system has been described above as consisting of three direct stresses, and three shear stresses. However there are some simplifications that occur reasonably frequently.

Two principal stresses ZERO
This is uniaxial tension or compression. The simple version of Hookes Law (to be explained further below), E=stress/strain, is applicable for this state only.

One principal stress ZERO
This is a biaxial system, eg plane stress and plane strain. Plane stress occurs when there are no stresses planes perpendicular to the plane being

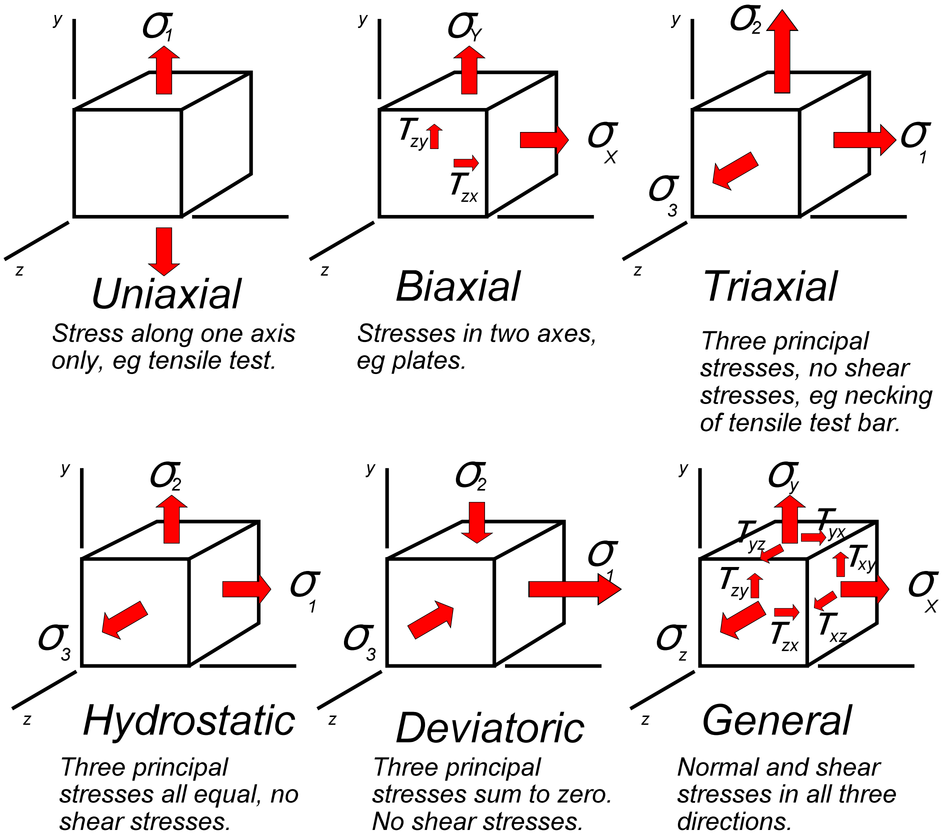
considered. Plane strain is when there is no strain in the perpendicular to the plane being considered. These are two dimensional systems.

No principal stresses zero

These are stress systems where all the principal stresses exist, but the shear stresses are zero. There are a number of sub-categories here.

- ***Triaxial*** **system, eg necking of a solid bar under tension. There are no shear stresses, and therefore all the direct stresses are principal stresses: σ_1, σ_2, σ_3, with all τ zero**
- ***Hydrostatic*** **(also called isotropic) stress, if all principal stresses are equal: $\sigma_1 = \sigma_2 = \sigma_3 = p$ (eg a cube dropped to the floor of the ocean). Distortion occurs without change in shape. Bulk modulus K is a measure of the pressure required per unit volume change:**

$$K = \frac{E}{3(1-2.v)} = \frac{-p}{\Delta V/V} = \frac{E.G}{9G-3E}$$



Loading cases, from the most simple to the most complex.

- ***Deviatoric*** **stress if sum of principal stresses is zero: $\sigma_1 + \sigma_2 + \sigma_3 = 0$ and change in volume is negligible. At some orientation all normal tresses**

vanish, and distortion is only shear. Shear modulus or Modulus of rigidity, G is a measure of the resistance to this type of deformation.

1.1.4 Deviatoric and isotropic stress system

Any stress system, including the general stress system of:

σ_x	τ_{xy}	τ_{xz}
	σ_y	τ_{yz}
		σ_z

can be resolved into a deviatoric component (changes shape) and an isotropic component (changes volume). The shear components are deviatoric (produce shape change only). The normal stresses may be resolved into isotropic plus deviatoric stresses.

The *isotropic* (pressure) stress tensor is

p	**0**	**0**
0	**p**	**0**
0	**0**	**p**

where

$$p = \frac{\sigma_x+\sigma_y+\sigma_z}{3}$$

The *deviatoric* stress field is

$(\sigma_x - p)$	τ_{xy}	τ_{xz}
0	$(\sigma_y - p)$	τ_{yz}
0	0	$(\sigma_z - p)$

Adding the two gives the original stresses. Therefore any stress system may be converted into an isotropic stress plus a deviatoric stress. The deviatoric stress is important in producing plastic flow in ductile materials.

1.1.5 Octahedral Stress

The common co-ordinate system for stress is the Cartesian system of three axes x, y and z, at right angles to each other. Up to now we have considered the stresses that act on the six surfaces of a cube. However there is an alternative mathematical system, based on the stresses on the surface of an octahedral (8 surface structure). It so happens that any x, y z cubic stress

system is much less complicated when transformed to the octahedral system. The octahedral surfaces have normal stress of:

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

(in terms of the principal stresses). The normal octahedral stresses are hydrostatic (change volume but not shape). The octahedral surfaces also have shear stresses. These are deviatoric (change shape only), and are:

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_2 - \sigma_1)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_3 - \sigma_2)^2 \right]^{0,5}$$

The equations may also be written:

$$\sigma_{oct} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

$$\tau_{oct} = \frac{1}{3}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2]^{0,5}$$

Therefore a six component cubic stress system may be reduced to just two stresses: octahedral normal stress, and octahedral shear stress, which are applied to all the surfaces of the octahedral.

A common criterion of failure is based on the octahedral shear stress, in that failure occurs when:

$$\tau_{oct} = \frac{\sqrt{2}}{3}.R_m$$

where R_m is the ultimate tensile strength of the material.

1.2 Strain

Strain refers to the physical deformation of an object. There are two types of strain:

(a) The first is normal strain, which is ϵ_i = (change in length)/(original length).

(b) Next is shear strain, which is the angular deformation. There are two possible measurements for this:
- **Engineering Shear strain γ_{ij} is the reduction of a right angle**
- **Tensor shear strain ϵ_{ij} is:**

$$\epsilon_{ij} = \frac{\gamma_{ij}}{2}$$

In other words, motion of a body is composed of RIGID BODY MOTION (translation and rotation) and DEFORMATION (dilation and distortion).

There are various other types of strain:
- **Volumetric strain is e = $\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$**
- **In cylindrical coordinates:**
  - **Radial strain ϵ_{rr}**
  - **Transverse strain $\epsilon_{r\theta}$**
  - **Shear strain $\gamma_{r\theta}$**

Like stress, a small block of material can have normal and shear strains on each of the six faces. Strains on a small cube are written with a double subscript, the first referring to the surface on which the stress acts, and the second to the direction.

$$\begin{matrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{matrix}$$

The tensor is symmetrical about the principal diagonal as the strains on opposite faces are numerically equal for a small block, eg $\epsilon_{xz} = \epsilon_{zx}$.

The strains ϵ_{xx}, ϵ_{yy}, and ϵ_{zz} are all direct strains. All the other strains are shear strains. Therefore it is often more convenient to rewrite the strains as

$$\begin{matrix} \epsilon_{x} & \gamma_{xy} & \gamma_{xz} \\ & \epsilon_{y} & \gamma_{yz} \\ & & \epsilon_{z} \end{matrix}$$

Therefore there are six possible strains on a small cube of material, three normal strains, and three shear strains. Tensile strains are taken as positive.

These are the strains on the faces of a cube. But if the cube was cut out of the material at a different orientation, then different values of strains would be

measured. Below are the equations that are used to determine the strains on other planes. Often the important part is to determine the maximum strain at the point, that is the principal strains. When the strains are at the principal values, then it will be found that there is no shear strain.

1.2.1 Principal Strains

The strains on some plane inclined at θ to that on which the values are known, are:

$$\epsilon_\theta = \epsilon_x \cos^2\theta + \gamma_{xy}\sin\theta\cos\theta + \epsilon_y \sin^2\theta$$

$$\frac{1}{2}\gamma_\theta = \frac{1}{2}\gamma_{xy}(\cos^2\theta - \sin^2\theta) + (\epsilon_y - \epsilon_x)\sin\theta\cos\theta$$

When the normal strains reach their maximum and minimum values, the shear strains become zero. The max and min normal strains are called principal strains and are:

$$\epsilon_{1,2} = \frac{1}{2}\left[(\epsilon_x + \epsilon_y) \pm \sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2} \right]$$

The angle at which these occur, relative to the original strain system, is:

$$\tan(2.\theta) = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

Conversely, maximum shear strain occurs at planes 45 deg from planes of max or min normal strain, and is given by:

$$\tau_3 = \pm\frac{\sigma_1 - \sigma_2}{2} = -\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$

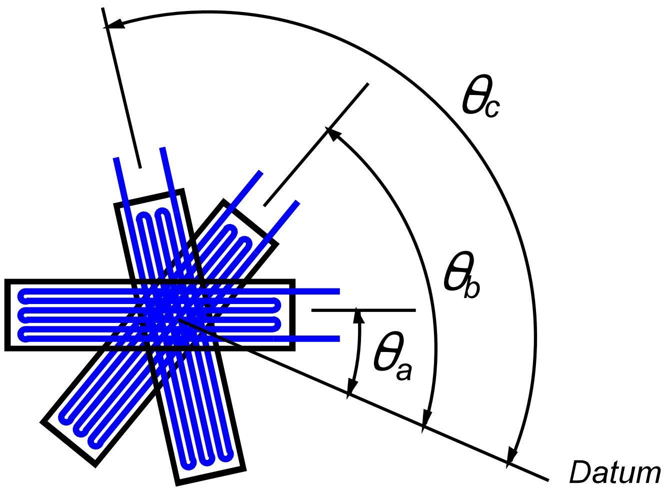
and similarly for the other axes.

1.2.2 Strain Gauges

Strain gauges are small strips of (essentially) wire that are stuck onto the surface of a part, and which measure the local strain. The strain is determined by the change in electrical resistance of the tiny wire as it is stretched. In practice the gauge is made of foil rather than wire, and the measuring part consists of tracks laid in parallel (but connected in series), to provide a greater

length and therefore greater sensitivity. Typical gauge dimensions are 3 mm wide and 10 mm long.

As gauges are placed on a free surface, one of the principal stresses must be



General case for three strain gauges at any angles relative to each other.

zero (there can be no stress coming out of the surface). The stresses are then biaxial or uniaxial. For biaxial strain systems there exist only ϵ_x, ϵ_y, and γ_{xy} at the point of interest. (Any strain with a z in it is zero.) Therefore there are three unknowns, and three values will be needed to solve the problem, i.e. three gauges. For uniaxial strain there is only one strain, and therefore only one gauge is needed.

The gauges read strain values in the direction of their long axis. If another gauge was to be placed on the same spot, but with a different orientation, then it would read a different strain. The problem is to determine the local strains ϵx, ϵy, and γxy in some reference direction. Then the principal strains can be found.

General case

If three gauges, a, b and c, are placed at angles of θ_a, θ_b, and θ_c from a datum line, and strain readings taken, then there are three equations which have to be solved simultaneously:

$$\epsilon_a = \epsilon_x \cos^2\theta_a + \gamma_{xy} \sin\theta_a \cos\theta_a + \epsilon_y \sin^2\theta_a$$
$$\epsilon_b = \epsilon_x \cos^2\theta_b + \gamma_{xy} \sin\theta_b \cos\theta_b + \epsilon_y \sin^2\theta_b$$
$$\epsilon_c = \epsilon_x \cos^2\theta_c + \gamma_{xy} \sin\theta_c \cos\theta_c + \epsilon_y \sin^2\theta_c$$

Assume that $\theta_a = 0$, so that

$\epsilon_x = \epsilon_a$

Then put $\theta = \theta_c - \theta_a$ and $\phi = \theta_b - \theta_a$ .Then the shear strain is

$$\gamma_{xy} = \frac{\epsilon_c - \epsilon_a \cos^2\theta - (\epsilon_b - \epsilon_a \cos^2\phi).\dfrac{\sin^2\theta}{\sin^2\phi}}{\sin\theta\cos\theta - \sin\phi\cos\phi.\dfrac{\sin^2\theta}{\sin^2\phi}}$$

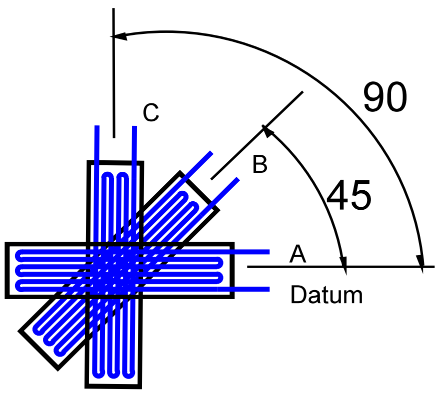
Then the normal strain in the y direction is

$$\epsilon_y = \frac{(\epsilon_b - \epsilon_a \cos^2\phi - \gamma_{xy}\sin\phi\cos\phi)}{\sin^2\phi}$$

Now the following are known: ϵ_x, ϵ_y, and γ_{xy}, and the next step is usually to determine the principal strains.

However practical applications of strain gauges are usually one of the following rather than the general case.

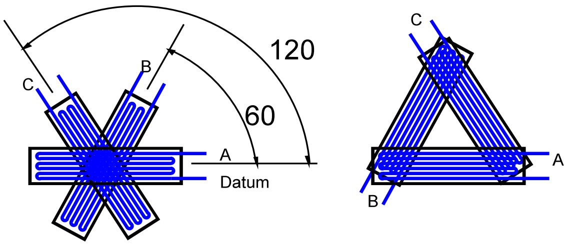
Rectangular rosette
The gauges are arranged with θ_a=0, θ_b=45, and θ_c=90 degrees.
Then the local strains are
$\epsilon_x = \epsilon_a$
$\epsilon_y = \epsilon_c$
$\gamma_{xy} = 2\epsilon_b - \epsilon_a - \epsilon_c$



Rectangular arrangement of strain gauges.

Of course these strains are with respect to the gauge chosen as the reference, i.e. gauge a, and this is an arbitrary decision. The maximum strain will not necessarily lie in this direction, so it will usually be necessary to find the principal strains.

This problem may also be solved by Mohr's strain circle, or by the general case given above.



Delta arrangement of strain gauges, showing two possible configurations.

Delta or equiangular rosette
In this case the gauges are equally oriented, that is θ_a=0, θ_b=60, and θ_c=120 degrees. To determine the principal strains, the following calculations are required:

$$\Delta = \frac{\epsilon_a+\epsilon_b+\epsilon_c}{3}$$

$$\Delta_1 = \epsilon_{med}-\epsilon_{small}$$

$$\Delta_2 = \epsilon_{large}-\epsilon_{med}$$

$$\alpha = atan\left[\frac{\sqrt{3}\Delta_2}{2\Delta_1+\Delta_2}\right]$$

$$R = \frac{2\Delta_1+\Delta_2}{3\cos\alpha}$$

The value of R is the radius of the Mohr strain circle, and Δ is the distance from the origin to the centre of the circle. The principal strains are then

$$\epsilon_1 = \Delta+R$$

$$\epsilon_2 = \Delta-R$$

This problem may also be solved by using the general case given above.

Determining stresses from strains
After determining the principal strains, the principle stresses are determined using the following:

$$\sigma_z = \frac{E}{1-\nu^2}(\epsilon_z+\nu.\epsilon_y)$$

and

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y+\nu.\epsilon_z)$$

where
E Modulus of elasticity of substrate
ν **Poisson's ratio for substrate**

It is important to note that it is NOT permissible just to multiply the principal strain with the Modulus of elasticity.

1.3 Linear Stress-strain Relations

It is often necessary to convert between stress and strain. Materials themselves tend to be influenced more by strain than by stress, since the change in spacing between molecules is the fundamental effect of external loads. However in engineering applications we tend to think rather of stress, and most of the mechanical tests on materials are done in terms of this parameter.

When converting between stress and strain, it is important to note the difference between linear (or elastic) and non-linear (or plastic) conditions. The linear range corresponds to loading below the yield of the material, and stress is proportional to strain. When the load is removed the material returns to its original shape. The non-linear range is for materials that are yielding in plastic deformation. The relationship between stress and strain is more complex than before, and is not one of proportionality. On removing the load, the material does not return to its original shape, but exhibits permanent deformation. The linear case is generally the more common in engineering, since it gives designers a larger safety margin.

1.3.1 Elastic Materials

Modulus of elasticity (Young's Modulus) E = stress/strain, but is only valid for uniaxial stress, and in the linear (elastic) portion of the stress-strain curve.

1.3.2 Poisson's Effect

Lengthening a bar causes the diameter to shrink. The coupling factor is denoted by λ or ν.

For isotropic materials it will be found that Poisson's ratio is in the range $0 \leq \nu \leq 1/2$.
There is no such coupling for shear.

1.3.3 Hooke's Law

The fundamental law relating stress and strain is:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha.\Delta T$$

and similarly

$$\epsilon_y = \frac{1}{E}[\sigma_y - v(\sigma_x + \sigma_z)] + \alpha.\Delta T$$

and

$$\epsilon_z = \frac{1}{E}[\sigma_z - v(\sigma_x + \sigma_y)] + \alpha.\Delta T$$

where
α **coefficient of thermal expansion**
ΔT **temperature change**

For shear the equations become:

$$\gamma_{xy} = 2\epsilon_{xy} = \frac{1}{G}\tau_{xy}$$

and similarly for yz and zx.

