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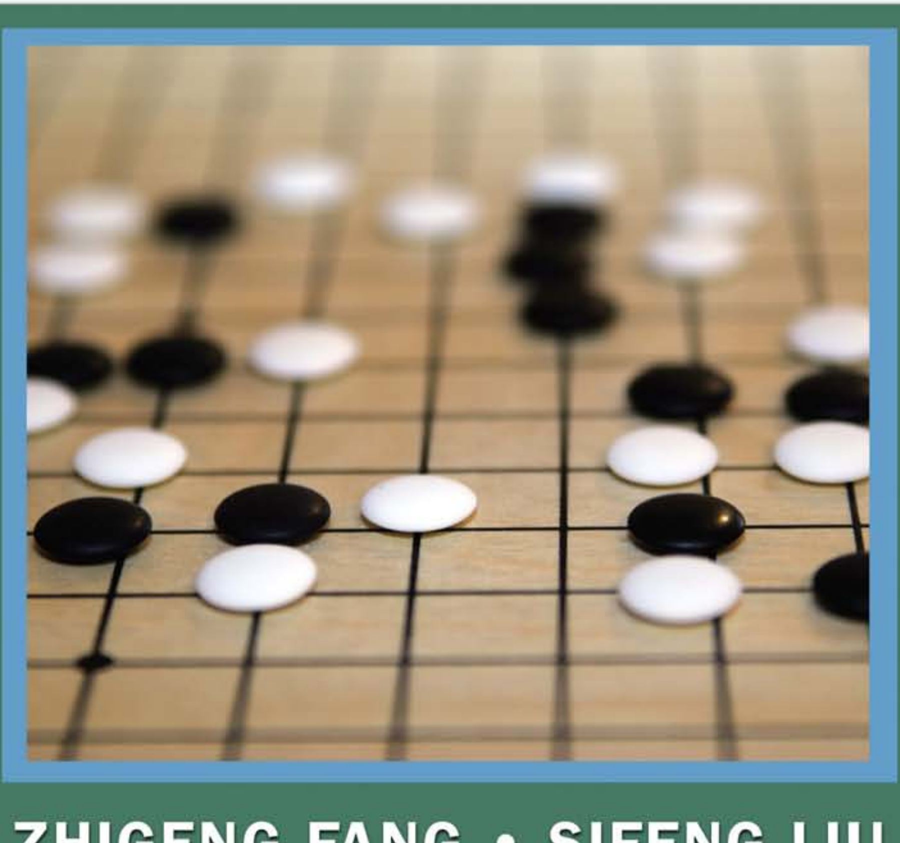

ZHIGENG FANG • SIFENG LIU
HONGXING SHI • YI LIN

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Preface

In game theory, in addition to theories such as incomplete information and bounded rationality, issues such as the uncertainty of future and limited knowledge (limited information or poor information) are also involved. However, based on the division of game-theory issues with respect to information, which is popular in the academic field, the completion or incompletion of information mainly refers to the symmetry and asymmetry information of players. Some deficiencies exist to some extent, such as ignorance of information loss. In the fact, due to many factors both stochastic and nonstochastic, the gain and loss in any two games cannot be constant even in the strictest conditions, which cannot be precise and obvious. Loss of information commonly exists when the analysis condition of classical game theory is hard to be satisfied.

People tend to understand systems incompletely, which is in most cases presented in the form of "grey" rather than that of "white." In the study of game theory, all the issues are almost grey. However, research on such information loss is rare in classical game theory, and most recent research has been carried out in a simplified method (by simply considering grey systems as white ones), resulting in a forecasting function that is far from satisfactory in a real situation.

This book mainly discusses issues of limited rationality as well as limited knowledge according to abundant theories of the grey system. Moreover, the grey matrix, grey bimatrix, and grey evolutional game model that are closer to real situations are established here, in addition to the effective design of concepts of solutions and their systems. The framework of grey bimatrix and grey evolutional game models is established so as to solve some economic problems more effectively.

The book consists of eleven chapters and one summary chapter, the first three of which are composed by Sifeng Liu, while Chapters 4–8 were written by Zhigeng Fang. Chapters 9 and 10 were written by Hongxing Shi, and Jeffrey Forrest wrote Chapter 11. The staff involved in this study also include Yong Tao, Chuanmin Mi, Aiqing Ruan, Zhendong Xu, Benhai Guo, Baohua Yang, Hui Wu, Bin Yu. Zhigeng Fang was also in charge of compiling this book.

The research in this book was sponsored by the Research Foundation for Philosophy and Social Sciences of the Chinese National Ministry, National Natural Science foundation, Postdoctoral Programs Foundations of China and Jiangnsu Province, Doctoral Programs Foundation of the Chinese National Ministry, the Research Foundation for the Excellent and Creative Teamwork in Science and Technology in Higher Institutions of Jiangsu Province and the Foundation for the Master's and Doctoral Programs in the Eleventh Five-Year Plan in Nanjing University of Aeronautics and Astronautics. Many references and research achievements of experts were investigated during the writing of this book. In addition, the authors very much appreciate the assistance rendered by the publisher.

Readers are welcome to point out and correct any errors and omissions in the book.

Abstract

This book introduces classic game theory into the realm of grey system theory with limited knowledge (or information loss), or the problem of information loss into the field of game theory. It greatly enriches the research of game theory as well as its research category and theory system, and boosts the philosophy and theory of grey systems.

Originating from real economic problems, the book has achieved the following results: establishing grey matrix game model based on pure and mixed strategies; proposing the concepts of grey saddle points, grey mixed strategy solutions and their corresponding structures; putting forward the models and methods of risk measurement and evaluation of optimal grey strategies; and raising and basically solving the problems of grey matrix games. Included are definitions of the test rules of information distortion experienced during calculation, the design of tokens based on new interval grey numbers, and new arithmetic laws to manipulate grey numbers. The method of backward induction presented here has solved the Centipede Game paradox that has perplexed academic circles for a long time, and greatly enriched and developed the core methods of subgame Nash perfect equilibrium analysis as a result.

The book caters to advanced undergraduate students and graduate students in such majors as economics, management, and system engineering. It can also be used as a reference for management staff, researchers, and engineering and technology professionals.

The Authors

Zhigeng Fang is professor and doctoral tutor at the College of Economics and Management, Nanjing University of Aeronautics and Astronautics (NUAA), China. From 1996 to 1999, he studied at Xi'an University of Technology, earning a master's degree in management science and engineering management. From 2003 to 2007, he attended NUAA, earning a doctoral degree in systems engineering.

His academic part-time positions include under-secretary-general of the IEEE International Professional Committee of Grey Systems; governor of the Chinese Society for Optimization, Overall Planning and Economic Mathematics; managing director of Complex Systems Research Committee; secretary-general of the Grey Systems Society of China; under-secretary-general of the Systems Engineering Society of Jiangsu Province, China; deputy director of the Post-Evaluation Research Center of Jiangsu Province; deputy director of the Grey Systems Research Institute of NUAA. He is a member of Services Science Global and the Services and Logistics Technical Committee of IEEE Intelligent Transportation Systems Society. His main research directions lie in the fields of theory and methods of uncertain prediction and decision, Grey game theory, applied economic, and energy economics.

Sifeng Liu earned a B.S. degree in mathematics at Henan University, China, 1981, an M.S. degree in economics and a Ph.D. in systems engineering at Huazhong University of Science and Technology, China, in 1986 and 1998, respectively. He has attended Slippery Rock University in Pennsylvania, and Sydney University in Australia as a visiting professor. Professor Liu is currently director of the Institute for Grey Systems Studies and dean of the College of Economics and Management at Nanjing University of Aeronautics and Astronautics, where he is a distinguished professor, academic leader, and doctor tutor in management science and systems engineering discipline.

Dr. Liu's main research activities are in grey systems theory and econometrics. He has published over 200 research papers and 16 books. He has been awarded 18 provincial and national prizes for his outstanding achievements in scientific research and applications, and was selected by the Personnel Ministry of China as a distinguished professor. Dr. Liu was recognized in 2002 by The World Organization of Systems and Cybernetics.

Dr. Liu is a member of teaching direct committee of management science and engineering of the Ministry of Education, China. He is also the expert of soft science at the Ministry of Science and Technology, China, and Department of Management Science of the National Natural Science Foundation of China (NSFC). Professor Liu currently serves as a chair of the Technical Committee on Grey Systems of the EEE Systems, Man, and Cybernetics (SMC) Society, is president of the Grey Systems Society of China (GSSC), vice president of the Chinese Society for Optimization, Overall Planning and Economic Mathematics (CSOOPEM), vice president of the Beijing Chapter of IEEE SMC, vice president of the Econometrics and Management Science Society of Jiangsu Province (EMSSJS), and vice president of the Systems Engineering Society of Jiangsu Province (SESJS). He is a member of the editorial board of over 10 professional journals, including *The Journal of Grey System* (UK); *Scientific Inquiry* (USA); *Journal of Grey System* (Taiwan, China); *Chinese Journal of Management Science; Systems Theory and Applications; Systems Science and Comprehensive Studies in Agriculture*; and *Journal of Nanjing University of Aeronautics and Astronautics*, among others.

Dr. Liu was selected as National Excellent Teacher in 1995, was awarded Expert Enjoying Government's Special Allowance by the State Council of China in 2000, National Expert with Prominent Contribution in 1998, and Outstanding Managerial Personnel of China in 2005.

Hongxing Shi is a Ph.D. candidate at the Institute for Grey Systems Studies, Nanjing University of Aeronautics and Astronautics. He is engaged in Grey system theory, science of science and management of science and technology. He has published two books and more than 20 papers and has participated in three projects at the national level and and five projects at the provincial level, and has been awarded for "high-quality social science application research of Jiangsu Province."

Yi Lin holds all his educational degrees in pure mathematics and had one-year post-doctorial experience in statistics at Carnegie Mellon University, Pittsburgh, Pennsylvania. Currently, he is a guest professor of several major universities in China, including the College of Economics and Management at NUAA, professor of mathematics of at Slippery Rock University, Pennsylvania, and president of the International Institute for General Systems Studies, Inc., Pennsylvania. He serves on the editorial boards of 11 professional journals, including *Kybernetes: The International Journal of Systems, Cybernetics and Management Science, Journal of Systems Science and Complexity,* and *International Journal of General Systems*. Some of Dr. Lin's research was funded by the United Nations, the State of Pennsylvania, the National Science Foundation of China, and the German National Research Center for Information Architecture and Software Technology. As of the end of 2007, he had published well over 200 research papers and 20 monographs and special topic volumes. Some of these monographs and volumes were published by such prestigious publishers as Springer, World Scientific, Kluwer Academic, and Academic Press. Over the years, Dr. Yi Lin's scientific achievements have been recognized by various professional organizations and academic publishers. In 2001, he was inducted into the Honorary Fellowship of the World Organization of Systems and Cybernetics. His research interests are wide ranging, covering areas such as mathematical and general systems theory and applications, foundations of mathematics, data analysis, predictions, economics and finance, management science, and philosophy of science.

Chapter 1

Introduction

This chapter reviews the research background and analyzes the current defects existing in game theory. Based on these findings, this chapter proposes research ideas, research content, and the framework of the full text.

1.1 Background, Meaning, and Purpose of the Research

1.1.1 Background

Game theory began in 1944, marked by the publication of *Game Theory and Economic Behavior*, written cooperatively by John Von Neumann and Oskar Morgenstern. By the 1950s, cooperative game theory had developed to its peak; noncooperative game theory also appeared at that time. The two papers of J. F. Nash ("The Equilibrium of *N* Persons' Game" [1950] and "Non-cooperative Game" [1951]) clearly put forward the Nash equilibrium, and Tucker formulated the example of the "Prisoners' Dilemma" (1950).

Their works laid the cornerstone for modern game theory. R. Seleten (1965) introduced dynamic analysis into game theory and put forward the first important improvement of the Nash equilibrium (or the subgame perfect Nash equilibrium) as well as its solution—backward induction. J. C. Harsany (1967) introduced incomplete information factors into game analysis and defined the basic equilibrium concept of the incomplete information static game—the Bayesian-Nash equilibrium; after that, he constructed the basic theory of the incomplete information game. Later on, the dynamic game theory of incomplete information developed rapidly. In 1991, Furdenberg and Tirole defined their basic concept as the perfect Bayesian-Nash equilibrium. Since the 1970s, game theory has been formed as a complete system. Roughly from the 1980s, game theory has gradually been accepted as a part of mainstream economics. Especially in aspects of modern oligopoly theory and information economics, the application of game theory to economics has received considerable success. To a certain extent, we can even treat it as the foundation of microeconomics. In 1994, the Nobel Prize in Economics was awarded to Nash, Harsany, and Seleten for their pioneering work in the development and application of game theory.

Reviewing the development of research on and application of game theory, we can now say that game theory has produced far-reaching effects by its successful application not only in economics but also in many other fields. Taking the "Prisoners' Dilemma" as an example, we can see that its definition and solution had a significant impact on disciplines in different fields such as economics, sociology, political science, and criminal psychology. The theory of the "Two-Level Game" is related to the interaction between domestic and international factors in international negotiations, and has made great contributions to resolving international problems on regional conflicts and relations between different countries with respect to economy and polity.

Game theory has had a profound impact on the field of management and is likely to form a new cross-discipline of the game theory of management (Hou Guangming and Li Cunjin, Game theory on management: An emerging cross-discipline, *Beijing Institute of Technology Journal*, Social Science Version, 2001). Noncooperative game theory is extremely likely to provide a support for the unified integration of different social sciences (Myerson, Roger B., 1999). Game theory has transformed the economy, and it will revolutionize sociology (Li Junlin, The rational, balanced, and the evolution of game theory: On the review of the development of game theory, Nankai Economic Research, 2000, 4).

We know that the rational hypothesis of mainstream economy has been treated as an extremely facilitated premise for analysis. As a branch of economics, game theory has targeted main actors' behaviors. In the game theory point of view, actors' behaviors are in line with the rational principle, which is the same as mainstream economics. Game theory, however, requires a higher and more stringent rationality on actors' behaviors than that of mainstream economics (Xie Shiyu, Evolutionary game theory under the conditions of limited rationality, *Journal of Shanghai University of Finance and Economics*, 2001, 10). Noncooperative game theory—based on *limited* rationality—has beautiful, rigorous mathematical reasoning and modeling; economists have found that almost all economic problems can be transformed into actors' interactive problems (Li Junlin, Rational, balanced, and the evolutionary game theory).

In reality, the concept of people's conduct as totally rational has understandably caused widespread suspicion. In the development of game theory, in order to verify the consistency between theory and reality, some scholars designed a number of game experiments for participation by many people; in the process of those experiments, the choice of the participants' actual strategies was observed to see whether they complied with game theory predictions—hence, it was called experimental game theory. Experimental results and theoretical prediction, however, are not always consistent with each other, mainly because of the hypothesis of actors' rational behaviors in game theory.

An important hypothesis of game theory is that both actors work from a base of common knowledge. For example, assuming that rationality is the common knowledge to each actor—that is, "all participants are rational, and all participants know that participants are all rational, and all participants know that all participants know that all participants are rational," and so on. This is an unimaginably infinite process, and it is too stringent to be guaranteed in the real world, which is one of the greatest confusions encountered by classic game theory. When facing complicated, decision-making issues, the rationality as performed by actors cannot meet the requirements of game theory. Not only will people's personal choices result in mistakes, but collective decision-making will likewise result in faulty choices. Frequent conflicts of human society, the blindness and low efficiency in the process of choosing leaders in enterprises, and so on, are all the evidence of incomplete rationality in collective decision-making.

Classic game theory cannot resolve the rational confused issues, and therefore evolutionary game theory was developed. Evolutionary game theory abandoned the hypothesis of total rationality in classic game theory. The players are regarded as having limited rationality, which results in their own evolution by means of intercompetition, and evolutionary game theory gives a reasonable explanation of the formation of players' habits (Maynard Smith, J., 1982). In the 1960s, biologists started to explain some biological phenomena using evolutionary game theory; in particular, Maynard and Price (1973) and Maynard (1974) put forward the basic concept of evolutionarily stable strategy (ESS). The theory gradually has been widely used in the fields of ecology, sociology, economics, and so forth.

Recent papers on theoretical research and the application of evolutionary game theory have taken up a major portion of all the game papers, and the theory will play an increasingly important role (Friedman, Daniel, On economy applications of evolutionary game theory, *Journal of Evolutionary Economics*, 1998, 8, 15–34). The main foreign journals that currently publish articles on evolutionary game theory and related contents are *Evolutionary Economics*, *Journal of Mathematical Biology*, *International Journal of Game Theory*, and the *Review of Economic Design*.

If the development of classic game theory, which made a great impact on the fields of economy and sociology, was unexpected, even higher expectations have been placed on evolutionary game theory. In the entire process of the development, game theory has gradually improved itself by means of resolving the problems raised by reality. Game theory has made surprising developments. Not long ago, we made a detailed search for articles published in recent years on the research and the application of game theory.

We found a total of more than 80 articles in Chinese and 442 articles in English that are related to several aspects of the research and the application of game theory. In a preliminary study of these articles, we found that both classic game theory and evolutionary game theory are based on classic mathematics, and the major problems that can be resolved are identified as follows: complete information static game, complete information dynamic game, incomplete information static game, incomplete information dynamic game, and evolutionary game based on limited rationality.

In reality, in addition to cases of incomplete information or limited rationality, there are situations of uncertainty or limited knowledge (or called limited information, poor information) of the future, as well as other issues (Liu Sifeng et al., *Grey System Theory and Its Applications*, Science Press, 1999). In game theory, "incomplete information" has a specific meaning: a game that meets the assumption of a payoff as a function based on common knowledge to all of the participants is called a complete information game, while others are called incomplete information games. Total rationality is where the participants in the game always aim for their own maximum interests whether in a certain or uncertain environment, they have the judgment and decision-making capabilities to pursue their own maximum interests, and they use their perfect judgment and decision-making capabilities in the interactive game environment. Participants not only have perfect rationality, but they also trust the rationality of each other. Except for rational common knowledge, the rationality that meets the requirements of the hypothesis is called limited rationality (Xie Shiyu, Evolutionary game theory under the conditions of limited rationality, *Journal of Shanghai University of Finance and Economics*, 2001, 10). These issues are not widely covered in current game theory, especially the limited knowledge of issues.

We know that—because of the aspects of cognitive ability, knowledge level, time, and other varieties of factors—the limited knowledge of people ("grey" information) is widespread. People cannot totally understand the system—that is, the systems confronted by people are often not "white," but "grey." In the following four general cases, system information manifests as incomplete information, or grey (Liu Sifeng et al., *Grey System Theory and Its Applications*, Science Press, 1999):

1. Incomplete information of elements (parameters).
2. Incomplete structural information.
3. Incomplete information of border.
4. Incomplete information of operational behaviors.

It can be inferred that many of the issues related to game theory are grey. Current game theory uses oversimplified approaches (a simplified grey system is a white system) to resolve the problems in reality, and therefore the forecast and guiding role of game theory is greatly reduced.

In the handling of some uncertain information issues, classic game theory has shown the application of a grey-system mind. Take the Grasp-Money Game as an example of an incomplete information issue (Zhang Weiying, *Game Theory and Economics of Information*, Shanghai People's Publishing House, 1996.6). This game

Table 1.1 Incomplete Information Grasp-Money Game

		Player 2	
Player 1		*Grasp*	*Not*
	Grasp	−1, −1	$1 + \theta_1$, 0
	Not	0,$1 + \theta_2$	0, 0

made the introduction of two types of participants, and assumes that in the interval of $[-\varepsilon, +\varepsilon]$, the θ_i obeys uniform distribution. The profit and loss value matrix is shown in Table 1.1. Until now, the literature resolving the game issues systematically by means of grey system theory have been extremely rare.

Compared with other uncertain mathematical theories (probability and statistics, fuzzy math, and so forth), grey system theory has its own unique research objective, research field, and research methods. To a certain extent, we may say that in the process of handling the issues as limited knowledge and limited rationality, the grey system theory could provide wealth theory and other certain methodologies. If we could combine grey game theory with other game theories (classic game and evolutionary game), the game model would be more in line with people's social and economic lives in reality, as well as with people's intuitive feelings, and then the model would be more widely applicable.

This book studies and solves the problems of limited rationality and limited knowledge in game theory by means of the wealth theory and the relevant methods of grey system theory. By creating a grey matrix, grey bimatrix (two-matrix), and grey evolutionary game model that are more connected with economic problems in reality, by designing a simple and efficient concept and structure system of the explanation, by completing some of its basic research, and by creating and designing the framework (system) of a grey matrix, grey bimatrix, and evolutionary game theory, this book contributes to solving the real economic issues by means of providing a powerful tool—the grey matrix, grey two-matrix, and evolutionary game theory.

1.1.2 Significance of the Topic

Academic circles have broadly accepted two ways to divide game theory research: one from the angle of the participants' action order—which divides game theory into static and dynamic games—and the other from the angle of one's knowledge about the characteristics, strategic space, and payment functions of the other related counterparts—which academic circles divided into complete information and incomplete information games. The latter mainly concerns the understanding of the competitors' related information in the game. To be more precise, complete and incomplete information in this situation mean symmetric and asymmetric information respectively.

Table 1.2 Game Profit and Loss Value of Two Color-TV Oligopoly Manufacturers Competing for Market Shares Based on Symmetric Loss Information

Manufacturer-1	*Manufacturer-2*			
		Strategy-1	*Strategy-2*	…
	Strategy-1	$[a_{11},b_{11}]$ $[c_{11},d_{11}]$	$[a_{12},b_{12}]$ $[c_{12},d_{12}]$	…
	Strategy-2	$[a_{21},b_{21}]$ $[c_{21},d_{21}]$	$[a_{22},b_{22}]$ $[c_{22},d_{22}]$	…
	…	……	……	

In addition to symmetric and asymmetric information, there possibly is another situation. Owing to grey information problems—such as uncertainty of the future, finite knowledge, and small samples (or poor information)—all counterparts may not clearly understand the characteristics, strategic space, and payment functions of the others or of themselves, even based on symmetric information. In this situation, we call it *lost information*. For example, in a comparably closed color-TV market, two oligopoly manufacturers compete for market share by action on a variety of strategies such as lower prices, better service, more product functions, and higher quality. For various reasons, the players cannot make an accurate estimate about the profit and loss in different situations, however. Even under strict conditions, the profit and loss value of any two games cannot be the same, as they are affected by regular and random factors. In reality, the profit and loss value matrix is grey and lacks information, and does not present clear and accurate resolution, as Table 1.2 shows.

Table 1.2 demonstrates the profit and loss value of two color-TV oligopoly manufacturers competing for market share based on symmetric loss information. In the table, $[a_{ij}, b_{ij}]$ and $[c_{ij}, d_{ij}]$, where ($i = 1, 2,\ldots, m$, and $j = 1, 2,\ldots, n$) mean the income intervals of grey manufacturer 1 and grey manufacturer 2, respectively, result from the lack of corresponding information of the profit and loss value; the ellipses represent the omitted strategies or profit and loss values.

Academic circles have not paid enough attention to lost information in the division of game information,[1,2] as shown in Table 1.3. Although in recent years some scholars have begun to study the game with lost information and have made some improvement in this field, the establishment of the whole theory system has a long way to go. Research in the following theory areas remains blank: static non-zero-sum grey games based on symmetric lost information, dynamic games based on symmetric lost information, and static and dynamic games based on asymmetric lost information, as shown in Table 1.3.

By using the rich ideological nutrition and related methods of grey system theory, this book will mainly study and solve grey game issues based on symmetric lost information, as well as find and perfect the methods for identifying the

Table 1.3 Classification of Game Theory and Corresponding Concepts of the Equilibrium

Action *Order Information*			
		Static	*Dynamic*
No loss	Symmetric information	Static game of symmetric information: Nash equilibrium (Nash 1950, 1951)	Dynamic game of symmetric information: subgame perfect Nash equilibrium (Seleten 1965)
	Asymmetric information	Static game of asymmetric information: Bayesian-Nash equilibrium (Harsany 1967, 1968)	Dynamic game of asymmetric information: perfect Bayesian-Nash equilibrium (Seleten 1975; Kreps and Wilson 1982; Furdenberg and Tirole 1991)
Loss	Symmetric loss information	Grey matrix game: grey saddle point (Zhigeng Fang and Sifeng Liu 2003, 2005) Static game of symmetric loss information: equilibrium=?	Dynamic game of symmetric loss information: equilibrium=?
	Asymmetric loss information	Static game of asymmetric loss information: equilibrium=?	Dynamic game of asymmetric loss information: equilibrium=?

Note: Question mark ("?") represents a theory blank.

characteristics, strategic space, and payment function of the game issues. After that, the book will construct the concept and the structural system of game results, using the Nash equilibrium and its approaches, which is one of the core concepts of noncooperative game theory, to fill in the blanks in this field. Furthermore, in some certain scope, research results should be expanded to fields of static game based on symmetric lost information and asymmetric lost information game theories, and further discussion is necessary before establishing game theory system based on lost information. Related research results of game theory will be applied to economic and social fields, and a game model with higher explanation and forecast abilities should be made to meet the requirement of reality.

1.1.3 Research Target

From the aspect of economic and social realities, this book will abstract grey matrix game theory based on lost information, by which we can resolve related problems. This book will also abstract the risk theory issues of grey game equilibrium results based on lost information, and establish the theory and method system of grey game equilibrium results based on aspects of risk measurement, prevention, and control. The capacity—which contains explanations and predictions of grey game theory when treating the problems in reality—will be upgraded.

A second target is an exploration of the fatal flaw in the classical Cournot oligopoly model as well as the related oligopoly output-making competition model. A descriptive game structure model that is of strong universality to the realistic decision-making situations will be constructed. The book also focuses on a kind of game equilibrium analysis and the existence of an equilibrium point under the condition of a game with lost information. Then, it will establish the concepts and resulting rules, like the framework of the incomplete static game of profit and loss value information and of grey potential equilibrium.

When focusing on the phenomenon that backward induction's paradox does not match the reality, this book also reveals the root of the paradox of backward induction, and will design a backward induction grey number structured algorithm of multistage static game theory. The paradox of the Centipede Game will be cracked, thus describing a third target of this book.

When considering the fact that the classic model cannot make a prediction in one-off game results or in periods of short-term economic equilibrium, this book constructs the game theory chain model, which is based on evolutional game theory in a symmetrical or asymmetrical situation. We found that the restrictions requested by the classical model are too strict to be satisfied in reality. Taking this factor into consideration, and based on the assumption of limited rationality, this book finds the optimal grey quotation model, which is based on ideal quotation and assessment with the precise value and experience.

1.2 Status of Research and Development

Game theory began in 1944, and by the 1950s, cooperative game theory had developed to its peak; noncooperative game theory also began to appear at that time. After nearly sixty years of development, noncooperative game theory was a completely theoretical system. It mainly involves the following aspects: complete static information, complete dynamic information, incomplete static information, incomplete dynamic information, and so on. At present, the application of game theory covers almost every field of the economic and social life of people, and it is of great success. In the entire process of development, game theory gradually improved itself by means of resolving the problems raised by reality. As in classic game theory, there are many issues that cannot be satisfactorily resolved; academic circles hasten the birth of many new theories, such as the following:

1. *Fuzzy Game.* In the process of decision-making, we bring inspired information and knowledge into the framework of classic game theory, and through the use of a strategy set fuzzy subordination, put the strategic selection reference—which cannot be disposed of by traditional game equilibrium—into the framework of the fuzzy game. Thus, we can get the equilibrium results of fuzzy game, and the best strategy for each of the participants in the end. All of this is a constant process of studying, judging, and adjusting.
2. *Evolutionary Game.* The theory is the combination of game analysis and a dynamic evolutionary process that is based on individuals with limited rationality, and treats the group as the research target. It holds the opinion that individuals in reality are not completely rational, and that decision-making is realized by means of various dynamic processes, such as imitating each other, study, and mutation. The theory of an evolutionary game is an innovation in the methods of economics, because the theory established a new framework for analysis based on denying the foundation of traditional theory, which is the hypothesis of people with rationality. Through the combination of the latest contributions in different theoretical fields, such as ecology, sociology, psychology, and economics, evolutional game theory provides a new vision for researching various economic behaviors and for investigating the equilibrium of the process.
3. *Experimental Game.* Experimental game theory is a special game theory. Experiment can provide people useful information by observation. This kind of information cannot be easily gotten from game theory. Because of the habit of analyzing the structure of the game by the concepts of equilibrium and refinement, experimental knowledge is always treated as insignificant; at the same time, the role of experimental knowledge in the players' reactions to a certain game is even ruled out. In a nonstrategic environment, people could make a reasonable strategy according to the principles of an optimal conditional expectation utility, but in the strategic environment only by means of relying on players' rationality can people easily make definite prediction—and even though the prediction is made, the credibility of it in reality is questionable. In the field of applied economics, the issues cannot be solved only by rationality. Only when rationality and experimental knowledge are combined can we get a solution. There is a strong complementarity between experiment and rationality: rationality can provide an analyzing framework for experimental information, and the experiment can offer a checking method for rationality.

Xu Jiuping, a professor in Sichuan University, published an academic paper named "Zero-Sum Two-Person Games with Grey Number Payoff Matrix in Linear

Table 1.4 Game Theory and Its Application Retrievals

Retrieval Source	*Search Terms*	*Retrieval Strategy*	*Retrieval Time*	*Retrieval Results*
Chinese academic periodical net	game	keywords	1999–2007	14489
Chinese master's thesis full-text database	game	keywords	1999–2007	2976
Chinese doctoral dissertations full-text database	game	keywords	1999–2007	913
Elsevier	game theory	keywords	1999–2007	547
Springer	game theory	summary	1999–2007	960
EI Village	game theory	subject/title/ abstract	1999–2007	13639
ISI Web of Science (SCI)	game theory	keywords	1999–2007	1624

Programming," which was related to the solution of grey payoff matrix, but the paper did not definitively lay out the concept of grey game theory and equilibrium results.

In the process of researching academic theses on game theory and its application from related resources, we recently reviewed thousands of related articles in Chinese and English. The research field has been extended to economics, sociology, the environment, politics, and so on. Table 1.4 shows the detail.

No matter whether it is domestic or foreign literature, treatment of the uncertain game theory issues—as a research target and as a definitive scientific theory of grey game—has never been found.

The thesis of this book mainly involves game theory, grey system theory, and other research domains that are the application of grey system ideas, theories, and methods in game theory. It is used to resolve the game problem of bounded knowledge (or lost information) that universally exists but is not involved (or cannot be solved) in classical game theory. We can say the subject pulls the problem of missing information into the game theory domain and makes game theory widespread to the grey system domain of bounded knowledge and poor information.

1.3 Research and Technology Roadmap

1.3.1 Main Content

From the aspect of the economic and social realities, this book abstracts grey matrix game theory based on lost information, by which we can resolve related problems, and initiates research on the mechanism of matrix game theory issues (grey matrix game) based on interval grey numbers. It also constructs the grey matrix game model based on pure strategy and puts forward the conception and structure system of grey saddle point and grey mixed strategy. This book abstracts the risk-theory issues of grey game equilibrium results based on lost information, and establishes the theory and method system of grey game equilibrium results based on aspects of risk measurement, prevention, and control. The capacity—which contains explanation and prediction of grey game theory when treating the problems in reality—will be upgraded. The grey matrix game theory will then almost be finished.

This book also reveals that there is a fatal flaw in the classical Cournot oligopoly model as well as in the related oligopoly output–making competition model. Some new suppositions of a duopoly game are re-established, including the game-purpose supposition, the time-order supposition, bounded rationality, and knowledge supposition. A descriptive game structure model that has strong universality to the realistic decision-making situations is constructed. This book brings forward the concepts of the damping loss and the total damping cost when the first decision-maker completely seizes the whole market, and designs related algorithms.

Grey Game Theory and Its Applications in Economic Decision-Making proposes one kind of game equilibrium analysis and the existence of its equilibrium point under conditions of incomplete game information. It also establishes the framework of a static game whose information of profit and loss value is correspondingly incomplete, as well as the concepts of equipollence, and superior and inferior potential degree of grey numbers in the range of gain and loss value; the Nash equilibrium of grey potential pure strategy; dominant pure strategy of grey potential; and the position judging rule of grey number relationships among ranges. This book proves:

If for the N-person stable game that has incomplete profit and loss information, there exists a Nash equilibrium of grey potential pure strategy, then we can apply the method of grey potential advantaged and disadvantaged strategy, grey potential marking method, and grey potential arrow method in different situations.
Then the equilibrium point can be conveniently found.

When focusing on the phenomenon that backward induction's paradox does not match reality, this book reveals the root of the paradox of backward induction, and constructs a new kind of dynamic game model structure. Then it designs a backward *grey number structured* algorithm, and establishes the concept system of the termination and guide Nash equilibrium of multistage dynamic game theory,

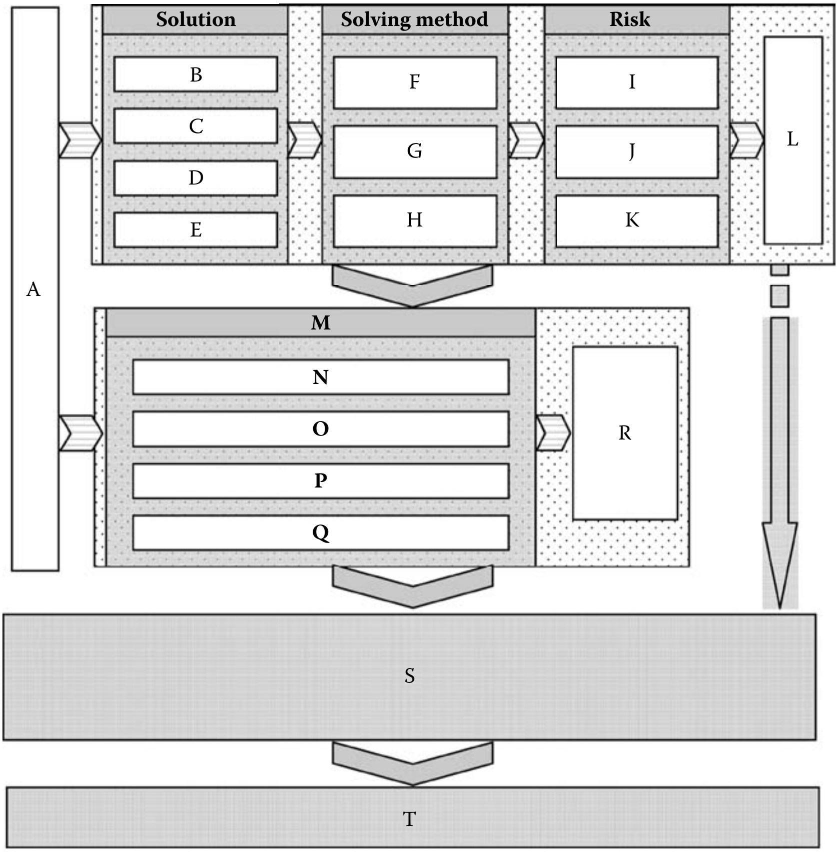


Figure 1.1 Technique route of the project. A: from the aspect of the economic and social realities with abstracted grey matrix game theory based on lost information; B: research on existence of the solution; C: grey absolute pure strategy equilibrium; D: grey potential pure strategy equilibrium; E: grey mix strategy equilibrium; F: psychological assumption of game based on incomplete information; G: solving grey absolute strategy, grey potential pure strategy, and grey mix strategy; H: grey matrix solution of game equilibrium; I: existence of game risk based on incomplete information; J: risk of grey absolute strategy, grey potential pure strategy, and grey mix strategy solutions; K: research on risk evaluation, prediction, and control; L: resolution of the theory problems of grey matrix game equilibrium; M: research on a static, nonmatrix game problem based on incomplete information; N: research on the existence of the solution; O: analysis of grey potential Nash equilibrium and grey mix strategy solution; P: risk analysis of grey potential Nash equilibrium and grey mix strategy solution; Q: research on risk evaluation, prediction, and control of a grey equilibrium solution; R: Resolution of the theory problems of grey matrix game equilibrium of static, nonmatrix game based on symmetrical incomplete information; S: design of a new kind of dynamic

while providing a convenient and effective balanced analyzing method that is of better use for cracking the paradox of the Centipede Game.

When focusing on the fact that the current model cannot make predictions to one-off game results or to periods of short-term economic equilibrium, this book constructs the game chain model, which is based on evolutional game theory in a symmetrical or asymmetrical situation, and the model profoundly reveals the relationship of players' interdependence and intertransformation in the process of the game. It also sets up the recurrent relation of each player's individual quantities and expected income on average in each step of the evolution game.

It has been found that there are some defects in the classical first-price sealed auction model, whose conditions are restricted too much to fit the real situation. This book designs grey correction factors of experiential ideal quotations, and according to the hypothesis of limited rationality, establishes an optimal grey quotation model based on accurate evaluation of value and experiential ideal quotation.

1.3.2 Technical Route

Supposed research program and feasibility analysis: The project is supposed to be divided into four steps for further discussion, from easy to hard, simple to complex, and theory research to application utility (see Figure 1.1).

In the first step, this book treats game theory issues in reality based on incomplete microeconomic information between individuals, which is abundant in economic and social life, as the research target, and abstracts grey matrix game theory of interval grey tokens based on incomplete symmetry information.

In the second step, for solving the issues of grey matrix game, the book constructs a game model and puts forward the conception and structure of grey equilibrium results, the theory and methods for solving it, and the approaches for risk

Figure 1.1 (Continued).
game model structure, and establishment of the concept system of termination and guide Nash equilibrium of a multistage dynamic game, while providing a convenient and effective balanced analysis method, which is of better use for cracking the paradox of the "Centipede Game." Furthermore, when focusing on the fact that the current model cannot make predictions in a one-off game or in a period of short-term economic equilibrium, we constructed the game chain model, which is based on the evolutional game in a symmetrical or asymmetrical situation. Then, according to the hypothesis of limited rationality, an optimal grey quotation model based on accurate evaluation of value and experiential ideal quotation was established; T: through the combination of theoretical research achievements and related economic and social practice, the information provided here can make a great contribution to the development of economics and society, while upgrading the theory and promoting the improvement of practice.

analysis and control. All of the contributions fill in the blanks of equilibrium results in the grey matrix and perfect the theory of the grey matrix game.

The third step, based on former theoretical research achievements into aspects of the grey matrix game, expands the research field to game issues of a static non-matrix based on incomplete information, and constructs the concept and the structural system of the game solution, using the Nash equilibrium and its approaches, which is one of the core concepts of noncooperative game theory—to fill in the blanks in equilibrium results of grey games. It perfects the theoretical system of a static grey game based on incomplete information.

The fourth step, so as to resolve the existing problems in the field of game theory with the application of grey systematic thought, designs a new kind of dynamic game model structure, and establishes the concept system of the termination and guide Nash equilibrium of multistage dynamic game theory, while providing a convenient and effective balanced analysis method, which is of better use for cracking the paradox of the Centipede Game. when focusing on the fact that the current model cannot make predictions in one-off game results or in periods of short-term economic equilibrium, this book constructs a game theory chain model that is based on the evolutional game in a symmetrical or asymmetrical situation. According to the hypothesis of limited rationality, it then establishes an optimal grey quotation model based on accurate evaluation of value and experiential ideal quotation.

During the fifth step, through the combination of theoretical research achievements and related economic and social practice, this book makes a great contribution to the development of economy and society, while upgrading the theory and promoting the improvement of practice.

1.4 Main Innovative Points and Characteristics

1.4.1 Main Innovative Points

This book, for the first time, and with the application of the thought and theory of grey systems, deals with game issues such as limited rationality, limited knowledge, and uncertainty in the future. Then it designs a simple, effective solution for grey linear programming and the grey matrix in a grey matrix game. This book abstracts the risk-theory issues of grey game equilibrium results based on lost information, and establishes the theory and method system of grey game equilibrium solutions based on the aspects of risk measurement, prevention, and control. The construction of grey matrix game theory has almost been finished.

Treating economic problems in reality as the research background, this book establishes the characteristics and strategy of a grey two-matrix game as well as the characterization methods of the pay-off function. It also establishes the framework of a static game whose information of profit and loss value is correspondingly incomplete, and investigates the concepts of equipollence, and superior and inferior potential degrees of grey numbers in the range of profit and loss values; the Nash equilibrium of grey potential pure strategy; the dominant pure strategy of grey potential; and the position judging rule of grey number relationships among ranges. The article proves that if, for the *N*-person stable game that has incomplete profit and loss information, there exists a Nash equilibrium of grey potential pure strategy, then we can apply the method of grey potential advantaged and disadvantaged strategy, the grey potential marking method, and the grey potential arrow method in different situations. Then the equilibrium point can be conveniently found.

Grey Game Theory reveals the root of the paradox of backward induction, and constructs a new kind of dynamic game model structure. Then it designs a backward *grey number structured* algorithm, as well as establishes the concept of the *termination* and *guide* Nash equilibrium of multistage dynamic game theory, while providing a convenient and effective balanced analyzing method that is of better use for cracking the paradox of the Centipede Game.

A descriptive game structure model that is of strong universality to the realistic decision-making situations of oligopoly in an output-making competition is constructed. This book also brings forward the concepts of the damping loss and the total damping cost when the first decision-maker completely seizes the whole market, and it designs related algorithms.

By constructing the game chain model, which is based on the evolutional game whether in a symmetrical or asymmetrical situation, this book reveals the evolutionary game phenomenon of biotic wrong-test. The book builds the optimal grey quotation model with limited rationality based on accurate evaluation of value and experiential ideal quotation, and finds that a bidder's optimal grey quotation depends not only on values the bidder itself estimates but also on values the rivals estimate and on a menace reflection grey coefficient. The optimal grey quotation of a bidder is not the half-value of goods at auction but is generally higher than half the value.

1.4.2 Main Characteristics

For the first time (according to the results of our search; related grey systems are rare) from the aspect of the economic and social practice, this book abstracts grey matrix game theory of an interval grey token based on incomplete symmetry information. Then it expands the classic game theory into the field of grey numbers, which enlarges the application scale of game theory and enriches the theory and thought of grey systems.

With the application of the thought and theory of a grey system, this book deals with game issues based on incomplete information, which is caused by factors such as limited rationality, limited knowledge, uncertainty in the future, and so forth. Three important theoretical issues solved are as follows:

First, a *game equilibrium of grey game* is well resolved.

Second, we provide a reasonable explanation for the *equilibrium of a grey matrix* of static nonmatrix game issues based on incomplete information.

Third, this book provides the solution of the Centipede Game, which has puzzled theory circles for a long time.

This book mainly researches the field of game theory, and solves a series of fundamental problems not many people are involved in—namely, game theory and its equilibrium, and the risks of an equilibrium solution on the condition of incomplete information. Game theory has produced far-reaching influence in many disciplinary fields, such as economics, sociology, political science, and criminal psychology.

From the aspect of the economic problems in reality, this book researches the mechanism and regulation of the game chain issues based on an interval grey number matrix, two-matrix games, and evolutionary games.

This book constructs the framework system of game theory based on grey data and applies game theory to economic life to solve realistic problems.

Chapter 2

Study of the Grey Matrix Game Model Based on Pure Strategy

2.1 Study of the Simple Grey Matrix Game Model Based on Pure Strategy

In a two-player finite zero-sum game, the revenue matrix the two players take is largely predetermined by the players. People never give precise judgments about game results in advance because of such factors as incomplete information, the system's design ability, and random fluctuation.

Judging by the result of the two-player finite zero-sum game—even though the total sum all players get from the game is zero, and the backward value of the game is clear, with the effect of all the actual random and nonrandom factors—the actual game results are not necessarily the same each time, even if the pure strategy were to be exactly repeated each time.

For example, two oligopolies, *A* and *B*, carve up a relatively closed TV market. In order to grab more market share, they act on all kinds of strategies (lowering the price, providing high-quality service, increasing product functions, and improving the quality of the products) to compete with each other. Due to a variety of reasons, the competing players cannot precisely estimate the revenues beforehand for all possible conditions. Even with relatively strict limitations, influenced by all the random and nonrandom factors, a player in a game won't have the exact revenue twice. In reality, the revenue matrix of a two-player finite zero-sum game is rather grey rather than fully clear and precise.

2.1.1 Construction of a Simple (Standard) Grey Matrix Game Model Based on Pure Strategy

2.1.1.1 Analysis of a Grey Game Profit and Loss Value Matrix

Definition 2.1 Grey matrix: If there is any grey number $a_{ij}(\otimes)$; $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ in elements of a matrix, this matrix is called a grey number matrix, or a grey matrix for short, expressed as $A^{l}(\otimes) = [a_{ij}(\otimes)]_{m\times n}$; $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

Definition 2.1 is used to explain that if there is one or more elements of a matrix that are grey numbers, this matrix will be called a grey matrix for short.

Definition 2.2 Grey matrix game: For the two-person limited zero-sum game, if the profit and loss value matrix for the players' assessment in advance is a gray matrix, we can then refer to the game as a grey matrix game, which is expressed as $\tilde{G}^{l}(\otimes) = \{S_1^{l}, S_2^{l}; A^{l}(\otimes)\}$, where $S_1^{l} = \{\alpha_1^{l}, \alpha_2^{l}, \ldots, \alpha_m^{l}\}$ is the strategy set of Player 1; $S_2^{l} = \{\beta_1^{l}, \beta_2^{l}, \ldots, \beta_n^{l}\}$ is the strategy set of Player 2; and $A^{l}(\otimes)$ is the grey profit and loss value matrix for players to judge in advance.

Since the assessment of the profit and loss value for various games could be different from each other, we will consider the following two situations. First, learning is the same for the two players, which means that they will make decisions according to the same predictive grey profit and loss value matrix. The grey profit and loss value matrix is common knowledge and can be formulated as Eq. (2.1):

$$A^{0}(\otimes) = \begin{bmatrix} [a_{11}^{0}, b_{11}^{0}] & [a_{12}^{0}, b_{12}^{0}] & \cdots & [a_{1n}^{0}, b_{1n}^{0}] \\ [a_{21}^{0}, b_{21}^{0}] & [a_{22}^{0}, b_{22}^{0}] & \cdots & [a_{2n}^{0}, b_{2n}^{0}] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{m1}^{0}, b_{m1}^{0}] & [a_{m2}^{0}, b_{m2}^{0}] & \cdots & [a_{mn}^{0}, b_{mn}^{0}] \end{bmatrix} \tag{2.1}$$

where $a_{ij}^{0} \le b_{ij}^{0}$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$.

Second, the assessment of the profit and loss value for any possible situation could be different from that determined by the two players, and the two cannot send the assessment to each other—that is, the grey profit and loss value matrix is *not* common knowledge but is the secret of the players, and the profit and loss value matrix of Player 1 and Player 2 can be formulated as

Eqs. (2.2) and (2.3):

$$A^1(\otimes) = \begin{bmatrix} [a_{11}^1, b_{11}^1] & [a_{12}^1, b_{12}^1] & \cdots & [a_{1n}^1, b_{1n}^1] \\ [a_{21}^1, b_{21}^1] & [a_{22}^1, b_{22}^1] & \cdots & [a_{2n}^1, b_{2n}^1] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{m1}^1, b_{m1}^1] & [a_{m2}^1, b_{m2}^1] & \cdots & [a_{mn}^1, b_{mn}^1] \end{bmatrix} \tag{2.2}$$

where $a_{ij}^1 \le b_{ij}^1$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$, and

$$A^2(\otimes) = \begin{bmatrix} [a_{11}^2, b_{11}^2] & [a_{12}^2, b_{12}^2] & \cdots & [a_{1n}^2, b_{1n}^2] \\ [a_{21}^2, b_{21}^2] & [a_{22}^2, b_{22}^2] & \cdots & [a_{2n}^2, b_{2n}^2] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{m1}^2, b_{m1}^2] & [a_{m2}^2, b_{m2}^2] & \cdots & [a_{mn}^2, b_{mn}^2] \end{bmatrix} \tag{2.3}$$

where $a_{ij}^2 \le b_{ij}^2$, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

2.1.1.2 Size Comparison of Interval Grey Numbers

In this book and in follow-up studies, in order to discuss the problem conveniently, the grey numbers we discuss will be limited to interval grey numbers (and we call them grey numbers for short) because the interval grey number is the most representative grey number of all kinds of grey numbers; according to actual situations, the upper or lower bound of grey numbers that have no upper or lower bounds can be assumed to be a prodigious number, and this kind of the grey number can be transformed to the interval grey number.

Because size comparison of the grey number will be involved in the solving process of the standard grey matrix game, we will study the size comparison of the simple interval grey number first.

Definition 2.3 Size comparison of the grey number: For two grey number $\otimes_1 \in [a,b] (a \le b)$ and $\otimes_2 \in [c,d]$ $(c \le d)$, if they can fit following four conditions, these two interval grey numbers are called simple interval grey numbers (simple grey numbers for short).

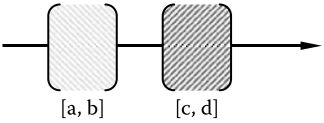


Figure 2.1 Grey number [*a*, *b*] < [*c*, *d*] represented on a number axis.

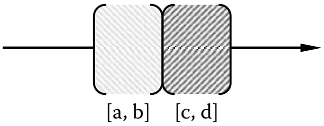


Figure 2.2 Grey number [*a*, *b*] ≤ [*c*, *d*] represented on a number axis.

1. When $b < c$, then $[a,b] < [c,d]$; that is, $[a,b](a \le b)$ is less than $[c,d]$ $(c \le d)$. This can be expressed in a number axis, as shown in Figure 2.1.
2. When $b \le c$, then $[a,b] \le [c,d]$; that is, $[a,b](a \le b)$ is less than or equal to $[c,d]$ $(c \le d)$. This can be expressed in a number axis, as shown in Figure 2.2.
3. When $a = c$, $b = d$, and the obtained numbers of two grey numbers are the same $(\gamma_1 = \gamma_2)$, it meets that $\tilde{\otimes}_1 = \gamma_1 a + (1-\gamma_1)b, \gamma_1 \in [0,1]$, $\tilde{\otimes}_2 = \gamma_2 c + (1-\gamma_2)d, \gamma_2 \in [0,1]$, and $\gamma_1 = \gamma_2$. Then $[a,b] = [c,d]$; that is, grey number $[a,b](a \le b)$ is equal to grey number $[c,d]$ $(c \le d)$. This can be expressed in a number axis, as shown in Figure 2.3.
4. When $a = c$, $b = d$, and the obtained numbers of two grey numbers are not the same $(\gamma_1 \ne \gamma_2)$:

If $\gamma_1 < \gamma_2$, then $\otimes_1 > \otimes_2$ (that is $[a,b] > [c,d]$), and grey number $[a,b](a \le b)$ is larger than grey number $[c,d]$ $(c \le d)$. This can be expressed in a number axis, as shown in Figure 2.4

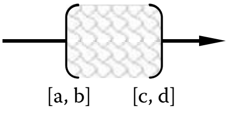


Figure 2.3 When obtained numbers are the same, grey number [*a*, *b*]=[*c*, *d*] represented on a number axis.

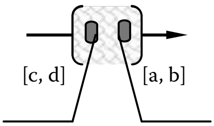


Figure 2.4 When $\gamma_1 > \gamma_2$ Cgrey number [*a*, *b*] > [*c*, *d*] represented on a number axis.

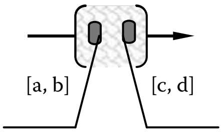


Figure 2.5 When $\gamma_1 < \gamma_2$ grey number $[a,b] < [c,d]$ represented on a number axis.

If $\gamma_1 > \gamma_2$, then $\otimes_1 < \otimes_2$ (that is $[a,b] < [c,d]$), and grey number $[a,b] (a \le b)$ is smaller than grey number $[c,d]$ $(c \le d)$. This can be expressed in a number axis, as shown in Figure 2.5

Definition 2.3 gives us some simple grey forms in which it is easy to compare sizes. If we want to compare sizes of other forms, we need more information, such as the concrete numerical value of the obtained numbers of every grey number. In the following study, in order to use these simplest grey forms with easy-to-compare sizes from Definition 2.3, we will give it a definition, as shown in Definition 2.4.

Definition 2.4 Comparable simple grey number set: For two arbitrary grey numbers of a grey number set $A(\otimes) = \{\otimes | \otimes_i = [a_{i1}, b_{i2}],\ i = 1, 2, \ldots, m\}$, where the two arbitrary grey numbers are represented in a number axis, there are just four conditions that can appear, which are regulated by Definition 2.3 (shown in Figures 2.1–2.5). That is, either two numbers are coincident and their obtained number is known or their intersection (except the endpoints) is a null set. We can call these grey number sets either comparable grey number sets or noncomparable grey number sets.

We have to judge maximum and minimum numbers from some grey numbers in many conditions. A definition of this problem is shown in Definition 2.5.

Definition 2.5 Maximum and minimum decidable grey number: For a grey number set $A(\otimes) = \{\otimes | \otimes_i = [a_{i1}, b_{i2}],\ i = 1, 2, \ldots, m\}$, where all the grey numbers of this set are represented on a number axis, if a grey number $[a_{k1}, b_{k2}]$ is either coincident with all the other grey numbers (and the obtained number is known) or their intersection (except the endpoints) is a null set, we can call this grey number a size-decidable grey number, or decidable grey number for short.

In all the decidable grey numbers, the grey number that is at the leftmost end of the number axis is called the minimum decidable grey number, and the grey number at the rightmost end of the number axis is called the maximum decidable grey number.

Example 2.1 For some grey numbers $\otimes_1 = [-1,2]$, $\otimes_2 = [-1,2]$, $\otimes_3 = [-1,0]$, $\otimes_4 = [3,4]$, we can determine that $\otimes_1 < \otimes_4$, $\otimes_2 < \otimes_4$, $\otimes_3 < \otimes_4$. If $\otimes_1$ and $\otimes_2$ have the same obtained number, then $\otimes_1 = \otimes_2$; if $\gamma_1 < \gamma_2$, then $\otimes_1 > \otimes_2$; and if $\gamma_1 > \gamma_2$, then $\otimes_1 < \otimes_2$. If the

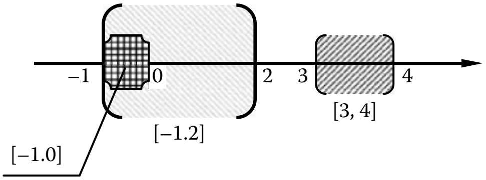


Figure 2.6 Number axis of decidable grey numbers.

obtained number of $\otimes_1$ and $\otimes_2$ is not decidable, then the size of $\otimes_1$ and $\otimes_2$ cannot be determined (it is therefore incomparable), and the size of $\otimes_1$, $\otimes_2$, and $\otimes_3$ cannot be determined (and it is also incomparable). In all these grey numbers, $\otimes_4 = [3,4]$ is the maximum decidable grey number. There is no minimum decidable grey number of these grey numbers; see Figure 2.6.

Theorem 2.1 The sufficient and necessary conditions of comparable grey number set: For m grey numbers $\otimes_i = [a_{i1}, b_{i2}], i = 1,2,\ldots,m$, the sufficient and necessary conditions of a comparable grey number set $A(\otimes) = \{\otimes | \otimes_i = [a_{i1}, b_{i2}],\ i = 1,2,\ldots,m\}$, which is composed of these m grey numbers, are that these m grey numbers are all decidable grey numbers.

Proof: Sufficiency: For m grey numbers $\otimes_i = [a_{i1}, b_{i2}],\ i = 1,2,\ldots,m$, if these m grey numbers are all decidable grey numbers, then from Definition 2.5, we know that in these m grey numbers $\otimes_i = [a_{i1}, b_{i2}],\ i = 1,2,\ldots,m$, any grey number $\otimes_i = [a_{i1}, b_{i2}],\ i = 1,2,\ldots,m$ is either coincident with all the other grey numbers (and the obtained number is known) or their intersection, except the endpoints, is a null set.

According to Definition 2.4, grey number set $A(\otimes) = \{\otimes | \otimes_i = [a_{i1}, b_{i2}], i = 1,2,\ldots,m\}$, which is composed of these m grey numbers $\otimes_i = [a_{i1}, b_{i2}],\ i = 1,2,\ldots,m$, must be a comparable grey number set.

Necessity: For m grey numbers $\otimes_j = [a_{j1}, b_{j2}], j = 1,2,\ldots,m$, they can compose a comparable grey number set $A(\otimes) = \{\otimes | \otimes_j = [a_{j1}, b_{j2}], j = 1,2,\ldots,m\}$.

According to Definition 2.4, any grey number $\otimes_k = [a_{k1}, b_{k2}], j = k$ of this set is either coincident with all the other grey numbers (and the obtained number is known) or their intersection, except the endpoints, is a null set, and according to Definition 2.5, this grey number $\otimes_k = [a_{k1}, b_{k2}], j = k$ is a decidable grey number.

When $k = 1,2,\ldots,m$, the situations are all the same, and therefore these m grey numbers are decidable grey numbers.

Theorem 2.2 Relationship of comparable grey number set and maximum and minimum decidable grey numbers: For a comparable grey number set $A(\otimes) = \{\otimes | \otimes_i = [a_j, b_j];\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n\}$, there must be maximum and minimum decidable grey numbers of the set; otherwise, it could not be true.

Proof: For a comparable grey number set $A(\otimes) = \{\otimes | \otimes_i = [a_j, b_j];\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n\}$, according to Definition 2.4, all the grey numbers of the set are comparable, so we can get the maximum and minimum decidable grey numbers after comparing every grey number with the others.

However, for a set $A(\otimes) = \{\otimes | \otimes_i = [a_j, b_j];\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n\}$ that has maximum and minimum decidable grey numbers, according to Definition 2.5, we can judge its maximum and minimum decidable grey numbers, but we cannot always compare the size of other grey numbers. Because of this, this set might not be a comparable grey number set.

2.1.1.3 Modeling of Standard Grey Matrix Game

In order to conveniently study the grey matrix game problem, we first consider the situation where learning is the same for the two players, which means that they will make decisions according to the same predictive grey profit and loss value matrix: the grey profit and loss value matrix is based on common knowledge. (A discussion about the situation where learning is not same for the two players will be studied in the following chapter.) In order to conveniently discuss the model and possibly keep its realistic meaning, we set some assumptions for the model discussion:

1. The game strategy sets for both players are all finite sets and are based on their mutual knowledge.
2. In the game, the preassessed price-margin matrices for both players are all interval grey matrices and are based on their mutual knowledge.
3. In the game, the real result of each game is of zero sum.
4. Both players participating in the game are rational people, and their game behaviors are likewise rational.

Definition 2.6 Grey game strategy: In the process of a grey game, the strategy used by players is called *grey game strategy.*

Definition 2.7 Rigorous standard grey game strategy: In the process of a grey game, if the game value grey set of all the players' grey game strategies is a comparable grey number set, the grey strategy in this grey game process is the rigorous standard grey game strategy.

Definition 2.7 regulates the rigorous standard grey game strategy. It shows that there is a characteristic of the grey game value of the rigorous standard grey game strategy of a player $\alpha_i (i = 1,2,\ldots,m)$ and the rigorous standard grey game strategy of another player $\beta_j (j = 1,2,\ldots,n)$; a pair of grey numbers are either coincident to each other (and the obtained number is known) or their intersection, except the endpoints, is a null set.

Definition 2.8 Rigorous standard grey game: In the process of a grey game, if strategies of the players are all rigorous standard grey game strategies, we call this game a *rigorous standard grey game.*

Definition 2.9 Rigorous standard grey matrix game: If the rigorous standard grey game is a two-player finite zero-sum game, we call this game a *rigorous standard grey matrix game*, and it is represented as $\tilde{G}^0(\otimes) = \{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$, where S_1^0 is the rigorous standard grey strategy set of the Player 1, S_2^0 is the rigorous standard grey strategy set of the Player 2, and $\tilde{A}^0(\otimes)$ is the predictive rigorous standard grey game profit and loss value matrix of Players 1 and 2.

Theorem 2.3 Comparable conditions of $\tilde{A}^0(\otimes)$: For a profit and loss value matrix $\tilde{A}^0(\otimes)$ of any rigorous standard grey matrix game $\tilde{G}^0(\otimes) = \{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$, every grey number of $\tilde{A}^0(\otimes)$ must meet the comparability condition: two arbitrary grey numbers in the matrix are either coincident to each other (and the obtained number is known) or their intersection, except the endpoints, is a null set.

Proof: Assume the game matrix of a rigorous standard grey matrix game $\tilde{G}^0(\otimes) = \{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ is shown as Eq. (2.1), that is, $\tilde{A}^0(\otimes) = A^0(\otimes)$.

According to Definition 2.6, the strategy of every player in the rigorous standard grey matrix game is the standard grey game strategy. According to Definition 2.4, in the process of a rigorous standard grey matrix game, grey game value grey numbers of Player 1's rigorous standard grey game strategy $\alpha_i^0 (i = 1, 2, \ldots, m)$ and Player 2's grey game strategy $\beta_j^0 (j = 1, 2, \ldots, n)$ are either coincident to each other (and the obtained number is known) or their intersection, except the endpoints, is a null set (that is, it meets the comparability conditions).

Example 2.2 Rigorous standard grey game matrix of Players 1 and 2 is shown as Eq. (2.4):

$$\tilde{A}^0(\otimes) = \begin{array}{c} \\ \alpha_1^0 \\ \alpha_2^0 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1^0 & \beta_2^0 & \beta_3^0 \end{array} \\ \begin{bmatrix} 1 & [2,3] & [3,4] \\ 1 & 3 & 3 \end{bmatrix} \end{array} \tag{2.4}$$

According to Eq. (2.4), we get the profit and loss value of Player 1 by a game with two rigorous standard grey strategies, α_1^0 and α_2^0, of Player 1 and three rigorous standard grey strategies, β_1^0, β_2^0, and β_3^0, of Player 2, where the grey numbers of the matrix are either coincident to each other (and the obtained number is known) or their intersection, except the endpoints, is a null set. To a determined whitenization number of Eq. (2.4), we can take it as a particular grey number; for example, whitenization number 3 can be taken as grey number [3,3].

Theorem 2.4 $\tilde{A}^0(\otimes)$ has the same form and comparability as A: Profit and loss value matrix $\tilde{A}^0(\otimes)$ of a rigorous standard grey matrix game $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ has the same form and comparability as profit and loss value matrix A of the common whitenization number matrix game $G(\otimes)=\{S_1, S_2; A\}$.

Proof: Assume $\tilde{A}^0(\otimes)$ of $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ is shown as Eq. (2.1), and A of $G(\otimes)=\{S_1, S_2; A\}$ is shown as Eq. (2.5).

$$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{2.5}$$

Equation (2.5) can be rewritten as the grey form of Eq. (2.6). By comparing Eq. (2.1) and Eq. (2.6), we find that the two matrices have the same form.

The essential difference of a common grey game matrix and a whitenization game matrix is that, from the view of the whitenization game matrix, every element of the matrix is a determined unique whitenization number, and there is no doubt that these whitenization numbers are either coincident to each other or their intersection is a null set; size comparisons of them are easy. They have a good property of comparability, and it is useful for game determination. From the view of a common grey game matrix, however, every element is the interval grey number that has endpoints that are determined unique whitenization numbers, so the grey numbers are not always coincident to each other (although the obtained number is known) or their intersection, except the endpoints, is a null set. So it is hard to compare the sizes of these grey numbers—that is, they do not have comparability, so we cannot get a game determination.

$$A'=\begin{pmatrix} [a_{11}, a_{11}] & [a_{12}, a_{12}] & \cdots & [a_{1n}, a_{1n}] \\ [a_{21}, a_{21}] & [a_{22}, a_{22}] & \cdots & [a_{2n}, a_{2n}] \\ \vdots & \vdots & \cdots & \vdots \\ [a_{m1}, a_{m1}] & [a_{m2}, a_{m2}] & \cdots & [a_{mn}, a_{mn}] \end{pmatrix} \tag{2.6}$$

According to Theorem 2.3, profit and loss value matrix $\tilde{A}^0(\otimes)$ of the rigorous standard grey matrix game $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ meets the conditions that two arbitrary grey numbers in the matrix are either coincident to each other (and the obtained number is known) or their intersection, except the endpoints, is a null set.

Therefore, $\tilde{A}^0(\otimes)$ of $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ and A of $G(\otimes)=\{S_1, S_2; A\}$ have the same property of comparability.

According to Theorem 2.4, $\tilde{A}^0(\otimes)$ of $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ has same form and comparability with A of $G(\otimes)=\{S_1, S_2; A\}$; in other words, $\tilde{A}^0(\otimes)$ is an extension or a more general form of A, and A is a particular form of $\tilde{A}^0(\otimes)$. We can consider that $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ is an extension of a more general form of $G(\otimes)=\{S_1, S_2; A\}$, and $G(\otimes)=\{S_1, S_2; A\}$ is a particular form of $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$.

In Definition 2.4, we get the concept of a rigorous standard grey matrix game; in the game of $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$, every grey element of $\tilde{A}^0(\otimes)$ meets the comparability condition, and they have a good property of comparability. Using this property, we can compare sizes of grey numbers, and we might continue to decide the solution in the sense of pure strategy. According to Theorem 2.2, however, $\tilde{A}^0(\otimes)$ needs a higher requirement. We can relax limitation of $\tilde{A}^0(\otimes)$ by considering that it does not affect deciding the solution of a grey matrix game in the sense of pure strategy.

Definition 2.10 Standard grey matrix game: For a grey matrix game $\tilde{G}^l(\otimes)=\{S_1^l, S_2^l; A^l(\otimes)\}$, there is a minimum (maximum) decidable grey number in every row (column) of $\tilde{A}^l(\otimes)$, and the maximum (minimum) decidable grey number exists in the minimum (maximum) grey number set that is gotten from every row (column). We call the grey matrix game the standard grey game strategy set, and $\tilde{A}(\otimes)$ is the standard grey profit and loss value matrix of the player 1.

According to Definitions 2.9 and 2.10, by comparing $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)\}$ and $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$, we know that the requirement of S_1, S_2, and $\tilde{A}(\otimes)$ in $\tilde{G}(\otimes)$ is much lower than the requirement of S_1^0, S_2^0, and $\tilde{A}^0(\otimes)$ in $\tilde{G}^0(\otimes)$; in fact, according to Definition 2.9, $\tilde{A}^0(\otimes)$ is a comparable grey number set [shown as Eq. (2.4)], and according to Definition 2.10, the requirement of $\tilde{A}(\otimes)$ is that it has maximum and minimum decidable grey numbers in the determined situation. $\tilde{A}(\otimes)$ is an extension of $\tilde{A}^0(\otimes)$, and $\tilde{A}^0(\otimes)$ is a particular form of $\tilde{A}(\otimes)$.

Theorem 2.5 Relationship of $\tilde{G}^0(\otimes)$ and $\tilde{G}(\otimes)$: For a rigorous standard grey matrix game $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$, it must be a standard grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)\}$; otherwise, it could not be true.

Proof: According to Definition 2.9, for a rigorous standard grey matrix game $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$, $\tilde{A}^0(\otimes)$ is a comparable grey number set (shown in Definition 2.4); however, according to Definition 2.10, for a standard grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)\}$, the requirement of $\tilde{A}(\otimes)$ is that it has maximum and minimum decidable grey numbers in the determined situation (shown as Definition 2.5).

According to Theorem 2.3, any $\tilde{A}^0(\otimes)$ must meet the requirement of Definition 2.10, so $\tilde{G}^0(\otimes)=\{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$ must be $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)\}$; otherwise, it could not be true.

2.1.2 Solution of a Simple (Standard) Grey Matrix Game Model Based on Pure Strategy

2.1.2.1 Concept of Pure Strategy Solution of a Standard Grey Matrix Game

The standard grey matrix $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ is an extension of the classical matrix game $G=\{S_1,S_2;A\}$. When we define the concept of a pure strategy solution for a standard grey matrix, we use the basic assumption of rational behavior in the classic matrix game G. In the process of a standard gray matrix game $\tilde{G}(\otimes)$, if both players have no flukes or adventures and each considers that the rival will try to make his opponent own the least, then each will surely try to make decisions based on the most profitable situation from a set of mutually worst situations. This is the safest and most acceptable manner. Hence, we get the concept of the pure strategy solution of a standard grey matrix game $\tilde{G}(\otimes)$, as shown in Definition 2.11.