1.3.4 Relations Between Elastic Constants

The fundamental constants are shear modulus G, and bulk modulus K, related as follows:

$$E = 2G(1+v) = 3K(1-2.v) = \frac{9.G.K}{3K+G}$$

1.3.5 Plane Stress

This is a thin plate of uniform thickness and no curvature (it is a two dimensional component). The loading is entirely in the plane: there can be no pressure or force trying to bulge the plate out of its plane. Plane stress is applicable to cases which consist of thin plates with loading along any of the edges. Plane stress requires that there be no stress in one of the axes (eg the x axis, arbitrarily. The plate is assumed to be thin, so that no stresses develop across the plate thickness (i.e. along the X axis). This is typically true for thin plates where there is no stress difference across the thickness: all depths (i.e. X axis) have the same stress distribution. There is no stress normal to the YZ plane, that is in the axis of the thickness. However there is still a strain in this direction (X), which is due to the Poisson effect from the main stresses Y and Z. (Thick plates used in pressure vessels would not be considered plane stress, because there are stresses through the thickness.)

Plane stress typically occurs for thin sheets and plates that are loaded only in the plane. There is no stress through the plate thickness. If the sheet is in the yz plane (arbitrary decision), then all the x stresses are zero:

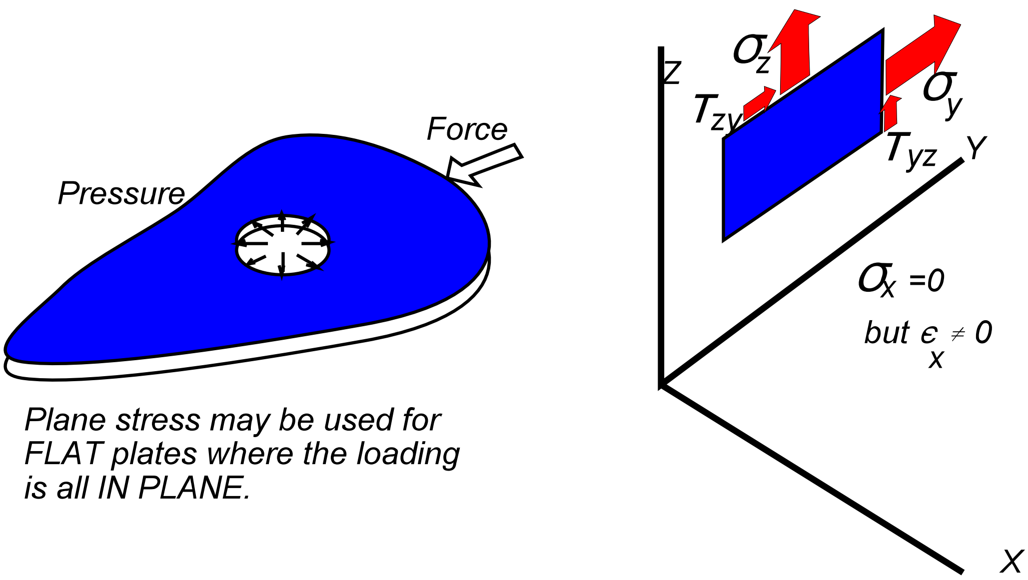
$$\sigma_x = \tau_{zx} = \tau_{xy} = 0$$

The strains are determined by substituting into Hooke's Law equations, which reduce to:

$$\epsilon_y = \frac{1}{E}(\sigma_y - v.\sigma_z) + \alpha.\Delta T$$

and similarly

$$\epsilon_z = \frac{1}{E}(\sigma_z - v.\sigma_y) + \alpha.\Delta T$$



Plane stress

However, just because there is no stress in the x direction does not mean that there is no x strain: it is generated indirectly by the Poisson effect. Therefore:

$$\epsilon_x = \frac{1}{E}(-v(\sigma_z + \sigma_y)) + \alpha.\Delta T$$

Solving these equations for stresses gives:

$$\sigma_z = \frac{E}{1-\nu^2}(\epsilon_z + \nu.\epsilon_y - \alpha.\Delta T)$$

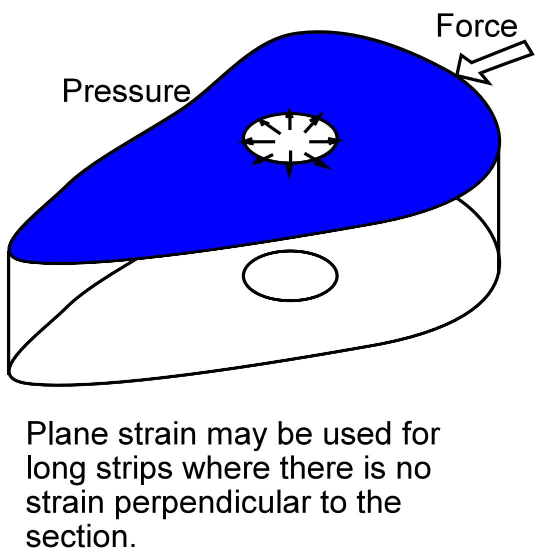
and similarly for y:

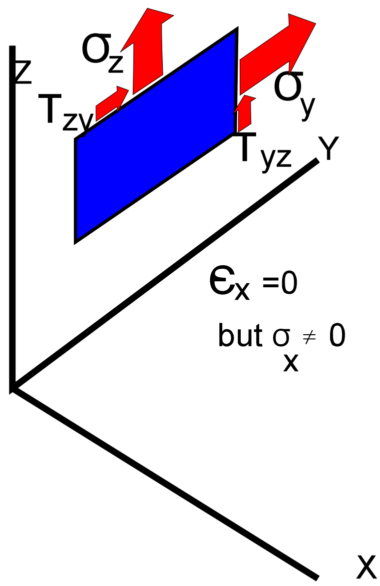
$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu.\epsilon_z - \alpha.\Delta T)$$

1.3.6 Plane Strain

Plane strain applies to long strips, the ends of which are constrained, that is there is no deflection in the long direction. If the sheet is in the yz plane (arbitrary decision), then all the x strains are zero:

$$\epsilon_{xz} \text{ or } \gamma_{xz} = \epsilon_{xy} \text{ or } \gamma_{xy} = \epsilon_{xx} = 0$$





Plane strain

Substitute into Hooke's Law equations, which reduce to:

$$\epsilon_z = \frac{1+v}{E}(\sigma_z(1-v) - v.\sigma_y) + E.\alpha.\Delta T$$

and similarly for yy:

$$\epsilon_y = \frac{1+v}{E}(\sigma_y(1-v) - v.\sigma_z) + E.\alpha.\Delta T$$

Solving these equations for stresses gives:

$$\sigma_z = \frac{E}{(1+v).(1-2v)}.[\ (1-v)\epsilon_z + v.\epsilon_y\ ] - \frac{E.\alpha.\Delta T}{1-2v}$$

and similarly for y:

$$\sigma_y = \frac{E}{(1+v).(1-2v)}.[\ (1-v)\epsilon_y + v.\epsilon_z\ ] - \frac{E.\alpha.\Delta T}{1-2v}$$

However, just because there is no strain in the x direction does not mean that there is no x stress: it is generated indirectly by the Poisson effect.

$$\sigma_x = \frac{E.v}{(1+v).(1-2v)}.[\ \epsilon_z + \epsilon_y\ ] - \frac{E.\alpha.\Delta T}{1-2v}$$

1.4 Non-linear Stress-strain

Non-linear stress refers to stress and strain in the plastic (non-linear) region. Hook's law applies only in the elastic (linear) region. Therefore the relationship between stress and strain is non-linear in the plastic region. The non-linearity is generally due to overload. However it can also be due to ageing, load rate dependence, and path dependence (effect of stresses before final stress).

1.4.1 Non-linear relationship

There are a number of ways of approaching the non-linear relationship.

Perfect plasticity
A common assumption is that the stress remains constant when it reaches the yield, while the strain carries on. Such a material would be "perfectly plastic". Although materials do not generally show perfect plasticity, the assumption is nonetheless reasonable since (1) it is conservative as the stress rises above the yield in most materials (due to strain hardening), and (2) the assumption makes the maths easier to deal with.

Curve fitting
A more accurate method is to fit a curve to experimental stress-strain data. This could be done for any material.

Ramberg-Osgood
The Ramberg-Osgood method is based on curve fitting. They investigated a number of materials and found that it was possible to write a generic equation instead of having to find a unique equation for each material. They give the relationship between stress and strain as follows:

$$\epsilon = \frac{\sigma}{E}\left[1+\frac{3}{7}\left(\frac{\sigma}{\sigma_{0,7}}\right)^{n-1}\right]$$

where
$\sigma_{0,7}$ **0,2% yield strength**
n shape factor, from tables or the equation below:

$$n = 1+\frac{\ln(\frac{17}{7})}{\ln(\frac{\sigma_{0,7}}{\sigma_{0,85}})}$$

where
$\sigma_{0,85}$ **is the stress at where the line σ = 0,85.E intersects the stress-strain curve of the material**
Note: n -> ∞ is an elastic, perfectly plastic material. Properties for select materials follow.

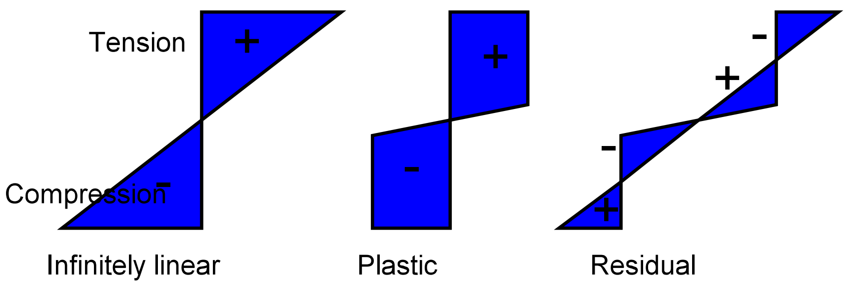
Material	*Modulus of elasticity E [GPa]*	*$\sigma_{0,7}$ or 0,2% yield strength [MPa]*	*Ramberg-Osgood Shape factor, n*
Aluminium alloys			
2014-T6	**73**	**410**	**20**
2024 T4	**73**	**330**	**10**
6061 T6	**69**	**280**	**30**
7075 T6	**72**	**500**	**20**

Material	***Modulus of elasticity E [GPa]***	***$\sigma_{0,7}$ or 0,2% yield strength [MPa]***	***Ramberg-Osgood Shape factor, n***
4139 steel			
Normalised	**200**	**510**	**20**
1/4 hard	**200**	**650**	**25**
1/2 hard	**200**	**790**	**35**
3/4 hard	**200**	**970**	**40**
Full hard	**200**	**1140**	**50**
Colled rolled Stainless steel 18Cr-8Ni *	**190**	**640**	**4**
1/4 hard	**190**	**810**	**5**
1/2 hard	**190**	**980**	**6**
3/4 hard	**190**	**1220**	**8**
Full hard			
Colled rolled and heat treated Stainless steel 18Cr-8Ni *			
1/4 hard	**190**	**450**	**5**
1/2 hard	**190**	**850**	**6**
3/4 hard	**190**	**1140**	**7**
Full hard	**190**	**1260**	**9**

Note *   Applicable to tension and transverse compression only.

1.4.2 Residual Stresses

Loading into the plastic region causes residual stresses even after the original load is removed. The effect is caused by the un-yielded material springing back when the load is removed, stressing the former regions in the opposite direction.



To determine residual stresses, use the principle of superposition: apply the load but assume that the material is infinitely elastic. Then apply the load, taking platic flow into account. Finally, the residual stresses are the difference between the two stress distributions. The process is illustrated for a beam in bending.

The first figure shows the application of a bending stresses across a beam section, with the assumption that the material remains elastic. The second figure shows the effect of plasticity, assuming a perfectly plastic material: the stress cannot exceed the yield, and so the maximum stresses of the elastic condition cannot be achieved. It is not simply a case of cutting off the peaks of the linear case, because the total area of the + region must stay the same. Therefore the remaining elastic regions of the section must carry extra load. (As the load is increased, so the plastic region approaches closer to the centre of the beam. If the entire beam section should become plastic, then a hinge would develop and the beam would collapse.)

The third figure shows the residual stress distribution in the beam once the bending moment causing case 2 has been removed. The elastic regions (towards the centre of the beam) try to revert to their original dimensions, but they cannot do this completely and in doing so they pull/push the formerly plastic regions into a compromise. The residual stresses have been shown on a sloped axis. The horizontal component is the one of interest.) Note that the resultant of the residual stresses must be zero: the part may not have a net tension or compression otherwise it would not be stationary!

1.5 Photoelasticity

The basic principle of photoelasticity is that certain transparent materials change the polarisation of light passing through them, where the amount of change depends on the strain (stress) in the material. In practice the effect is observed by making a part out of the (special) plastic, and subjecting it to loads. Often it is necessary to scale the model, and also the loading. The model is illuminated with light, and a series of filters. The light can either be monochromatic, or white light. The path that the light takes before being observed is light source, followed by polarising filter, then model, then another polarising filter, and then only the viewer. Stresses in the model show up as coloured and dark bands. These are called isochromatics and isoclinics. The Isoclinics show the orientation of the principal stresses. The Isochromatics give the magnitude of the (difference in) principal stresses ($\sigma_1 - \sigma_2$), in terms of a number of fringes.

Isoclinics

Appearance: **dark bands**

Use: **orientation of principal stresses; lines connect points with same inclination. Will be perpendicular to free boundary, since one of the principal stresses will be zero.**

***Method*:** **Light source : polariser at θ : model : analyzer at θ + 90° . This gives an isoclinic for angle θ. Repeat with other angles to get a family of curves. Then construct stress trajectories (also called isostatics).**

Isochromatics

Appearance: **coloured bands in white light, dark bands in monochromatic light**

Use: **Magnitude of principal stresses; gives $\sigma_1 - \sigma_2$. Note that the maximum shear stress is $\tau_{max} = (\sigma_1 - \sigma_2)/2$. In use, count the fringe order (eg 5th green fringe) and determine $\sigma_1 - \sigma_2 = n.F$ where n is the fringe order and F is the fringe value in Mpa/fringe. At a free edge one of the principal stresses will be zero so $\sigma_1 = n.F$**

***Method*:** **Light source : polariser at θ : model : analyzer at θ + 90° . This will give both isochromatics and isoclinics. The isoclinics may be removed by using cross polarised light (also called 1/4 wave or circular polarised light). The method is Light source : polariser at θ : 1/4 wave plate at θ + 45° : model : 1/4 wave plate at θ - 45° : analyzer at θ + 90°. Changing the angle θ does not affect the isochromatics.**

2 STRAIN ENERGY

The previous section looked at the basic principles of stress and strain, and how they are related. These concepts are important because many engineering design principles are based on determining the stress in the part, and comparing it to the permissible value for the material. However there is another design approach, and it is in many ways even more powerful. It is the method of strain energy. This energy is the stored energy in a part under strain. The strain energy method assumes that the part will fail when the strain energy in the part equals the strain energy in a tensile test piece. The value of the method is that the strain energy at a point in a part is determined from ALL the stresses acting there, and this can then be compared to the simple uniaxial stress case of a tensile test piece for which data are readily available.

The strain energy methods are applied in a number of powerful tools, such as Von Mises (distortion energy), Castigliano's theorems, influence coefficients, and virtual work.

2.1 Simple Stresses

The strain energy of a gradually increasing tensile load is:

simple tension:

$$U = \frac{\sigma^2}{2E} \text{ per unit volume}$$

For shear the strain energy is:

simple shear:

$$U = \frac{\tau^2}{2G} \text{ per unit volume}$$

For bending, the strain energy is:

$$\text{pure bending:} \quad U = \int_0^L \frac{M^2}{2E.I}.dx$$

where L and x refer to the length of the beam.

Strain energy due to torsion of a solid shaft is:

$$U = \frac{\tau^2}{4G} \quad \textit{per unit volume}$$

which is the total energy for which stress varies from 0 at the axis to τ at the outside.

For a hollow shaft:

$$U = \frac{\tau^2}{4G}.\frac{(D^2+d^2)}{D^2} \quad \textit{per unit volume}$$

2.2 Complex Stresses

The total strain energy of the part, per unit volume, is:

$$U = \frac{1}{2E}[\sigma_x^2+\sigma_y^2+\sigma_z^2 - 2\lambda(\sigma_x.\sigma_y+\sigma_x.\sigma_z+\sigma_y.\sigma_z)] + \frac{1}{2G}[\tau_{xy}^2+\tau_{xz}^2+\tau_{yz}^2]$$

where G is the MODULUS OF RIGIDITY (eg 82,6 x 10^9 Pa for steel).
In terms of principal stresses this is:

$$U = \frac{1}{2E}[\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\nu(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_2\sigma_3)]$$

2.3 Components of strain energy

Total strain energy consists of two parts, dilation energy and distortion energy:

Dilation energy, this changes the volume, but not the shape

$$U_{dilation} = \frac{1}{18K}[\sigma_1+\sigma_2+\sigma_3]^2$$

Distortion energy, which changes the shape but not the volume, (also called shear strain energy)

$$U_{distortion} = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2]$$

The distortion energy is the basis of the Von Mises failure criterion, which is discussed later.

2.4 Impact loads

For a suddenly applied load (i.e. live load without shock), the simple tension stress in a supporting bar is *twice the static stress.*

For a live load with shock, the tension in a supporting bar is

$$\sigma = \frac{W}{A} \pm \sqrt{[\frac{W}{A}]^2 + \frac{2WhE}{AL}}$$

where

W applied weight
A cross sectional area of bar
h height that weight drops
L length of supporting bar

These equations only apply to bars that are carrying the load in tension.

These stresses may be used to determine the strain energy. For bars of different sections, determine the strain energy for the individual sections, and then add the component strain energies.

2.5 Shock loading

The stress when a load is dropped from some height onto a structure, may be determined from the following. The equation applies to beams and springs especially.

$$\sigma = \sigma_{static}.[1 + \sqrt{(1 + \frac{2h}{y})}\ ]$$

where

σ_{static} **stress when the same load acts statically**
y **deflection under static load**
h **height that weight drops**

2.6 Influence coefficients

If n forces (P_1 ..P_i ..P_n) act on a structure, at n different points, then the deflection at some point I is

$$\delta_i = a_{i1}.P_1 + a_{i2}.P_2 + \ldots + a_{ij}.P_j + \ldots + a_{\epsilon}.P_n$$

where
a_{ij} **Influence coefficient, the displacement at point I, in the direction of the force P_i, which is produced by a unit load acting at point j**
P_j **force at point j**

The matrix of influence coefficients is symmetrical, as $a_{ij} = a_{ji}$

2.6.1 Reciprocal theorem

For two superposed load systems F_P and P_Q producing respective displacements of δ_P and δ_Q, then the following holds:

$$\sum_i F_{P_i}.\delta_{Q_i} = \sum_i F_{Q_i}.\delta_{P_i}$$

2.6.2 First theorem of Castigliano

If the total strain energy U, expressed in terms of the external loads P, be partially differentiated with respect to any one of these external loads, then the result gives the displacement δ of that load in the direction of the line of action of the load:

$$\frac{\partial U}{\partial P_i} = \delta_i$$

To use this theorem, first determine the force P in each member of the structure:

$$P = \sum_j b_j \, .W_j$$

where b_j is the force in the member when a unit force of W_j acts alone.