Definition 2.11 Pure strategies solution of standard grey matrix game: For a standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, if there are pure strategies like α_{i^*},β_{j^*} that are composed of situation $(\alpha_{i^*},\beta_{j^*})$, this can make $[a_{ij^*},b_{ij^*}]\le[a_{i^*j^*},b_{i^*j^*}]\le[a_{i^*j},b_{i^*j}]$ come into existence, where $i=1,2,\ldots,m$ and $j=1,2,\ldots,n.$; situation $(\alpha_{i^*},\beta_{j^*})$ is called the solution of a standard grey matrix game in the sense of pure strategy.

α_{i^*},β_{j^*} is the grey optimum pure strategy of Player 1 and Player 2, respectively.

The defrayal of Player 1 $[a_{i^*j^*},b_{i^*j^*}]$ is called the value of a standard grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, signed as $V_G(\otimes)$.

According to Definition 2.11, we know that the standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ must have a pure strategy α_{i^*},β_{j^*} if the solution would exist in the sense of pure strategy. The corresponding number of rows i^* and number of columns j^* in the strategy α_{i^*},β_{j^*} can make the game value $[a_{i^*j^*},b_{i^*j^*}]$ fit for $[a_{ij^*},b_{ij^*}]\le[a_{i^*j^*},b_{i^*j^*}]\le[a_{i^*j},b_{i^*j}]$, where $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. The positive integer (i^*,j^*) is called the grey saddle point of a grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ because the game value of this saddle point is a grey number.

The game composed of $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ is the grey matrix game with a grey saddle point. The grey profit and loss value matrix $\tilde{A}(\otimes)$ must therefore have a grey saddle point if the grey matrix game has a solution in the sense of pure strategy. Otherwise, if the grey profit and loss value matrix $\tilde{A}(\otimes)$ has a grey saddle point (i^*,j^*), the solution of the grey matrix game is the grey situation $(\alpha_{i^*},\beta_{j^*})$, which is composed of optimal grey pure strategies α_{i^*},β_{j^*} corresponding to grey saddle point (i^*,j^*).

The grey game value $V_G(\otimes)$ is just $[a_{i^*j^*},b_{i^*j^*}]$, which is in the corresponding grey saddle point (i^*,j^*) in the grey profit and loss value matrix $\tilde{A}(\otimes)$. It is the smallest grey number in i^* rows and the biggest in j^* columns of grey matrix $\tilde{A}(\otimes)$.

Table 2.1 Decision Table of Grey Matrix Game

	β_1	β_2	β_3	$\min_j[a_{ij},b_{ij}]$
α_1	[−1,1]	[2,3]	[3,4]	[−1,1]
α_2	[1,2]	[1,2]	[4,5]	[1,2]*
$\max_i[a_{ij},b_{ij}]$	[1,2]*	[2,3]	[4,5]	

We can investigate some solutions of a simple grey matrix game to explain the above point. The grey profit and loss value matrix $\tilde{A}(\otimes)$ of a standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ has grey elements $[a_{ij},b_{ij}]$; (i,j) is the grey saddle point of $\tilde{A}(\otimes)$ and $[a_{ij},b_{ij}]$ is game value of grey matrix $\tilde{A}(\otimes)$. α_i is the optimal pure strategy of Player 1, β_j is optimal pure strategy of Player 2, and $(\alpha_{i^*},\beta_{j^*})$ is the solution of grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$.

Example 2.3 The profit and loss value matrix of a standard grey game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ of Players 1 and 2 is shown in Eq. (2.7); and in the second row of $\tilde{A}(\otimes)$, the obtained numbers of grey number [1,2] are the same:

$$\tilde{A}(\otimes)=\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array}\begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} [-1,1] & [2,3] & [3,4] \\ [1,2] & [1,2] & [4,5] \end{bmatrix} \end{array} \tag{2.7}$$

From Eq. (2.7), we can get results shown in Table 2.1. Hence, $\max_i \min_j [a_{ij},b_{ij}]=\min_j \max_i [a_{ij},b_{ij}]=[a_{21},b_{21}]=[1,2]$.

The game value of the grey matrix $V_G(\otimes)=[1,2]$ solution of the grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ is (α_2,β_1), where α_2 and β_1 are the grey optimal pure strategies of Player 1 and Player 2 respectively.

2.1.2.2 The Sufficient and Necessary Term and the Property of Pure Strategy Solution of Standard Grey Matrix Game

From Example 2.3, we know that whether the standard grey matrix game has a solution in the sense of pure strategy (or the grey profit and loss value matrix $\tilde{A}(\otimes)$ has a grey saddle point) is directly related to whether elements of the grey profit and loss value matrix $\tilde{A}(\otimes)$ meets the conditions $\max_i \min_j [a_{ij},b_{ij}]=\min_j \max_i [a_{ij},b_{ij}]$. We will extend the fact to a more general situation.

Theorem 2.6 Profit and loss relationship of players: In a standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, $\tilde{A}(\otimes)=\{[a_{ij},b_{ij}]\}$ is an $m\times n$ order grey matrix, then $\max_i \min_j [a_{ij},b_{ij}]\le\min_j \max_i [a_{ij},b_{ij}]$.

Proof: For every $i = 1, 2, \ldots, m$; there must be a $\min_j [a_{ij}, b_{ij}] \le \max_i [a_{ij}, b_{ij}]$; that is. $\min_j b_{ij} \le \max_i a_{ij}$. Hence, $\max_i \min_j [a_{ij}, b_{ij}] \le \max_i [a_{ij}, b_{ij}]$; that is, $\max_i \min_j b_{ij} \le \max_i a_{ij}$.

We get $\max_i \min_j [a_{ij}, b_{ij}] \le \min_j \max_i [a_{ij}, b_{ij}]$; that is, $\max_i \min_j b_{ij} \le \min_j \max_i a_{ij}$.

The significance of Theorem 2.6 is that it demonstrates that the minimum profit of Player 1 is less than maximum loss of Player 2.

In a following section, we will discuss the existence of the sufficient and necessary terms for a standard grey matrix game solution in the sense of pure strategy.

Theorem 2.7 The sufficient and necessary conditions of a pure strategy solution: For a standard grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, the sufficient and necessary conditions that having a solution (or that the grey profit and loss value matrix has a saddle point) in the sense of pure strategy are $\max_i \min_j [a_{ij}, b_{ij}] = \min_j \max_i [a_{ij}, b_{ij}]$.

Proof: Sufficiency: Let $\min_j [a_{ij}, b_{ij}]$ be the biggest position where $i = i^*$ and let $\max_i [a_{ij}, b_{ij}]$ be the smallest where $j = j^*$. Then

$$\left.\begin{aligned} \min_j [a_{i^*j}, b_{i^*j}] &= \max_i \quad \min_j [a_{ij}, b_{ij}] \\ \max_i [a_{ij^*}, b_{ij^*}] &= \min_j \quad \max_i [a_{ij}, b_{ij}] \end{aligned}\right\} \tag{2.8}$$

From $\max_i \min_j [a_{ij}, b_{ij}] = \min_j \max_i [a_{ij}, b_{ij}]$, we get

$$\min_j [a_{i^*j}, b_{i^*j}] = \max_i [a_{ij^*}, b_{ij^*}] \tag{2.9}$$

According to the definition of minimum, $\min_j [a_{i^*j}, b_{i^*j}] \le [a_{i^*j^*}, b_{i^*j^*}]$, and $\max_i [a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}]$.

From the definition of maximum, we get that for any $i = 1, 2, \ldots, m$, there is $a_{ij^*} \le a_{i^*j^*}$.

From the definition of maximum, we get that

$$\max_i [a_{ij^*}, b_{ij^*}] \ge a_{i^*j^*} \tag{2.10}$$

From Eqs. (2.9) and (2.10), we obtain $\min_j [a_{i^*j}, b_{i^*j}] \ge [a_{i^*j^*}, b_{i^*j^*}]$.

According to the definition of minimum, for any $j = 1, 2, \ldots, n$, there is $[a_{i^*j}, b_{i^*j}] \ge [a_{i^*j^*}, b_{i^*j^*}]$. So for any $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$, we get $[a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le [a_{i^*j}, b_{i^*j}]$.

By Definition 2.11, we know that the grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ has its solution $(\alpha_{i^*}, \beta_{j^*})$.

Necessity: If the grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ has a solution, from Definition 2.11 we know that there is a $\alpha_{i^*}, \beta_{j^*}$ that can make $[a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le [a_{i^*j}, b_{i^*j}]$ reasonable for any $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

Therefore,

$$\max_i [a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le \min_j [a_{i^*j}, b_{i^*j}] \tag{2.11}$$

Because of $\min_j \max_i [a_{ij}, b_{ij}] \le \max_i [a_{ij^*}, b_{ij^*}]$, $\min_j [a_{i^*j}, b_{i^*j}] \le \max_i \min_j [a_{ij}, b_{ij}]$, from Eq. (2.11)

$$\min_j \max_i [a_{ij}, b_{ij}] \le \max_i \min_j [a_{ij}, b_{ij}] \tag{2.12}$$

and from the definitions of maximum and minimum, we get $\min_j [a_{ij}, b_{ij}] \le [a_{ij}, b_{ij}] \le \max_i [a_{ij}, b_{ij}]$, then

$$\min_j [a_{ij}, b_{ij}] \le \min_j \max_i \tag{2.13}$$

Choosing the maximum of i from both sides of Eq. (2.13), we get

$$\max_i \min_j [a_{ij}, b_{ij}] \le \min_j \max_i [a_{ij}, b_{ij}] \tag{2.14}$$

From Eqs. (2.12) and (2.13), we get $\max_i \min_j [a_{ij}, b_{ij}] = \min_j \max_i [a_{ij}, b_{ij}]$

Inference 2.1 If grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ has a solution $(\alpha_{i^*}, \beta_{j^*})$ in the sense of pure strategy, then the game value $V_G(\otimes) = [a_{i^*j^*}, b_{i^*j^*}]$ must meet the condition $[a_{i^*j^*}, b_{i^*j^*}] = \max_i \min_j [a_{ij}, b_{ij}] = \min_j \max_i [a_{ij}, b_{ij}]$.

Proof: From Definition 2.11, we get

$$[a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le [a_{i^*j}, b_{i^*j}], i = 1, 2, \ldots, m; j = 1, 2, \ldots, n \tag{2.15}$$

Choosing the maximum and minimum from both sides of Eq. (2.15), then $\max_i [a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le \min_j [a_{i^*j}, b_{i^*j}]$, and according to Eq. (2.8), we get

$$\min_j \max_i [a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le \max_i \min_j [a_{i^*j}, b_{i^*j}] \tag{2.16}$$

The grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ has a solution, and according to Theorem 2.7, we know that both sides of Eq. (2.16) must be equal, so $[a_{i^*j^*}, b_{i^*j^*}] = \max_i \min_j [a_{ij}, b_{ij}] = \min_j \max_i [a_{ij}, b_{ij}]$.

Theorem 2.7 explains the sufficient and necessary term that shows the existence of the standard grey matrix game solution in the sense of pure strategy. This theorem can be restated with the concept of grey saddle point: the sufficient and necessary term that a standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ has a solution is that there must be at least one grey saddle point. However, there might be more than one saddle point.

Theorem 2.8 Indifference and interchangeability of grey saddle points: Assuming that $(i_{k_1},j_{k_1}),(i_{k_2},j_{k_2})$ are grey saddle points of the standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, then $(i_{k_1},j_{k_2}),(i_{k_2},j_{k_1})$ are grey saddle points of the grey matrix game, and game values in the grey saddle points are the same; that is,

$$\left[a_{i_{k_1}j_{k_1}},b_{i_{k_1}j_{k_1}}\right]=\left[a_{i_{k_2}j_{k_2}},b_{i_{k_2}j_{k_2}}\right]=\left[a_{i_{k_1}j_{k_2}},b_{i_{k_1}j_{k_2}}\right]=\left[a_{i_{k_2}j_{k_1}},b_{i_{k_2}j_{k_1}}\right]$$

Proof: Because $(i_{k_1},j_{k_1}),(i_{k_2},j_{k_2})$ are grey saddle points of the game, we get

$$\left[a_{ij_{k_1}},b_{ij_{k_1}}\right]\le\left[a_{i_{k_1}j_{k_1}},b_{i_{k_1}j_{k_1}}\right]\le\left[a_{i_{k_1}j},b_{i_{k_1}j}\right];\quad \left[a_{ij_{k_2}},b_{ij_{k_2}}\right]\le\left[a_{i_{k_2}j_{k_2}},b_{i_{k_2}j_{k_2}}\right]\le\left[a_{i_{k_2}j},b_{i_{k_2}j}\right],$$

$i=1,2,\ldots,m;j=1,2,\ldots,n,$ where k_1 and k_2 are codes of the grey saddle points, respectively.

So, $[a_{i_{k_1}j_{k_2}},b_{i_{k_1}j_{k_2}}]\le[a_{i_{k_2}j_{k_2}},b_{i_{k_2}j_{k_2}}]\le[a_{i_{k_2}j_{k_1}},b_{i_{k_2}j_{k_1}}]\le[a_{i_{k_1}j_{k_1}},b_{i_{k_1}j_{k_1}}]\le[a_{i_{k_1}j_{k_2}},b_{i_{k_1}j_{k_2}}]$.

Then, $[a_{i_{k_1}j_{k_1}},b_{i_{k_1}j_{k_1}}]=[a_{i_{k_2}j_{k_2}},b_{i_{k_2}j_{k_2}}]=[a_{i_{k_1}j_{k_2}},b_{i_{k_1}j_{k_2}}]=[a_{i_{k_2}j_{k_1}},b_{i_{k_2}j_{k_1}}]$.

And we get $[a_{ij_{k_2}},b_{ij_{k_2}}]\le[a_{i_{k_1}j_{k_2}},b_{i_{k_1}j_{k_2}}]\le[a_{i_{k_1}j},b_{i_{k_1}j}]$; $i=1,2,\ldots,m;j=1,2,\ldots,n$.

Then (i_{k_1},j_{k_2}) is a grey saddle point of grey game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$. We can prove that (i_{k_2},j_{k_1}) is a grey saddle point in the same way.

Theorem 2.8 explains that a grey saddle point of the grey matrix game has two properties: (1) interchangeability of a grey saddle point; that is, if $(\alpha_{i_{k_1}},\beta_{j_{k_1}})$ and $(\alpha_{i_{k_2}},\beta_{j_{k_2}})$ are solutions of grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, then $(\alpha_{i_{k_1}},\beta_{j_{k2}})$ and $(\alpha_{i_{k_2}},\beta_{j_{k_1}})$ are solutions too; and (2) indifference of grey saddle point; that is, if $(\alpha_{i_{k_1}},\beta_{j_{k_1}})$ and $(\alpha_{i_{k_2}},\beta_{j_{k2}})$ are solutions of grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, then the grey game value $V_G(\otimes)=[a_{i_{k_1}j_{k_1}},b_{i_{k_1}j_{k_1}}]=[a_{i_{k_2}j_{k_2}},b_{i_{k_2}j_{k_2}}]$.

2.1.2.3 Relationship between Pure Strategies of Standard and Rigorous Standard Grey Matrix Games

We have proved that the rigorous standard grey matrix game $\tilde{G}^0(\otimes)=\{S_1^0,S_2^0;\tilde{A}^0(\otimes)\}$ is a particular form of the standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, and $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$ is an extension of $\tilde{G}^0(\otimes)=\{S_1^0,S_2^0;\tilde{A}^0(\otimes)\}$. We will now prove that the pure strategy solution of $\tilde{G}(\otimes)$ is the pure strategy solution of $\tilde{G}^0(\otimes)$.

Lemma 2.1 Position of pure strategy solution: In any standard grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)\}$, if a solution in the sense of pure strategy is in existence, then we can get its game value (grey number) from the minimum-maximum (maximum-minimum) decidable grey number in the row (column).

Proof: Assume the grey game matrix $\tilde{A}(\otimes)$ of $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)\}$ is as shown in Eq. (2.11):

$$\tilde{A}(\otimes)=\begin{array}{cc}
 & \underset{j}{\min}[a_{ij},b_{ij}] \\
\left[\begin{array}{cccc}
[a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] \\
[a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] \\
\cdots & \cdots & \cdots & \cdots \\
[a_{m1},b_{m1}] & [a_{m2},b_{m2}] & \cdots & [a_{mn},b_{mn}]
\end{array}\right] & \begin{array}{c} [a_{1q},b_{1q}] \\ [a_{2r},b_{2r}] \\ \vdots \\ [a_{ms},b_{ms}] \end{array} \\
\begin{array}{cccc} \underset{i}{\max}\{[a_{ij},b_{ij}] & [a_{v2},b_{v2}] & \cdots & [a_{wn},b_{wn}]\} \end{array} &
\end{array} \tag{2.17}$$

According to Eq. 2.17, if the sequence

$$\underset{j}{\min}[a_{ij},b_{ij}]=\underset{i\in q,r,\ldots,s}{\min}\{[a_{1q},b_{1q}],[a_{2r},b_{2r}],\ldots,[a_{ms},b_{ms}]\} \tag{2.18}$$

is the minimum decidable grey number in every row of $\tilde{A}(\otimes)$; and the sequence

$$\underset{i}{\max}[a_{ij},b_{ij}]=\underset{i\in u,v,\ldots,w}{\max}\{[a_{u1},b_{u1}],[a_{vr},b_{vr}],\ldots,[a_{wn},b_{wn}]\} \tag{2.19}$$

is the maximum decidable grey number in every column of $\tilde{A}(\otimes)$, then according to Definition 2.1 and Theorem 2.3, we know that game value $V_G(\otimes)$ of grey matrix game $\tilde{G}(\otimes)$ in the sense of pure strategy must be the maximum of Eq. (2.18) and the minimum of Eq. (2.19), and the two are equal, that is: $\max_i \min_j [a_{ij},b_{ij}]=\min_j \max_i [a_{ij},b_{ij}]$, and then

$$\begin{aligned} V_G(\otimes) &= \underset{i\in 1,2,\cdots,m}{\max}\ \underset{i\in q,r,\cdots,s}{\min}\{[a_{1q},b_{1q}],[a_{2r},b_{2r}],\ldots,[a_{ms},b_{ms}]\} \\ &= \underset{j\in 1,2,\cdots,n}{\min}\ \underset{i\in u,v,\cdots,w}{\max}\{[a_{u1},b_{u1}],[a_{vr},b_{vr}],\ldots,[a_{wn},b_{wn}]\} \end{aligned}$$

In this way, the pure strategy $\alpha_{i^*}, \beta_{j^*}$, which corresponds to row and column i^*, j^*, where the game values $V_G(\otimes)$ are called the grey optimal pure strategies of Players 1 and 2, respectively, the situation $(\alpha_{i^*}, \beta_{j^*})$ is the solution of the standard grey matrix game in the sense of pure strategy.

Hence, if $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ has a solution in the sense of pure strategy, we can get game value (grey number) from minimum-maximum (maximum-minimum) decidable grey number in the row (column).

Theorem 2.9 Relationship between pure strategies of $\tilde{G}(\otimes)$ and $\tilde{G}^0(\otimes)$: A pure strategy solution of the standard grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ is surely a pure strategy solution of the rigorous standard grey matrix game $\tilde{G}^0(\otimes) = \{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$.

Proof: Assume the grey game matrix $\tilde{A}(\otimes)$ of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ is as shown in Eq. (2.11).

According to Lemma 2.1, if $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ has a solution in the sense of pure strategy. We can get its game value (grey number) from the minimum-maximum (maximum-minimum) decidable grey number in the row (column); the pure strategy $\alpha_{i^*}, \beta_{j^*}$, which corresponds with row and column i^*, j^*, where the game values $V_G(\otimes)$ are called grey optimal pure strategy of Player 1 and Player 2, respectively. The situation $(\alpha_{i^*}, \beta_{j^*})$ is the solution of a standard grey matrix game in the sense of pure strategy.

Is the pure strategy solution of $\tilde{G}(\otimes)$ the pure strategy solution of $\tilde{G}^0(\otimes)$?

We consider the problem where a standard grey game matrix $\tilde{A}(\otimes)$ [as shown in Eq. (2.17)] is not a rigorous standard grey matrix game matrix $\tilde{A}^0(\otimes)$. There must be some elements in $\tilde{A}(\otimes)$ where the relationship between these elements cannot meet the requirement of a decidable grey number. These grey numbers are either totally coincident (the obtained numbers are known) or their intersection is a zero set.

We can determine that in $\tilde{A}(\otimes)$, these undetermined grey numbers are not grey elements in the set of minimum grey numbers in every row and column in $\tilde{A}(\otimes)$ [Eq. (2.17), and they are not grey elements in the set of maximum grey numbers in every row and column in $\tilde{A}(\otimes)$ [Eq. 2.11]; otherwise, it contradicts the definition of a standard grey game matrix.

We can use some way to transform $\tilde{A}(\otimes)$, into $\tilde{A}^0(\otimes)$,and this will not change the solution of the former standard grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$.

To sum up, a pure strategy solution of the standard grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ is surely a pure strategy solution of the rigorous standard grey matrix game $\tilde{G}^0(\otimes) = \{S_1^0, S_2^0; \tilde{A}^0(\otimes)\}$.

The significance of Theorem 2.9 is that it explains, in the sense of pure strategy, a pure strategy solution of standard grey matrix game $\tilde{G}(\otimes)$, which is a lower requirement of the players, is the same with a pure strategy solution of the rigorous standard grey matrix game $\tilde{G}^0(\otimes)$. This means that if we want to judge whether there is a solution to a grey matrix game in the sense of pure strategy, we just need to determine the strategy of the standard grey matrix game, and we do not need to accurately determine the strategy of the rigorous standard grey matrix game.

To sum up the above studies, we know that the relationship between a classical whitenization number matrix, and standard and rigorous standard grey matrix games is that the former is a special case of the latter, and the latter is an more general extension of the former. If we can get a pure strategy solution to the standard grey matrix game $\tilde{G}(\otimes)$, we can get the pure strategy solution of the rigorous standard grey matrix game $\tilde{G}^0(\otimes)$. We also know that no matter what the obtained number of every grey element (where all the obtained number of elements in $\tilde{A}(\otimes)$ are a certain number, game problem $\tilde{G}(\otimes)$ becomes a classical matrix game problem G), and the game value is $V_G(\otimes)=[a_{i^*j^*},b_{i^*j^*}]$, where $[a_{i^*j^*},b_{i^*j^*}]=\max_i \min_j [a_{ij},b_{ij}]=\min_j \max_i [a_{ij},b_{ij}]$.

2.2 Study of a Pure Strategy Solution of a Grey Matrix Game Based on Unilateral Strategy Analysis

2.2.1 Analysis of Grey Game Revenue Matrix

Theorem 2.10 The matrix consisting of the extracted left terminal points from the grey game matrix $\tilde{A}^k, k=0,1,2$ is called the game decision-making matrix that is most conservative for Player A and most optimistic for Player B, while the matrix consisting of extracted right terminal points is called the game decision-making matrix most optimistic for A and most conservative for B.

Proof: A grey matrix $\tilde{A}^k, k=0,1,2$ is made up of regional grey numbers. If there are whitened elements, we can make each a grey number whose left and right terminal points are the same whitened number and thus change them into regional grey numbers. The regional grey numbers have the property that $a_{ij}^k \le b_{ij}^k, (k=0,1,2; i=1,2,\ldots,m; j=1,2,\ldots,n)$ —that is, each regional grey number's left terminal point has a smaller value than the right terminal point.

We can extract the left terminal points of all the elements in the grey game matrix ($\tilde{A}^k, k=0,1,2$) to form a game decision-making matrix whose right terminal points form another matrix that is the most conservative for Player A and most optimistic for Player B, and most optimistic for A and most conservative for B, respectively.

Considering the limited length of this chapter, we will only discuss the kind of matrix as shown in Eq. (2.1) and we will discuss the matrices in Eqs. (2.2) and (2.3) in other chapters.

2.2.2 Model Construction

In a grey matrix game, where each player knows that his rival is also rational and will try to make him get the least rather than expecting any fluke, he will then choose the best from the most disadvantageous outcomes that are probable for him and will make his decisions based upon those choices. This is the sensible behavior and is also a safe way that both players can accept in reality. According to this hypothesis of sensible behavior, the players can only make decisions based on the matrix of the most conservative values in grey matrix games.

Definition 2.12 Assume that $\tilde{G} = \{S_1, S_2; \tilde{A}\}$ is a grey matrix game, where $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}, \tilde{A} = \{a_{ij}, b_{ij}\}_{m\times n}$; if Players A and B both make game decisions according to their respective most conservative value matrix $\tilde{A}^1 = \{a_{ij}\}_{m\times n}$ and $\tilde{A}^2 = \{b_{ij}\}_{m\times n}$, then the game decisions are called the unilateral conservative game decisions of A and B, expressed as $\tilde{G}^1 = \{S_1, S_2; \tilde{A}^1\}$and $\tilde{G}^2 = \{S_1, S_2; \tilde{A}^2\}$ respectively.

Definition 2.13 Assume that $\tilde{G}^1 = \{S_1, S_2; \tilde{A}^1\}$ and $\tilde{G}^2 = \{S_1, S_2; \tilde{A}^2\}$ are the unilateral conservative game decisions of Players A and B respectively in the process of making the players' unilateral conservative game decision, and

$$\max_i \ \min_j a_{ij} = \min_i \ \max_j a_{ij} = a_{i^*j^*} \tag{2.20}$$

$$\max_i \ \min_j b_{ij} = \min_i \ \max_j b_{ij} = b_{i^*j^*} \tag{2.21}$$

If Eqs. (2.20) and (2.21) are true, and we also assume that $V_G^1 = a_{i^*j^*}$, $V_G^2 = b_{i^*j^*}$, and thus call V_G^1 and V_G^2 the value of games $\tilde{G}^1$ and $\tilde{G}^2$ respectively, then the pure conditions $(\alpha^1_{i^*}, \beta^1_{j^*})$ and $(\alpha^2_{i^*}, \beta^2_{j^*})$ that satisfy Eqs. (2.20) and (2.21) respectively are called the solutions (or balance conditions) of games $\tilde{G}^1$ and $\tilde{G}^2$ with the pure strategic sense, respectively. $\alpha^1_{i^*}$, $\beta^1_{j^*}$ and $\alpha^2_{i^*}$, $\beta^2_{j^*}$ are the best pure strategies of Player A of game $\tilde{G}^1$ and Player B of game $\tilde{G}^2$.

Theorem 2.11 Assume that $\tilde{G} = \{S_1, S_2; \tilde{A}\}$ is a grey matrix game. Then for the unilateral conservative game decisions $\tilde{G}^k, k = 1,2$ of Players A and B, where $\tilde{G}^1$ stands for the unilaterally conservative decisions of Player A and $\tilde{G}^2$ stands for the unilaterally conservative decisions of Player B, the sufficient and necessary conditions for a solution with the pure strategic sense is that there exist pure conditions $(\alpha^k_{i^*}, \beta^k_{j^*}), k = 1,2$ under which for any $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,m$, the inequality $a^k_{ij^*} \le a^k_{i^*j^*} \le a^k_{i^*j}$ in which $k = 1,2$ is true.

Theorem 2.11 mainly explains the sufficient and necessary conditions for a solution to $\tilde{G}^k, k = 1,2$ of pure strategic sense when each player of the grey matrix game $\tilde{G} = \{S_1, S_2; \tilde{A}\}$ takes unilaterally conservative game decisions. (See Theorem 1 of Ref. 5 for a similar proof of this theorem.)

Theorem 2.12 The sufficient and necessary conditions for an overall condition (integrated) best solution to the grey matrix game $\tilde{G} = \{S_1, S_2; \tilde{A}\}$ of pure strategic sense are as follows.

The players' single unilateral conservative game decisions are solvable under the pure strategic sense. From each unilateral conservative game decision $\tilde{G}^k, k = 1, 2$, Players A and B can acquire their own best grey pure strategy, $\alpha^1_{i^*}$ and $\beta^2_{j^*}$, respectively.

Proof of this theorem is divided into two steps.

First, we prove the sufficiency of the theorem. We assume that the grey revenue matrix of the grey matrix game $\tilde{G} = \{S_1, S_2; \tilde{A}\}$ is expressed as

$$\tilde{A} = \begin{bmatrix} [a_{11}, b_{11}] & [a_{12}, b_{12}] & \cdots & [a_{1n}, b_{1n}] \\ [a_{21}, b_{21}] & [a_{22}, b_{22}] & \cdots & [a_{2n}, b_{2n}] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{m1}, b_{m1}] & [a_{m2}, b_{m2}] & \cdots & [a_{mn}, b_{mn}] \end{bmatrix} \tag{2.22}$$

where $a_{ij} \le b_{ij}$ $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

From Eq. (2.22) we can get the conservative value matrix $\tilde{B}^1$ and $\tilde{B}^2$ of Players A and B respectively as

$$\tilde{B}^1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{2.23}$$

$$\tilde{B}^2 = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} \tag{2.24}$$

Judging from the known conditions, Player A can acquire his best grey pure strategy α_{i^*} and β_{j^0}, $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$ from Eq. (2.23) with corresponding game values $V_G^1 = a_{i^* j^0}$; Player B can also acquire his best strategy α_{k^0} and β_{l^*}, $k = 1, 2, \ldots, m; l = 1, 2, \ldots, n$ from Eq. (2.24) with game values of $V_G^2 = b_{k^0 l^*}$.

With the best grey pure strategy of each player as α_{i^*} and β_{l^*}, where $i = 1, 2, \ldots, m; l = 1, 2, \ldots, n$, we get a grey overall pure condition $(\alpha_{i^*}, \beta_{l^*})$ and the game values under this condition $[a_{i^* l^*}, b_{i^* l^*}]$ for the grey revenue matrix Eq. (2.22), as shown in Table 2.2.

Table 2.2 Best Unilateral and Overall Conservative Solutions of Grey Matrix Game $\tilde{G}=\{S_1,S_2;\tilde{A}\}$ of Pure Strategy

B A	β_1	β_2	…	β_{j^0}	…	β_{l^*}	…	β_n	Min
α_1	$[a_{11}, b_{11}]$	$[a_{11}, b_{11}]$	…	$[a_{1j^0}, b_{1j^0}]$	…	$[a_{1l^*}, b_{1l^*}]$	…	$[a_{1n}, b_{1n}]$	..
α_2	$[a_{21}, b_{21}]$	$[a_{22}, b_{22}]$	…	$[a_{2j^0}, b_{2j^0}]$	…	$[a_{2l^*}, b_{2l^*}]$	…	$[a_{2n}, b_{2n}]$	..
…	…	…	…	…	…	…	…	…	..
α_{s^0}	$[a_{s^01}, b_{s^01}]$	$[a_{s^02}, b_{s^02}]$	…	$[a_{s^0j^0}, b_{s^0j^0}]$	…	$\underline{[a_{s^0l^*}, b_{s^0l^*}]}$	…	$[a_{s^0n}, b_{s^0n}]$	$\underline{[a_{s^0l^*}, b_{s^0l^*}]^{\underline{*}}}$
…	…	…	…	…	…	…	…	…	..
α_{i^*}	$[a_{i^*1}, b_{i^*1}]$	$[a_{i^*2}, b_{i^*2}]$	…	$\underline{[a_{i^*j^0}, b_{i^*j^0}]}$	…	$\boxed{[a_{i^*l^*}, b_{i^*l^*}]}$	…	$[a_{i^*n}, b_{i^*n}]$	$\underline{[a_{i^*j^0}, b_{i^*j^0}]^{\underline{*}}}$ $\boxed{[a_{i^*l^*}, b_{i^*l^*}]^{**}}$
…	…	…	…	…	…	…	…	…	..
α_m	$[a_{m1}, b_{m1}]$	$[a_{m2}, b_{m2}]$	…	$[a_{mj^0}, b_{mj^0}]$	…	$[a_{ml^*}, b_{ml^*}]$	…	$[a_{mn}, b_{mn}]$	..
Max	..	..	..	$\underline{[a_{i^*j^0}, b_{i^*j^0}]^{\underline{*}}}$	..	$\underline{[a_{s^0l^*}, b_{s^0l^*}]^{\underline{*}}}$ $\boxed{[a_{i^*l^*}, b_{i^*l^*}]^{**}}$	..	..	..

In Table 2.2, the grey numbers underlined and noted with an asterisk (*) are the unilaterally conservative solutions for the players, and the shaded and those with a double asterisk (**) are the game solutions for grey overall pure conditions.

Table 2.2 shows that, for the grey matrix game $\tilde{G}=\{S_1,S_2;\tilde{A}\}$, the game value $a_{i^*l^*}$ in $[a_{i^*l^*}, b_{i^*l^*}]$ of its grey overall pure condition is an improvement on the pure strategic conservative solution $a_{i^*j^0}$ of Player A. Player A will be satisfied with solution $a_{i^*l^*}$ until more game information becomes clearer or new information can improve the solution. Similarly, for Player B, the game value $b_{i^*l^*}$ in $[a_{i^*l^*}, b_{i^*l^*}]$ of the grey overall pure condition is an improvement on the pure strategic conservative solution $b_{s^0l^*}$. Player B will be satisfied with solution $b_{i^*l^*}$ until other game information gets clearer or new information can improve the solution. For the grey matrix game $\tilde{G}=\{S_1,S_2;\tilde{A}\}$, the gaming solution $[a_{i^*l^*}, b_{i^*l^*}]$ of the grey overall pure condition is like a saddle point called the grey saddle point under grey overall pure conditions.

Second, we prove the necessity of the theorem.

If the grey matrix game $\tilde{G} = \{S_1, S_2; \tilde{A}\}$ has $[a_{i^*l^*}, b_{i^*l^*}]$ as the best overall (integrated) solution of pure strategy—that is, there exists a grey saddle point of grey overall pure condition—then, for Player A, according to the above analysis, the grey overall pure strategy solution must be greater than or equal to his unilaterally conservative pure strategic solution α_{i^*}, that is, $\min\{a_{ij}\} = a_{i^*j^0} \le a_{i^*l^*}$, $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$.

Similarly, for Player B, the solution to unilaterally conservative pure strategy β_{l^*} is gotten by $\min\{b_{ij}\} = b_{s^0l^*} \le b_{i^*l^*}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$.

In this way, based on the unilateral conservative value matrix, we finish the demonstration by getting unilaterally conservative solutions of Players *A* and *B*, pure strategically, expressed as $[a_{i^*j^0}, b_{i^*j^0}]$ and $[a_{s^0l^*}, b_{s^0l^*}]$.

The demonstration for Theorem 2.12 is constructive; it explains the process of solving the problem of the grey matrix two-player finite zero-sum games based on pure strategy.

2.2.3 Case Study

A purchasing agent of a certain unit makes decisions on the coal stock in autumn for heating in the following winter. It is known that minimum and maximum coal consumption are 15 tons and 17 tons for a normal winter season, 10 and 11 tons for a warmer winter, and 20 and 22 tons for a colder winter, respectively. Assume that coal prices in winter change with the extent of cold, which are 10–11, 15–16, and 20–22 Yuan per ton for the warmer, normal, and colder climates respectively, and assume that the coal price in autumn is 10 Yuan per ton. What stock should the agent make in autumn in order to keep the lowest expense of the unit, on the condition that there is no exact winter weather forecast for the upcoming season?

This stock problem can be taken as a game problem. Taking the purchasing agent as Player *A*, he has three strategies (expressed as grey numbers): purchasing [10,11], [15,16], or [20, 22] tons in autumn, expressed as α_1, α_2, and α_3 respectively.

Taking Nature as Player B (which might be dealt with as a rational player), it (the winter climate) has three strategies too: being warmer (the coal price is [10,11] for each ton), being normal ([15,16]), or being colder ([20,22]), expressed as β_1, β_2, and β_3, respectively.

Make the actual expense of coal for winter heating (which is the total expense of the autumn purchase and the winter replenishment) stand for the revenue of Player *A*, and we get the matrix shown in Table 2.3.

In Table 2.3, the grey numbers with an asterisk (*) are the unilaterally conservative solutions of the players, while those with a double asterisk (**) are the game solutions to the grey overall pure conditions.

The process to get the grey revenue matrix in Table 2.3 is as follows:

For a warmer winter (when Player *B* chooses strategy β_1), by choosing strategy α_1, Player *A* has the revenue - $\{[10,11] \cdot [10,11]\} = [-121,-100]$.

Table 2.3 Grey Revenue Matrix of the Unit's Winter Heating and the Grey Matrix Game

A \ B	β_1	β_2	β_3	*Min*
α_1	[–121,–100]	[–233,–160]	[–385,–280]	[–385,–280]
α_2	[–170,–150]	[–170,–150]	[–324,–210]	[–324,–210]
α_3	[–220,–200]	[–220,–200]	[–220,–200]	$[-220^*,-200^*]^{**}$
Max	[–121,–100]	[–170,–150]	$[-220^*,-200^*]^{**}$	

For a normal winter (when Player *B* chooses strategy β_2), by choosing strategy α_1, Player *A* has the revenue – {[10,11] · [10,11] + {[15,17] – [10,11]} · [15,16]} = [–233,–160].

For a colder winter (when Player B chooses strategy β_3), by choosing strategy α_1, Player A has the revenue – {[10,11] · [10, 11] + {[20, 22] – [10,11]} · [20,22]} = [–385,–280].

By the same process, we get the additional data in Table 2.3.

According to Theorem 2.12 and judging by Table 2.3, the grey matrix game solution of this problem is (α_3,β_3)—that is, it is reasonable to stock [20,22] tons of coal in autumn.

2.3 Example Analysis of the Grey Matrix Game Model in Stock Speculation for Immediate Price-Margin Based on Pure Strategies

An investor has $200,000 in October 2003 and wants to buy stocks of a listed company in Xi City, China. There are three investing strategies to choose from: buying $200,000 of stocks early in October; buying $100,000 of stocks early in October and investing the extra money otherwise, for which the monthly earning ratio is 0.1%; and buying no dollar stock early in October and using all the money for another investment, for which the monthly earning ratio is also 0.1%. However, there are three possible occasions for the stock at the end of October, which are: a good market with stock price $P_1^{\otimes} = [5.5, 6]$ (dollars per stock); a better market with stock price $P_2^{\otimes} = [5, 5.5]$ (dollars per stock); and a worse market with stock price $P_3^{\otimes} = [4, 5]$. .In early October 2003, if the investor bought $200,000 of stocks with $P_1^{\prime\otimes} = [5, 5.2]$, or bought $100,000 of stocks with $P_2^{\prime\otimes} = [5, 5.1]$, how could the investor earn the largest profits in the stock market, neglecting all other fees?

This stock investment can be regarded as a grey game problem. We refer to the investor as Player 1, who has three game strategies as mentioned before,

Table 2.4 Grey Price-Margin Value Matrix and Grey Matrix Game of an Investor in October 2003

S F	$\beta_1^{\otimes}$	$\beta_2^{\otimes}$	$(\beta_3^{\otimes})$	*Min*
$\alpha_1^{\otimes}$	[1.154, 4.000]	[–0.770, 2.000]	[–4.615, 0.000]]	[–4.615, 0.000]
$\alpha_2^{\otimes}$	[0.884, 2.100]	[–0.096, 1.100]	[–2.057, 0.100]	[–2.057, 0.100]
$\alpha_3^{\otimes}$	[0.200, 0.200]	[0.200, 0.200]	$[0.200, 0.200]^{**}$	$[0.200^{*}, 0.200^{*}]$
Max	[1.154, 4.000]	[0.200, 2.000]	$[0.200^{*}, 0.200^{*}]$	

Note: F: the first; S: the second. The grey value with * represents the one-sided conservative solution for the player, and the value with ** is the game solution under a globally grey pure strategy game.

$\alpha_1^{\otimes}, \alpha_2^{\otimes}$, and $\alpha_3^{\otimes}$, and we regard the market variation as Player 2 (a rational player), and treat the possible three occasions as three grey game strategies, $\beta_1^{\otimes}, \beta_2^{\otimes}, \beta_3^{\otimes}$.

We treat the gained profit by the investor in October 2003 as the earnings of Player 1 in the game, and then obtain the matrix as shown in Table 2.4.

Chapter 3

Pure Strategy Solution of a Grey Matrix Game Based on an Interval Grey Number Not to Be Determined Directly

3.1 Study of a Pure Strategy Solution and Risk of a Grey Matrix Game Based on Interval Grey Number Not to Be Determined Directly

3.1.1 Background

Using the interval grey number, we can easily express every player's game value, which is difficult to represent accurately in white numbers. So we call the game profit and loss value matrix that consists of those interval grey numbers as a grey profit and loss value matrix, $A(\otimes)$, and we call the matrix game decided by the grey profit and loss value matrix a grey matrix game, $G(\otimes) = \{S_1, S_2, A(\otimes)\}$.[4,12,13] $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ stands for a player's strategies, $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ stands for another player's strategies, and $A(\otimes)$ is the grey profit and loss value matrix that is determined by the players using the interval number in advance [as shown by Eq. (3.1)].

$$A(\otimes)=\begin{bmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [a_{m1},b_{m1}] & [a_{m2},b_{m2}] & \cdots & [a_{mn},b_{mn}] \end{bmatrix} \tag{3.1}$$

Previous studies have generalized the classical matrix game method to a matrix game problem based on interval grey numbers, established the model of a matrix game problem based on a standard interval grey number, and discussed its pure strategies. The foundation of the discussions is that the large and small relations of grey numbers in $A(\otimes)$ can be determined directly, but in case where the large and small relations of grey numbers in $A(\otimes)$ cannot be judged directly, no effective solution was provided.

However, it is very common when big and small relations of interval grey numbers cannot be determined directly in $A(\otimes)$ (as shown by Figure 3.1). We call this kind of matrix, $A(\otimes)$, a substandard grey profit and loss value matrix,[5,14] and we call the game decided by the $A(\otimes)$ a substandard grey profit and loss value matrix game, $G(\otimes)=\{S_1,S_2,A(\otimes)\}$. $G(\otimes)$ is one of the most widespread, extensive, and generalized ones of all grey matrix games. Based on previous studies about the standard grey profit and loss value matrix game $\tilde{G}^0(\otimes)$, this chapter presents a detailed study on the pure strategy solution of $G(\otimes)$.

3.1.2 Judgment on the Relationship of the Superior, Inferior, and Equipollence Position Degrees That Include Mixed Ranges

For judging the optimum pure strategy solution of a substandard grey profit and loss value matrix game when some internal number information of the interval grey numbers is unknown, we must find a method that can easily judge the big and small grey elements in $A(\otimes)$. Here, we define conceptions of superior, inferior, and equipollence position degrees between two grey numbers so that we can easily judge them under other situations.

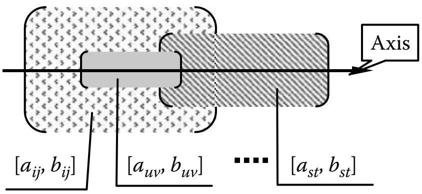


Figure 3.1 Big and small relations of interval grey numbers could not be judged directly.

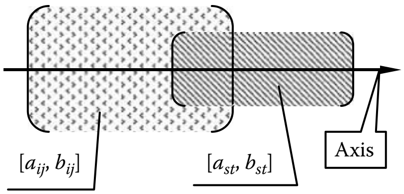


Figure 3.2 Big and small relations of interval grey numbers.

Definition 3.1 Conceptions of superior, inferior, and equipollence position degrees: For two interval grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$, $\otimes_{st} \in [a_{st}, b_{st}]$, and $b_{st} \geq b_{ij} \geq a_{st}$ (as shown Figure 3.2), we can divide their intersection range $\otimes_{ij} \cup \otimes_{st}$ into three parts, $[a_{ij}, a_{st}]$, $[a_{st}, b_{ij}]$, and $[b_{ij}, b_{st}]$, according to the position of the two endmost grey numbers on the axis. $[a_{st}, b_{ij}]$ is the intersection range, $\otimes_{ij} \cup \otimes_{st}$, between the two interval numbers, so:

1. We call this mixed range, $\otimes_{ij} \cup \otimes_{st}$, an equipollence position range, $EPD_{ij\to st} = \frac{b_{ij}-a_{st}}{b_{ij}-a_{ij}} \geq 0$ (where *ij*→*st* stands for $\otimes_{ij}$ compared to $\otimes_{st}$) is the equipollence position degree of the interval number $\otimes_{ij}$ compared to $\otimes_{st}$, and $EPD_{st\to ij} = \frac{b_{ij}-a_{st}}{b_{st}-a_{st}} \geq 0$ is the equipollence position degree of the interval number $\otimes_{st}$, compared to $\otimes_{ij}$.
2. We call the range $[b_{ij}, b_{st}]$, which is divided by the two endmost grey numbers at the right of the mixed range on the axis, as the superior position range of one grey number $[a_{st}, b_{st}]$ compared to another one $[a_{ij}, b_{ij}]$; $SPD_{st\to ij} = \frac{b_{st}-b_{ij}}{b_{st}-a_{st}} \geq 0$ is the superior position degree of interval number $\otimes_{st}$, compared to $\otimes_{ij}$.
3. We call the range $[b_{ij}, b_{st}]$, which is divided by the two endmost grey numbers at the left of the mixed range on the axis, as the inferior position range of one grey number $[a_{st}, b_{st}]$ compared to another one $[a_{ij}, b_{ij}]$; $IPD_{ij\to st} = -\frac{a_{st}-a_{ij}}{b_{ij}-a_{ij}} \leq 0$ is the inferior position degree of interval number $\otimes_{ij}$ compared to $\otimes_{st}$.

We have given the position relationship of two grey numbers in Definition 3.1, but what relationship exists among the three positions for one grey number compared to another?

Theorem 3.1 The relationship of superior, inferior, and equipollence position degrees: For two interval numbers $\otimes_{ij}$ and $\otimes_{st}$, the sum of the superior, inferior, and equipollence positions' absolute values of one grey number compared to another among them is 1, as $|EDP_{ij\to st}|+|SDP_{ij\to st}|+|IDP_{ij\to st}| = 1$, $|EDP_{st\to ij}|+|SDP_{st\to ij}|+|IDP_{st\to ij}| = 1$.

Proof: For two interval grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$ and $\otimes_{st} \in [a_{st}, b_{st}]$, they have only two relationships on the axis as follows:

1. $\otimes_{ij}$ is not mixed with $\otimes_{st}$; namely, $b_{ij} \leq a_{st}$ or $b_{st} \leq a_{ij}$.
2. $\otimes_{ij}$ is mixed with $\otimes_{st}$. Because of its complexity, we divide this case into four parts:
   (a) $b_{st} \geq b_{ij} \geq a_{st}$ (as shown in Figure 3.2)
   (b) $b_{ij} \geq b_{st} \geq a_{ij}$ (as shown Figure 3.2, changing the position of $\otimes_{ij}$ and $\otimes_{st}$ in this case)
   (c) $\otimes_{ij}$ becomes a subset of $\otimes_{st}$; namely, $a_{ij} \leq a_{st}$, $b_{ij} \leq b_{st}$
   (d) $\otimes_{st}$ becomes a subset of $\otimes_{ij}$; namely, ast ≤ aij, bst ≤ bij.

Taking the case 2(a), for instance, we give the proof process, which is the same as the others. As shown in Figure 3.2, we write superior, inferior, and equipollence position degrees of $\otimes_{ij}$ compared to $\otimes_{st}$, and give the sum of its absolute value [as shown Eq. (3.2)].

$$|EPD_{ij\to st}| + |SDP_{ij\to st}| + |IDP_{ij\to st}| = \left|\frac{b_{ij} - a_{st}}{b_{ij} - a_{ij}}\right| + 0 + \left|-\frac{a_{st} - a_{ij}}{b_{ij} - a_{ij}}\right| = \left|\frac{b_{ij} - a_{ij}}{b_{ij} - a_{ij}}\right| = 1 \qquad (3.2)$$

Definition 3.2 The relationships' judgment rules of interval grey number position: For two grey numbers, we call the sum of the superior and inferior position degrees of one compared to the other as the number's position, and it is abbreviated to position. If the position is positive, we call it a positive position, and call the relevant grey number a superior grey number; if the position is minus, we call it a minus position, and call the number an inferior position grey number; and if position is zero, we call it an equal position, and call the number an equipollence position grey number.

From Definition 3.2, we know, for two grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$ and $\otimes_{st} \in [a_{st}, b_{st}]$,

If $SDP_{ij\to st} + IDP_{ij\to st} > 0$, we say that $\otimes_{ij}$ compared to $\otimes_{st}$ exists in a positive position; $\otimes_{ij}$ is the superior grey number, and $\otimes_{st}$ is the inferior grey number, marked as $\otimes_{ij} > \otimes_{st}$

If $SDP_{ij\to st} + IDP_{ij\to st} < 0$, we say that $\otimes_{ij}$ compared to $\otimes_{st}$ exists in a minus position; $\otimes_{ij}$ is the inferior grey number, and $\otimes_{st}$ is the superior grey number, marked as $\otimes_{ij} < \otimes_{st}$

If $SDP_{ij\to st} + IDP_{ij\to st} = 0$, we say that $\otimes_{ij}$ has an equal position with $\otimes_{st}$; $\otimes_{ij}$ and $\otimes_{st}$ are the equipollence position grey number, marked as $\otimes_{ij} = \otimes_{st}$

In Definition 3.2, this chapter uses some mathematical operators such as >, <, and =, which stand for the position relationships among grey numbers rather than

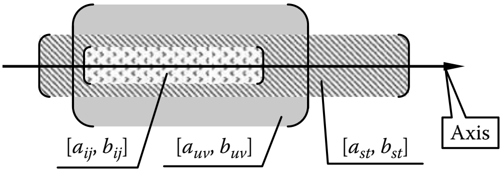


Figure 3.3 Transferable nature of the position.

their meaning in math, and we similarly define the big and small judgment rules on two grey numbers. However, whether the relationships among grey numbers can pass down is very important for judging, so we define Theorem 3.2 as follows.

Theorem 3.2 The transferable nature of big and small position relationship among interval grey numbers: Given two interval grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$, $\otimes_{uv} \in [a_{uv}, b_{uv}]$,and $\otimes_{st} \in [a_{st}, b_{st}]$, where $\otimes_{ij} \le \otimes_{uv}$ and $\otimes_{uv} \le \otimes_{st}$(as shown in Figure 3.3), then $\otimes_{ij} \le \otimes_{st}$.

Proof: According to Definition 3.1 and $\otimes_{ij} \le \otimes_{uv}$, $\otimes_{uv} \le \otimes_{st}$, we know that the margin between superior and inferior position degrees of $\otimes_{uv}$ compared to $\otimes_{ij}$ is more than or equal to zero [as shown Eq. (3.3)], and the margin between superior and inferior position degrees of $\otimes_{st}$ compared to $\otimes_{uv}$ is less than or equal to zero [as shown Eq. (3.4)]:

$$SDP_{uv\to ij} - IDP_{uv\to ij} = \frac{b_{uv} - b_{ij}}{b_{uv} - a_{uv}} - \frac{a_{ij} - a_{uv}}{b_{uv} - a_{uv}} = \frac{(b_{uv} + a_{uv}) - (a_{ij} + b_{ij})}{b_{uv} - a_{uv}} \ge 0 \quad (3.3)$$

$$SDP_{st\to uv} - IDP_{st\to uv} = \frac{b_{st} - b_{uv}}{b_{st} - a_{st}} - \frac{a_{st} - a_{uv}}{b_{st} - a_{st}} = \frac{(b_{st} + a_{st}) - (a_{uv} + b_{uv})}{b_{st} - a_{st}} \ge 0 \quad (3.4)$$

The margin between the superior and inferior position degrees of $\otimes_{st}$ compared to $\otimes_{ij}$ is shown in Eq. (3.5) as follows:

$$SDP_{st\to ij} - IDP_{st\to ij} = \frac{b_{st} - b_{ij}}{b_{st} - a_{st}} - \frac{a_{st} - a_{ij}}{b_{st} - a_{st}} = \frac{(b_{st} + a_{st}) - (a_{ij} + b_{ij})}{b_{st} - a_{st}} \quad (3.5)$$

From Eq. (3.3), we get $(b_{uv} + a_{uv}) \ge (a_{ij} + b_{ij})$, then,

$$(b_{st} + a_{st}) - (a_{ij} + b_{ij}) \ge (b_{st} + a_{st}) - (a_{uv} + b_{uv}) \quad (3.6)$$

From Eq. (3.4), we get

$$(b_{st} + a_{st}) - (a_{uv} + b_{uv}) \geq 0 \tag{3.7}$$

From Eqs. (3.5), (3.6), and (3.7), we get

$$SDP_{st\to ij} - IDP_{st\to ij} \geq 0 \tag{3.8}$$

So, using Eq. (3.8) and Definition 3.2, we obtain $\otimes_{ij} \leq \otimes_{st}$.

According to the rules in Definition 3.2, we can easily make a judgment on some grey numbers in $A(\otimes)$ about the position. Although the position is not always the grey numbers' sizes, the judgment rules bring us a method for solving the pure strategy solution position in $G(\otimes)$.

3.1.3 The Position Optimum Pure Strategy Solutions and the Answers

Lightly judging some interval grey numbers' sizes can easily resolve the pure strategy solution in a substandard grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$. Sutra matrix game[3–7] considered people's "rational behavior"; namely, if both players do not want to run a risk or do not feel lucky, they instead think of how the other side would try to make his opponent get less. They would choose the most favorable case that would appear from among the worst. This is a method both sides can accept, so we can study conceptions such as optimum pure strategies, the solution, pure situation, and so on, under the grey number position. Here, we define some conceptions about $G(\otimes)$.

Definition 3.3 Position optimum pure strategy solution: Given a substandard grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, where $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}, S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$, and $A(\otimes) = ([a_{ij}, b_{ij}])_{m\times n}$, there is a pure strategy like $\alpha_{i^*}, \beta_{j^*}$ under the position of grey number that can make Eq. (3.9) come into existence as follows:

$$[a_{ij^*}, b_{ij^*}] \leq [a_{i^*j^*}, b_{i^*j^*}] \leq [a_{i^*j}, b_{i^*j}] \tag{3.9}$$

where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

The situation $(\alpha_{i^*}, \beta_{j^*})$ is called the solution of $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ under the position pure strategies; $\alpha_{i^*}, \beta_{j^*}$ is the players' grey position optimum pure strategies, signed position optimum pure strategy for short; and the defrayal of Player 1 $[a_{i^*j^*}, b_{i^*j^*}]$ is called the grey game value $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, signed by $V_G(\otimes)$.

From Definition 3.3 we know that $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ must have a solution $\alpha_{i^*}, \beta_{j^*}$ if the solution would exist. An opposite number of rows i^* and number of columns j^* in the solution $\alpha_{i^*}, \beta_{j^*}$ can make the game value $[a_{i^*j^*}, b_{i^*j^*}]$ fit for

Eq. (3.9). The pair of positive whole numbers (i^*, j^*) is called the saddle point of a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ under the grey position optimum pure strategies, signed position saddle point; the grey number $[a_{i^*j^*}, b_{i^*j^*}]$ in $A(\otimes)$, which is determined by (i^*, j^*), is called the position saddle point of grey matrix $A(\otimes)$.

The grey game value $V_G(\otimes)$ is just $[a_{i^*j^*}, b_{i^*j^*}]$, which is in the opposite rows and columns of (i^*, j^*) in $A(\otimes)$. It is the smallest grey number on position in i^* rows, and the biggest in j^* columns.

Theorem 3.3 The sufficient and necessary term that the position optimum pure strategy solution is in existence: For a substandard grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, the sufficient and necessary term that has a solution or a saddle point under the position pure strategy is:[15–18]

$$\max_i \quad \min_j [a_{ij}, b_{ij}] = \min_j \quad \max_i [a_{ij}, b_{ij}] \tag{3.10}$$

Proof: Sufficiency: Let $\min_j [a_{ij}, b_{ij}]$ be the biggest position when $i = i^*$, and let $\max_i [a_{ij}, b_{ij}]$ be the smallest when $j = j^*$, then

$$\left.\begin{aligned} \min_j \quad [a_{i^*j}, b_{i^*j}] &= \max_i \quad \min_j [a_{ij}, b_{ij}] \\ \max_j \quad [a_{i^*j}, b_{i^*j}] &= \min_j \quad \max_i [a_{ij}, b_{ij}] \end{aligned}\right\} \tag{3.11}$$

From $\max_i \quad \min_j [a_{ij}, b_{ij}] = \min_j \quad \max_i [a_{ij}, b_{ij}]$, we get

$$\min_j [a_{i^*j}, b_{i^*j}] = \max_i [a_{ij^*}, b_{ij^*}] \tag{3.12}$$

According to the definition of minimum, we get $\min_j [a_{i^*j}, b_{i^*j}] \le [a_{i^*j^*}, b_{i^*j^*}]$, then $\max_i [a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}]$.

And from the definition of maximum, we get that for any $i = 1, 2, \ldots, m$, there is $a_{ij^*} \le a_{i^*j^*}$ and

$$\max_i [a_{ij^*}, b_{ij^*}] \ge a_{i^*j^*} \tag{3.13}$$

From Eqs. (3.12) and (3.13), we obtain $\min_j [a_{i^*j}, b_{i^*j}] \ge [a_{i^*j^*}, b_{i^*j^*}]$. According to the definition of minimum again, for any $j = 1, 2, \ldots, n$, there is $[a_{i^*j}, b_{i^*j}] \ge [a_{i^*j^*}, b_{i^*j^*}]$; so for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$, then $[a_{ij^*}, b_{ij^*}] \le [a_{i^*j^*}, b_{i^*j^*}] \le [a_{i^*j}, b_{i^*j}]$.

By Definition 3.3, we know that the grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ has its solution, $(\alpha_{i^*}, \beta_{j^*})$.

Necessity: If the grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$ has a solution, from Definition 3.3 we know that there is a α_{i^*},β_{j^*} that can make $[a_{ij^*},b_{ij^*}]\le[a_{i^*j^*},b_{i^*j^*}]\le[a_{i^*j},b_{i^*j}]$ reasonable for any $i=1,2,\ldots,m;\quad j=1,2,\ldots,n.$

Therefore,

$$\max_i[a_{ij^*},b_{ij^*}]\le[a_{i^*j^*},b_{i^*j^*}]\le\min_j[a_{i^*j},b_{i^*j}] \tag{3.14}$$

Because of $\min_j\ \max_i[a_{ij},b_{ij}]\le\max_i[a_{ij^*},b_{ij^*}]$ and $\min_j[a_{i^*j},b_{i^*j}]\le\max_i\ \min_j$ $[a_{ij}, b_{ij}]$ from Eq. (3.14),

$$\min_j\quad\max_i\,[a_{ij},b_{ij}]\le\max_i\quad\min_j[a_{ij},b_{ij}] \tag{3.15}$$

and from the definitions of maximum and minimum, we get $\min_j[a_{ij}, b_{ij}]\le[a_{ij},b_{ij}]\le\max_i[a_{ij},b_{ij}]$, then

$$\min_j[a_{ij},b_{ij}]\le\min_j\quad\max_i[a_{ij},b_{ij}] \tag{3.16}$$

Choosing the maximum for i from both sides of Eq. (3.16), we get

$$\max_i\quad\min_j[a_{ij},b_{ij}]\le\min_j\quad\max_i[a_{ij},b_{ij}] \tag{3.17}$$

From Eqs. (3.15) and (3.17), we get $\max_i\quad\min_j[a_{ij},b_{ij}]=\min_j\ \max_i[a_{ij},b_{ij}]$.

We must make one thing clear: We used the signs for choosing big or small that belong to sutra math, but we mean the size of the grey numbers' positions different from their real sense of large or small.

3.1.4 Case Study

After a systemic study of the solution of the substandard grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$ under position pure strategies, we would resolve a real matter using the above theory of a pure strategy solution based on its grey position.

Example 3.1 One buyer wants to know how much coal he could store.[9] In natural conditions, 25 tons of coal is necessary in a normal winter, 20 tons in a warmer winter, and 40 tons in a colder winter. We assume that the price of coal would vary depending on the weather. It would move between [1,2] (century/ton), [1,1.5], and [1,3] in those three conditions. And if the price moved between 1 and 1.2 in autumn, how much should the buyer store without an exact weather forecast of the coming winter?

Table 3.1 Position Optimum Pure Strategy Solution of $G(\otimes)$

Strategies	β_1	β_2	β_3	$\min_j[a_{ij},b_{ij}]$
α_1	[–24,–20]	[–34,–25]	[–69,–35]	[–69,–35]
α_2	[–25,–15]	[–30,–25]	[–60,–35]	[–60,–35]
α_3	[–42,–35]	[–42,–35]	[–42,–35]	[–42,–35]*
$\max_j[a_{ij},b_{ij}]$	[–25,–15]	[–30,–25]	[–42,–35]*	

Note: * = position optimum pure strategy solution

Solution: This problem can be regarded as a countermeasure or a game. The buyer is Player 1 and he has three strategies: buying 20, 25, or 40 tons, signed α_1, α_2, and α_3. We can take Nature as Player 2, which also has three strategies: natural, warmer, or colder, signed β_1 ,β_2, and β_3. From the topic, we can get a grey profit and loss value matrix as shown in Eq. (3.18):

$$A(\otimes) = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccc} \beta_1 & \beta_2 & \beta_3 \end{array} \\ \begin{bmatrix} [-24,-20] & [-34,-25] & [-69,-35] \\ [-25,-15] & [-30,-25] & [-60,-35] \\ [-42,-35] & [-42,-35] & [-42,-35] \end{bmatrix} \end{array} \tag{3.18}$$

From Table 3.1, we know that the optimum pure strategy solution of this game is (α_3,β_3); namely, storing 40 tons of coal is suitable in autumn and the game value is [−42,–35].

3.1.5 Summary

A key step in solving for a pure strategy of the grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ is that the interval grey number in the $A(\otimes)$ cannot be put in order directly in light of its values. In fact, we cannot solve the problem if the relationship of the interval grey number cannot be determined accurately and easily. After studying characteristics and laws in $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ this section deems that we determine the pure strategy solution mainly based on the hypothesis of players' rational behaviors and the key precondition is that we can arrange the grey elements of the profit and loss value matrix $A(\otimes) = ([a_{ij}, b_{ij}])_{m\times n}$.

Considering people's finite knowledge and logos in the grey matrix game, this chapter found a judgment system about the position relationship of interval grey numbers that well satisfies the judgment request of $G(\otimes)$ under the grey numbers position. There are many differences about big and small relationships between a grey number and its position. If the number satisfies the size relation, it also satisfies the position; the same is true in reverse. So it can be seen that, compared with position relationship, the number relationship has a stricter request on the judgment condition. It is the direct spread of sutra math in the grey field. The latter has the looser request as it is more generic in spread about the number size.

3.2 Study on Strategy Dominance and Its Optimum Solution of Grey Matrix Game Based on Interval Grey Numbers Not to Be Determined Directly

3.2.1 *The Dominance Analysis of Position Optimum Pure Strategy*

Definition 3.4 Position dominance: Suppose there is a substandard grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)^2\}$. where $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$, $A(\otimes) = ([a_{ij}, b_{ij}])_{m\times n}$, and for all $j = 1, 2, \ldots, n$, we have

$$[a_{i^0 j}, b_{i^0 j}] \geq [a_{k^0 j}, b_{k^0 j}] \tag{3.19}$$

If all positions of elements in row i^0 in $A(\otimes)$ are beyond or equal to them, we say that Player 1's pure strategy α_{i^0} is dominant to α_{k^0}.

Likewise, to all $i = 1, 2, \ldots, m$, we have

$$[a_{ij^0}, b_{ij^0}] \leq [a_{il^0}, b_{il^0}] \tag{3.20}$$

If all positions of elements in column j^0 in $A(\otimes)$ are less than or equal to themselves, we say that Player 2's pure strategy β_{j^0} is dominant to β_{l^0}.

Theorem 3.4 The rule of position strategy dominance of a substandard grey matrix: Suppose there is a substandard grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, where $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$, and $A(\otimes) = ([a_{ij}, b_{ij}])_{m\times n}$. If pure strategy α_1 is dominant by one of the other pure strategies $\alpha_2, \ldots, \alpha_m$, a new matrix solution G' : $G' = (S_1', S_2', A')$ can be obtained, where $S_1' = \{\alpha_2, \alpha_3, \ldots, \alpha_m\}$, $A' = ([a_{ij}', b_{ij}'])_{(m-1)\times n}$, $[a_{ij}', b_{ij}'] = [a_{ij}, b_{ij}]$, $i = 2, 3, \ldots, m$; and $j = 1, 2, \ldots, n$. Therefore, we have

1. $V_{G'(\otimes)} = V_{G(\otimes)}$
2. The optimum pure strategy of Players 1 and 2 in $G'(\otimes)$ are the same as in $G(\otimes)$.

Proof: Let α_2 be dominant to α_1, as

$$[a_{2j}, b_{2j}] \ge [a_{1j}, b_{1j}],\ j = 1, 2, \ldots, n \tag{3.21}$$

$G'(\otimes)$ has a solution; that is to say, there is a $[a'_{i^*j^*}, b'_{i^*j^*}]$, making

$$[a'_{ij^*}, b'_{ij^*}] \le [a'_{i^*j^*}, b'_{i^*j^*}] \le [a'_{i^*j}, b'_{i^*j}] \tag{3.22}$$

where $i = 2, 3, \ldots, m \qquad j = 1, 2, \ldots, n$

Since α_2 is dominant to α_1, from Eq. (3.21), we get

$$[a_{1j}, b_{1j}] \le [a_{2j}, b_{2j}] = [a'_{2j}, b'_{2j}] \le [a'_{i^*j^*}, b'_{i^*j^*}]$$

Therefore, $[a_{i^*j^*}, b_{i^*j^*}] = [a'_{i^*j^*}, b'_{i^*j^*}]$.

So we have $V_{G(\otimes)} = V_{G'(\otimes)}$, which implies the optimum pure strategy of Players 1 and 2 in $G'(\otimes)$ are the same as in $G(\otimes)$.

Deduction: If α_1 is not dominant by one of the other pure strategies $\alpha_2, \ldots, \alpha_m$ but by a convex linear combination of $\alpha_2, \ldots, \alpha_m$, the conclusion of Theorem 3.4 is still tenable.

3.2.2 Case Study

In order to stabilize the financial system and the macroeconomy, the banking industry needs a forward-looking method to lift loan-loss provisions. During a period of economic expansion and rapid growth of credit, adequate provisions are needed to cushion losses from the current recession. The policy of dynamic provisioning is such a method. Dynamic provisioning is able to release the cycle-dependent behaviors of merchant banks to some extent, and the impact on profit sheets and balance sheets caused by loan loss will be more moderate. Through the lens of a merchant bank lifting its loan-loss provisions, we will apply the method of the grey matrix game to lift the appropriate level of loss provisions dynamically.

Example 3.2 Consider a certain merchant bank that needs to lift its loan-loss provisions. We know that loans requiring loss provisions are 100 units in normal time, while during economic booms and stagnant periods they are 90 and 110 units respectively. In order to stabilize the financial system and the macroeconomy, the banking industry needs a forward-looking method to lift its loan-loss provisions. Lifting proportions vary dynamically. When the economy is doing well, the proportion and opportunity costs are higher. Considering these factors, we can suppose the unit price as 1 to 1.4 in a boom period, 1 to 1.2 in a normal period, and 0.5 to 0.8 in a stagnant period. Assume the current lifting unit price of this bank is 1. What is the optimum amount of loan-loss provisions the bank needs to lift at present?

This can be regarded as a matrix game problem, in which the bank is Player 1. The three strategies, α_1, α_2, and α_3, are methods to lift its loan-loss provisions due to different situations of economy: rising, normal, and stagnant. Let the economic situation be Player 2, and the three strategies β_1, β_2, and β_3 be a rising economy, a normal economy, and a stagnant economy. Now taking the expense this merchant bank spends on lifting its loan-loss provisions as a gain of Player 1, we can get a grey game matrix:

$$A(\otimes) = \begin{bmatrix} [-90,-90] & [-102,-100] & [-106,-100] \\ [-100,-100] & [-100,-100] & [-108,-105] \\ [-110,-110] & [-110,-110] & [-110,-110] \end{bmatrix}$$

According to Theorem 3.4, since the second row is dominant to the third, we know that the third row can be ruled out. We get:

$$A_1(\otimes) = \begin{bmatrix} [-90,-90] & [-102,-100] & [-106,-100] \\ [-100,-100] & [-100,-100] & [-108,-105] \end{bmatrix}$$

For $A_1(\otimes)$, since the second column is dominant to the first, we can rule out the first column. We get:

$$A_2(\otimes) = \begin{bmatrix} [-102,-100] & [-106,-100] \\ [-100,-100] & [-108,-105] \end{bmatrix}$$

According to rules of judgment of the relations among interval grey number positions, we know that the second column of $A_2(\otimes)$ is dominant to the first, and so we rule out the first column. We get:

$$A_3(\otimes) = \begin{bmatrix} [-106,-100] \\ [-108,-105] \end{bmatrix}$$

Since we have $\max_j [a_{ij}, b_{ij}] = [-106,-100]$ in $A_3(\otimes)$, the optimum solution of a position pure strategy is (α_1, β_3) : the bank should lift 90 units of its loan-loss provisions, which is suitable for a rising economy.

3.2.3 Summary

We are unable to solve the grey matrix game problem based on interval grey numbers that are not directly determined. The sticking point is how to judge their big and small relations. After studying the characteristics and laws in $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, applying the ideas of a grey system, and considering that players in a grey matrix game are of finite knowledge and logos, we have constructed the judgment rule system, and given a rule of strategy dominance under a grey number position and a pure strategy solution. As a grey matrix game based on interval grey numbers not to be judged directly, there are differences between a pure strategy solution under a grey number position and sutra math, which means that risk exists when making decision by means of the pure strategy solution under a grey number position. Further study is required if we want to judge and measure risk. Otherwise, if a pure strategy solution of $G(\otimes)$ under a position of pure strategy doesn't exist, the existence of a mixed strategy solution also needs further research.

3.3 Study of Risk of Position Optimum Pure Strategy Solution Based on a Grey Interval Number Matrix Game

3.3.1 Identity and Definition of Overrated and Underestimated Risks of Position Optimum Pure Strategy Solution

Example 3.3 Two oligarchic manufacturers carve up a relatively closed TV market of a certain area. They compete with each other in order to seize more market share. They can choose one of the following two strategies: Strategy 1, depreciating and providing high-quality service, or Strategy 2, adding new functions to the product and improving its quality. However, due to a variety of reasons, the manufacturers (players) cannot make precise estimates of the increasing market shares that would result from practicing the two strategies (game profit and loss value) beforehand. In fact, even under certain restrictions, it is impossible to make any two profit and loss values of such a game completely the same, due to random and nonrandom factors. However, based on past experience, they can obtain a relatively accurate estimate about the interval grey number of a profit and loss value for this contest aiming at market shares, as shown in Eq. (3.23). In Eq. (3.23),α_1, β_1 represents Strategy 1 used by Players 1 and 2 respectively, and α_2, β_2 represents Strategy 2. In this way, Eq. (3.23) becomes a problem of the type $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, which is a grey matrix game.

$$A(\otimes)=\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array}\begin{array}{c} \begin{array}{cc} \beta1 & \beta2 \end{array} \\ \begin{bmatrix} [5,7] & [4,9] \\ [1,8] & [1,10] \end{bmatrix} \end{array} \tag{3.23}$$

Solution: We can get the position pure strategy solution of this problem after applying the relevant method provided by Ref. 21, as shown in Table 3.2.

Table 3.2 Position Optimum Pure Strategy Solutions of G(⊗)

Player 2		β_1	β_2	$\min_j[a_{ij}, b_{ij}]$
Player 1	α_1	$a_{11}(\otimes) = [5,7]$	$a_{12}(\otimes) = [4,9]$	$[5,7]^*$
	α_2	$a_{21}(\otimes) = [1,8]$	$a_{22}(\otimes) = [1,10]$	$[1,8]$
	$\max_j[a_{ij}, b_{ij}]$	$[5,7]^*$	$[4,9]$	

Note: * = Position optimum pure strategy solution

Table 3.2 indicates that in such game processes, as described in Example 3.3, both players have pure strategy solutions under a grey position. When both players apply pure Strategy 1, the optimum game value for them is $V_G^*(\otimes) = a_{11}(\otimes) = [5, 7]$.

For problems of a grey matrix game, $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, there are differences between meanings of pure strategy solutions under a grey number position and sutra math. This means that risk exists when making decisions by means of the pure strategy solution under a grey number position.

Example 3.3 indicates that such kinds of equilibriums, or a grey position saddle point, exist, based on the meaning of the grey position. If the two players can only get a certain amount of grey information, this is a makeshift equilibrium that enables players' profits to maximize. If no further information can be obtained by the player, such a game decision is the optimum one to him. However, there are differences between meanings of pure strategy solution under grey number positions and sutra math that, in a sense, reflect the risk of game decision-making under a grey position.

In Example 3.3, if the game profit and loss value matrix is composed of $A(\otimes)$'s right and left extreme points respectively, as shown in Eqs. (3.23) and (3.24), it becomes the classical problem of matrix games, G_1 and G_2. The game solution of these two can be easily gotten based on matrix game theories: When Players 1 and 2 get their optimum pure strategy, α_1, β_2, respectively, the game value is $V_{G1} = 4$; when α_2, β_1, , the game value is $V_{G2} = 8$.)

$$A_1 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} \end{array} \tag{3.24}$$

$$A_2 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix} \end{array} \tag{3.25}$$

From the viewpoint of after the event, by comparing a pure strategy solution of a given grey matrix game problem based on the meaning of position, with the solution of G_1 and G_2, we can know that:

1. G_1's game value is $V^*_{G1} = 4$, and the left extreme point value of $G(\otimes)$'s grey game value is $LV_G^*(\otimes) = \mathrm{Min}\{V_G^*(\otimes)\} = 5$, which is one unit more. This means for $G(\otimes)$, if the same situation as G_1 happens, the player can only get four rather than five units of market share. Risk comes, which overestimates by one unit the game profit value.
2. G_2's game value is $V^*_{G1} = 8$, and the right extreme point value of $G(\otimes)$'s grey game value is $RV_G^*(\otimes) = \mathrm{Max}\{V_G^*(\otimes)\} = 7$, which is one unit less. This means for $G(\otimes)$, if the same situation as G_2 happens, the player can get as much as eight rather than seven units of market share. Risk comes, which underestimates by one unit the game profit value.

The risk analysis about pure strategy solution in Example 3.3, based on the meaning of position, is universal in a sense. For problems of a grey matrix game, there are no more forms of uncertainties of pure strategy solutions based on the meaning of position. Therefore, we will give the definition of these two forms of uncertainty.

Definition 3.5 The risk of overestimation for Player 1: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, then from an ex-post viewpoint, we can call $\sigma_{high.1}^k = -(V_{G^*}^k - LV_G^*(\otimes))$, $\sigma_{high.1}^k \geq 0$, the margin between the optimum game value $V_{G^*}^k$, and the value of its left extreme point $LV_G^*(\otimes) = a_{ij}^*$, as the overestimation risk of game k for Player 1, the value of which can be measured by $\sigma_{high.1}^k$.

Definition 3.6 The greatest risk of overestimation for Player 1: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, we call the greatest of all the overestimated values $\sigma_{high.1}^k$ $(k = 1,2,\ldots,\infty)$, and the greatest risk of overestimation for Player 1 as $\sigma_{high.1}^M = Max_{k=1}^{\infty}\{\sigma_{high.1}^k\}$.

Definition 3.7 The risk of underestimation for Player 1: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, then from an ex-post viewpoint, we can call $\sigma_{low.1}^k = V_{G^*}^k - RV_G^*(\otimes)$, $\sigma_{low.1}^k \geq 0$, the margin between the optimum game value $V_{G^*}^k$, and the value of its right extreme point $RV_G^*(\otimes) = b_{ij}^*$, as the underestimated risk of game k for Player 1, the value of which can be measured by $\sigma_{low.1}^k$.

Definition 3.8 The greatest risk of underestimation for Player 1: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, we call the greatest of all the underestimated values $\sigma_{low.1}^k$ $(k = 1,2,\ldots,\infty)$, and the greatest risk of underestimation for Player 1 as $\sigma_{low.1}^M = Max_{k=1}^{\infty}\{\sigma_{low.1}^k\}$.

Theorem 3.5 The risk relationship among players: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, then for the optimum solution of this group, the risk of overestimation for Player 1 is equivalent to the risk underestimated for Player 2, and vice versa.

Proof: According to characteristics of the matrix game problem, we can know that the profit of Player 1 is the loss of Player 2. Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, the optimum position game value is $V^*_G(\otimes) = [a^*_{ij}, b^*_{ij}]$; this means the game profit value for Player 1 is $[a^*_{ij}, b^*_{ij}]$, and $[-b^*_{ij}, -a^*_{ij}]$ for Player 2.

The overestimated risk value in game k (where $k = 1,2,\ldots,\infty$, which in reality may arise) for Player 1 implies: The exact optimum game value of this game for him is $\sigma^k_{high.1}$ units less than the extreme left point value $LV^*_G(\otimes) = a^*_{ij}$ of $V^*_G(\otimes) = [a^*_{ij}, b^*_{ij}]$, as $y^*_{k.1} = a^*_{ij} - \sigma^k_{high.1}$; for Player 2, the exact optimum game value of this game for him is $\sigma^k_{high.1}$ units more than the extreme right point value $RV^*_{G.2}(\otimes) = -a^*_{ij}$ of $V^*_{G.2}(\otimes) = [-b^*_{ij}, -a^*_{ij}]$, as $y^*_{k.2} = \sigma^k_{high.1} - a^*_{ij}$. Player 2 here undervalued his own game profit value, leading to the risk of underestimation, the value of which is equivalent to the risk of overestimation for Player 1, as $\sigma^k_{low.2} = \sigma^k_{high.1}$.

Similarly, if, in game k (where $k = 1,2,\ldots,\infty$ that in reality may arise), the underestimated risk value for Player 1 is $\sigma^k_{low.1}$ ($k = 1,2,\ldots,\infty$), then it means that the overestimated risk value for Player 2 is $\sigma^k_{high.2} = \sigma^k_{low.1}$.

In the process of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, the overestimation risk value for Player 1 is equivalent to the risk of underestimation for Player 2, and vice versa.

Deduction 3.1 Relationship of the greatest overestimated and underestimated risk among players: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V^*_G(\otimes) = [a^*_{ij}, b^*_{ij}]$, then the greatest overestimation risk value for Player 1 is equivalent to the greatest underestimation risk value for Player 2, shown as $\sigma^M_{high.1} = Max^{\infty}_{k=1}\{\sigma^k_{high.1}\} = Max^{\infty}_{k=1}\{\sigma^k_{low.2}\} = \sigma^M_{low.2}$; the greatest underestimated risk value for Player 1 is equivalent to the greatest overestimated risk value for Player 2, as $\sigma^M_{low.1} = Max^{\infty}_{k=1}\{\sigma^k_{low.1}\} = Max^{\infty}_{k=1}\{\sigma^k_{high.2}\} = \sigma^M_{high.2}$.

The proof of this deduction is similar to Theorem 3.4, which is relatively simple, so further details of it are unnecessary here.

The risks of optimum position pure strategy solution for players are mainly due to the following facts: from the viewpoint of ex-post, in a real-game problem $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, decision-making is based on the meaning of grey position in its grey individual value matrix, which is considered to have few possibilities.

3.3.2 Judgment of Position Optimum Pure Strategy Solution Risk

If optimum position pure strategy solution of $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ exists, we are concerned about whether risk exists when practicing this strategy and how much risk we need to bear. In order to solve the problem, we will give the following lemma.

Lemma 3.1 The most pessimistic and most optimistic value for Player 1: Given a problem of grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, if two new game individual value matrices are composed by the left and right extreme point values of the grey interval factors in $A(\otimes)_{m\times n}$ respectively, then the game values of $G_L=\{S_1,S_2,A_L\}$ and $G_R=\{S_1,S_2,A_R\}$, determined by $\{A_L\}_{m\times n}$ and $\{A_R\}_{m\times n}$, are equivalent to the most pessimistic (least) game value and the most optimistic (greatest) game value for Player 1 in $G(\otimes)$.

Proof: To prove $V_L^*=V_{L(\otimes)}^*$ considering the problem of grey matrix game $G(\otimes)$, if a new game profit and loss value matrix $\{A_L\}_{m\times n}$ is composed of the left extreme point value of its grey interval number, then the game problem, determined by $\{A_L\}_{m\times n}$, becomes a classical game problem of mathematics. Based on the theory of matrix games, we can get its solution easily. The optimum game strategies for Players 1 and 2 are $T_L\ (x)$ and $T_L\ (y)$ respectively; the game value is V_L^*.

Now what we need to prove is that V_L^* is the most optimistic (least) game value $V_{L(\otimes)}^*$ for a $G(\otimes)$ problem, shown as $V_L^*=V_{L(\otimes)}^*$.

Assuming that V_L^* is not the most pessimistic (least) game value for $G(\otimes)$ problem, then $V_{L(\otimes)}^*<V_L^*$ exists; that is to say, among $A(\otimes)_{m\times n}$ of $G(\otimes)$, at least one matrix $\{A_L\}'_{m\times n}$ exists that is not composed of the left extreme point value of $A(\otimes)_{m\times n}$'s grey factors completely [as shown in Eq. (3.27)]; the game value of the game problem, determined by $\{A_L\}'_{m\times n}$, is the most pessimistic (least) game value of $G(\otimes)$, shown as $V_{L'}^*=V_{L(\otimes)}^*$.

$$A_L=\begin{bmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} & a_{i-1,k} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,k-1} & a_{i,k} & \cdots & a_{i,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,k-1} & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \tag{3.26}$$

$$A_L'=\begin{bmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} & a_{i-1,k} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,k-1} & a_{i,k}' & \cdots & a_{i,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,k-1} & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \tag{3.27}$$

Keeping universality, assume that there are only $a_{ik}^{/}$ in $\{A_L\}_{m\times n}^{/}$, which is not equivalent to a_{ik} in $\{A_L\}_{m\times n}$, shown as $a_{ik} < a_{ik}^{/} < b_{ik}$. According to the theory of partitioned matrix games, we can disassemble factors in $\{A_L\}_{m\times n}$ and $\{A_L\}_{m\times n}^{/}$ into sub-blocks. For convenience, we take the two neighboring factors of a_{ik} and $a_{ik}^{/}$ to constitute the two sub-block matrix [as shown In Eqs. (3.28) and (3.29)]. Then, after a proper compartmentalization of other factors, we can get partitioned matrices A_{LL} and $A_{LL}^{/}$, as shown in Eqs. (3.30) and (3.31).

$$A_{ik} = [a_{i,k-1} \quad a_{i,k}] \tag{3.28}$$

$$A_{ik}^{/} = \begin{bmatrix} a_{i,k-1} & a_{i,k}^{/} \end{bmatrix} \tag{3.29}$$

In this way, only game values V_{LL}^{*} and $V_{LL}^{*/}$, determined by partitioned matrices A_{LL} and $A_{LL}^{/}$, may not be equivalent, while the game values of the other partitioned matrices must be the same. In fact, after this matrix sub-block, and based on the theory of it, we switch the problem that focuses on comparing whether V_L^{*} and $V_{L(\otimes)}^{*}$ are equivalent to the problem that focuses on the comparison between V_{LL}^{*} and $V_{LL}^{*/}$. If $V_{LL}^{*} \le V_{LL}^{*/}$, then it indicates that, when a_{ik} turns into $a_{ik}^{/}$ ($a_{ik} < a_{ik}^{/}$), the game value of this problem won't decrease; and if $V_{LL}^{*} > V_{LL}^{*/}$, then it indicates that when a_{ik} turns into $a_{ik}^{/}$ ($a_{ik} < a_{ik}^{/}$), the game value of this problem will decrease.

$$A_{LL} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1s} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ik} & \cdots & A_{is} \\ \vdots & & \vdots & & \vdots \\ A_{t1} & \cdots & A_{tk} & \cdots & A_{ts} \end{bmatrix} \tag{3.30}$$

$$A_{LL}^{/} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1s} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ik}^{/} & \cdots & A_{is} \\ \vdots & & \vdots & & \vdots \\ A_{t1} & \cdots & A_{tk} & \cdots & A_{ts} \end{bmatrix} \tag{3.31}$$

$$V_{ik}^{*} = \begin{cases} a_{i,k-1}, & a_{i,k-1} \le a_{i,k} \\ \\ a_{i,k}, & a_{i,k} \le a_{i,k-1} \end{cases} \tag{3.32}$$

$$V_{ik}^{*'} = \begin{cases} a_{i,k-1}, & a_{i,k-1} \le a_{i,k}' \\ \\ a_{i,k}', & a_{i,k}' \le a_{i,k-1} \end{cases} \tag{3.33}$$

According to the theory of matrix games, we can get the game values V_{ik}^{*} and $V_{ik}^{*'}$ for Eqs. (3.28) and (3.29), as shown by Eqs. (3.23) and (3.33). Then, through Eqs. (3.23) and (3.33), we can know that, because $a_{ik} < a_{ik}'$, then $V_{ik}^{*'}$ is not less than V_{ik}^{*}. As long as $a_{ik} < a_{ik}'$, $V_{ik}^{*} \le V_{ik}^{*'}$ will be tenable consequentially. According to the theory of the partitioned matrix, game values V_{LL}^{*} and $V_{LL}^{*'}$ for the profit and loss value matrices $\{A_L\}_{m\times n}$ and $\{A_L\}_{m\times n}'$ can be worked out through Eqs. (3.34) and (3.35).

$$A_{LL.V} = \begin{bmatrix} V_{11}^{*} & \cdots & V_{1k}^{*} & \cdots & V_{1s}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{i1}^{*} & \cdots & V_{ik}^{*} & \cdots & V_{is}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{t1}^{*} & \cdots & V_{tk}^{*} & \cdots & V_{ts}^{*} \end{bmatrix} \tag{3.34}$$

$$A_{LL.V}' = \begin{bmatrix} V_{11}^{*} & \cdots & V_{1k}^{*} & \cdots & V_{1s}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{i1}^{*} & \cdots & V_{ik}^{*'} & \cdots & V_{is}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{t1}^{*} & \cdots & V_{tk}^{*} & \cdots & V_{ts}^{*} \end{bmatrix} \tag{3.35}$$

Similarly, according to the theory of matrix games, we can get the conclusion that, in Eqs. (3.34) and (3.35), as long as $V_{ik}^{*} \le V_{ik}^{*'}$, then V_{L}^{*} is not less than $V_{L'}^{*}$; that is to say, $V_{L}^{*} \le V_{L'}^{*}$ is tenable.

As long as $a_{ik} \le a_{ik}'$, we can have $V_{L}^{*} \le V_{L'}^{*}$ consequentially. Similarly, by using the same method, we can get that such relationship exists, $V_{L}^{*} \le V_{L'}^{*}$, between the game value of the game problem, which takes $\{A_L\}_{m\times n}$ and $\{A_L\}_{m\times n}'$ as its profit and loss value matrix, as long as some factor of $\{A_L\}_{m\times n}'$ is not less than that of $\{A_L\}_{m\times n}$. Game value V_{L}^{*}, determined by $\{A_L\}_{m\times n}$, the profit and loss value matrix composed of the left extreme point value, is equivalent to the most pessimistic (least) $V_{L(\otimes)}^{*}$ of $G(\otimes)$, as $V_{L'}^{*} = V_{L(\otimes)}^{*}$.

Therefore, the assumption is false, and then the original proposition is true.

By using similar methods as above, we can prove that, given a problem of the grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if a new game profit and loss value matrix $\{A_R\}_{m\times n}$ is composed of the right extreme point value of $A(\otimes)_{m\times n}$'s grey interval factors, then the game value of the game problem $G_R = \{S_1, S_2, A_R\}$, determined by $\{A_R\}_{m\times n}$, is equivalent to the most optimistic (greatest) game value $V^*_{R(\otimes)}$ for Player 1 in $G(\otimes)$.

Definition 3.2 The most pessimistic and most optimistic value for Player 2: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, the most optimistic (greatest) and most pessimistic (least) value for Player 2 equals the opposite number of the most pessimistic (least) and most optimistic (greatest) value for Player 1, marked as $V^*_{opt(\otimes)} = -V^*_{L(\otimes)}$, $V^*_{pes(\otimes)} = -V^*_{R(\otimes)}$.

Theorem 3.6 The greatest risk of overestimation for Player 1: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V^*_G(\otimes) = [a^*_{ij}, b^*_{ij}]$, then the greatest overestimation risk value $\sigma^M_{high.1}$ for Player 1 equals the margin between a^*_{ij} and the most pessimistic (least) game value $V^*_{L(\otimes)}$ of $G(\otimes)$, marked as $\sigma^M_{high.1} = (a^*_{ij} - V^*_{L(\otimes)})$.

Proof of Theorem 3.6 is simple; we can prove it only based on Definitions 3.6 and 3.7 and the relevant conclusions of Lemma 3.1. No further details are necessary.

Theorem 3.7 The greatest risk of underestimation for Player 1: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V^*_G(\otimes) = [a^*_{ij}, b^*_{ij}]$, then the greatest underestimation risk value $\sigma^M_{low.1}$ for Player 1 equals the margin between b^*_{ij} and the most optimistic (greatest) game value $V^*_{R(\otimes)}$ of $G(\otimes)$, marked as $\sigma^M_{low.1} = (V^*_{R(\otimes)} - b^*_{ij})$.

Proof of Theorem 3.7 is simple: We can prove it only based on Definitions 3.6 and 3.7 and the relevant conclusions of Lemma 3.1; no more details are necessary.

Deduction 3.3 The greatest overestimation and underestimation risk for Player 2: Given a problem of grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, if its optimum position pure strategy solution exists, and the optimum game value is $V^*_G(\otimes) = [a^*_{ij}, b^*_{ij}]$, then the greatest overestimation risk value $\sigma^M_{high.2}$ for Player 2 equals the margin between $(-b^*_{ij})$ and the most pessimistic (least) game value $V^*_{pes(\otimes)} = -V^*_{R(\otimes)}$ of $G(\otimes)$, marked as $\sigma^M_{high.2} = (V^*_{R(\otimes)} - b^*_{ij})$; the greatest underestimation risk value $\sigma^M_{low.2}$ for Player 2 equals the sum between a^*_{ij} and the most optimistic (greatest) game value $V^*_{opt(\otimes)} = -V^*_{L(\otimes)}$ of $G(\otimes)$, marked as $\sigma^M_{low.2} = (a^*_{ij} - V^*_{L(\otimes)})$.

Proof of Deduction 3.3 is simple; we can prove it only based on Definitions 3.6 and 3.7, the relevant conclusion of Lemma 3.1, and the relationship between profit for players, indicated by the profit and loss value matrix game. No more details are necessary.

Example 3.4 In Example 3.3, a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ [as shown in Eq. (3.23)], the optimum position pure strategy solution, if such solution exists, is to try to work out the greatest overestimated and underestimated risk of optimum position pure strategy solutions for Player 1 and 2.

Solution: In Example 3.3, the game value of optimum position pure strategy solution is $V^*_G(\otimes) = a^*_{11}(\otimes) = [a^*_{11}, b^*_{11}]$ = [5,7], while the most optimistic (greatest) and the most pessimistic (least) game profit is $V^*_{R(\otimes)} = V^*_{G2} = 8$, $V^*_{L(\otimes)} = V^*_{G1} = 4$ respectively.

In this way,

the greatest overestimated risk for Player 1 is $\sigma^M_{high.1} = a^*_{ij} - V^*_{L(\otimes)} = 5 - 4 = 1$;
the greatest underestimated risk for Player 1 is $\sigma^M_{low.1} = V^*_{R(\otimes)} - b^*_{ij} = 8 - 7 = 1$;
the greatest overestimated risk for Player 2 is $\sigma^M_{high.2} = V^*_{R(\otimes)} - b^*_{ij} = 8 - 7 = 1$;
the greatest underestimated risk for Player 2 is $\sigma^M_{low.2} = a^*_{ij} - V^*_{L(\otimes)} = 5 - 4 = 1$.

3.3.3 Summary

Using the ideas of grey systems and a theory of systems engineering, this section reveals the laws of grey game decision-making for people with finite knowledge edge and rationality, and designs the decision-making rule of people in the process of a grey game: Both players choose the situation of maximum (most advantage) grey game value position from among the possible situations of minimum (most disadvantageous) grey game value positions, which is a compromise method accepted by both sides. Based on this, this section has identified and defined the overestimated and underestimated risks of position optimum pure strategy solutions of the grey matrix game, and has designed the arithmetic for measurement thereof, which can measure the risk of a position optimum pure strategy solution.

Chapter 4

Grey Matrix Game Model Based on Grey Mixed Strategy

4.1 Grey Mixed Strategy and Grey Mixed Situation

4.1.1 Background

Taking the requirement of game study into consideration, we put forward and define basic concepts such as grey random events, grey probabilities, and so forth. We then study the relations between grey events as well as their operation rules, and prove six important properties of grey interval probabilities. Based on this, we define the concepts of grey mixed strategies, grey mixed situations, and grey mathematical expectations, and we construct the framework of solutions to both grey mixed extensions and grey mixed strategies, which paves the way for further research on grey matrix games based on mixed strategies.[24]

In Chapter 2, we studied problems such as the solution to a grey matrix game, which has a pure strategic solution; the properties of the solution; the method adopted to find the solution; and the relationships between standard and nonstandard grey matrix games together with their transition. However, not all grey matrix game problems have saddle points. In cases where no grey saddle point exists, players do not simply select a certain strategy but select one strategy with a certain grey interval probability. This is the grey matrix game problem based on grey mixed strategy.

In the study of this problem, we need to use concepts of grey events, grey interval probabilities, and so forth, as well as their properties. These concepts are discussed briefly in order to lay down the foundations for further research.

4.1.2 Relation and Operation of Grey Events

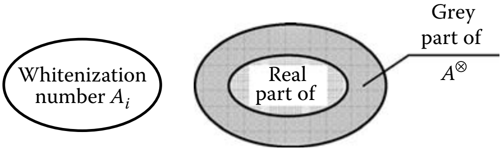
In classical probability theory, an event can be expressed by an accurate whitenization number, and its set has a definite boundary. As illustrated in Figure 4.1(a), the set of event A has a definite boundary. Nevertheless, in the system of grey probability, we cannot describe an event with an exact whitenization number. Therefore, we introduce the concept of grey event.

Definition 4.1 Grey event: We call a subset of experiment E's sample space S as E's random event, or simply, event. If the subset of S can only be described with a grey number $\otimes_i = [a_i, b_i], i = 1, 2, \ldots$, then we call the grey subset of the sample space S as E's grey random event, and grey event in abbreviation. In every experiment, when and only when one sample point of the grey subset appears, we say that a grey event happens.

From Definition 4.1, grey event $A^{\otimes}$ is expressed by a grey number, and the set consisting of grey event $A^{\otimes}$ has a grey boundary. In fact, given any grey number $\otimes_i = [a_i, b_i], i = 1, 2, \ldots$, it can be expressed as Eq. (4.1). In this formula, a_i is called the white part of $\otimes_i$, while $c_i \cdot \gamma_i$ is called the grey part of $\otimes_i$; in the grey part $c_i \cdot \gamma_i$, we define $c_i = \frac{1}{b_i - a_i}$ as a grey coefficient, and γ_i as a unit grey number (or grey number unit).

$$\begin{aligned}\otimes_i &= [a_i, b_i] = [a_i, b_i] + a_i - a_i = a_i + [0, b_i - a_i] \\ &= a_i + \frac{1}{b_i - a_i}[0,1] = a_i + \frac{1}{b_i - a_i}.\gamma_i \\ &= a_i + c_i \bullet \gamma_i; \quad c_i = \frac{1}{b_i - a_i}, \gamma_i \in [0,1], i = 1, 2, \ldots\end{aligned} \tag{4.1}$$

Since any grey number can be expressed as Eq. (4.1), the grey set expressed by grey numbers can be illustrated by Figure 4.1(b). If unit grey number γ_i is an



(a) (b)

Figure 4.1 (a) Set of whitenization numbers Ai. (b) Grey set Ai of grey numbers $\otimes_i$.

exact number, then $\otimes_i$ is whitened as a whitenization number, and Figure 4.1(b) is transformed into Figure 4.1(a). The set of whitenization numbers in Figure 4.1(a) is a special case of the grey set in Figure 4.1(b) where we let Υ_i be an exact number. Figure 4.1(b) can also be said to be an extension of Figure 4.1(a).

Assume the sample space of experiment E is S and $A^\otimes, B^\otimes, A_k^\otimes (k = 1,2,\ldots)$ are grey subsets of S.

1. If $A^\otimes \subset B^\otimes$, we say that grey event $B^\otimes$ contains grey event $A^\otimes$; the occurrence of event $A^\otimes$ leads to the occurrence of event $B^\otimes$ inevitably, as illustrated in Figure 4.2(a). If $A^\otimes \subset B^\otimes$ and $B^\otimes \subset A^\otimes$—that is, if $A^\otimes = B^\otimes$—we say that grey event $A^\otimes$ is equal to grey event $B^\otimes$.
2. Grey event $A^\otimes \bigcup B^\otimes = \{x|\in A^\otimes orx \in B^\otimes x\}$. It is called the sum event of grey events $A^\otimes$ and $B^\otimes$, as illustrated in Figure 4.2(b). When and only when at least one of $A^\otimes$ and $B^\otimes$ occurs, grey event $A^\otimes \bigcup B^\otimes$ occurs. Similarly, we call

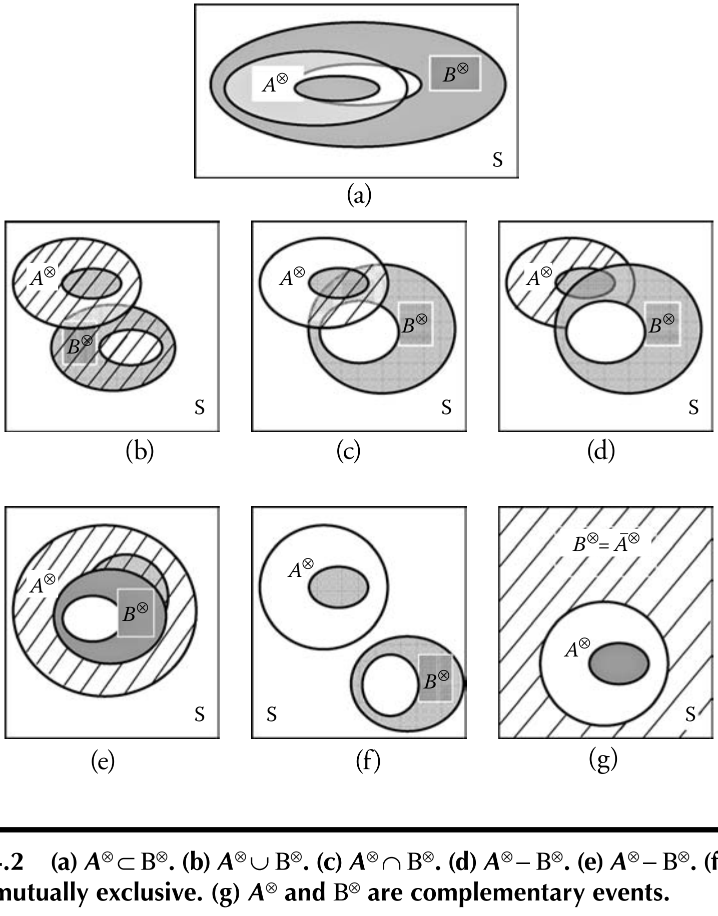


Figure 4.2 (a) $A^\otimes \subset B^\otimes$. (b) $A^\otimes \cup B^\otimes$. (c) $A^\otimes \cap B^\otimes$. (d) $A^\otimes - B^\otimes$. (e) $A^\otimes - B^\otimes$. (f) $A^\otimes$ and $B^\otimes$ are mutually exclusive. (g) $A^\otimes$ and $B^\otimes$ are complementary events.

$\bigcup_{k=1}^{n} A_k^{\otimes}$ the sum event of n grey events $A_1^{\otimes}, A_2^{\otimes}, \ldots, A_n^{\otimes}$; we say that $\bigcup_{k=1}^{\infty} A_k^{\otimes}$ is the sum event of denumerable grey events $A_1^{\otimes}, A_2^{\otimes}, \ldots, A_n^{\otimes}$.

3. Grey event $A^{\otimes} I B^{\otimes} = \{x | x \in A^{\otimes} \text{and } x \in B^{\otimes}\}$ It is called the product event of grey events $A^{\otimes}$ and $B^{\otimes}$, as illustrated in Figure 4.2(c). When and only when both $A^{\otimes}$ and $B^{\otimes}$ occur, grey event $A^{\otimes} \bigcap B^{\otimes}$ (also denoted as $A^{\otimes}B^{\otimes}$) occurs. Similarly, $\bigcap_{k=1}^{n} A_k^{\otimes}$ is the product event of n grey events $A_1^{\otimes}, A_2^{\otimes}, \ldots, A_n^{\otimes}$, and $\bigcap_{k=1}^{\infty} A_k^{\otimes}$ is the product event of denumerable grey events $A_1^{\otimes}, A_2^{\otimes}, \ldots, A_n^{\otimes}$.
4. Grey event $A^{\otimes} - B^{\otimes} = \{x | x \in A^{\otimes} \text{and } x \notin B^{\otimes}\}$ is the difference event of grey events $A^{\otimes}$ and $B^{\otimes}$, as illustrated in Figures 4.2(d) and 4.2(e). When and only when $A^{\otimes}$ occurs and $B^{\otimes}$ does not occur, grey event $A^{\otimes} - B^{\otimes}$ occurs.
5. If $A^{\otimes} \bigcap B^{\otimes} = \phi$, we say grey events $A^{\otimes}$ and $B^{\otimes}$ are mutually exclusive. It means grey events $A^{\otimes}$ and $B^{\otimes}$ cannot happen at the same time. Their basic grey events are mutually exclusive, as shown in Figure 4.2(f).
6. If $A^{\otimes} \bigcup B^{\otimes} = S$ and $A^{\otimes} \bigcap B^{\otimes} = \phi$, then grey events $A^{\otimes}$ and $B^{\otimes}$ are converse grey events or complementary events, as showed in Figure 4.2(g). It means in every experiment, one and only one grey event of $A^{\otimes}$ and $B^{\otimes}$ will definitely happen. The complementary grey event of $A^{\otimes}$ is denoted $\bar{A}^{\otimes}$ and $\bar{A}^{\otimes} = S - A^{\otimes}$.

As a matter of fact, in Figure 4.2, for any grey number, if only its unit grey number specifies a fixed value, then the relation will be transformed into that of sets of whitenization numbers. The major difference between the above relation of grey events and that of classical events lies in that the boundary of a grey event depends on the specified value of its grey number unit. In Figure 4.2, the relation of grey events is only formal. When their grey number units are specified, a form of relation may be translated into another one. For example, in Figure 4.2(c), if the units of grey events $A^{\otimes}$ and $B^{\otimes}$ are specified at a fixed value, the intersection set of $A^{\otimes}$ and $B^{\otimes} - A^{\otimes} \bigcap B^{\otimes}$ may be an empty set. As a result, the relation of $A^{\otimes}$ and $B^{\otimes}$ in Figure 4.2(c) is transformed into that of mutually exclusive events showed in Figure 4.2(f).

In the operation of grey events, we usually use the following laws. Suppose $A^{\otimes}$, $B^{\otimes}$, $C^{\otimes}$ are grey events and we have:

Grey Commutative Law: $A^{\otimes} \bigcup B^{\otimes} = B^{\otimes} \bigcup A^{\otimes}; A^{\otimes} \bigcap B^{\otimes} = B^{\otimes} \bigcap A^{\otimes}$

Grey Associative Law:
$$A^{\otimes} \bigcup (B^{\otimes} \bigcup C^{\otimes}) = (A^{\otimes} \bigcup B^{\otimes}) \bigcup C^{\otimes}$$
$$A^{\otimes} \bigcap (B^{\otimes} \bigcap C^{\otimes}) = (A^{\otimes} \bigcap B^{\otimes}) \bigcap C^{\otimes}$$

Grey De Morgan's Law: $\overline{A^{\otimes} \bigcup B^{\otimes}} = \bar{A}^{\otimes} \bigcap \bar{B}^{\otimes}; \overline{A^{\otimes} \bigcap B^{\otimes}} = \bar{A}^{\otimes} \bigcup \bar{B}^{\otimes}$

Here, we only generally extend the operation rules of events in classical probability theory to grey events' operation fields, in order to make full use of these classical events' fine mathematical properties. If only the unit grey number γ of a grey event is specified as an accurate number, the operation of grey events can be transformed into that of classical events. Therefore, the difference of the above operation rules of grey events and classical events is only exhibited in that form.

4.1.3 Basic Concepts and Properties of Grey Interval Probability

In classical probability theory, when $n \to \infty$ (n is the sample space of a random experiment), we regard the occurrence frequency of event A-$f_n(A)$ as its occurrence probability $P(A)$, and probability $P(A)$ is used to measure the possibility of event A's occurrence in an experiment. In reality, however, due to the limitations of people's time, energy, ability, and other conditions, the sample space n of a random experiment is not able to reach infinity. That is to say, $P(A)$ is a theoretical value that is highly abstract.

Although the occurrence probability of some random events can be inferred according to the principle of arrangement and combination, it is not true for many events in the real world, as people can only estimate according to limited statistical material and cannot find a certain principle or infer the probability of such events precisely. In classical probability theory, we usually use the average value of an event's occurrence time in an experiment to estimate its occurrence probability, and this surely leads to considerable errors.

Based on the above fact, when estimating the occurrence probability of an event in reality, we consider the minimum frequency $\min(f_i(A)), i = 1, 2, \ldots, m$ and maximum frequency $\max(f_i(A)), i = 1, 2, \ldots, m$ of an event's occurrence in a statistical period as the less possible probability (or called left probability) and the more possible probability (or called right probability). Then the left and right probability constitute a most possible probability interval of the event's occurrence, expressed as $P(A^{\otimes}) = [\min(f_i(A)), \max(f_i(A))], i = 1, 2, \ldots, m.$ We say that $P(A^{\otimes})$ is the grey interval probability of event A. Here, we give a general description of this concept. Later, we will give a more accurate definition.

Example 4.1 In a certain place, the rainfall conditions in spring for the past 100 years are sorted into rare, acceptable, considerable, and excessive, and we count these four conditions in this 100 years grouped by every successive 20 years. Then we get five groups in sum, as illustrated in Table 4.1.

As shown in Table 4.1, the grey interval probabilities of four spring rainfall conditions—rare, acceptable, considerable, and excessive—in this place are [0.45,0.65], [0.15,0.40], [0.05,0.20], and [0.00,0.10] respectively.

This shows that for an experimental period, when an accurate occurrence probability of a certain event and its distribution are not exactly known, we can estimate the event's occurrence probability with its minimum and maximum interval frequency, which is more precise than adopting the simple average value estimation method.

Table 4.1 Statistical and Classification Table of Spring Rainfall Conditions in 100 Years

Type of Rainfall		*Rare*		*Acceptable*		*Considerable*		*Excessive*	
Group by Time		*Times*	*Frequency*	*Times*	*Frequency*	*Times*	*Frequency*	*Times*	*Frequency*
	1–20	10	0.50	5	0.25	3	0.15	2	0.10
	21–40	9	0.45	7	0.35	4	0.20	1	0.05
	41–60	15	0.75	3	0.15	1	0.05	1	0.05
	61–80	10	0.50	8	0.40	1	0.05	1	0.05
	81–100	13	0.65	6	0.30	1	0.05	0	0.00

Definition 4.2 Grey interval frequency or simply grey frequency: Under the same conditions, we conduct $K,(k=1,2,\ldots,K)$ groups of an experiment in $N(n=1,2,\ldots,N)$ times each. We can determine the times of event A's occurrence in these experiment as follows: in all K groups of experiments, we consider the minimum times of A's occurrence $LA=\min_{k=1,2,\ldots,K}\{n_k^i(A), i=1,2,\ldots,N\}$ and the maximum times of its occurrence $RA=\max_{k=1,2,\ldots,K}\{n_k^i(A), i=1,2,\ldots,N\}$ as the left and right endpoint values of event A's occurrence frequency respectively, which is denoted $n_A^{\otimes}=[LA,RA]$. $n_A^{\otimes}$ is called the grey interval frequency (grey frequency in abbreviation) of event A's occurrence; LA is the left frequency and RA is the right frequency.

The ratio of this interval grey number $n_A^{\otimes}$ and N- $^{n_A^{\otimes}}/_{N}={}^{[LA,RA]}/_{N}=[{}^{LA}/_{N},{}^{RA}/_{N}]$ is called the grey interval frequency of event A's occurrence, or simply grey frequency, and we designate it as $f_n(A^{\otimes})=[Lf,Rf]$, where $Lf={}^{LA}/_{N}, Rf={}^{RA}/_{N}, Lf\le Rf$.

The frequency in any whitenization number form n_A can be written in the form of a grey interval frequency $n_A^{\otimes}=[LA,RA]$, where $LA=RA$; it can also be written as $f_n(A^{\otimes})=[Lf,Rf]$, where $Lf={}^{LA}/_{N}, Rf={}^{RA}/_{N}, Lf=Rf$.

From Definition 4.2, we see easily that grey frequency has the following basic properties:

1. $0\le f_n(A^{\otimes})=[Lf,Rf]\le 1$; that is, $0\le LF\le 1, 0\le RF\le 1$.
2. In the experiment of group k, if K is the sample space, then $f_n(K^{\otimes})=1$, which is also written in the form of $f_n(K^{\otimes})=1=[1,1]$.
3. If $A_1^{\otimes},A_2^{\otimes},\ldots,A_k^{\otimes}$ are mutually exclusive events, then $f_n(A_1^{\otimes}\cup A_2^{\otimes}\cup\cdots\cup A_k^{\otimes})=f_n(A_1^{\otimes})+f_n(A_2^{\otimes})+\cdots+f_n(A_k^{\otimes})$.

Definition 4.3 Grey interval probability or simply grey probability: Assume E is a random experiment and S is its sample space. For every grey event A of E, we specify an interval grey number $A^{\otimes}=[a,b]$ in positive real number range, denoted

$P(A^{\otimes}) = [LA, RA], LA \le RA$; we call it the grey interval probability of grey event $A^{\otimes}$ (grey probability in abbreviation). If the set function satisfies the following conditions:

1. For every grey event $A^{\otimes}$, we have $P(A^{\otimes}) = [LA, RA] \ge 0$; that is, $LA \ge 0$, $RA \ge 0$;
2. P(S) = 1;
3. Suppose $A_1^{\otimes}, A_2^{\otimes}, \ldots$ are mutually exclusive events—that is, for $i \ne j$, $A_i^{\otimes} A_j^{\otimes} = \phi; i, j = 1, 2, \ldots$—and we have

$$P(A_1^{\otimes} \cup A_2^{\otimes} \cup \cdots) = P(A_1^{\otimes}) + P(A_2^{\otimes}) + \cdots \tag{4.2}$$

Equation (4.2) is called the denumerable and additive property of grey probability.

Definition 4.4 The sum grey probability of sample space S is 1: Assume the sample space S of a random experiment E consists of m events, $A_1^{\otimes}, A_2^{\otimes}, \ldots, A_n^{\otimes}$, that are mutually exclusive—that is, for $i \ne j, A_i^{\otimes} A_j^{\otimes} = \phi; i, j = 1, 2, \ldots, m$. The sum grey probability of sample space S equals 1.

$$\begin{aligned}
P(S^{\otimes}) &= \sum_{i=1}^{m} P(A_i^{\otimes}) = [LA_1, RA_1] + [LA_2, RA_2] + \cdots + [LA_m, RA_m] \\
&= LA_1 + \tfrac{1}{RA_1 - LA_1}[0,1] + LA_2 + \tfrac{1}{RA_2 - LA_2}[0,1] + \cdots + LA_m + \tfrac{1}{RA_m - LA_m}[0,1] \\
&= LA_1 + \tfrac{1}{RA_1 - LA_1} \cdot \gamma_1 + LA_2 + \tfrac{1}{RA_2 - LA_2} \cdot \gamma_1 + \cdots + LA_m + \tfrac{1}{RA_m - LA_m} \cdot \gamma_m \\
&= [1,1] \big|_{\gamma_1 = c_1, \gamma_2 = c_2, \ldots, \gamma_m = c_m, 0 \le c_i \le 1, i = 1, 2, \ldots, m} = 1 \big|_{\gamma_1 = c_1, \gamma_2 = c_2, \ldots, \gamma_m = c_m, 0 \le c_i \le 1, i = 1, 2, \ldots, m}
\end{aligned}$$

The equation $P(S^{\otimes}) = [1,1] \big|_{\gamma_1 = c_1, \gamma_2 = c_2, \ldots, \gamma_m = c_m, 0 \le c_i \le 1, i = 1, 2, \ldots, m} = 1 \big|_{\gamma_1 = c_1, \gamma_2 = c_2, \ldots, \gamma_m = c_m, 0 \le c_i \le 1, i = 1, 2, \ldots, m}$ can be simply written as $P(S^{\otimes}) = [1,1]_{\otimes} = 1_{\otimes}$.

Definition 4.4 shows that for a random experiment E, if only the unit grey number $\gamma_i, i = 1, 2, \ldots, m$ of m mutually exclusive grey events $A_i^{\otimes}, i = 1, 2, \ldots, m$, which constitutes the sample space S, specifies an accurate constant $c_i, i = 1, 2, \ldots, m$, the sum grey probability of S equals 1. We make such a definition in order not to go against the classical definition, which provides grey probability with mathematical properties as well as classical probability.

From Definitions 4.3 and 4.4 regarding grey probability, we can deduce some properties of grey probability.

Property 4.1 $P(\varphi^{\otimes}) = [LP(\varphi), RP(\varphi)] = [0, 0]$.

Proof: Set $A_n^{\otimes} = \phi, (n = 1, 2, \ldots)$, and we rewrite it in the form of interval grey number as

$$A_n^{\otimes} = [L\varphi, R\varphi] = [\varphi, \varphi]; \bigcup_{n=1}^{\infty} A_n^{\otimes} = [\varphi, \varphi], \quad \text{and}$$

$$A_i^{\otimes} A_j^{\otimes} = [LA_i \cdot LA_j, RA_i \cdot RA_j] = [\phi \cdot \phi, \phi \cdot \phi] = [\phi, \phi], i \neq j$$

According to the denumerable and additive property of grey interval probability [see Eq. (4.2)], we have

$$P(\varphi^{\otimes}) = [LP(\varphi), RP(\varphi)] = \left[\bigcup_{n=1}^{\infty} L(A_n), \bigcup_{n=1}^{\infty} R(A_n)\right] = \left[\sum_{n=1}^{\infty} LP_n, \sum_{n=1}^{\infty} RP_n\right]$$

$$= \left[\sum_{n=1}^{\infty} LP_n(\varphi), \sum_{n=1}^{\infty} RP_n(\varphi)\right]$$

Since the two real numbers satisfy $LP_n(\varphi) \geq 0, RP_n(\varphi) \geq 0,$ we deduce from the above equation that

$$P(\varphi^{\otimes}) = [LP(\varphi), RP(\varphi)] = [0, 0]$$

Property 4.2 If $A_1^{\otimes}, A_2^{\otimes}, \ldots, A_n^{\otimes}$ are mutually exclusive grey events, we have

$$P\left(A_1^{\otimes} \cup A_2^{\otimes} \cup \cdots \cup A_n^{\otimes}\right) = P\left(A_1^{\otimes}\right) + P\left(A_2^{\otimes}\right) + \cdots + P\left(A_n^{\otimes}\right)$$

$$= [LP(A_1), RP(A_1)] + \cdots + [LP(A_n), RP(A_n)] \quad (4.3)$$

Equation (4.3) is called the finite and additive property of grey probability.

Proof: Set $A_{n+1}^{\otimes} = A_{n+2}^{\otimes} = \cdots = \phi$, and we have $A_i^{\otimes} A_j^{\otimes} = \phi, i \neq j; i, j = 1, 2, \ldots$. Deduced from Eq. (4.2), we have

$$P\left(A_1^{\otimes} \cup A_2^{\otimes} \cup \cdots A_n^{\otimes}\right) = P\left(\bigcup_{k=1}^{\infty} A_k^{\otimes}\right) = \sum_{k=1}^{\infty} P\left(A_k^{\otimes}\right) = \sum_{k=1}^{n} P\left(A_k^{\otimes}\right) + 0$$

$$= P\left(A_1^{\otimes}\right) + P\left(A_2^{\otimes}\right) + \cdots + P\left(A_n^{\otimes}\right)$$

Property 4.3 If $A^{\otimes}$ and $B^{\otimes}$ are two grey events and $A^{\otimes} \subset B^{\otimes}$, then

$$P(B^{\otimes} - A^{\otimes}) = P(B^{\otimes}) - P(A^{\otimes}) \tag{4.4}$$

$$P(B^{\otimes}) \geq P(A^{\otimes}) \tag{4.5}$$

Proof: From $A^{\otimes} \subset B^{\otimes}$, we know $B^{\otimes} = A^{\otimes} \bigcup (B^{\otimes} - A^{\otimes})$ and $A^{\otimes}(B^{\otimes} - A^{\otimes}) = \phi$; from the finite and additive property of grey probability [see Eq. (4.2)], we have

$$P(B^{\otimes}) = P(A^{\otimes}) + P(B^{\otimes} - A^{\otimes})$$

Thus, Eq. (4.4) is proved.

Then according to the definition of probability, $P(B^{\otimes} - A^{\otimes}) = P([LB, RB] - [LA, RA]) \geq 0$, and we deduce that $P(B^{\otimes}) \geq P(A^{\otimes})$.

Property 4.4 For any grey event $A^{\otimes}$, $P(A^{\otimes}) = P[LA, RA] \leq 1$; that is, $P(LA) \leq 1$, $P(RA) \leq 1$.

Proof: Since $A^{\otimes} \subset S$, from Property 4.3, we have $P(A^{\otimes}) \leq P(S^{\otimes}) = 1$, that is,

$$P(A^{\otimes}) = P[LA, RA] \leq P[LS, RS] = [1,1]$$

Property 4.5 For any grey event $A^{\otimes}$,

$$\begin{aligned} P(\bar{A}^{\otimes}) &= P[L\bar{A}, R\bar{A}] = [P(L\bar{A}), P(R\bar{A})] = 1 - P(A^{\otimes}) = 1 - P[LA, RA] \\ &= [1 - P(LA), 1 - P(RA)] \end{aligned}$$

Proof: Since $A^{\otimes} \bigcup \bar{A}^{\otimes} = S$ and $A^{\otimes}\bar{A}^{\otimes} = \phi$, from Eq. (4.3), we have

$$\begin{aligned} [1,1] &= P(S^{\otimes}) = P(A^{\otimes} \cup \bar{A}^{\otimes}) = P(A^{\otimes}) = P(\bar{A}^{\otimes}) = P[LA, RA] = P[L\bar{A}, R\bar{A}] \\ &= [P(LA), P(RA)] + [P(L\bar{A}), P(R\bar{A})] \end{aligned}$$

Property 4.6 For any two events $A^{\otimes}$, $B^{\otimes}$,

$$P(A^{\otimes} \bigcup B^{\otimes}) = P(A^{\otimes}) + P(B^{\otimes}) - P(A^{\otimes}B^{\otimes}) \tag{4.6}$$

Proof: Since $A^{\otimes} \subset S$ from Property 4.3, it follows that $P(A^{\otimes}) \leq P(S^{\otimes}) = 1$.

As $A^{\otimes} \bigcup B^{\otimes} = A^{\otimes} \bigcup (B^{\otimes} - A^{\otimes} \cdot B^{\otimes})$ [see Fig. 4.2(b)], and $A^{\otimes} \bigcup (B^{\otimes} - A^{\otimes} \cdot B^{\otimes}) = \phi$, $A^{\otimes}B^{\otimes} \subset B^{\otimes}$ from Properties 4.2 and 4.3, we deduce that

$$P(A^{\otimes} \bigcup B^{\otimes}) = P(A^{\otimes}) + P(B^{\otimes} - A^{\otimes}B^{\otimes}) = P(A^{\otimes}) + P(B^{\otimes}) - P(A^{\otimes}B^{\otimes})$$

In fact, Eq. (4.6) can also be extended to the condition of several grey events. For instance, let $A_1^\otimes, A_2^\otimes, A_3^\otimes$ be any three grey events, then

$$P\left(A_1^\otimes \cup A_2^\otimes \cup A_3^\otimes\right) = P\left(A_1^\otimes\right) + P\left(A_2^\otimes\right) + P\left(A_3^\otimes\right) - P\left(A_1^\otimes A_2^\otimes\right) - P\left(A_1^\otimes A_3^\otimes\right) - P\left(A_2^\otimes A_3^\otimes\right) + P\left(A_1^\otimes A_2^\otimes A_3^\otimes\right) \tag{4.7}$$

Generally, for any n grey events $A_1^\otimes, A_2^\otimes, \ldots, A_n^\otimes$, we can use induction to prove that

$$P\left(A_1^\otimes \cup A_2^\otimes \cup \cdots \cup A_3^\otimes\right) = \sum_{i=1}^{n} P\left(A_i^\otimes\right) - \sum_{1\le i<j\le n} P\left(A_i^\otimes A_j^\otimes\right) + \sum_{1\le i<j<k\le n} P\left(A_i^\otimes A_j^\otimes A_k^\otimes\right) + \cdots + (-1)^{n-1} P\left(A_1^\otimes A_2^\otimes \cdots A_n^\otimes\right) \tag{4.8}$$

4.1.4 Grey Mixed Strategy and Related Theorems

If there is no pure strategic solution existing in standard grey matrix game $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$, it is critical for a player to keep his own strategy secret, or he will give an advantage to his opponent. How can he keep his opponent from knowing his strategy? The best method is to select his strategy in a random way. In a game, the players would use some random equipment to determine which pure strategy they should choose. For every pure strategy, there is a possibility of choosing it—players select every pure strategy with a grey interval probability. Thus, it is impossible for the opponent to know the player's pure strategy in advance. Even a player himself does not know the result of his selection, for the pure strategy is selected at the last moment with the help of random equipment. Therefore, a player's former selection of pure strategy is replaced with the selection of grey interval probability for every pure strategy.

Definition 4.5 Grey mixed strategy and grey mixed situation: Given a grey matrix game $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$, where $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, $S_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$, and $\tilde{A}(\otimes) = (a_{ij}(\otimes))_{m\times n}$, we say that the corresponding grey probability vectors $X^\otimes = (x_1^\otimes, x_2^\otimes, \ldots, x_m^\otimes)$ where $(x_i \ge 0, i = 1, 2, \ldots, m, \Sigma_{i=1}^{m} x_i^\otimes = [1,1]_\otimes)$ and $Y^\otimes = (y_1^\otimes, y_2^\otimes, \ldots, y_n^\otimes)$ where $(y_j \ge 0, j = 1, 2, \ldots, n, \Sigma_{j=1}^{n} y_j^\otimes = [1,1]_\otimes)$ are in pure strategic set S_1, S_2 for grey mixed strategies for Players 1 and 2 respectively. $(X^\otimes, Y^\otimes)$ is called the grey mixed situation.

We denote the set consisting of all of Player 1's grey mixed strategies as $S_1^{\otimes}$, that is, $S_1^{\otimes} = \{X^{\otimes}\}$; and the one consisting of all of Player 2's grey mixed strategies is $S_2^{\otimes}$, that is, $S_2^{\otimes} = \{Y^{\otimes}\}$.

In reality, Player 1's (or Player 2's) grey mixed strategy $X^{\otimes}$(or $Y^{\otimes}$) is the distribution of grey interval probabilities in set S$_1$ (or S$_2$), where $x_i^{\otimes}$(or $y_j^{\otimes}$) is the grey probability of Player 1 (or Player 2) selecting pure strategy α_i (or β_j). Therefore, pure strategy can also be seen as the special situation of a grey mixed strategy. For example, as Player 1 selects pure strategy α_i, it is a grey mixed strategy $X_i^{\otimes} = (0,\ldots 0, x_i^{\otimes} = [1,1]_{\otimes}, 0,\ldots,0)$.

If Player 1 selects strategy $X^{\otimes} = (x_1^{\otimes}, x_2^{\otimes},\ldots,x_m^{\otimes})$, and Player 2 selects strategy $Y^{\otimes} = (y_1^{\otimes}, y_2^{\otimes},\ldots,y_n^{\otimes})$, the pure strategies α_i and β_j the two players select respectively can be regarded as independent grey events. Thus, the grey interval probabilities of situation (α_i, β_j)'s occurrence are $x_i^{\otimes}$ and $y_j^{\otimes}$ respectively, and the revenue of Player 1 is $a_{ij}(\otimes)$. According to the concept of mathematical expectation, we introduce the notion of grey payment function.

Definition 4.6 Grey payment function: The grey mathematical expectation $E^{\otimes}(X^{\otimes},Y^{\otimes}) = \Sigma_{i=1}^{m}\Sigma_{j=1}^{n} a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes} = [LE, RE], LE \le RE$ is called the grey payment (or grey payment function) of Player 1.

$E^{\otimes}(X^{\otimes},Y^{\otimes})$ is the function of the grey mixed situation $(X^{\otimes},Y^{\otimes})$.

Definition 4.7 Grey mixed extension: Assume in a grey matrix game $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$, $X^{\otimes}, Y^{\otimes}$ are strategies used by Players 1 and 2 respectively.

$$S_1^{\otimes} = \left\{ X^{\otimes} \in R^m \,|x_i^{\otimes} \ge 0, i = 1,2,\ldots,m, \sum_{i=1}^{m} x_i^{\otimes} = [1,1]_{\otimes} \right\}$$

$$S_2^{\otimes} = \left\{ Y^{\otimes} \in R^n \,|y_j^{\otimes} \ge 0, j = 1,2,\ldots,n, \sum_{i=1}^{m} y_j^{\otimes} = [1,1]_{\otimes} \right\}$$

$$E^{\otimes} = E^{\otimes}(X^{\otimes},Y^{\otimes}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes} = [LE, RE], LE \le RE$$

We call $\tilde{G}^{\otimes} = \{S_1^{\otimes}, S_2^{\otimes}, E^{\otimes}\}$ the grey mixed extension of $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$.

In Definition 4.7, Player 1's one grey mixed strategy is a grey m-dimensional vector:

$$X^{\otimes} = \left\{ x_1^{\otimes}, x_2^{\otimes},\ldots,x_m^{\otimes} \right\}, \sum_{i=1}^{m} x_1^{\otimes} = \sum_{i=1}^{m} [Lx_i, Rx_i] = [1,1]_{\otimes}, 0 \le Lx_i, Rx_i 1, i = 1,2,\ldots,m;$$

Player 2's one mixed strategy is a grey n-dimensional vector:

$$\sum_{j=1}^{n} y_j^{\otimes} = \sum_{j=1}^{n} [Ly_j, Ry_j] = [1,1]_{\otimes}, 0 \le Ly_j, Ry_j \le 1, \ j = 1,2,\ldots,n$$

Definition 4.8 Solution for a grey mixed strategy: Suppose $\tilde{G}^{\otimes} = \{S_1^{\otimes}, S_2^{\otimes}, E^{\otimes}\}$ is the grey mixed extension of grey matrix game $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$. If for any $X^{\otimes} \in S_1^{\otimes}, Y^{\otimes} \in S_2^{\otimes}$ there exists a grey mixed condition $(X^{*\otimes}, Y^{*\otimes})$ in $S_1^{\otimes} = \{X^{\otimes}\}$, $S_2^{\otimes} = \{Y^{\otimes}\}$ such that

$$E^{\otimes}(X^{\otimes}, Y^{*\otimes}) \le E^{\otimes}(X^{*\otimes}, Y^{*\otimes}) \le E^{\otimes}(X^{*\otimes}, Y^{\otimes}) \tag{4.9}$$

Then the grey mixed situation $(X^{*\otimes}, Y^{*\otimes})$ is the solution of $\tilde{G}$ under a grey mixed strategy. $X^{*\otimes}, Y^{*\otimes}$ are the optimal grey (mixed) strategies of Players 1 and 2 respectively. The payment of Player 1 $E^{\otimes}(X^{*\otimes}, Y^{*\otimes})$ is called the grey game value of $\tilde{G}$ under a grey mixed strategy, denoted $V_{\tilde{G}^{\otimes}}^{*}$.

The direct meaning of Definition 4.8 is if $X^{*\otimes}, Y^{*\otimes}$ are strategies satisfying Eq. (4.9), then Player 1 can adopt strategy $X^{*\otimes}$ whatever strategy $Y^{\otimes}$ that Player 2 selects such that the least revenue of Player 1 is $E^{\otimes}(X^{*\otimes}, Y^{*\otimes})$. $E^{\otimes}(X^{*\otimes}, Y^{*\otimes})$ is thus the minimum grey expectation value (revenue) Player 1 can get. Player 2's selection of strategy $Y^{\otimes}$ makes the maximum grey expectation revenue of Player 1 $E^{\otimes}(X^{*\otimes}, Y^{*\otimes})$ whatever strategy $X^{\otimes}$ that Player 1 selects.

In the grey matrix game $\tilde{G}$ for Player 1, he always hopes that he can have the largest value of $E_{\otimes}(X^{\otimes}, Y^{\otimes})$ by selecting grey mixed strategy $X^{\otimes}$. Through the discussion of pure strategy, we assume that every player adopts rational action, which means Player 1 should select grey mixed strategy $X^{\otimes}$ such that the following equation holds:

$$\max_{X^{\otimes} \in S_1^{\otimes}} \ \min_{Y^{\otimes} \in S_2^{\otimes}} \ E^{\otimes}(X^{\otimes}, Y^{\otimes}) = v_1(\otimes)$$

Likewise, Player 2 should select $Y^{\otimes}$ such that his loss is the lowest; that is,

$$\min_{Y^{\otimes} \in S_2^{\otimes}} \ \max_{X^{\otimes} \in S_1^{\otimes}} \ E^{\otimes}(X^{\otimes}, Y^{\otimes}) = v_2(\otimes)$$

We can get $v_1(\otimes), v_2(\otimes)$, which satisfy the conditions above, for $S_1^{\otimes}, S_2^{\otimes}$, which are bounded and closed sets in R^m, R^n; $E^{\otimes}(X^{\otimes}, Y^{\otimes})$ is the continuous function of $X^{\otimes}, Y^{\otimes}$. Then we have the following theorem.

Theorem 4.1 $v_1(\otimes) \le v_2(\otimes)$: Suppose $\tilde{A}(\otimes)$ is the payment matrix of a standard grey matrix game $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$, then

$$v_1(\otimes) = \max_{X^{\otimes} \in S_1^{\otimes}} \ \min_{Y^{\otimes} \in S_2^{\otimes}} \ E^{\otimes}(X^{\otimes}, Y^{\otimes}) \le \min_{Y^{\otimes} \in S_2^{\otimes}} \ \max_{X^{\otimes} \in S_1^{\otimes}} \ E^{\otimes}(X^{\otimes}, Y^{\otimes}) \le v_2(\otimes)$$

Proof: Firstly, for any $X^{\otimes} \in S_1^{\otimes}, Y^{\otimes} \in S_2^{\otimes}$, we have $\min_{Y^{\otimes} \in S_2^{\otimes}} E^{\otimes}(X^{\otimes}, Y^{\otimes}) \le E^{\otimes}(X^{\otimes}, Y^{\otimes})$, and $\max_{X^{\otimes} \in S_1^{\otimes}} \min_{Y^{\otimes} \in S_2^{\otimes}} E^{\otimes}(X^{\otimes}, Y^{\otimes}) \le \max_{X^{\otimes} \in S_1^{\otimes}} E^{\otimes}(X^{\otimes}, Y^{\otimes})$. Since the left side of the inequality is a constant, we have

$$v_1(\otimes) = \max_{X^{\otimes} \in S_1^{\otimes}} \min_{Y^{\otimes} \in S_2^{\otimes}} E^{\otimes}(X^{\otimes}, Y^{\otimes}) \le \min_{Y^{\otimes} \in S_2^{\otimes}} \max_{X^{\otimes} \in S_1^{\otimes}} E^{\otimes}(X^{\otimes}, Y^{\otimes}) \le v_2$$

In 1928, Von Neumann proved that in a matrix game of ordinary whitenization numbers, the two players have equal game values in any matrix game; that is, $v_1 = v_2$. This is the famous fundamental theorem on matrix games of accurate whitenization numbers, namely, the theorem of maximum and minimum values. In such a case, does a relation such as $v_1(\otimes) = v_2(\otimes)$ still exist in a grey matrix game?

Having limited space, we will elaborate the proof of this important theorem in another place.

4.1.5 Summary

We have proposed basic concepts of grey random events, grey probabilities, and so forth in this section, studied the relations of grey events and their operational rules, and proved six important properties of grey interval probabilities. Based on this, we put forward the concepts of grey mixed strategies, grey mixed situations, and grey mathematical expectations, and constructed the framework of the solution to grey mixed extensions and grey mixed strategies, which paved the way for further study on grey matrix game problems based on mixed strategies.

4.2 Characterization of an Interval Grey Number and Improvement of Its Operation

4.2.1 Background

Example 4.2 Given interval grey numbers $a_{11}(\otimes) = 1$; $a_{12}(\otimes) = 5$; $a_{21}(\otimes) = [2,3]$; $a_{22}(\otimes) = [0,1]$ and using current operational rules of grey numbers,[25,26,27] we have the result shown in Eq. (4.10).

However, if we use an ordinary mathematical computation method, the maximum and minimum values of Eq. (4.10) are shown in Eq. (4.11).[17] That is to say, the possible range (interval grey number) of Eq. (4.10) should be $x'(\otimes) = [\frac{1}{5}, \frac{3}{7}]$ when using the classic mathematical computation method.

$$\begin{aligned} x(\otimes) &= \frac{a_{22}(\otimes) - a_{21}(\otimes)}{(a_{11}(\otimes) + a_{22}(\otimes)) - (a_{12}(\otimes) + a_{21}(\otimes))} = \frac{[0,1] - [2,3]}{(1 + [0,1]) - (5 + [2,3])} \\ &= \frac{-[1,3]}{-[5,7]} = \left[\frac{1}{7}, \frac{3}{5}\right] \end{aligned} \tag{4.10}$$

$$\max\{x(\otimes)\} = \frac{3}{7}\bigg|_{a_{21} = 3, a_{22} = 0} \qquad \min\{(\otimes)\} = \frac{1}{5}\bigg|_{a_{21} = 2, a_{22} = 1} \tag{4.11}$$

As we know from the operational results of Eqs. (4.10) and (4.11), $\frac{1}{7} < [\frac{1}{5}, \frac{3}{7}] < \frac{3}{5}$, which shows it will result in disaccord with the operational results of classical mathematics when using current operational rules of interval grey numbers.

According to the definition of a grey number's degree of greyness, we can compute that the degrees of greyness for $x(\otimes)$ and $x^{/}(\otimes)$ are respectively $g_1^o(\otimes) = \frac{l(\otimes)}{|\hat{\otimes}|} = 1.2308$ and $g_2^o(\otimes) = \frac{l(\otimes)}{|\hat{\otimes}|} = 0.7273$.

Compared to classical mathematics, this will result in a larger degree of greyness when using current operational rules of an interval grey number. The major reason lies in that it may cause the same grey number to specify different values under the same computational conditions when we use the traditional computation method of grey numbers. For example, in Eq. (4.9), when $a_{22}(\otimes) = 1, a_{21}(\otimes) = 2$ in the numerator and $a_{22}(\otimes) = 0, a_{21}(\otimes) = 3$ in the denominator, the left endpoint value of $x(\otimes)$ is $\frac{1}{7}$. Thus, the value is reduced abnormally. For the same reason, under other conditions, the right endpoint value is also enlarged abnormally using the traditional computation method of grey numbers.

Therefore, in order to avoid the drawback of the current grey algebraic computation method, here we only carry out a limited discussion regarding this problem.

4.2.2 Standard Interval Grey Number and Its Operation

Definition 4.9 Standard interval grey number: If a grey number can be written as in Eq. (4.12) and (4.3), a_i is called the white part of $\otimes_i$,

$$\otimes_i = a_i + c_i.\gamma_i, \quad i = 1, 2, \ldots, n \tag{4.12}$$

where $c_i \cdot \gamma_i$ is called the grey part of $\otimes_i$, where c_i is the grey coefficient and γ_i is the unit grey number (or grey number unit), and then we say the grey number expressed in Eq. (4.12) is the standard interval grey number.[25,26,27]

Definition 4.9 regulates the standard expression of an interval grey number. According to Definition 4.9, if there exist two standard interval grey numbers $\otimes_K = a_K + c_K \cdot \gamma_K$ and $\otimes_L = -a_K - c_K.\gamma_K$ that satisfy $\otimes_K + \otimes_L = (a_K + c_K.\gamma_K) + (-a_K - c_K \cdot \gamma_K) = 0$ $\otimes_K$ and $\otimes_L$ are opposite numbers.