Then determine the total strain energy:

$$U = \sum_i \frac{P_i^2 .L_i}{2AE}$$

where
A cross sectional area of structure
E modulus of elasticity
L_i length of member I

2.6.3 Second theorem of Castigliano

The partial derivative of the total strain energy U of a framework, with respect to the force F in a redundant member, equals the initial lack of fit of that member:

$$\frac{\partial U}{\partial F} = \delta_{lack\ of\ fit}$$

Alternatively, the magnitudes of statically indeterminate reactive forces R are such as to make the strain energy of the system a minimum. Therefore, at a redundant support it may be written:

$$\frac{\partial U}{\partial R} = 0$$

The method used to apply the theorem to redundant frames is as follows:
1 Remove the redundant member, and replace it by tension forces F
2 Determine forces in all members, due to external loads P, with F = 0
3 Determine forces in all members, due to tension F only
4 Determine total strain energy of all members

$$U = \sum_i \frac{P_i^2 .L_i}{2AE}$$

5 Partially differentiate the total strain energy with respect to F, and equate to the lack of fit (which should be a known amount, and may be zero)
6 Solve for F
7 Solve for forces in other members by summing forces determined in (2) and (3) above.

Angles and moments may also be included in the strain energy.

2.7 Virtual work

A series of loads P_i produce stresses σ_i in a structure. By some means (not necessarily P_i), deformations u_i are produced in the same structure at the point of application of the loads, in the direction of P_i, and resulting in strains ϵ_i . These two (independent) systems are related by the virtual work theorem:

$$\sum_i P_i \,.u_i = \int_{volume} \sigma.\epsilon \; dV$$

The special form for beams becomes

$$\sum_i P_i \,.u_i = \int_L M.\theta \; ds$$

where
M bending moment as a function of distance along beam
s distance along beam (varies from 0 to L)
L length of beam

The special form for frame members becomes

$$\sum_i P_i \,.u_i = \sum_k F_k \,.\Delta_k$$

where
F_k force in member k
Δ_k extension of member k
To find forces in a frame, allow unit extension of the member (virtual displacement), while all other members are unchanged. To find deflection, apply a single point load at the point. Substitute into the virtual work equation and solve.

3 TORSION

Torsion refers to the twisting of a part about its long axis. This is different from bending, where the axis of rotation is usually transverse to the long axis of the part. However there are several similarities between torsion and bending, not the least of which is the use of the same units [force x distance, eg Nm]. In this book the symbol T has been used for torsion, and M for bending, so as to avoid unnecessary confusion.

3.1 Torsion equation

The relationship between change in helix angle (called shear strain ϕ) and shear stress τ is:

$$G = \frac{\tau}{\phi}$$

where G is the Torsional Modulus of the material, also called the modulus of rigidity. (G = 80 000 MPa for steel). In general:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{L}$$

where

T	**torque**
J	**polar moment of area**
τ	**shear stress**
r	**radius of interest (usually outside radius)**
G	**torsional modulus**
Θ	**angle of twist (viewed on end of shaft at L)**
L	**length of shaft**

3.2 Torsion of Circular Shafts

The fundamental relationship between torque and shear stress for round shafts is as already given:

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{L}$$

This equation may be used to design a shaft only if torsion is the only loading present (no bending).

The Polar moment of area is

$$J = \int r^2 dA$$

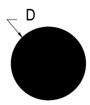
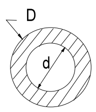
For a solid shaft:

$$J = \frac{\pi}{32} D^4$$

For a hollow shaft:

$$J = \frac{\pi}{32}(D^4 - d^4)$$

The polar moment of area J is also called the polar moment of inertia , although this is an inaccurate term. However it is NOT the same as the second moment of area. The second moment of area usually uses the symbol I, and is used in the equation for bending stress. Some typical values of I and J are given below. The cases illustrated are for solid and hollow round shafts.

Section	Second Moment of Area I	Polar Moment of Area J
 **Solid**	$\frac{\pi . D^4}{64}$	$\frac{\pi . D^4}{32}$
 **Hollow**	$\frac{\pi(D^4 - d^4)}{64}$	$\frac{\pi(D^4 - d^4)}{32}$

For an applied torsional moment T, on a shaft, the shear stress at radius r, is

$$\tau = \frac{T.r}{J}$$

Note that Shear stress is greatest at the outside of the shaft.

The change in angle (twist) over a given length L is:

$$\theta = \int_0^L \frac{T}{G.J}.dz$$

where z is distance along the shaft axis.

For shafts of uniform section, this simplifies to

$$\theta = L\frac{T}{G.J}$$

These equations are valid only for SOLID & HOLLOW circular shafts. Other shafts warp out of the rΘ plane as they twist, and the analysis is more complex.

Shafts are typically loaded in both torsion and bending. However for the special cases where only torsion or bending is applied, then the fundamental equations may be used. For combined torque and bending moment, or fluctuating loads, it is necessary to use other equations to determine the stresses.

3.3 Torsion of Non-Circular Shafts

For sections other than solid or hollow round bars, the shear stress and torsional deflection have to be determined using the equations given on the next page. Although a polar moment of inertia can be calculated for non-round sections, it may not be used. Instead a shear constant K is used in the place of J, but K is not calculated as J.

Table: Shear stress and torsional deflection for various sections, including non-round.

Section	***Torsional deflection*** $\theta = \frac{TL}{KG}$ ***where K is as follows***	***Shear stress, maximum value, at the position given.***
Round Solid **D outer diameter**	$K = \frac{\pi D^4}{32}$	$\tau = \frac{16T}{\pi D^3}$ **at outside diameter**
Round hollow **D outer diameter** **d inner diameter**	$K = \frac{\pi}{32} . (D^4 - d^4)$	$\tau = \frac{16TD}{\pi (D^4 - d^4)}$ **at outside diameter**
C shaped Open tube: (round hollow with slit) **r mean radius** **t wall thickness**	$K = \frac{2\pi}{3} . r\ t^3$	$\tau = \frac{3T}{2\pi r\ t^2} + \frac{1{,}8Tt}{4\pi^2 r^2 t^2}$ **opposite ends**
Solid ellipse **a half of major axis** **b half of minor axis**	$K = \frac{\pi a^3 b^3}{a^2 + b^2}$	$\tau = \frac{2T}{\pi a b^2}$ **on outside of minor axis**
Hollow ellipse **a half of major axis outside** **b half of minor axis outside** **a1 half of major axis inside** **b1 half of minor axis inside**	$K = \frac{\pi a_1^3 b_1^3}{a_1^2 + b_1^2} . [(1+q)^4 - 1]$ $where\ \ q = \frac{a - a_1}{a_1} = \frac{b - b_1}{b_1}$	$\tau = \frac{2T}{\pi a_1 b_1^2 [(1+q)^4 - 1]}$ **on outside of minor axis**
Equiangular triangle **b length of one side**	$K = \frac{\sqrt{3}}{80} . b^4$	$\tau = \frac{20T}{b^3}$ **on middle of each side**

Section	***Torsional deflection*** $\theta = \frac{TL}{KG}$ ***where K is as follows***	***Shear stress, maximum value, at the position given.***
Heaxgon **b** **length of one side (NOT across flats)**	$K = 2{,}69.b^4$	$\tau = \frac{1{,}09T}{b^3}$ **on middle of each side**
Solid Rectangle **a** **length** **b** **width**	$K = \frac{ab^3}{16}.[\frac{16}{3} - 3{,}36\frac{b}{a}(1 - \frac{b^4}{12a^4})]$	$\tau = \frac{(3a + 1{,}8b)T}{a^2b^2}$ **on middle of longer side**
Hollow rectangle **a** **length outside** **b** **width outside** **t1** **wall thickness (one side) in width** **t2** **wall thickness (one side) in length**	$K = \frac{2t_1t_2(a-t_2)^2(b-t_1)^2}{at_2 + bt_1 - t_2^2 - t_1^2}$	$\tau = \frac{T}{2t\ (a-t_2).(b-t_1)}$ **put t = t1 for stress at middle of longer side, and t = t2 for stress at shorter side (NB 1)**
Solid square **b** **length of one side**	$K = 0{,}1406b^4$	$\tau = \frac{4{,}8T}{b^3}$ **on middle of each side**

Note 1: **Hollow rectangle: this equation permits different wall thicknesses in the length and width. If the wall thickness is constant then the stress on the longer side is greater. However, if the width wall thickness is increased to compensate, then the shorter side may become more highly stressed. Anyway, the stress is highest at the inside corners, but depends on the fillet geometry.**

3.4 Torsional Strain Energy

The strain energy for torsion is:

$$U = \frac{1}{2G}\int \tau^2.dV = \frac{1}{2G}\int \frac{T^2}{J}.dz$$

For constant shaft section this becomes

$$U = \frac{1}{2G}\frac{T^2}{J}.L$$

3.5 Torsion of Springs

The springs discussed here are axial coils of wire, that are used to generate axial force. In the process of generating this axial force, the wire in the spring itself is subject to torsion, and hence the inclusion of springs in this section.

3.5.1 Close Coiled Springs

Close coiled spring with wire radius r, overall coil radius R, and number of coils N, the force F is:

$$F = k.\Delta$$

where spring constant k is:

$$k = \frac{G.r^4}{4R^3N}$$

3.5.2 Open Coiled Helical Springs

The spring has a helix angle of α, n coils, wire radius of r, coil radius of R, and axial force P. The axial deflection is:

$$\delta = \frac{L.R^2.P}{E.I}(1 + \text{v}.\cos^2\alpha)$$

where
l = 2πR.n
and I is the second moment of area of the spring wire.

For a closed coil spring α = 0.

The angular displacement of a suspended load P is:

$$\theta = \frac{L.P.R.\text{v}.\sin(2\alpha)}{2E.I}$$

3.6 Thin Walled Non-circular Shafts

Thin walled tubes are sometimes used for torsion. The closed circular tubes may be analysed with the basic torsion equation. However the non-circular tubes require more special treatment. The shear stress may change radially in the wall thickness. The total shear force (per unit length), also called *shear flow*, is the same everywhere along the periphery, and is:

$$q = \int \tau . dt$$

The shear stress is:

$$\tau = \frac{T}{2A.t} = \frac{q}{t}$$

where A is the area enclosed by the mean perimeter (t/2) of the tube.

The torsional deflection of the tube is:

$$\theta = \int_0^L \left(\frac{T}{4G.A^2} \oint \frac{1}{t}.ds \right).dz$$

where s is distance along the periphery from some arbitrary starting point.

For constant M, G and A this becomes:

$$\theta = \frac{L.M}{4.G.A^2} \oint \frac{1}{t}.ds = \frac{1}{2G.A} \oint \tau ds$$

3.7 Plastic Torsion

For elastic torsion the shear strain varies linearly with the radius. So the centre of a round section carries no shear strain, and the outside carries the greatest. For elastic behaviour, the shear stress is proportional to the shear strain, so the shear stress is also zero at the centre and maximum at the outside. If the torque on the part is increased sufficiently, there will come a point when the shear stress on the outside gets as far as it can go: the yield stress in shear. Any further increase in loading cannot increase the stress any higher, but regions increasingly closer to the centre become stressed to the yield point instead. Shear (angular strain) still varies linearly with radius (as for the elastic case), and is:

$$\gamma(r) = \frac{r.\gamma(R)}{R}$$

where r is radius within the shaft and R is outer radius. The shear at the outer radius is in rad/[metre shaft length]. However the shear stress no longer varies linearly (E) with shear. Knowing γ(R) (or guessing it iteratively), γ(r) may be determined. Then the shear stress may be determined at each radius, if the

shape of the stress-strain curve is known. The Ramberg-Osgood representation of the stress-strain curve may be used. In this case shear stress is determined by solving the following relationship numerically:

$$\gamma = \frac{\tau}{G}[1 + \frac{3}{7}(\frac{\tau}{\tau_{yield}})^{n-1}]$$

where

n Ramberg-Osgood Shape factor, see tables

and the torsional yield strength (by von Mises) is:

$$t_{yield} = \frac{\sigma_{yield}}{\sqrt{3}}$$

The torque transmitted is found by integrating along the radius:

$$T = \int 2.\pi.r^2\tau(r).dr$$

If necessary this torque must be compared to the imposed torque, and the calculation repeated with a modified value of surface shear.

3.8 Elastic Buckling of Thin Walled Cylinders Under Torsion

Thin walled tubes may exhibit a mode of failure which is torsional buckling. The critical shear stress at which failure depends on the length of the cylinder. In the equations below:

R cylinder radius
L cylinder length
t cylinder thickness
v Poisson's ratio

Incipient buckling may occur at stresses 0,84 of the predicted critical stress. For non-circular cylinders a conservative stress is obtained by letting R be the largest radius of curvature in the section.

3.8.1 Long Cylinders

These are cylinders where $L/R > 3(R/t)^{1/2}$. The critical stress is:

$$\tau_{cr} = 0,272.\frac{E}{(1 - \nu^2)^{3/4}}(\frac{t}{R})^{3/2}$$

3.8.2 Moderate Length Cylinders

The critical stress is:

$$\tau_{cr} = \pi^2 k . \frac{E}{12(1 - \nu^2)}\left(\frac{t}{L}\right)^2$$

where the buckling coefficient is:

$$k = 0{,}85 . Z^{3/4}$$

and the curvature parameter is:

$$Z = \frac{L^2}{R.t}(1 - \nu^2)^{1/2}$$

3.8.3 Short Length Cylinders

These are cylinders for which Z < 50. The critical stress is determined from the equation above for moderate length cylinder, except that the buckling coefficient is:

k = 5,35 for simply supported edges
k = 8,98 for clamped edges

3.9 Torsion of Rectangular Cross Sections

Rectangular sections are sometimes used under torsion, for example as a type of torsion spring. For longer side length a and shorter side b, the maximum stress is:

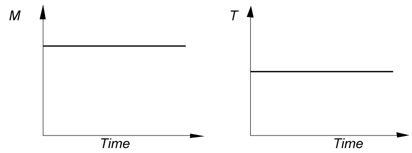
$$\tau_{max} = \frac{T}{\alpha . a . b^3}$$

The angle of twist per unit length is:

$$\theta = \frac{T}{\beta . a . b^3 . G}$$

The constants are:

a/b	α	β
1	0,208	0,141
1,2	0,219	0,166
1,5	0,231	0,196
2	0,246	0,229
2,5	0,258	0,249
3	0,267	0,263
4	0,282	0,281
5	0,291	0,291
10	0,312	0,312
∞	0,333	0,333



Static loading

Alternatively

$$\beta = \frac{1}{16}\left[\frac{16}{3} - 3{,}36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$$

3.10 Torsion of Open Sections Made Up of Narrow Rectangles

There are applications where a number of rectangular sections are bundled together and subject to torsion. An example is the torsion spring in the suspension of the VW Beetle. The rectangles have longer side length a_i and shorter side b_i. The maximum stress occurs at the centre of the long side (a) of the widest (b) rectangle, and is:

$$\tau_{max} = \frac{3T.b_{max}}{\sum_i a_i.b_i^3}$$

The angle of twist per unit length is the same for each rectangle, and is:

$$\theta = \frac{3T}{G.\sum_i a_i.b_i^3}$$

3.11 Combined Bending and Twisting of Shafts

When bending and torque are applied to a shaft (or any structure), then it is not permissible to apply just the torque equation, or just the bending equation. Neither can the shear and bending stresses just be added together. Instead it is necessary to combine the two stresses using the basic principles of structural mechanics. Depending on the criteria of failure used, there arise several different equations for this purpose. They will not all give the same answer, and the designer must chose carefully among the options. The chapter on shafts describes these in detail.

The most general loading of all for a shaft is with bending moment, torque (twisting moment), and axial load, each with an alternating (a) and mean (m) component.

$$d = \sqrt[3]{\frac{32.n}{\pi R_f}\sqrt{(M_a + P_a\,.\frac{d}{2})^2 + 3\frac{T_a^2}{4}} + \frac{32.n}{\pi R_m}\sqrt{(M_m + P_m\,.\frac{d}{2})^2 + 3\frac{T_m^2}{4}}}$$

where

d	**shaft diameter**
n	**reserve factor**
M	**bending moment [Nm]**
P	**axial force [N]**
T	**torque [Nm]**
m	**subscript refers to mean value: (max value + min value)/2**
a	**subscript refers to alternating value: (max value - min value)/2**
R_m	**maximum material stress (subscript s for shear)**
R_f	**fully corrected fatigue strength**

4 BENDING OF BEAMS

4.1 Introduction

Beams are solid elements that are supported at one or more places, and subject to forces. Typical examples of beams are bridges, roof trusses, and diving board. Shafts are another type of beam, although they are subjected to torsion as well.

Beams are subject to bending, which creates tensile stress on the outer radius of curvature, and compressive stress on the inner radius. These stresses may cause failure, and thus are of concern to the engineer. In addition, beams may be considered to fail when a given deflection is exceeded. Thus we are interested in the stress and deflection of beams. The stress and deflection depend on the loading, beam geometry, beam orientation, and beam material.

A frequently encountered problem is the need to design the section and material of a beam, where the span and applied loads are known. The design task is to determine the maximum stress or deflection in the beam, and to compare this with the material properties etc. To do this it is necessary to determine the bending moment at various parts of the beam. The method of determining this is given below.

4.2 Bending Moments

The first problem in most beam design is to determine the bending moment. Bending moment has units of force x length, eg N.m. These are the same as for torque, and in fact the two terms are

4.2.1 Bending Moment diagrams

The method used to determine bending moment for a beam is as follows:

1 Convert actual physical structure to a structural model of point and distributed loads. The supports of a beam may be modelled as one of the following: simply supported (rotation at the support is possible) or built in (no rotation possible at support, also called encastre).

2 Draw approximate deflected shape

3 Calculate reactions by applying Newton's Laws:
(1) $\Sigma T=0$ about any point. It is usually best to use one of the supports (eg that on the far left) as the origin. Decide whether to make clockwise

or anticlockwise moments positive, and stick to the choice. Reactions (unknown at this point) must be included as variables.

(2) $\Sigma F=0$ in the vertical direction, taking eg downwards as positive. Reactions (unknown at this point) must be included as variables.