If there exist two standard interval grey numbers $\otimes_K = a_K + c_K \cdot \gamma_K$ and $\otimes L = \frac{1}{\otimes_K} = \frac{1}{a_K + c_K \cdot \gamma_K}$, $\otimes_K$ and $\otimes_L$ are reciprocal numbers.

Theorem 4.2 Standard expression of grey number: Any interval grey number $\otimes_i = [a_i, b_i], a_i \le b_i, i = 1$, can be expressed as the standard grey number form in Definition 4.5 [see Eq. (4.12)].

Proof: Without loss of generality, we can transform (the process is called standard transformation) any interval grey number $\otimes_i = [a_i, b_i], a_i \le b_i, i = 1, 2, \ldots, n$ shown in Eq. (4.13) and obtain its standard interval grey number

$$\begin{aligned}\otimes_i &= [a_i,b_i] = [a_i,b_i] + a_i - a_i = a_i + [0,b_i - a_i] = a_i + (b_i - a_i)[0,1] = a_i + (b_i - a_i).\gamma_i \\ &= a_i + c_i.\gamma_i; \\ c_i &= (b_i - a_i), \gamma_i \in [0,1], i = 1,2,\ldots\end{aligned} \tag{4.13}$$

In Eq. (4.13), a_i is called the white part of $\otimes_i$ while $c_i \cdot \gamma_i$ is called the grey part of $\otimes_i$; in the grey part $c_i \cdot \gamma_i$, we call $c_i = \frac{1}{b_i - a_i}$ the grey coefficient and γ_i the unit grey number (or grey number unit).

The constructive proof of Theorem 4.2 shows that any interval grey number can be expressed in the standard form shown in Eq. (4.13) through standard transformation.

Definition 4.10 Operation rules of standard interval grey number: If we still adopt classic mathematical operational rules when doing algebraic operations of any of several standard interval grey numbers $F(\Theta) = \Theta\{\otimes_1, \otimes_2, \ldots, \otimes_n\}$, and let these grey numbers' obtained numbers $\gamma_i, \gamma_i \in [0,1], i = 1,2,\ldots,n$ be exact constants, the minimum and maximum values $\min\{F(\Theta)\} = \min \Theta\{\otimes_1, \otimes_2, \ldots, \otimes_n\}|_{\gamma_i = c_i, c_i \in [0,1], i=1,2,\ldots,n}$ and $\max\{F(\Theta)\} = \max \Theta\{\otimes_1, \otimes_2, \ldots, \otimes_n\}|_{\gamma_i = c_i, c_i \in [0,1], i=1,2}$ we obtain are respectively considered the left and right endpoint values of $F(\Theta) = \Theta\{\otimes_1, \otimes_2, \ldots, \otimes_n\}$. This operation process is called the standard interval grey number's operation process.

Example 4.3 Given interval grey numbers $a_{11}(\otimes) = 1, a_{12}(\otimes) = 5, a_{21}(\otimes) = [2,3]$, $a_{22}(\otimes) = [0,1]$, if we compute Eq. (4.13) using the operational rules of grey numbers in Definition 4.10, the result is shown in Eq. (4.14).

Since $a_{21}(\otimes) = 2 + [0,1]_{21} = 2 + \gamma_{21}, a_{22}(\otimes) = [0,1]_{22} = \gamma_{22}$, we have

$$\begin{aligned} x(\otimes) &= \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{\gamma_{22} - (2 + \gamma_{21})}{(1 + \gamma_{22}) - (5 + 2 + \gamma_{21})} = \frac{\gamma_{22} - \gamma_{21} - 2}{\gamma_{22} + -6 - \gamma_{21}} \\ \max\{x(\otimes)\} &= \max\left\{\frac{\gamma_{22} - \gamma_{21} - 2}{\gamma_{22} + -6 - \gamma_{21}}\right\} |\gamma_{21} = 1, \gamma_{22} = 0 = \frac{3}{7}; \\ \min\{x(\otimes)\} &= \min\left\{\frac{\gamma_{22} - \gamma_{21} - 2}{\gamma_{22} + -6 - \gamma_{21}}\right\} |\gamma_{21} = 0, \gamma_{22} = 1 = \frac{1}{5} \\ x(\otimes) &= [\min\{x(\otimes)\}, \max\{x(\otimes)\}] = \left[\frac{1}{5}, \frac{3}{7}\right] \end{aligned} \tag{4.14}$$

Adopting the classic mathematical method of finding extreme values, we similarly obtain the maximum and minimum values of Eq. (4.14): $\max\{x(\otimes)\} = \frac{3}{7}, \min\{x(\otimes)\} = \frac{1}{5}$. Compared with Eq. (4.14), we know the computation results of the two different methods are the same.

From Definition 4.2, we know that since the operational rules of standard interval grey numbers use classic mathematical methods of finding extreme value to determine final results, their computation results certainly equal that of classical mathematics, without causing the grey number's degree of greyness in the result to be enlarged abnormally.

4.2.3 The First and Second Standard Grey Numbers

Definition 4.11 The first and second standard grey numbers: Given any grey number $\otimes_i = [a_i, b_i], (a_i \le b_i, i = 1, 2, \ldots)$, if we conduct a standard transformation on the left endpoint value [as in Eq. (4.15)], the standard grey number we get is called a grey number in the first standard expression, or simply the first standard grey number, where a_i, $(b_i - a_i)\gamma_i^{(1)}$ and $\gamma_i^{(1)}$ are called the white part, grey part, and unit grey number (grey coefficient) respectively. If we conduct a standard transformation on the right endpoint value [as in Equation (4.15)], the standard grey number we get is called a grey number in the second standard expression, or simply the second standard grey number, where b_i, $-(b_i - a_i)\gamma_i^{(2)}$ and $\gamma_i^{(2)}$ are called the white part, grey part, and unit grey number (grey coefficient) respectively.

$$\begin{aligned}\otimes_i^1 &= [a_i, b_i] = a_i - a_i + [a_i, b_i] = a_i + [0, b_i - a_i] = a_i + (b_i - a_i)[0,1] \\ &= a_i + (b_i - a_i)\gamma_i^{(1)}, \left(0 \le \gamma_i^{(1)} \le 1\right)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\otimes_i^2 &= [a_i, b_i] = b_i - b_i + [a_i, b_i] = b_i - [0, b_i - a_i] = b_i + (b_i - a_i)[0,1] \\ &= b_i - (b_i - a_i)\gamma_i^{(2)}, \left(0 \le \gamma_i^{(2)} \le 1\right)\end{aligned} \tag{4.16}$$

Any standard grey number can be standardized into expressions of the first and second standard grey numbers. Although these two expressions of standard grey numbers are different in form, they are the same in essence, for they express the same grey number.

Theorem 4.3 The sum of the first and second standard grey numbers' obtained number is 1: Given any grey number $\otimes_i = [a_i, b_i], (a_i \le b_i, i = 1, 2, \ldots)$, if we express it as the first standard grey number $(\otimes_i^1)$ and the second standard grey number $(\otimes_i^2)$, where $\gamma_i^{(1)}, (0 \le \gamma_i^{(1)} \le 1)$ and $\gamma_i^{(2)}, (0 \le \gamma_i^{(2)} \le 1)$ stand for the two standard grey numbers respectively, the sum of the standard grey numbers in the two different expressions equals 1—that is, $\gamma_i^{(1)} + \gamma_i^{(2)} = 1$.

Proof: Without the loss of generality, any given grey number $\otimes_i = [a_i, b_i]$, $(a_i \le b_i, i = 1, 2, \ldots)$ can be expressed in the forms of the first and second standard grey numbers according to Definition 4.5, which is shown in Eqs. (4.15) and (4.16).

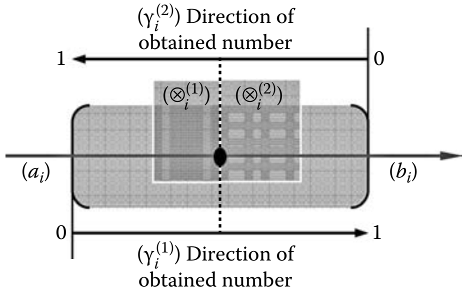


Figure 4.3 Relation between the first and second standard grey number's obtained number $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$.

As Eqs. (4.15) and (4.16) express the same grey number, in spite of the formal difference between the two expressions of standard grey numbers, they should be equal in essence, as in Eq. (4.17):

$$\otimes_i^{(1)} = \otimes_i^{(2)}, \quad a_i + (b_i - a_i)\gamma_i^{(1)} = b_i - (b_i - a_i)\gamma_i^{(2)}, \quad \gamma_i^{(1)} + \gamma_i^{(2)} = 1 \qquad (4.17)$$

As we know from Theorem 4.2, the different expressions of the same grey number have no distinction in essence. We can illustrate this point by Figure 4.3. Even though the two expressions of standard grey numbers have no difference in essence, we should adopt one expression in the computational process of one problem in order to make it comprehensible and avoid the confusion of obtained number, especially for the same grey number.

4.2.4 Judgment of Quantitative Relations of Standard Grey Numbers

In Chapter 2, we gave several simple forms of grey numbers whose quantitative relations can be easily determined by general conditions and the method of their judgment. Next, we will discuss the forms of compound standard grey numbers and the judgments of their quantitative relations.

Definition 4.12. Compound standard grey number: If a grey number ($\otimes_k$) is obtained from several standard grey numbers $(\otimes_{ij})$ after several times of adding, subtracting, multiplying, dividing, and other forms of mathematical operations, it is called a compound standard grey number, or compound grey number for short, and designated $\otimes_k = f(\otimes_{ij})_{\gamma_{ij}} = \min_{\gamma_{ij}} f(\bullet), \max_{\gamma_{ij}} f(\bullet)]$; $(0 \le \gamma_{ij} \le 1, i, j = 1,2,\ldots,n)$.

From Definition 4.12, we know that a compound standard grey number is obtained from a single standard grey number after several mathematical operations. Also, we can get compound standard grey numbers from a compound standard grey number or together with a single standard grey number after such mathematical operations.

Compound standard grey numbers and the first and second standard grey numbers have the same expression, which has the white part, grey part, and grey coefficient. Yet the grey coefficient of its grey part is the compound of several single grey numbers.

Definition 4.13 Judgment of quantitative relation of compound grey numbers: Given two compound grey numbers $\otimes_K = f(\otimes_{ij})_{\gamma_{ij}}, (0 \le \gamma_{ij} \le 1, i, j = 1, 2, \ldots$ and $\otimes_L = f(\otimes_{uv})_{\gamma_{uv}}$, $(0 \le \gamma_{uv} \le 1, u, v = 1, 2, \ldots)$, if they satisfy $\otimes_K \ge \otimes_L$ whatever the obtained numbers of $\gamma_{ij}, (0 \le \gamma_{ij} \le 1, i, j = 1, 2, \ldots$ and $\gamma_{uv}, (0 \le \gamma_{uv} \le 1, u, v = 1, 2, \ldots)$ are, we say that the compound grey number $\otimes_K$ is smaller than $\otimes_L$. If they satisfy $\otimes_K = \otimes_L$ whatever the obtained numbers of $\gamma_{ij}, (0 \le \gamma_{ij} \le 1, i, j = 1, 2, \ldots)$ and $\gamma_{uv}, (0 \le \gamma_{uv} \le 1, u, v = 1, 2, \ldots)$ are, we say that the compound grey number $\otimes_K$ is equal to $\otimes_L$. Otherwise, we say the relationship between compound grey number $\otimes_K$ and $\otimes_L$ cannot be determined.[3,4,5,6]

Example 4.4 Given two single standard grey numbers $\otimes_1 = a + (b-a)\gamma_1$, $(0 \le \gamma_1 \le 1); \otimes_2 = c + (d-c)\gamma_2, (0 \le \gamma_2 \le 1)$, the compound standard grey numbers obtained by addition and subtraction of the two standard grey numbers are $\otimes_{12}^{(1)}$ and $\otimes_{12}^{(2)}$ respectively:

$$\otimes_{12}^{(1)} = \otimes_1 + \otimes_2 = a + (b-a)\gamma_1 + c + (d-c)\gamma_2 = (a+c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1 + \frac{1}{(d-c)}\gamma_2\right)$$

$$\otimes_{12}^{(2)} = \otimes_1 - \otimes_2 = a + (b-a)\gamma_1 - c - (d-c)\gamma_2 = (a-c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1 - \frac{1}{(b-a)}\gamma_2\right)$$

If

$$\left\{(a+c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1 + \frac{1}{(d-c)}\gamma_2\right)\right\}_{\gamma_i, (0 \le \gamma_i \le 1, i=1,2)}$$

$$-\left\{(a-c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1 - \frac{1}{(b-a)}\gamma_2\right)\right\}_{\gamma_i, (0 \le \gamma_i \le 1, i=1,2)} \ge 0,$$

then $\otimes_{12}^{(1)} \ge \otimes_{12}^{(2)}$.

If

$$\left\{(a+c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1 + \frac{1}{(d-c)}\gamma_2\right)\right\}_{\gamma_i, (0 \le \gamma_i \le 1, i=1,2)}$$

$$-\left\{(a-c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1 - \frac{1}{(b-a)}\gamma_2\right)\right\}_{\gamma_i, (0 \le \gamma_i \le 1, i=1,2)}$$

,

then $\otimes_{12}^{(1)} < \otimes_{12}^{(2)}$.

If

$$\left\{(a+c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1+\frac{1}{(d-c)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le1,i=1,2)}$$

$$-\left\{(a-c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1-\frac{1}{(b-a)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le1,i=1,2)}=0$$

then $\otimes_{12}^{(1)}=\otimes_{12}^{(2)}$.

Otherwise, we say $\otimes_{12}^{(1)}$ and $\otimes_{12}^{(2)}$ are compound grey numbers whose quantitative relationship cannot be determined.

4.2.5 Case Study

According to the characterization and operational rules of standard grey numbers, we take a 2 × 2 zero-sum matrix game based on the grey matrix form as an example and compute the solution of its mixed strategy as shown in Example 4.5.

Example 4.5. Find the solution of grey matrix game $\tilde{G}(\otimes)=\{S_1^{\otimes},S_2^{\otimes},\tilde{A}(\otimes)\}$ where

$$\tilde{A}(\otimes)=\begin{pmatrix}[0,1] & [2,3]\\ 4 & 2\end{pmatrix}=\begin{pmatrix}\gamma_{11} & 2+\gamma_{12}\\ 4 & 2\end{pmatrix},(\gamma_{11}\in[0,1],\gamma_{12}\in[0,1]).$$

Solution: Using the formula of finding a solution to the 2 × 2 zero-sum matrix game's mixed strategy (Gan Ying'ai et al., Operation Research, Qinghua University Publish House, 1990) and the characterization and operational rules of standard grey numbers provided by this chapter, we have $a_{11}(\otimes)=\gamma_{11},a_{12}(\otimes)=2+\gamma_{12},a_{21}(\otimes)=4,a_{22}(\otimes)$ $=2$, so the optimal grey mixed strategies and grey game values for Players 1 and 2 are respectively shown in Eqs. (4.18), (4.19), (4.20), (4.21), and (4.22). Comparison of computation methods of original and standard grey numbers are demonstrated in Table 4.2.

$$x_1^*(\otimes)=\frac{a_{22}(\otimes)-a_{21}(\otimes)}{(a_{11}(\otimes)+a_{22}(\otimes))-(a_{12}(\otimes)+a_{21}(\otimes))}=\frac{2-4}{(\gamma_{11}+2)-(2+\gamma_{12}+4)}$$

$$=\frac{-2}{\gamma_{11}-\gamma_{12}-4}=\begin{cases}\frac{2}{3}, & if,\gamma_{11}=1,\gamma_{12}=0\\ \frac{2}{5}, & if,\gamma_{11}=0,\gamma_{12}=1\end{cases}$$

that is,

$$x_1^*(\otimes)=\left[\frac{2}{5},\frac{2}{3}\right] \tag{4.18}$$

Table 4.2 Comparison of Computation Method of Original and Standard Grey Number

Method / *Index*	*Computation Method of Original Grey Number*		*Computation Method of Standard Grey Number*	
	Result	*Degree of Greyness*	*Result*	*Degree of Greyness*
$x_1^*(\otimes)$	[5/2, 2/3]	0.5000	[5/2, 2/3]	0.5000
$x_2^*(\otimes)$	[1/5, 1]	1.330	[1/3, 3/5]	0.5700
$y_1^*(\otimes)$	[0, 1/3]	2.0000	[0, 1/4]	2.0000
$y_2^*(\otimes)$	[3/5, 4/3]	0.4889	[5/4, 1]	0.2222
$v_G^*(\otimes)$	[6/5, 4]	1.0769	[2, 5/2]	0.4444

$$x_2^*(\otimes)=\frac{a_{11}(\otimes)-a_{12}(\otimes)}{(a_{11}(\otimes)+a_{22}(\otimes))-(a_{12}(\otimes)+a_{21}(\otimes))}=\frac{\gamma_{11}-(2+\gamma_{12})}{(\gamma_{11}+2)-(2+\gamma_{12}+4)}=\frac{\gamma_{11}-2-\gamma_{12}}{\gamma_{11}-\gamma_{12}-4}$$

$$=\begin{cases}\dfrac{1}{3}, & if,\gamma_{11}=1,\gamma_{12}=0\\[2ex] \dfrac{3}{5}, & if,\gamma_{11}=0,\gamma_{12}=1\end{cases}$$

that is,

$$x_2^*(\otimes)=\left[\frac{1}{3},\frac{3}{5}\right] \tag{4.19}$$

$$y_1^*(\otimes)=\frac{a_{22}(\otimes)-a_{12}(\otimes)}{(a_{11}(\otimes)+a_{22}(\otimes))-(a_{12}(\otimes)+a_{21}(\otimes))}=\frac{2-(2+\gamma_{12})}{(\gamma_{11}+2)-(2+\gamma_{12}+4)}=\frac{-\gamma_{12}}{\gamma_{11}-\gamma_{12}-4}$$

$$=\begin{cases}0, & if,\gamma_{12}=0\\[2ex] \dfrac{1}{4}, & if,\gamma_{11}=1,\gamma_{12}=1\end{cases}$$

that is,

$$y_1^*(\otimes)\left[0,\frac{1}{4}\right] \tag{4.20}$$

$$y_2^*(\otimes)=\frac{a_{11}(\otimes)-a_{21}(\otimes)}{(a_{11}(\otimes)+a_{22}(\otimes))-(a_{12}(\otimes)+a_{21}(\otimes))}=\frac{\gamma_{11}-4}{(\gamma_{11}+2)-(2+\gamma_{12}+4)}=\frac{\gamma_{11}-4}{\gamma_{11}-\gamma_{12}-4}$$

$$=\begin{cases} 1, & if,\gamma_{12}=0 \\ \frac{4}{5}, & if,\gamma_{11}=0,\gamma_{12}=1 \end{cases}$$

that is,

$$y_2^*(\otimes)=\left[\frac{4}{5},1\right] \tag{4.21}$$

$$v_G^*(\otimes)=\frac{a_{11}(\otimes)\circlearrowright a_{22}(\otimes)-a_{12}(\otimes)\circlearrowright a_{21}(\otimes)}{(a_{11}(\otimes)+a_{22}(\otimes))-(a_{12}(\otimes)+a_{21}(\otimes))}=\frac{2\circlearrowright\gamma_{11}-(2+\gamma_{12})\circlearrowright 4}{(\gamma_{11}+2)-(2+\gamma_{12}+4)}=\frac{2\circlearrowright\gamma_{11}-8-4\circlearrowright\gamma_{12}}{\gamma_{11}-\gamma_{12}-4}$$

$$=\begin{cases} 2, & if,\gamma_{12}=0 \\ \frac{5}{2}, & if,\gamma_{11}=1,\gamma_{12}=1 \end{cases}$$

that is,

$$v_G^*(\otimes)=\left[2,\frac{5}{2}\right] \tag{4.22}$$

4.2.6 Summary

This section aimed at the severe problems in current characterizations and operations of interval grey numbers that result in unnecessarily enlarged degrees of greyness and heavily distorted information. This section defined the standard interval grey number and the first and second standard grey numbers, and analyzed the relationship between the first and second standard grey numbers. Furthermore, it designed the transformation rule from a general interval grey number to a standard interval grey number, and provided the comparison and operational rules of standard interval grey numbers, which fairly solves the problems of quantitative comparisons and operations of interval grey numbers.

4.3 The Maximum-Minimum Grey Game Value and the Grey Saddle Point of Grey Mixed Strategy

4.3.1 Theorem of the Maximum-Minimum Grey Game Value

In the grey matrix game $\tilde{G}=\{S_1,S_2,\tilde{A}(\otimes)\}$, Player 1 always hopes that by selecting grey mixed strategy $X^{\otimes}$ he can get the largest game expectation value $E_{\otimes}(X^{\otimes},Y^{\otimes})$. Through the discussion of a grey matrix game with pure strategy, we know that every player will adopt rational action—that is, Player 1 should select grey mixed strategy $X^{\otimes}$such that Eq. (4.23) holds:

$$\max_{X^{\otimes}\in S_1^{\otimes}} \quad \min_{Y^{\otimes}\in S_2^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) = v_1(\otimes) \tag{4.23}$$

Player 2 should select $Y^{\otimes}$such that his loss is at most equal to

$$\min_{Y^{\otimes}\in S_2^{\otimes}} \quad \max_{X^{\otimes}\in S_1^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) = v_2(\otimes) \tag{4.24}$$

We can obtain $v_1(\otimes), v_2(\otimes)$ satisfying the above conditions, for $S_1^{\otimes}, S_2^{\otimes}$are respectively bounded and a closed set in R^m, R^n, and $E^{\otimes}(X^{\otimes},Y^{\otimes})$ is a successive function. Therefore, we have the following theorems.

Theorem 4.4 $v_1(\otimes) \le v_2(\otimes)$: Assuming that $\tilde{A}(\otimes)$ is the payment matrix of standard grey matrix game $\tilde{G} = \{S_1, S_2, \tilde{A}(\otimes)\}$, it is true that

$$v_1(\otimes) = \max_{X^{\otimes}\in S_1^{\otimes}} \quad \min_{Y^{\otimes}\in S_2^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) \le \min_{Y^{\otimes}\in S_2^{\otimes}} \quad \max_{X^{\otimes}\in S_1^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) \le v_2(\otimes) \tag{4.25}$$

Proof: For any $X^{\otimes} \in S_1^{\otimes}, Y^{\otimes} \in S_2^{\otimes}$, there exists $\min_{Y^{\otimes}\in S_2^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) \le E^{\otimes}(X^{\otimes},Y^{\otimes})$, so $\max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) \le \max_{X^{\otimes}\in S_1^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes})$. As the left side of the inequality is a constant, we have

$$v_1(\otimes) = \max_{X^{\otimes}\in S_1^{\otimes}} \quad \min_{Y^{\otimes}\in S_2^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) \le \min_{Y^{\otimes}\in S_2^{\otimes}} \quad \max_{X^{\otimes}\in S_1^{\otimes}} E^{\otimes}(X^{\otimes},Y^{\otimes}) \le v_2(\otimes)$$

Theorem 4.5 Theorem of the maximum-minimum grey game value: Assuming the payment matrix of a grey matrix game is $\tilde{A}(\otimes) = (a_{ij}(\otimes))_{m\times n}$ $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, then the game values of players are equal—that is,

$$\begin{aligned} v_1(\otimes) &= \max_{X^{\otimes}\in S_1^{\otimes}} \quad \min_{Y^{\otimes}\in S_2^{\otimes}} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(\otimes) x_i^{\otimes} y_j^{\otimes} \\ &= \min_{Y^{\otimes}\in S_2^{\otimes}} \quad \max_{X^{\otimes}\in S_1^{\otimes}} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(\otimes) x_i^{\otimes} y_j^{\otimes} = v_2(\otimes) \end{aligned} \tag{4.26}$$

Proof: For Theorem 4.4, we only need to prove that $v_2(\otimes) \le v_1(\otimes)$ is true. From two important theorems of grey inequality,[3] one of the following two propositions is surely true:

Proposition 4.1 There exists $Y^{\otimes}=(y_1^{\otimes},y_2^{\otimes},\ldots,y_n^{\otimes})\in S_2^{\otimes}$ such that $\Sigma_{j=1}^{n}a_{ij}(\otimes)y_j^{\otimes}\le 0, i=1,2,\ldots,m.$ Therefore, for any $X^{\otimes}=(x_1^{\otimes},x_2^{\otimes},\ldots,x_m^{\otimes})\in S_1^{\otimes}$, there exists $\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes}\le 0$, that is, $\max_{X^{\otimes}\in S_1^{\otimes}}\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes}\le 0$. So we have $v_2(\otimes)=\min_{Y^{\otimes}\in S_2^{\otimes}}\max_{X^{\otimes}\in S_1^{\otimes}}\Sigma_{i=1}^{m}\Sigma_{j=1}^{n}a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes}\le 0.$

Proposition 4.2 There exists $X^{\otimes}=(x_1^{\otimes},x_2^{\otimes},\ldots,x_m^{\otimes})\in S_1^{\otimes}$ such that $\Sigma_{i=1}^{m}a_{ij}(\otimes)x_i^{\otimes}>0, j=1,2,\ldots,n.$ Therefore, for any $Y^{\otimes}=(y_1^{\otimes},y_2^{\otimes},\ldots,y_m^{\otimes})\in S_2^{\otimes}$, there exists $\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes}>0$ that is, $\min_{Y^{\otimes}\in S_1^{\otimes}}\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes}\ge 0$. So $v_2(\otimes)=\max_{X^{\otimes}\in S_1^{\otimes}}\min_{Y^{\otimes}\in S_2^{\otimes}}\Sigma_{j=1}^{n}\Sigma_{i=1}^{m}a_{ij}(\otimes)x_i^{\otimes}y_j^{\otimes}\ge 0$ is true.

Considering both Propositions 4.1 and 4.2, we have $v_1(\otimes)\ge 0$, that is, $v_1(\otimes)=[v_1^1,v_1^2], 0\le v_1^1\le v_1^2$, or $v_2(\otimes)\le 0$, that is, $v_2(\otimes)=[v_2^1,v_2^2], v_2^1\le v_2^2\le 0$, and then $v_1(\otimes)<0<v_2(\otimes)$ does not hold.

Furthermore, we can construct a matrix $\tilde{B}(\otimes)=(a_{ij}(\otimes)-k)_{m\times n}=(a_{ij}(\otimes)-[k,k])_{m\times n}$ as Eq. (4.27).

$$\tilde{B}(\otimes)=\begin{pmatrix} \left[a_{11}^1,a_{11}^2\right]-[k,k] & \left[a_{12}^1,a_{12}^2\right]-[k,k] & \cdots & \left[a_{1n}^1,a_{1n}^2\right]-[k,k] \\ \left[a_{21}^1,a_{21}^2\right]-[k,k] & \left[a_{22}^1,a_{22}^2\right]-[k,k] & \cdots & \left[a_{2n}^1,a_{2n}^2\right]-[k,k] \\ \vdots & \vdots & \cdots & \vdots \\ \left[a_{m1}^1,a_{m1}^2\right]-[k,k] & \left[a_{m2}^1,a_{m2}^2\right]-[k,k] & \cdots & \left[a_{mn}^1,a_{mn}^2\right]-[k,k] \end{pmatrix} \tag{4.27}$$

With the above method, we have Eq. (4.28) where k is any real number:

$$v_1(\otimes)-k<0<v_2(\otimes)-k \tag{4.28}$$

Then $[v_1^1,v_1^2]-[k,k]<[0,0]<[v_2^1,v_2^2]-[k,k],[v_1^1-k,v_1^2-k]<[0,0]<[v_2^1-k,v_2^2-k]$ does not hold. Therefore, $v_1(\otimes)<k<v_2(\otimes)$ is not true and neither is the inequality $v_1(\otimes)<v_2(\otimes)$. Hence, we have $v_1(\otimes)\ge v_2(\otimes)$.

To sum up, by the results of Theorem 4.1 $v_1(\otimes)\le v_2(\otimes)$ and $v_1(\otimes)\ge v_2(\otimes)$, we deduce that

$$v_1(\otimes)\big|_{\gamma_{ij}=c_{ij},0\le c_{ij}\le 1,i=1,2,\ldots,m,j=1,2,\ldots,n}=v_2(\otimes)\big|_{\gamma_{ij}=c_{ij},0\le c_{ij}\le 1,i=1,2,\ldots,m,j=1,2,\ldots,n}$$

that is,

$$v_1(\otimes)=\left[v_1^1,v_1^2\right]_{\gamma_{ij}=c_{ij},0\le c_{ij}\le 1,i=1,2,\ldots,m,j=1,2,\ldots,n}=\left[v_2^1,v_2^2\right]_{\gamma_{ij}=c_{ij},0\le c_{ij}\le 1,i=1,2,\ldots,m,j=1,2,\ldots,n}=v_2(\otimes)$$

4.3.2 Grey Saddle Point of Grey Mixed Strategy

Having the theorem of the maximum-minimum grey number as the basic theorem of a grey matrix game, we can discuss the grey saddle point of a grey mixed strategy.

Definition 4.14 Grey saddle point of grey mixed strategy: Let the payment matrix of grey matrix game be $\tilde{A}(\otimes)=(a_{ij}(\otimes))_{m\times n}$. If there exists $X^{*\otimes}(\otimes)\in S_1^{\otimes}, Y^{*\otimes}\in S_2^{\otimes}$ such that Eq. (4.29) holds,

$$E(X^{*\otimes},Y^{*\otimes})=\max_{X^{\otimes}\in S_1^{\otimes}}\ \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes})=\min_{Y^{\otimes}\in S_2^{\otimes}}\ \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes}) \tag{4.29}$$

we say $X^{*\otimes},Y^{*\otimes}$ is a grey saddle point of the standard grey matrix game under a grey mixed strategy, and $X^{*\otimes},Y^{*\otimes}$ are respectively the optimal grey mixed strategies for Players 1 and 2. $E(X^{*\otimes},Y^{*\otimes})$ is called the grey game's value, denoted $v(\otimes)$.

Theorem 4.6 The sufficient and necessary condition for grey saddle point's existence: The sufficient and necessary condition for $X^{*\otimes},Y^{*\otimes}$ being the grey saddle point of grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2,\tilde{A}(\otimes)\}$ is

$$E(X^{\otimes},Y^{*\otimes})\le E(X^{*\otimes},Y^{*\otimes})\le E(X^{*\otimes},Y^{\otimes}),\quad \forall \quad X^{\otimes}\in S_1^{\otimes},Y^{\otimes}\in S_2^{\otimes} \tag{4.30}$$

Proof: Sufficiency: As for any $X^{\otimes},Y^{\otimes}$, there exists

$$E(X^{\otimes},Y^{*\otimes})\le E(X^{*\otimes},Y^{*\otimes})\le E(X^{*\otimes},Y^{\otimes}),\quad \forall \quad X^{\otimes}\in S_1^{\otimes},Y^{\otimes}\in S_2^{\otimes}$$

So we have $\max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{*\otimes})\le E(X^{*\otimes},Y^{*\otimes})\le \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{*\otimes},Y^{\otimes})$
Also, the following inequalities are true:

$$\min_{Y^{\otimes}\in S_2^{\otimes}}\max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes})\le \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{*\otimes}),\ \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{*\otimes},Y^{\otimes})\le \max_{X^{\otimes}\in S_1^{\otimes}}\min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes})$$

Therefore,

$$\min_{Y^{\otimes}\in S_2^{*}}\max_{X^{\otimes}\in S_1^{*}} E(X^{\otimes},Y^{\otimes})\le E(X^{*\otimes},Y^{*\otimes})\le \max_{X^{\otimes}\in S_1^{*}}\min_{Y^{\otimes}\in S_2^{*}} E(X^{\otimes},Y^{\otimes}) \tag{4.31}$$

On the other side, for $\forall\ X^{\otimes} \in S_1^{\otimes}, Y^{\otimes} \in S_2^{\otimes}$ there exists

$$\min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes}) \leq E(X^{\otimes},Y^{\otimes}) \leq \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes})$$

Thus,

$$\max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes}) \leq \min_{Y^{\otimes}\in S_2^{\otimes}} \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes}) \tag{4.32}$$

By inequalities (4.31) and (4.32), we have

$$\max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes}) = \min_{Y^{\otimes}\in S_2^{\otimes}} \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes}) = E(X^{*\otimes},Y^{*\otimes})$$

and its standard grey matrix game value $v(\otimes) = E(X^{*\otimes},Y^{*\otimes})$.

Necessity: Suppose there exist $X^{*\otimes} \in S_1^{\otimes}, Y^{*\otimes} \in S_2^{\otimes}$ such that the following equations hold:

$$\min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{*\otimes},Y^{\otimes}) = \max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes})$$

$$\max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{*\otimes}) = \min_{Y^{\otimes}\in S_2^{\otimes}} \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes})$$

Since $\max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{\otimes},Y^{\otimes}) = \min_{Y^{\otimes}\in S_2^{\otimes}} \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes})$, then we have

$$\begin{aligned}\max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{*\otimes}) &= \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{*\otimes},Y^{\otimes}) \leq E(X^{*\otimes},Y^{*\otimes}) \leq \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{*\otimes}) \\ &= \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{*\otimes},Y^{\otimes})\end{aligned}$$

Hence, for $\forall\ X^{\otimes} \in S_1^{\otimes}, Y^{\otimes} \in S_2^{\otimes}$, there exists

$$E(X^{\otimes},Y^{\otimes}) \leq \max_{X^{\otimes}\in S_1^{\otimes}} E(X^{\otimes},Y^{\otimes}) \leq E(X^{*\otimes},Y^{*\otimes}) \leq \min_{Y^{\otimes}\in S_2^{\otimes}} E(X^{*\otimes},Y^{\otimes}) \leq E(X^{*\otimes},Y^{\otimes})$$

Theorem 4.7 Commutability of the optimal grey mixed strategy: Assuming $(X_1^{*\otimes},Y_1^{*\otimes})$ and $(X_2^{*\otimes},Y_2^{*\otimes})$ are respectively the grey saddle points of a standard grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2, \tilde{A}(\otimes)\}$, then so are $(X_1^{*\otimes},Y_2^{*\otimes})$ and

$(X_1^{*\otimes}, Y_2^{*\otimes})$. In addition, the game values in grey saddle points are equal, that is, $v_{11}(\otimes) = v_{12}(\otimes) = v_{21}(\otimes) = v_{22}(\otimes)$, where

$$v_{11}(\otimes) = E\left(X_1^{*\otimes}, Y_1^{*\otimes}\right),$$

$$v_{12}(\otimes) = E\left(X_1^{*\otimes}, Y_2^{*\otimes}\right), \quad v_{21}(\otimes) = E\left(X_2^{*\otimes}, Y_1^{*\otimes}\right), \quad v_{22}(\otimes) = E\left(X_2^{*\otimes}, Y_2^{*\otimes}\right)$$

Proof: As $(X_1^{*\otimes}, Y_1^{*\otimes})$ and $(X_2^{*\otimes}, Y_2^{*\otimes})$ are all grey saddle points of grey mixed strategy in a standard grey matrix game, we have

$$E(X^{\otimes}, Y_1^{*\otimes}) \le E(X_1^{*\otimes}, Y_1^{*\otimes}) \le E(X_1^{*\otimes}, Y^{\otimes}), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$$

$$E(X^{\otimes}, Y_2^{*\otimes}) \le E(X_2^{*\otimes}, Y_2^{*\otimes}) \le E(X_2^{*\otimes}, Y^{\otimes}), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$$

Therefore,

$$E\left(X_1^{*\otimes}, Y_2^{*\otimes}\right) \le E\left(X_2^{*\otimes}, Y_2^{*\otimes}\right) \le E\left(X_2^{*\otimes}, Y_1^{*\otimes}\right) \le E\left(X_1^{*\otimes}, Y_1^{*\otimes}\right) \le E\left(X_1^{*\otimes}, Y_2^{*\otimes}\right)$$

that is,

$$E\left(X_1^{*\otimes}, Y_2^{*\otimes}\right) = E\left(X_2^{*\otimes}, Y_2^{*\otimes}\right) = E\left(X_2^{*\otimes}, Y_1^{*\otimes}\right) = E\left(X_1^{*\otimes}, Y_1^{*\otimes}\right) = E\left(X_1^{*\otimes}, Y_2^{*\otimes}\right)$$

So we deduce $E(X^{\otimes}, Y_2^{*\otimes}) \le E(X_1^{*\otimes}, Y_2^{*\otimes}) \le E(X_1^{*\otimes}, Y^{\otimes}), i = 1, 2, \ldots, m, j = 1, 2, \ldots,$ n; that is, $E(X_1^{*\otimes}, Y_2^{*\otimes})$ is a grey saddle point of this standard grey matrix game.

For the same reason, $E(X_2^{*\otimes}, Y_1^{*\otimes})$ is also a grey saddle point.

Theorem 4.7 explains that standard grey matrix games having a grey saddle point in a grey mixed strategy sense possess two properties that games of other forms do not have: one is the commutability of grey saddle points, and the other is that the game values in grey saddle points are equal.

4.3.3 Summary

This section proved the theorem of the maximum-minimum grey game value with the theory of a grey system, and laid the theoretical fundament for a grey matrix game. Based on this, we proposed the concept of grey saddle points of grey mixed strategy and demonstrated the sufficient and necessary condition for the existence of grey saddle points and the commutability of a grey mixed optimal strategy.

4.4 Properties of a Grey Mixed Strategy and Its Grey Linear Program Model

4.4.1 Properties of a Grey Mixed Strategy

Theorem 4.8 Commutability of the optimal grey mixed strategy: Assume a grey matrix game $A(\otimes), B(\otimes)$ satisfies $A(\otimes) = (a_{ij}^{\otimes})_{m\times n}$ $B(\otimes) = (a_{ij}^{\otimes} + k)_{m\times n}$, where k is any constant. The relationship between $v_A(\otimes)$ and $B(\otimes)$, which are the game values of $A(\otimes), B(\otimes)$, satisfies Eq. (4.33):

$$v_B(\otimes) = v_A(\otimes) + k \tag{4.33}$$

The sufficient and necessary condition for $(X^{*\otimes}, Y^{*\otimes})$ to be a grey saddle point of $A(\otimes)$ is for $(X^{*\otimes}, Y^{*\otimes})$ to also be a saddle point of $B(\otimes)$.

Proof:

1. Demonstration of Eq. (4.33). From the theorem of the maximum-minimum grey game value,[3] we have

$$\begin{aligned} v_B(\otimes) &= \max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} X^{\otimes}B(\otimes)(Y^{\otimes})^T = \max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} \sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij}^{\otimes} + k\right)x_i^{\otimes}y_j^{\otimes} \\ &= \max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} \left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij}^{\otimes}\right)x_i^{\otimes}y_j^{\otimes} + k\cdot\sum_{i=1}^{m}\sum_{j=1}^{n}x_i^{\otimes}y_j^{\otimes}\right\} \\ &= \max_{X^{\otimes}\in S_1^{\otimes}} \min_{Y^{\otimes}\in S_2^{\otimes}} \left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(a_{ij}^{\otimes}\right)x_i^{\otimes}y_j^{\otimes} + k\right\} \\ &= v_A(\otimes) + k \end{aligned} \tag{4.34}$$

2. Demonstration of the sufficient and necessary condition for $(X^{*\otimes}, Y^{*\otimes})$ being a saddle point of $A(\otimes)$. Because the process is similar to the proof of the sufficient and necessary condition for a grey saddle point's existence,[1] here we omit it for the limited space.

Theorem 4.9 The sufficient and necessary condition for optimal grey mixed strategy:

1. Suppose that $v(\otimes)$ is the game value of a grey matrix game $A(\otimes)$, then the sufficient and necessary condition for $X^{*\otimes}$ to be the optimal grey mixed

strategy of Player 1 is that

$$v(\otimes) \le E(X^{*\otimes}, Y^{\otimes}), \quad \forall \quad Y^{\otimes} \in S_2^{\otimes}.$$

2. Supposing $v(\otimes)$ is the game value of grey matrix game $A(\otimes)$, then the sufficient and necessary condition for $Y^{*\otimes}$ to be the optimal grey mixed strategy of Player 2 is that

$$v(\otimes) \le E(X^{\otimes}, Y^{*\otimes}), \quad \forall \quad X^{\otimes} \in S_1^{\otimes}$$

Proof:

1. Suppose $X^{*\otimes}$ is the optimal grey mixed strategy of Player 1, from the theorem of the maximum-minimum grey game value,[1] we know there exists the optimal grey mixed strategy $Y^{1\otimes}$ such that $E(X^{*\otimes}, Y^{1\otimes})(\otimes) \le E(X^{*\otimes}, Y^{\otimes}), \forall\, Y^{\otimes} \in S_2^{\otimes}$ holds; that is,

   $$v(\otimes) = E(X^{*\otimes}, Y^{1\otimes})(\otimes) \le E(X^{*\otimes}, Y^{\otimes}), \quad \forall \quad Y^{\otimes} \in S_2^{\otimes}$$

   Whereas, suppose $X^{*\otimes}$ satisfies $v(\otimes) \le E(X^{*\otimes}, Y^{\otimes}), \forall\, Y^{\otimes} \in S_2^{\otimes}$. According to the theorem of the maximum-minimum grey game value, there exists $(X^{1\otimes}, Y^{1\otimes})$ such that

   $$E(X^{\otimes}, Y^{1\otimes}) \le E(X^{1\otimes}, Y^{1\otimes}) = v(\otimes) \le E(X^{1\otimes}, Y^{\otimes})$$

   Therefore, $E(X^{*\otimes}, Y^{1\otimes}) \le E(X^{1\otimes}, Y^{1\otimes}) = v(\otimes) \le E(X^{*\otimes}, Y^{\otimes})$ and $E(X^{\otimes}, Y^{1\otimes}) \le E(X^{1\otimes}, Y^{1\otimes}) = E(X^{*\otimes}, Y^{1\otimes}) \le E(X^{*\otimes}, Y^{\otimes})$ is true for $\forall\, X^{\otimes} \in S_1^{\otimes}, Y^{\otimes} \in S_2^{\otimes}$; that is to say, $X^{*\otimes}$ is the optimal grey mixed strategy of Player 1.
2. With the same method, supposing that $v(\otimes)$ is the game value of grey matrix game $A(\otimes)$, we can prove that the sufficient and necessary condition for $Y^{*\otimes}$ to be the optimal grey mixed strategy of Player 2 is that $v(\otimes) \le E(X^{\otimes}, Y^{*\otimes}), \forall\, X^{\otimes} \in S_1^{\otimes}$.

Theorem 4.10 Grey game value in grey saddle points: Let $(X^{*\otimes}, Y^{*\otimes})$ be a grey saddle point of grey matrix game $A(\otimes)$, and then

$$\max_{1 \le i \le m} E(\alpha_i, Y^{*\otimes}) = \min_{1 \le j \le n} E(X^{*\otimes}, \beta_j) = v(\otimes) \tag{4.35}$$

where α_i is the unit vector whose ith component's value is 1 in R^m and β_j is the unit vector whose jth component's value is 1 in R^n.

Proof: According to Theorem 4.2, we know for any $j, 1 \le j \le n$, there exists $v(\otimes) \le E(X^{*\otimes}, \beta_j)$, so $v(\otimes) \leq \min\limits_{1\leq j\leq n} E(X^{*\otimes},\beta_j)$.

If $v(\otimes) \leq \min_{1\leq j\leq n} E(X^{*\otimes},\beta_j)$ then $v(\otimes) \leq E(X^{*\otimes},\beta_j), j=1,2,\ldots,n$.

Therefore, $v(\otimes) = v(\otimes)\cdot(\Sigma_{j=1}^{n} y_j^{*\otimes}) < \Sigma_{j=1}^{n} E(X^{*\otimes},\beta_j)\cdot y_j^{*\otimes} = E(X^{*\otimes},Y^{*\otimes}) = v(\otimes)$.

This contradiction indicates that $v(\otimes) = \min_{1\leq j\leq n} E(X^{*\otimes},\beta_j)$.

With the same method, we can prove that $v(\otimes) = \max_{1\leq i\leq m} E(\alpha_i, Y^{*\otimes})$.

4.4.2 Grey Linear Program Model of Grey Matrix Game

Theorems 4.8, 4.9, and 4.10 provide the grey linear program method with which we can find the players' optimal grey mixed strategies.

Theorem 4.11 Grey linear program model of grey matrix game: Given grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, if $(X^{*\otimes},Y^{*\otimes})$ is the optimal grey mixed strategy of the matrix game, then $(X^{*\otimes},Y^{*\otimes})$ can be found by solving a grey linear program problem.

Proof: Without loss of generality, assume grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ where $\tilde{A}(\otimes) = (a_{ij}^{\otimes})_{m\times n}$.

We can find the optimal grey mixed strategy of Player 1, $x_i^{\otimes}, i=1,2,\ldots,m$. According to Theorem 4.11, we assume the grey game value of $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ satisfies $v(\otimes) \geq 0$.

Group of inequality:

$$\begin{cases} \sum\limits_{i=1}^{m} a_{ij}^{\otimes} x_i^{\otimes} \geq v(\otimes)\,\big|_{\gamma_i=c_i,1\leq c_i\leq 1, i=1,2,\ldots,m} & j=1,2,\ldots,n \\ x_i^{\otimes} \geq 0 \qquad i=1,2,\ldots,m & \\ \sum\limits_{i=1}^{m} x_i^{\otimes} = [1,1]_{\gamma_i=c_i,1\leq c_i\leq 1, i=1,2,\ldots,m} & \end{cases} \tag{4.36}$$

has the solution that stands for the optimal strategy for Player 1. Since $x_i^{/\otimes} = \frac{x_i^{\otimes}}{v(\otimes)}$, $i=1,2,\ldots,m$, we can transform the group of inequality (4.36) into

$$\begin{cases} \sum\limits_{i=1}^{m} a_{ij}^{\otimes} x_i^{/\otimes} \geq [1,1]_{\gamma_i=c_i,1\leq c_i\leq 1, i=1,2,\ldots,m} & j=1,2,\ldots,n \\ \sum\limits_{i=1}^{m} x_i^{/\otimes} = \dfrac{1}{v(\otimes)} & \\ x_i^{/\otimes} \geq 0 & i=1,2,\ldots,m \end{cases} \tag{4.37}$$

where $v(\otimes) = \max_{X^{\otimes}\in S_1^{\otimes}} \min_{1\leq j\leq n} E(X^{\otimes},\beta_j) = \max_{X^{\otimes}\in S_1^{\otimes}} \min_{1\leq j\leq n} \Sigma_{i=1}^{m} a_{ij}^{\otimes} x_i^{\otimes}$.

Therefore, the problem of finding Player 1's optimal grey mixed strategy can be replaced by finding a solution to the following linear program [see Eq. (4.38)]:

$$\min\left\{x_1^{/\otimes}+x_2^{/\otimes}+\cdots+x_m^{/\otimes}\right\}$$

$$s.t.\begin{cases}\sum_{i=1}^{m} a_{ij}^{\otimes}x_i^{/\otimes}\geq[1,1]_{\gamma_i=c_i\,,0\leq c_i\leq 1,i=1,2,\ldots,m} & j=1,2,\ldots,n\\ x_i^{/\otimes}\geq 0 & i=1,2,\ldots,m\end{cases}\tag{4.38}$$

Likewise, the problem of finding Player 2's optimal grey mixed strategy can be replaced by finding a solution to the following linear program [see Eq. (4.39)]:

$$\max\left\{y_1^{/\otimes}+y_2^{/\otimes}+\cdots+y_n^{/\otimes}\right\}$$

$$s.t.\begin{cases}\sum_{j=1}^{n} a_{ij}^{\otimes}y_j^{/\otimes}\leq[1,1]_{\gamma_j=c_j,0\leq c_j\leq 1,j=1,2,\ldots,n} & i=1,2,\ldots,m\\ y_j^{/\otimes}\geq 0 & j=1,2,\ldots,n\end{cases}\tag{4.39}$$

The proof of Theorem 4.11 is constructive. We can translate the problem of finding the optimal grey mixed strategy of any grey matrix game $\tilde{G}(\otimes)=\{S_1\,,S_2\,;\tilde{A}(\otimes)\}$ into a grey linear program problem.

By the method of finding a solution to the linear program, we can transform Eqs. (4.38) and (4.39) into a standard grey linear program problem. Here, we take the latter as an example; its transformation into a standard form can be seen in Eq. (4.40) [Eq. (4.38) can be dealt with in the same way.].

$$\max\{z\}=\sum_{j=1}^{n} y_j^{/\otimes} \qquad 4.40.1$$

$$s.t.\begin{cases}\sum_{j=1}^{n} a_{ij}^{\otimes}y_j^{/\otimes}=[1,1]_{\gamma_j=c_j,0\leq c_j\leq 1,j=1,2,\cdots,n}, \quad i=1,2,\ldots,m & 4.40.2\\ y_j^{/\otimes}\geq 0, j=1,2,\ldots,n & 4.40.3\end{cases}\tag{4.40}$$

4.4.3 The Concept of a Grey Linear Programming Model Solution of a Grey Matrix Game

Before discussing how to get the solution of a grey linear programming model of a grey matrix game, we need to learn the concept of solution of such model.

Definition 4.15 Grey practical solution: For the grey linear programming problem of a grey matrix game [such as Eq. (4.8)], we call the solution $Y^{\otimes} = (Y_1^{\otimes}, Y_2^{\otimes}, \ldots, Y_3^{\otimes})^T$, which satisfies constraint condition (4.40.2), the grey practical solution of this grey linear programming problem. Moreover, the grey practical solution that maximizes the objective function is defined as the optimum grey solution.

Definition 4.16 Grey radix: For the grey linear programming problem of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ [as shown in Eq. (4.40)], if $\tilde{A}(\otimes) = (a_{ij}^{\otimes})_{m\times n}$ is the $m \times n$–dimensional coefficient matrix, the order of which is m; and $B(\otimes) = (a_{ij}^{\otimes})_{m\times n}$ is the $m \times n$–rank nonsingular grey sub matrix, then $\tilde{B}(\otimes) = (a_{ij}^{\otimes})_{m\times n}$ is regarded as a grey radix of this grey linear programming problem.

From Definition 4.16, we know that grey matrix $\tilde{B}(\otimes) = (a_{ij}^{\otimes})_{m\times n}$ is composed of m grey column vectors that are linearly independent of each other. To keep universality, we can assume $\tilde{B}(\otimes)$ as shown in Eq. (4.41):

$$\tilde{B}(\otimes) = \left(a_{ij}^{\otimes}\right)_{m\times m} = \begin{pmatrix} a_{11}^{\otimes} & a_{12}^{\otimes} & \cdots & a_{1m}^{\otimes} \\ a_{21}^{\otimes} & a_{22}^{\otimes} & \cdots & a_{2m}^{\otimes} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}^{\otimes} & a_{m2}^{\otimes} & \cdots & a_{mm}^{\otimes} \end{pmatrix} = \left(P_1^{\otimes}, \quad P_2^{\otimes}, \cdots, P_j^{\otimes}, \quad \cdots, \quad P_m^{\otimes}\right) \tag{4.41}$$

Therefore, we can call $P_j^{\otimes} (j = 1, 2, \ldots, m)$ the grey radix vector. The grey variable corresponding to it can be called the grey basis variable, or the non-grey-based variable.

Definition 4.17 Grey feasible basis: For the grey linear programming problem of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ [as shown in Eq. (4.40)], we call the grey basis solution the grey basis feasible solution, and the grey basis corresponding to it can be called the grey feasible basis.

From the above definitions, we can get a simple structure that depicts the relationships among solutions of the grey linear programming problem of a grey matrix game, as shown by Figure 4.4. In the figure, part of the set between the grey practical solution and the grey radix solution is vivid, while the boundary is not. As long as the obtained number of all grey numbers in these grey sets is certain, the boundary will be vivid. The relationship among solutions of such grey linear programming problems, shown by Figure 4.4, should be the congruence of solutions of some whitened linear programming problem, after the grey ones have been whitened.

Theorem 4.12 The feasible domain of $\tilde{G}(\otimes)$'s linear programming is a grey convex set: If the grey feasible domain exists for the grey linear programming problem of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, as shown in

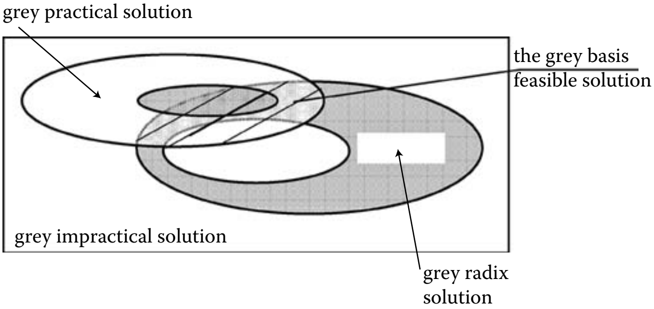


Figure 4.4 The relationships among solutions of the grey linear programming problem.

Eq. (4.42):

$$D(\otimes)=\left\{Y^{/\otimes}\left|\sum_{j=1}^{n}P_j^{\otimes}y_j^{/\otimes}=[1,1]_{\gamma_j=c_j,0\le c_j\le 1,j=1,2,\ldots,n}=[1,1]_{\otimes},0\le y_j^{/\otimes}\le 1\right.\right\}\quad(4.42)$$

and there into

$$\tilde{A}(\otimes)=\left(a_{ij}^{\otimes}\right)_{m\times m}=\begin{pmatrix} a_{11}^{\otimes} & a_{12}^{\otimes} & \cdots & a_{1n}^{\otimes}\\ a_{21}^{\otimes} & a_{22}^{\otimes} & \cdots & a_{2n}^{\otimes}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1}^{\otimes} & a_{m2}^{\otimes} & \cdots & a_{mn}^{\otimes}\end{pmatrix}=\left(P_1^{\otimes},P_2^{\otimes},\ldots,P_j^{\otimes},\ldots,\quad P_n^{\otimes}\right)$$

then its feasible domain [as shown in Eq. (4.42)] is a grey convex set.[24]

Proof: If we want to prove the set composed of all grey points, which satisfies the constraint condition of linear programming problem as well, is grey convex, we only need give the proof that a grey point on the line connected by random two grey points in $D(\otimes)$ is included in it.

$$\sum_{j=1}^{n}P_j^{\otimes}y_j^{/\otimes}=[1,1]_{\gamma_j=c_j,0\le c_j\le 1,j=1,2,\ldots,n}=[1,1]_{\otimes},0\le y_j^{/\otimes}\le 1,j=1,2,\ldots,n\quad(4.43)$$

Assume that $Y_1^{/\otimes}=(y_{11}^{/\otimes},y_{21}^{/\otimes},\ldots,y_{n1}^{/\otimes})^T,Y_2^{/\otimes}=(y_{12}^{/\otimes},y_{22}^{/\otimes},\ldots,y_{n2}^{/\otimes})^T$ are two random points in $D(\otimes);Y_1^{/\otimes}\neq Y_2^{/\otimes}$.

Therefore, we have

$$\sum_{j=1}^{n}P_j^{\otimes}y_{j1}^{/\otimes}=[1,1]_{\gamma_{j1}=c_{j1},0\le c_{j1}\le 1,j=1,2,\ldots,n}=[1,1]_{\otimes},0\le y_{j1}^{/\otimes}\le 1,j=1,2,\ldots,n$$

$$\sum_{j=1}^{n}P_j^{\otimes}y_{j2}^{/\otimes}=[1,1]_{\gamma_{j2}=c_{j2},0\le c_{j2}\le 1,j=1,2,\ldots,n}=[1,1]_{\otimes},0\le y_{j2}^{/\otimes}\le 1,j=1,2,\ldots,n$$

Let $Y_{3j}^{/\otimes} = (y_{13}^{/\otimes}, y_{23}^{/\otimes}, \ldots, y_{n3}^{/\otimes})^T$ be a random point on the connection of $Y_1^{/\otimes}, Y_2^{/\otimes}$, as $Y_3^{/\otimes} = \alpha \cdot Y_1^{/\otimes} + (1-\alpha) \cdot Y_2^{/\otimes} \quad (0 \le \alpha \le 1)$

Each component of $Y_{3j}^{/\otimes}$ is $Y_{j3}^{/\otimes} = \alpha \cdot y_{j1}^{/\otimes} + (1-\alpha) \cdot Y_{j2}^{/\otimes}$. Put it into the constraint condition, and we get

$$\sum_{j=1}^{n} P_j^{\otimes} \cdot y_{j3}^{/\otimes} = \sum_{j=1}^{n} P_j^{\otimes} \cdot \left[\alpha \cdot y_{j1}^{/\otimes} + (1-\alpha) \cdot y_{j2}^{/\otimes}\right]$$

$$= \alpha \cdot \sum_{j=1}^{n} P_j^{\otimes} \cdot y_{j1}^{/\otimes} + \sum_{j=1}^{n} P_j^{\otimes} \cdot y_{j2}^{/\otimes} - \alpha \cdot \sum_{j=1}^{n} P_j^{\otimes} \cdot y_{j2}^{/\otimes}$$

$$= \alpha \cdot [1,1]_{\gamma_{j1}=c_{j1},0\le c_{j1}\le 1,j=1,2,\ldots,n} + [1,1]_{\gamma_{j2}=c_{j2},0\le c_{j2}\le 1,j=1,2,\ldots,n} - \alpha \cdot [1,1]_{\gamma_{j2}=c_{j2},0\le c_{j2}\le 1,j=1,2,\ldots,n}$$

$$= [1,1]_{\gamma_{j2}=c_{j2},0\le c_{j2}\le 1,j=1,2,\ldots,n} = [1,1]_{\otimes}$$

because $y_{j1}^{/\otimes}, y_{j2}^{/\otimes} \ge 0, \alpha \ge 0, 1-\alpha \ge 0, j = 1,2,\ldots,n,\ y_{j3}^{/\otimes} \ge 0, j = 1,2,\ldots,n$

Thus it can be seen that $y_3^{/\otimes} \in D(\otimes), D(\otimes)$ is a grey convex set.

4.4.4 Summary

In this section, we have studied the exchangeability of a grey mixed optimum strategy, have given the proof of the necessary and sufficient condition for the existence of grey mixed optimum strategy, and have drawn a vital conclusion that the grey game values of Players 1 and 2 are equal. On the basis of these, we have also proved that, for any grey matrix game problem, we can work out the solution of its grey mixed optimum strategy by working out the solution to a grey linear programming problem. We have built the structure and system for the solution of the grey matrix game's grey linear programming problem based on the theory of grey systems and operations.

4.5 Seeking Solutions of Grey Linear Programming Model of a Grey Matrix Game

4.5.1 Grey Basis Feasible Solution Corresponds to the Vertex of a Grey Feasible Domain

Lemma 4.1 The necessary and sufficient condition for the existence of grey basis feasible solution: Given the grey linear programming model of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, as shown in Eq. (4.44), then a practical solution of this grey linear programming problem $Y'^{\otimes} = (y_1'^{\otimes}, y_2'^{\otimes}, \ldots, y_m'^{\otimes})^T$ is a basis feasible solution, which is that the grey coefficient column vector corresponding to the positive component of $Y'^{\otimes}$ is linearity independent.

$$\max\{z\} = \sum_{j=1}^{n} y_j'^{\otimes} \qquad 4.44.1$$

$$s.t. \begin{cases} \sum_{j=1}^{n} a_{ij}^{\otimes} y_j'^{\otimes} = [1,1]_{\gamma_j = c_j, 0 \le c_j \le 1, j=1,2,\ldots,n} = [1,1]_{\otimes}, i = 1,2,\ldots,m & 4.44.2 \\ y_j'^{\otimes} \ge 0, j = 1,2,\ldots,n & 4.44.3 \end{cases}$$

Proof:

1. Necessity: We can prove it according to the definition of a grey basis feasible solution.[3]
2. Sufficiency: If grey vectors $P_1^{\otimes}, P_2^{\otimes}, \ldots, P_k^{\otimes}$ are linearity independent, then it is certain that $k \le m$.

Theorem 4.13 Grey basis feasible solution corresponds to the grey vertex of grey feasible domain: Given the grey linear programming model of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, as shown in Eq. (4.44), then the grey basis feasible solution $Y'^{\otimes}$ of this grey linear programming problem corresponds to the grey vertex of grey feasible domain $D(\otimes)$.

Proof: To keep universality, we can assume the forward m parts are positive. Therefore,

$$\sum_{j=1}^{m} P_j^{\otimes} \cdot y_j'^{\otimes} = [1,1]_{\gamma_j = c_j, 0 \le c_j \le 1, j=1,2,\ldots,m} = [1,1]_{\otimes} \qquad (4.45)$$

Now we will discuss it through two steps, using reduction to absurdity respectively.

1. If $Y'^{\otimes}$ is not the grey basis practical solution, then it will never be the grey vertex of grey practical domain $D(\otimes)$.

According to Lemma 4.1, we know that, if $Y'^{\otimes}$ is not the grey basis feasible solution, the n-dimensional grey coefficient column vectors $P_1^{\otimes}, P_2^{\otimes}, \ldots, P_m^{\otimes}$, ,which correspond to its grey positive component, are grey linear correlated. There is a group of grey coefficients, which is not 0 entirely, making:

$$k_1^{\otimes} P_1^{\otimes} + k_2^{\otimes} P_2^{\otimes} + \cdots + k_m^{\otimes} P_m^{\otimes} = 0_{\gamma_{ki}, \gamma_{Pij}, i=1,2,\ldots,m, j=1,2,\ldots,n} \qquad (4.46)$$

Using a grey number multiple Eq. (4.46) in which $\mu^{\otimes} > 0$, then by adding and subtracting with Eq. (4.45), we can get:

$$(y_1^{/\otimes} - \mu^{\otimes} k_1^{\otimes}) P_1^{\otimes} + (y_2^{/\otimes} - \mu^{\otimes} k_2^{\otimes}) P_2^{\otimes} + \cdots + (y_m^{/\otimes} - \mu^{\otimes} k_m^{\otimes}) P_m^{\otimes} = 0_{\gamma_{ki}, \gamma_{Pij}, \gamma_{\eta} i=1,2,\cdots,m, j=1,2,\cdots,n}$$

$$(y_1^{/\otimes} + \mu^{\otimes} k_1^{\otimes}) P_1^{\otimes} + (y_2^{/\otimes} + \mu^{\otimes} k_2^{\otimes}) P_2^{\otimes} + \cdots + (y_m^{/\otimes} + \mu^{\otimes} k_m^{\otimes}) P_m^{\otimes} = 0_{\gamma_{ki}, \gamma_{Pij}, \gamma_{\eta} i=1,2,\cdots,m, j=1,2,\cdots,n}$$

Now choose:

$$Y_1^{/\otimes} = \left\{ \left(y_1^{/\otimes} - \mu^{\otimes} k_1^{\otimes} \right), \left(y_2^{/\otimes} - \mu^{\otimes} k_2^{\otimes} \right), \ldots, \left(y_m^{/\otimes} - \mu^{\otimes} k_m^{\otimes} \right), 0, 0, \ldots, 0 \right\}$$

$$Y_2^{/\otimes} = \left\{ \left(y_1^{/\otimes} - \mu^{\otimes} k_1^{\otimes} \right), \left(y_2^{/\otimes} - \mu^{\otimes} k_2^{\otimes} \right), \ldots, \left(y_m^{/\otimes} - \mu^{\otimes} k_m^{\otimes} \right), 0, 0, \ldots, 0 \right\}$$

Through $Y_1^{/\otimes}, Y_2^{/\otimes}$, we can get $Y^{/\otimes} = \frac{1}{2} Y_1^{/\otimes} + \frac{1}{2} Y_2^{/\otimes} : Y^{/\otimes}$ is the midpoint of the connection linked by $Y_1^{/\otimes}, Y_2^{/\otimes}$.

When $\mu^{\otimes}$ Is sufficiently small, it is can be guaranteed that $y_i^{/\otimes} \pm \mu^{\otimes} k_i^{\otimes} \geq 0, i = 1, 2, \ldots, m$; $Y_1^{/\otimes}, Y_2^{/\otimes}$ are grey practical solutions. Then, $Y^{/\otimes}$ is not the grey vertex of grey practical domain $D(\otimes)$.

2. If $Y^{/\otimes}$ is not the grey vertex of grey feasible domain $D(\otimes)$, then it will never be the grey feasible solution.

Because $Y^{/\otimes}$ is not the grey vertex of grey feasible domain $D(\otimes)$, we can find out two different points in feasible domain $D(\otimes)$

$$Y_1^{/\otimes} = \left(y_{11}^{/\otimes}, y_{12}^{/\otimes}, \ldots, y_{1n}^{/\otimes} \right)^T, \quad Y_2^{/\otimes} = \left(y_{21}^{/\otimes}, y_{22}^{/\otimes}, \ldots, y_{2n}^{/\otimes} \right)^T$$

making $Y^{/\otimes} = \eta^{\otimes} Y_1^{/\otimes} + (1 - \eta^{\otimes}) Y_2^{/\otimes}, 0 < \eta^{\otimes} < 1$

Assume $Y^{/\otimes}$ is a grey radix feasible solution, and the corresponding grey vector group $P_1^{\otimes}, P_2^{\otimes}, \ldots, P_m^{\otimes}$ grey linearity-independent. When $j > m$, $y_j^{/\otimes} = y_{1j}^{/\otimes} = y_{2j}^{/\otimes} = 0$, because $Y_1^{/\otimes}, Y_2^{/\otimes}$ are the two grey points of feasible domain, they should satisfy Eqs. (4.47) and (4.48):

$$\sum_{j=1}^{m} P_j^{\otimes} \cdot y_{1j}^{/\otimes} = [1,1]_{\gamma_{Pj}, \gamma_{yj}, j=1,2,\ldots,m} = [1,1]_{\otimes} \tag{4.47}$$

$$\sum_{j=1}^{m} P_j^{\otimes} \cdot y_{2j}^{/\otimes} = [1,1]_{\gamma_{Pj}, \gamma_{yj}, j=1,2,\ldots,m} = [1,1]_{\otimes} \tag{4.48}$$

We can get Eq. (4.49) after Eq. (4.47) minus Eq. (4.48):

$$\sum_{j=1}^{m} P_j^{\otimes} \left(y_{1j}^{/\otimes} - y_{2j}^{/\otimes} \right) = 0_{\gamma_{Pj}, \gamma_{y1j}, \gamma_{y2j} j=1,2,\ldots,m} = 0_{\otimes} \tag{4.49}$$

Because $Y_1^{/\otimes} \neq Y_2^{/\otimes}$, the grey coefficient $(y_{1j}^{/\otimes} - y_{2j}^{/\otimes})$ in Eq. (4.49) won't be 0 entirely. Therefore, the grey vector group $P_1^{\otimes}, P_2^{\otimes}, \ldots, P_m^{\otimes}$ is grey correlated, contrary to the assumption. $Y^{/\otimes}$ is not the basis feasible solution.

4.5.2 The Optimum Grey Game Value Corresponds to the Vertex of Grey Linear Programming Feasible Domain

Lemma 4.2 The expression of grey vertex for $Y^{\otimes} \in A(\otimes)$: If $A(\otimes)$ is a bounded grey convex set, then any point $Y^{\otimes} \in A(\otimes)$ can be expressed by the grey convex combination of $A(\otimes)$'s grey vertex.

Proof: Apply induction to prove:

1. Given a bounded grey convex set $A(\otimes)$ is a grey triangle A_1 $(\otimes)$ with three grey vertexes, as shown in Figure 4.5.

To keep universality, we may assume $Y^{1,\otimes}$ is one grey point of grey triangle A_1 $(\otimes)$, shown as $Y^{1,\otimes} \in A_1$ $(\otimes)$, and $Y_1^{\otimes}, Y_2^{\otimes}, Y_3^{\otimes}$ are the three grey vertexes of this triangle, as shown in Figure 4.5. Try to express grey point $Y^{1,\otimes}$ with the coordinates of these three grey vertexes.

Choose one vertex $Y_2^{\otimes}$, make a connection $\overline{Y^{1,\otimes}Y_2^{\otimes}}$, and extend to intersect line $\overline{Y_1^{\otimes}Y_3^{\otimes}}$ with point $Y_{13}^{\otimes}$. Because $Y_{13}^{\otimes}$ Is a point of connection between $Y_1^{\otimes}, Y_3^{\otimes}$, we can use $Y_1^{\otimes}, Y_3^{\otimes}$ to express linearity as shown by Eq. (4.50):

$$Y_{13}^{\otimes} = \lambda_1^{\otimes} Y_1^{\otimes} + \left(1 - \lambda_1^{\otimes}\right) Y_3^{\otimes}, \quad 0 < \lambda_1^{\otimes} < 1 \tag{4.50}$$

Because $Y^{1,\otimes}$ is a grey point on the connection between $Y_{13}^{\otimes}$ and $Y_2^{\otimes}$, we get that Eq. (4.51) is tenable:

$$Y^{1,\otimes} = \lambda_2^{\otimes} Y_{13}^{\otimes} + \left(1 - \lambda_2^{\otimes}\right) Y_2^{\otimes}, \quad 0 < \lambda_2^{\otimes} < 1 \tag{4.51}$$

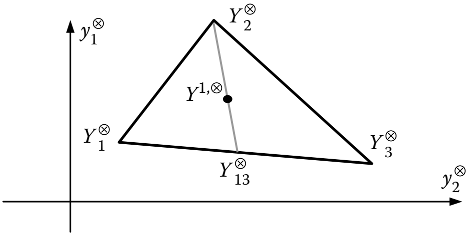


Figure 4.5 Triangle's grey inside point.

Putting Eq. (4.50) into (4.51), we may get that

$$\begin{aligned} Y^{1,\otimes} &= \lambda_2^\otimes\left(\lambda_1^\otimes Y_1^\otimes + \left(1-\lambda_1^\otimes\right)Y_3^\otimes\right) + \left(1-\lambda_2^\otimes\right)Y_2^\otimes \\ &= \lambda_2^\otimes\,\lambda_1^\otimes\,Y_1^\otimes + \left(1-\lambda_2^\otimes\right)Y_2^\otimes + \lambda_2^\otimes\left(1-\lambda_1^\otimes\right)Y_3^\otimes \end{aligned}$$

then $\mu_1^{3,\otimes} = \lambda_2^\otimes\lambda_1^\otimes, \mu_2^{3,\otimes} = (1-\lambda_2^\otimes), \mu_3^{3,\otimes} = \lambda_2^\otimes(1-\lambda_1^\otimes)$ and we get that

$$Y^{1,\otimes} = \mu_1^{1,\otimes}\,Y_1^\otimes + \mu_2^{1,\otimes}\,Y_2^\otimes + \mu_3^{1,\otimes}\,Y_3^\otimes,\quad \sum_{i=1}^{3}\mu_i^{1,\otimes} = [1,1]_{\gamma_i\,,i=1,2,3}$$

$$= [1,1]_\otimes, [0,0]_\otimes < \mu_i^{1,\otimes} < [1,1]_\otimes$$

Thus it can be seen, when bounded grey convex set $A(\otimes)$ is grey triangle $A_1\,(\otimes)$ with three grey vertexes, for any $Y^{1,\otimes} \in A_1\,(\otimes)$, $Y^{1,\otimes}$ can be expressed as the grey convex combination of $A_1\,(\otimes)$'s grey vertexes.

2. Given bounded grey convex set $A(\otimes)$ is a grey quadrangle $A_2\,(\otimes)$ with four grey vertexes, as shown in Figure 4.6.

To keep universality, we may assume $Y^{2,\otimes}$ as one grey point of grey quadrangle $A_2\,(\otimes)$, shown as $Y^{2,\otimes} \in A_2\,(\otimes)$, and $Y_1^\otimes, Y_2^\otimes, Y_3^\otimes, Y_4^\otimes$ are the four grey vertexes of this grey quadrangle $A_2\,(\otimes)$, as shown In Figure 4.6. Try to express grey point $Y^{2,\otimes}$ with the coordinates of these four grey vertexes.

Choose one grey point in grey quadrangle $Y^{2,\otimes} \in A_2\,(\otimes)$, make a line $\overline{Y_{13}^\otimes Y_{24}^\otimes}$ through $Y^{2,\otimes}$, intersecting with $\overline{Y_1^\otimes Y_3^\otimes}$ on $Y_{13}^\otimes$, and $\overline{Y_2^\otimes\,Y_4^\otimes}$ on $Y_{13}^\otimes$.

Similarly, we can prove,

$$Y^{2,\otimes} = \mu_1^{2,\otimes}\,Y_1^\otimes + \mu_2^{2,\otimes}\,Y_2^\otimes + \mu_3^{2,\otimes}\,Y_3^\otimes + \mu_4^{2,\otimes}\,Y_4^\otimes,\quad \sum_{i=1}^{4}\mu_i^{2,\otimes} = [1,1]_{\gamma_i\,,i=1,2,3,4}$$

$$= [1,1]_\otimes, [0,0]_\otimes < \mu_i^{2,\otimes} < [1,1]_\otimes$$

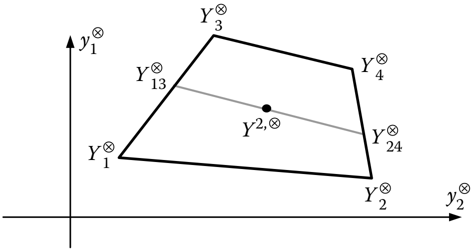


Figure 4.6 Quadrangle's grey inside point.

3. Given bounded grey convex set $A(\otimes)$ is a grey n-polygon with n grey points, the original claim is tenable, as the following formula:

$$Y^{n-2,\otimes} = \mu_1^{n-2,\otimes} Y_1^\otimes + \mu_2^{n-2,\otimes} Y_2^\otimes + \cdots + \mu_n^{n-2,\otimes} Y_n^\otimes,$$

$$\sum_{i=1}^{4} \mu_i^{n-2,\otimes} = [1,1]_{\gamma_i, i=1,2,\ldots,n} = [1,1]_\otimes, [0,0]_\otimes < \mu_i^{n-2,\otimes} < [1,1]_\otimes$$

4. Given bounded grey convex set $A(\otimes)$ is a grey $n + 1$ polygon with $n + 1$ grey points, the original claim is tenable as well.

Assume $Y^{n-1,\otimes}$ is a grey inside point of $A(\otimes)$, connect $Y^{n-2,\otimes}, Y^{n-1,\otimes}$, and extend $Y^{n-2,\otimes}, Y^{n-1,\otimes}$ intersecting with line $Y_{n+1}^\otimes, Y_k^\otimes$ on $Y_{n+1,k}^\otimes$; therefore,

$$Y_{n+1,k}^\otimes = \varepsilon_1^\otimes Y_{n+1}^\otimes + \left(1-\varepsilon_1^\otimes\right) Y_k^\otimes \quad [0,0]_\otimes < \varepsilon_2^\otimes < [1,1]_\otimes \tag{4.52}$$

Thus,

$$Y^{n-1,\otimes} = \varepsilon_2^\otimes Y^{n-2,\otimes} + \left(1-\varepsilon_2^\otimes\right) Y_{n+1,k}^\otimes, \quad [0,0]_\otimes < \varepsilon_2^\otimes < [1,1]_\otimes \tag{4.53}$$

Putting Eq. (4.52) into Eq. (4.53), we can get:

$$\begin{aligned} Y^{n-1,\otimes} &= \varepsilon_2^\otimes Y^{n-2,\otimes} + \left(1-\varepsilon_2^\otimes\right)\left(\varepsilon_1^\otimes Y_{n+1}^\otimes + \left(1-\varepsilon_1^\otimes\right) Y_k^\otimes\right) = \varepsilon_2^\otimes Y^{n-2,\otimes} + \left(1-\varepsilon_2^\otimes\right)\varepsilon_1^\otimes Y_{n+1}^\otimes \\ &\quad + \left(1-\varepsilon_2^\otimes\right)\left(1-\varepsilon_1^\otimes\right) Y_k^\otimes \, [0,0]_\otimes < \varepsilon_1^\otimes, \varepsilon_2^\otimes < [1,1]_\otimes \end{aligned} \tag{4.54}$$

Putting Eq. (4.51) into Eq. (4.54), we can get:

$$\begin{aligned} Y^{n-1,\otimes} &= \varepsilon_2^\otimes \left(\mu_1^\otimes Y_1^\otimes + \mu_2^\otimes Y_2^\otimes + \mu_3^\otimes Y_3^\otimes + \cdots + \mu_k^\otimes Y_k^\otimes + \cdots + \mu_n^\otimes Y_n^\otimes\right) + \left(1-\varepsilon_2^\otimes\right)\varepsilon_1^\otimes Y_{n+1}^\otimes + \\ &\quad + \left(1-\varepsilon_1^\otimes\right)\left(1-\varepsilon_2^\otimes\right) Y_k^\otimes \\ &= \varepsilon_2^\otimes \mu_1^\otimes Y_1^\otimes + \varepsilon_2^\otimes \mu_2^\otimes Y_2^\otimes + \varepsilon_2^\otimes \mu_3^\otimes Y_3^\otimes + \cdots + \varepsilon_2^\otimes \mu_k^\otimes Y_k^\otimes + \cdots + \varepsilon_2^\otimes \mu_n^\otimes Y_n^\otimes + \\ &\quad + \left(1-\varepsilon_2^\otimes\right)\varepsilon_1^\otimes Y_{n+1}^\otimes + \left(1-\varepsilon_2^\otimes\right)\left(1-\varepsilon_1^\otimes\right) Y_k^\otimes \\ &= \varepsilon_2^\otimes \mu_1^\otimes Y_1^\otimes + \varepsilon_2^\otimes \mu_2^\otimes Y_2^\otimes + \varepsilon_2^\otimes \mu_3^\otimes Y_3^\otimes + \cdots + \left(\varepsilon_2^\otimes \mu_k^\otimes + \left(1-\varepsilon_2^\otimes\right)\left(1-\varepsilon_1^\otimes\right)\right) Y_k^\otimes + \\ &\quad + \cdots + \varepsilon_2^\otimes \mu_n^\otimes Y_n^\otimes + \left(1-\varepsilon_2^\otimes\right)\varepsilon_1^\otimes Y_{n+1}^\otimes \\ &\qquad [0,0]_\otimes < \varepsilon_1^\otimes, \varepsilon_2^\otimes, \mu_i^\otimes < [1,1]_\otimes, i = 1,2,\ldots,n+1 \end{aligned} \tag{4.55}$$

In Eq. (4.55), if we let the coefficient of $Y_i^{\otimes}, i = 1,2,\ldots,n+1$ be $\mu_i^{n-1,\otimes}, i = 1,2,\ldots,$ n + 1 respectively, then we can express Eq. (4.55) by Eq. (4.56):

$$Y^{n-1,\otimes} = \mu_1^{n-1,\otimes} Y_1^{\otimes} + \mu_2^{n-1,\otimes} Y_2^{\otimes} + \cdots + \mu_{n+1}^{n-1,\otimes} Y_{n+1}^{\otimes}, \quad \sum_{i=1}^{n+1} \mu_i^{n-1,\otimes} = [1,1]_{\gamma_i, i=1,2}$$

$$= [1,1]_{\otimes}, [0,0]_{\otimes} < \mu_i^{n-1,\otimes} < [1,1]_{\otimes} \tag{4.56}$$

Thus it can be seen, given bounded grey convex set $A(\otimes)$ is a grey $n + 1$ polygon with $n + 1$ grey point $A_{n-1}(\otimes)$, the original claim is tenable as well.

To sum up, the original claim is proved.

Theorem 4.14 The optimum value of grey objective function corresponds to the vertex of a grey convex set: Given the grey linear programming model of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, shown as Eq. (4.44). If its feasible domain $D(\otimes)$ is bounded, then the grey objective function of this grey linear programming problem is bound to reach optimum on the grey vertex of its grey feasible domain.

Proof: To keep universality, we can assume $Y_1^{/\otimes}, Y_2^{/\otimes}, \ldots, Y_k^{/\otimes}$ is the grey vertex of its grey feasible domain. If $Y_0^{/\otimes}$ is not, and the objective function reaches optimum on it, as

$$z_{\otimes}^{*} = C \cdot Y_0^{/\otimes}, C = 1,1,\ldots,1), Y_0^{/\otimes} = \left(y_{01}^{/\otimes}, y_{02}^{/\otimes}, \ldots, y_{0n}^{/\otimes}\right)^T$$

(The standard type is $z_{\otimes}^{*} = \max\{z\} = \Sigma_{j=1}^{n} y_j^{/\otimes}$.)

Because $Y_0^{/\otimes}$ is not a grey vertex, it can be grey linearly expressed as Eq. (4.57) by the grey vertex of this grey feasible domain $D(\otimes)$.

$$Y_0^{/\otimes} = \sum_{i=1}^{k} \lambda_i^{\otimes} Y_i^{/\otimes}, \qquad \lambda_i^{\otimes} \geq 0, \sum_{i=1}^{k} \lambda_i^{\otimes} = [1,1]\Big|_{\gamma_{\lambda i}, i=1,2,\ldots,k} \tag{4.57}$$

Therefore,

$$C \cdot Y_0^{/\otimes} = C \cdot \sum_{i=1}^{k} \lambda_i^{\otimes} Y_i^{/\otimes} = \sum_{i=1}^{k} \lambda_i^{\otimes} C Y_i^{/} \tag{4.58}$$

We are sure to find out a grey vertex $Y_m^{/\otimes}$ among all grey vertexes, which can make $Y_m^{/\otimes}$ the maximum among all $Y_i^{/\otimes}$. Replace all $Y_i^{/\otimes}$ with $Y_m^{/\otimes}$, and we get

$$\sum_{i=1}^{k} \lambda_i^{\otimes} C Y_i^{/\otimes} \leq \sum_{i=1}^{k} \lambda_i^{\otimes} C Y_m^{/\otimes} = C Y_m^{/\otimes} = Y_m^{/\otimes}$$

Then we can get $C Y_0^{/\otimes} \leq C Y_m^{/\otimes}$ as $Y_0^{/\otimes} \leq Y_m^{/\otimes}$.

According to the assumption, $C \cdot Y_0^{/\otimes}$ is maximum, then we can only have $C \cdot Y_0^{/\otimes} = C \cdot Y_m^{/\otimes}$. That is to say, the objective function reaches maximum on $Y_m^{/\otimes}$.

The objective function will reach maximum on several vertexes sometimes. Then the grey convex combination of these grey vertexes also reaches grey maximum. Here, we regard such grey linear programming problems as having infinite grey optimum solutions.

Assume $\hat{Y}_1^{/\otimes}, \hat{Y}_2^{/\otimes}, \ldots, \hat{Y}_k^{/\otimes}$ are the grey vertexes when the grey objective function reaches grey optimum value. If $\hat{Y}^{/\otimes}$ is the grey convex combination of these grey vertexes, as

$$\hat{Y}^{/\otimes} = \sum_{i=1}^{k} \lambda_i^{\otimes} \hat{Y}_i^{/\otimes}, \quad \lambda_i^{\otimes} \ge 0, \sum_{i=1}^{k} \lambda_i^{\otimes} = [1,1]\Big|_{\gamma_{\lambda_i}, i=1,2,\ldots,k}$$

Thus, $C \cdot \hat{Y}^{/\otimes} = C \cdot \Sigma_{i=1}^{k} \lambda_i^{\otimes} \hat{Y}_i^{/\otimes} = \Sigma_{i=1}^{k} \lambda_i^{\otimes} C \hat{Y}_i^{/\otimes}$.

Assume $C \cdot \hat{Y}_i^{/\otimes} = M, i = 1,2,\ldots,k$; thus, $C \cdot \hat{Y}^{/\otimes} = \Sigma_{i=1}^{k} \lambda_i^{\otimes} \cdot M = M \cdot [1,1]_{\otimes} = M$.

Otherwise, if the feasible domain is unbounded, then it may not have the optimum solution. If it has, it must be on a certain grey vertex. According to the above discussion, we can get the following conclusion:

Given a set, composed of all feasible solutions of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$'s linear programming problem, it is a grey convex set, maybe an unbounded domain, it has limited grey vertexes, and then the basis feasible solution of this grey linear programming problem corresponds to a grey vertex in its grey feasible domain; if it has the optimum solution, it is bound to be gotten on this grey vertex.

4.5.3 Grey Linear Programming Solution Seeking of Optimum Grey Game Value

Theorem 4.15 $Lz_{\otimes}^{*}$ and $Rz_{\otimes}^{*}$ must correspond to the obtained number of some grey factors' units grey number: Given the grey linear programming model of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, shown by Eq. (4.59), then the left extreme value $Lz_{\otimes}^{*}$ of this grey linear programming problem's grey optimum solution $z_{\otimes}^{*}$ $(\min\{\Sigma_{j=1}^{n} y_j^{/\otimes}\})$, and the right extreme value $Rz_{\otimes}^{*}$ $(\max\{\Sigma_{j=1}^{n} y_j^{/\otimes}\})$ [as shown In Eq. (4.11)], is bound to correspond to the obtained number of some grey factors' units grey number:

$$z_{\otimes}^{*} = \left\{ \max(z) = \sum_{j=1}^{n} y_j^{/\otimes} \right\} = [Lz_{\otimes}^{*}, Rz_{\otimes}^{*}] = \left[\min\left\{ \sum_{j=1}^{n} y_j^{/\otimes} \right\}, \max\left\{ \sum_{j=1}^{n} y_j^{/\otimes} \right\} \right] \tag{4.59}$$

Proof: Given the grey linear programming model of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$, we can transfer it as Eq. (4.60):

$$z_\otimes^* = \left\{ \max(z) = \sum_{j=1}^{n} y_j^\otimes \right\} = [Lz_\otimes^*, Rz_\otimes^*] = \left[\min\left\{ \sum_{j=1}^{n} y_j^{\prime\otimes} \right\}, \max\left\{ \sum_{j=1}^{n} y_j^{\prime\otimes} \right\} \right] \qquad 4.60.1$$

$$s.t.\begin{cases} a_{11}^\otimes y_1^\otimes + a_{12}^\otimes y_2^\otimes + \cdots + a_{1n}^\otimes y_n^\otimes = [1,1]_\otimes & 4.60.2 \\ a_{21}^\otimes y_1^\otimes + a_{22}^\otimes y_2^\otimes + \cdots + a_{2n}^\otimes y_n^\otimes = [1,1]_\otimes & 4.60.3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{k1}^\otimes y_1^\otimes + a_{k2}^\otimes y_2^\otimes + \cdots + a_{kn}^\otimes y_n^\otimes = [1,1]_\otimes & 4.60.K \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{L1}^\otimes y_1^\otimes + a_{L2}^\otimes y_2^\otimes + \cdots + a_{Ln}^\otimes y_n^\otimes = [1,1]_\otimes & 4.60.L \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}^\otimes y_1^\otimes + a_{m2}^\otimes y_2^\otimes + \cdots + a_{mn}^\otimes y_n^\otimes = [1,1]_\otimes & 4.60.n \\ y_j^{\prime\otimes} \geq 0, j = 1,2,\cdots,n & 4.60.n+1 \end{cases}$$

(4.60)

According to grey curve Eqs. (4.60.2), (4.60.3),…, (4.60.*n*) in Eq. (4.60), we can draw their corresponding grey curves $L_1^\otimes, L_2^\otimes, \ldots, L_K^\otimes, \ldots L_L^\otimes, \ldots L_n^\otimes$. These curves compose a grey polygon, as shown in Figure 4.7.

To keep universality, in order to illustrate the problem conveniently, here we only prove it with a two-dimensional polygon. In Figure 4.7, grey point $P_K^{i,\otimes}$ of a grey convex polygon is composed of the intersection points of grey curve $L_K^\otimes, L_L^\otimes$. When expressing the grey coefficient in Eq. (4.60) by the form of standard grey numbers, we can transfer Eqs. (4.60.*K*) and (4.60.*L*) into the form of Eqs. (4.61.*K*) and (4.61.*L*).

$$(La_{k1} + (Ra_{k1} - La_{k1})\gamma_{K1})y_1^\otimes + \cdots + (La_{ki} + (Ra_{ki} - La_{ki})\gamma_{Ki})y_i^\otimes +$$

$$+ \cdots + (La_{kn} + (Ra_{kn} - La_{kn})\gamma_{Kn})y_n^\otimes = [1,1]_\otimes \qquad 4.61.K$$

$$(La_{L1} + (Ra_{L1} - La_{L1})\gamma_{L1})y_1^\otimes + \cdots + (La_{Li} + (Ra_{Li} - La_{Li})\gamma_{Li})y_i^\otimes +$$

$$+ \cdots + (La_{Ln} + (Ra_{Ln} - La_{Ln})\gamma_{Ln})y_n^\otimes = [1,1]_\otimes \qquad 4.61.L$$

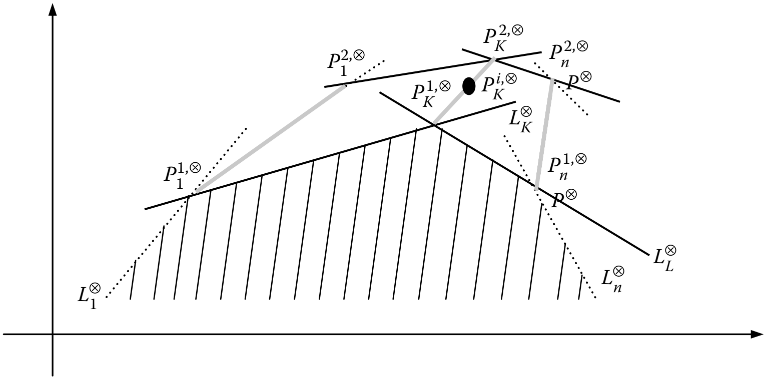


Figure 4.7 Grey convex polygon sketch map of grey linear programming problem.

For grey vertex $P_K^{\otimes}$, from Figure 4.7 we know that it is actually a curve, which is a point set of countless whitenization numbers, formed through the intersecting of grey curve $L_K^{\otimes}, L_L^{\otimes}$. These points continuously distribute on a certain numerical interval.

From Eqs. (4.61.*K*) and (4.61.*L*), we know for the units grey number γ_{Li}, $\gamma_{yi}, i = 1, 2, \ldots, n$ of all grey numbers.

If the optimum solution $z_{\otimes}^{*} = [Lz_{\otimes}^{*}, Rz_{\otimes}^{*}]$ of this problem is reached on grey vertex $P_K^{\otimes}$, then the left and right extreme point values of this problem's grey optimum solution are bound to be determined by the obtained number of units' grey numbers in Eq. (4.16).

To sum up, suppose there is a grey linear programming problem of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ [as shown in Eq. (4.44)]. If its grey feasible domain is bounded, then the objective function is sure to reach optimum on the grey vertex of its feasible domain. However, the bounded grey feasible domain $D(\otimes)$ is determined by the obtained number.

For some grey numbers' units grey number in $\tilde{A}(\otimes)$, the left and right extreme point values of this problem's grey optimum solution must be determined by the obtained number of some grey numbers' units grey number in $\tilde{A}(\otimes)$.

Example 4.6. Solve the grey saddle point, based on the meaning of mixed strategies, of grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ [and from there into $\tilde{A}(\otimes)$, as Eq. (4.62) shows]:

$$\tilde{A}(\otimes) = \begin{pmatrix} [1,2] & 0 & 2 \\ 0 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix} \tag{4.62}$$

$$\min(z_{\otimes}) = \min\left(x_1^{\otimes} + x_2^{\otimes} + x_3^{\otimes}\right)$$

$$s.t.\begin{cases}(1+\gamma_{11})x_1^{\otimes} + x_3^{\otimes} \geq 1_{\otimes} \\ 3x_2^{\otimes} + 2x_3^{\otimes} \geq 1_{\otimes} \\ 2x_1^{\otimes} + x_2^{\otimes} + x_3^{\otimes} \geq 1_{\otimes} \\ 0 \leq \gamma_{11} \leq 1 \\ x_i^{\otimes} \geq 0_{\otimes}, i = 1,2,3\end{cases} \tag{4.63}$$

$$\max(z_{\otimes}) = \max\left(y_1^{\otimes} + y_2^{\otimes} + y_3^{\otimes}\right)$$

$$s.t.\begin{cases}(1+\gamma_{11})y_1^{\otimes} + 2y_3^{\otimes} \leq 1_{\otimes} \\ 3y_2^{\otimes} + y_3^{\otimes} \leq 1_{\otimes} \\ y_1^{\otimes} + 2y_2^{\otimes} + y_3^{\otimes} \leq 1_{\otimes} \\ 0 \leq \gamma_{11} \leq 1 \\ y_i^{\otimes} \geq 0_{\otimes}, j = 1,2,3\end{cases} \tag{4.64}$$

Solution: Transfer this problem into two grey linear programming problems, as shown by Eqs. (4.60) and (4.61).

Aiming at Eq. (4.64), choose grey relaxed variables $y_4^{\otimes}, y_5^{\otimes}, y_6^{\otimes}$ as grey basis variables, transfer it into a standardized grey linear programming problem, and search optimization through twiddle iteration by this table.

$$\begin{cases}y_1^{\otimes} = \dfrac{1}{1+\gamma_{11}} \\ y_2^{\otimes} = \dfrac{1}{2} - \dfrac{1}{2+2\gamma_{11}} \\ y_3^{\otimes} = 0 \\ 0 \leq \gamma_{11} \leq 1\end{cases} \tag{4.65}$$

$$\begin{aligned} v^{\otimes} &= \frac{1}{y_1^{\otimes}+y_2^{\otimes}+y_3^{\otimes}} \\ &= \frac{2+2\gamma_{11}}{2+\gamma_{11}}\Bigg|_{0 \leq \gamma_{11} \leq 1} \\ &= \left[\frac{4}{3}, 2\right] \end{aligned} \tag{4.66}$$

$$\begin{cases} y_1^{*\otimes} = v^{\otimes} \cdot y_1^{\otimes} = \dfrac{2(1+\gamma_{11})}{2+\gamma_{11}} \cdot \dfrac{1}{1+\gamma_{11}} = \dfrac{2}{2+\gamma_{11}} \Big|_{0\le\gamma_{11}\le 1} = \left[\dfrac{2}{3}, 1\right] \\ y_2^{*\otimes} = v^{\otimes} \cdot y_2^{\otimes} = \dfrac{2(1+\gamma_{11})}{2+\gamma_{11}} \cdot \left(\dfrac{1}{2} - \dfrac{1}{2(1+\gamma_{11})}\right) \Big|_{0\le\gamma_{11}\le 1} = \left[0, \dfrac{1}{3}\right] \\ y_3^{*\otimes} = v^{\otimes} \cdot y_3^{\otimes} = \dfrac{2(1+\gamma_{11})}{2+\gamma_{11}} \cdot 0 = 0_{\otimes} \end{cases} \tag{4.67}$$

The optimum grey strategy for Player 2 is indicated by Eq. (4.67). Obviously, in Eq. (4.64), the sum of all optimum grey strategies for Player 2 is [1,1], shown by Eq. (4.68):

$$\sum_{i=1}^{3} y_i^{*\otimes} = y_1^{*\otimes} + y_2^{*\otimes} + y_3^{*\otimes} = \left[\frac{2}{3}, 1\right] + \left[0, \frac{1}{3}\right] + 0_{\otimes} = [1,1]_{\gamma_{yi}=c_{yi},\, 0\le c_{yi}\le 1,\, i=1,2,3} = [1,1]_{\otimes} \tag{4.68}$$

for grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)\}$ [and there into $\tilde{A}(\otimes)$,as Eq. (4.62) shows].

Because the grey matrix game between Players 1 and 2 is a problem of grey linear programming, which is mutually dual, from the row of test numbers in Table 4.3 we can get the grey optimum solution of Player 1's grey linear programming problem, shown by Eq. (4.66). From Eqs. (4.66) and (4.69), we can get the optimum grey strategy for Player 1, shown by Eq. (4.70):

$$\begin{cases} x_1^{\otimes} = \dfrac{1}{2+2\gamma_{11}} \\ x_2^{\otimes} = 0 \\ x_3^{\otimes} = \dfrac{1}{2} \\ 0 \le \gamma_{11} \le 1 \end{cases} \tag{4.69}$$

$$\begin{cases} x_1^{*\otimes} = v^{\otimes} \cdot x_1^{\otimes} = \dfrac{2+2\gamma_{11}}{2+\gamma_{11}} \cdot \dfrac{1}{2+2\gamma_{11}} = \dfrac{1}{2+\gamma_{11}} \Big|_{0 \le \gamma_{11} \le 1} = \left[\dfrac{1}{3}, \dfrac{1}{2}\right] \\ x_2^{*\otimes} = v^{\otimes} \cdot x_1^{\otimes} = 0_{\otimes} \\ x_3^{*\otimes} = v^{\otimes} \cdot x_3^{\otimes} = \dfrac{2+2\gamma_{11}}{2+\gamma_{11}} \cdot \dfrac{1}{2} = \dfrac{1+\gamma_{11}}{2+\gamma_{11}} \Big|_{0\le\gamma_{11}\le 1} = \left[\dfrac{1}{2}, \dfrac{2}{3}\right] \\ \displaystyle\sum_{i=1}^{3} x_i^{*\otimes} = [1,1]_{\otimes} \end{cases} \tag{4.70}$$

4.5.4 Summary

This section proved two significant theorems: that the feasible domain of $\tilde{G}(\otimes)$'s linear programming is a grey convex set and that the grey basis feasible solution corresponds to the grey vertex of a grey feasible domain. Based on the proof that any

Table 4.3 Grey Linear Programming Table Optimization of the Grey Saddle Point

C_j			1	1	1	0	0	0	θ
C_B	Y_B	b	$y_1^\otimes$	$y_2^\otimes$	$y_3^\otimes$	$y_4^\otimes$	$y_5^\otimes$	$y_6^\otimes$	
0	$y_4^\otimes$	1	$\boxed{(1+\gamma_{11})}$	0	2	1	0	0	$\frac{1}{1+\gamma_{11}}$
0	$y_5^\otimes$	1	0	3	1	0	1	0	—
0	$y_6^\otimes$	1	1	2	2	0	0	1	1
$-z_\otimes$		0	1	1	1	0	0	0	
Because when $0 \le \gamma_{11} \le 1$, $\frac{1}{1+\gamma_{11}} \le 1$, choose $y_1^\otimes$ as entering variable, $y_4^\otimes$ as leaving variable.									θ
1	$y_1^\otimes$	$\frac{1}{1+\gamma_{11}}$	1	0	$\frac{2}{1+\gamma_{11}}$	$\frac{1}{1+\gamma_{11}}$	0	0	—
0	$y_5^\otimes$	1	0	3	1	0	1	0	$\frac{1}{3}$
0	$y_6^\otimes$	$1-\frac{1}{1+\gamma_{11}}$	0	2	$2-\frac{2}{1+\gamma_{11}}$	$-\frac{1}{1+\gamma_{11}}$	0	1	$\frac{1}{2}(1-\frac{1}{1+\gamma_{11}})$
$-z_\otimes$		$\frac{1}{1+\gamma_{11}}$	0	1	$1-\frac{2}{1+\gamma_{11}} \le 0$	$-\frac{1}{1+\gamma_{11}} \le 0$	0	1	
Because when $0 \le \gamma_{11} \le 1$, $\frac{1}{2}(1-\frac{1}{1+\gamma_{11}}) \le \frac{1}{3}$, choose $y_2^\otimes$ as entering variable, $y_6^\otimes$ as leaving variable.									θ
1	$y_1^\otimes$	$\frac{1}{1+\gamma_{11}}$	1	0	$\frac{2}{1+\gamma_{11}}$	$\frac{1}{1+\gamma_{11}}$	0	0	
0	$y_5^\otimes$	$\frac{-1}{2}+\frac{3}{2+2\gamma_{11}}$	0	0	$-2+\frac{3}{1+\gamma_{11}}$	$\frac{3}{2+2\gamma_{11}}$	1	$-\frac{3}{2}$	
1	$y_2^\otimes$	$\frac{1}{2}-\frac{1}{2+2\gamma_{11}}$	0	1	$1-\frac{1}{1+\gamma_{11}}$	$-\frac{1}{2+2\gamma_{11}}$	0	$\frac{1}{2}$	
$-z_\otimes$		$\frac{1}{2}+\frac{1}{2+2\gamma_{11}}$	0	0	$-\frac{1}{1+\gamma_{11}} \le 0$	$-\frac{1}{2+2\gamma_{11}} \le 0$	0	$-\frac{1}{2}$	

point in a grey convex set can be linearly expressed with the grey vertex of its convex set, we proved that the grey optimum value of the objective function of the linear programming for a grey matrix game corresponds to the grey vertex of its convex set. We also proved that the grey optimum value of it is bound and corresponds to the obtained number of some grey factors' unit grey number in grey game matrix $\tilde{A}(\otimes)$.

Chapter 5

Study of Elementary Transformations of the Grey Matrix and the Invertible Grey Matrix

5.1 Grey Vector Groups and Grey Linear Correlations

5.1.1 Basic Concept of Grey Vectors and Grey Linear Combinations

Definition 5.1 Grey vector: $n \times 1$ grey matrices are called n-dimensional grey column vectors, and $1 \times n$ grey matrices are called n-dimensional grey row vectors; n-dimensional grey column vectors and n-dimensional grey row vectors are together called n-dimensional grey vectors, and they are also called grey vectors for short.

Definition 5.2 Grey linear combinations: Assume a grey vector group $\alpha^{\otimes}, \alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ in which $\alpha_i^{\otimes} = \{\alpha_{i1}^{\otimes}, \alpha_{i2}^{\otimes}, \ldots, \alpha_{ij}^{\otimes}, \ldots, \alpha_{in}^{\otimes}\}$, $\alpha_{ij}^{\otimes} = [L\alpha_{ij}^{\otimes}, R\alpha_{ij}^{\otimes}]$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ are all n-dimensional grey vectors and $k_1^{\otimes}, k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_m^{\otimes}$ is a group of grey numbers. If every grey component $\gamma_{\alpha i}, \gamma_{ki}, \gamma_{\alpha ij}, i = 1, 2, \cdots, m$, $j = 1, 2, \ldots, n$ of $\alpha^{\otimes}, \alpha_i^{\otimes}, k_i^{\otimes} i = 1, 2, \ldots, m$ meets the conditions, then

$$\alpha^{\otimes} \Big|_{\gamma_{ai}} \left\{ k_1^{\otimes} \alpha_1^{\otimes} + k_2^{\otimes} \alpha_2^{\otimes} + \cdots + k_m^{\otimes} \alpha_m^{\otimes} \right\}_{\gamma_{ki}, \gamma_{aij}} \Big| \tag{5.1}$$

We can say that grey vector $\alpha^{\otimes}$ can be grey linear represented by the grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$, or $\alpha^{\otimes}$ is a grey linear combination of $\alpha_1^{\otimes}$, $\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$.

Theorem 5.1 Necessary conditions of $\gamma_{\alpha i},\gamma_{ki},\gamma_{\alpha ij}, i=1,2,\ldots,m, j=1,2,\ldots,n$ For an arbitrary grey vector $\alpha^{\otimes}$ and an arbitrary grey vector group $\alpha_1^{\otimes}$, $\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ in which $\alpha_i^{\otimes}=\{\alpha_{i1}^{\otimes},\alpha_{i2}^{\otimes},\ldots,\alpha_{ij}^{\otimes},\ldots,\alpha_{in}^{\otimes}\}$, if $\alpha^{\otimes}$ can be grey linear represented by a grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_n^{\otimes}$ that has a grey coefficient group of $k_1^{\otimes},k_2^{\otimes},\ldots,k_i^{\otimes},\ldots,k_m^{\otimes}$, we can meet the conditions of $\gamma_{\alpha i},\gamma_{ki},\gamma_{\alpha ij}, i=1,2,\ldots,m, j=1,2,\ldots,n$, which is the obtained number of the unit grey numbers of $\alpha^{\otimes},\alpha_i^{\otimes},k_i^{\otimes} i=1,2,\ldots,m$.

Proof: To make the proof general, for an arbitrary n-dimensional grey vector $\alpha^{\otimes}$, a grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$, and a grey coefficient group $k_1^{\otimes},k_2^{\otimes},\ldots,k_i^{\otimes},\ldots,k_m^{\otimes}$, we assume that the obtained number of the unit grey numbers of this grey coefficient group meets the conditions that make Eq. (5.1) tenable.

Then we can represent Eq. (5.1) as the form of Eq. (5.2):

$$k_1^{\otimes}\alpha_1^{\otimes}+k_2^{\otimes}\alpha_2^{\otimes}+\cdots+k_m^{\otimes}\alpha_m^{\otimes}-\alpha^{\otimes}=\{0,0,\ldots,0\}$$

$$\begin{aligned}&[Lk_1,Rk_1]\cdot\{[L\alpha_{11},R\alpha_{11}],\ldots,[L\alpha_{1n},R\alpha_{1n}]\}+\ldots+[Lk_m,Rk_m]\cdot\{[L\alpha_{m1},R\alpha_{m1}],\ldots,\\&\quad[L\alpha_{mn},R\alpha_{mn}]\}-\{[L\alpha_1,R\alpha_1],\ldots,[L\alpha_n,R\alpha_n]\}=\{0,0,\ldots,0\}\end{aligned}$$

$$\begin{aligned}&(Lk_1+(Rk_1-Lk_1)\gamma_{k1})\cdot\{(L\alpha_{11}+(R\alpha_{11}-L\alpha_{11})\gamma_{\alpha 11}),\ldots,\\&\quad(L\alpha_{1n}+(R\alpha_{1n}-L\alpha_{1n})\gamma_{\alpha 1n})\}+\cdots+(Lk_m+(Rk_m-Lk_m)\gamma_{km})\cdot\\&\{(L\alpha_{m1}+(R\alpha_{m1}-L\alpha_{m1})\gamma_{am1}),\ldots,(L\alpha_{mn}+(R\alpha_{mn}-L\alpha_{mn})\gamma_{\alpha mn})\}-\\&\quad-\{(L\alpha_1+(R\alpha_1-L\alpha_1)\gamma_{\alpha 1}),\ldots,(L\alpha_n+(R\alpha_n-L\alpha_n)\gamma_{an})\}=\{0,0,\ldots,0\}\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{m}(Lk_i+(Rk_i-Lk_i)\gamma_{ki})\cdot\{(L\alpha_{i1}+(R\alpha_{i1}-L\alpha_{i1})\gamma_{ai1}),\ldots,\\&(L\alpha_{in}+(R\alpha_{in}-L\alpha_{in})\gamma_{ain})\}-\{(L\alpha_1+(R\alpha_1-L\alpha_1)\gamma_{a1}),\ldots,(L\alpha_n+\\&\quad+(R\alpha_n-L\alpha_n)\gamma_{an})\}=\{0,0,\ldots,0\}\end{aligned}$$

$$\begin{aligned}&\left\{\left[\sum_{i=1}^{m}Lk_i\cdot L\alpha_{i1}+Lk_i\cdot(Rk_i-Lk_i)\cdot\gamma_{ai1}+L\alpha_{i1}\cdot(Rk_i-Lk_i)\cdot\gamma_{ki}+\right.\right.\\&\quad\left.+(Rk_i-Lk_i)\cdot(R\alpha_{i1}-L\alpha_{i1})\cdot\gamma_{ai1}\cdot\gamma_{ki}-(L\alpha_1+(R\alpha_1-L\alpha_1)\gamma_{a1}\right]+\cdots+\end{aligned}$$

$$
+\left[\sum_{i=1}^{m} Lk_i \cdot L\alpha_{in} + Lk_i \cdot (Rk_i - Lk_i)\cdot \gamma_{ain} + L\alpha_{in}\cdot (Rk_i - Lk_i)\cdot \gamma_{ki} + \right.
$$

$$
\left. + (Rk_i - Lk_i)\cdot (R\alpha_{in} - L\alpha_{in})\cdot \gamma_{ain}\cdot \gamma_{ki} - (L\alpha_n + (R\alpha_n - L\alpha_n)\gamma_{an})\right]\Bigg\}\Bigg\} = \{0,0,\ldots,0\}
$$

If we make γ_{ki}, $\gamma_{\alpha i}$, $\gamma_{\alpha ij}$, $0 \le \gamma_{ki} \le 1, 0 \le \gamma_{\alpha i} \le 1, 0 \le \gamma_{\alpha ij} \le 1, i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ as the seeking knowledge variables respectively, we can get Eqs. (5.2.1), (5.2.2),…,(5.2.n) from the formula above.

$$
\begin{cases}
\sum_{i=1}^{m} Lk_i \cdot L\alpha_{i1} + Lk_i \cdot (Rk_i - Lk_i)\cdot \gamma_{\alpha i1} + L\alpha_{i1}\cdot (Rk_i - Lk_i)\cdot \gamma_{ki} + \\
\quad + (Rk_i - Lk_i)\cdot (R\alpha_{i1} - L\alpha_{i1})\cdot \gamma_{\alpha i1}\cdot \gamma_{ki} + = (L\alpha_1 + (R\alpha_1 - L\alpha_1)\gamma_{\alpha 1} \quad 5.2.1 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{i=1}^{m} Lk_i \cdot L\alpha_{in} + Lk_i \cdot (Rk_i - Lk_i)\cdot \gamma_{\alpha in} + L\alpha_{in}\cdot (Rk_i - Lk_i)\cdot \gamma_{ki} + \\
\quad + (Rk_i - Lk_i)\cdot (R\alpha_{in} - L\alpha_{in})\cdot \gamma_{\alpha in}\cdot \gamma_{ki} = (L\alpha_n + (R\alpha_n - L\alpha_n)\gamma_{\alpha n}) \quad 5.2.n
\end{cases}
\tag{5.2}
$$

where $0 \le \gamma_{ki} \le 1, 0 \le \gamma_{\alpha i} \le 1, 0 \le \gamma_{\alpha ij} \le 1, i = 1,2,\ldots,m, j = 1,2,\ldots,n.$

According to the knowledge of linear algebra, we can get the solution by solving the nonhomogeneous linear equation set (5.2.1)–(5.2.n), which meets the above conditions.

We can then get the conditions of $\gamma_{\alpha i}, \gamma_{ki}, \gamma_{\alpha ij}, i = 1,2,\ldots,m, j = 1,2,\ldots,n$, which is the obtained number of the unit grey numbers of $\alpha^{\otimes}, \alpha_i^{\otimes}, k_i^{\otimes} i = 1,2,\ldots,m.$

The grey linear correlation between grey vectors is regulated by some obtained number conditions. The proof of Theorem 5.1 is constructive, and it explains the way to solve obtained number conditions for every grey component when the grey vectors are grey linear correlated.

For a grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ and the grey vector $\alpha^{\otimes}$, whether $\alpha^{\otimes}$ can be linear represented by $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ must be decided in the actual situation. However, when $\gamma_{\alpha i}, \gamma_{ki}, \gamma_{\alpha ij}, i = 1,2,\ldots, m, j = 1,2,\ldots,n$ meets the conditions, $\alpha^{\otimes}$ must be linear represented or not—there is no the third choice.

Definition 5.3 *N*-dimensional basic grey vector group: We call the *n*-dimensional grey vector group

$$e_1^{\otimes}=\begin{pmatrix}[1,1]\\ [0,0]\\ [0,0]\\ \vdots\\ [0,0]\end{pmatrix},\quad e_2^{\otimes}=\begin{pmatrix}[0,0]\\ [1,1]\\ [0,0]\\ \vdots\\ [0,0]\end{pmatrix},\ldots,\quad e_n^{\otimes}=\begin{pmatrix}[0,0]\\ [0,0]\\ [0,0]\\ \vdots\\ [1,1]\end{pmatrix}$$

an *n*-dimensional basic grey vector group, and *n*-dimensional grey basic group for short.

Obviously, for an *n*-dimensional grey vector $\alpha^{\otimes}$, no matter what the obtained number of the unit grey numbers, $\gamma_{\alpha j}, 0 \le \gamma_{\alpha j} \le 1, j = 1,2,\ldots,n$ is any number between 0 and 1, $\alpha^{\otimes}$ can be linear represented by an *n*-dimensional basic grey vector group; if we assume $\alpha^{\otimes} = (\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_n^{\otimes})^T$, then we get $\alpha^{\otimes} = \alpha_1^{\otimes} e_1^{\otimes} + \alpha_2^{\otimes} e_2^{\otimes} + \cdots + \alpha_n^{\otimes} e_n^{\otimes}$.

Definition 5.4 Equivalent grey vector groups: If every grey vector $\alpha_i^{\otimes}$, $(i = 1,2,\ldots,m)$ in the grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ can be grey linear represented by the grey vector group $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$, we can say that grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ can be grey linear represented by the grey vector group $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$. If two grey vector groups can be linear represented with each other, then we say that the two grey vector groups are equivalent.

If grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ can be grey linear represented by grey vector group $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$, and grey vector group $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$ can be grey linear represented by grey vector group $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_t^{\otimes}$, then we can say that grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ can be grey linear represented by grey vector group $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_t^{\otimes}$. This conclusion is called the transitivity of grey linear representation.

Equivalency of grey vectors groups is a relationship between two grey vector groups. It is not difficult to prove that there are three properties of the relationship.

1. Reflexivity: Every grey vector group is equivalent to itself.
2. Symmetry: If $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ is equivalent to $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$, then $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$ is equivalent to $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$.
3. Transitivity: If $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ is equivalent to $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$, and $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_s^{\otimes}$ is equivalent to $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_t^{\otimes}$, then $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ is equivalent to $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_t^{\otimes}$.

5.1.2 Grey Linear Correlation of Grey Vectors

Definition 5.5 Grey linear independence: For an n-dimensional grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_i^\otimes,\ldots,\alpha_m^\otimes$, if there is a group of grey numbers $k_1^\otimes,k_2^\otimes,\ldots,k_i^\otimes,\ldots,k_m^\otimes$ that are not zero so that

$$k_1^\otimes\alpha_1^\otimes+k_2^\otimes\alpha_2^\otimes+\cdots+k_m^\otimes\alpha_m^\otimes=[0,0]_{\gamma_{\alpha i},\gamma_{ki},\, i=1,\,2,\ldots,\,m}=[0,0]_\otimes \tag{5.3}$$

then we say grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_i^\otimes,\ldots,\alpha_m^\otimes$ is grey linear correlated, or we say the grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_i^\otimes,\ldots,\alpha_m^\otimes$ is grey linear independent.

Example 5.1 Grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\alpha_3^\otimes,$ as shown in Eq. (5.5)

$$\alpha_1^\otimes=\begin{pmatrix}[1,1]\\ [0,0]\\ [1,1]\end{pmatrix},\alpha_2^\otimes=\begin{pmatrix}[2,4]\\ [1,2]\\ [1,1]\end{pmatrix},\alpha_3^\otimes=\begin{pmatrix}[0,0]\\ [-2,-1]\\ [1,3]\end{pmatrix} \tag{5.4}$$

Because there is a group of grey numbers $k_1^\otimes=[-4,-2],k_2^\otimes=[1,1],k_3^\otimes=[1,1],$ we get

$$\begin{aligned}
&k_1^\otimes\cdot\alpha_1^\otimes+k_2^\otimes\cdot\alpha_2^\otimes+k_3^\otimes\cdot\alpha_3^\otimes\\
&=[-4,-2]\cdot\{[1,1],[0,0],[1,1]\}^T+[1,1]\cdot\{[2,4],[1,2],[1,1]\}^T+[1,1]\cdot\{[0,0],[-2,-1],[1,3]\}^T\\
&=(-4+2\gamma_{k1})\{1,0,1\}^T+1\cdot\{(2+2\gamma_{\alpha21}),(1+\gamma_{\alpha22}),1\}^T+1\cdot\{0,(-2+\gamma_{\alpha32}),(1+2\gamma_{\alpha33})\}^T\\
&=\{(-4+2\gamma_{k1}),0,(-4+2\gamma_{k1})\}^T+\{(2+2\gamma_{\alpha21}),(1+\gamma_{\alpha22}),1\}^T+\{0,(-2+\gamma_{\alpha32}),(1+2\gamma_{\alpha33})\}^T\\
&=\{(-2+2(\gamma_{k1}+\gamma_{\alpha21})),(-1+\gamma_{\alpha22}+\gamma_{\alpha32}),(-2+2(\gamma_{k1}+\gamma_{\alpha33}))\}^T
\end{aligned} \tag{5.5}$$

In Eq. (5.2), when every component of $k_1^\otimes\cdot\alpha_1^\otimes+k_2^\otimes\cdot\alpha_2^\otimes+k_3^\otimes\cdot\alpha_3^\otimes$ is 0, that is,

$$\{(-2+2(\gamma_{k1}+\gamma_{\alpha21})),(-1+\gamma_{\alpha22}+\gamma_{\alpha32}),(-2+2(\gamma_{k1}+\gamma_{\alpha33}))\}^T=\{0,0,0\}^T \tag{5.6}$$

to get Eq. (5.6) tenable, we must get Eq. (5.7) tenable. If we assume that $x_1=\gamma_{k1}$, $x_2=\gamma_{\alpha21},x_3=\gamma_{\alpha22},x_4=\gamma_{\alpha32},x_2=\gamma_{\alpha33}$, and $0\le x_i\le1,i=1,2,3,4,5,$ then Eq. (5.7) can be expressed as Eq. (5.8):

$$\begin{cases}-2+2(\gamma_{k1}+\gamma_{\alpha21})=0\\ -1+\gamma_{\alpha22}+\gamma_{\alpha32}=0\\ -2+2(\gamma_{k1}+\gamma_{\alpha33})=0\end{cases} \tag{5.7}$$

$$0\le\gamma_{k1},\gamma_{\alpha21},\gamma_{\alpha22},\gamma_{\alpha32},\gamma_{\alpha33}\le1$$

$$\begin{cases} x_1 = 1 - x_2 \\ x_3 = 1 - x_4 \\ x_2 = x_5 \end{cases} \tag{5.8}$$

$$0 \le x_i \le 1,\ i = 1,2,3,4,5$$

According to Eq. (5.8), we know $x_2, x_4, (0 \le x_2, x_4 \le 1)$ are free variables; once the values of $x_i, 0 \le x_i \le 1, i = 1,2,3,4,5$ meet the conditions of Eq. (5.8), grey vectors $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \alpha_3^{\otimes}$ are linear correlated.

According to Definition 5.5, we know if a grey vector group is not grey linear correlated, we call it grey linear independent. This concept can be expressed in two equivalent ways, as follows:

1. In any grey number group $k_1^{\otimes}, k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_m^{\otimes}$ where the vectors are not all 0, no matter what the obtained number of $\gamma_{ki}, \gamma_{aij}, 0 \le \gamma_{ki}, \gamma_{aij} \le 1$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ is, Eq. (5.6) cannot be tenable.
2. If we want to get Eq. (5.6) tenable, $k_1^{\otimes}, k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_m^{\otimes}$ must be 0 for all.

It is easier to use the two ways above to prove whether a grey vector group is linear independent or not.

Example 5.2 Proving that an *n*-dimensional basic grey vector group $e_1^{\otimes} = (1, 0, \ldots, 0)^T, e_2^{\otimes} = (0, 1, \ldots, 0)^T, \ldots, e_m^{\otimes} = (0, \ldots, 0, 1)^T$ is linear independent:

Proof 1: For grey numbers $k_1^{\otimes}, k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_m^{\otimes}$ that are not 0 for all, we can get

$$k_1^{\otimes} \cdot e_1^{\otimes} + k_2^{\otimes} \cdot e_2^{\otimes} + \cdots + k_m^{\otimes} \cdot e_m^{\otimes} = \left(k_1^{\otimes}, k_2^{\otimes}, \ldots, k_m^{\otimes}\right)^T \ne (0, 0, \ldots, 0)$$

So, we can get that $e_i^{\otimes}, i = 1, 2, \ldots, m$ is linear independent.

Proof 2: Assume that grey number group $k_1^{\otimes}, k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_m^{\otimes}$ makes $k_1^{\otimes} \cdot e_1^{\otimes} + k_2^{\otimes} \cdot e_2^{\otimes} + \cdots + k_m^{\otimes} \cdot e_m^{\otimes} = (0, 0, \ldots, 0$ then from $k_i^{\otimes} = 0, i = 1, 2, \ldots, m,$ we can get that $k_1^{\otimes} = k_2^{\otimes} = k_i^{\otimes} = \cdots = k_m^{\otimes} = [0, 0]$. So $e_i^{\otimes}, i = 1, 2, \ldots, m$ is linear independent.

5.1.3 Theorems about Grey Vectors Grey Linear Correlation

According to definitions of grey vectors grey linear correlation and linear independence, we can get some simple and useful conclusions.

1. Any grey vector group that concludes with zero vectors is grey linear correlated.
2. To a grey vector group of just one grey vector $\alpha^{\otimes}$, if $\alpha^{\otimes} = 0$, then the grey vector group is grey linear correlated; if $\alpha^{\otimes} \ne 0$, the grey vector group is linear independent.

3. Any extended group of a grey linear correlation grey vector group must be a grey linear correlation group. That is, if $\alpha_1^\otimes, \alpha_2^\otimes, \ldots, \alpha_i^\otimes, \ldots, \alpha_m^\otimes$ is a grey linear correlation group, after adding several grey vectors $\alpha_{m+1}^\otimes, \alpha_{m+2}^\otimes, \ldots, \alpha_s^\otimes$ the new grey vector group $\alpha_1^\otimes, \alpha_2^\otimes, \ldots, \alpha_{m+1}^\otimes, \alpha_{m+2}^\otimes, \ldots, \alpha_s^\otimes$: is grey linear correlated.
4. Any nonzero partial group of a linear independent grey vector group must be a linear independent group.

Because it is easy to prove these conclusions, we omit the process of proof here but we proceed to some propositions that are related to grey vector group grey linear correlations.

Theorem 5.2 Grey linear correlation of *n*-1–dimensional grey vector group: If an *n*-dimensional grey vector group

$$\alpha_1^\otimes = \left(a_{11}^\otimes, a_{12}^\otimes, \ldots, a_{1,n-1}^\otimes, a_{1n}^\otimes\right)^T,$$

$$\alpha_2^\otimes = \left(a_{21}^\otimes, a_{22}^\otimes, \ldots, a_{2,n-1}^\otimes, a_{2n}^\otimes\right)^T, \ldots, \alpha_m^\otimes = \left(a_{m1}^\otimes, a_{m2}^\otimes, \ldots, a_{m,n-1}^\otimes, a_{mn}^\otimes\right)^T$$

is grey linear correlated, then the *n*-1–dimensional grey vector group that is produced by reducing the last grey component of every grey vector

$$\beta_1^\otimes = \left(a_{11}^\otimes, a_{12}^\otimes, \cdots, a_{1,n-1}^\otimes\right)^T, \beta_2^\otimes = \left(a_{21}^\otimes, a_{22}^\otimes, \ldots, a_{2,n-1}^\otimes\right)^T, \ldots, \beta_m^\otimes = \left(a_{m1}^\otimes, a_{m2}^\otimes, \ldots, a_{m,n-1}^\otimes\right)^T$$

is also grey linear correlated.

Proof: Because *n*-dimensional grey vector group $\alpha_1^\otimes, \alpha_2^\otimes, \ldots, \alpha_i^\otimes, \ldots, \alpha_m^\otimes$ is grey linear correlated, there is a group of grey numbers $k_1^\otimes, k_2^\otimes, \ldots, k_i^\otimes, \ldots, k_m^\otimes$ where

$$k_1^\otimes \cdot \alpha_1^\otimes + k_2^\otimes \cdot \alpha_2^\otimes + \cdots + k_m^\otimes \cdot \alpha_m^\otimes = (0,0,\ldots,0)_{\gamma_{ki}, \gamma_{\alpha ij}, i=1,2,\ldots,m, j=1,2,\ldots,n}$$

According to the definition, if the vectors are equal, every component of the vector is equal, and we can get Eq. (5.9):

$$k_1^\otimes \cdot \left(a_{11}^\otimes, a_{12}^\otimes, \ldots, a_{1,n-1}^\otimes\right)^T + k_2^\otimes \cdot \left(a_{21}^\otimes, a_{22}^\otimes, \ldots, a_{2,n-1}^\otimes\right)^T + \cdots + k_m^\otimes \cdot \left(a_{m1}^\otimes, a_{m2}^\otimes, \ldots, a_{m,n-1}^\otimes\right)^T$$

$$= (0,0,\ldots,0)_{\gamma_{ki}, \gamma_{\alpha ij}, i=1,2,\ldots,m, j=1,2,\ldots,n-1} \qquad (5.9)$$

or

$$k_1^{\otimes} \cdot \beta_1^{\otimes} + k_2^{\otimes} \cdot \beta_2^{\otimes} + \cdots + k_m^{\otimes} \cdot \beta_m^{\otimes} = (0,0,\ldots,0)_{\gamma_{ki},\gamma_{\alpha ij},\, i=1,2,\ldots,m,\, j=1,\,2,\ldots,\,n-1}$$

For cases where $k_1^{\otimes}, k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_m^{\otimes}$ are not all 0, the *n*-1–dimensional grey vector group $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_i^{\otimes}, \ldots, \beta_m^{\otimes}$ is grey linear correlated.

Theorem 5.3 Grey linear independence of an *n* + 1–dimensional grey vector group: If *n*-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ is linear independent, the *n*+1–dimensional grey vector group $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_m^{\otimes}$ that is produced by increasing a component to the end of every grey vector is grey linear independent.

Proof: We may use an inverse proof to demonstrate this theorem, hypothesizing that there is an *n*-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ that is linear independent, and the *n*+1–dimensional grey vector group $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_m^{\otimes}$ that is produced by increasing a component to the end of every grey vector is grey linear correlated.

If $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_m^{\otimes}$ is grey linear correlated, according to Theorem 5.2, and if we reduce the last component of $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_m^{\otimes}$, we can get the *n*-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$, and therefore this *n*-dimensional grey vector group must be grey linear correlated. This is contradictory to the hypothesis; so the hypothesis is wrong, and the original proposition is right.

It is easy to find that, in Theorems 5.2 and 5.3, the grey component we can reduce or add is not just the last component, and the number of components we can reduce or add is not just 1. However, for every vector, the number and the location of the grey component we reduce or add must be the same. In this way, we can extend Theorems 5.2 and 5.3 as follows:

Theorem 5.4 Grey linear correlation of an *n*–*r*–dimensional grey vector group: If *n*-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ is grey linear correlated, then the *n*-*r*–dimensional grey vector group that is produced by reducing *r* components of every grey vector is also grey linear correlated.

Theorem 5.5 Grey linear independence of an *n* + *r*–dimensional grey vector group: If *n*-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ is linear independent, then the *n* + *r*–dimensional grey vector group $\eta_1^{\otimes}, \eta_2^{\otimes}, \ldots, \eta_i^{\otimes}, \ldots, \eta_m^{\otimes}$ that is produced by increasing *r* components of every grey vector is grey linear independence.

We can prove Theorems 5.4 and 5.5 in the same way we proved Theorems 5.2 and 5.3.

5.1.4 Summary

In this part, we put forward some concepts of grey vectors, grey linear correlations, and grey linear independence. We constructed a preliminary theoretical frame system of these concepts, and provided a good basis for the study of grey matrix game models based on grey mixed strategy.

5.2 Maximum Grey Vector Groups and the Rank of Grey Matrix

5.2.1 Basic Theorems about Grey Vector Group Grey Linear Correlations

Theorem 5.6 The sufficient and necessary conditions of grey vector group grey linear correlations: For a grey vector group that has more than one grey vector, the sufficient and necessary conditions of this grey vector group of grey linear correlations are that there is at least one grey vector that can be grey linear represented by other grey vectors.

Proof: Assume that $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ is an n-dimensional grey vector group ($m \geq 2$).

If $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ is a grey linear correlation, there will be a group of grey numbers $k_1^{\otimes},k_2^{\otimes},\ldots,k_i^{\otimes},\ldots,k_m^{\otimes}$ that is not 0 for all, so that

$$k_1^{\otimes}\cdot\alpha_1^{\otimes}+k_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots+k_m^{\otimes}\cdot\alpha_m^{\otimes}=(0,0,\ldots,0)_{\gamma_{ki},\gamma_{\alpha ij},i=1,2,\ldots,m,j=1,2,\ldots,n} \tag{5.10}$$

If we assume that $k_1^{\otimes}\neq[0,0]$, then $\alpha_1^{\otimes}=(-\frac{k_2^{\otimes}}{k_1^{\otimes}})\cdot\alpha_2^{\otimes}+(-\frac{k_3^{\otimes}}{k_1^{\otimes}})\cdot\alpha_3^{\otimes}+\cdots+(-\frac{k_m^{\otimes}}{k_1^{\otimes}})\cdot\alpha_m^{\otimes}$.

So $\alpha_1^{\otimes}$ can be grey linear represented by other grey vectors of this grey vector group. Generally, in Eq. (5.10), if $k_i^{\otimes}\neq[0,0]$, $\alpha_i^{\otimes}$ can certainly be grey linear represented by other vectors.

If any grey vector (assume as $\alpha_i^{\otimes}$) in the grey vector group can be grey linear represented by other vectors, that is, if

$$\alpha_i^{\otimes}=k_1^{\otimes}\cdot\alpha_1^{\otimes}+k_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots k_{i-1}^{\otimes}\cdot\alpha_{i-1}^{\otimes}+k_{i-1}^{\otimes}\cdot\alpha_{i-1}^{\otimes}\cdots+k_m^{\otimes}\cdot\alpha_m^{\otimes}$$

$$\begin{aligned}&k_1^{\otimes}\cdot\alpha_1^{\otimes}+k_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots k_{i-1}^{\otimes}\cdot\alpha_{i-1}^{\otimes}+(-1)\alpha_i^{\otimes}+k_{i-1}^{\otimes}\cdot\alpha_{i-1}^{\otimes}\cdots+\\&+k_m^{\otimes}\cdot\alpha_m^{\otimes}=(0,0,\ldots,0)_{\gamma_{ki},\gamma_{\alpha ij},i=1,2,\ldots,m,j=1,2,\ldots,n}\end{aligned} \tag{5.11}$$

where the coefficient of $\alpha_i^{\otimes}$ is $-1\neq0$, then Eq. (5.11) shows that $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ is grey linear correlated.

Theorem 5.7 Representation method of grey vector group grey linear correlation: If an n-dimensional grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ is linear independent, but $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes},\beta^{\otimes}$ is grey linear correlated, then $\beta^{\otimes}$ can surely be grey linear represented by $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$, because the obtained number of every grey component and their grey coefficients have to meet some conditions, and the representation method (grey coefficient) is unique.

Proof: Assuming that there is a group of numbers $k_1^{\otimes},k_2^{\otimes},\ldots,k_i^{\otimes},\ldots,k_m^{\otimes},l^{\otimes}$ that is not 0 for all, where

$$k_1^{\otimes}\cdot\alpha_1^{\otimes}+k_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots+k_m^{\otimes}\cdot\alpha_m^{\otimes}+l^{\otimes}\cdot\beta^{\otimes}=(0,0,\ldots,0)_{\gamma_{ki},\gamma_{li},\gamma_{\alpha ij},\gamma_{\beta i}i=1,2,\ldots,m,j=1,2,\ldots,n}$$

then we can find that $l^{\otimes}\neq[0,0]$. If not, $k_1^{\otimes},k_2^{\otimes},\ldots,k_i^{\otimes},\ldots,k_m^{\otimes}$ will not be 0 for all, and

$$k_1^{\otimes}\cdot\alpha_1^{\otimes}+k_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots+k_m^{\otimes}\cdot\alpha_m^{\otimes}=(0,0,\ldots,0)_{\gamma_{ki},\gamma_{\alpha ij},i=1,2,\ldots,m,j=1,2,\ldots,n}$$

This contradicts the theory that $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ is linear independent. In this way, we can prove that $l^{\otimes}\neq[0,0]$, and $\beta^{\otimes}$ can be grey linear represented by $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$.

Then we will prove that representation method (grey coefficient) is unique.

Assume that

$$\beta^{\otimes}\Big|_{\gamma_{\beta ij},i=1,2,\ldots,m,j=1,2,\ldots,n}=\left\{k_1^{\otimes}\cdot\alpha_1^{\otimes}+k_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots+k_m^{\otimes}\cdot\alpha_m^{\otimes}\right\}_{\gamma_{ki},\gamma_{\alpha ij},i=1,2,\ldots,m,j=1,2,\ldots,n} \tag{5.12}$$

and

$$\beta^{\otimes}\Big|_{\gamma_{\beta ij},i=1,2,\ldots,m,j=1,2,\ldots,n}=\left\{l_1^{\otimes}\cdot\alpha_1^{\otimes}+l_2^{\otimes}\cdot\alpha_2^{\otimes}+\cdots+l_m^{\otimes}\cdot\alpha_m^{\otimes}\right\}_{\gamma_{li},\gamma_{\alpha ij},i=1,2,\ldots,m,j=1,2,\ldots,n} \tag{5.13}$$

By subtracting Eq. (5.12) from Eq. (5.13), we find

$$\begin{aligned}&\left(k_1^{\otimes}-l_1^{\otimes}\right)\cdot\alpha_1^{\otimes}+\left(k_2^{\otimes}-l_2^{\otimes}\right)\cdot\alpha_2^{\otimes}+\cdots+\left(k_m^{\otimes}-l_m^{\otimes}\right)\cdot\alpha_m^{\otimes}\\&=(0,0,\ldots,0)_{\gamma_{ki},\gamma_{li},\gamma_{\alpha ij},i=1,2,\ldots,m,j=1,2,\ldots,n}\end{aligned}$$

and because $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$ is grey linear independent, we get that

$$k_1^{\otimes}-l_1^{\otimes}=0,k_2^{\otimes}-l_2^{\otimes}=0,\cdots,k_i^{\otimes}-l_i^{\otimes}=0,\ldots,k_m^{\otimes}-l_m^{\otimes}=0$$

We can find that $\beta^{\otimes}$ can be grey linear represented by $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_m^{\otimes}$, and the representation method (grey coefficient) is unique.

Theorem 5.8 The sufficient and necessary conditions of grey vector group grey linear independence: The sufficient and necessary conditions of *n*-dimensional grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_n^{\otimes}$ [shown as Eq. (5.14)] that is linear independent are that the determinant rank of the grey matrix $A(\otimes)=(\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_n^{\otimes})$ is 0:

$$\alpha_1^{\otimes}=\left(a_{11}^{\otimes},a_{12}^{\otimes},\ldots,a_{1n}^{\otimes}\right)^T,\alpha_2^{\otimes}=\left(a_{21}^{\otimes},a_{22}^{\otimes},\ldots,a_{2n}^{\otimes}\right)^T,\ldots,\alpha_n^{\otimes}=\left(a_{n1}^{\otimes},a_{n2}^{\otimes},\ldots,a_{nn}^{\otimes}\right)^T \tag{5.14}$$

Proof: Necessity: When , $n=1$ the conclusion is tenable; we just have to prove the situation of $n\geq 2$. When $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_n^{\otimes}$ is grey linear correlated, at least one vector can be grey linear represented by other grey vectors. Assume that $\alpha_1^{\otimes}=k_2^{\otimes}\cdot\alpha_2^{\otimes}+k_3^{\otimes}\cdot\alpha_3^{\otimes}\ldots+k_n^{\otimes}\cdot\alpha_n^{\otimes}$.

If we add $(-k_2^{\otimes})$ times of the second column, $(-k_3^{\otimes})$ times of the third column,…, $(-k_n^{\otimes})$ times of the n column of the grey matrix $A(\otimes)$ to the first column, we can get the grey matrix $A(\otimes)$ as shown in Eq. (5.15):

$$A_1(\otimes)=\begin{pmatrix} 0 & a_{21}^{\otimes} & \cdots & a_{n1}^{\otimes} \\ 0 & a_{22}^{\otimes} & \cdots & a_{n2}^{\otimes} \\ \vdots & \vdots & & \vdots \\ 0 & a_{2n}^{\otimes} & \cdots & a_{nn}^{\otimes} \end{pmatrix} \tag{5.15}$$

$$\begin{aligned} |A(\otimes)|\Big|_{\gamma_{aij}=c_{aij},0\leq c_{aij}\leq 1,i,j=1,2,\ldots,n} &= |A_1(\otimes)|\Big|_{\gamma_{aij}=c_{aij},0\leq c_{aij}\leq 1,i,j=1,2,\ldots,n} \\ &= 0_{\gamma_{aij}=c_{aij},0\leq c_{aij}\leq 1,i,j=1,2,\ldots,n} \end{aligned} \tag{5.16}$$

According to the property of determinants, if the values of determinants of grey matrices $A(\otimes)$ and $A_1(\otimes)$ are the same, then Eq. (5.16) is tenable.