Solve these two equations for the reactions.

4 Calculate shear force at various points along the beam, at least at the point loads. Draw shear force diagram (SFD). Note that shear force at any point along the beam is the sum of all the FORCES (including reactions) to the left of the point. General convention is that positive shear force is when the right part of the beam tends to move downwards compared to the left, although this convention is arbitrary. Between point loads, the shear force will be seen to remain constant. Along uniformly distributed loads the shear force will change linearly.

5 Calculate bending moment at various points along the beam, at least at the point loads. Draw bending moment diagram (BMD). Note that bending moment at any point along the beam is the sum of all the BENDING MOMENTS (FORCE x moment arm) to the left of that point (or the right). Include the reactions. For uniformly distributed loads, temporarily assume the total load acting at the mass centre, and use the moment arm of the mass centre. For non-uniform distributed loads it will be necessary to integrate the product of force and moment arm. This is equivalent to the area under the SFD to the left. The bending moment will always be zero at the free end of a simply supported beam. Built in beams will have end fixing moments, described below. General convention is that positive bending moment is that which causes the beam to sag in the middle (i.e. clockwise moment to the left of the point of interest), although this convention is arbitrary. Between point loads, the bending moment will change linearly. Along uniformly distributed loads the bending moment will change parabolically.

The general bending moment equation for beams is:

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = -w$$

where

M Moment,
F Shear force,
w beam loading, [N/m]
x axial distance,

The equation is not very useful in this form, but it gives rise to the following characteristics.

*** local maximum or minimum bending moment (M) occurs at the point where the shear force (F) changes sign**

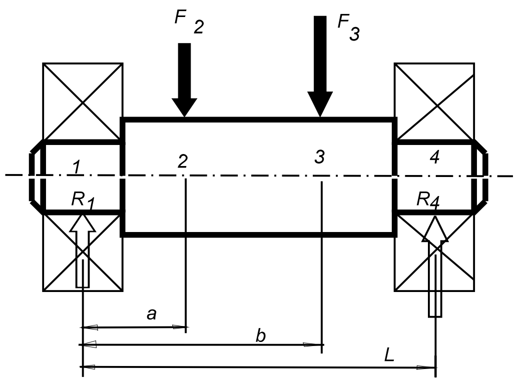
* **bending moment at any point x is equivalent to the area under the shear force diagram to the left.**

4.2.2 Common Bending cases

Common bending moment cases are shown below. These are simply supported cases, and in the notation used below reactions are R and applied forces F.

Central loads

Supports (eg shaft bearings) at each end, and two central loads. With applied forces (➧) in the directions shown, the magnitudes of the reaction forces (⇨) are:

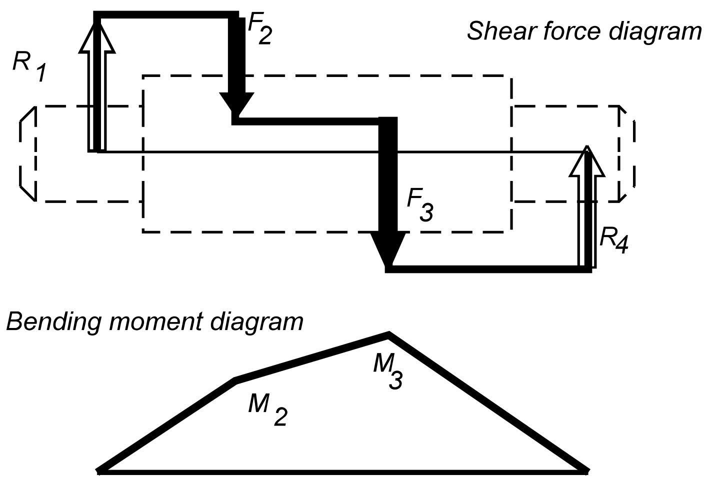


$$R_4 = \frac{(a F_2 + b F_3)}{L}$$

$$R_1 = F_2 + F_3 - R_4$$

where
- **R_1 support reaction**
- **R_4 support reaction**
- **F_2 force**
- **F_3 force**

The shear force and bending moment diagrams are shown in the diagram. The bending moments are
M_2 = a. R_1
M_3 = (L-b).R_4



Central loads are typically found in gearboxes, each central load corresponding to a gear, and the reactions are carried by the bearings.

Note 1 **The directions of the reaction forces will be in the direction shown in the diagram. If a negative reaction force is obtained, then this indicates that the direction is opposite to that shown, i.e. downwards.**

Note 2 **If one or more of the applied forces is in a direction opposite to that shown, then use the negative value of the force in the equations.**

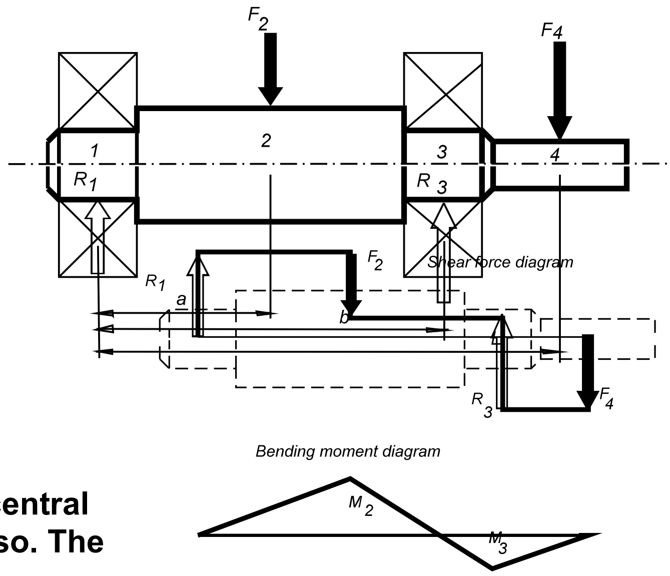
These notes apply also to the other bending cases shown below.

For a *single* point load F2 acting at a, the maximum lateral deflection of the shaft is given by

$$\delta_{max} = \frac{F_2 a(L^2 - a^2).\sqrt{3(L^2 - a^2)}}{27E.I.L}$$

Note that this is only for a single point load.

Overhung load and central load
Overhung load at one end, and load between supports (eg shaft bearings). With applied forces (➧) in the directions shown, the magnitudes of the reaction forces (⇨) are:



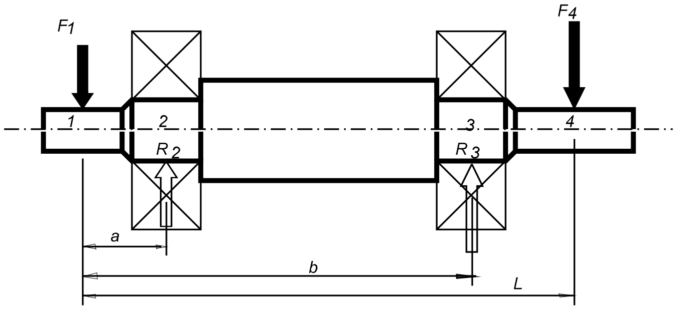
$$R_3 = \frac{aF_2 + LF_4}{b}$$

$$R_1 = F_2 + F_4 - R_3$$

where the notation, is as for the central load case. Notes 1 and 2 apply also. The shear force and bending moment diagrams are shown in the diagram.

The loading is typical of gearboxes with protruding shaft, and electric motors.

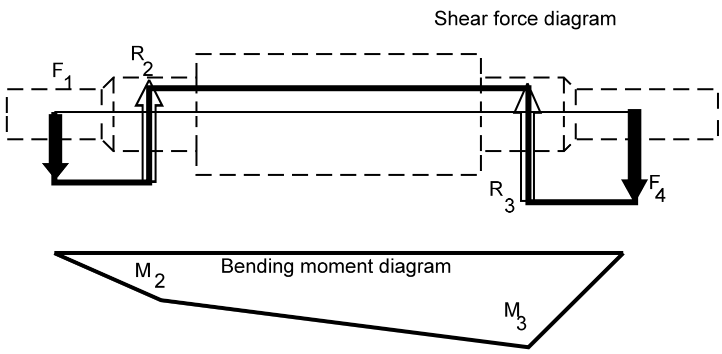
Overhung loads
Loads at each end, and two supports (eg shaft bearings) in between. With applied forces (➧) in the directions shown, the magnitudes of the reaction forces (⇨) are:



$$R_3 = \frac{(L-a)F_4 - a\,F_1}{(b-a)}$$

$R_2 = F_4 + F_1 - R_3$

where the notation, is as for the central load case. Notes 1 and 2 apply also. The shear force and bending moment diagrams are shown in the diagram.



The loading is typical of intermediate and lay shafts. Some electric motors also have protruding shafts at both ends like this.

Bending Moments

For each of the shaft cases described above, the bending moment is zero at the free ends of the beam (whether there is a support there or not). The bending moments at 2 and 3 are given by
$M_2 = a.R_1$ **(or = a.F_1 as the case may be)**
$M_3 = (L - b).R_4$ **(or = (L - b).F_4 as the case may be)**

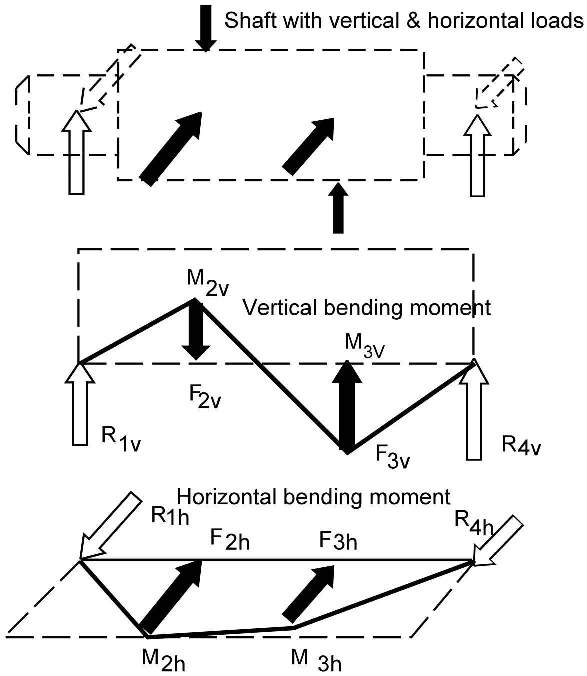
In the case of beams or shafts with end couplings, there may exist bending moments at the coupling. The value will depend on the type of coupling (some couplings transmit bending moment, while other do not), and on the load on the other side of the coupling.

4.2.3 Horizontal and vertical loads

If loads exist in horizontal and vertical planes, then it is necessary to determine the horizontal and vertical components of the load, and then the horizontal and vertical reactions.

Thereafter the horizontal and vertical bending moments may be determined. The resultant moment at any one beam section x is given by

$$M_x = (M_x^{2} + M_y^{2})^{0,5}$$



4.3 Internal Bending Moments

Bending creates tensile strain (and strain)on the outer radius of curvature, and compressive strain on the inner radius. The strain is greatest at the surfaces, and reduces linearly towards the geometric (area) centre of the beam, regardless of whether the material is elastic or in the plastic condition. Strain can be converted to stress, at which step it will be necessary to account for any plastic behaviour. For most engineering applications the materials are not loaded into the plastic region, and thus stress will be proportional to strain ($E=\sigma/\epsilon$).

Thus the outer radius of curvature is under tensile stress, and the inner radius under compression. The total tensile force may be determined by integrating

the product of local stress and elemental area. The total tensile stress must balance the total compressive stress, to permit the evident non-acceleration of the beam. This may be shown to require that the changeover from tensile to compressive stress must occur at a plane passing through the centre of the cross sectional area. This stress free surface in the beam is called the neutral surface. The neutral axis is the longitudinal line through the centroid of the section. The centroid is at:

$$\bar{y} = \frac{\int y.dA}{A}$$

where
y distance from a reference point
A (elemental) cross sectional area of beam

The internal moment is the sum of the internal forces:

$$M = \int \sigma_y . y.dA$$

and this resolves to give the bending stress:

$$\sigma = \frac{M.y}{I} = \frac{M}{Z} = \frac{E.y}{R}$$

where
σ stress
y distance to outside of beam, from neutral axis,
I second moment of area of beam section,
Z section modulus, Z = I/y
R radius of curvature

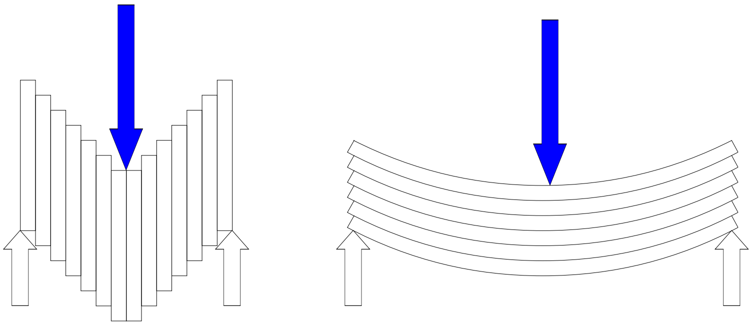
The second moment of area (about the neutral axis) is:

$$I = \int y^2 dA$$

where the integral is in the depth of the beam.

4.4 Internal Shear

Shear occurs along the axis of the beam and perpendicular to it. This can be noticed in the two orientations of the loose composite beams shown below.



Vertical shear ***Horizontal shear***
Note: shear is greatest at the ends.

Shear stress cannot cross a free boundary.

4.4.1 Simple Beam Sections

Thus a beam has to resist both vertical and horizontal shear forces. The shear stress at some distance y_1 from the neutral axis is given by:

$$\tau = \frac{F}{I.b_1}\int_{y_1}^{d_1} b.y.dy = \frac{F}{I.b_1}A.\bar{y}$$

where

A **area of the section lying between y_1 and the outer fibre**
b_1 **the width of the section where the shear stress is being evaluated**
y **distance (depth direction) from NA**
d_1 **distance from NA to outer fibre**
$\bar{y}$ **the distance of the centroid of this area from the neutral axis**
I **second moment of area of whole section about NA**
F **transverse shear force (from weight loads etc)**

For a rectangular section of width b and depth d, the shear stress is:

$$\tau = \frac{6F}{b.d^3}\left(\frac{d^2}{4} - y_1^2\right)$$

The maximum stress occurs at the NA and is:

$$\tau_{max} = \frac{3}{2}\frac{F}{b.d}$$

For a solid circular section:

$$\tau = \frac{4}{3}\frac{F}{\pi r^4}(r^2 - h^2)$$

where

h distance (along y, the depth direction) from the neutral axis (h varies from 0 to r)

r radius of the section

The maximum is at the NA (h=0):

$$\tau_{max} = \frac{4}{3}\frac{F}{\pi.r^2}$$

For a thin circular tube the maximum shear stress is:

$$\tau_{max} = \frac{F}{\pi R.t}$$

4.4.2 Shear Centre

The resultant of all the shear forces acting over the cross section passes through the shear centre. Only if the external force is applied through this same point will there be no twisting of the beam.

The principle is:

(1) Determine the shear stresses over the area. For open sections apply:

$$\tau = \frac{F}{I.b_1}A.\bar{y}$$

Alternatively use:

$$\tau = \tau_o - \frac{F}{I_{xx}} \int_{0_{along\ perimeter}}^{s} y.ds$$

where τ_o is zero at the top and bottom of symmetrical sections and is otherwise found from:

$$\int \tau . \frac{\partial \tau}{\partial \tau_o} ds = 0$$

(2) Determine the combined moment of all the shear forces

$$shear\ moment = \int_{around\ section} L.\tau.dA$$

where L is the moment arm of each shear stress about the centroid of the section.

(3) Balance the shear moment with the external force

$$F_{applied}.h = shear\ moment\ as\ above$$

where h is the lateral distance of the shear centre form the centroid.

4.5 Composite and Flitched Beams

For beams made of different materials that are symmetrically arranged around the centroid, and adequately bonded, then
the stress at distance y_2 from the neutral axis of material 2, is:

$$\sigma_2 = \frac{M.y_2}{I_2 + \frac{E_1}{E_2}.I_1}$$

where M is the total moment carried by the beam.

For a built up beam to behave as a compound beam rather than as a number of separate beams (each with its own neutral axis), it is necessary for the joint to be sufficiently strong to resist the shearing forces (complementary shear). (No bonding is only acceptable if there is pure bending.) Use the complementary shear equation to determine shear stress. Otherwise, to determine shear force over some distance x (say the pitch of some rivets), use the following:

$$F = \tau . b_1 . x = \frac{M_2 - M_1}{I} . A . \bar{y}$$

where

M subscript 1 and 2 are moments at the beginning and end of length x, (chose x wisely!)

b_1 width of the section where the shear stress is being evaluated

ў distance of the centroid of this area

4.6 Reinforced Concrete Beams

4.6.1 Beams

Concrete is weak in tension, so reinforcing rods are placed on the tension side. Assume that steel takes all the force on the tension side. The compressive force in the concrete equals the tensile force in the steel, as follows:

$$(Force\ in\ concrete)\ \sigma_c . B . \frac{h}{2}$$
$$= (Force\ in\ steel)\ \sigma_s . a$$

where

B beam width

h distance from neutral axis to outermost fibre of concrete

a cross sectional area of steel

The above pair of equations is used with the following one to solve for the two stresses and the depth of the neutral axis, h:

$$\frac{\sigma_c}{\sigma_s} = \frac{E_c}{E_s} . \frac{h}{(d - h)}$$

An optimum design is such that maximum allowable stresses in the concrete and the steel are reached together. Allow a minimum of 25 mm cover over the steel.

Approximate values for concrete E = 17 GPa, R_d = 7 MPa, ρ = 2400 kg/m^3

4.6.2 Slabs

If the slab is supported on all four sides then design for the shorter span. Place additional bars ("distribution steel") at right angles, about 20% of the main reinforcement. Minimum cover of 12 mm for slabs.

4.6.3 Bonding of Reinforcing Rods

Safe adhesive stress between concrete and steel is 0,7 MPa. Extend bar (diameter d) by length:

$$L = \frac{\sigma_s}{\tau_{adh}} \cdot \frac{d_s}{4}$$

Also provide a semi-circular hook at the end.

4.6.4 Shear of Reinforced Concrete

Need not be considered in slabs. May be important in beams (provide stirrups or bent-up bars).