Sufficiency: We can prove that $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_n^{\otimes}$ is grey linear correlated from $|A(\otimes)|\Big|_{\gamma_{aij}=c_{aij},0\leq c_{aij}\leq 1,i,j=1,2,\ldots,n}=0_{\gamma_{aij}=c_{aij},0\leq c_{aij}\leq 1,i,j=1,2,\ldots,n}$.

We can use the mathematical induction method to prove dimensional n.

When $n=1$, the conclusion is surely tenable. Assuming that the conclusion is tenable to n-1–dimensional, we discuss the situation of an n-dimensional grey vector group. If there is a 0 vector in n-dimensional grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_n^{\otimes}$, this grey vector group is a grey linear correlated group. Or assuming that $a_1^{\otimes}\neq 0$ and $a_{11}^{\otimes}\neq 0$, then $A(\otimes)$ can be transformed by the grey dispelling transform of the column to Eq. (5.17).

$$B(\otimes)=\begin{pmatrix} a_{11}^{\otimes} & 0 & \cdots & 0 \\ a_{12}^{\otimes} & b_{22}^{\otimes} & \cdots & b_{n2}^{\otimes} \\ \vdots & \vdots & & \vdots \\ a_{1n}^{\otimes} & b_{2n}^{\otimes} & \cdots & b_{nn}^{\otimes} \end{pmatrix} \tag{5.17}$$

$$\begin{pmatrix} b_{22}^{\otimes} & \cdots & b_{n2}^{\otimes} \\ \vdots & & \vdots \\ b_{2n}^{\otimes} & \cdots & b_{nn}^{\otimes} \end{pmatrix} = 0_{\gamma_{bij} = c_{bij}\, 0 \le c_{bij} \le 1,\, i,\, j = 1, 2, \ldots, n} \tag{5.18}$$

Because $|B(\otimes)|_{\gamma_{aij}=c_{aij}, \gamma_{bij}=c_{bij}\, 0 \le c_{aij}, c_{bij} \le 1, i, j=1,2,\ldots,n} = |A(\otimes)|_{\gamma_{aij}=c_{aij}, 0 \le c_{aij} \le 1, i, j=1,2,\ldots,n} = 0_{\gamma aij} = c_{aij}, {}_{0 \le c_{aij} \le 1, i, j = 1,2,\ldots, n}$, we can prove that Eq. (5.18) is tenable.

According to induction hypothesis, n-1 n-1–dimensional grey vectors $\beta_2^{\otimes} = (b_{22}^{\otimes}, \ldots, b_{2n}^{\otimes})^T, \ldots, \beta_n^{\otimes} = (b_{n2}^{\otimes}, \ldots, b_{nn}^{\otimes})^T$ are grey linear correlated. We find that there is a group of grey numbers that is not 0 for all so that $k_2^{\otimes}\beta_2^{\otimes} + \cdots + \beta_n^{\otimes}k_n^{\otimes} = 0_{\gamma_{bij}=c_{bij}, 0 \le c_{bij} \le 1, i, j=2,3,\ldots,n}$ for an n-dimensional grey vector group $\beta_2^{/\otimes} = (0, b_{22}^{\otimes}, \ldots, b_{2n}^{\otimes})^T, \ldots, \beta_n^{/\otimes} = (0, b_{n2}^{\otimes}, \ldots, b_{nn}^{\otimes})^T$, then

$$k_2^{\otimes}\beta_2^{/\otimes} + \cdots + k_n^{\otimes}\beta_n^{/\otimes} = 0_{\gamma_{bij}=c_{bij}, 0 \le c_{bij} \le 1, i, j=1,2,\ldots,n} \tag{5.19}$$

According to the practical process of dispelling transform from $A(\otimes)$ to $B(\otimes)$, we can find that $\beta_j^{/\otimes} = \alpha_j^{\otimes} - \frac{a_{j1}^{\otimes}}{a_{11}^{\otimes}} a_1^{\otimes}$, and $j = 1, 2, \ldots, n$, if we put it to Eq. (5.19).

We can therefore get that

$$\left(-\frac{1}{a_{11}^{\otimes}} \sum_{j=2}^{n} k_j^{\otimes} \cdot a_{j1}^{\otimes} \right) \alpha_1^{\otimes} + k_2^{\otimes} \cdot \alpha_2^{\otimes} + \cdots + k_n^{\otimes} \cdot \alpha_n^{\otimes} = (0, 0, \ldots, 0)_{\gamma_{ki}, \gamma_{\alpha ij}, i = 1,2,\ldots, m, j = 1,2,\ldots,n}$$

Because $k_2^{\otimes}, \ldots, k_i^{\otimes}, \ldots, k_n^{\otimes}$ are not all 0, the formulas above prove that $\alpha_1^{\otimes}$, $\alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_n^{\otimes}$ are not grey linear correlated.

Inference 5.1 The sufficient and necessary condition that n-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_n^{\otimes}$ [shown in Eq. (5.14)] is linear independent is that the value of the determinant of grey matrix $A(\otimes) = (\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_n^{\otimes})$ is not 0.

Inference 5.2 For m n-dimensional grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_m^{\otimes}$ ($m < n$), if there is an m-order non-[0,0] subdeterminant in grey matrix $A(\otimes) = (\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_n^{\otimes})$, the grey vector group $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_i^{\otimes}, \ldots, \alpha_n^{\otimes}$ is linear independent.

The conclusion of Inference 5.2 can be proved by Theorem 5.8 and its properties.

Theorem 5.9 Grey linear correlation of the grey vector in which the number of gray vectors is larger than the dimensional number: An n-dimensional grey vector group in which the number of grey vectors is larger than n must be the grey linear correlation grey vector group.

Proof: We just need to prove that the grey vector group that is composed by $n+1$ n-dimensional grey vectors is grey linear correlated. Assume that $\alpha_1^{\otimes}=(a_{11}^{\otimes},a_{12}^{\otimes},\ldots,a_{1n}^{\otimes})^T,\alpha_2^{\otimes}=(a_{21}^{\otimes},a_{22}^{\otimes},\ldots,a_{2n}^{\otimes})^T,\ldots,\alpha_{n+1}^{\otimes}=(a_{n+1,1}^{\otimes},a_{n+1,2}^{\otimes},\ldots,a_{n+1,n}^{\otimes})^T$. If we add a zero component to the end of every grey vector, we will get $n+1$ $n+1$–dimensional grey vectors $\alpha_1'^{\otimes}=(a_{11}^{\otimes},a_{12}^{\otimes},\ldots,a_{1n}^{\otimes},0)^T,\alpha_2'^{\otimes}=(a_{21}^{\otimes},a_{22}^{\otimes},\ldots,a_{2n}^{\otimes},0)^T,\ldots,$ $\alpha_{n+1}'^{\otimes}=(a_{n+1,1}^{\otimes},a_{n+1,2}^{\otimes},\ldots,a_{n+1,n}^{\otimes},0)^T$.

For $A'(\otimes)=(\alpha_1'^{\otimes},\alpha_2'^{\otimes},\cdots,\alpha_{n+1}'^{\otimes})$, we can get that $|A'(\otimes)|=0$. According to Theorem 5.8, we find that $\alpha_1'^{\otimes},\alpha_2'^{\otimes},\ldots,\alpha_i'^{\otimes},\ldots,\alpha_{n+1}'^{\otimes}$ is grey linear correlaed. Because $\alpha_i'^{\otimes}$ has just one more zero component than $\alpha_i^{\otimes}$, we prove that $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_i^{\otimes},\ldots,\alpha_{n+1}^{\otimes}$ is also a grey linear correlation.

Theorem 5.10 Grey linear correlation between grey vector groups: If grey vector group $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_r^{\otimes}$ can be grey linear represented by grey vector group $\beta_1^{\otimes},\beta_2^{\otimes},\ldots,\beta_s^{\otimes}$, and $r>s$, then $\alpha_1^{\otimes},\alpha_2^{\otimes},\ldots,\alpha_r^{\otimes}$ is surely a grey linear correlation group.

Proof: According to the known conditions, we assume that

$$\begin{cases}\alpha_1^{\otimes}=k_{11}^{\otimes}\beta_1^{\otimes}+k_{12}^{\otimes}\beta_2^{\otimes}+\cdots+k_{1s}^{\otimes}\beta_s^{\otimes}\\ \alpha_2^{\otimes}=k_{21}^{\otimes}\beta_1^{\otimes}+k_{22}^{\otimes}\beta_2^{\otimes}+\cdots+k_{2s}^{\otimes}\beta_s^{\otimes}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \alpha_r^{\otimes}=k_{r1}^{\otimes}\beta_1^{\otimes}+k_{r2}^{\otimes}\beta_2^{\otimes}+\cdots+k_{rs}^{\otimes}\beta_s^{\otimes}\end{cases} \tag{5.20}$$

Grey coefficients in Eq. (5.20) are composed of r s-dimensional grey vectors:

$$B_1(\otimes)=\left(k_{11}^{\otimes},k_{12}^{\otimes},\ldots,k_{1s}^{\otimes}\right)^T,B_2(\otimes)=\left(k_{21}^{\otimes},k_{22}^{\otimes},\ldots,k_{2s}^{\otimes}\right)^T,\ldots,B_r(\otimes)=\left(k_{r1}^{\otimes},k_{r2}^{\otimes},\ldots,k_{rs}^{\otimes}\right)^T$$

For $r>s$, this group of coefficients is grey vectors linear correlated; that is, if there is a group of grey numbers $q_1^{\otimes},q_2^{\otimes},\ldots,q_r^{\otimes}$, then

$$q_1^{\otimes}B_1(\otimes)+q_2^{\otimes}B_2(\otimes)+\cdots+q_r^{\otimes}B_r(\otimes)=0_{\gamma_{qi},\gamma_{Bi},i=1,2,\ldots,r} \tag{5.21}$$

So we can get Eq. (5.22)

$$\begin{aligned}&q_1^{\otimes}\alpha_1^{\otimes}+q_2^{\otimes}\alpha_2^{\otimes}+\cdots+q_r^{\otimes}\alpha_r^{\otimes}\\ &=q_1^{\otimes}\left(k_{11}^{\otimes}\beta_1^{\otimes}+k_{12}^{\otimes}\beta_2^{\otimes}+\cdots+k_{1s}^{\otimes}\beta_s^{\otimes}\right)+\cdots+q_r^{\otimes}\left(k_{r1}^{\otimes}\beta_1^{\otimes}+k_{r2}^{\otimes}\beta_2^{\otimes}+\cdots+k_{rs}^{\otimes}\beta_s^{\otimes}\right)\\ &=\left(q_1^{\otimes}k_{11}^{\otimes}+q_2^{\otimes}k_{21}^{\otimes}+\cdots+q_r^{\otimes}k_{r1}^{\otimes}\right)\beta_1^{\otimes}+\cdots+\left(q_1^{\otimes}k_{1s}^{\otimes}+q_2^{\otimes}k_{2s}^{\otimes}+\cdots+q_r^{\otimes}k_{rs}^{\otimes}\right)\beta_s^{\otimes}\end{aligned} \tag{5.22}$$

According to Eq. (5.21), if the grey coefficients of $\beta_1^\otimes,\beta_2^\otimes,\ldots,\beta_s^\otimes$ in Eq. (5.22) are all 0, then

$$q_1^\otimes\alpha_1^\otimes + q_2^\otimes\alpha_2^\otimes + \cdots + q_r^\otimes\alpha_r^\otimes = 0_{\gamma_{qi},\gamma_{\alpha i},\, i=1,2,\ldots,r}$$

$q_1^\otimes,q_2^\otimes,\ldots,q_r^\otimes$ are not all 0, so $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_r^\otimes$ is a grey linear correlation.

Inference 5.3 Grey linear independence between grey vector groups: If grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_r^\otimes$ is linear independent and it can be grey linear represented by grey vector group $\beta_1^\otimes,\beta_2^\otimes,\ldots,\beta_s^\otimes$, then we can get that $r < s$.

Inference 5.4 Numbers of grey vectors between equivalent grey vector groups: The number of grey vectors is the same as the number of equivalent grey linear independent grey vector groups.

Proof: To make the proof general, assume that $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_r^\otimes$ and $\beta_1^\otimes,\beta_2^\otimes,\ldots,\beta_s^\otimes$ are both grey linear independent grey vector groups, and they are equivalent.

According to Inference 5.3, we know that $r \le s$ and $s \le r$, so, $r = s$.

5.2.2 Grey Vector Groups and Grey Rank of a Grey Matrix

Definition 5.6 Maximum grey independent group: If a part of grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_s^\otimes$, which is assumed as $\alpha_{i1}^\otimes,\alpha_{i2}^\otimes,\ldots,\alpha_{ir}^\otimes$, meets that (1) $\alpha_{i1}^\otimes,\alpha_{i2}^\otimes,\ldots,\alpha_{ir}^\otimes$ is grey linear independent, and (2) every grey vector of $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_s^\otimes$ can be grey linear represented by $\alpha_{i1}^\otimes,\alpha_{i2}^\otimes,\ldots,\alpha_{ir}^\otimes$, then $\alpha_{i1}^\otimes,\alpha_{i2}^\otimes,\ldots,\alpha_{ir}^\otimes$ is a maximum grey linear independent group of grey vector group $\alpha_1^\otimes,\alpha_2^\otimes,\ldots,\alpha_s^\otimes$, or grey maximum independent group for short.

A maximum grey independent group of a grey linear independent vector group is the grey vector group itself. If every grey vector of a grey vector group is a zero vector, this grey vector group has no grey maximum independent group.

According to Definition 5.6, a grey vector group and its grey maximum independent group are equivalent.

Example 5.3 Assuming a three-dimensional grey vector group

$$\alpha_1^\otimes = ([1,2],0,0),\alpha_2^\otimes = (0,[2,4],0),\alpha_3^\otimes = ([1,2],[-4,-2],0)$$

where $\alpha_1^\otimes,\alpha_2^\otimes$ are grey linear independent, and

$$\begin{aligned}
&\alpha_3^\otimes = \alpha_1^\otimes - \alpha_2^\otimes\\
&\left\{\left(1+\gamma_{\alpha31}^\otimes\right),\left(-4+2\gamma_{\alpha32}^\otimes\right),0\right\} = \left\{\left(1+\gamma_{\alpha11}^\otimes\right),0,0\right\} - \left\{0,\left(2+2\gamma_{\alpha22}^\otimes\right),0\right\}\\
&\left\{\left(1+\gamma_{\alpha31}^\otimes\right),\left(-4+2\gamma_{\alpha32}^\otimes\right),0\right\} = \left\{\left(1+\gamma_{\alpha11}^\otimes\right),\left(-2-2\gamma_{\alpha22}^\otimes\right),0\right\}
\end{aligned} \tag{5.23}$$

According to Eq. (5.23), when $\gamma_{\alpha 31}^{\otimes} = \gamma_{\alpha 11}^{\otimes}$ and $\gamma_{\alpha 32}^{\otimes} = 1 - \gamma_{\alpha 22}^{\otimes}$, $\alpha_3^{\otimes}$ can be grey-linear represented by $\alpha_1^{\otimes}, \alpha_2^{\otimes}$, then $\alpha_1^{\otimes}, \alpha_2^{\otimes}$ is a grey maximum independent group, and $\alpha_1^{\otimes}, \alpha_3^{\otimes}$ and $\alpha_2^{\otimes}, \alpha_3^{\otimes}$ are grey maximum independent groups.

Example 5.3 shows that the grey maximum independent group of a grey vector group is not definitely unique. However, any two maximum independent groups of the same grey vector groups are equivalent with the former grey vector group; according to transitivity of equivalence, these two maximum independent groups are equivalent, and they have the same number of grey vectors.

Definition 5.7 Rank of maximum grey vector group: The numbers of grey vectors in the maximum grey independent group of a grey vector group is called the grey rank of the grey vector group (or grey rank number).

Theorem 5.11 Rank of equivalent grey vector groups: Equivalent grey vector groups have the same rank.

Proof: Assume a grey vector group

$$\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_s^{\otimes} \tag{5.24}$$

and a grey vector group

$$\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_t^{\otimes} \tag{5.25}$$

are equivalent. We also assume $\alpha_{i1}^{\otimes}, \alpha_{i2}^{\otimes}, \ldots, \alpha_{ir}^{\otimes}$ is the maximum grey independent group of Eq. (5.24), and $\beta_{j1}^{\otimes}, \beta_{j2}^{\otimes}, \ldots, \beta_{jl}^{\otimes}$ is the maximum grey independent group of Eq. (5.25). Because $\alpha_{i1}^{\otimes}, \alpha_{i2}^{\otimes}, \ldots, \alpha_{ir}^{\otimes}$ is equivalent to Eq. (5.24), Eq. (5.24) is equivalent with Eq. (5.25), and Eq. (5.25) is equivalent to $\beta_{j1}^{\otimes}, \beta_{j2}^{\otimes}, \ldots, \beta_{jl}^{\otimes}$. With the transitivity of equivalence, we can find that $\alpha_{i1}^{\otimes}, \alpha_{i2}^{\otimes}, \ldots, \alpha_{ir}^{\otimes}$ is equivalent to $\beta_{j1}^{\otimes}, \beta_{j2}^{\otimes}, \ldots, \beta_{jl}^{\otimes}$. And because they are a grey independent group, they will have the same number of grey vectors—that is, $r = l$. The ranks of Eqs. (5.24) and (5.25) are also the same.

We will put out the concept of grey matrix grey rank. Assume an $m \times n$ grey matrix, shown as Eq. (5.26). Every row of grey matrix $A(\otimes)$ is an n-dimensional row vector. Shown as Eq. (5.27), then $\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_m^{\otimes}$ is the grey row vector group of the grey matrix $A(\otimes)$

$$A(\otimes) = \begin{pmatrix} a_{11}^{\otimes} & a_{12}^{\otimes} & \cdots & a_{1n}^{\otimes} \\ a_{21}^{\otimes} & a_{22}^{\otimes} & \cdots & a_{2n}^{\otimes} \\ \vdots & \vdots & & \vdots \\ a_{m1}^{\otimes} & a_{m2}^{\otimes} & \cdots & a_{mn}^{\otimes} \end{pmatrix} \tag{5.26}$$

$$\begin{aligned} \alpha_1^{\otimes} &= \left(a_{11}^{\otimes}, a_{12}^{\otimes}, \ldots, a_{1n}^{\otimes}\right) \\ \alpha_2^{\otimes} &= \left(a_{21}^{\otimes}, a_{22}^{\otimes}, \ldots, a_{2n}^{\otimes}\right) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_m^{\otimes} &= \left(a_{m1}^{\otimes}, a_{m2}^{\otimes}, \ldots, a_{mn}^{\otimes}\right) \end{aligned} \tag{5.27}$$

In the same way, every column of grey matrix $A(\otimes)$ is an m-dimensional column vector, and these m-dimensional column vectors are called the grey column vector group.

Definition 5.8 Grey row rank and grey column rank: The grey rank of a grey row vector group of grey matrix $A(\otimes)$ is called a grey row rank of $A(\otimes)$; the grey rank of a grey column vector group of grey matrix $A(\otimes)$ is called a grey column rank of $A(\otimes)$.

Definition 5.9 Grey determinant rank of $A(\otimes)$: The maximum order of a nonzero grey subdeterminant of grey matrix $A(\otimes)$ is called a grey determinant rank of $A(\otimes)$.

The grey determinant rank of a zero vector is 0.

Theorem 5.12 Grey row rank, grey column rank, and grey determinant rank: The grey column rank and the grey determinant rank of any grey matrix $A(\otimes)$ are the same.

Proof: If $A(\otimes)=0$, the theorem is tenable, because grey column rank and grey determinant rank of $A(\otimes)$ are both 0.

Assume that $A(\otimes) = (a_{ij}^{\otimes})_{m\times n}$ is a nonzero grey matrix, and the grey determinant rank of $A(\otimes)$ is r. Then the value of one of the grey r-order subdeterminants in $A(\otimes)$ is not zero, and any other values of the grey subdeterminants that are higher than r order are 0. So we assume values of the left upper angular r-order grey subdeterminants are not 0; that is,

$$D(\otimes) = \begin{vmatrix} a_{11}^{\otimes} & a_{12}^{\otimes} & \cdots & a_{1r}^{\otimes} \\ a_{21}^{\otimes} & a_{22}^{\otimes} & \cdots & a_{2r}^{\otimes} \\ \vdots & \vdots & & \vdots \\ a_{r1}^{\otimes} & a_{r2}^{\otimes} & \cdots & a_{rr}^{\otimes} \end{vmatrix} \neq 0 \,|\, \gamma_{aij}, 0\le\gamma_{aij}\le 1, i,j=1,2,\ldots,r \tag{5.28}$$

$$D(\otimes) = \begin{vmatrix} a_{11}^{\otimes} & \cdots & a_{1r}^{\otimes} & a_{1j}^{\otimes} \\ \vdots & & \vdots & \vdots \\ a_{r1}^{\otimes} & & a_{rr}^{\otimes} & a_{rj}^{\otimes} \\ a_{i1}^{\otimes} & \cdots & a_{ir}^{\otimes} & a_{ij}^{\otimes} \end{vmatrix}, i = 1,2,\ldots,m \tag{5.29}$$

$\beta_1^{\otimes},\beta_2^{\otimes},\ldots,\beta_n^{\otimes}$ is the grey column vector group of $A(\otimes)$, and we will prove that forward columns of the *r*th column $\beta_1^{\otimes},\beta_2^{\otimes},\ldots,\beta_r^{\otimes}$ of $D(\otimes)$ in $A(\otimes)$ make up the grey maximum independent set of grey column vectors in $A(\otimes)$.

Because $D(\otimes) \neq 0$ [shown in Eq. (5.28)], according to Inference 5.2 of Theorem 5.10, we obtain that $\beta_1^{\otimes},\beta_2^{\otimes},\ldots,\beta_r^{\otimes}$ is linear independent. We will then prove that any column $\beta_j^{\otimes}$ of $A(\otimes)$ can be grey linear represented by $\beta_1^{\otimes},\beta_2^{\otimes},\ldots,\beta_r^{\otimes}$. In order to prove that, we just need to prove that any $\beta_j^{\otimes}(j > r)$ from backward $n - r$ columns of $A(\otimes)$ can always be grey linear represented by $\beta_1^{\otimes},\beta_2^{\otimes},\ldots,\beta_r^{\otimes}$.

Let us consider $r + 1$–order determinant, shown in Eq. (5.29), when $i = 1,2,\ldots,r$, and two rows of $D_i(\otimes)$ are the same; hence, $D_i(\otimes) = 0$. When $i = r+1, r+2,\ldots,m$, $D_i(\otimes)$ is an $r + 1$-order subdeterminant of $A(\otimes)$, and we can get $D_i(\otimes) = 0$. Now we can expand $D_i(\otimes)$ according to the last row, if we sign the algebraic complement value of grey elements $a_{i1}^{\otimes},a_{i2}^{\otimes},\ldots,a_{ir}^{\otimes}$ in $D_i(\otimes)$ as $k_1^{\otimes},k_2^{\otimes},\ldots,k_r^{\otimes}$, and then pay attention to the algebraic complement of grey element $a_{ij}^{\otimes}$ in $D_i(\otimes)$, which is shown in Eq. (5.30). Hence, we get Eq. (5.31). If $D_i(\otimes) = 0$, we can make Eq. (5.32) tenable.

$$(-1)^{r+1+r+1}\begin{vmatrix} a_{11}^{\otimes} & \cdots & a_{1r}^{\otimes} \\ \vdots & & \vdots \\ a_{r1}^{\otimes} & \cdots & a_{rr}^{\otimes} \end{vmatrix} = D(\otimes) \tag{5.30}$$

$$D_i(\otimes) = k_1^{\otimes}a_{i1}^{\otimes} + \cdots + k_r^{\otimes}a_{ir}^{\otimes} + D(\otimes)\cdot a_{ij}^{\otimes} \tag{5.31}$$

$$k_1^{\otimes}a_{i1}^{\otimes} + \cdots + k_r^{\otimes}a_{ir}^{\otimes} + D(\otimes)\cdot a_{ij}^{\otimes} = 0, i = 1,2,\ldots,m \tag{5.32}$$

According to the construction of $D_i(\otimes)$, when *j* is fixed no matter how we change *i*, $k_1^{\otimes},k_2^{\otimes},\ldots,k_r^{\otimes}$ in Eq. (5.32) are always constants that have no relationship to *i*. Hence, we can get Eq. (5.33) tenable.

$$k_1^{\otimes}\left(a_{11}^{\otimes},a_{21}^{\otimes},\ldots a_{m1}^{\otimes}\right)^T + k_2^{\otimes}\left(a_{12}^{\otimes},a_{22}^{\otimes},\ldots a_{m2}^{\otimes}\right)^T + \cdots + k_r^{\otimes}\left(a_{1r}^{\otimes},a_{2r}^{\otimes},\cdots a_{mr}^{\otimes}\right)^T +$$

$$+D(\otimes)\left(a_{1j}^{\otimes},a_{2j}^{\otimes},\ldots a_{mj}^{\otimes}\right)^T = (0,0,\ldots,0)^T \,\Big|_{\gamma_{ais},\gamma_{kt},0\le\gamma_{ais},\gamma_{ks}\le 1,i=1,\ldots,r,s=1,\ldots,r}$$

then

$$k_1^{\otimes}\beta_1^{\otimes} + k_2^{\otimes}\beta_2^{\otimes} + \cdots + k_r^{\otimes}\beta_r^{\otimes} + D(\otimes)\cdot\beta_j^{\otimes} = (0,0,\ldots,0)^T \,\Big|_{\gamma_{ais},\gamma_{kt},0\le\gamma_{ais},\gamma_{ks}\le 1,\, i=1,\ldots,r,\, s=1,\ldots,r} \tag{5.33}$$

Because $D(\otimes) \neq 0$, according to Eq. (5.33), we know that $\beta_j^{\otimes}$ can be linear represented by $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_r^{\otimes}$.

In this way, we prove that $\beta_1^{\otimes}, \beta_2^{\otimes}, \ldots, \beta_r^{\otimes}$ is the grey maximum independent group of the grey column vector set of $A(\otimes)$. Hence, the grey column rank of $A(\otimes)$ is r, which is equal to the grey determinant rank of $A(\otimes)$.

We can prove that the grey row rank of any grey matrix $A(\otimes)$ is the same as the grey determinant rank in the same way. So grey row rank, grey column rank, and grey determinant rank of $A(\otimes)$ are the same number, and we can get Definition 5.10.

Definition 5.10 Grey rank $(A)^{\mathrm{R}}$ of $A(\otimes)$: For a grey matrix $A(\otimes)$, we define its grey row rank (grey column rank or grey determinant rank) as the grey rank of this grey matrix, denoted $(A)^{\mathrm{R}}$.

According to the above analysis, for any grey matrix $A(\otimes)$:

$$\text{Rank } (A)^{\mathrm{R}} = \text{grey row rank of } A(\otimes) = \text{grey column rank of } A(\otimes) = \text{grey determinant rank of } A(\otimes) \tag{5.34}$$

The continued equality form of Eq. (5.34) opens lots of ways to get the rank of a grey matrix, it provides a way to solve the problem of getting the rank of a vector group using grey matrix rank, and it can also be used to determine the grey linear correlation of a grey vector group.

Let us think of a practical way to get the grey rank of a grey matrix. Under simple conditions, using grey row rank, grey column rank, or grey determinant rank to get the grey matrix rank numbers is convenient and feasible. If there are too many grey elements and few of them are 0, it will be difficult to solve the problem in the above way. Theorem 5.13 provides the theoretical basis of practical solutions to solve grey rank of the grey matrix.

Theorem 5.13 Elementary transformation of $A(\otimes)$ will not change $(A)^{\mathrm{R}}$: For any grey matrix $A(\otimes)$, if we do elementary transformation to $A(\otimes)$, the grey rank $(A)^{\mathrm{R}}$ of $A(\otimes)$ will not change.

Proof: We just need to prove that after one elementary transformation the new matrix has the same grey rank as the rank of the former grey matrix. To make the proof general, we can prove it with row transformation.

Assume that after a row elementary transformation, grey matrix $A(\otimes)$ is changed to grey matrix $B(\otimes)$. If the elementary transformation is a multiplying transform, the grey row vector group of $A(\otimes)$ and the grey row vector group of $B(\otimes)$ are equivalent, and hence their grey row rank are the same; that is, they have equal grey ranks. If the elementary transformation is a row dispelling transform, such as adding $k^{\otimes}$ times of the second row to the first row, then if the grey row vector group of $A(\otimes)$ is

$$\alpha_1^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_m^{\otimes} \tag{5.35}$$

and the grey row vector group of $B(\otimes)$ will be

$$\alpha_1^{\otimes} + k^{\otimes}\alpha_2^{\otimes}, \alpha_2^{\otimes}, \ldots, \alpha_m^{\otimes} \tag{5.36}$$

It is easy to find that Eqs. (5.35) and (5.36) are equal, so grey rank $(A)^{\mathrm{R}}$ = grey rank $(B)^{\mathrm{R}}$.

The situation of column elementary transformation is totally the same with row elementary transformation.

Inference 5.5 Elementary transformation of $A(\otimes)$ will not change $(A)^{\mathrm{R}}$: Assuming an $m \times n$ grey matrix $A(\otimes)$, to any m-order grey inverse matrix $P(\otimes)$ and an n-order grey inverse matrix $Q(\otimes)$, we can get that

$$\text{grey rank } (A)^{\mathrm{R}} = \text{grey rank } (PA)^{\mathrm{R}} = \text{grey rank } (QA)^{\mathrm{R}} = \text{grey rank } (PAQ)^{\mathrm{R}}$$

The sufficient and necessary conditions that n-order grey matrix $A(\otimes)$ is invertible is that $A(\otimes)$ can be represented as the product of some elementary grey matrices. So there must be a grey elementary matrix $P_1^{\otimes}, P_2^{\otimes}, \ldots, P_s^{\otimes}$ where $P^{\otimes} = P_1^{\otimes} P_2^{\otimes} \ldots P_s^{\otimes}$. In this way, $P^{\otimes} A(\otimes) = P_1^{\otimes} P_2^{\otimes} \ldots P_s^{\otimes} A(\otimes)$, which is equal to having s times of elementary transformations, and the transformations will not change the grey rank $(A)^{\mathrm{R}}$ of grey matrix $A(\otimes)$. Grey rank $(PA)^{\mathrm{R}}$ = grey rank $(A)^{\mathrm{R}}$.

We can prove that grey rank $(QA)^{\mathrm{R}}$ = grey rank $(A)^{\mathrm{R}}$, grey rank $(PAQ)^{\mathrm{R}}$ = grey rank $(A)^{\mathrm{R}}$ in the same way.

The grey rank number $(A)^{\mathrm{R}}$ of grey matrix $A(\otimes)$ is just the occurrence number r of 1, which appeared in the standard form of the elementary transformation of $A(\otimes)$.

Example 5.4 Solve the grey matrix $A(\otimes)$ that is shown as Eq. (5.37).

$$A(\otimes) = \begin{pmatrix} [1,2] & 0 & 0 \\ 0 & [2,4] & 0 \\ [1,2] & [-4,-2] & 0 \end{pmatrix} \tag{5.37}$$

Solution: Change all the grey numbers of Eq. (5.37) to the standard form, shown as Eq. (5.38).

After the elementary transformation, we can get the standard form of $A(\otimes)$, shown as Eq. (5.39). We can therefore get the grey rank of $A(\otimes)$; that is, $(A)^{\mathrm{R}} = 2$.

$$A(\otimes) = \begin{pmatrix} (1+\gamma_{\alpha 11}^{\otimes}) & 0 & 0 \\ 0 & \left(2+2\gamma_{\alpha 22}^{\otimes}\right) & 0 \\ \left(1+\gamma_{\alpha 31}^{\otimes}\right) & \left(-4+2\gamma_{\alpha 32}^{\otimes}\right) & 0 \end{pmatrix} \tag{5.38}$$

$$A(\otimes) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.39}$$

> To sum up, according to the definition of the grey matrix grey rank number, to a $m \times n$ grey matrix $A(\otimes)$, we know that the grey rank $(A)^R \le \min\{m, n\}$. That is, the grey rank of the grey matrix will be less than the number of its row and column. We call the grey matrix in which the grey rank number is equal with the row number the *grey full-row rank matrix*; we call the grey matrix in which the grey rank number is equal with column number a *grey full-column rank matrix*; and we call the grey matrix that has no full rank the *grey order reducing matrix*. It is easy to find that the grey full-rank matrix is just the grey invertible matrix.

5.2.3 Summary

This section proved the sufficient and necessary conditions of grey vector group grey linear correlation; constructed a representation method of grey vector group grey linear correlation; put forward the concepts of rank of maximum grey vector groups, grey row ranks of grey matrices, grey column ranks, and grey determinant ranks; and proved that the elementary transformation of the grey matrix $A(\otimes)$ will not change grey rank $(A)^R$. Hence, this part of the book provided a solid foundation for the study of a grey matrix game model based on grey mixed strategy.

5.3 The Elementary Transformation of the Grey Matrix and Its Grey Invertible Matrix

5.3.1 The Elementary Transformation of the Grey Matrix

Definition 5.11 The elementary transformation of the grey matrix: For a grey matrix $A(\otimes)$, there are three kinds of transforms, which are called |, ||, and ||| elementary row (column) transformations of the grey matrix:

1. Using nonzero grey number multiples of a row (column) of the grey matrix
2. Adding some number of times of a row (column) to another row (column)
3. Interchanging two rows (columns) of the grey matrix with each other

The first elementary transformation is called a *grey multiplying transform*, the second elementary transformation is called a *grey dispelling transform*, and the third elementary transformation is called a *grey interchanging transform*. All three kinds of elementary transformations are called *elementary transformations of the grey matrix*.

After the elementary transformation, a grey matrix will be changed to another grey matrix. We call the grey matrix $B(\otimes)$ that is gotten by using elementary transformation of grey matrix $A(\otimes)$ the equivalent grey matrix of $A(\otimes)$, signed as $A(\otimes) \cong B(\otimes)$. To a nonzero grey matrix $A(\otimes)$, the number of equivalent grey matrices of $A(\otimes)$ is infinite.

Theorem 5.14 $A(\otimes)$ can be changed to a standard form: Any grey matrix $A(\otimes)$ can be changed to the form of Eq. (5.40) after the elementary transformation, where $1^{\otimes} = [1,1], 0^{\otimes} = [0,0]$

$$C(\otimes) = \begin{pmatrix} 1^{\otimes} & & & & & \\ & \vdots & & & & \\ & & 1^{\otimes} & & & \\ & & & 0^{\otimes} & & \\ & & & & \vdots & \\ & & & & & 0^{\otimes} \end{pmatrix} \tag{5.40}$$

$$B(\otimes) = \begin{pmatrix} 1^{\otimes} & 0^{\otimes} & \cdots & 0^{\otimes} \\ 0^{\otimes} & b_{22}^{\otimes} & \cdots & b_{2n}^{\otimes} \\ \vdots & \vdots & & \vdots \\ 0^{\otimes} & b_{m2}^{\otimes} & \cdots & b_{mn}^{\otimes} \end{pmatrix} \tag{5.41}$$

$$D(\otimes) = \begin{pmatrix} 1^{\otimes} & 0^{\otimes} & 0^{\otimes} & \cdots & 0^{\otimes} \\ 0^{\otimes} & 1^{\otimes} & 0^{\otimes} & \cdots & 0^{\otimes} \\ 0^{\otimes} & 0^{\otimes} & c_{33}^{\otimes} & \cdots & c_{3n}^{\otimes} \\ \vdots & \vdots & \vdots & & \vdots \\ 0^{\otimes} & 0^{\otimes} & c_{m3}^{\otimes} & \cdots & c_{mn}^{\otimes} \end{pmatrix} \tag{5.42}$$

Proof: If $A(\otimes)$ is a zero matrix, it has been in the form of the grey matrix ($r = 0$), shown as Eq. (5.40). Otherwise, we have to make nonzero grey element $d^{\otimes}$ in the first row and the first column of the grey matrix by using the grey interchanging transform of the rows and columns. Then, multiplying $\frac{1}{d^{\otimes}}$ to the first row to change grey element $d^{\otimes}$, which is in the first row and first column (1,1) to [1,1] (or signed as $1^{\otimes}$). After some proper dispelling transforms of rows and columns, we can change the form of the matrix to the form shown as Eq. (5.41).

If $b_{ij}^{\otimes}$ $(i = 1,2,\cdots,m; j = 1,2,\cdots,n)$ are all 0, then $B(\otimes)$ is the grey matrix form ($r = 1$) of Eq. (5.40). Otherwise, do some familiar elementary transformations in the

2–m rows and 2–n columns of $B(\otimes)$; that is, change the nonzero grey element in (2,2) to $1^{\otimes}$. After some proper dispelling transforms, we can change other nonzero elements of the second row and second column to $0^{\otimes}$. We should pay attention that these elementary transformations do not change the value of the first row and first column. We have changed the form of $A(\otimes)$ to the form of Eq. (5.42) until now.

If we go on, we can get the form of the grey matrix as the form in Eq. (5.40).

The grey matrix $C(\otimes)$, which changes into a new form after elementary transformation of $A(\otimes)$, has the same form as Eq. (5.40), where $C(\otimes)$ is called the grey standard form after the elementary transformations of $A(\otimes)$. We will then prove that, in the grey matrix standard form, the number r of $1^{\otimes}$ in the grey matrix is uniquely determined by grey matrix $A(\otimes)$.

Definition 5.12 Unit grey matrix: For a grey matrix $E(\otimes)$, the elements in its diagonal are all $1^{\otimes}$, and other elements are all $0^{\otimes}$. We can call $E(\otimes)$ a unit grey matrix.

Definition 5.13 Elementary grey matrix: After the elementary transformations of rows and columns, the new matrix of unit grey matrix $E(\otimes)$ is called an elementary grey matrix.

Obviously, elementary grey matrices are all grey square matrices. We can get an elementary grey matrix after some elementary transformations of $E(\otimes)$. Three elementary transformations corresponding to the three elementary grey matrices are shown as Eqs. (5.43), (5.44), and (5.45), respectively.

$$P^{\otimes}(i[k^{\otimes}])=\begin{bmatrix} 1^{\otimes} & & & & & & \\ & \vdots & & & & & \\ & & 1^{\otimes} & & & & \\ \cdots & \cdots & \cdots & k^{\otimes} & \cdots & \cdots & \cdots \\ & & & & 1^{\otimes} & & \\ & & & & & \vdots & \\ & & & & & & 1^{\otimes} \end{bmatrix}\begin{matrix} \\ \\ \\ i^{th}\,row(k^{\otimes}\neq 0) \\ \\ \\ \\ \end{matrix} \tag{5.43}$$

$$P^{\otimes}(i,j[k^{\otimes}])=\begin{bmatrix} 1^{\otimes} & & & & & & \\ & \vdots & & & & & \\ & & 1^{\otimes} & \cdots & k^{\otimes} & \cdots & \cdots \\ & & & \vdots & \cdots & & \\ & & & & 1^{\otimes} & \cdots & \cdots \\ & & & & & \vdots & \\ & & & & & & 1^{\otimes} \end{bmatrix}\begin{matrix} \\ \\ i^{th}\,row \\ \\ j^{th}\,row \\ \\ \\ \end{matrix} \tag{5.44}$$

$$
P^{\otimes}(i,j)=\left[\begin{array}{ccccccccccc}
1^{\otimes} & & & & & & & & & & \\
 & \vdots & & & & & & & & & \\
 & & 1^{\otimes} & & & & & & & & \\
 & & & 0^{\otimes} & \cdots & \cdots & 1^{\otimes} & \cdots & \cdots & \cdots & \cdots \\
 & & & \vdots & 1^{\otimes} & & \vdots & & & & \\
 & & & \vdots & & \vdots & \vdots & & & & \\
 & & & 1^{\otimes} & \cdots & \cdots & 1^{\otimes} & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & & 0^{\otimes} & & & \\
 & & & & & & & & 1^{\otimes} & & \\
 & & & & & & & & & \vdots & \\
 & & & & & & & & & & 1^{\otimes}
\end{array}\right]
\begin{array}{l}
\\ \\ \\ i^{th}\,row \\ \\ \\ j^{th}\,row \\ \\ \\ \\ \\
\end{array}
\tag{5.45}
$$

In three kinds of elementary grey matrices, $P^{\otimes}(i[k^{\otimes}])$ is the elementary grey matrix by multiplying $k^{\otimes}$ to the ith row of $E(\otimes)$ (or multiplying $k^{\otimes}$ to the ith column), and it is called a grey multiplying matrix; $P^{\otimes}(i,j[k^{\otimes}])$ is the elementary grey matrix by adding $k^{\otimes}$ times of the jth row to the ith row (or adding $k^{\otimes}$ times of the ith column to the jth column), and it is called a grey dispelling matrix; $P^{\otimes}(i,j)$ is the elementary grey matrix by changing the ith and jth rows with each other (or changing the ith and jth columns with each other), and it is called grey interchanging matrix.

Theorem 5.15 $A(\otimes)$ can be interchanged with the standard form: Using an elementary grey matrix to left multiply grey matrix $A(\otimes)$ is the same as doing some proper elementary transformations of rows; using an elementary grey matrix to right multiply grey matrix $A(\otimes)$ is the same as doing some proper elementary transformations of columns.

Proof: From Definition 5.13, we have three kinds of elementary grey matrices. According to these three kinds of elementary grey matrices, we can prove respectively the following:

1. Using an m-order multiplying grey matrix $P^{\otimes}(i[k^{\otimes}])$ to left multiply grey matrix $A(\otimes)_{m\times n}$ enlarges the ith row of $A(\otimes)$ $k^{\otimes}$ times; using an n order multiplying grey matrix $P^{\otimes}(i[k^{\otimes}])$ to right multiply grey matrix $A(\otimes)_{m\times n}$ enlarges the ith column of $A(\otimes)$ $k^{\otimes}$ times.
2. Using an m-order dispelling grey matrix $P^{\otimes}(i[k^{\otimes}])$ to left multiply grey matrix $A(\otimes)_{m\times n}$ adds $k^{\otimes}$ times of the jth row of $A(\otimes)$ to the ith row; using an n-order dispelling grey matrix $P^{\otimes}(i[k^{\otimes}])$ to right multiply grey matrix $A(\otimes)_{m\times n}$ adds $k^{\otimes}$ times of the ith column of $A(\otimes)$ to the jth column.

3. Using an m-order interchanging grey matrix $P^{\otimes}(i[k^{\otimes}])$ to left multiply grey matrix $A(\otimes)_{m\times n}$ exchanges the ith and jth rows with each other; using an n-order interchanging grey matrix $P^{\otimes}(i[k^{\otimes}])$ to right multiply grey matrix $A(\otimes)_{m\times n}$ interchanges the ith and jth columns with each other.

Therefore, using an elementary grey matrix to left multiply grey matrix $A(\otimes)$ is the same as doing some proper elementary transformations of rows; using an elementary grey matrix to right multiply grey matrix $A(\otimes)$ is the same as doing some proper elementary transformations of columns.

According to Theorem 5.15, a row (column) elementary transformation of $A(\otimes)$ can be seen as the same as left (right) multiplying a proper elementary grey matrix to $A(\otimes)$.

We have to pay attention to the effect of elementary transformation that dispelling grey matrix $P^{\otimes}(i, j[k^{\otimes}])$ left or right multiplies with a grey matrix is not the same as to the row and the column. The change of left multiplication is the ith row and the change of right multiplication is the jth column.

Definition 5.14 Grey determinant $|A(\otimes)|$: The determinant that is constructed by elements of an n-order grey square matrix $A(\otimes)$ that stays in the former position is called the *determinant of grey square matrix* $A(\otimes)$, grey determinant for short, signed as $|A(\otimes)|$ or $\det A(\otimes)$.

Taking a grey determinant can be seen as a kind of calculation. According to the property of determinants, we know that to any n-order grey square matrix $A(\otimes)$ and its grey constant $k^{\otimes}$, Eqs. (4.46) and (4.47) are established.

$$|A^{T}(\otimes)| = |A(\otimes)| \tag{5.46}$$

$$|k^{\otimes}\cdot A(\otimes)| = (k^{\otimes})^{n}\,|A(\otimes)| \tag{5.47}$$

Lemma 5.1 Dispelling transform of grey square matrix cannot change the value of the determinant: If after many times of grey dispelling transforms (including row transforms and column transforms) the grey square matrix $A(\otimes)$ is changed to grey matrix $B(\otimes)$, then $|A(\otimes)| = |B(\otimes)|$.

Proof: From Definition 5.13, we know several times of grey dispelling transforms of $A(\otimes)$ is to add several times of some row (column) to another row (column); in this way, we get the grey square matrix $B(\otimes)$.

According to the property of grey determinants, if we add several grey number times of a row (column) of a determinant to another row (column), the value of the determinant will not be changed.

Because of this, we know that the value of the grey determinant that is constructed by $A(\otimes)$ is the same as the grey determinant that is constructed by $B(\otimes)$; that is, $|A(\otimes)| = |B(\otimes)|$.

Lemma 5.2 By using grey dispelling transforms of the row or column, we can get an upper triangular matrix of $A(\otimes)$: Any grey square matrix $A(\otimes)$ can be transformed into an upper triangular matrix by using a grey dispelling transform of the row or column.

Proof: Assume that $A(\otimes)=(a_{ij}^{\otimes})_{n\times n}$. If $a_{11}^{\otimes}\neq 0^{\otimes}$, we can do some proper grey dispelling transforms of the row to transform the grey elements in the first column to $0^{\otimes}$, expecting the grey element in the position of (1,1).

If $a_{11}^{\otimes}=0^{\otimes}$ and grey elements of the first column of $A(\otimes)$ are not all $0^{\otimes}$ and if we add a row in which a nonzero grey element of the first column to the first row, we could make sure that the element in the position of (1,1) is not $0^{\otimes}$. We could add some proper times of the first row to all other rows, and in this way we can make sure that the elements in the first row that are in the position of (1,1) are all $0^{\otimes}$. After some proper grey dispelling transforms, $A(\otimes)$ can be transformed to the form that is shown in Eq. (5.48).

We could do some proper grey dispelling transforms to other 2–n rows of $A(\otimes)$ in the same way and change $A_1(\otimes)$ to the grey matrix in which every grey element below the position of (2,2) in the second row is $0^{\otimes}$.

Continuing grey dispelling transforming of the row, we can transform $A(\otimes)$ to the grey upper triangular matrix that is shown in Eq. (5.49). In Eq. (5.49), $d_{11}^{\otimes}=c_{11}^{\otimes},\ldots$, and we cannot eliminate the possibility that some elements in the leading diagonal are $0^{\otimes}$.

$$A_1(\otimes)=\begin{pmatrix} c_{11}^{\otimes} & c_{12}^{\otimes} & \cdots & c_{1n}^{\otimes} \\ 0^{\otimes} & c_{22}^{\otimes} & \cdots & c_{2n}^{\otimes} \\ \vdots & \vdots & & \vdots \\ 0^{\otimes} & c_{n2}^{\otimes} & \cdots & c_{nn}^{\otimes} \end{pmatrix} \tag{5.48}$$

$$D(\otimes)=\begin{pmatrix} d_{11}^{\otimes} & & & * \\ & d_{22}^{\otimes} & & \\ & & \vdots & \\ & & & d_{nn}^{\otimes} \end{pmatrix} \tag{5.49}$$

In the same way, we could transform a grey matrix to a grey upper triangular matrix by dispelling transform of the column, but we must do it from the last row.

Theorem 5.16 The determinant on the product of any two grey square matrices is equal to the product of the determinants of two grey square matrices: Assume that $A(\otimes)$ and $B(\otimes)$ are both n-order grey matrices, then we get that $|A(\otimes)\cdot B(\otimes)|=|A(\otimes)|\cdot|B(\otimes)|$.

Proof: According to Lemma 5.2, there must be the grey dispelling matrix $P_1(\otimes), P_2(\otimes), \ldots, P_s(\otimes)$ that could make Eq. (5.50) tenable:

$$
\begin{aligned}
& P_s(\otimes)\cdots P_2(\otimes)P_1(\otimes)A(\otimes) \\
& = D(\otimes) = \begin{pmatrix} d_{11}^{\otimes} & & & * \\ & d_{22}^{\otimes} & & \\ & & \vdots & \\ & & & d_{nn}^{\otimes} \end{pmatrix}
\end{aligned}
\tag{5.50}
$$

$$
\begin{aligned}
& B(\otimes)Q_t(\otimes)\cdots Q_2(\otimes)Q_1(\otimes) \\
& = F(\otimes) = \begin{pmatrix} f_{11}^{\otimes} & & & * \\ & f_{22}^{\otimes} & & \\ & & \vdots & \\ & & & f_{nn}^{\otimes} \end{pmatrix}
\end{aligned}
\tag{5.51}
$$

From Lemma 5.1, we know that $|A(\otimes)| = |D(\otimes)| = d_{11}^{\otimes} d_{22}^{\otimes} \cdots d_{nn}^{\otimes}$ and $B(\otimes)| = ||F(\otimes)| = f_{11}^{\otimes} f_{22}^{\otimes} \cdots f_{nn}^{\otimes}$.

According to Eqs. (5.50) and (5.51), we get that Eq. (5.52) is tenable.

$$
P_s(\otimes)\cdots P_2(\otimes)P_1(\otimes)A(\otimes)B(\otimes)Q_t(\otimes)\cdots Q_2(\otimes)Q_1(\otimes) = D(\otimes)F(\otimes) \tag{5.52}
$$

If we sign that $H(\otimes) = D(\otimes)F(\otimes)$, then $H(\otimes)$ is a grey upper triangular matrix, and we could get that Eqs. (5.53) and (5.54) are tenable:

$$
H(\otimes) = D(\otimes)F(\otimes) = \begin{pmatrix} d_{11}^{\otimes} f_{11}^{\otimes} & & & * \\ & d_{22}^{\otimes} f_{22}^{\otimes} & & \\ & & \vdots & \\ & & & d_{nn}^{\otimes} f_{nn}^{\otimes} \end{pmatrix} \tag{5.53}
$$

$$
P_s(\otimes)\cdots P_2(\otimes)P_1(\otimes)A(\otimes)B(\otimes)Q_t(\otimes)\cdots Q_2(\otimes)Q_1(\otimes) = H(\otimes) \tag{5.54}
$$

This explains that the grey matrices $A(\otimes)$ and $B(\otimes)$ can be transformed to $H(\otimes)$ by grey dispelling transformation. Hence, we can prove that Eq. (5.55) is tenable.

$$
\begin{aligned}
|A(\otimes)B(\otimes)| = |H(\otimes)| &= c_{11}^{\otimes} d_{11}^{\otimes} c_{22}^{\otimes} d_{22}^{\otimes} \cdots c_{nn}^{\otimes} d_{nn}^{\otimes} = \left(c_{11}^{\otimes} c_{22}^{\otimes} \cdots c_{nn}^{\otimes}\right)\left(d_{11}^{\otimes} d_{22}^{\otimes} \cdots d_{nn}^{\otimes}\right) \\
&= |A(\otimes)|\,|B(\otimes)|
\end{aligned}
\tag{5.55}
$$

5.3.2 Grey Invertible Matrix

Definition 5.15 Grey invertible matrix: For an n-order grey matrix $A(\otimes)$, there is an n-order grey matrix $B(\otimes)$ that could make

$$A(\otimes)B(\otimes)=B(\otimes)A(\otimes)=E(\otimes) \tag{5.56}$$

This means that $A(\otimes)$ is a grey invertible matrix, $B(\otimes)$ is called the grey invertible matrix of $A(\otimes)$; $B(\otimes)$ is called the grey invertible of $A(\otimes)$ for short. If we cannot find a grey matrix that could make Eq. (5.56) tenable, we will say that $A(\otimes)$ is an irreversible matrix.

It is easy to prove that any elementary matrix can be inverted, and the invertible matrix of every grey elementary matrix is an elementary matrix that is the same type as the former grey matrix. The grey invertible of $P^{\otimes}(i[k^{\otimes}])$ is $P^{\otimes}(i[\frac{1}{k^{\otimes}}])$, the grey invertible of $P^{\otimes}(i,j[k^{\otimes}])$ is $P^{\otimes}(i,j[-k^{\otimes}])$, and the grey invertible of $P^{\otimes}(i,j)$ is itself.

A grey invertible matrix must be a grey square matrix. Generally, we do not discuss the invertible matrix of a nongrey square matrix. For grey invertible matrices, we draw the following conclusions:

1. Because $A(\otimes)$ and $B(\otimes)$ are symmetric in the definition, if $A(\otimes)$ is invertible, its grey invertible $B(\otimes)$ is invertible, and the invertible matrix of $B(\otimes)$ is $A(\otimes)$. We can say that $A(\otimes)$ and $B(\otimes)$ are invertible of each other.
2. The grey invertible of a invertible grey matrix is unique. If $B(\otimes)$, $C(\otimes)$ are both invertible grey matrices of $A(\otimes)$, then

$$A(\otimes)B(\otimes)=B(\otimes)A(\otimes)=E(\otimes),\ \ A(\otimes)C(\otimes)=C(\otimes)A(\otimes)=E(\otimes)$$

so

$$\begin{aligned}C(\otimes)&=C(\otimes)E(\otimes)=C(\otimes)(A(\otimes)B(\otimes))=(C(\otimes)A(\otimes))B(\otimes)\\&=E(\otimes)B(\otimes)=B(\otimes)\end{aligned}$$

Because of this, we can prove that the grey invertible matrix is unique.

After we prove the uniqueness of the grey invertible, we can sign the invertible grey matrix of $A(\otimes)$ as $A^{-1}(\otimes)$. Accordingly, $A(\otimes)$ and $A^{-1}(\otimes)$ are invertible of each other, and we could make the conclusion that

3. If $A^{-1}(\otimes)$ is invertible, then $(A^{-1}(\otimes))^{-1}=A(\otimes)$ and
4. If n-order grey matrix $A(\otimes)$ and $B(\otimes)$ are invertible, then $A(\otimes)B(\otimes)$ is invertible and $(A(\otimes)B(\otimes))^{-1}=B^{-1}(\otimes)A^{-1}(\otimes)$.

In fact,

$$\begin{aligned}&(A(\otimes)B(\otimes))(B^{-1}(\otimes)A^{-1}(\otimes))=A(\otimes)(B(\otimes)B^{-1}(\otimes))A^{-1}(\otimes)\\&=A(\otimes)E(\otimes)A^{-1}(\otimes)=A(\otimes)A^{-1}(\otimes)=E(\otimes)\end{aligned}$$

We can prove that $(B^{-1}(\otimes)A^{-1}(\otimes))(A(\otimes)B(\otimes))=E(\otimes)$ in the same way. $A(\otimes)B(\otimes)$ is therefore invertible and $(A(\otimes)B(\otimes))^{-1}=B^{-1}(\otimes)A^{-1}(\otimes)$.

We can extend the result that if n-order grey matrices $A_1(\otimes), A_2(\otimes), \ldots, A_s(\otimes)$ are all invertible, then $A_1(\otimes)A_2(\otimes)\cdots A_s(\otimes)$ is invertible, and $(A_1(\otimes)A_2(\otimes)\cdots A_s(\otimes))^{-1} = A_s^{-1}(\otimes)\cdots A_2^{-1}(\otimes)A_1^{-1}(\otimes)$.

We can also get that, when $A(\otimes)$ is invertible, $A(\otimes)A(\otimes)\cdots A(\otimes) = A^m(\otimes)$ is also invertible, and $(A^m(\otimes))^{-1} = (A^{-1}(\otimes))^m$. If we regulate that $A^{-m}(\otimes) = (A^{-1}(\otimes))^m$, then we get $(A^m(\otimes))^{-1} = A^{-m}(\otimes)$.

5. If n-order grey matrix $A(\otimes)$ is invertible, then its transpose $A^T(\otimes)$ is also invertible, and $(A^T(\otimes))^{-1} = (A^{-1}(\otimes))^T$. because when $A(\otimes)$ is invertible, $A(\otimes)A^{-1}(\otimes) = A^{-1}(\otimes)A(\otimes) = E(\otimes)$, if we transpose them we can get

$$(A^{-1}(\otimes))^T A^T(\otimes) = A^T(\otimes)(A^{-1}(\otimes))^T = E(\otimes)$$

According to Definition 5.5, $A^T(\otimes)$ is invertible, and the invertible matrix of $A^T(\otimes)$ is just $(A^{-1}(\otimes))^T$.

Definition 5.16 Adjoint grey matrix: Assume that $A(\otimes) = (a_{ij}^{\otimes})$ is an n-order grey square matrix and assume that the grey algebraic cofactor of $a_{ij}^{\otimes}$ in $|A(\otimes)|$ is $A_{ij}^{\otimes}(i, j = 1, 2, \ldots, n)$. Then we say that the grey matrix $A^*(\otimes)$ [as shown in Eq. (5.57)] is the adjoint grey matrix of $A(\otimes)$.

According to the properties shown in Eqs. (5.58) and (5.59), we can get Eq. (5.60):

$$A^*(\otimes) = \begin{pmatrix} A_{11}^{\otimes} & A_{21}^{\otimes} & \cdots & A_{n1}^{\otimes} \\ A_{12}^{\otimes} & A_{22}^{\otimes} & \cdots & A_{n2}^{\otimes} \\ \vdots & \vdots & & \vdots \\ A_{1n}^{\otimes} & A_{2n}^{\otimes} & \cdots & A_{nn}^{\otimes} \end{pmatrix} \tag{5.57}$$

$$a_{i1}^{\otimes}A_{j1}^{\otimes} + a_{i2}^{\otimes}A_{j2}^{\otimes} + \cdots + a_{in}^{\otimes}A_{jn}^{\otimes} = \begin{cases} |A(\otimes)|, & (i = j) \\ 0^{\otimes}, & (i \neq j) \end{cases} \tag{5.58}$$

$$a_{1i}^{\otimes}A_{1j}^{\otimes} + a_{2i}^{\otimes}A_{2j}^{\otimes} + \cdots + a_{ni}^{\otimes}A_{nj}^{\otimes} = \begin{cases} |A(\otimes)|, & (i = j) \\ 0^{\otimes}, & (i \neq j) \end{cases} \tag{5.59}$$

$$A(\otimes)A^*(\otimes) = A^*(\otimes)A(\otimes) = \begin{pmatrix} |A(\otimes)| & & & \\ & |A(\otimes)| & & \\ & & \vdots & \\ & & & |A(\otimes)| \end{pmatrix} = |A(\otimes)|\,E(\otimes) \tag{5.60}$$

Theorem 5.17 The sufficient and necessary conditions ① of the inverse of $A(\otimes)$: The sufficient and necessary conditions of the inverse of an n-order grey matrix are that $|A(\otimes)| \neq 0^{\otimes}$.

Proof: If $A(\otimes)$ is invertible, there must be an $A^{-1}(\otimes)$ that makes $A(\otimes)A^{-1}(\otimes) = E(\otimes)$. Picking determinants from two ends, we can get that $|A(\otimes)||A^{-1}(\otimes)| = 1^{\otimes}$; therefore, $|A(\otimes)| \neq 0^{\otimes}$.

Therefore, if $|A(\otimes)| \neq 0^{\otimes}$, according to Eq. (5.60), we could find that Eq. (5.61) is tenable.

$$A(\otimes)\left(\frac{1}{A(\otimes)}A^{*}(\otimes)\right) = \left(\frac{1}{A(\otimes)}A^{*}(\otimes)\right)A(\otimes) = E(\otimes) \tag{5.61}$$

According to the definition of inverse, from Eq. (5.61), we know that $A(\otimes)$ is invertible and $A^{-1}(\otimes) = \frac{1}{|A(\otimes)|}A^{*}(\otimes)$.

An invertible grey matrix is also called a nonsingular grey matrix, and an irreversible grey matrix is also called a singular grey matrix. Whether the value of a grey determinant is $0^{\otimes}$ or not is the main difference between the two.

The conclusion of Theorem 5.17 is simple and important. The process of its proof points out a way to find the invertible matrix—the adjoint grey matrix.

Example 5.5 A known grey matrix $A(\otimes)$ is shown in Eq. (5.62), and we must determine whether $A(\otimes)$ is invertible or not and try to get $A^{-1}(\otimes)$.

$$A(\otimes) = \begin{pmatrix} 1 & [2,3] \\ 0 & [3,4] \end{pmatrix} \tag{5.62}$$

$$A^{*}(\otimes) = \begin{pmatrix} [3,4] & -[2,3] \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3+\gamma_{22} & -(2+\gamma_{12}) \\ 0 & 1 \end{pmatrix} \tag{5.63}$$

Solution: Because $|A(\otimes)| = [3,4] = 3+\gamma_{22}, \gamma_{22} \in [0,1]$, $A(\otimes)$ is invertible. We can calculate the adjoint grey matrix shown in Eq. (5.63). According to Theorem 5.17, we can get the invertible matrix of $A(\otimes)$ that is shown in Eq. (5.64):

$$A^{-1}(\otimes) = \frac{1}{|A(\otimes)|}A^{*}(\otimes) = \frac{1}{3+\gamma_{22}}\begin{pmatrix} 3+\gamma_{22} & -(2+\gamma_{12}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2+\gamma_{12}}{3+\gamma_{22}} \\ 0 & \frac{1}{3+\gamma_{22}} \end{pmatrix} \tag{5.64}$$

$(\gamma_{12} \in [0,1], \gamma_{22} \in [0,1])$

Theorem 5.18 The sufficient and necessary conditions ② of the inverse of $A(\otimes)$: The sufficient and necessary conditions of an inverse of an n-order grey matrix are that $A(\otimes)$ can be transformed to an n-order grey unit matrix by a series of elementary transformations.

Proof: According to Theorem 5.14, $A(\otimes)$ can be transformed to some standard forms by a series of elementary transformations that is shown in Eq. (5.40). For a square matrix, any elementary transformation will not change the properties of any non-$0^{\otimes}$ of the grey determinant; hence, $|A(\otimes)|$ and $|C(\otimes)|$ have to both be $0^{\otimes}$ or not. Because of that, if $A(\otimes)$ is invertible, the grey elements in the leading diagonal of $C(\otimes)$ will all be $1^{\otimes}$. This explains that $A(\otimes)$ could be transformed into grey unit matrix $E(\otimes)$ by grey elementary transformations.

If $A(\otimes)$ is irreversible, $|C(\otimes)|$ must be $0^{\otimes}$, and $A(\otimes)$ cannot be transformed to a grey unit matrix.

Theorem 5.19 The sufficient and necessary conditions ③ of the inverse of $A(\otimes)$: The sufficient and necessary conditions of an inverse of an n-order grey matrix are that $A(\otimes)$ can be represented as the product of several grey elementary matrices.

Proof: Because of the inverse of the grey elementary matrices, sufficiency of the theorem is obviously right. To the necessity, we will prove it as follows.

Doing a grey elementary transformation once is the same as multiplying a grey elementary matrix. The fact that an invertible grey matrix can be transformed to a grey unit matrix by a series of elementary transformations could be likened to the fact that, for a invertible grey matrix $A(\otimes)$, there must be the grey elementary matrices $P_1(\otimes), P_2(\otimes), \ldots, P_s(\otimes)$, $Q_1(\otimes), Q_2(\otimes), \ldots, Q_t(\otimes)$, which could make Eq. (5.65) tenable:

$$A(\otimes) = P_1^{-1}(\otimes), P_2^{-1}(\otimes), \ldots, P_s^{-1}(\otimes) Q_1^{-1}(\otimes), Q_2^{-1}(\otimes), \ldots, Q_t^{-1}(\otimes) \tag{5.65}$$

This means that $A(\otimes)$ can be transformed to the unit grey matrix $E(\otimes)$ by grey elementary row transformations that correspond to the elementary grey matrices $P_1^{-1}(\otimes), P_2^{-1}(\otimes), \ldots, P_k^{-1}(\otimes)$.

We can prove the transformation of the columns in the same way.

The inference of Theorem 5.19 provides a useful way to get the invertible grey matrix. Assuming that $A(\otimes)$ is an invertible grey matrix, there must be elementary grey matrices $Q_1(\otimes), Q_2(\otimes), \ldots, Q_k(\otimes)$ that could make Eq. (5.66) tenable. If we right multiply $A^{-1}(\otimes)$ to two ends of Eq. (5.66), we will get Eq. (5.67):

$$Q_k(\otimes) \cdots Q_1(\otimes) Q_2(\otimes) A(\otimes) = E(\otimes) \tag{5.66}$$

$$Q_k(\otimes) \cdots Q_1(\otimes) Q_2(\otimes) E(\otimes) = A^{-1}(\otimes) \tag{5.67}$$

By comparing Eqs. (5.66) and (5.67), we know that if we add the grey elementary transformations of rows that could transform $A(\otimes)$ to the grey unit matrix to the grey unit matrix $E(\otimes)$ one by one, we can get $A^{-1}(\otimes)$ finally.

The concrete way to solve a grey invertible matrix according to the invertible grey matrix follows.

Writing the same order unit grey matrix $E(\otimes)$ to the right side of n-order grey matrix $A(\otimes)$ gets a grey matrix $(A(\otimes), E(\otimes))$ that has n rows and $2n$ columns.

By doing row grey elementary transformation to the grey matrix $(A(\otimes), E(\otimes)$, the process of the transformations must do so by transforming $A(\otimes)$ to $E(\otimes)$ effectively with only row transformations.

After several transformations, when $A(\otimes)$ is transformed to $E(\otimes)$, the right sub-block (the position of the former $E(\otimes)$) is just $A^{-1}(\otimes)$.

Example 5.6 For a known grey matrix shown in Eq. (5.68), we must solve the invertible grey matrix $A^{-1}(\otimes)$.

$$A(\otimes) = \begin{pmatrix} 1 & [2,3] \\ 2 & [3,5] \end{pmatrix} \tag{5.68}$$

$$A(\otimes) = \begin{pmatrix} 1 & 2+\gamma_{12} \\ 2 & 3+2\gamma_{22} \end{pmatrix} \tag{5.69}$$

Solution: Transforming the grey numbers in Eq. (5.68) to the standard grey numbers that are shown in Eq. (5.69), $\gamma_{12} \in [0,1], \gamma_{22} \in [0,1]$ represent the obtained number of grey numbers [2,3] and [3,5] in Eq. (5.68).

According to Eq. (5.69), we can construct grey matrix $(A(\otimes), E(\otimes))$ and do some elementary transformations to it. (Note that the formulas above the arrows represent the transformations.)

$$(A(\otimes), E(\otimes)) = \begin{pmatrix} 1 & 2+\gamma_{12} & 1 & 0 \\ 2 & 3+2\gamma_{22} & 0 & 1 \end{pmatrix} \xrightarrow{\boxed{2}-2\cdot\boxed{1}} \begin{pmatrix} 1 & 2+\gamma_{12} & 1 & 0 \\ 0 & 2\gamma_{22}-2\gamma_{12}-1 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{\boxed{2}\cdot\frac{1}{2\gamma_{22}-2\gamma_{12}-1}} \begin{pmatrix} 1 & 2+\gamma_{12} & 1 & 0 \\ 0 & 1 & \dfrac{-2}{2\gamma_{22}-2\gamma_{12}-1} & \dfrac{1}{2\gamma_{22}-2\gamma_{12}-1} \end{pmatrix}$$

$$\xrightarrow{\boxed{1}-\boxed{2}\cdot(2+\gamma_{12})} \begin{pmatrix} 1 & 0 & \dfrac{3+2\gamma_{22}}{2\gamma_{22}-2\gamma_{12}-1} & \dfrac{-2-\gamma_{12}}{2\gamma_{22}-2\gamma_{12}-1} \\ 0 & 1 & \dfrac{-2}{2\gamma_{22}-2\gamma_{12}-1} & \dfrac{1}{2\gamma_{22}-2\gamma_{12}-1} \end{pmatrix}$$

In this way, we can get the grey invertible matrix of Eq. (5.68), which is shown in Eq. (5.70):

$$A^{-1}(\otimes) = \begin{pmatrix} \dfrac{3+2\gamma_{22}}{2\gamma_{22}-2\gamma_{12}-1} & \dfrac{-2-\gamma_{12}}{2\gamma_{22}-2\gamma_{12}-1} \\ \dfrac{-2}{2\gamma_{22}-2\gamma_{12}-1} & \dfrac{1}{2\gamma_{22}-2\gamma_{12}-1} \end{pmatrix} \tag{5.70}$$

5.3.3 Summary

In this part of the chapter, we gave strict definitions to the elementary transformation and invertible grey matrix, and we proved some properties and theorems of the elementary grey matrix and grey elementary transformations. On the basis of that, we proved the sufficient and necessary conditions of the inverse of any grey matrix $A(\otimes)$ and designed many kinds of algorithms to solve the problem of an invertible grey matrix. We therefore provided convenient and feasible tools for the following study on grey matrix games.

Chapter 6

Matrix Solution Method of a Grey Matrix Game

6.1 Matrix Solution Method of a Grey Matrix Game Based on a Grey Full-Rank Expanding Square Matrix

In the solution process of the grey matrix game model based on a grey mixed strategy, the grey full-rank matrix is a brief and available method that contains abundant information for solution. We prove in the following that in the grey inverse matrix of a grey full-rank expansion square matrix, grey elements in the last row and column correspond respectively to solutions of an optimal grey game strategy and the grey game values of Players 1 and 2.

6.1.1 Concept of an Expanding Square Matrix and Its Grey Inverse Matrix

Definition 6.1 Grey game expanding square matrix of players: There is a problem of grey matrix game $G(\otimes)=\{S_1,S_2;A(\otimes)\}$ when the grey profit and loss matrix is the square matrix $A(\otimes)_{n\times n}$. If we add the column vector $(1,1,\ldots,1,0)^{\mathrm{T}}$ and the row vector $(-1,-1,\ldots,-1,0)$ respectively to the right column and the last row of $A(\otimes)_{n\times n}$, there is a new square matrix $B_1(\otimes)_{(n+1)\times(n+1)}$. Here, we call $B_1(\otimes)$ the grey game expanding square matrix of Player 1, or for short the grey expanding matrix of Player 1, such as shown in Eq. (6.1). If we add the column vector $(-1,-1,\ldots,-1,0)^{\mathrm{T}}$ and row vector $(1,1,\ldots,1,0)$ respectively to the right column and the last row of $A(\otimes)_{n\times n}$, then we build a new square matrix $B_2(\otimes)_{(n+1)\times(n+1)}$. We call

$B_2(\otimes)$ the grey game expanding square matrix of Player 2, or for short the grey expanding matrix of Player 2, such as shown in Eq. (6. 2).

$$B_1(\otimes)=\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & 1 \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & 1 \\ -1 & -1 & \cdots & -1 & 0 \end{pmatrix} \tag{6.1}$$

$$B_2(\otimes)=\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & -1 \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & -1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{6.2}$$

Example 6.1 The grey profit and loss matrix of grey game $G(\otimes)=\{S_1,S_2;A(\otimes)\}$ is shown in Eq. (6.3). The grey expanding square matrices of Players 1 and 2 are respectively shown in Eqs. (6.4) and (6.5).

$$\tilde{A}(\otimes)=\begin{pmatrix} 2 & 0 & [2,3] \\ 0 & [3,4] & 1 \\ 1 & 2 & 1 \end{pmatrix} \tag{6.3}$$

$$\tilde{B}(\otimes)^1=\begin{pmatrix} 2 & 0 & [2,3] & 1 \\ 0 & [3,4] & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \tag{6.4}$$

$$B_2(\otimes)=\begin{pmatrix} 2 & 0 & [2,3] & -1 \\ 0 & [3,4] & 1 & -1 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{6.5}$$

Definition 6.2 Grey full-rank expanding square matrices of Players 1 and 2: In a grey matrix game $G(\otimes)=\{S_1,S_2;A(\otimes)_{n\times n}\}$ problem corresponding to the grey expanding square matrices $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$ of Players 1 and 2, if the grey circuit rank of $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$ is $(B)_1^{\odot}=(B)_2^{\odot}=n+1$, then we call grey expanding square matrices $B_1(\otimes)_{(n+1)\times(n+1)}$

and $B_2(\otimes)_{(n+1)\times(n+1)}$ respectively the grey full-rank expanding square matrices of Players 1 and 2 of $G(\otimes)$.

Theorem 6.1 A grey full-rank expanding square matrix is an inverse matrix: In grey matrix game $G(\otimes)=\{S_1, S_2; A(\otimes)_{n\times n}\}$, the grey full-rank expanding square matrices $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$ [shown in Eqs. (6.2) and (6.3)] are inverse matrices without fail.

Proof: In the problem of a grey matrix game $G(\otimes)=\{S_1, S_2; A(\otimes)_{n\times n}\}$, the grey circuit rank $(B)_1^{\odot}=(B)_2^{\odot}=n+1$ of the full-rank expanding square matrices $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$ is equal to its module.

Because $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$ are grey full rank, the value of the determinants is not zero; namely, $|B_1(\otimes)_{(n+1)\times(n+1)}|\neq 0$, $|B_2(\otimes)_{(n+1)\times(n+1)}|\neq 0$.

Because a necessary and sufficient condition of an inverse matrix is that the value of the determinant is not equal to zero, the grey full-rank expanding square matrices $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$ of $G(\otimes)$ are inverse matrices without fail.

According to Theorem 6.1, based on this method of solving the inverse matrix of a grey matrix, we can work out the grey inverse matrices $B_1^{-1}(\otimes)_{(n+1)\times(n+1)}$ and $B_2^{-1}(\otimes)_{(n+1)\times(n+1)}$ [Eqs. (6.6) and (6.7)] of the grey full-rank expanding square matrices $B_1(\otimes)_{(n+1)\times(n+1)}$ and $B_2(\otimes)_{(n+1)\times(n+1)}$.

$$\tilde{B}^{-1}(\otimes)^1=\begin{pmatrix} \left[c^1_{1,1},d^1_{1,1}\right] & \left[c^1_{1,2},d^1_{1,2}\right] & \cdots & \left[c^1_{1,n},d^1_{1,n}\right] & \left[c^1_{1,n+1},d^1_{1,n+1}\right] \\ \left[c^1_{2,1},d^1_{2,1}\right] & \left[c^1_{2,2},d^1_{2,2}\right] & \cdots & \left[c^1_{2,n},d^1_{2,n}\right] & \left[c^1_{2,n+1},d^1_{2,n+1}\right] \\ \vdots & \vdots & & \vdots & \vdots \\ \left[c^1_{n,1},d^1_{n,1}\right] & \left[c^1_{n,2},d^1_{n,2}\right] & \cdots & \left[c^1_{n,n},d^1_{n,n}\right] & \left[c^1_{n,n+1},d^1_{n,n+1}\right] \\ \left[c^1_{n+1,1},d^1_{n+1,1}\right] & \left[c^1_{n+1,2},d^1_{n+1,2}\right] & \cdots & \left[c^1_{n+1,n},d^1_{n+1,n}\right] & \left[c^1_{n+1,n+1},d^1_{n+1,n+1}\right] \end{pmatrix} \tag{6.6}$$

$$\tilde{B}^{-1}(\otimes)^2=\begin{pmatrix} \left[c^2_{1,1},d^2_{1,1}\right] & \left[c^2_{1,2},d^2_{1,2}\right] & \cdots & \left[c^2_{1,n},d^2_{1,n}\right] & \left[c^2_{1,n+1},d^2_{1,n+1}\right] \\ \left[c^2_{2,1},d^2_{2,1}\right] & \left[c^2_{2,2},d^2_{2,2}\right] & \cdots & \left[c^2_{2,n},d^2_{2,n}\right] & \left[c^2_{2,n+1},d^2_{2,n+1}\right] \\ \vdots & \vdots & & \vdots & \vdots \\ \left[c^2_{n,1},d^2_{n,1}\right] & \left[c^2_{n,2},d^2_{n,2}\right] & \cdots & \left[c^2_{n,n},d^2_{n,n}\right] & \left[c^2_{n,n+1},d^2_{n,n+1}\right] \\ \left[c^2_{n+1,1},d^2_{n+1,1}\right] & \left[c^2_{n+1,2},d^2_{n+1,2}\right] & \cdots & \left[c^2_{n+1,n},d^2_{n+1,n}\right] & \left[c^2_{n+1,n+1},d^2_{n+1,n+1}\right] \end{pmatrix} \tag{6.7}$$

Example 6.2 Given a grey matrix game $G(\otimes)=\{S_1, S_2; A(\otimes)_{n\times n}\}$ problem where the grey expanding square matrix of Player 1 is instanced by Eq. (6.8), solve the grey circuit rank and inverse matrix $B_1^{-1}(\otimes)$ of this expanding square matrix $B_1(\otimes)$.

Solution:

$$\tilde{B}(\otimes)^1 = \begin{pmatrix} 1 & [3,4] & 1 \\ [4,6] & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3+\gamma_{12} & 1 \\ 4+2\gamma_{21} & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix} \tag{6.8}$$

$$(B(\otimes)^1, E(\otimes)) = \left(\begin{array}{ccc|ccc} 1 & 3+\gamma_{12} & 1 & 1^{\otimes} & 0 & 0 \\ 4+2\gamma_{21} & 2 & 1 & 0 & 1^{\otimes} & 0 \\ -1 & -1 & 0 & 0 & 0 & 1^{\otimes} \end{array}\right) \xrightarrow[\boxed{3}+\boxed{1}]{\boxed{2}-(4+2\gamma_{21})\cdot\boxed{1}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3+\gamma_{12} & 1 & 1 & 0 & 0 \\ 0 & -(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}) & -(3+2\gamma_{21}) & -(4+2\gamma_{21}) & 1 & 0 \\ 0 & 2+\gamma_{12} & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\frac{1}{2+\gamma_{12}}\cdot\boxed{3}]{-\frac{1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}\cdot\boxed{2}} \left(\begin{array}{ccc} 1 & 3+\gamma_{12} & 1 \\ 0 & 1 & \frac{(3+2\gamma_{21})}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} \\ 0 & 1 & \frac{1}{2+\gamma_{12}} \end{array}\right|$$

$$\left|\begin{array}{ccc} 1 & 0 & 0 \\ \frac{(4+2\gamma_{21})}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & \frac{-1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & 0 \\ \frac{1}{2+\gamma_{12}} & 0 & \frac{1}{2+\gamma_{12}} \end{array}\right)$$

$$\xrightarrow{\boxed{3}-\boxed{2}} \left(\begin{array}{ccc} 1 & 3+\gamma_{12} & 1 \\ 0 & 1 & \frac{(3+2\gamma_{21})}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} \\ 0 & 0 & \frac{4+\gamma_{12}+2\gamma_{21}}{(2+\gamma_{12})\bullet(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} \end{array}\right|$$

$$\left|\begin{array}{ccc} 1 & 0 & 0 \\ \frac{(4+2\gamma_{21})}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & \frac{-1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & 0 \\ \frac{2+2\gamma_{21}}{(2+\gamma_{12})\bullet(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & \frac{1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & \frac{1}{2+\gamma_{12}} \end{array}\right)$$

$$\xrightarrow{\frac{(2+\gamma_{12})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})}{4+\gamma_{12}+2\gamma_{21}}\cdot\boxed{3}}$$

$$\left(\begin{array}{ccc|c}
1 & 3+\gamma_{12} & 1 & 1 \\
0 & 1 & \frac{3+2\gamma_{21}}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & \frac{4+2\gamma_{21}}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} \\
0 & 0 & 1 & \frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right.\times$$

$$\times\left.\begin{array}{cc}
0 & 0 \\
\frac{-1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & 0 \\
\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & \frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right)$$

$$\xrightarrow[\boxed{2}-\left(\frac{3+2\gamma_{21}}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}\right)\cdot\boxed{3}]{\boxed{1}-\boxed{3}}$$

$$\left(\begin{array}{ccc|c}
1 & 3+\gamma_{12} & 0 & \frac{2+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\
0 & 1 & 0 & \frac{10+2\gamma_{12}+8\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} \\
0 & 0 & 1 & \frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right.-$$

$$\left.\begin{array}{cc}
-\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & -\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \\
-\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & -\frac{3+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \\
\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & \frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right)$$

$$\xrightarrow{\boxed{1}-(3+\gamma_{12})\cdot\boxed{2}}$$

$$\left(\begin{array}{ccc|c}
1 & 0 & 0 & \dfrac{2+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\[2ex]
0 & 1 & 0 & \dfrac{10+2\gamma_{12}+8\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\bullet(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} \quad - \\[2ex]
0 & 0 & 1 & \dfrac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right.$$

$$\left.\begin{array}{cc}
-\dfrac{1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & -\dfrac{1+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\[2ex]
-\dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\bullet(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & -\dfrac{3+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \\[2ex]
\dfrac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & \dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right)$$

The grey expanding square matrix of Player 1 is a grey full-rank square matrix, its grey circuit rank is $(B)_1^{\odot}=3$, and the grey inverse matrix is shown in Eq. (6.9):

$$B_1^{-1}(\otimes)=\left\{\begin{array}{c}
\dfrac{2+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\[2ex]
\dfrac{10+2\gamma_{12}+8\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} \quad - \\[2ex]
\dfrac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right.$$

$$\left.\begin{array}{cc}
-\dfrac{1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & -\dfrac{1+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\[2ex]
-\dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & -\dfrac{3+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \\[2ex]
\dfrac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & \dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}
\end{array}\right\}
\tag{6.9}$$

6.1.2 Optimum Grey Game Strategy and Grey Game Value of Player 1

Theorem 6.2 The optimum grey game strategy and grey game value of Player 1: In grey matrix game $G(\otimes)=\{S_1, S_2; A(\otimes)_{n\times n}\}$, if the grey inverse matrix $B_1^{-1}(\otimes)_{(n+1)\times(n+1)}$ [Eq. (6.6)] of grey expanding square matrix $B_1(\otimes)_{(n+1)\times(n+1)}$ of Player 1 exists, and the value of each grey element of the last row of $B_1^{-1}(\otimes)_{(n+1)\times(n+1)}$ is not less than zero (that is, $[c_{n+1,1}^1, d_{n+1,1}^1]\geq 0^{\otimes}, [c_{n+1,2}^1, d_{n+1,2}^1]\geq 0^{\otimes}, \ldots, [c_{n+1,n+1}^1, d_{n+1,n+1}^1]\geq 0^{\otimes}$),then these grey elements respectively correspond to each optimum grey game strategy and grey game value of Player 1, namely,

$$x_1^{\otimes}=\left[c_{n+1,1}^1, d_{n+1,1}^1\right], x_2^{\otimes}=\left[c_{n+1,2}^1, d_{n+1,2}^1\right], \ldots, x_n^{\otimes}=\left[c_{n+1,n}^1, d_{n+1,n}^1\right],$$

$$v_{\otimes}^{*}=\left[c_{n+1,n+1}^1, d_{n+1,n+1}^1\right]$$

Proof: There is a grey matrix game $G(\otimes)=\{S_1, S_2; A(\otimes)_{n\times n}\}$, where a grey system of inequalities of optimum grey mix strategy $x_i(\otimes), i=1,2,\ldots,n$ and optimum grey game value $v^{*}(\otimes)$ of Player 1, such as shown in Eq. (6.10):

$$\begin{cases}\sum_{i=1}^{n} a_{ij}(\otimes)x_i(\otimes)\geq v^{*}(\otimes), \quad j=1,2,\ldots,n \\ x_i(\otimes)=[Lx_i, Rx_i]\geq[0,0]=0^{\otimes}, \quad i=1,2,\ldots,n \\ \sum_{i=1}^{n} x_i(\otimes)=\sum_{i=1}^{n}[Lx_i, Rx_i]=[1,1]_{\gamma_i=c_i, 0\leq c_i\leq 1, i=1,2,\ldots,n}=1^{\otimes}\end{cases} \tag{6.10}$$

$$\begin{cases}[a_{11}, b_{11}]\cdot x_1(\otimes)+[a_{21}, b_{21}]\cdot x_2(\otimes)+\cdots+[a_{n1}, b_{n1}]\cdot x_n(\otimes)-v^{*}(\otimes)=0^{\otimes} \\ [a_{12}, b_{12}]\cdot x_1(\otimes)+[a_{22}, b_{22}]\cdot x_2(\otimes)+\cdots+[a_{n2}, b_{n2}]\cdot x_n(\otimes)-v^{*}(\otimes)=0^{\otimes} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [a_{1n}, b_{1n}]\cdot x_1(\otimes)+[a_{2n}, b_{2n}]\cdot x_2(\otimes)+\cdots+[a_{nn}, b_{nn}]\cdot x_n(\otimes)-v^{*}(\otimes)=0^{\otimes} \\ x_1(\otimes)+x_2(\otimes)+\cdots+x_n(\otimes)-0\cdot v^{*}(\otimes)=1^{\otimes}\end{cases} \tag{6.11}$$

If we do not consider the restriction request in $x_i(\otimes)=[Lx_i, Rx_i]\geq[0,0], i=1,2,\ldots,n$ in Eq. (6.10) and we replace the inequality to the equation, counterchange properly, then we get Eq. (6.11). For Eq. (6.11), if we consider $x_i(\otimes), i=1,2,\ldots,n$

and $v^*(\otimes)$ as unknown variables of this linear equation, we can overwrite Eq. (6.10) as Eq. (6.11). Then we hold Eq. (6.11) as a grey matrix, such as shown in Eqs. (6.12) and (6.13).

$$(x_1(\otimes), x_2(\otimes), \ldots, x_n(\otimes), v^*(\otimes)) \cdot \begin{pmatrix} [a_{11}, b_{11}] & [a_{12}, b_{12}] & \cdots & [a_{1n}, b_{1n}] & 1^{\otimes} \\ [a_{21}, b_{21}] & [a_{22}, b_{22}] & \cdots & [a_{2n}, b_{2n}] & 1^{\otimes} \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{n1}, b_{n1}] & [a_{n2}, b_{n2}] & \cdots & [a_{nn}, b_{nn}] & 1^{\otimes} \\ -1^{\otimes} & -1^{\otimes} & \cdots & -1^{\otimes} & 0^{\otimes} \end{pmatrix}$$

$$= (0^{\otimes}, 0^{\otimes}, \ldots, 0^{\otimes}, 1^{\otimes}) \tag{6.12}$$

$$(x_1(\otimes), x_2(\otimes), \ldots, x_n(\otimes), v^*(\otimes)) \cdot B_1(\otimes) = (0^{\otimes}, 0^{\otimes}, \ldots, 0^{\otimes}, 1^{\otimes}) \tag{6.13}$$

In Eq. (6.13), $B_1(\otimes)_{(n+1)\times(n+1)}$ is the grey expanding matrix of Player 1. According to the foregoing condition, $B_1(\otimes)_{(n+1)\times(n+1)}$ has an inverse grey matrix $B_1^{-1}(\otimes)_{(n+1)\times(n+1)}$, the rank $(B)_1^{\odot} = n+1$ of which is grey full-rank. According to the necessary and sufficient condition of a single solution of inhomogeneous linear systems, Eq. (6.13) has only the solution:

$$(x_1(\otimes), x_2(\otimes), \ldots, x_n(\otimes), v^*(\otimes)) \cdot B(\otimes) \cdot B_1^{-1}(\otimes) = (0^{\otimes}, 0^{\otimes}, \ldots, 0^{\otimes}, 1^{\otimes}) \cdot B_1^{-1}(\otimes)$$

$$(x_1(\otimes), x_2(\otimes), \ldots, x_n(\otimes), v^*(\otimes)) = (0^{\otimes}, 0^{\otimes}, \ldots, 0^{\otimes}, 1^{\otimes}) \cdot B_1^{-1}(\otimes)$$

$$(x_1(\otimes), x_2(\otimes), \ldots, x_n(\otimes), v^*(\otimes)) = (0^{\otimes}, 0^{\otimes}, \ldots, 0^{\otimes}, 1^{\otimes}) \cdot$$

$$\cdot \begin{pmatrix} \left[c_{1,1}^1, d_{1,1}^1\right] & \left[c_{1,2}^1, d_{1,2}^1\right] & \cdots & \left[c_{1,n}^1, d_{1,n}^1\right] & \left[c_{1,n+1}^1, d_{1,n+1}^1\right] \\ \left[c_{2,1}^1, d_{2,1}^1\right] & \left[c_{2,2}^1, d_{2,2}^1\right] & \cdots & \left[c_{2,n}^1, d_{2,n}^1\right] & \left[c_{2,n+1}^1, d_{2,n+1}^1\right] \\ \vdots & \vdots & & \vdots & \vdots \\ \left[c_{n,1}^1, d_{2,1}^1\right] & \left[c_{n,2}^1, d_{n,2}^1\right] & \cdots & \left[c_{n,n}^1, d_{n,n}^1\right] & \left[c_{n,n+1}^1, d_{n,n+1}^1\right] \\ \left[c_{n+1,1}^1, d_{n+1,1}^1\right] & \left[c_{n+1,2}^1, d_{n+1,2}^1\right] & \cdots & \left[c_{n+1,n}^1, d_{n+1,n}^1\right] & \left[c_{n+1,n+1}^1, d_{n+1,n+1}^1\right] \end{pmatrix}$$

$$(x_1(\otimes), x_2(\otimes), \ldots, x_n(\otimes), v^*(\otimes))$$

$$= \left(\left[c_{n+1,1}^1, d_{n+1,1}^1\right], \left[c_{n+1,2}^1, d_{n+1,2}^1\right], \ldots, \left[c_{n+1,n}^1, d_{n+1,n}^1\right], \left[c_{n+1,n+1}^1, d_{n+1,n+1}^1\right]\right) \tag{6.14}$$

According to Eq. (6.14), when coring

$$x_1(\otimes)=\left[c^1_{n+1,1},d^1_{n+1,1}\right],x_2(\otimes)=\left[c^1_{n+1,2},d^1_{n+1,2}\right],\ldots,x_n(\otimes)=\left[c^1_{n+1,n},d^1_{n+1,n}\right],v^*_\otimes$$

$$=\left[c^1_{n+1,n+1},d^1_{n+1,n+1}\right]$$

the request of Eq. (6.11) would be fulfilled. But Eq. (6.11) is obtained by Eq. (6.10), which neglects the nonnegativity of $x_i(\otimes),i=1,2,\ldots,n,$ so according to the value of $x_i(\otimes)$ fulfills Eq. (6.11), which could fulfill the system of Eq. (6.10) only when nonnegative.

This shows that, in Eq. (6.14), when $x_i(\otimes)$ is nonnegative, the optimum grey game strategy and grey game value of Player 1 are respectively

$$x_1^\otimes=\left[c^1_{n+1,1},d^1_{n+1,1}\right],x_2^\otimes=\left[c^1_{n+1,2},d^1_{n+1,2}\right],\ldots,x_n^\otimes$$

$$=\left[c^1_{n+1,n},d^1_{n+1,n}\right],v^*_\otimes=\left[c^1_{n+1,n+1},d^1_{n+1,n+1}\right].$$

Example 6.3 In the following problem of grey matrix game $G(\otimes)=\{S_1,S_2;A(\otimes)_{n\times n}\}$, the grey expanding square matrix of Player 1 is shown by Eq. (6.8). Solve the optimum grey game strategy and grey game value of Player 1 in the process of grey matrix game $G(\otimes)$.

Solution: According to Theorem 6.2 and Eq. (6.9), the optimum grey game strategy and grey game value of Player 1 are respectively:

$$x_1^\otimes=\frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}=$$

$$\left[min\left(\frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1},\quad max\left(\frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1}\right]=[0.40,0.67]$$

$$x_2^\otimes=\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}}=$$

$$\left[min\left(\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1},max\left(\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1}\right]=[0.33,0.60]$$

$$v^*_\otimes=\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}=\left[min\left(\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1},\right.$$

$$\left.max\left(\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1}\right]=[2.50,3.14]$$

6.1.3 Optimum Grey Game Strategy and Grey Game Value of Player 2

Theorem 6.3 The optimum grey game strategy and grey game value of Player 2: In grey matrix game $G(\otimes)=\{S_1, S_2; A(\otimes)\}$, if the grey inverse matrix $B_2^{-1}(\otimes)_{(n+1)\times(n+1)}$ [Eq. (6.7)] of grey expanding square matrix $B_2(\otimes)_{(n+1)\times(n+1)}$ of Player 2 exists, and the value of each grey element of the right column of $B_2^{-1}(\otimes)_{(n+1)\times(n+1)}$ is not less than zero (namely, $[c^2_{n+1,1}, d^2_{n+1,1}] \ge 0^{\otimes}, [c^2_{n+1,2}, d^2_{n+1,2}] \ge 0^{\otimes}, \ldots, [c^2_{n+1,n+1}, d^2_{n+1,n+1}] \ge 0^{\otimes}$), then these grey elements respectively correspond to each optimum grey game strategy and grey game value of Player 2.

Proof: In grey matrix game $G(\otimes)$ there exists a grey system of inequalities of optimum grey mix strategy $y_j(\otimes), j=1,2,\ldots,n$ and optimum grey game value $v^*_{\otimes}$ of players [Eq. (6.15)]:

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij}(\otimes) y_j(\otimes) \le v^*(\otimes), i=1,2,\ldots,n \\ y_j(\otimes)=[Ly_j, Ry_j] \ge [0,0], j=1,2,\ldots,n \\ \sum\limits_{j=1}^{n} y_j(\otimes) \sum\limits_{j=1}^{n} [Ly_j, Ry_j] = [1,1]_{\gamma_j=c_j, 0\le c_j\le 1, j=1,2,\ldots,n} = 1^{\otimes} \end{cases} \tag{6.15}$$

$$\begin{cases} [a_{11},b_{11}]\cdot y_1(\otimes)+[a_{12},b_{12}]\cdot y_2(\otimes)+\cdots+[a_{1n},b_{1n}]\cdot y_n(\otimes)-v^*(\otimes)=0^{\otimes} \\ [a_{21},b_{21}]\cdot y_1(\otimes)+[a_{22},b_{22}]\cdot y_2(\otimes)+\cdots+[a_{2n},b_{2n}]\cdot y_n(\otimes)-v^*(\otimes)=0^{\otimes} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [a_{n1},b_{n1}]\cdot y_1(\otimes)+[a_{n2},b_{n2}]\cdot y_2(\otimes)+\cdots+[a_{nn},b_{nn}]\cdot y_n(\otimes)-v^*(\otimes)=0^{\otimes} \\ y_1(\otimes)+y_2(\otimes)+\cdots+y_n(\otimes)-0\cdot v^*(\otimes)=1^{\otimes} \end{cases} \tag{6.16}$$

If we do not consider the restriction request $y_j(\otimes)=[Ly_j, Ry_j]\ge[0,0], j=1,2,\ldots,n$ in Eq. (6.15) and replace the inequality in the equation, counterchange properly, then we get Eq. (6.16). For Eq. (6.15), if we consider $y_j^{\otimes}$, $j=1,2,\cdots,n$ and $v^*_{\otimes}$ as unknown variables of this linear equation and overwrite Eq. (6.15) as Eq. (6.16), then we can hold Eq. (6.14) as a grey matrix, such as shown in Eqs. (6.17) and (6.18):

$$\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & -1^{\otimes} \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & -1^{\otimes} \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & -1^{\otimes} \\ 1^{\otimes} & 1^{\otimes} & \cdots & 1^{\otimes} & 0^{\otimes} \end{pmatrix} \cdot \begin{pmatrix} y_1^{\otimes} \\ y_2^{\otimes} \\ \vdots \\ y_n^{\otimes} \\ y^*_{\otimes} \end{pmatrix} = \begin{pmatrix} 0^{\otimes} \\ 0^{\otimes} \\ \cdots \\ 0^{\otimes} \\ 1^{\otimes} \end{pmatrix} \tag{6.17}$$

$$\tilde{B}(\otimes)^2 \cdot \begin{pmatrix} y_1^{\otimes} \\ y_2^{\otimes} \\ \vdots \\ y_n^{\otimes} \\ \overset{*}{y}_{\otimes} \end{pmatrix} = \begin{pmatrix} 0^{\otimes} \\ 0^{\otimes} \\ \vdots \\ 0^{\otimes} \\ 1^{\otimes} \end{pmatrix} \tag{6.18}$$

In Eq. (6.18), $B_2(\otimes)_{(n+1)\times(n+1)}$ is expanded from the grey profit and loss value matrix $A(\otimes)_{n\times n}$. From known conditions, $B_2(\otimes)_{(n+1)\times(n+1)}$ exists, as does invertible grey matrix $B_2^{-1}(\otimes)_{(n+1)\times(n+1)}$ whose grey rank $(B)_2^{\odot} = n+1$ is full grey rank. According to the necessary and sufficient condition of a single solution of an inhomogeneous linear system, Eq. (6.18) has only the solution

$$B_2^{-1}(\otimes)\cdot B_2(\otimes)\cdot(y_1(\otimes), y_2(\otimes),\ldots, y_n(\otimes), v^*(\otimes))^T = B_2^{-1}(\otimes)\cdot(0^{\otimes}, 0^{\otimes},\ldots, 0^{\otimes}, 1^{\otimes})^T$$

$$(y_1(\otimes), y_2(\otimes),\ldots, y_n(\otimes), v^*(\otimes))^T = B^{-1}(\otimes)^2 \cdot (0^{\otimes}, 0^{\otimes},\ldots, 0^{\otimes}, 1^{\otimes})^T$$

$$\begin{pmatrix} y_1(\otimes) \\ y_2(\otimes) \\ \vdots \\ y_n(\otimes) \\ v^*(\otimes) \end{pmatrix}$$

$$= \begin{pmatrix} \left[c_{1,1}^2, d_{1,1}^2\right] & \left[c_{1,2}^2, d_{1,2}^2\right] & \cdots & \left[c_{1,n}^2, d_{1,n}^2\right] & \left[c_{1,n+1}^2, d_{1,n+1}^2\right] \\ \left[c_{2,1}^2, d_{2,1}^2\right] & \left[c_{2,2}^2, d_{2,2}^2\right] & \cdots & \left[c_{2,n}^2, d_{2,n}^2\right] & \left[c_{2,n+1}^2, d_{2,n+1}^2\right] \\ \vdots & \vdots & & \vdots & \vdots \\ \left[c_{n,1}^2, d_{n,1}^2\right] & \left[c_{n,2}^2, d_{n,2}^2\right] & \cdots & \left[c_{n,n}^2, d_{n,n}^2\right] & \left[c_{n,n+1}^2, d_{n,n+1}^2\right] \\ \left[c_{n+1,1}^2, d_{n+1,1}^2\right] & \left[c_{n+1,2}^2, d_{n+1,2}^2\right] & \cdots & \left[c_{n+1,n}^2, d_{n+1,n}^2\right] & \left[c_{n+1,n+1}^2, d_{n+1,n+1}^2\right] \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0^{\otimes} \\ 0^{\otimes} \\ \cdots \\ 0^{\otimes} \\ 1^{\otimes} \end{pmatrix}$$

$$(y_1(\otimes), y_2(\otimes), \ldots, y_n(\otimes), v^*(\otimes))^T$$

$$= \left(\left[c^2_{1,n+1}, d^2_{1,n+1}\right], \left[c^2_{2,n+1}, d^2_{2,n+1}\right], \ldots, \left[c^2_{n,n+1}, d^2_{n,n+1}\right], \left[c^2_{n+1,n+1}, d^2_{n+1,n+1}\right]\right)^T \tag{6.19}$$

According to Eq. (6.19), when coring

$$y_1(\otimes) = \left[c^2_{1,n+1}, d^2_{1,n+1}\right], y_2(\otimes) = \left[c^2_{2,n+1}, d^2_{2,n+1}\right], \ldots, y_n(\otimes) = \left[c^2_{n,n+1}, d^2_{n,n+1}\right]$$

$$v^*(\otimes) = \left[c^2_{n+1,n+1}, d^2_{n+1,n+1}\right]$$

the request of Eq. (6.16) would be fulfilled. But Eq. (6.16) is obtained from Eq. (6.15), which neglects the nonnegativity of $y_j(\otimes), j = 1, 2, \ldots, n,$ so according to the value of $y_j(\otimes), j = 1, 2, \ldots, n,$ fulfills Eq. (6.16), which could fulfill the system of Eq. (6.15) only when nonnegative.

This shows that, in Eq. (6.19), when $y_j(\otimes), j = 1, 2, \ldots, n$ is nonnegative, the optimum grey game strategy and grey game value of Player 2 are respectively

$$y_1(\otimes) = \left[c^2_{1,n+1}, d^2_{1,n+1}\right], y_2(\otimes) = \left[c^2_{2,n+1}, d^2_{2,n+1}\right], \ldots, y_n(\otimes) = \left[c^2_{n,n+1}, d^2_{n,n+1}\right]$$

$$v^*(\otimes) = \left[c^2_{n+1,n+1}, d^2_{n+1,n+1}\right]$$

6.1.4 Optimum Grey Game Strategy and Grey Game Value of Players Based on a Combined Grey Full-Rank Expanding Square Matrix

Definition 6.3 Combined grey full-rank expanding square matrix of players and its grey inverse matrix: In grey matrix game $G(\otimes) = \{S_1, S_2; A(\otimes)\}$ where the grey profit and loss matrix is the square matrix $A(\otimes)_{n\times n}$, if we add the column vector $(1,1,\ldots,1,0)^T$ and the row vector $(-1,-1,\ldots,-1,0)$ respectively to the right column and the last row of $A(\otimes)_{n\times n}$, there is a new square matrix $B_1(\otimes)_{(n+1)\times(n+1)}$. We call $B_1(\otimes)$ the player in a grey full-rank expanding square matrix, or grey full expanding matrix for short, such as shown in Eq. (6.18); if $B(\otimes)_{(n+1)\times(n+1)}$ grey rank value is $(B)^{\odot} = n+1$, we call $B(\otimes)_{(n+1)\times(n+1)}$ the grey full-rank expanding square matrix.

According to correlation theorems, we can solve the problem of grey matrix game $G(\otimes)$ and the inverse matrix $\tilde{B}^{-1}(\otimes)_{(n+1)\times(n+1)}$ of a combined grey full-rank expanding square matrix $B(\otimes)_{(n+1)\times(n+1)}$, such as shown in Eq. (6.21):

$$B(\otimes)=\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & 1 \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{6.20}$$

$$B^{-1}(\otimes)=\begin{pmatrix} [c_{1,1},d_{1,1}] & [c_{1,2},d_{1,2}] & \cdots & [c_{1,n},d_{1,n}] & [c_{1,n+1},d_{1,n+1}] \\ [c_{2,1},d_{2,1}] & [c_{2,2},d_{2,2}] & \cdots & [c_{2,n},d_{2,n}] & [c_{2,n+1},d_{2,n+1}] \\ \vdots & \vdots & & \vdots & \vdots \\ [c_{n,1},d_{n,1}] & [c_{n,2},d_{n,2}] & \cdots & [c_{n,n},d_{n,n}] & [c_{n,n+1},d_{n,n+1}] \\ [c_{n+1,1},d_{n+1,1}] & [c_{n+1,2},d_{n+1,2}] & \cdots & [c_{n+1,n},d_{n+1,n}] & [c_{n+1,n+1},d_{n+1,n+1}] \end{pmatrix} \tag{6.21}$$

Theorem 6.4 The optimum grey game strategy and grey game value of Player 1: In grey matrix game $G(\otimes)=\{S_1,S_2;A(\otimes)\}$, with a grey inverse matrix $B^{-1}(\otimes)_{(n+1)\times(n+1)}$ of a combined grey expanding square matrix $B_1(\otimes)_{(n+1)\times(n+1)}$ of Player 1 [such as shown in Eq. (6.21)], if the value of each grey element of the last row of $B_1^{-1}(\otimes)_{(n+1)\times(n+1)}$ is not less than zero (namely $[c_{n+1,1},d_{n+1,1}]\geq 0^{\otimes}$, $[c_{n+1,2},d_{n+1,2}]\geq 0^{\otimes},\ldots,[c_{n+1,n+1},d_{n+1,n+1}]\geq 0^{\otimes}$), then these grey elements respectively correspond to each optimum grey game strategy of Player 1 (namely, $x_1\otimes=$ $[c_{n+1,1},d_{n+1,1}]$, $x_2(\otimes)=[c_{n+1,2},d_{n+1,2}]$, $\ldots,x_n(\otimes)=[c_{n+1,n},d_{n+1,n}]$). If the value of each grey element of the last right-hand column of $B_1^{-1}(\otimes)_{(n+1)\times(n+1)}$ is not less than zero (namely, $[c_{n+1,1},d_{n+1,1}]\geq 0^{\otimes},[c_{n+1,2},d_{n+1,2}]\geq 0^{\otimes},\ldots,[c_{n+1,n+1},d_{n+1,n+1}]\geq 0^{\otimes}$), then these grey elements respectively correspond to each optimum grey game strategy of Player 2 (namely, $y_1(\otimes)=[c_{1,n+1},d_{1,n+1}]$, $y_2(\otimes)=[c_{2,n+1},d_{2,n+1}],\ldots,$ $y_n(\otimes)=[c_{n,n+1},d_{n,n+1}]$), and the grey game value of the grey game is $v^*(\otimes)=$ $-u^*(\otimes)=-[c_{n+1,n+1},d_{n+1,n+1}]$.

Proof: There is a grey matrix game $G(\otimes)=\{S_1,S_2;A(\otimes)_{n\times n}\}$ where there exists a grey system of inequalities of optimum grey mix strategy $x_i(\otimes),i=1,2,\ldots,n$, $y_j(\otimes),j=1,2,\ldots,n$ and optimum grey game value $v^*(\otimes)$ of Players 1 and 2, such as shown in Eqs. (6.10) and (6.15).

If we do not consider the nonnegative restriction request $x_i(\otimes), i = 1,2,\ldots,n$, $y_j(\otimes), j = 1,2,\ldots,n$ in Eqs. (6.10) and (6.15), order $v_\otimes^* = -u_\otimes^*$, and replace the inequality to the equation, counterchange properly, then we get Eqs. (6.22) and (6.23):

$$\begin{cases} [a_{11}, b_{11}] \cdot x_1(\otimes) + [a_{21}, b_{21}] \cdot x_2(\otimes) + \cdots + [a_{n1}, b_{n1}] \cdot x_n(\otimes) + u^*(\otimes) = 0^\otimes \\ [a_{12}, b_{12}] \cdot x_1(\otimes) + [a_{22}, b_{22}] \cdot x_2(\otimes) + \cdots + [a_{n2}, b_{n2}] \cdot x_n(\otimes) + u^*(\otimes) = 0^\otimes \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [a_{1n}, b_{1n}] \cdot x_1(\otimes) + [a_{2n}, b_{2n}] \cdot x_2(\otimes) + \cdots + [a_{nn}, b_{nn}] \cdot x_n(\otimes) + u^*(\otimes) = 0^\otimes \\ x_1(\otimes) + x_2(\otimes) + \cdots + x_n(\otimes) + 0 \cdot u^*(\otimes) = 1^\otimes \end{cases} \tag{6.22}$$

$$\begin{cases} [a_{11}, b_{11}] \cdot y_1(\otimes) + [a_{12}, b_{12}] \cdot y_2(\otimes) + \cdots + [a_{1n}, b_{1n}] \cdot y_n(\otimes) + u^*(\otimes) = 0^\otimes \\ [a_{21}, b_{21}] \cdot y_1(\otimes) + [a_{22}, b_{22}] \cdot y_2(\otimes) + \cdots + [a_{2n}, b_{2n}] \cdot y_n(\otimes) + u^*(\otimes) = 0^\otimes \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ [a_{n1}, b_{n1}] \cdot y_1(\otimes) + [a_{n2}, b_{n2}] \cdot y_2(\otimes) + \cdots + [a_{nn}, b_{nn}] \cdot y_n(\otimes) + u^*(\otimes) = 0^\otimes \\ y_1(\otimes) + y_2(\otimes) + \cdots + y_n(\otimes) + 0 \cdot u^*(\otimes) = 1^\otimes \end{cases} \tag{6.23}$$

For Eqs. (6.22) and (6.23), if we consider $x_i(\otimes), i = 1,2,\ldots,n$, $y_j(\otimes), j = 1,2,\ldots,n$ and $v_\otimes^*$ as unknown variables of this linear equation, and overwrite Eqs. (6.22) and (6.23) as Eqs. (6.24) and (6.25), then for Eqs. (6.24) and (6.25) matrix counterchange, such as shown in Eqs. (6.26) and (6.27).

According to Eqs. (6.26) and (6.27), when coring

$$x_1^\otimes = [c_{n+1,1}, d_{n+1,1}], x_2^\otimes = [c_{n+1,2}, d_{n+1,2}], \ldots, x_n^\otimes = [c_{n+1,n}, d_{n+1,n}]; y_1^\otimes = [c_{1,n+1}, d_{1,n+1}],$$

$$y_2^\otimes = [c_{2,n+1}, d_{2,n+1}], \ldots, y_n^\otimes = [c_{n,n+1}, d_{n,n+1}]; v_\otimes^* = -u_\otimes^* = -[c_{n+1,n+1}, d_{n+1,n+1}]$$

the requests of Eqs. (6.22) and (6.23) would be fulfilled. But Eqs. (6.22) and (6.23) are obtained by Eqs. (6.10) and (6.15), which neglect the nonnegativity of $x_i^\otimes, i = 1,2,\ldots,n$, $y_j^\otimes, j = 1,2,\ldots,n$, so according to the value of $x_i^\otimes, i = 1,2,\cdots,n$, $y_j^\otimes, j = 1,2,\ldots,n$, it fulfills Eqs. (6.22) and (6.23), which could fulfill the system of inequalities shown in Eqs. (6.10) and (6.15) only when meeting nonnegativity conditions:

$$\left(x_1^{\otimes},x_2^{\otimes},\ldots,x_n^{\otimes},u_{\otimes}^{*}\right)\cdot\begin{pmatrix}[a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & 1^{\otimes}\\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & 1^{\otimes}\\ \vdots & \vdots & & \vdots & \vdots\\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & 1^{\otimes}\\ 1^{\otimes} & 1^{\otimes} & \cdots & 1^{\otimes} & 0^{\otimes}\end{pmatrix} \tag{6.24}$$

$$=(0^{\otimes},0^{\otimes},\ldots,0^{\otimes},1^{\otimes})$$

$$\begin{pmatrix}[a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & 1^{\otimes}\\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & 1^{\otimes}\\ \vdots & \vdots & & \vdots & \vdots\\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & 1^{\otimes}\\ 1^{\otimes} & 1^{\otimes} & \cdots & 1^{\otimes} & 0^{\otimes}\end{pmatrix}\cdot\begin{pmatrix}y_1^{\otimes}\\ y_2^{\otimes}\\ \vdots\\ y_n^{\otimes}\\ u_{\otimes}^{*}\end{pmatrix}=\begin{pmatrix}0^{\otimes}\\ 0^{\otimes}\\ \cdots\\ 0^{\otimes}\\ 0^{\otimes}\end{pmatrix} \tag{6.25}$$

$$(x_1^{\otimes},x_2^{\otimes},\ldots,x_n^{\otimes},u_{\otimes}^{*})=(0^{\otimes},0^{\otimes},\ldots,0^{\otimes},1^{\otimes})\cdot B^{-1}(\otimes)=(0^{\otimes},0^{\otimes},\ldots,0^{\otimes},1^{\otimes})$$

$$\begin{pmatrix}[c_{1,1},d_{1,1}] & [c_{1,2},d_{1,2}] & \cdots & [c_{1,n},d_{1,n}] & [c_{1,n+1},d_{1,n+1}]\\ [c_{2,1},d_{2,1}] & [c_{2,2},d_{2,2}] & \cdots & [c_{2,n},d_{2,n}] & [c_{2,n+1},d_{2,n+1}]\\ \vdots & \vdots & & \vdots & \vdots\\ [c_{n,1},d_{n,1}] & [c_{n,2},d_{n,2}] & \cdots & [c_{n,n},d_{n,n}] & [c_{n,n+1},d_{n,n+1}]\\ [c_{n+1,1},d_{n+1,1}] & [c_{n+1,2},d_{n+1,2}] & \cdots & [c_{n+1,n},d_{n+1,n}] & [c_{n+1,n+1},d_{n+1,n+1}]\end{pmatrix} \tag{6.26}$$

$$\begin{pmatrix}y_1^{\otimes}\\ y_2^{\otimes}\\ \vdots\\ y_n^{\otimes}\\ u_{\otimes}^{*}\end{pmatrix}=B^{-1}(\otimes)\cdot\begin{pmatrix}0^{\otimes}\\ 0^{\otimes}\\ \cdots\\ 0^{\otimes}\\ 1^{\otimes}\end{pmatrix}$$

$$=\begin{pmatrix}[c_{1,1},d_{1,1}] & [c_{1,2},d_{1,2}] & \cdots & [c_{1,n},d_{1,n}] & [c_{1,n+1},d_{1,n+1}]\\ [c_{2,1},d_{2,1}] & [c_{2,2},d_{2,2}] & \cdots & [c_{2,n},d_{2,n}] & [c_{2,n+1},d_{2,n+1}]\\ \vdots & \vdots & & \vdots & \vdots\\ [c_{n,1},d_{n,1}] & [c_{n,2},d_{n,2}] & \cdots & [c_{n,n},d_{n,n}] & [c_{n,n+1},d_{n,n+1}]\\ [c_{n+1,1},d_{n+1,1}] & [c_{n+1,2},d_{n+1,2}] & \cdots & [c_{n+1,n},d_{n+1,n}] & [c_{n+1,n+1},d_{n+1,n+1}]\end{pmatrix}\cdot\begin{pmatrix}0^{\otimes}\\ 0^{\otimes}\\ \cdots\\ 0^{\otimes}\\ 1^{\otimes}\end{pmatrix} \tag{6.27}$$

Therefore, when $x_i(\otimes), i = 1,2,\ldots,n,\ \ y_j(\otimes), j = 1,2,\ldots,n,$ Eqs. (6.24) and (6.25) meet the nonnegativity constraint, and the optimum grey game strategy of Players 1 and 2 in grey matrix game respectively are $x_1(\otimes) = [c_{n+1,1}, d_{n+1,1}]$, $x_2(\otimes) = [c_{n+1,2}, d_{n+1,2}], \ldots, x_n(\otimes) = [c_{n+1,n}, d_{n+1,n}]$ and $y_1(\otimes) = [c_{1,n+1}, d_{1,n+1}], y_2(\otimes) = [c_{2,n+1}, d_{2,n+1}], \ldots, y_n(\otimes) = [c_{n,n+1}, d_{n,n+1}]$, the grey game value of grey matrix game is

$$v^*(\otimes) = -u^*(\otimes) = -[c_{n+1,n+1}, d_{n+1,n+1}].$$

Example 6.4 In grey matrix game $G(\otimes) = \{S_1, S_2; A(\otimes)\}$, where the grey game profit and loss value matrix $A(\otimes)$ is shown by Eq. (6.8), solve the optimum grey game strategy of Players 1 and 2 $x_i(\otimes), i = 1,2,\ldots,n,\ \ y_j(\otimes), j = 1,2,\cdots,n$ in the grey game.

Solution: Using the same method as shown in Examples 6.2 and 6.3, solve the grey inverse matrix $\tilde{B}^{-1}(\otimes)$ of a combined grey expanding square matrix $\tilde{B}(\otimes)$, such as shown in Eqs. (6.28) and (6.29):

$$(B(\otimes), E(\otimes)) = \left(\begin{array}{ccc|ccc} 1 & 3+\gamma_{12} & 1 & 1^{\otimes} & 0 & 0 \\ 4+2\gamma_{21} & 2 & 1 & 0 & 1^{\otimes} & 0 \\ 1 & 1 & 0 & 0 & 0 & 1^{\otimes} \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3+\gamma_{12} & 1 & 1 & 0 & 0 \\ 0 & -(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}) & -(3+2\gamma_{21}) & -(4+2\gamma_{21}) & 1 & 0 \\ 0 & -2-\gamma_{12} & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} & -\frac{1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & \frac{1+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\ 0 & 1 & 0 & \frac{10+2\gamma_{12}+8\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & -\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & \frac{3+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \\ 0 & 0 & 1 & \frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} & \frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & -\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \end{array}\right) \tag{6.28}$$

$$\tilde{B}^{-1}(\otimes)=\begin{pmatrix} \dfrac{2+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} & -\dfrac{1}{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}} & \dfrac{1+\gamma_{12}}{4+\gamma_{12}+2\gamma_{21}} \\ \dfrac{10+2\gamma_{12}+8\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & -\dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{(4+\gamma_{12}+2\gamma_{21})\cdot(10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21})} & \dfrac{3+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \\ \dfrac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} & \dfrac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}} & -\dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}} \end{pmatrix} \tag{6.29}$$

According to Eq. (6.29), we can get the optimum grey game strategy and grey game value of Players 1 and 2 in the grey matrix game as shown In Eqs. (6.30), (6.31), and (6.32):

$$x_1^{\otimes}=\frac{2+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}=[0.40,0.67],x_2^{\otimes}=\frac{2+\gamma_{12}\cdot}{4+\gamma_{12}+2\gamma_{21}}=[0.33,0.60], \tag{6.30}$$

$$y_1^{\otimes}=\frac{19+7\gamma_{12}+12\gamma_{21}+3\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}=[4.75,6.83],\ y_2^{\otimes}=\frac{3+2\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}=[0.60,0.83] \tag{6.31}$$

$$v_{\otimes}^{*}=\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}=\left[\min\left(\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1},\right.$$

$$\left.\max\left(\frac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le1}\right]=[2.50,3.14] \tag{6.32}$$

6.1.5 Summary

This section defined the concept of a grey full-rank expanding square matrix for Players 1 and 2 in a grey matrix game, and proved that if the last row and right column in the grey inverse matrix of these grey full-rank expanding square matrix meet nonnegativity conditions, then the values of these grey elements of a grey inverse matrix correspond to each optimum grey game strategy and grey game value of Players 1 and 2. On the basis of the above, we studied further the combined grey full-rank expanding square matrix based on Players 1 and 2, and proved the following as well: if the last row and right column (excluding the grey elements in the right corner) in a grey inverse matrix of these combined grey full-rank expanding square matrices meet nonnegativity conditions, then the value of these grey elements of the grey inverse matrix correspond to each optimum grey game strategy and grey game value of Players 1 and 2.

6.2 Construction of an Analogous Grey Full-Rank Expanding Square Matrix and Its Necessary and Sufficient Conditions

During the process of a grey matrix game in reality, the grey full-rank expanding square matrix does not always exist. Generally speaking, a grey full-rank expanding square matrix does not exist in the following two occasions:

- when the grey profit and loss matrix is not a square matrix
- when the expanding square matrix of a grey profit and loss matrix is not of full rank

In this situation, how should we construct its grey expanding square matrix and make sure it is full rank? This section studies how to construct a grey full-rank expanding square matrix, including information regarding optimum solutions when the grey profit and loss matrix is not a square matrix. We also research the necessary and sufficient conditions for the existence of a grey full-rank expanding square matrix.

6.2.1 Construction of an Analogous Grey Full-Rank Expanding Square Matrix

Definition 6.4 Long row-dimensional and long column-dimensional grey matrix: A grey matrix $\tilde{A}(\otimes)_{m\times n}$, is called a long row-dimension grey matrix (or column grey matrix for short) if its row dimensions are more than its column dimensions ($m > n$), and long column-dimension grey matrix (or row-dimension grey matrix for short) if its column dimensions are more than its row dimensions ($n > m$).

Definition 6.5 Analogous grey expanding square matrix: There is a grey matrix $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix, namely, $m \neq n$.

If $\tilde{A}(\otimes)_{m\times n}$ is a long row-dimension grey matrix ($m > n$), we can consider n grey strategy vectors of Player 2 as column strategy vectors, then choose n grey vectors from m grey strategy vectors of Player 1 at random as row vectors. An $n\times n$ dimensional grey square matrix $\tilde{A}_i(\otimes)_{n\times n}, i = 1,2,\ldots$ analogous to the form of a grey profit and loss value matrix comes into shape. We can call the grey square matrix $\tilde{A}_i(\otimes)_{n\times n}, i = 1,2,\ldots$ the column-dimension grey profit and loss matrix of grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, expressed as $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$. The combined grey expanding square matrix constructed by $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$ is called column together grey expanding square matrix, column grey expanding square matrix for short, expressed as $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots$. If the square matrix is grey full rank, it is called a column grey full-rank expanding square matrix; otherwise, it is called grey rank-decreased.

If $\tilde{A}(\otimes)_{m\times n}$ is a long column-dimension grey matrix ($m<n$), we can consider m grey strategy vectors of Player 1 as row strategy vectors, then choose n grey vectors from m grey strategy vectors of Player 2 at random as column vectors. An $m\times m$ dimensional grey square matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ analogous to the form of a grey profit and loss value matrix comes into shape. We can call the grey square matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ the row-dimension grey profit and loss matrix of grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, expressed as $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$. The combined grey expanding square matrix constructed by $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ is called a row together grey expanding square matrix, row grey expanding square matrix for short, expressed as $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots$. If the square matrix is grey full rank, it is called row grey full-rank expanding square matrix; otherwise, it is called *grey rank-decreased*.

The column and row combined grey expanding square matrix are called analogous together grey expanding square matrix, analogous grey expanding square matrix for short. If they are grey full rank, we call them analogous grey full-rank expanding square matrix; otherwise, they are called *grey rank-decreased.*

Example 6.5 There is a grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix ($m\neq n$).

If $\tilde{A}(\otimes)_{m\times n}$ is a long column-dimension grey matrix ($m<n$), then according to Definition 6.5, we can list the row together grey expanding square matrix, such as Eq. (6.33). If $\tilde{A}(\otimes)_{m\times n}$ is a long row-dimension grey matrix ($m>n$), then according to Definition 6.5, we can list the column together grey expanding square matrix, such as Eq. (6.34) (where $i=1,2,\ldots$):

$$\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}=\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1m},b_{1m}] & 1 \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2m},b_{2m}] & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{m1},b_{m1}] & [a_{m2},b_{m2}] & \cdots & [a_{mm},b_{mm}] & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{6.33}$$

$$\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}=\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1n},b_{1n}] & 1 \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2n},b_{2n}] & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{n1},b_{n1}] & [a_{n2},b_{n2}] & \cdots & [a_{nn},b_{nn}] & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{6.34}$$

Theorem 6.5 The number of analogous grey expanding square matrices: In a grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, if $\tilde{A}(\otimes)_{m\times n}$ is a long row-dimension grey matrix ($m > n$), then we can construct a $C_m^n=\frac{m!}{n!(m-n)!}$ column grey expanding square matrix; if $\tilde{A}(\otimes)_{m\times n}$ is a long column-dimension grey matrix ($m < n$), then, we can construct a $C_n^m=\frac{n!}{m!(n-m)!}$ row grey expanding square matrix.

Proof: Generally, a grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix.

If $\tilde{A}(\otimes)_{m\times n}$ is a long row-dimension grey matrix ($m > n$), according to Definition 6.5, we can choose n vectors from m row vectors at random, together with n column vectors, to construct an $n\times n$ column-dimension grey profit and loss matrix $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$. According to the grey profit and loss matrix, we can list the corresponding analogous together grey full-rank expanding matrix $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots$. Because it is a combination problem, the number of analogous together grey full-rank expanding matrices that can be formed is a combination of C_m^n, namely, $C_m^n=\frac{m!}{n!(m-n)!}$.

If $\tilde{A}(\otimes)_{m\times n}$ is a long column-dimension grey matrix ($m < n$), we can list the corresponding analogous together grey full-rank expanding matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots$. Because it is a combination problem, the number of analogous together grey full-rank expanding matrices that can be formed is a combination of C_n^m, namely, $C_n^m=\frac{n!}{m!(n-m)!}$.

6.2.2 Necessary and Sufficient Conditions of Full-Rank for an Analogous Grey Expanding Square Matrix

Only when the inverse matrix of the analogous grey expanding square matrix exists can we solve the grey game conveniently according to the grey expanding square matrix of this kind. According to Definition 6.5, we can list the analogous grey expanding square matrix of the game. Does an inverse matrix of the analogous grey expanding square matrix exist? Are they grey full-ranked or not, and in what situation? Theorem 6.6 mainly solves this.

Theorem 6.6 The necessary and sufficient conditions of full-rank for an analogous grey expanding square matrix: In the grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, the necessary and sufficient conditions that the analogous grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots$ [such as Eqs. (6.33) and (6.34)] are grey full-rank expanding square matrices are that there are no two-row (column) vectors whose grey elements are correspondingly equal in the row- and column-dimension grey profit and loss matrices $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ and $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$.

Proof: Sufficient conditions: If there are no two-row (column) vectors whose grey elements are correspondingly equal in $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ and $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$, then we should prove that $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots$ are full-ranked.

Because the value of the determinant of grey matrix $\tilde{B}(\otimes)$ is the highest rank of grey subdeterminant not zero, the grey rank $(B)^{\mathrm{R}}$ of grey matrix $\tilde{B}(\otimes)$ is equal to the rank of row or column and determinant of $\tilde{B}(\otimes)$. We can infer that if analogous together grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$ are grey rank-decreased, the value of the subdeterminant of the highest rank of the square matrix ($m + 1$ and $n + 1$ rank determinant) is sure to be equal to 0.

According to the characters of determinants, if the value of ($m + 1$) and ($n + 1$) rank grey determinants of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$ are required to be 0, then among ($m + 1$) and ($n + 1$) rank grey determinants made up of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$ at least one of the three following situations is sure to exist:

Circumstance 1: Two rows (columns) of grey elements are all zero in the determinant.

Circumstance 2: Two rows (columns) of grey elements are in proportion correspondingly to the determinant, and the value of that proportion is equal.

Circumstance 3: Two rows (columns) of grey elements are completely equal correspondingly to the determinant.

In reality, Circumstance 1 obviously does not exist in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$. This is determined by the structure and character of the grey expanding square matrix, such as shown in Eqs. (6.33) and (6.34).

For Circumstance 2, if two rows (columns) of grey elements are in proportion correspondingly and have the same proportion in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$, then the proportion can only be 1, determined by the structure of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$. The two rows (columns) of grey elements are completely equal correspondingly in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$.

From the above, for $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$, Circumstances 2 and 3 belong to the same circumstance, namely, Circumstance 3.

We can infer, then, if the circumstance where in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$ two rows (columns) of grey elements are completely equal doesn't exist, then $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$ can't be grey rank-decreased expanding square matrices but are sure to be grey full ranked.

Secondly necessary condition: If $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}$, $i = 1,2,\ldots$ are grey full ranked, it is impossible that the three circumstances referred to in grey rank-decreased square matrices would exist. We know from the structure of the $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$ that the three circumstances don't exist in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots$, so there are no two rows (columns) of grey elements that are completely equal in the corresponding row- and column-dimension grey profit and loss matrices $\tilde{A}_i(\otimes)_{m\times m}, i = 1,2,\ldots$ and $\tilde{A}_i(\otimes)_{n\times n}, i = 1,2,\ldots$.

According to Theorem 6.6, in grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, even if $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix, we still can construct analogous together grey full-rank square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$ if there are no two row (column) vectors whose grey elements are completely correspondingly equal in row- and column-dimension grey profit and loss matrices $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ and $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$.

Example 6.6 There is a grey profit and loss matrix [such as Eq. (6.35)] for a given grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$. The analogous grey expanding square matrices are as shown in Eqs. (6.36), (6.37), and (6.38). Known from the three formulas, there are no two row (column) vectors whose grey elements are completely correspondingly equal in its column-dimension grey profit and loss matrix $\tilde{A}_i(\otimes)_{2\times 2}, i=1,2,3$. As a result, Eqs. (6.36), (6.37), and (6.38) are grey full ranked and are called analogous grey full-rank expanding square matrices.

$$\tilde{A}(\otimes)=\begin{pmatrix} 2 & 7 \\ 6 & [5,6] \\ 11 & 2 \end{pmatrix} \tag{6.35}$$

$$\tilde{B}_1(\otimes)=\begin{pmatrix} 2 & 7 & 1 \\ 6 & [5,6] & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.36}$$

$$\tilde{B}_2(\otimes)=\begin{pmatrix} 2 & 7 & 1 \\ 11 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.37}$$

$$\tilde{B}_3(\otimes)=\begin{pmatrix} 6 & [5,6] & 1 \\ 11 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.38}$$

6.2.3 Full-Rank Treatment of an Analogous Grey Rank-Decreased Expanding Square Matrix

Theorem 6.6 has proved that for a grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$, where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix if its row or column grey gain and loss matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ or $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$ can fulfill the necessary and sufficient conditions of analogous grey full rank expanding square matrix, we can construct analogous grey full rank expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$. If there are two row (column) vectors whose grey elements are correspondingly equal in $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ or $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots$, the necessary and sufficient conditions for the existence of an analogous grey full-rank expanding square matrix cannot be fulfilled. Thus, the analogous grey expanding square matrix is grey rank-decreased according to $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots$ or $\tilde{A}_i(\otimes)_{n\times n}, i=1,2,\ldots,$. Because the inverse of the grey rank-decreased square matrix doesn't exist, we should treat this kind of matrix as full rank in order to work out the grey optimum solutions of the grey game.

Theorem 6.7 Full-rank treatment of an analogous grey rank-decreased expanding square matrix: There is a grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$, where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix. If there exists a certain analogous grey expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ or $\tilde{B}_L(\otimes)_{(n+1)\times(n+1)}$ that is grey rank-decreased, we can reconstruct the corresponding analogous grey full-rank expanding square matrix $\tilde{B}_{Ki}(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots$ or $\tilde{B}_{Li}(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,$ which has lower dimensions, by eliminating the same game strategy in the corresponding row- or column-dimension grey profit and loss matrix $\tilde{A}_K(\otimes)_{m\times m}$ or $\tilde{A}_L(\otimes)_{n\times n}$.

Proof: There is a grey matrix game $\tilde{G}(\otimes)=\{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix. Suppose that a certain analogous grey expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$, which is grey rank-decreased, exists. (Other forms of the analogous grey rank-decreased expanding square matrix $\tilde{B}_L(\otimes)_{(n+1)\times(n+1)}$ are proved similarly.)

According to the structural character of the analogous grey expanding square matrix, if the grey expanding square matrix is grey rank-decreased, there definitely exist two row (or column) vectors whose grey elements are correspondingly equal in the square matrix. As shown in Eq. (6.39), we suppose the circumstance above: the grey vector of column s is equal to that of column t, and both of them are $\{[a_{1K}, b_{1K}], [a_{2K}, b_{2K}], \ldots, [a_{mK}, b_{mK}]\}^T$.

$$\tilde{B}_K(\otimes)_{(m+1)\times(m+1)} = \begin{pmatrix} [a_{11},b_{11}] & \cdots & [a_{1K},b_{1K}] & \cdots & [a_{1K},b_{1K}] & \cdots & [a_{1m},b_{1m}] & 1 \\ [a_{21},b_{21}] & \cdots & [a_{2K},b_{2K}] & \cdots & [a_{2K},b_{2K}] & \cdots & [a_{2m},b_{2m}] & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & \vdots \\ [a_{m1},b_{m1}] & \cdots & [a_{mK},b_{mK}] & \cdots & [a_{mK},b_{mK}] & \cdots & [a_{mm},b_{mm}] & 1 \\ 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{6.39}$$

$$\tilde{A}_K(\otimes)_{m\times m} = \begin{pmatrix} [a_{11},b_{11}] & \cdots & [a_{1K},b_{1K}] & \cdots & [a_{1K},b_{1K}] & \cdots & [a_{1m},b_{1m}] \\ [a_{21},b_{21}] & \cdots & [a_{2K},b_{2K}] & \cdots & [a_{2K},b_{2K}] & \cdots & [a_{2m},b_{2m}] \\ \vdots & & \vdots & & \vdots & & \vdots \\ [a_{m1},b_{m1}] & \cdots & [a_{mK},b_{mK}] & \cdots & [a_{mK},b_{mK}] & \cdots & [a_{mm},b_{mm}] \end{pmatrix} \tag{6.40}$$

According to the construction theorem of an analogous grey expending square matrix, since there are two columns of vectors whose grey elements are correspondingly equal in $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$, then the same situation definitely exists in the corresponding row-dimension grey profit and loss matrix $\tilde{A}_K(\otimes)_{m\times m}$, such as in Eq. (6.40), supposing that $\beta_s = \beta_t = \{[a_{1K},b_{1K}],[a_{2K},b_{2K}],\ldots,[a_{mK},b_{mK}]\}^T$.

Now we can get rid of one of the two same column vectors in Eq. (6.40), then we get am $m\times(m-1)$ long dimension grey profit and loss matrix, as shown in Eq. (6.41).

According to Eq. (6.41) and the methods of Definition 6.5 and Theorem 6.5, we can construct C_m^{m-1} analogous grey expanding square matrices whose dimensions are 1 less than Eq. (6.39), such as that shown in Eq. (6.42) (where $i=1,2,\cdots,C_m^{m-1}$). If the necessary and sufficient conditions of a grey full-rank expanding square matrix can be fulfilled for Eq. (6.41), then Eq. (6.42) should be grey full-ranked; otherwise, we can take similar steps. We can always get the grey full-rank expanding square matrix needed.

$$\tilde{A}_K(\otimes)^{(1)}_{m\times m-1} = \begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1K},b_{1K}] & \cdots & [a_{1m-1},b_{1m-1}] \\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2K},b_{2K}] & \cdots & [a_{2m-1},b_{2m-1}] \\ \vdots & \vdots & & \vdots & & \vdots \\ [a_{m1},b_{m1}] & [a_{m2},b_{m2}] & \cdots & [a_{mK},b_{mK}] & \cdots & [a_{mm-1},b_{mm-1}] \end{pmatrix} \tag{6.41}$$

$$\tilde{B}_{Ki}(\otimes)^{(1)}_{m\times m} = \begin{pmatrix} [a_{1,1},b_{1,1}] & \cdots & [a_{1,K},b_{1,K}] & \cdots & [a_{1,m-1},b_{1,m-1}] & 1 \\ [a_{2,1},b_{2,1}] & \cdots & [a_{2,K},b_{2,K}] & \cdots & [a_{2,m-1},b_{2,m-1}] & 1 \\ \vdots & & \vdots & & \vdots & \vdots \\ [a_{m-1,1},b_{m-1,1}] & \cdots & [a_{m-1,K},b_{m-1,K}] & \cdots & [a_{m-1,m-1},b_{m-1,m-1}] & 1 \\ 1 & \cdots & 1 & \cdots & 1 & 0 \end{pmatrix}, \quad i=1,2,\ldots,C_m^{m-1} \tag{6.42}$$

Theorem 6.8 Full-rank treatment does not change the previous grey optimum solution: There is a grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix. If a certain analogous grey rank-decreased expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ or $\tilde{B}_L(\otimes)_{(n+1)\times(n+1)}$ needs full-rank treatment according to Theorem 6.7, then the treatment doesn't change the grey optimum solution of the previous grey game.

Proof: There is a grey matrix game $\tilde{G}(\otimes)=\{S_1\ ,S_2\ ;\tilde{A}(\otimes)_{m\times n}\}$ where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix. Suppose that a certain analogous grey rank-decreased expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ needs full-rank treatment according to Theorem 6.7 (other forms of the analogous grey rank-decreased expanding square matrix $\tilde{B}_L\,(\otimes)_{(n+1)\times(n+1)}$ are proved similarly).

An analogous grey rank-decreased expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ is such as that shown in Eq. (6.39); here, for convenience, we suppose that the grey vector of column s is equal to that of column t, both of them are $\{[a_{1K},b_{1K}],[a_{2K},b_{2K}],\ldots,[a_{mK},b_{mK}]\}^T$, and other situations are proved similarly. The solutions contained in the grey game only have the following two circumstances:

Circumstance 1: $\tilde{B}_K(\otimes)_{(m\ 1)\times(m+1)}$ doesn't contain the information for grey optimum solutions. The information definitely exists in other analogous grey expanding square matrices. Thus, the grey optimum solutions of the previous grey game wouldn't be changed however we do full-rank treatment.

Circumstance 2: $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ contains the information for grey optimum solutions. According to the theorem for the structure and construction, there definitely exist two columns of vectors whose grey elements are equal correspondingly in the row-dimension grey profit and loss matrix $\tilde{A}_K(\otimes)_{m\times m}$ [such as shown in Eq. (6.40)], supposing $\beta_s=\beta_t=\{[a_{1K},b_{1K}],[a_{2K},b_{2K}],\ldots,[a_{mK},b_{mK}]\}^T$.

In Eq. (6.40), as for the strategy of grey game player, there are two identical strategies among M from the grey game strategies of Player 2, $\beta_s=\beta_t=\{[a_{1K},b_{1K}],[a_{2K},b_{2K}],\ldots,[a_{mK},b_{mK}]\}^T$.