4.7 Leaf Springs

Leaf springs are laminated plates, all initially bent into semicircular arcs with the same radius of curvature (touch at ends only). The ends have triangular taper. As load increases the leaves continue to be bent into semi-circular arcs of same curvature, and contact still occurs only at ends of leaves. The maximum fibre stress remains constant from end to end, and is:

$$\sigma = \frac{3W.L}{2n.b.t^2}$$

where

a	**length over which the end taper occurs**
b	**width of the leaves**
t	**thickness of one leaf**
n	**number of leaves**
W	**centrally applied load**
L	**end to end length of longest spring initially**

The central deflection is:

$$\delta = \frac{3W.L^3}{8E.n.b.t^3}$$

The proof load (load required to straighten the leaves) depends on the initial central deflection δ_o:

$$W_o = \frac{8E.n.b.t^3.\delta_o}{3L^3}$$

4.8 Deflection of Beams

4.8.1 Beam Equation

For central point loading or uniform loading, a double integration of the equation below gives the deflection at any point:

$$\frac{d^2y}{dx^2} = -\frac{M}{E.I}$$

where
y deflection
x distance along beam
M bending moment
E modulus of elasticity
I second moment of area

4.8.2 Mcaulay's Method

For other beams use Mcaulay's method. Write the moment as a function of distance x along the beam, where x is measured from one end of the beam. Terms that must not be included until x gets to them are put in [] brackets. If necessary Distributed loads should be extended to the end of the beam, and the extension compensated by a negative load. Use:

$$\frac{d^2y}{dx^2} = -\frac{M_x}{EI}$$

Integrate this expression twice, to get displacement y, introducing constants of integration along the way. Terms in [] brackets must be treated as single terms in the integration (no integration inside [] brackets). Solve the two constants of integration with known values of x and y. Maximum deflection occurs where dy/dx = 0, calculate x at this position (ignoring [] bracket terms that you haven't come to). From this calculate displacement at this position. The method may be used to determine reaction forces and moments in built-in beams.

4.8.3 Moment-area Method

Another method is Moment-Area. The change in slope between any two points on the deformed beam equals the area of the BM diagram lying between the two points:

$$\Delta\theta = \int_P^Q \frac{M}{E.I}.dx$$

The intercept on the y axis, between the tangents drawn from points P and Q on the deflection curve, is the moment of the bending moment diagram.

$$y = \int_{point\ P}^{point\ Q} \frac{M.x}{E.I} dx$$

In use: for example place the y axis at the point where the deflection is required. Account must be taken of + and - BM areas. Can be applied to built in beams.

4.8.4 Finite Element Analysis

This is a method in which the beam in divided into small regions, and the deflections (and stresses) determined in each. A computer is used for the calculations. Complicated shapes are accommodated. Accuracy is usually adequate (providing the model is correct).

4.8.5 Beam Tables

Tables are available with deflections at critical points. However the tables are usually limited to commonly encountered standard cases, and cannot reliably be applied to applications that deviate from the the case shown. Tables for shafts are apparently unavailable.

4.9 Statically Indeterminate Beams

The principle of superposition: remove one support and determine the deflection at that point. Then remove the load and determine the deflection at the support caused by the support reaction. The deflection at the support is zero however, so add these to deflection, equate to zero and solve for the support reaction and moment as required.

4.10 Principal Second Moments of Area

The second moment of area of a section about an axis is generally:

$$I_x = \int R^2.dA$$

Knowing the second moments of area about two perpendicular axes, the principal maximum second moment of area is:

$$I_{u,v} = \frac{1}{2}[\ (I_x+I_y)\ \pm\ \sqrt{(I_x-I_y)^2+4I_{xy}}\ ]$$

at an angle of:

$$\tan(2.\theta) = \frac{-2I_{xy}}{I_x-I_y}$$

where the area product of area is:

$$I_{xy} = \int\int x.y.dx.dy$$

4.11 Unsymmetrical Bending of Beams

Given a plane of loading at some angle φ from the I_X principal axis of second moment of area, then first Find the position of the neutral axis, α from the I_X axis:

$$\tan(\alpha) = -\frac{I_X}{I_Y}.\frac{1}{\tan\ \phi}$$

The most highly stressed (bending) point is that furthest from the NA. The stress is:

$$\sigma = \pm M\ [\frac{\cos\phi}{I_Y}.X + \frac{\sin\phi}{I_X}.Y]$$

where X and Y are coordinates relative to the principal axes.

4.12 Encastre Beams

These are beams that are built in at both ends. The method is:

1) **Assume the beam to be simply supported and draw the SF and BM diagrams.**

2) **Consider the fixing moments at ends A and B alone: (use symbols as they will not be known yet). The fixing reactions will be opposite and equal in magnitude, given by:**

$$R_F = \frac{M_B - M_A}{L}$$

Draw SF and BM diagrams using these variables.

3) **Determine moments A and B (based on the Moment-Area theorems) as follows:**
The area of the two BM diagrams are equated:
$A_{\text{BM simple support}} = A_{\text{BM fixing reactions}}$

Also the centroids of these areas are equated:
$x_{\text{BM simple support}} = x_{\text{BM fixing reactions}}$

These equations allow moments A and B to be solved.

4) **Superpose the two SF and the two BM diagrams to obtain the resultant SF and BM diagrams.**

4.13 Continuous Beams

These are beams on more than two supports. The methods of solving them are given below.

4.13.1 Clapeyron's Theorem

(Theorem of Three Moments). Relates the moments at three successive supports of the beam. When a continuous beam has many spans, the equation is applied to each pair of consecutive spans in turn (1&2, 2&3, ..). As many equations will be obtained as there are unknown support reactions. The general form of the equations is:

$$\frac{1}{L_1}\int_0^{L_1}\int_0^{x} M^{/}.dx.dx$$

$$- \frac{1}{L_2}\int_0^{L_2}\int_0^{x} M^{/}.dx.dx$$

$$- \int_0^{L_1} M^{/}.dx$$

$$= \frac{1}{6}[M_A.L_1 + 2M_B(L_1 + L_2) + M_C.L_2]$$

where

M_A unknown bending moment at support A
M_B unknown bending moment at support B
M' moment at section x from A when the beam is assumed to be simply supported at A and B
M" moment at section x due to the support moments M_A and M_B
L span lengths

For point loads or where the loading is discontinuous between spans:

$$M_A.L_1 + 2M_B(L_1 + L_2) + M_C.L_1$$

$$= -6[\frac{A_1.\overline{x_1}}{L_1} + \frac{A_2.\overline{x_2}}{L_2}]$$

If the second moment of area differs between spans, and if the supports are at different levels, the equation becomes:

$$M_A.\frac{L_1}{I_1} + 2M_B(\frac{L_1}{I_1} + \frac{L_2}{I_2}) + M_C.\frac{L_1}{I_2}$$

$$= -6[\frac{A_1.\overline{x_1}}{L_1.I_1} + \frac{A_2.\overline{x_2}}{L_2.I_2}] + 6E[\frac{\delta_{AB}}{L_1} + \frac{\delta_{BC}}{L_2}]$$

where

δ_{AB} distance support B is below support A
x_1 centroid of BM diagram, measured from A
x_2 centroid of BM diagram, measured from C

4.13.2 Shearing Forces in a Continuous Beam

For each span, sum the shear under simple support, and the support reaction due to the bending moments at each end of the beam. A support in the middle will end up with shear forces from the spans on either side.

4.13.3 Conjugate Beam

An imaginary beam corresponding to to the given real beam. It is a maths model used to solve slope and deflection of the real beam.

1) **work out bending moment diagram**
2) **divide all bending moments by EI**
3) **apply this as loading on another beam, simply supported**

The slope at any point on the real beam is given by the shear at the corresponding point on the conjugate beam.

The deflection at any point on the real beam is given by the bending moment at the corresponding point on the conjugate beam. See references for further details.

4.14 Thick Curved Bars

The distance between centre line (centroid) and neutral axis is:

$$e = R - \frac{A}{\int \frac{1}{R-y}.dA}$$

where
R **radius of centre line**
y **distance of element from centre line**

Alternatively:

$$e = \frac{m}{1+m}.R$$

where

$$m = \frac{1}{A.R}\int (1 + \frac{y}{R} + \frac{y^2}{R^2} + ...).y.dA$$

The bending stress is:

$$\sigma_B = \frac{M.(y-e)}{A.e.(R-y)}$$

4.14.1 Deflection of Thick Curved Bars

For a bar that is encastre at one end, formed into an arc of (midline) radius R over 90 degrees, with a load applied to the end in a direction parallel to the bar at its root. The deflection in the direction of the load is:

$$\delta = \frac{\pi}{4}.\frac{P.R}{A.E}.(\frac{R}{e} - 1 + \frac{2\alpha.E}{G})$$

where α = 3/5 for a bar of uniform rectangular section.

4.14.2 Thick Ring

For a thick ring of midline radius R, with a tension force P, the moment in the ring is:

$$M = \frac{P.R}{2}(\sin\theta + \frac{2e}{\pi R} - \frac{2}{\pi})$$

where θ is angle measured from the line of action of P, and e is determined as for thick curved bars.

4.15 Plastic Bending

In the elastic bending of a beam, the strain varies linearly from zero at the neutral axis, to a maximum at the outermost fibre. The stress is proportional to the strain (by E), and thus also varies linearly. If the bending moment is increased, there comes a time when the outer fibres are at yield. Any further increase in bending moment does not cause an increase in stress, since the fibres are already at yield. Instead, the fibres next inside start to yield. As the load increases, so more of the cross section goes into yield.

Therefore, while the strain distribution across the section remains LINEAR, the stresses do not vary linearly.

4.15.1 Calculation of Plastic moment

The strains at the top and bottom surfaces of the beam are ϵ_1 and ϵ_2. The procedure is:

(1) assume a value of ϵ_1

(2) knowing $\sigma(\epsilon)$, find

$$\int_0^{\epsilon_1} \sigma(\epsilon) d\epsilon = \int_{-\epsilon_2}^{0} \sigma(\epsilon) d\epsilon$$

(3) calculate moment:

$$M = \frac{b.d^2}{(\epsilon_1 + \epsilon_2)^2} . \int_{-\epsilon_2}^{\epsilon_1} \epsilon.\sigma(\epsilon).d\epsilon$$

(4) if this equals the applied moment then ϵ_1 was chosen correctly, otherwise repeat

(5) if necessary find

$$R = \frac{d}{\epsilon_1 + \epsilon_2}$$

(6) repeat for other points along the beam

4.15.2 Elastic-perfectly Plastic Materials

For a beam of rectangular section, of material that is elastic-perfectly plastic, the Fully Plastic bending moment (when the whole beam section is plastic) is:

$$M_p = \frac{1}{4} b.d^2.\sigma_{yield}$$

At such a point a frictional plastic hinge forms and the structure fails. When the fully plastic moment is reached, the stress across the section is uniformly

equal to the yield strength. The neutral axis is positioned to divide the section into two EQUAL areas.

At a lower stress, only the outer fibres go plastic:

$$M_y = \frac{1}{6} b.d^2.\sigma_{yield}$$

but the fully plastic moment is generally the more useful one.

4.15.3 Limit Design

The accuracy of elastic beam analysis depends heavily on the correctness of the end fixation conditions. Furthermore, the analysis is often complicated if there are several supports (statically indeterminate). Limit (or plastic) design is an alternative approach, which is simpler to implement and less sensitive to fixation conditions. The method assumes that the structure cannot fail until the entire cross section has gone plastic at each of several critical places. When loading increases to the fully plastic moment, a number of friction hinges develop (one more than the degree of redundancy) and the structure collapses as a mechanism. Design uses an increase in load of 1,75 before failure.

Plastic modulus

Limit design does not use the second moment of area I, or the section modulus (elastic) Z = I/y, since the stresses are not linear. Instead, Z_p the plastic modulus of the section is used. This is given by the sum of the first moments of area about the equal-area axis. Values of Z_p for various sections may be found from tables, eg YOUNG WC, 1989, Roark's Formulas for stress and strain, McGraw-Hill. **For example**

Rectangle $Z_p = 0,25\ bd^2$

Hollow circle $Z_p = 1,333\ (R_2^3 - R_1^3)$

Shape factor

The second moment of area and section modulus (elastic) is often widely known, and the plastic modulus can be determined by applying a factor called the shape factor SF to it:

$$SF = \frac{M_p}{M_y} = \frac{Z_p}{Z}$$

For a rectangular beam the factor is 1,5, for a solid circular beam 1,7, and for an I beam about 1,15.

Fully plastic moment
The fully plastic moment is given by

$$M_p = Z_p .R_m = SF.Z.R_m$$

where R_m is the yield strength of the material.

Application to design
To use the method to determine the maximum load that a structure can take before collapse,

- ☐ **Determine Z_p for the section concerned, eg using tables for various sections. Otherwise use the second moment of area multiplied by the shape factor.**
- ☐ **Determine R_m the yield strength of the material**
- ☐ **Determine the fully plastic moment $M_p = Z_p R_m$**
- ☐ **Find by inspection (or determine) the positions where the maximum bending moments are going to occur (one more than the degree of redundancy). Tables may be used, but these are not the normal tables for elastic bending moment cases.**
- ☐ **Put the moment at these places equal to M_p**
- ☐ **The problem will now have been simplified to several separate statically determinate cases, which can be readily solved in terms of M_p and the applied load. The Beam tables for plastic collapse will often provide this information.**
- ☐ **Ensure that an adequate safety factor exists.**

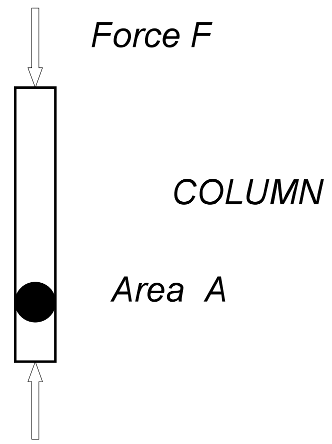
Cautions
The method is not compatible with the pre-solved equations for the statically indeterminate case, since these other methods do not take plasticity into account. For the same reason, the method does not permit the superposition of results, and all loads acting on the structure must be considered at once. Furthermore the method does not consider the effects of low cycle fatigue.

5 BUCKLING OF COLUMNS

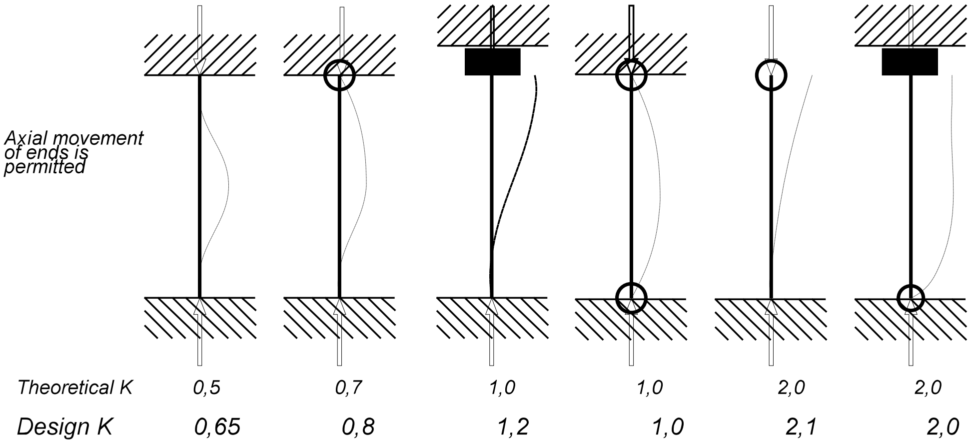
Columns are members that are loaded with an axial compressive force. The nominal stress is

$$\sigma = \frac{F}{A}$$

where
F **compressive force**
A **cross sectional area**



However the column may not fail by direct compression, but rather by bending sideways to the load. This is called buckling, and various formulae relate the critical axial stress σ_c at which buckling occurs, to the geometry of the column.



5.1 Long Columns

These are columns for which $\sigma_c \leq R_e/2$. Euler's theory of elastic buckling applies. The critical stress is:

$$\sigma_c = \frac{\pi^2.E}{(L'/R_g)^2}$$

where

R_e Yield strength,

E Modulus of elasticity, E eg 206 x 10^9 Pa

L' Effective length L'= L.K

- **L Actual length of column**
- **K Effective length factor**
  - **K = 1/(number of half sine waves in buckled mode)**
  - **= or see table**

R_g radius of gyration,

$$R_g = \sqrt{\frac{I}{A}}$$

where

I second moment of area of column section

Note:

I = $\pi d^4/64$ for a solid round section of diameter d

$I = \pi(D^4 - d^4)/64$ for a hollow round section of outer diameter D and inner diameter d

$I = b\,d^3/12$ for a rectangular section of breadth b and depth d

A cross sectional area of column

The Euler equation may also be expressed in terms of the critical force F_c (also referred to as F_e):

$$F_c = \frac{\pi^2 E.I}{L^2}$$

A safety factor s_f may be included, eg 23/12 (AISC), in which case the equation for stress becomes

$$\sigma_{allow} = \frac{\sigma_c}{s_f}$$

The factors that make a column "long" are:
- **large length L**
- **large K (that is less constraint)**
- **small cross section (eg diameter)**
- **low modulus of elasticity E**

For the particular case of a solid round section of diameter d, subject to compressive load F:

$$d = \left[\frac{64 s_f.F.L^2.K^2}{\pi^3 E}\right]^{\frac{1}{4}}$$

although it will be necessary to determine σ_c to ensure that the column is long. If σ_c shows that the column is intermediate, then an appropriate intermediate column equation will need to be used to determine the diameter.

Column design should account for the possibility of the column buckling in both lateral directions. In some cases (eg Pin jointed columns) there may be a different effective length factor (K) for each lateral direction. In these cases it is possible to change the depth and width of the column so that it resists buckling equally in both directions.

5.2 Intermediate Columns

These are columns for which $\sigma_c > R_e/2$, where σ_c is determined from the Euler equation for long columns. There are different ways of solving the problem, and the better methods are given first.

5.2.1 Tangent Modulus Theory

The Tangent Modulus Theory may be used, with the Ramberg-Osgood form for representing the stress-strain curve. Plastic and elastic buckling are accommodated.

The critical stress is solved from:

$$\sigma_c[1 + 3\frac{n}{7}(\frac{\sigma_c}{\sigma_{0,7}})^{n-1}] = \frac{\pi^2.E}{(L^{/}/R_g)^2}$$

where

n — **Ramberg-Osgood Shape factor n, see tables**

$\sigma_{0,7}$ — **Ramberg-Osgood strength $\sigma_{0,7}$, see tables, or approx equal $\sigma_{0,2\%}$ (yield) strength**

and other parameters are as defined for long columns.

Material	$\sigma_{0.7}$ [MPa]	n
ALUMINIUM		
2014 T6	410	20
2024 T4	330	10
6061 T6	280	30
7075 T6	500	20
STEEL 4130		
Normalised	510	20
Full hard	1140	50
STAINLESS STEEL		
18Cr 8 Ni		
1/4 hard	450	5
Full hard	1260	9
Table: Ramberg-Osgood factors. From EISENBERG "Introduction to the mechanics of solids.		

5.2.2 AISC Empirical Formula

This is a design code from the American Institute of Steel Constructors, which is valid under the following conditions:

- **steels**
- **yield $R_e \leq 690$ MPa**
- **$L'/r \leq 200$ (L' and r are defined above)**

The allowable stress is:

$$\sigma_{allow} = \left(1 - \frac{(L'/R_g)^2}{4\pi^2 E/R_m}\right).\frac{R_e}{s_f}$$

where safety factor is:

$$s_f = \frac{5}{3} + \frac{3}{8}\frac{(L'/R_g)}{(2\pi^2 E/R_e)^{0,5}} - \frac{1}{8}\frac{(L'/R_g)^3}{(2\pi^2 E/R_e)^{1,5}}$$

The safety factor accounts for load and fabrication eccentricities.

For the particular case of a solid round section of diameter d, subject to compressive load F, the AISC equation becomes:

$$d = \left[\frac{s_f}{R_e}\frac{4}{\pi}.F + \frac{4}{\pi^2}.L^2.K^2\frac{R_m}{E}\right]^{\frac{1}{2}}$$

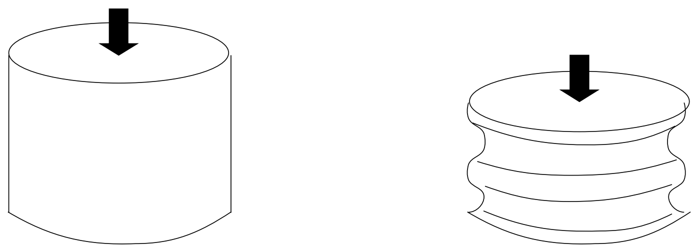
5.2.3 Rankine Formula

The allowable stress in the column is

$$\sigma_{allow} = \frac{R_e}{1 + \frac{R_e}{\pi^2 E}.\left[\frac{L^{/}}{R_g}\right]^2}.\frac{1}{s_f}$$

5.3 Short Columns

Short columns fail by localised buckling (crippling), or direct compression (crushing or shear).



Cylinders (eg soft drink cans) often fail in this mode. The mechanics are complex and are not discussed here.

5.4 Struts with Eccentric Load

The maximum stress is

$$\sigma_{max} = \frac{F}{A}\left(1 + \frac{e.d}{k^2.\sin\theta}\right)$$

where

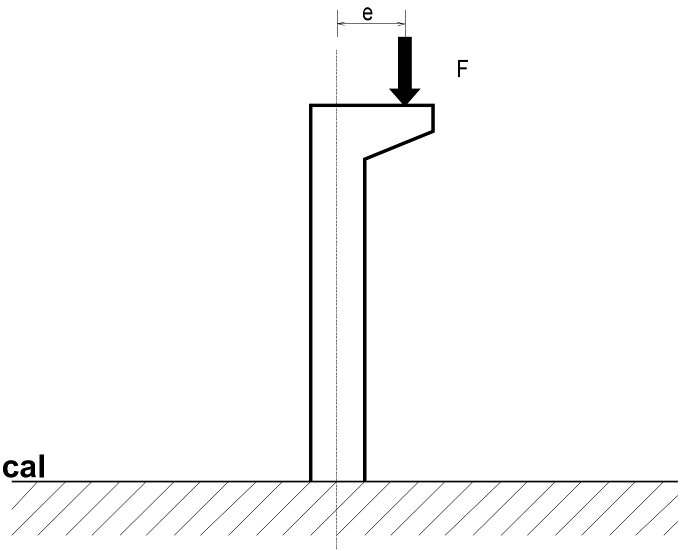
F applied compressive force
A cross sectional area of strut
e eccentricity of load
k radius of gyration, where

$$k = \sqrt{\frac{I}{A}}$$

θ **Euler angle, where**

$$\theta = \frac{L}{2}\sqrt{\frac{P}{E.I}}$$

The above equations require numerical methods to solve.

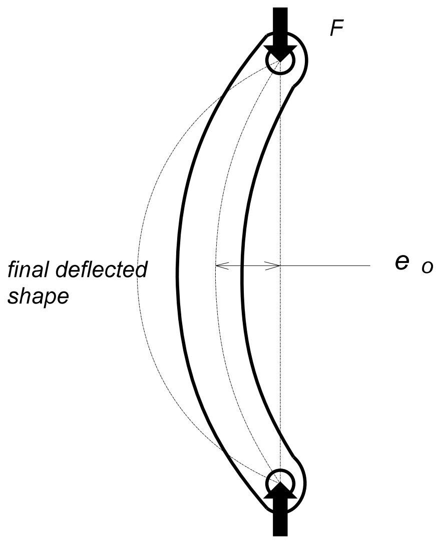


5.5 Struts with Initial Curvature

The following equation assumes that the initial curvature is the shape of a sine curve, with a central (lateral) offset of e_o. The maximum stress is

$$\sigma_{max} = \frac{F}{A}\left(1 + \frac{d}{k^2}.e_o.\frac{F_c}{(F_c - F)}\right)$$

where F_c is the Euler critical load.

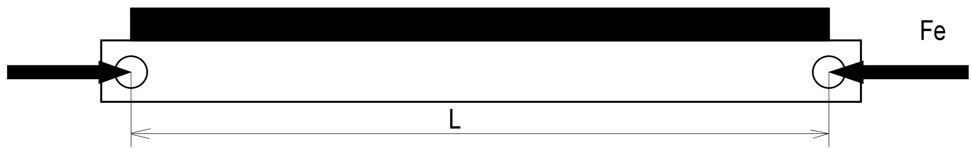


5.6 Struts with Transverse Loading (Beam-struts)

For pin jointed ends, and transverse loading of w N/m (which may be a linear function of length), the maximum bending moment is:

$$M_{max} = \frac{1{,}02F_c}{(F_c - F)} \cdot \frac{w.L^3}{8}$$

where F_c is the Euler critical load.



6 CYLINDERS AND STRUCTURES WITH AXIAL SYMMETRY

This section covers thin and thick walled rings, cylinders, and pressure vessels. There is an important structural difference between thin and thick cylinders, and this needs to be noted carefully while reading this section. The thick cylinders equations are more complex.

6.1 Thin Curved Bars and Rings

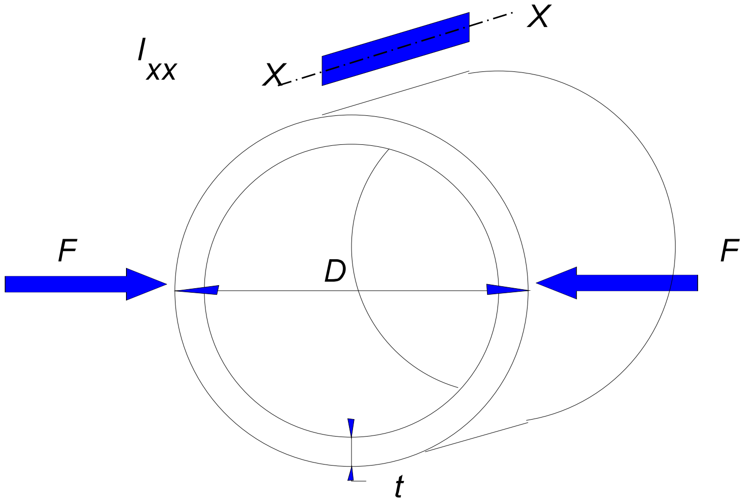
Thin vs Thick

The following equations apply to thin rings, that is rings where the thickness is very much less than the diameter. Quite where the boundary is between "thick" and "thin" is left to the designer's judgement, and if it becomes a disputable matter then you should probably rather be using thick ring/cylinder equations.

6.1.1 Thin Ring

This case is for a thin (thickness much less than diameter) ring (short in the axial direction), under point loading. For thin rings, the bending stress under tension/compression is:

$$\sigma = \frac{F.D.t}{4\pi.I}$$



Thin ring under point loading

where

D outside diamter of ring,

t depth of ring section, where t << D

F force
I second moment of area of ring section, about an axis parallel to the axis of the cylinder

The increase in the diameter (δ) in the line of action of the force is:

$$\delta = \left(\frac{\pi}{4} - \frac{2}{\pi}\right)\frac{F.D^3}{8E.I}$$

where
E modulus of elasticity

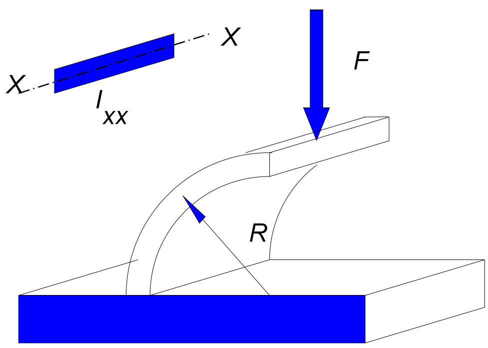
This structure is sometimes used as a load cell. Either strain gauges are placed on the ring, or a clock gauge (micrometer) placed across the inner diameter.

6.1.2 Thin Bar Bent Into Circular Arc Initially

The bar is encastre at one end, formed into an arc of radius R over 90 degrees, with a load applied to the end in a direction parallel to the bar at its root. The deflection in the direction of the load is:

$$\delta_x = \frac{\pi}{4}.\frac{F.R^3}{E.I}$$

The moment at the root is M = F.R



Thin curved bar

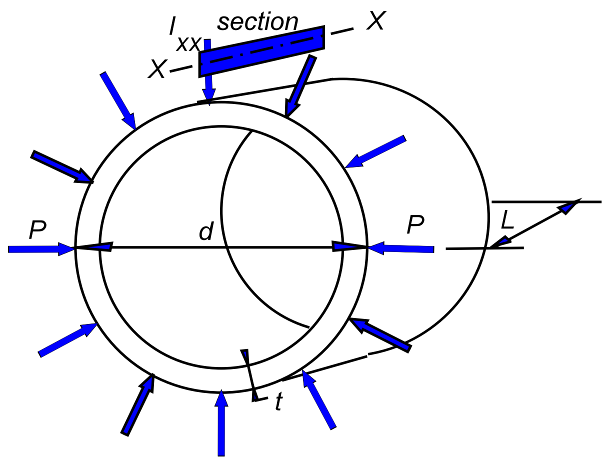
The deflection perpendicular to the direction of the load is:

$$\delta_y = \frac{F.R^3}{2E.I}$$

6.1.3 Thin Circular Ring or Tube under External Pressure

Ring or tube under uniform external pressure

Rings under external pressure (or internal suction) may collapse under elastic



Thin tube under external pressure

instability (buckling), at a critical pressure given by:

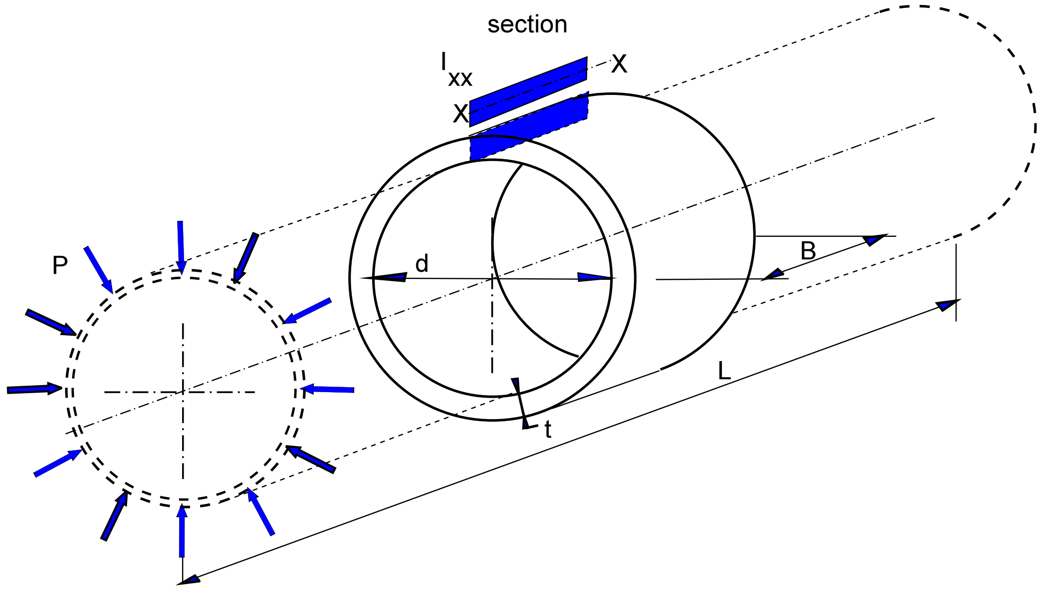
$$P_c = \frac{3E.I}{L.R^3} = \frac{24E.I}{L.d^3}$$

where

P_c external pressure per unit cylinder length
R radius of centre line of ring section
d diameter of ring
I second moment of area of ring section, along an axial axis
E modulus of elasticity
L axial length of cylinder on which pressure is applied (or ring centre to ring centre if stiffening rings are used)

Stiffened tube under uniform external pressure

The above equation may also be applied to stiffening rings. It is conservatively assumed that the ring takes the full pressure load for the cylinder.



Tube with stiffening ring

Note that in the case of stiffening rings, L is the axial length of cylinder on which pressure is applied, that is ring centre to ring centre.

6.1.4 Long cylinder under External Pressure

For a long cylinder:

$$P_c = \frac{2E}{(1-\nu^2)} . (\frac{t}{d})^3$$

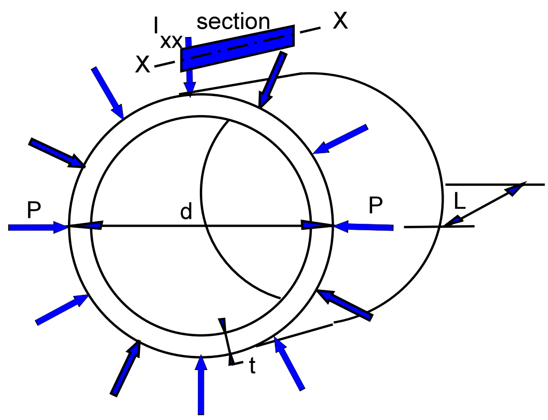
where

t wall thickness
d diameter of the tube d = 2R
ν **Poisson's ratio**

Use safety factor of 4 to accommodate initial errors in cylindricity.

This applies for cylinder lengths L that fit:

$$L > \frac{4\pi\sqrt{6}}{27} . (1-\nu^2)^{0,25} . d . \sqrt{\frac{d}{t}}$$



Long cylinder under external pressure

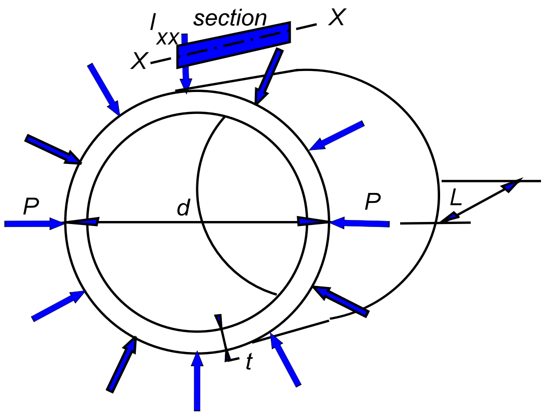
6.1.5 Short cylinder under External Pressure

This case is for a cylinder that does not fall into the "long" category above, and is also not a ring. The part of a cylinder between supporting rings is also often a "short" cylinder. There are two equations, of which the Von Mises is considered the better.

Southwell equation

This equation only takes radial pressure into account (not axial pressure). The critical pressure (at which buckling may be expected) is

$$p_c = \frac{1}{3}.(n^2-1).\frac{2E}{1-v^2}.[\frac{t}{d}]^3 + \frac{2E.\frac{t}{d}}{(n^2-1).n^4.[\frac{2L}{\pi d}]^4}$$



Short cylinder under external pressure

where

n number of lobes at collapse

Von Mises equation

Von Mises equation may be used for vessels subject to combined radial and axial pressure, or radial only. The critical pressure is

$$P_c = [\ ([n^2+(\frac{\pi d}{2L})^2]^2-2k_1.n^2+k_2).\ x\frac{1}{3}.\frac{2E}{1-v^2}.(\frac{t}{d})^3 + \frac{2E.t/d}{[(\frac{2n.L}{\pi.d})^2+1]^2}\ ].\frac{1}{n^2-1+(\frac{\pi.d}{8L})^2}$$

where

$$k_1 = (1\ +\ (1+v)\rho).(2\ +\ (1-v)\rho)$$

$$k_2 = [1-\rho.v].[1+\rho(1+2v) -\rho^2.(1-v^2).(1+\frac{1+v}{1-v}.\rho)]$$

where

$$\rho = \frac{1}{\left(\frac{2n.L}{\pi.d}\right)^2 + 1}$$

where for both the Southwell and Von Mises equations,

n **number of lobes at collapse (usually 2 or 4). This corresponds to the number of dents around the circumference. For example, if the cylinder buckles into an elliptical shape, then n=2.**

d **diameter of cylinder (often the mid plane of the wall)**

t **wall thickness**

v **Poisson's ratio (usually 0.3 for steel)**

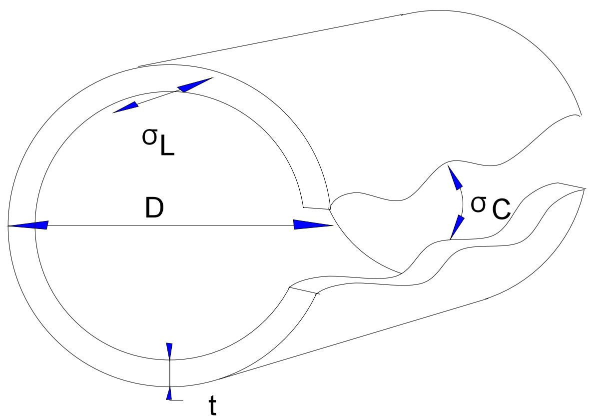
L **length of cylinder**

ρ **constant as given above, NOT density!**

6.2 Thin Cylindrical Pressure Vessels

These equations are for vessels with positive outward pressure. This is a common industrial structure which is used for a variety of silos, vessels and tanks containing liquids or solids. If a cylinders has a sufficiently thin wall, then there will be no variation of stress through the wall. An internal pressure will cause the wall to stretch in its circumference. The circumferential (or hoop) stress is:

$$\sigma_c = \frac{p.d}{2t}$$



Thin pressure vessel

where

p **radial pressure**

d **cylinder diameter**

t **wall thickness**

The end walls of the cylinder create a longitudinal stress given by:

$$\sigma_L = \frac{p.d}{4t}$$

Diametral strain is the same as circumferential strain = δd/d:

$$\frac{\delta d}{d} = \frac{1}{E}(\sigma_c - \nu\sigma_L) = \frac{p.d}{4E.t}(2 - \nu)$$

where ν is Poisson's ratio
Longitudinal strain is:

$$\delta L = \frac{1}{E}(\sigma_L - \nu.\sigma_c) = \frac{p.d}{4E.t}(1 - 2\nu)$$

6.3 Thin Spherical Pressure Vessels

The stress is:

$$\sigma = \frac{p.d}{4t}$$

where
p radial pressure
d sphere diameter
t wall thickness

6.4 Thick Cylinders

These are cylinders with pressure on the inside or outside. However the walls are too thick for the thin wall case to be applied. Stresses exist in the radial and circumferential (hoop) directions, and they vary with radius. These are principal stresses (there is no radial shear as small segments move together). Lame's equations give the stresses as functions of radius:

$$\sigma_c = a + \frac{b}{r^2} \qquad \text{Circumferential}$$

$$\sigma_r = a - \frac{b}{r^2} \qquad \text{Radial}$$

The constants a and b are determined from boundary conditions, as below.

6.4.1 Open Ended Cylinder with Internal Pressure

The inner radius is R_1 and the outer radius is R_2. Internal pressure is p. Boundary conditions are that the radial stress must be zero at the outside, and equal to the container pressure at the inside. Positive pressure results in a compressive stress which has a negative sign.

$$\sigma_r = -p \qquad at\ r = R_1$$

$$\sigma_r = 0 \qquad at\ r = R_2$$

Substituting into the equation for radial stress and solving for the constants gives:

$$a = \frac{R_1^2}{R_2^2 - R_1^2} p$$

$$b = \frac{R_1^2 . R_2^2}{R_2^2 - R_1^2} p$$

$$\sigma_r = p . \frac{R_1^2}{R_2^2 - R_1^2} . [1 - \frac{R_2^2}{r^2}] \qquad \text{Radial}$$

$$\sigma_c = p . \frac{R_1^2}{R_2^2 - R_1^2} . [1 + \frac{R_2^2}{r^2}] \qquad \text{Circumferential}$$

These constants are substituted into Lame's equations to give: where p is the internal pressure, and is positive for positive gauge pressures.

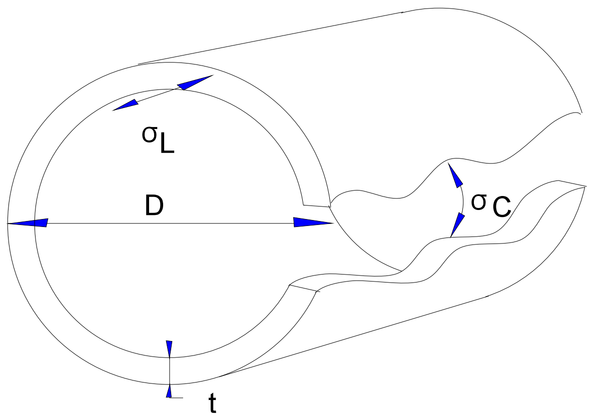
Notes:

(1) The constants were determined from the known boundary conditions for the radial stress, and then the same constants were used for the circumferential stress even though its boundary conditions were unknown.

(2) The radial stress is negative, that is compressive, throughout the wall. The greatest stress is at the inner wall.

(3) The circumferential stress is positive (tensile). The stress is highest at the inner wall.

6.4.2 Closed Ended Cylinder with Internal Pressure



Closed ended cylinder

Radial and circumferential stresses exist as for the open ended cylinder above. In addition there is an axial (longitudinal) stress (assumed to be uniform) given by:

$$\sigma_a = \frac{R_1^2}{R_2^2 - R_1^2} p$$

$$= \frac{\sigma_c + \sigma_r}{2}$$

$$= \frac{p.d}{4t} + \frac{\sigma_r}{2}$$

where

d internal diameter

t wall thickness

6.5 Design of Pressurised Thick Cylinders

The magnitude of the circumferential stress is greater than that of the radial stress. Both stresses are principal stresses. There are a number of ways in which cylinders could fail, some materials tending to fail more readily in certain of the ways. The methods of failure are given below.

6.5.1 Design for Maximum Principal Stress

For brittle materials the maximum circumferential stress may not exceed the ultimate tensile strength of the material (using a safety factor). This rearranges to:

$$\frac{t}{d} = \frac{1}{2}\left[\sqrt{\frac{\sigma_d + p}{\sigma_d - p}} - 1\right] \qquad \text{Lame (brittle)}$$

where
σ_d **design strength based on <u>ultimate tensile</u> strength**
p internal pressure (positive for positive gauge pressure)
d internal diameter

6.5.2 Design for Maximum Principal Strain

The maximum strain is given by

$$\epsilon = \frac{1}{E}[\sigma_c - \nu(\sigma_r + \sigma_a)]$$

Consequently there are different equations for open and closed cylinders:

$$\frac{t}{d} = \frac{1}{2}\left[\sqrt{\frac{\sigma_d - p(\nu - 1)}{\sigma_d - p(\nu + 1)}} - 1\right] \qquad \text{Open}$$

$$\frac{t}{d} = \frac{1}{2}\left[\sqrt{\frac{\sigma_d + p(1 - 2\nu)}{\sigma_d - p(1 + \nu)}} - 1\right] \qquad \text{Closed}$$

where
σ_d **design strength based on the <u>yield</u> strength.**
p internal pressure (positive for positive gauge pressure)

d internal diameter

6.5.3 Design for Distortion Energy

The distortion energy theory is:

$$U_d = \frac{\sigma_d^2}{6G} = \frac{1}{12G}[(\sigma_c - \sigma_r)^2 + (\sigma_c - \sigma_a)^2 + (\sigma_r - \sigma_a)^2]$$

Alternatively, the distortion energy stress (usually called von Mises stress) is

$$\sigma_d = [\frac{1}{2}[(\sigma_c - \sigma_r)^2 + (\sigma_c - \sigma_a)^2 + (\sigma_r - \sigma_a)^2]]^{\frac{1}{2}}$$

This gives:

$$\frac{t}{d} = \frac{1}{2}[\sqrt{\frac{\sigma_d}{\sigma_d - \sqrt{3}.p}} - 1]$$

where

σ_d **design strength based on the yield or the fatigue strength**
p internal pressure (positive for positive gauge pressure)
d internal diameter

Rearranging to get the pressure for a given thickness, diameter and von Mises stress:

$$p = \frac{\sigma_d}{\sqrt{3}}.(1 - \frac{1}{(1 + \frac{2t}{d})^2})$$

This theory is suitable for ductile materials.

6.5.4 Cylinder ends

The end (or head) of a cylinder has pressure applied to it, and this creates an axial stress in the cylinder. The value of this stress is quite simply determined, as the pressure times cylinder cross sectional area, divided by the area of the cylinder wall in cross section. Not all the design equations take this axial stress into account, and this is the fundamental difference between formulae for open and closed cylinders.

The shape of the cylinder end also has an effect on localised stresses. If the end is a simple flat plate, then the pressure tends to blow it out in the middle, and this creates compensating bending moment all around the edge. Therefore such an end introduces bending moment into the cylinder.

Hemispherical ends
It is possible to reduce these bending stresses by changing the shape of the end. A common design is using a hemispherical end: a half sphere. If the circumferential strains in the end and the cylinder are to be the same, then the relationship between the thicknesses is

$$\frac{t_1}{t_2} = \frac{2-\nu}{1-\nu}$$

where
t_1 wall thickness of cylinder
t_2 wall thickness of hemispherical end
ν Poisson's ratio

For a Poisson's ratio of 0.3 (common for steels), the relationship works out to be $t_2 = t_1/2.4$ that is the hemispherical end may be thinner than the cylinder.

Elliptical ends
Reduced bending stress between the end and the cylinder may also be obtained by using ends that are elliptically shaped.

6.5.5 Pressure Vessel Codes

Codes exist for the design, fabrication, inspection, and testing of pressure vessels. Vessels that are considered to be pressure vessels include fired and unfired steam boilers, evaporators and heat exchangers. Cylinders and tanks for the storage of compressed gases are also pressure vessels. The pressure vessel is considered to stop at the first circumferential joint to an end connection (ASME).

The ASME Boiler and pressure vessel code contains the following sections:

Section I	**Power boilers**
Section II	**Material specifications**
Section III	**Nuclear power plant components**
Section IV	**Heating boilers**
Section V	**Nondestructive testing**
Section VI	**Recommended rules for care and operation of heating boilers.**
Section VII	**Recommended rules for care of power boilers.**
Section VIII	**Rules for construction of pressure vessels.**
Section IX	**Welding qualifications**
Section X	**Fiberglass reinforced plastic pressure vessels**
Section XI	**Rules for in service inspection of nuclear reactor coolant systems.**

There is also an extensive BS code (BS 1515) similar to the ASME code, and these two are widely used internationally. A design will often state that the pressure vessel has been designed in accordance with a certain code.

The American Society of Mechanical Engineers (ASME) does NOT consider the following vessels to be pressure vessels for purposes of complying with regulations:

(1) Federal controlled vessels

(2) Vessels containing less than or equal to 120 gallons of water under pressure

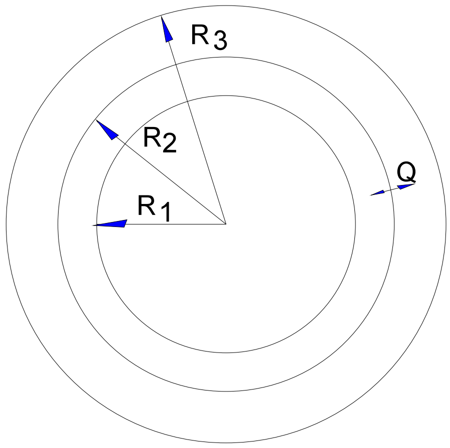
(3) Hot water tanks heated by any means where the heat input is less than or equal to 2 x 10^5 Btu/hr, the temperature less than or equal to 200°F and the capacity is less than or equal to 120 gallons.

(4) Where the internal or external pressure is less than or equal to 15 psi, regardless of size.

(5) Where the diameter is less than or equal to 6 inches, regardless of pressure.

6.6 Built Up Cylinders

Built up cylinders consist of two cylinders. These may be of the same or different materials. The purpose of the arrangement is either to locate two parts semi permanently, or to to provide a pressure vessel that is stronger than a single cylinder of the same material.



Built up cylinder

6.6.1 Stress distribution in Interference Fits

Two parts are fitted together concentrically, with an interference fit. This arrangement is used to provide a semi permanent connection that is able to transmit force and torque. The parts are either forced together axially, or the outer part expanded by heating (or inner part shrunk by cooling).

Geometry is:
R_1 inner radius
R_2 interface radius (or diameter D)
R_3 outer radius R_3
Q common pressure at interface

The inner diameter of the outer cylinder is slightly smaller than the outer diameter of the inner cylinder, the amount being the interference fit, δ (or shrinkage allowance):

$$\delta = D_{2_{inner}} - D_{2_{outer}}$$

Then the circumferential strain ϵ (also called diametral strain) is:

$$\delta = D(\epsilon_o - \epsilon_i) = \frac{D}{E}(\sigma_{c_o} - \sigma_{c_i})$$

where
D nominal interface diameter
E modulus of elasticity

After assembly the inside cylinder (subscript i) is in compression and the outside cylinder (subscript o) in tension. Apply Lame's equations, and express

the constants a_i, b_i, a_o, and b_o in terms of Q. For the inner cylinder the radial and circumferential stresses as functions of radius r are:

$$\sigma_{r_i} = Q.\frac{R_2^2}{R_1^2 - R_2^2}.[1 - \frac{R_1^2}{r^2}]$$

$$\sigma_{c_i} = Q.\frac{R_2^2}{R_1^2 - R_2^2}.[1 + \frac{R_1^2}{r^2}]$$

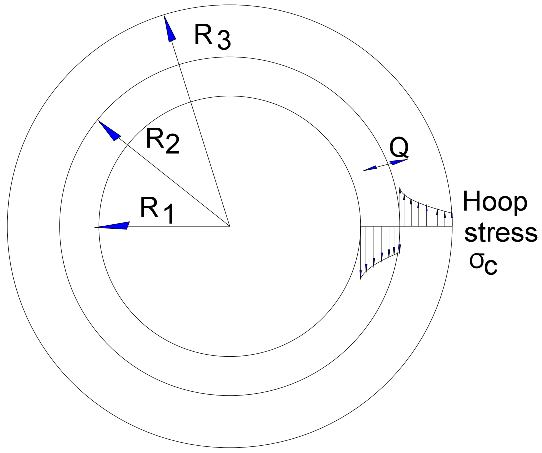
For the outer cylinder the radial and circumferential stresses as functions of radius r are:

$$\sigma_{r_o} = Q.\frac{R_2^2}{R_3^2 - R_2^2}.[1 - \frac{R_3^2}{r^2}]$$

$$\sigma_{c_o} = Q.\frac{R_2^2}{R_3^2 - R_2^2}.[1 + \frac{R_3^2}{r^2}]$$

These equations are substituted into that for circumferential strain. For the case of positive internal pressure and two cylinders of the same material:

$$\frac{\delta}{D} = \frac{Q}{E}[\frac{R_3^2 + R_2^2}{R_3^2 - R_2^2} + \frac{R_2^2 + R_1^2}{R_2^2 - R_1^2}]$$



Hoop stress distribution in built up cylinder

For positive internal pressure and two cylinders of dissimilar material:

$$\frac{\delta}{D} = Q[\frac{R_3^2 + R_2^2}{E_o(R_3^2 - R_2^2)} + \frac{R_2^2 + R_1^2}{E_i(R_2^2 - R_1^2} - \frac{\nu_i}{E_i} + \frac{\nu_o}{E_o}]$$

where ν is Poisson's ratio.

The appropriate equation may be solved for the interface pressure or geometry as required, eg:

$$Q=\frac{\delta}{D}\cdot\frac{1}{\frac{R_3^2+R_2^2}{E_o(R_3^2-R_2^2)}+\frac{R_2^2+R_1^2}{E_i(R_2^2-R_1^2)}-\frac{\nu_i}{E_i}+\frac{\nu_o}{E_o}}$$

The interface pressure will not usually be the greatest stress in the assembly. Instead it will be necessary to determine at least the circumferential stresses at the inside and outside of the inner and outer cylinders (four values, inside cylinder negative due to compression). Radial stresses may also be determined, and an appropriate failure mechanism used.

The above equations are valid for R_1 = 0, that is a solid shaft with a hub shrunk on.

The axial force required for pressed assembly is

$$F = \mu\pi 2R_2L.Q$$

where
μ **coefficient of friction**
L axial length of contact surface

The torque that the joint can take before slip is

$$T = 2\mu\pi R_2^2L.Q$$

6.6.2 Typical method for Interference Fits

Heavy press fits are basically a permanent assembly. The parts are either forced together axially, or the outer part expanded by heating (or inner part shrunk by cooling). The tighter the fit, and the larger the shaft diameter, the greater the torque that can be taken. A common requirement is to determine the axial force required to make/loosen the press fit, and the maximum torque that may be transmitted.

- **The following information is required:**
  R_1 inner radius of shaft (zero for solid shaft)
  R_2 interface radius (or diameter D), eg nominal shaft diameter at hub or gear blank

R_3 outer radius R_3. i.e. outer radius of gear hub. For solid blank use the pitch radius.
ν Poisson's ratio for shaft (inner, i) and hub (outer, o)
E modulus of elasticity for shaft (inner, i) and hub (outer, o)

- ☐ **Select a fit (see standard recommendations)**
- ☐ **The inner diameter of the outer cylinder is slightly smaller than the outer diameter of the inner cylinder, the amount being the interference fit, δ (or shrinkage allowance):**

$$\delta = D_{2_{inner}} - D_{2_{outer}}$$

This may be determined from the fits. In this case use the minimum interference (see tables for standard fits).

- ☐ **After assembly the inside cylinder (subscript i) is in compression and the outside cylinder (subscript o) in tension. The interface pressure is**

$$Q = \frac{\delta}{D} \cdot \frac{1}{\left[\frac{R_3^2+R_2^2}{E_o(R_3^2-R_2^2)} + \frac{R_2^2+R_1^2}{E_i(R_2^2-R_1)^2} - \frac{\nu_i}{E_i} + \frac{\nu_o}{E_o}\right]}$$

> *Interface stresses*
> *The interface pressure will not usually be the greatest stress in the assembly, so don't use this for failure analysis. You will need to do more work if you want that information too: determine the circumferential stresses at the inside and outside of the inner and outer cylinders (four values, inside cylinder negative due to compression). Radial stresses may also be determined, and an appropriate failure mechanism used. Consult a reference in structural mechanics for the details.*

- ☐ **The axial force required for pressed assembly (both parts at the same temperature) is**

$$F = \mu \pi 2 R_2 L . Q$$

where
μ coefficient of friction
L axial length of contact surface (hub length)

- ☐ **The torque that the joint can take before slip is**

$$T = 2 \mu \pi R_2^2 L . Q$$

6.6.3 Heated press fits

For heavy fits, it is common to heat the outer part and possibly also cooling the shaft. For a *uniform temperature rise* (axially symmetric temperature distribution) in a thick walled elastic cylindrical part, the radial strain as a function of radius is

$u(r) = \alpha \cdot (1+v) \cdot \Delta T \cdot r$

where

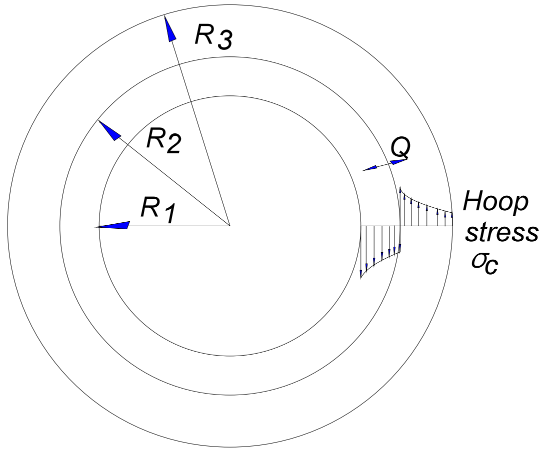
E modulus of elasticity
α coefficient of thermal expansion
ΔT change in temperature (relative to stress free condition)
v Poisson's ratio
r radius (variable)
R_1 inner radius of cylinder

The equation may be used to determine how much the inside of the hub expands. This may then be subtracted from the deviation due to the fit. In some cases there will even be a clearance fit where before there was interference. Determine the interface pressure Q with this new fit (if it is still interference), and from that get the required axial assembly force.

6.6.4 Pressurised Built Up Cylinders

The inside cylinder (subscript i) is initially in compression and the outside cylinder (subscript o) in tension. Where positive internal pressure is applied the compression of the inner first has to be overcome. This is more efficient use of the material, as by comparison a solid cylinder develops tension at the inner surface as soon as pressure is applied.

Geometry is inner radius R_1, interface radius R_2 (or D), outer radius R_3, common pressure at interface Q. The solution uses superposition of stresses from shrinkage and container pressure.



Hoop stress with no internal pressure

Step A Determine shrinkage stresses, without internal pressure

The method here is identical to that described above for shrink and interference fits (internal pressure is zero), and the outcome is the interface pressure Q.

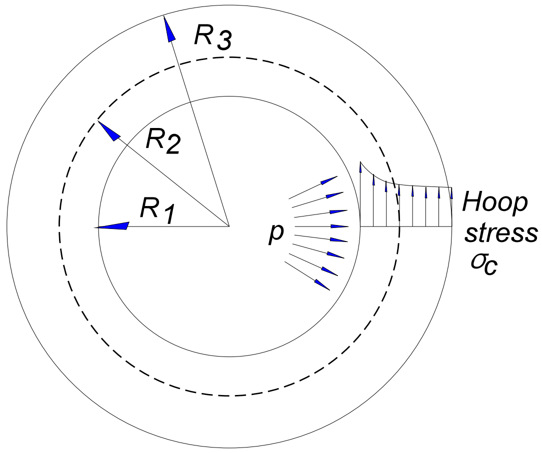
Then solve circumferential stresses at the inside and outside of the inner and outer cylinders (four values, inside cylinder negative due to compression).

Step B Treat the compound cylinder as a single cylinder (subscript j) free from shrinkage stresses, but subject to the containment pressure p.

Determine a and b from boundary conditions, and then circumferential stress:

$$\sigma_{r_j} = p.\frac{R_1^2}{R_3^2 - R_1^2}.[1 - \frac{R_3^2}{r^2}]$$

$$\sigma_{c_j} = p.\frac{R_1^2}{R_3^2 - R_1^2}.[1 + \frac{R_3^2}{r^2}]$$



Hoop stress due to pressure

Step C Superpose circumferential stresses from A and B above.

Stresses will be highest at the inner surfaces of the two cylinders.

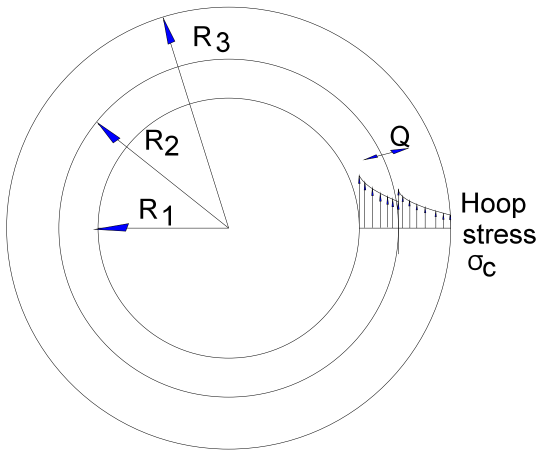
The optimal design of a compound cylinder is when the maximum principal stresses (brittle materials) or distortion energy (ductile materials) for the two cylinders are the same.

The following equation is based on the same value of maximum shear stress on the inside walls. It can be used for closed cylinders too, as the axial stress does not affect the maximum shear stress. Use the following:

$$p_{max} = \sigma_d.\frac{R_3 - R_1}{R_3}$$

$$\frac{\delta}{2R_2} = \frac{p}{E}$$

$$R_2 = \sqrt{R_1.R_3}$$



Resultant hoop stress due to shrinkage & pressure

where
σ_d design stress
p internal pressure

6.6.5 Wire Wound Cylinders

Winding a cylinder with wire creates a compressive hoop stress in the cylinder, and tension in the wire. The design procedure is given below. It assumes that the outer winding behaves as a cylinder.

Step A Determine the stresses in the inner cylinder and the outer winding due to winding only (without internal pressure).

The hoop stress in the wire due to winding only is

$$\sigma_{c_{wire}} = F[1 - \frac{r^2 - R_1^2}{2r^2}.\ln[\frac{R_3^2 - R_1^2}{r^2 - R_1^2}]]$$

The hoop stress in the cylinder due to winding only is

$$\sigma_{c_{cyl}} = -\frac{F}{2}[1 + \frac{R_1^2}{r^2}].\ln[\frac{R_3^2 - R_1^2}{R_2^2 - R_1^2}]$$

where
r radius (variable)
R_1 inner radius of inner cylinder
R_2 common radius of cylinders (nominal)
R_3 outer radius of windings
F tension in wire during winding

Stresses are determined at the inside and outside of each cylinder. The inner surfaces of each will be the most severely loaded.

Step B Determine stresses due to pressure only.

The internal pressure creates stresses in the cylinder. For this step the winding tension is ignored, and hoop stresses are determined based only on the pressure, and treating the compound as one unit. The hoop stresses are:

$$\sigma_{c_j} = p.\frac{R_1^2}{R_3^2 - R_1^2}.[1 + \frac{R_3^2}{r^2}]$$

and values will need to be determined at least at the inside of the assembly, and at the interface.

Step C Superpose stresses from winding and internal pressure

The stresses determined in steps A and B above are added. The highest values will be at the inner surfaces of the two cylinders. The optimal design of a compound cylinder is when the maximum principal stresses (brittle materials) or distortion energy (ductile materials) for the two cylinders are the same. Generally circumferential stresses dominate radial stresses and an approximate design may be reached by considering only the circumferential stresses.

6.7 Plastic Flow in Thick Cylinders

The analyses so far have assumed that stress and strain are linear, that is that deformation is elastic. In most cases engineers are happy to keep their designs in this region. But some high performance designs make use of the additional strength reserve that is available in permitting the stresses to go into the plastic (or non-linear) zone. Obviously these designs are more critical than elastic ones, and you need to be sure that your loading data are accurate if you are going to use the plastic methods. Since general industrial applications have a large amount of uncertainty in the loading data, it is common practice to stay with elastic analysis for such cases.

Plastic analysis usually assumes that the material goes into yield at a sharply defined stress, and that the stress stays constant regardless of the strain applied after that. In other words, it neglects any work hardening in the material.

The Elastic-Breakdown pressure is that needed to initiate yield on the inside wall. The cylinder will not burst at this loading, since the surrounding parts of the cylinder are still elastic and resist the yielded part at the inside. When the pressure is further increased the yield will eventually reach the outside wall, and bursting will then occur.

The Tresca yield criterion for thick cylinders is:

$$\sigma_c - \sigma_r = \sigma_y$$

where σ_y is the yield strength (also called R_e).

The radial and circumferential stresses as functions of radius are:

$$\sigma_r = \sigma_y.(\ln(\frac{r}{R_p}) - \frac{R_2^2 - R_p^2}{2R_2^2})$$

$$\sigma_c = \sigma_y.(\ln(\frac{r}{R_p}) + \frac{R_2^2+R_p^2}{2R_2^2})$$

where
R_1 inner radius
R_2 outer radius
r radius (variable)
R_p radius of boundary of plastic zone

For a given internal pressure p, the radius of yield R_p is determined by putting σ_r = -p at r = R_1, giving an equation that must be solved numerically for R_p:

$$\ln\frac{R_1}{R_p} - \frac{R_2^2-R_p^2}{2R_2^2} + \frac{p}{\sigma_y} = 0$$

The pressure at which elastic breakdown first occurs p_e, is obtained from the above equation by putting R_p = R_1, giving:

$$p_e = \sigma_y(\frac{R_2^2-R_1^2}{2R_2^2})$$

The bursting pressure is with r = R_1 and R_p = R_2, giving:

$$p_{\max} = \sigma_y.\ln\frac{R_2}{R_1}$$

6.8 Thermal stress and strain in Thick Cylinders

When a cylinder is heated, it expands in axially, circumferentially, and radially, and this can introduce stresses.

For a *uniform temperature rise* (axially symmetric temperature distribution) in a thick walled elastic cylinder, which is constrained to prevent axial expansion or contraction, the stresses are
*** radial stress is zero**
*** circumferential stress is zero**
*** axial stress is**

$$\sigma_z = -E.\alpha.\ \Delta T_m$$

where
E modulus of elasticity
α coefficient of thermal expansion
ΔT_m change in temperature (relative to stress free condition)

The equation for radial strain as a function of radius is

$$u(r) = \alpha\ .\ (1+v).\ \Delta T.\ r$$

where
v Poisson's ratio
r radius (variable)
R_1 inner radius of cylinder

These equations only apply if the cylinder is at uniform temperature. For solutions where the temperature varies with radius (as in heat exchanger applications), a more general solution needs to be used, and the reference below could be consulted. In such cases there will be stress in the radial, circumferential and axial directions too.

For more details refer to EISENBERG MA, 1980, Introduction to the mechanics of solids, Addison Wesley.

7 INERTIAL LOADS

Rotating a part generates stresses, since the material tries to move outwards. Radial acceleration at a constant rotation is

$$a = \frac{v^2}{r} = \omega^2 r$$

where
a radial acceleration
v peripheral velocity
r radius of interest
ω angular velocity

The force on a small piece of material is F = ma, where m is the mass. Therefore, the general conclusions are that
*** stresses will depend on density (m = ρV)**

*** stresses increase with radius**
*** stresses increase with the second power of the angular velocity**
These general principles are used to determine stresses in various structural geometries, as given below (methods are not shown).

7.1 Thin Rotating Ring

For a thin rotating ring, the hoop stress is:

$$\sigma_c = \rho . v^2$$

where
v **peripheral speed (m/s)**
ρ **density of the ring (eg kg.m^{-3})**

For steel with a bursting stress of 270 MPa and density of 7849 kg/m^3, the maximum peripheral speed is 186 m/s.

7.2 Thin Rotating Discs and Cylinders

The circumferential and radial stresses as functions of radius are:

$$\sigma_c = A + \frac{B}{r^2} - (1+3\nu)\frac{\rho}{8}.\omega^2.r^2$$

$$\sigma_r = A - \frac{B}{r^2} - (3+\nu)\frac{\rho}{8}.\omega^2.r^2$$

Warning
Be very careful to use mass density in the above and the following equations. DO NOT use weight density (N/m^3) as this will give wrong answers. It is common to see weight density being used in textbooks. This is wrong, as may be shown by both dimensional analysis and derivation of the equations. The reason the error persist is that mass density and weight density have the same numerical value in the imperial system. They are not the same in the metric system.

where
A, B constants determined from boundary conditions, see below
ν **Poisson's ratio**
ρ **mass density of rotating material eg in kg/m^3**
ω **angular velocity in rad/s**

7.2.1 Disc Without Central Hole

At r=0 stress cannot be infinite, so B=0
At r=R, radial stress is zero so:

$$A = (3+\nu)\frac{\rho}{8}.\omega^2.R^2$$

The maximum stress occurs at the centre.

7.2.2 Disc with Central Hole

Disc has inner radius r_1 and outer radius r_2. Radial Stresses are zero at inner and outer radii. Therefore:

$$A = (3+v)\frac{\rho}{8}.\omega^2.(r_1^2+r_2^2)$$

$$B = (3+v)\frac{\rho}{8}.\omega^2.r_1^2.r_2^2$$

As r_1 -> 0, so σ_c -> twice that of no hole.

7.3 Rotating Long Cylinders

Assume plane sections remain plane after stressing. The general equations are:

$$\sigma_c = A + \frac{B}{r^2} - \frac{(1+2v)}{(1-v)}.\frac{\rho}{8}.\omega^2.r^2$$

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3-2v)}{(1-v)}.\frac{\rho}{8}.\omega^2.r^2$$

7.3.1 Solid Cylinder

At r=0 stress cannot be infinite, so B=0
At r=R, radial stress is zero so:

$$A = \frac{(3-2v)}{(1-v)}.\frac{\rho}{8}.\omega^2.R^2$$

The maximum stress occurs at the centre.

7.3.2 Rotating Hollow Cylinder

Cylinder has inner radius r_1 and outer radius r_2. Radial Stresses are zero at inner and outer radii. Therefore:

$$A = \frac{(3-2\nu)}{(1-\nu)}.\frac{\rho}{8}.\omega^2.(r_1^2+r_2^2)$$

$$B = \frac{(3-2\nu)}{(1-\nu)}.\frac{\rho}{8}.\omega^2.r_1^2.r_2^2$$

As r_1 -> 0, so σ_c -> twice that of no hole.

Maximum radial stress occurs where:

$$r = \sqrt{r_1.r_2}$$

7.4 Discs of Uniform Strength

The thickness of the disc is:

$$z = z_1.e^{-\rho.\frac{\omega^2}{2\sigma_d}.(r^2-r_1^2)}$$

where z_1 is the thickness at root (at radius r_1)

7.5 Blade Loading

For a disc that is slotted to carry blades, with inner radius r_1, base of slot radius r_2, and outer radius r_3, the radial stress at r_2 is approximately as follows (assumes slots are filled but not able to carry circumferential stress):

$$\sigma_{r_2} = \rho.\omega^2.\frac{r_3^3-r_2^3}{3r_2} + \frac{m_{blade}.n.\omega^2.r_{blade}}{2\pi.r_2}$$

where
m_{blade} mass of single blade
n number of blades
r_{blade} radius of mass centre of blade

Use this equation to determine boundary condition at r_2.

8 FLAT PLATES AND MEMBRANES

8.1 Rectangular Plates

8.1.1 Rectangular Plates Subjected to Pure Bending in One Direction Only

Use Modified Young's Modulus in beam equation, where:

$$E^{/} = \frac{E}{1-v^2}$$

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E^{/}}{R}$$

8.1.2 Rectangular Plates Subjected to Pure Bending in Two Directions

The stress is

$$\sigma = \frac{6M}{t^2}$$

where M is the moment about the stress axis, and t the thickness of the plate.

8.2 Circular Plates

For a simply supported plate with a uniform load:

$$\sigma = \frac{3}{32}.p.(\frac{d}{t})^2.(3+v)$$

where
p pressure on plate
d outer (exposed) diameter of plate
t plate thickness
v **Poisson's ratio**

For a plate clamped about its circumference, with a uniform load P, the radial stress is:

$$\sigma_r = 3\frac{P}{8t^2}.[\,(1+v)R^2 - (3+v)x^2]$$

and the circumferential stress is:

$$\sigma_c = 3\frac{P}{8t^2}.[\,(1+v)R^2 - (1+3v)x^2]$$

where R is the radius of the plate and x is the radial distance from the centre.

The maximum deflection is:

$$\delta_{max} = \frac{3}{16}.\frac{1-v^2}{E}.\frac{p.R^4}{t^3}$$

8.3 Membrane Stresses

For membranes that are shells of revolution, subject to loading which is symmetrical about the axis:

$$\frac{\sigma_l}{R_l}+\frac{\sigma_c}{R_c} = \frac{p}{t}$$

where
l refers to the longitudinal plane
c refers to the circumferential plane
R radius of curvature
p pressure
t thickness of membrane

9 FRAMES

If m is the number of members and j the number of joints, then the forces may be solved if:

- **m = 2j - 3 (for a two dimensional frame)**
- **m = 3j - 6 (for a three dimensional frame)**

Otherwise the system is redundant (has too many members).

To solve forces, select a joint, and sum horizontal and vertical forces, and equate each to zero. Start at a joint where there are only two unknown forces. For redundant frames, see the section in STRAIN ENERGY.

10 CONTACT STRESSES

Contacting surfaces under load, generate stresses in and under the surfaces. The Hertz analysis assumes frictionless contact (no sliding friction). Elastic deformation occurs when curved surfaces are pressed together, so that small contact areas develop. Given two surfaces 1 and 2, and the following:

F contact force
R radius of surface
E modulus of elasticity
λ **Poisson's ratio**

and

$$\Delta = \frac{1-\lambda_1^2}{E_1} + \frac{1-\lambda_2^2}{E_2}$$

The values of interest are the maximum pressure p_o (which occurs in the middle of the load area), and the radius of the load area.

For two SPHERES, the maximum contact pressure is

$$p_o = 0{,}578\left[\frac{F(1/R_1+1/R_2)^2}{\Delta^2}\right]^{1/3}$$

the radius of contact is

$$a = 0{,}908\left[\frac{F.\Delta}{1/R_1 + 1/R_2}\right]^{1/3}$$

For a SPHERE and FLAT PLATE, use the above equations, with radius R_2 for the plate as infinite.

For a SPHERE and SPHERICAL SOCKET, use the above equations, with radius R_2 for the socket as negative.

For two PARALLEL CYLINDERS, the maximum contact pressure is

$$p_o = 0{,}564\left[\frac{F(1/R_1 + 1/R_2)}{L.\Delta}\right]^{1/2}$$

and the width of contact is

$$b = 1{,}13\left[\frac{F.\Delta}{L(1/R_1 + 1/R_2)}\right]^{1/2}$$

where L is the length of the cylinders.

For a CYLINDER and FLAT PLATE, use the above equations, with radius R_2 for the plate as infinite.

For a CYLINDER and CYLINDRICAL GROOVE, use the above equations, with radius R_2 for the groove as negative.

The maximum pressure is the surface compressive stress σ_z. Due to the Poisson effect the material also tries to expand in the other directions, and thus creates stresses σ_x and σ_y. These are actually principal stresses. The maximum shear stress occurs about 0,5b under the surface, and it reverses sign as a rotating load approaches and then passes. This alternating stress induces fatigue failure.

Hertz contact pressure alone is not entirely adequate for situations where sliding friction also occurs. The friction introduces a tangential normal force and a tangential shear force. The normal force reverses sign, and the surface tensile stress is the more damaging.

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