According to the theorem of strategy superiority in a grey game, we can infer that the grey game strategy β_s (or β_t) of Player 2 is not better than strategy β_t(or β_s); if Player 2 has taken strategy β_t (or β_s), then he would not change β_t (or β_s) for β_s (or β_t) easily, because what he would do would not improve the situation (and of course, vice versa). Thus, during such processes of a grey game, we can optimize the grey game strategy β_s (or β_t) of Player 2 and make it superior, and then only β_s (or β_t) remains. We can get an $m\times(m-1)$ long row-dimension grey profit and loss matrix, such as Eq. (6.14). Equation (6.14) still keeps the information of grey optimum solutions of the previous grey game.

The structure of Theorem 6.7 proves the method of listing analogous grey full-rank expanding square matrix $\tilde{B}_{Ki}(\otimes)^{(1)}_{m\times m}$ $i=1,2,\ldots,C_m^{m-1}$ or $\tilde{B}_{Li}(\otimes)^{(1)}_{n\times n}, i=1,2,\ldots,C_n^{n-1}$ according to some analogous grey rank-decreased expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ or $\tilde{B}_L\,(\otimes)_{(n+1)\times(n+1)}$.

Theorem 6.8 has proved that full-rank treatment of the analogous grey rank-decreased expanding square matrix according to Theorem 6.7 would not change the optimum solutions of the previous grey game.

6.2.4 Summary

This section mainly designed the method of constructing the analogous grey expanding square matrix for nonsquare matrix $\tilde{A}(\otimes)_{m\times n}$ $(m \neq n)$, and proved the necessary and sufficient conditions for the existence of the analogous grey full-rank expanding square matrix. The section also designed the method of full-rank treatment, which can keep the information for the optimum solutions of the grey game reliable for an analogous grey rank-decreased expanding square matrix.

6.3 Compound Standard Grey Number and $\tilde{G}(\otimes)$'s Infeasible Solution, Feasible Solution, and Optimal Solution Matrix

In the process of finding a solution by the grey matrix method that is based on the game problem of a nonsquare grey profit and loss value matrix, we need to involve the concept of compound grey numbers and how to determine them, so we need to establish the concepts of an infeasible solution, feasible solution, and optimal solution matrix, and grasp their relationship and their ability to express the grey game optimal solution.

6.3.1 Concept of Compound Standard Grey Numbers and Their Determination

Definition 6.6 The first and second standard grey numbers: Given a grey number $\otimes_i = [a_i, b_i], (a_i \le b_i, i = 1,2,\ldots)$, if we change its left endpoint number in a standard way [shown in Eq. (6.43)], the standard number we get is called the first form of the grey number, or the first standard grey number for short. In it, a_i, $(b_i - a_i)\gamma_i^{(1)}$, and $\gamma_i^{(1)}$ are called the white part, grey part, and unit grey number (grey coefficient) respectively. If we change its left endpoint number in a standard way [shown in Eq. (6.44)], the standard number we get is called the second form of the grey number, or the second standard number in short, in which b_i, $-(b_i - a_i)\gamma_i^{(2)}$, and $\gamma_i^{(2)}$ are called the white part, grey part, and unit grey number (grey coefficient) respectively.

$$\otimes_i^{(1)} = [a_i, b_i] = a_i - a_i + [a_i, b_i] = a_i + [0, b_i - a_i] = a_i + (b_i - a_i)[0,1]$$
$$= a_i + (b_i - a_i)\gamma_i^{(1)}, (0 \le \gamma_i^{(1)} \le 1) \tag{6.43}$$

$$\otimes_i^{(2)} = [a_i, b_i] = b_i - b_i + [a_i, b_i] = b_i - [0, b_i - a_i] = b_i - (b_i - a_i)[0,1]$$
$$= b_i - (b_i - a_i)\gamma_i^{(2)}, (0 \le \gamma_i^{(2)} \le 1) \tag{6.44}$$

Any standard number can be transformed into the form of the first and second standard numbers. Although they differ in form, they are the same by nature, for they express the same grey number.

Theorem 6.9 The sum of the specific value of the first and second standard grey numbers is 1: Given a grey number $\otimes_i = [a_i, b_i], (a_i \le b_i, i = 1, 2, \ldots)$, if we express it in the form of the first grey number $(\otimes_i^1)$ and second standard grey number $(\otimes_i^2)$, in which $\gamma_i^{(1)}, (0 \le \gamma_i^{(1)} \le 1)$ and $\gamma_i^{(2)}, (0 \le \gamma_i^{(2)} \le 1)$ express the two obtained standard grey numbers respectively, then with regard to the same grey number, the sum of these two kinds of standard grey numbers is 1, namely, $\gamma_i^{(1)} + \gamma_i^{(2)} = 1$.

Proof: Given one grey number $\otimes_i = [a_i, b_i], (a_i \le b_i, i = 1, 2, \ldots)$, by Definition 6.6, express it in the form of the first and the second standard grey numbers, shown by Eqs. (6.43) and (6.44).

Equations (6.43) and (6.44) indicate the same grey number, so they are equal intrinsically regardless of their different forms, shown by Eq. (6.45):

$$\otimes_i^{(1)} = \otimes_i^{(2)}, a_i + (b_i - a_i)\gamma_i^{(1)} = b_i - (b_i - a_i)\gamma_i^{(2)}, \quad \gamma_i^{(1)} + \gamma_i^{(2)} = 1 \qquad (6.45)$$

From Theorem 6.9, the expression form of the same grey number with different standards differs in form, but they are equal intrinsically; we can illustrate this with Figure 6.1. Even though the two kinds of grey numbers are of no difference in expression, however, in the process of calculating the practical problems, we had better adopt one kind of standard grey number for the same problem in order to understand the problem and avert the confusion of the obtained numbers, especially to the same grey number.

As for the determination problem of a single standard grey number, we have had a detailed explanation in the theses "Grey Matrix Game Model Research (3) Based on Pure Strategy: Nonstandard Grey Matrix Game Process Analysis" and "Grey Matrix Game Model Research (2) Based on Grey Mix Strategy: Grey Surpass Plane Support Theorem and Two Important Grey Inequalities." Now we mainly research the determination problems of compound grey numbers that are calculated by a

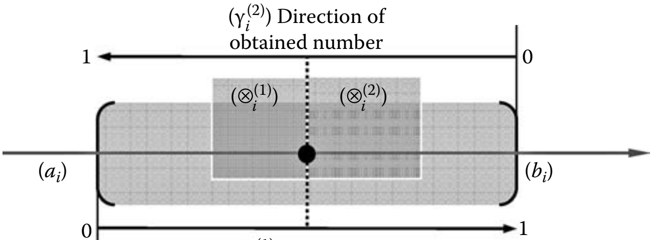


Figure 6.1 Relationship of first and second standard grey numbers.

single standard grey number through various kinds of mathematic calculations in the following text.

Definition 6.7 Compound standard grey number: If a grey number $(\otimes_k)$ is obtained through several times of addition, subtraction, multiplication, division, and other kinds of mathematic calculations, then we call the grey number $(\otimes_k)$ a compound standard grey number, or a compound grey number, expressed by $\otimes_k = f(\otimes_{ij})_{\gamma ij} = [\min_{\gamma ij} f(\bullet), \max_{\gamma ij} f(\bullet)]$, $(0 \le \gamma_{ij} \le 1, i, j = 1,2,\ldots)$.

From Definition 6.7, the grey number that is calculated by a single standard grey number through several times of mathematical calculations is a compound standard grey number. Also the grey number calculated by compound standard grey numbers or compound standard grey numbers and a single standard grey number for several times are also called compound grey numbers.

Compound standard grey numbers have the same expression as the first and the second standard grey numbers, having a white part, grey part, and grey coefficient. The only difference is that the grey coefficient of its grey part is compounded by several grey coefficients of a single grey number.

Definition 6.8 Determination of a compound grey number: Given any two compound grey numbers $\otimes_K = f(\otimes_{ij})_{\gamma_{ij}}, (0 \le \gamma_{ij} \le 1, i, j = 1,2,\ldots)$ and $\otimes_L = f(\otimes_{uv})_{\gamma_{uv}}$, $(0 \le \gamma_{uv} \le 1, u, v = 1,2,\ldots)$,

- If no matter what $\gamma_{ij}, (0 \le \gamma_{ij} \le 1, i, j = 1,2,\ldots)$ and γ_{uv},$(0 \le \gamma_{uv} \le 1$, $u, v = 1,2,\ldots)$ are the obtained numbers they both meet $\otimes_K \ge \otimes_L$, then we say that compound grey number $\otimes_K$ is not smaller than $\otimes_L$.
- If no matter what $\gamma_{ij}, (0 \le \gamma_{ij} \le 1, i, j = 1,2,\ldots)$ and γ_{uv}, $(0 \le \gamma_{uv}) \le 1, u, v = 1,2,\ldots)$ are the obtained numbers they can meet $\otimes_K < \otimes_L$, then we say that compound grey number $\otimes_K$ is smaller than $\otimes_L$.
- If no matter what value $\gamma_{ij}, (0 \le \gamma_{ij} \le 1, i, j = 1,2,\ldots)$ and γ_{uv}, $(0 \le \gamma_{uv}) \le 1$, $u, v = 1,2,\ldots)$ are the obtained numbers they can meet $\otimes_K = \otimes_L$, then we say that compound grey number $\otimes_K$ equals $\otimes_L$.

Otherwise, we cannot judge the compound grey numbers $\otimes_K$ and $\otimes_L$.

Example 6.7 Given two single standard grey numbers $\otimes_1 = a + (b-a)\gamma_1, (0 \le \gamma_1 \le 1)$ and $\otimes_2 = c + (d-c)\gamma_2, (0 \le \gamma_2 \le 1)$, then through addition and subtraction, the two standard grey numbers can be compounded to be compounded standard grey numbers $\otimes_{12}^{(1)}$ and $\otimes_{12}^{(2)}$ respectively.

$$\otimes_{12}^{(1)} = \otimes_1 + \otimes_2 = a + (b-a)\gamma_1 + c + (d-c)\gamma_2 = (a+c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1 + \frac{1}{(d-c)}\gamma_2\right) \tag{6.46}$$

$$\otimes_{12}^{(2)} = \otimes_1 - \otimes_2 = a + (b-a)\gamma_1 - c - (d-c)\gamma_2 = (a-c) + \frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1 - \frac{1}{(b-a)}\gamma_2\right) \tag{6.47}$$

If

$$\left\{(a+c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1+\frac{1}{(d-c)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le 1,i=1,2)} -$$

$$-\left\{(a-c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1-\frac{1}{(b-a)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le 1,i=1,2)} \ge 0,$$

then $\otimes_{12}^{(1)} \ge \otimes_{12}^{(2)}$.

If

$$\left\{(a+c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1+\frac{1}{(d-c)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le 1,i=1,2)} -$$

$$-\left\{(a-c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1-\frac{1}{(b-a)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le 1,i=1,2)} < 0,$$

then $\otimes_{12}^{(1)} < \otimes_{12}^{(2)}$.

If

$$\left\{(a+c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(b-a)}\gamma_1+\frac{1}{(d-c)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le 1,i=1,2)}$$

$$-\left\{(a-c)+\frac{1}{(b-a)(d-c)}\left(\frac{1}{(d-c)}\gamma_1-\frac{1}{(b-a)}\gamma_2\right)\right\}_{\gamma_i,(0\le\gamma_i\le 1,i=1,2)} = 0,$$

then $\otimes_{12}^{(1)} = \otimes_{12}^{(2)}$.

Or we say that $\otimes_{12}^{(1)}$ and $\otimes_{12}^{(2)}$ are compound grey numbers we cannot determine.

6.3.2 Concepts of Grey Optimal Solutions, Feasible Solutions, and Infeasible Solution Matrix

Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, where there is a profit and loss value matrix $A(\otimes)_{m\times n}$ in the form of a nonsquare matrix. According to the constructing method of analogous grey full-rank expanding square matrix in Section 6.2, we can construct several analogous grey full-rank expanding square matrices. What is the meaning of constructing these grey full-rank expanding square matrices and what is the relationship between the inverse matrix and the optimal grey game strategy and grey game value of Players 1 and 2? Here we first define the concepts of a grey optimal solution, a feasible solution, and an infeasible solution matrix.

Definition 6.9 Nonhomogeneous grey linear equation group of grey game: Given any two grey inequality groups of grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem [shown in Eqs. (6.48) and (6.49)], if we neglect the nonminus constraint of variables in the grey inequality groups and make the inequality into an equality, we obtain the game players' grey strategy variables as $x_1^{\prime\otimes}, x_2^{\prime\otimes}, \ldots, x_m^{\prime\otimes}$, $y_1^{\prime\otimes}, y_2^{\prime\otimes}, \ldots, y_n^{\prime\otimes}, -v_\otimes^\prime$; then we can make the inequality group into the form of a grey matrix equation, as shown in Eqs. (6.50) and (6.51), and we call them nonhomogeneous grey linear equation groups of a grey game (matrix equations).

$$\begin{cases} \sum_{i=1}^{n} a_{ij}^{\otimes} x_i^{\otimes} \geq v_\otimes^*, j=1,2,\ldots,n \\ x_i^{\otimes} = [Lx_i, Rx_i] \geq [0,0], i=1,2,\ldots,n \\ \sum_{i=1}^{n} x_i^{\otimes} = \sum_{i=1}^{n} [Lx_i, Rx_i] = [1,1]_{\gamma_i = c_i, 0 \leq c_i \leq 1, i=1,2,\ldots,n} = 1^{\otimes} \end{cases} \tag{6.48}$$

$$\begin{cases} \sum_{j=1}^{n} a_{ij}^{\otimes} y_j^{\otimes} \leq v_\otimes^*, i=1,2,\ldots,n \\ y_j^{\otimes} = [Ly_j, Ry_j] \geq [0,0], j=1,2,\ldots,n \\ \sum_{j=1}^{n} y_i^{\otimes} = \sum_{j=1}^{n} [Ly_j, Ry_j] = [1,1]_{\gamma_j = c_j, 0 \leq c_j \leq 1, j=1,2,\ldots,n} = 1^{\otimes} \end{cases} \tag{6.49}$$

$$\left(x_1^{\prime\otimes}, x_2^{\prime\otimes}, \ldots, x_m^{\prime\otimes}, -v_\otimes^\prime\right) \bullet \tilde{B}(\otimes) = (0,0,\ldots,0,1)$$

$$\left(x_1^{\prime\otimes}, x_2^{\prime\otimes}, \ldots, x_m^{\prime\otimes}, -v_\otimes^\prime\right) \bullet \begin{pmatrix} [a_{11}, b_{11}] & [a_{12}, b_{12}] & \cdots & [a_{1n}, b_{1n}] & 1 \\ [a_{21}, b_{21}] & [a_{22}, b_{22}] & \cdots & [a_{2n}, b_{2n}] & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{m1}, b_{m1}] & [a_{m2}, b_{m2}] & \cdots & [a_{mn}, b_{mn}] & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}^T \tag{6.50}$$

$$\tilde{B}(\otimes) \bullet \left(y_1^{\prime\otimes}, y_2^{\prime\otimes}, \ldots, y_n^{\prime\otimes}, -v_\otimes^\prime\right)^T = (0,0,\ldots,0,1)^T$$

$$\begin{pmatrix} [a_{11}, b_{11}] & [a_{12}, b_{12}] & \cdots & [a_{1n}, b_{1n}] & 1 \\ [a_{21}, b_{21}] & [a_{22}, b_{22}] & \cdots & [a_{2n}, b_{2n}] & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ [a_{m1}, b_{m1}] & [a_{m2}, b_{m2}] & \cdots & [a_{mn}, b_{mn}] & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \bullet \begin{pmatrix} y_1^{\prime\otimes} \\ y_2^{\prime\otimes} \\ \vdots \\ y_n^{\prime\otimes} \\ -v_\otimes^\prime \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \tag{6.51}$$

We select $x_1^{/\otimes}, x_2^{/\otimes}, \ldots, x_m^{/\otimes}, y_1^{/\otimes}, y_2^{/\otimes}, \ldots, y_n^{/\otimes}, -v_\otimes^{/}$ as the variables of the matrix equation rather than the variables of inequality groups of Eqs. (6.48) and (6.49) because of the structure constraint of the matrix equations. In matrix Eqs. (6.50) and (6.51), we did not take the grey game strategy variables' nonminus constraint into consideration.

Definition 6.10 $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$'s solution row and solution column: Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; A(\otimes)_{m\times m}\}$ problem, if the inverse matrix of its grey full-rank expanding square matrix exists, then we say its last row and the right column is Player 1's solution row (solution row in short) and Player 2's solution column respectively.

Definition 6.11 $\tilde{G}(\otimes)$'s infeasible solution, feasible solution, and the optimal solution matrix: Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times m}\}$ problem, where $\tilde{A}(\otimes)_{m\times m}$ is a square matrix, the problem's grey full-rank expanding matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$'s inverse matrix $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ exists.

If in the solution row and solution column of the grey inverse matrix $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ (the right down grey element of $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ excluded) and all the grey elements are larger than or equal to 0, then we call the grey full-rank expanding square matrix $\tilde{B}(\otimes)_{(m\pm 1)\times(m+1)}$ the grey feasible solution matrix of a grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; A(\otimes)_{m\times m}\}$ problem.

In the solution row and solution column of the grey inverse matrix $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ (the right down grey element of $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ excluded), if there exist grey elements that are smaller than 0, then we call the grey full-rank expanding square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ the grey infeasible solution matrix of a grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; A(\otimes)_{m\times m}\}$ problem.

Example 6.8 Given a grey matrix game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times m}\}$ where its profit and loss value matrix $\tilde{A}(\otimes)_{m\times m}$ is shown as Eq. (6.52), we can get its grey full-rank expanding square matrix, as shown by Eq. (6.53). Trying to solve the inverse matrix $\tilde{B}^{-1}(\otimes)$ of Eq. (6.53), we can get the solution (the grey element values of $\tilde{B}^{-1}(\otimes)$'s solution row and solution column) of a matrix equation that meets the grey full-rank expanding square matrix [see Eqs. (6.54) and (6.55)], as shown by Eq. (6.56):

$$\tilde{A}(\otimes) = \begin{pmatrix} 2 & 3 & 11 \\ 7 & 5 & [2,3] \\ 7 & 6 & 2 \end{pmatrix} \tag{6.52}$$

$$\tilde{B}(\otimes) = \begin{pmatrix} 2 & 3 & 11 & 1 \\ 7 & 5 & 2+\gamma_{23} & 1 \\ 7 & 6 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{6.53}$$

$$\left(x_1^{/\otimes},x_2^{/\otimes},x_3^{/\otimes},-v_\otimes^{/}\right)\begin{pmatrix} 2 & 3 & 11 & 1\\ 7 & 5 & 2+\gamma_{23} & 1\\ 7 & 6 & 2 & 1\\ 1 & 1 & 1 & 0 \end{pmatrix}=(0,0,0,1) \tag{6.54}$$

$$\begin{pmatrix} 2 & 3 & 11 & 1\\ 7 & 5 & 2+\gamma_{23} & 1\\ 7 & 6 & 2 & 1\\ 1 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} y_1^{/\otimes}\\ y_2^{/\otimes}\\ y_3^{/\otimes}\\ -v_\otimes^{/} \end{pmatrix}=\begin{pmatrix} 0\\ 0\\ 0\\ 1 \end{pmatrix} \tag{6.55}$$

$$\begin{aligned} &x_1^{/\otimes}=[0.3750,0.3751], x_2^{/\otimes}=[-0.2857,-0.2500],\\ &x_3^{/\otimes}=[0.8750,0.9286], y_1^{/\otimes}=[0.3750,0.6429],\\ &y_2^{/\otimes}=[0,0.3125], y_3^{/\otimes}=[0.3125,0.3571]; v_\otimes^{/}=[5.1250,5.2143]. \end{aligned} \tag{6.56}$$

From Eq. (6.56), the solution that satisfies matrix Eqs. (6.54) and (6.55) is not necessarily the game player's optimal grey strategy and grey game value of the grey game. This is mainly because we did not give the corresponding constraint of a nonminus requirement aiming at $x_i^{\otimes}, i=1,2,\ldots,m$ and $y_j^{\otimes}, j=1,2,\ldots,n$ in the matrix equation. Therefore, we can come to a conclusion that Eq. (6.53) is a grey infeasible solution matrix.

Theorem 6.10 There are negative grey elements in the solution row and solution column of an inverse matrix of an infeasible solution equation: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times m}\}$ problem, if the inverse matrix $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ of the infeasible solution matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ exists, then in the inverse matrix's solution row and solution column (the right down elements of $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ excluded), there must be one or more negative grey elements.

Proof: By applying the method of disproof to a given grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times m}\}$ problem, assume its infeasible solution square matrix is $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$, as shown in Eq. (6.57):

$$\tilde{B}(\otimes)_{(m+1)\times(m+1)}=\begin{pmatrix} [a_{11},b_{11}] & [a_{12},b_{12}] & \cdots & [a_{1m},b_{1m}] & 1\\ [a_{21},b_{21}] & [a_{22},b_{22}] & \cdots & [a_{2m},b_{2m}] & 1\\ \vdots & \vdots & & \vdots & \vdots\\ [a_{m1},b_{m1}] & [a_{m2},b_{m2}] & \cdots & [a_{mm},b_{mm}] & 1\\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix} \tag{6.57}$$

The inverse matrix of the infeasible solution square matrix is $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$, shown in Eq. (6.58), and there are no negative grey elements in its solution row and solution column.

$$\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$$

$$= \begin{pmatrix} [c_{1,1}, d_{11}] & [c_{1,2}, d_{1,2}] & \cdots & [c_{1,m}, d_{1,m}] & [c_{1,m+1}, d_{1,m+1}] \\ [c_{2,1}, d_{2,1}] & [c_{2,2}, d_{2,2}] & \cdots & [c_{2,m}, d_{2,m}] & [c_{2,m+1}, d_{2,m+1}] \\ \vdots & \vdots & & \vdots & \vdots \\ [c_{m,1}, d_{m,1}] & [c_{m,2}, d_{m,2}] & \cdots & [c_{m,m}, d_{m,m}] & [c_{m,m+1}, d_{m,m+1}] \\ [c_{m+1,1}, d_{m+1,1}] & [c_{m+1,2}, d_{m+1,2}] & \cdots & [c_{m+1,m}, d_{m+1,m}] & [c_{m+1,m+1}, d_{m+1,m+1}] \end{pmatrix} \tag{6.58}$$

Then, according to the assumption of Eq. (6.58), we can know from the conclusion of Section 6.1 that Eq. (6.58)'s solution row and solution column correspond to the solution of the grey matrix problem. According to Definition 6.10, Eq. (6.58) is not a grey infeasible solution matrix but the solution matrix. This contradicts the assumption. Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times m}\}$, in its grey infeasible solution square matrix's inverse matrix, there must be one or more negative grey elements in its solution row and solution column (the right down grey element values of $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ excluded). So the original proposition is right.

6.3.3 Sufficient and Necessary Condition of the Existence of a Grey Optimal Solution Square Matrix

After establishing the concepts of the optimal solution, feasible solution, and infeasible solution matrices, we will study their relationship further.

Theorem 6.11 The sufficient and necessary condition that the unique grey expanding square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the optimal solution square matrix: Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times m}\}$ problem, if its grey profit and loss value matrix $\tilde{A}(\otimes)_{m\times m}$ is a square matrix and its unique grey expanding square matrix is $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$, then the sufficient and necessary condition that $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the optimal solution square matrix is that $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the feasible square matrix.

Proof: First prove the sufficiency. Namely, if a certain feasible solution square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the unique expanding square matrix of grey game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times m}\}$, the task is to prove that the feasible square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is a grey optimal solution square matrix.

According to Definition 6.9, with the grey profit and loss value matrix of the grey matrix game $\tilde{A}(\otimes)_{m\times m}$, we can get two grey constraint inequality groups [as Eqs. (6.48) and (6.49) show] and their corresponding grey matrix equations [as Eqs. (6.50) and (6.51) show].

Because $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the feasible solution square matrix of $\tilde{A}(\otimes)_{m\times m}$, according to Definition 6.11, the inverse matrix $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$ of square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ exists. Then by Eqs. (6.50) and (6.51), we can get the solution of the matrix equation's unknown variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}\ y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_n^{\prime\otimes},-v_{\otimes}^{\prime}$. When $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is a feasible solution matrix, the solutions of matrix equation $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}\ y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_n^{\prime\otimes},-v$ are the grey numbers that are not smaller than 0; $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the unique grey expanding square matrix of $\tilde{A}(\otimes)_{m\times m}$, so the solution of $\tilde{B}^{-1}(\otimes)_{(m+1)\times(m+1)}$'s solution row and solution column is the grey optimal solution of the grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times m}\}$ problem.

Proof: Necessity: If a certain feasible solution square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the unique expanding square matrix of grey game problem $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times m}\}$, the task is to prove that if $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is a grey optimal solution square matrix, then it must be the feasible square matrix.

Apply the method of disproof, and assume $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the grey optimal solution square matrix, but it is not a feasible solution square matrix.

From Definition 6.11, if $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is not the feasible solution square matrix, then there must be negative grey elements in $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$'s solution row and solution column. Thus from Definition 6.11 for the optimal solution square matrix, $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is certainly not the optimal solution square matrix, and this is contradictory to the assumption.

So the assumption is incorrect, and original proposition is correct.

Theorem 6.12 The necessary condition that the optimal solution matrix of a non-square matrix exists: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, if $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix, then the necessary condition that a certain square matrix of its analogous grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$ is the optimal solution square matrix is that the square matrix is a feasible solution square matrix.

Proof: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, where $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix.

According to the construction definition related to analogous grey expanding square matrix in Section 6.2, if $\tilde{A}(\otimes)_{m\times n}$ is a long column- or long row-dimensional grey matrix, then we can get its analogous grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$. Here, we only take grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ for an example to prove the theorem; the proof of grey square matrix $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$ is similar.

Each analogous grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$ can be seen as a subsidiary game problem. As for every subsidiary game problem, we can regard its analogous grey expanding square matrix as unique. If each subsidiary game problem can obtain its corresponding subsidiary problem's local grey optimal solution, then according to the presumption of rational behavior in the grey game process for the long row-dimensional matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem, its grey optimal solution is the smallest of the local optimal solutions.

From this we can see that, in order to make the grey matrix $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem have a grey optimal solution, we should first make its subsidiary game problem $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$ have a grey optimal solution, and later make each subsidiary game problem find a grey optimal solution in its grey expanding square matrix. Then in terms of Theorem 6.11, the analogous grey expanding square matrix of the subsidiary game problem is unique, and the sufficient and necessary condition that the analogous grey expanding square matrix is the grey optimal solution is if it is a grey feasible square matrix.

In reality, if there is subsidiary game problem $\tilde{G}_K(\otimes) = \{S_{K1}, S_{K2}; \tilde{A}_K(\otimes)_{m\times m}\}$ that has a unique analogous grey expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ and its grey expanding square matrix is not a feasible solution square matrix, then according to Theorem 6.10, the grey infeasible solution square matrix's inverse matrix $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$ has negative grey elements in its solution row or solution column. Later we will prove that if grey elements exist, then the subsidiary grey game problem must have a zero strategy variable and redundant constraint equation. Having omitted these zero strategy variables and redundant constraints properly, we can transform the subsidiary grey game problem into a less dimensional subsidiary game problem $\tilde{B}_{Ks}(\otimes), s = 1,2,\ldots$ so as to make it have a unique analogous grey expanding square matrix.

The subsidiary grey game problem that has a grey infeasible solution matrix can always be transferred into a grey game problem of an analogous grey expanding square matrix that has a unique feasible solution. According to Theorem 6.11, if the subsidiary game problem has a unique analogous expanding square matrix, then the sufficient and necessary condition that the analogous grey expanding square matrix is the grey optimal solution matrix is that it is a grey feasible matrix.

6.3.4 Summary

This section defined standard grey numbers and compound standard grey numbers and their determination rules. We established the concepts of infeasible solutions, feasible solutions, and the optimal solution square matrix and proved the sufficient and necessary condition that $\tilde{A}(\otimes)_{m\times m}$'s unique grey expanding square matrix $\tilde{B}(\otimes)_{(m+1)\times(m+1)}$ is the optimal solution square matrix and the necessary condition of the existence of the optimal solution matrix. In this way, it paves the way for a grey matrix solving method of grey game problem.

6.4 $\tilde{G}(\otimes)$'s Redundant Constraint, Zero Strategy Variables, and Grey Matrix Solving Method

Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, its grey full-rank expanding square matrix's inverse matrix exists. However, if the grey elements of its solution row or solution column cannot guarantee the requirement of not being negative, then the grey elements of its row and column cannot be the solution of the grey game problem. This section researches the grey matrix solving method when we cannot get the game player's optimal grey game strategy and optimal grey game value of the grey matrix game directly if they take advantage of the grey full-rank expanding square matrix only.

6.4.1 *Zero Strategy Variables in an Infeasible Solution Square Matrix*

According to the inverse matrix of an infeasible solution matrix, we cannot find the grey optimal solution to the grey matrix game problem. How, then, do we deal with the problem? When given more general conditions, how can we solve the grey game problem according to an infeasible solution square matrix that consists of the information of grey game optimal solution?

Definition 6.12 $\tilde{G}(\otimes)$'s zero strategy variable: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, in the optimal grey mix strategies $S_1^{\otimes}=\{x_1^{\otimes},x_2^{\otimes},\ldots,x_m^{\otimes}\},S_2^{\otimes}=\{y_1^{\otimes},y_2^{\otimes},\ldots,y_n^{\otimes}\}$ of Players 1 and 2, one or more grey optimal strategy variables are zero, and we call these grey strategy variables the zero strategy variables.

Theorem 6.13 The zero strategy variable in an infeasible solution matrix: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem's analogous grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)},i=1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)},i=1,2,\ldots,C_m^n$, if one or more analogous grey expanding square matrices are infeasible solution square matrices, then there must be a zero strategy variable in the grey inequality group constraint that is represented by these infeasible solution square matrices.

Proof: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem where its analogous grey expanding square matrix is $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)},i=1,2,\ldots,C_n^m$ ($\tilde{B}_i(\otimes)_{(n+1)\times(n+1)},i=1,2,\ldots,C_m^n$ has a similar proof), if the $B_K(\otimes)_{(m+1)\times(m+1)}$ (or more) square matrix in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)},i=1,2,\ldots,C_n^m$ is an infeasible solution square matrix, then there must be a zero strategy variable in the grey inequality group constraint that is represented by the infeasible solution square matrix.

In the process of the grey matrix game, each analogous grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)},i=1,2,\ldots,C_n^m$ corresponds to one grey profit and loss value matrix $\tilde{A}_i(\otimes)_{m\times m},i=1,2,\ldots,C_n^m$; in fact, we can regard every grey game problem

$\tilde{G}_i(\otimes)=\{S_1,S_2;\tilde{A}_i(\otimes)_{m\times m}\}$, determined by $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots,C_n^m$, as the *i*th subsidiary game problem of grey game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$.

According to the sufficient and necessary condition that the unique grey expanding square matrix is the optimal solution square matrix in Section 6.3, as for the *i*th subsidiary game, if every $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ is a unique feasible solution square matrix, then every $\tilde{B}_i^{-1}(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ corresponds to the *i*th subsidiary game problem's optimal grey game solution; for the given grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, its optimal solution should be the optimal grey game strategy and game value after considering all the optimal grey game solutions of subsidiary game problems that are determined by $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$.

If $B_K(\otimes)_{(m+1)\times(m+1)}$ in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ is an infeasible solution square matrix, then there must be negative grey elements in $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$'s solution row and solution column; we cannot find the optimal solution of grey matrix game problem that corresponds to $\tilde{A}_K(\otimes)_{m\times m}$ in $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$ directly.

According to Definition 6.10, we know that the solutions in $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$'s solution row and solution column must correspond to the solution of variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes},-v'_\otimes$ in its grey matrix equation. However, in a matrix equation, because of its structure constraints, we overlook the nonminus requirement of grey strategy variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes},-v'_\otimes$ in these grey inequality groups. Therefore, the solution of variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes},-v'_\otimes$ of the matrix equation may not be the solution of variables of inequality groups; only when a matrix equation takes over the nonminus requirement of variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes},-v'_\otimes$ can the two have the same solution.

Here we can see that the solution determined by an infeasible square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ is only the solution of the matrix equation's variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes},-v'_\otimes$ rather than the solution of variables $x_1^{\otimes},x_2^{\otimes},\ldots,x_m^{\otimes}$, $y_1^{\otimes},y_2^{\otimes},\ldots,y_m^{\otimes},-v_\otimes^{*}$. In the matrix equation, variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes}$ cannot obtain proper values when among the interval [0,1] (the range of grey probability value); they must obtain negative values by Definition 6.10. However, strategy variables $x_1^{\otimes},x_2^{\otimes},\ldots,x_m^{\otimes}$, $y_1^{\otimes},y_2^{\otimes},\ldots,y_m^{\otimes}$ of a grey game problem can only obtain grey probability values, so in order to transform variables $x_1^{\prime\otimes},x_2^{\prime\otimes},\ldots,x_m^{\prime\otimes}$, $y_1^{\prime\otimes},y_2^{\prime\otimes},\ldots,y_m^{\prime\otimes}$ into a grey game problem's feasible solution, we must make those grey strategy variables that correspond to negative ones be zero. In order to find the optimal solution of a grey matrix game problem's correspondence to $\tilde{A}_K(\otimes)_{m\times m}$ in an infeasible solution square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$, we must obtain some variables from the grey inequality group's strategy variables $x_1^{\otimes},x_2^{\otimes},\ldots,x_m^{\otimes}$, $y_1^{\otimes},y_2^{\otimes},\ldots,y_m^{\otimes}$ that are determined by $\tilde{A}_K(\otimes)_{m\times m}$ as zero strategy variable.

This proves that if one or more analogous grey expanding square matrices are infeasible solution square matrices, then there must be zero strategy variables in the grey inequality group constraint that are represented by infeasible solution square matrix.

6.4.2 Redundant Constraint Equation and Zero Strategy Variables in a Nonsquare Matrix

Definition 6.13 $\tilde{G}(\otimes)$'s grey redundant constraint equation: Given any two grey inequality groups of a grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, after getting its grey optimal solution, if we put the solutions back to regional inequality groups, there will be one or more strict grey inequality constraint equations. We call these grey inequality constraint equations the grey redundant constraint equations.

Theorem 6.14 The redundant constraint equation and zero strategy variable of $\tilde{A}(\otimes)_{m\times n}$'s grey inequality groups: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem that has a unique optimal solution, if $\tilde{A}(\otimes)_{m\times n}$ is a long column-dimensional grey matrix, then there must be a redundant constraint equation in the grey inequality group constraint of Player 1 of the grey game problem, and Player 2's strategy variable has a zero strategy variable. If $\tilde{A}(\otimes)_{m\times n}$ is long row-dimensional grey matrix, Player 1's strategy variable has a zero strategy variable.

Proof: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, $\tilde{A}(\otimes)_{m\times n}$ is not a square matrix. According to $\tilde{A}(\otimes)_{m\times n}$, we can give the grey inequality constraint equation of the grey game, shown in Eqs. (6.59) and (6.60):

$$\begin{cases}\sum_{i=1}^{n} a_{ij}^{\otimes} x_i^{\otimes} \geq v_{\otimes}^{*}, j=1,2,\ldots,n \\ x_i^{\otimes}=[Lx_i,Rx_i]\geq[0,0], i=1,2,\ldots,n \\ \sum_{i=1}^{n} x_i^{\otimes}=\sum_{i=1}^{n}[Lx_i,Rx_i]=[1,1]_{\gamma_i=c_i,0\leq c_i\leq 1,i=1,2,\ldots,n}=1^{\otimes}\end{cases} \tag{6.59}$$

$$\begin{cases}\sum_{j=1}^{n} a_{ij}^{\otimes} y_j^{\otimes} \leq v_{\otimes}^{*}, i=1,2,\ldots,n \\ y_j^{\otimes}=[Ly_j,Ry_j]\geq[0,0], j=1,2,\ldots,n \\ \sum_{j=1}^{n} y_j^{\otimes}=\sum_{j=1}^{n}[Ly_j,Ry_j]=[1,1]_{\gamma_j=c_j,0\leq c_j\leq 1,j=1,2,\ldots,n}=1^{\otimes}\end{cases} \tag{6.60}$$

We will first prove the situation when $\tilde{A}(\otimes)_{m\times n}$ (where $n > m$) is a long column-dimensional grey matrix. According to $\tilde{A}(\otimes)_{m\times n}$ ($n > m$), we can write its row-dimensional expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$. Based on Theorem 6.12, we can solve every $\tilde{A}_i(\otimes)_{m\times m}, i = 1,2,\ldots,C_n^m$'s subsidiary game problem corresponding to $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$, and thus we can find the grey game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$'s grey optimal solution; for the grey game problem that has a unique optimal solution, we can assume that the grey optimal solution exists in the *k*th row in the grey expanding square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$; that is to say, it can be found in the grey inequality groups (*m* groups in all) that are determined by the *k*th row in the grey profit and loss square matrix $\tilde{A}_K(\otimes)_{m\times m}$.

We know that, given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem, if $\tilde{A}(\otimes)_{m\times n}$ ($n > m$) is a long column-dimensional grey matrix, then Player 1 has n($n > m$) constraint grey inequality groups. According to Theorem 6.12, Player 1's rational behavior makes him obtain a group of grey game strategies $x_1^{\otimes}, x_2^{\otimes}, \ldots, x_m^{\otimes}$, and its grey game value $v_{\otimes}^{*}$ is the minimum of the grey game value determined by grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$ (determined by $\tilde{B}_K(\otimes)_{(m+1)\times(m\pm1)}$'s inverse matrix $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$).

Now if $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ is the optimal solution square matrix, then in $\tilde{A}_K(\otimes)_{m\times m}$'s grey subsidiary game there is no redundant constraint equation. The *m* inequalities of Players 1 and 2 determine the optimal solution of their respective grey games. However, the number of Player 1's grey constraint inequalities determined by $\tilde{A}(\otimes)_{m\times n}$ ($n > m$) is *n*; thus for Player 1, the *n* inequalities have only used *m* constraint grey game optimal solutions, and $n - m > 0$ have not been used.

From the obtained numbers of a game player's rational behavior, Player 1's $n - m > 0$ grey inequalities that have not been used to constitute grey game's optimal inequalities are all redundant constraint equations.

In the grey matrix game that is determined by a long column-dimensional grey matrix $\tilde{A}(\otimes)_{m\times n}$ ($n > m$), the number of Player 2's strategy variable is $n + 1$, namely, $y_1^{\otimes}, y_2^{\otimes}, \ldots, y_n^{\otimes}, v_{\otimes}^{*}$. However, in the grey game, the number of constraint grey inequalities that determine Player 2's optimal grey game strategy is only $m + 1$; we can determine Player 2's optimal grey game solution only by these $m + 1$ grey inequalities (*m* grey strategy variables and one grey game value variable, $y_1^{\otimes}, y_2^{\otimes}, \ldots, y_m^{\otimes}, v_{\otimes}^{*}$); so the $n - m > 0$ strategy variables that have not obtained values in Player 2's $m + 1$ grey inequalities obtain 0.

From this we can see that in the grey matrix game determined by a long column-dimensional grey matrix $\tilde{A}(\otimes)_{m\times n}$ ($n > m$), where a unique solution exists, Player 2 has $n - m > 0$ strategy variables at least.

Similarly we can prove that when $\tilde{A}(\otimes)_{m\times n}$ ($n > m$), where a unique solution exists, it is a long row-dimensional grey matrix; in grey inequality group's constraint of Player 2, there must be $m - n > 0$ redundant constraint equations, and Player 1's strategy variables have $m - n > 0$ zero strategy variables at least.

6.4.3 Optimal Grey Game Solution in a Nonsquare Matrix

Theorem 6.15 The optimal grey game solution in a nonsquare matrix: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, if $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long column-dimensional grey matrix, then the grey game's optimal game value must be the minimum of a grey game value determined by all the rows' grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$. If $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long row-dimensional grey matrix, then the grey game's optimal game value must be the maximum of a grey game value determined by all the columns' grey expanding square matrix $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$.

Proof: Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, we first consider the situation when analogous grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ and $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$ are both feasible solution square matrices (here, we only take $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ for an example; the proof is similar for $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$).

When $\tilde{A}(\otimes)_{m\times n}(n > m)$ is a long column-dimensional grey matrix, in terms of Definition 6.5, we can take Player 1's *m* grey strategy vectors as row strategy vectors, and then choose *m* grey vectors from Player 2's *n* grey strategy vectors arbitrarily as column vectors. In this way, an $m \times m$ dimensional grey square matrix that is similar in form with a grey profit and loss value matrix is formed; we use the row-dimensional grey profit and loss value matrix $\tilde{A}_i(\otimes)_{m\times m}, i=1,2,\ldots,C_n^m$ to construct C_n^m row grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$.

From the perspective of grey game strategy, in the grey matrix game process, both players adopt rational behavior; Player 1 applies *m* grey strategies to play the game with Player 2's *n* grey strategies. Because $m>n$, Player 2 can only use all his *n* strategies to play the game with Player 1. He can select from the *n* strategies' probability adoption appropriately, while Player 1 can select *n* more advantageous strategies from his *m* grey strategies. Player 1 can therefore select from the *m* strategies' probability adoption more advantageously.

The inevitable result of Player 1 and 2's grey game strategy choice is that Player 1 selects *n* strategies from *m* ones arbitrarily and based on the combinations of the plan, he selects one that produces the minimal grey game value in the game process with Player 2's *n* strategies; doing so accords with game players' rational behaviors.

We know that row grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ includes in the long column-dimensional grey matrix $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ game all the possible strategy combinations that Player 2 may adopt to play the game with Player 1; what strategy Player 2 will select to play the game with Player 1's *m* grey strategies is completely determined by both sides' rational behaviors.

So the optimal grey game solution of the grey game has two sides: the optimal grey probability of Player 1's rational choosing *m* grey strategies $x_1^{\otimes}, x_2^{\otimes}, \ldots, x_m^{\otimes}$ is most $v_{\otimes}^{*}$ when in the worst situation, and the optimal grey probability of Player

2's rational choosing m strategies from m grey strategies $y_1^{\otimes}, y_2^{\otimes}, \ldots, y_n^{\otimes}$ is least $v_{\otimes}^{*}$ when in the worst situation.

From this we can see that in the game process when $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long column-dimensional matrix, the optimal grey game value of the grey game must be the smallest $v_{\otimes}^{*}$ of the grey game values that are determined by all the rows' grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$. The grey game strategy that determines the optimal grey game value $v_{\otimes}^{*}$ of both Players 1 and 2 is the optimal grey game strategy.

For the grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ $(n \neq m)$ problem, according to the dimensions of its rows and columns, we can divide it into several subsidiary grey game problems:

- If $n > m$, then divide it into C_n^m subsidiary grey game problems of $\tilde{A}_i(\otimes)_{m\times m}, i = 1,2,\ldots,C_n^m$.
- If $n < m$, then divide it into C_m^n subsidiary grey game problems of $\tilde{A}_i(\otimes)_{m\times m}, i = 1,2,\ldots,C_n^m$.
- The grey optimal solution of the grey game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ $(n \neq m)$ problem is the whole grey optimal solution after considering that each subsidiary grey game problem's solution as the local grey optimal solution.

When $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long row-dimensional grey matrix, similarly we can prove that if $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long row-dimensional grey matrix, then the optimal grey game value of the grey game must be the maximum $v_{\otimes}^{*}$ of the grey game values that are determined by all the columns' grey expanding square matrices $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots,C_m^n$ (they are all feasible solution square matrices). Meanwhile, the grey game strategy of both Players 1 and 2 that determines the optimal grey game value $v_{\otimes}^{*}$ is the optimal grey game strategy.

Secondly, we discuss the situation in $\tilde{A}(\otimes)_{m\times n}$'s analogous grey expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1,2,\ldots,C_m^n$, where there exist one or more analogous grey expanding square matrices $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ or $\tilde{B}_L(\otimes)_{(n+1)\times(n+1)}$ that are not feasible solution square matrices (here, we only take $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ for an example; the proof for $\tilde{B}_L(\otimes)_{(n+1)\times(n+1)}$ is similar).

Because $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$ is not a feasible solution, we cannot find the solution of each subsidiary grey game problem in the inverse matrix of all the analogous grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}$, $i = 1,2,\ldots,C_n^m$ directly. Therefore, we cannot find an optimal solution for the grey game problem $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ $(n \neq m)$.

According to Theorem 6.14, for an infeasible square matrix $\tilde{B}_K(\otimes)_{(m+1)\times(m+1)}$, in $\tilde{B}_K^{-1}(\otimes)_{(m+1)\times(m+1)}$, if there are negative grey elements in the solution row and solution column of the game player matrix, then in the subsidiary grey game

$\tilde{G}_K(\otimes) = \{S_{K1}, S_{K2}; \tilde{A}_K(\otimes)_{m\times m}\}$, the game players' strategy variables must be zero variables.

Assuming that we remove s zero strategy variables from Player 1's m strategies (other situations of removing the zero strategy variables are similar), this equals removing s rows of grey elements. Thus, we transform a subsidiary game problem of $\tilde{A}_K(\otimes)_{m\times m}$ in the form of a square matrix into a long column-dimensional profit and loss value matrix $\tilde{A}_{Ks}(\otimes)_{(m-s)\times m}$ subsidiary problem; its game process and obtaining the grey game optimal solutions are similar to the above situation. According to this method, we can finally find the infeasible solution square matrix's local optimal solution.

After gaining the local optimal solution of these feasible and infeasible solution square matrices, we can obtain the optimal grey game strategy and grey game value of the grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem.

Similarly we can prove that if $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long column-dimensional grey matrix, then the optimal grey game value of the grey game must be the minimum $v^*_\otimes$ of the grey game values that are determined by all the rows' grey expanding square matrices $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i = 1, 2, \ldots, C_n^m$, and if $\tilde{A}(\otimes)_{m\times n}$ $(n > m)$ is a long row-dimensional grey matrix, then the optimal grey game value of the grey game must be the maximum of the grey game values that are determined by all the columns' grey expanding square matrices $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i = 1, 2, \ldots, C_m^n$

The proof of Theorem 6.15 is constructive; the proposition provides a grey matrix method that can solve the grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ $(n \neq m)$ problem.

6.4.4 $\tilde{A}(\otimes)_{m\times n}$'s Inferior Strategy and Its Redundant Constraint and Zero Strategy Variable

Definition 6.14 $\tilde{G}(\otimes)$'s inferior grey strategy: Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem, if any game player's grey game strategy is surpassed by any other better strategy, then we call that game player's grey game strategy an inferior strategy.

Theorem 6.16 Inferior strategy's zero strategy variable and redundant constraint: Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem, if a grey game strategy of the game player is an inferior strategy, then the game player's corresponding strategy's grey strategy variables must be zero strategy ones.

Proof: Given any grey matrix game $\tilde{G}(\otimes) = \{S_1, S_2; \tilde{A}(\otimes)_{m\times n}\}$ problem, we first discuss the situation when $\tilde{A}(\otimes)_{m\times n}$ is a long column-dimensional grey matrix, and assume the Hth column grey vector in $\tilde{A}(\otimes)_{m\times n}$ is Player 2's inferior grey strategy vector [shown by Eq. (6.61); the discussion of Player 1's inferior grey strategy is similar]. According to Eq. (6.61), we can get Player 1's constraint grey inequality group, illustrated in Eq. (6.62):

$$\tilde{A}(\otimes)_{m\times n}=\begin{pmatrix} [a_{11},b_{11}] & \cdots & [a_{1H},b_{1H}] & \cdots & [a_{1n},b_{1n}] \\ [a_{21},b_{21}] & \cdots & [a_{2H},b_{2H}] & \cdots & [a_{2n},b_{2n}] \\ \vdots & & \vdots & & \vdots \\ [a_{m1},b_{m1}] & \cdots & [a_{mH},b_{mH}] & \cdots & [a_{mn},b_{mn}] \end{pmatrix} \tag{6.61}$$

$$\begin{cases} \sum_{i=1}^{n} a_{ij}^{\otimes} x_i^{\otimes} \geq v_{\otimes}^{*}, j=1,2,\ldots,n \\ x_i^{\otimes}=[Lx_i,Rx_i]\geq[0,0], i=1,2,\ldots,n \\ \sum_{i=1}^{n} x_i^{\otimes} \sum_{i=1}^{n}[Lx_i,Rx_i]=[1,1]_{\gamma_i=c_i,0\leq c_i\leq 1,i=1,2,\ldots,n}=1^{\otimes} \end{cases} \tag{6.62}$$

As for Player 2, because the *H*th grey strategy is an inferior strategy, according to the game player's rational behavior in the grey game process, Player 2 will not adopt the inferior grey game strategy, and the strategy variable corresponding to the inferior strategy should obtain 0—namely, the zero strategy variable.

From Eq. (6.61), the *H*th column grey strategy vector is Player 2's inferior strategy vector. Then each grey element in the *H*th column grey vector is not necessarily smaller than that of one (or any) column grey strategy vectors that determine Player 2's grey optimal strategy. So as for Player 1, in the grey inequality determined by the *H*th column grey strategy vector, if we put Player 1's grey optimal strategy variable value and optimal grey game value in it, then it must be the strict inequality.

According to Theorem 6.15, Player 1's optimal grey game value is determined by the grey profit and loss value matrix $\tilde{A}(\otimes)'_{m\times(n-1)}$ that had removed the *H*th column grey strategy vector. In detail, $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$'s optimal grey game value is $v_{\otimes}^{*}$, the smallest of the local optimal solutions of subsidiary grey game problems that are determined by $\tilde{A}(\otimes)'_{m\times(n-1)}$, while in the inverse matrix of the analogous grey expanding square matrix corresponding to $v_{\otimes}^{*}$, the grey elements in the solution row and solution column are the game player's optimal grey strategy. Because of the game players' rational behaviors, their choices of grey game strategies must make Player 2 establish all the grey game strategy's numbers, which obtains the grey matrix equation corresponding to the optimal solution matrix. By Theorem 6.14, the grey inequality's constraint equation of Player 1 determined by the *H*th column grey strategy vector is a redundant constraint equation.

6.4.5 $\tilde{G}(\otimes)$'s Grey Matrix Method Solving Steps

We can conclude grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem's grey matrix method solving process as follows:

1. Examine and omit the inferior strategy variables for Players 1 and 2 in the grey profit and loss value matrix $\tilde{A}(\otimes)_{m\times n}$.
2. Construct $\tilde{A}(\otimes)_{m\times n}$'s analogous grey full-rank expanding square matrix $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$.
3. Solve the grey inverse matrix $\tilde{B}_i^{-1}(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ or $\tilde{B}_i^{-1}(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$ of $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}$, $i=1,2,\ldots,C_m^n$.
4. If $(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$) are all feasible solution square matrices, then in all the solution rows and solution columns of grey inverse matrices $\tilde{B}_i^{-1}(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ (or $\tilde{B}_i^{-1}(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$), the minimum (or the maximum) of its corresponding grey game value $v_\otimes^*$ is the optimal grey game value of the grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem; the grey elements of the solution matrix's solution rows and solution columns for the grey game value exist, corresponding to the game player's optimal grey game strategy.
5. If one (or any) square matrix in $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$) is an infeasible solution square matrix, remove the game player's zero strategy variables, reconstruct the subsidiary grey game problem's analogous grey expanding square matrix, and try to solve the optimal solution of the subsidiary grey game problem.
6. After getting the local optimal solution of the subsidiary grey game problem that was determined by all the analogous grey expanding square matrices of the grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem, try to find the minimum (or the maximum) $v_\otimes^*$ of the local optimal solution of the subsidiary grey game problem determined by all $\tilde{B}_i(\otimes)_{(m+1)\times(m+1)}, i=1,2,\ldots,C_n^m$ (or $\tilde{B}_i(\otimes)_{(n+1)\times(n+1)}, i=1,2,\ldots,C_m^n$) as the grey optimal solution of the grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem. The grey elements of the solution matrix's solution row and solution column that exist in the grey game value correspond to the game player's optimal grey game strategy.

Example 6.9 Given any grey matrix game $\tilde{G}(\otimes)=\{S_1,S_2;\tilde{A}(\otimes)_{m\times n}\}$ problem's grey profit and loss value matrix shown by Eq. (6.63), the grey mix strategy of Players 1 and 2 is $S_1^\otimes=\{x_1^\otimes,x_2^\otimes\}, S_2^\otimes=\{y_1^\otimes,y_2^\otimes,y_3^\otimes\}$ respectively. Try to disguise the grey redundant constraint equation and zero strategy variable.

$$\tilde{A}(\otimes)=\begin{pmatrix} [1,2] & 3 & 11 \\ 7 & 5 & [2,4] \end{pmatrix} \tag{6.63}$$

$$\begin{cases} (1+\gamma_{11})x_1^{\otimes} + 7x_2^{\otimes} \geq v_{\otimes}^{*} & 6.64.1 \\ 3x_1^{\otimes} + 5x_2^{\otimes} \geq v_{\otimes}^{*} & 6.64.2 \\ 11x_1^{\otimes} + (2+2\gamma_{23})x_2^{\otimes} \geq v_{\otimes}^{*} & 6.64.3 \\ x_1^{\otimes} + x_2^{\otimes} = 1^{\otimes} & 6.64.4 \end{cases} \tag{6.64}$$

$$\begin{cases} (1+\gamma_{11})y_1^{\otimes} + 3y_2^{\otimes} + 11y_3^{\otimes} \leq v_{\otimes}^{*} \\ 7y_1^{\otimes} + 5y_2^{\otimes} + (2+2\gamma_{23})y_3^{\otimes} \leq v_{\otimes}^{*} \\ y_1^{\otimes} + y_2^{\otimes} + y_3^{\otimes} = 1^{\otimes} \end{cases} \tag{6.65}$$

According to Eq. (6.63), we can write the grey game's grey inequality group [as Eqs. (6.64) and (6.65) show], and the analogous expanding square matrix shown by Eqs. (6.66), (6.67), and (6.68):

$$\tilde{B}_1(\otimes) = \begin{pmatrix} [1,2] & 3 & 1 \\ 7 & 5 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.66}$$

$$\tilde{B}_2(\otimes) = \begin{pmatrix} [1,2] & 11 & 1 \\ 7 & [2,4] & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.67}$$

$$\tilde{B}_3(\otimes) = \begin{pmatrix} 3 & 11 & 1 \\ 5 & [2,4] & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{6.68}$$

$$\begin{aligned} x_1^{\otimes} &= \frac{2\gamma_{23}-3}{2\gamma_{23}-11} = \left[\frac{1}{9}, \frac{3}{11}\right] && 69.1 \\ x_2^{\otimes} &= \frac{-8}{2\gamma_{23}-11} = \left[\frac{8}{11}, \frac{3}{9}\right] && 69.2 \\ v_{\otimes}^{*} &= \frac{-49+6\gamma_{23}}{2\gamma_{23}-11} = \left[\frac{49}{11}, \frac{43}{9}\right] && 69.3 \end{aligned} \tag{6.69}$$

$$\begin{aligned} &(1+\gamma_{11})x_1^{\otimes} + 7x_2^{\otimes} = (1+\gamma_{11})\frac{2\gamma_{23}-3}{2\gamma_{23}-11} - \frac{56}{2\gamma_{23}-11} \\ &= \frac{-56+2\gamma_{23}-3+2\gamma_{11}\gamma_{23}-3\gamma_{11}}{2\gamma_{23}-11} = \left[\frac{59}{11}, \frac{62}{9}\right]_{\gamma_{11},\gamma_{23},0\leq\gamma_{11},\gamma_{23}\leq 1} \\ &> v_{\otimes}^{*} = \frac{-49+6\gamma_{23}}{2\gamma_{23}-11} = \left[\frac{49}{11}, \frac{43}{9}\right]_{\gamma_{11},\gamma_{23},0\leq\gamma_{11},\gamma_{23}\leq 1} \end{aligned} \tag{6.70}$$

$$\begin{aligned} & y_1^{\otimes} = 0 && 6.71.1 \\ & y_2^{\otimes} = \frac{2\gamma_{23} - 9}{2\gamma_{23} - 11} = \left[\frac{7}{9}, \frac{9}{11}\right] && 6.71.2 \\ & y_3^{\otimes} = \frac{-2}{2\gamma_{23} - 11} = \left[\frac{2}{11}, \frac{2}{9}\right] && 6.71.3 \end{aligned} \qquad (6.71)$$

Through the solving process of analogous grey expanding square matrix Eqs. (6.66), (6.67), and (6.68), we can get the solution as shown in Eq. (6.69). Put Eqs. (6.69.1) and (6.69.2) into Eq. (6.64.1), we know that Eq. (6.64.1) is a strict inequality, shown by Eq. (6.70). We can see from this that Eq. (6.64.1) is a redundant constraint equation.

Similarly, we can get the optimal grey game strategy of Player 2, shown by Eq. (6.71). From Eq. (6.71) we know that $y_1^{\otimes}$ is the zero strategy variable of Player 2.

6.4.6 Summary

This section has defined the zero strategy variable and redundant constraint equation, and we also have studied the relationship of zero strategy variables in an infeasible solution square matrix, a redundant constraint equation, and a zero strategy variable in a nonsquare matrix $\tilde{A}(\otimes)_{m\times n}$. On this basis, we designed the basic steps of solving grey game problem by grey matrix method.

Chapter 7

Potential Optimal Strategy Solution's Venture and Control Problems Based on the Grey Interval Number Matrix Game

7.1 Study of the Venture Problem of a Potential Optimal Pure Strategy Solution for the Grey Interval Number Matrix Game

7.1.1 Optimal Potential Pure Strategy Solution of a Grey Interval Number Matrix Game

In order to resolve the optimal pure strategy problem of the grey matrix game on the condition that inner number distribution information is unknown, it is necessary to find a way to measure the grey numbers of $A(\otimes)$. Considering that people without clear inner distribution information of interval numbers might compare interval grey numbers in a way of potential comparison, here we make use of the conception of grey potential to compare the frames of interval grey numbers.

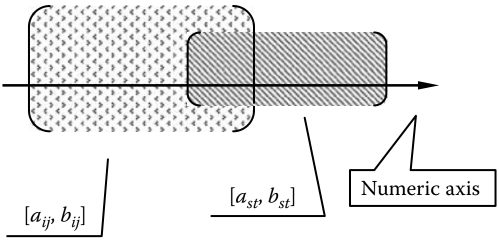


Figure 7.1 Relationship between interval grey numbers.

Definition 7.1 Conceptions of superior, inferior, and equipollent potential degrees: For a random two interval grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$, $\otimes_{st} \in [a_{st}, b_{ij}]$ and $b_{st} \geq b_{ij} \geq a_{st}$ (as shown in Figure 7.1), according to the potential of the endpoints of two grey numbers in the axis, we can divide the sum set range $\otimes_{ij} \bigcup \otimes_{st}$ into three parts, $[a_{ij}, a_{st}]$, $[a_{st}, b_{ij}]$, and $[b_{ij}, b_{st}]$, where $[a_{st}, b_{ij}]$ is the intersection range $\otimes_{ij} \bigcap \otimes_{st}$ between the two interval numbers.

1. We call the intersection range $\otimes_{ij} \bigcap \otimes_{st}$ the equipollent potential zone; $EPD_{ij \to st} = \frac{b_{ij}-a_{st}}{b_{ij}-a_{ij}} \geq 0$ is the equipollent potential degree of interval number $\otimes_{ij}$ compared to $\otimes_{st}$ (where $ij \to st$ means that $\otimes_{ij}$ is compared to $\otimes_{st}$), and $EPD_{st \to ij} = \frac{b_{ij}-a_{st}}{b_{st}-a_{st}} \geq 0$ is the equipollent potential degree of interval number $\otimes_{st}$ compared to $\otimes_{ij}$.
2. The range $[b_{ij}, b_{st}]$, which is divided by two right endpoints of two grey numbers and is at the right zone of intersection, is called the superior potential zone of grey number $[a_{st}, b_{st}]$ compared to $[a_{ij}, b_{ij}]$. $SPD_{st \to ij} = \frac{b_{st}-b_{ij}}{b_{st}-a_{st}} \geq$ is the superior potential degree of the interval number $\otimes_{st}$ compared to $\otimes_{ij}$.
3. The range $[a_{ij}, a_{st}]$, which is decided by the two left endpoints of two grey numbers and is on the left of the intersection, is defined as the inferior potential zone of grey number $[a_{ij}, b_{ij}]$ compared to $[a_{st}, b_{st}]$. $IPD_{ij \to st} = \frac{-a_{st}-a_{ij}}{b_{ij}-a_{ij}} \leq 0$ is the inferior potential degree of interval number $\otimes_{ij}$ compared to $\otimes_{st}$.

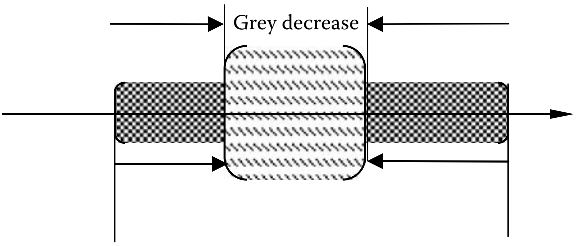


Figure 7.2 Change of the right and left values with grey decrease.

On the basis of these conceptions, we can now make a determination of the potential degrees among interval grey numbers without difficulty, based on the following determination rules.

Definition 7.2 The determination rules of potential relations of interval grey numbers: For two random grey numbers, the sum of the superior and inferior potential degrees of one compared to the other is called the potential difference of this grey number to the other, or potential in brief. If the potential is positive, it is called a positive potential, and the corresponding grey number is called a superior grey number; if the potential is minus, we call it a minus potential, and call the corresponding number an inferior potential grey number; and if the potential is zero, we call it an equal potential, and call these numbers equipollent potential grey numbers.

From Definition 7.2, we know that, for two grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$ and $\otimes_{st} \in [a_{st}, b_{st}]$,

If $SDP_{ij\to st} + IDP_{ij\to st} > 0$, we say that, compared to $\otimes_{st}$, $\otimes_{ij}$ is a positive potential; $\otimes_{ij}$ is the superior grey number and $\otimes_{st}$ is the inferior grey number, marked as $\otimes_{ij} > \otimes_{st}$.

If $SDP_{ij\to st} + IDP_{ij\to st} < 0$, we say that, compared to $\otimes_{st}$, $\otimes_{ij}$ is a minus potential; $\otimes_{ij}$ is the inferior grey number and $\otimes_{st}$ is the superior grey number, denoted as $\otimes_{ij} < \otimes_{st}$.

If $SDP_{ij\to st} + IDP_{ij\to st} = 0$, we say that $\otimes_{ij}$ and $\otimes_{st}$ have equal potential, and $\otimes_{ij}$ and $\otimes_{st}$ are equipollent potential grey numbers, marked as $\otimes_{ij} = \otimes_{st}$.

In Definition 7.2, some classical mathematical operators, such as >, <, and =, are used to describe the potential relationships among grey numbers, but their meanings are not the same as the greater and lesser judgments in classical mathematics.

Therefore, according to the rules in Definition 7.2, we can easily make a determination on some grey numbers in $A(\otimes)$ with respect to their potential meaning. Although the potential is not always the value of the grey number, the determination rules bring us a method to find the optimal potential pure strategies solution of $G(\otimes)$.

Obviously, lightly judging some interval grey numbers' size can make us easily resolve the pure strategy solution in substandard grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$. Following the theory of classical matrix games[34,35] and considering people's rational behavior, if both players do not want to run an unnecessary risk or do not feel lucky, they instead think of how the other side would try to make his opponent get less. They would therefore choose the most favorable case from among the worst possible cases. This is a way both sides can accept. It is obvious that these discussions on concepts such as optimum pure strategies, the solution, pure situation, and so on are all potential conditions. Here, we define some conceptions about $G(\otimes)$.

Definition 7.3 Potential pure strategies solution: Given a substandard grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, where $S_1=\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$, $S_2=\{\beta_1,\beta_2,\ldots,\beta_n\}$, and $A(\otimes)=([a_{ij},b_{ij}])_{m\times n}$, if there is a pure strategy, α_{i^*},β_{j^*}, under the potential meaning, which can make

$$[a_{ij^*},b_{ij^*}]\le[a_{i^*j^*},b_{i^*j^*}]\le[a_{i^*j},b_{i^*j}],\quad i=1,2,\ldots,m;\quad j=1,2,\ldots,n$$

come into existence, then situation $(\alpha_{i^*},\beta_{j^*})$ is called the potential pure strategy solution of $G(\otimes)=\{S_1,S_2,A(\otimes)\}$. α_{i^*},β_{j^*} is respectively the grey potential optimum pure strategy of Players 1 and 2, or potential optimum pure strategy for short; and the payoff of Player 1 $[a_{i^*j^*},b_{i^*j^*}]$ is called the grey game value of $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, denoted as $V_G(\otimes)$.

From Definition 7.3 we know that, if $G(\otimes)=\{S_1,S_2,A(\otimes)\}$ has a solution, there should exist a potential optimum pure strategy, α_{i^*},β_{j^*}, whose corresponding rows i^* and columns j^* can make the game value $[a_{i^*j^*},b_{i^*j^*}]$ meet the conditions of the optimal potential pure strategy solution. The pair of positive whole numbers (i^*,j^*) is called the saddle point of a grey matrix game under the grey potential optimum pure strategy, or potential saddle point in short, and the grey number $[a_{i^*j^*},b_{i^*j^*}]$ in $A(\otimes)$ that is determined by (i^*,j^*) is called a potential saddle point of grey matrix $A(\otimes)$. The grey game value $V_G(\otimes)$ is just the $[a_{i^*j^*},b_{i^*j^*}]$ that is the smallest potential grey number in i^* rows, and biggest potential in j^* columns.

Example 7.1 There is a comparatively closed color-TV market with two suppliers in the area. In order to get a larger market share, they compete with each other by using two strategies: Strategy 1, reducing the price and offering better service, or Strategy 2, offering more product functions and higher quality. For whatever reason, these suppliers cannot make an accurate estimate of future profit and loss values when they select one of the strategies. In fact, even under strict conditions, because of the stochastic and unstochastic system factors, the incomes of two random times of the closed-TV market (that is, the games) are not always the same. According to their history of competition experiences, players could make a relatively accurate judgment about the game results in advance [see Eq. (7.1)]. In Eq. (7.1), α_1, β_1 expresses Strategy 1 of Players 1 and 2 respectively, and α_2, β_2 expresses Strategy 2 of Players 1 and 2 respectively. So Eq. (7.1) constructs a grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$.

$$A(\otimes)=\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array}\begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} [5,7] & [4,9] \\ [1,8] & [1,10] \end{bmatrix} \end{array} \tag{7.1}$$

Solution: According to the method for grey potential pure strategy solution, we solve this potential pure strategy problem. If Players 1 and 2 both take Strategy 1, the payoff value is $V_G^*(\otimes)=a_{11}(\otimes)=[5,7]$ (for details, see Table 7.1).

Table 7.1 Optimal Potential Pure Strategy Solution of $G(\otimes)$

Player 2		β_1	β_2	$\min\limits_j[a_{ij},b_{ij}]$
Player 1	α_1	$a_{11}(\otimes) = [5,7]$	$a_{12}(\otimes) = [4,9]$	$[5,7]^*$
	α_2	$a_{21}(\otimes) = [1,8]$	$a_{22}(\otimes) = [1,10]$	$[1,8]$
	$\max\limits_j[a_{ij},b_{ij}]$	$[5,7]^*$	$[4,9]$	

Table 7.1 shows that there exists a potential pure strategy solution for both players in a game like that of Example 7.1; that is, if Players 1 and 2 both take Strategy 1, then their optimal payoff value is $V_G^*(\otimes) = a_{11}(\otimes) = [5,7]$.

Given a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, the potential meaning is different from the pure strategy solution in a classic mathematics meaning. It means that risk exists when using grey potential pure strategy solutions for decision-making.

Example 7.1 shows that there is an equilibrium, or grey saddle point, when players use certain information; it is an expedient measure for players to gain the most. In other words, unless players use more information, their decisions are optimal. However, there is a difference between decisions in grey potential meanings and decisions in classic math situations. The difference shows the risk of game decisions in grey potential cases to a certain extent.

In Example 7.1, if the payoff matrices are composed of the values of left and right endpoints in $A(\otimes)$, it constructs two classic matrix games, G_1 and G_2. From matrix game theory, it is easy to know that the solutions of G_1 and G_2 respectively are that if both Players 1 and 2 take their optimal pure strategy α_1, β_2, then the payoff value is $V_{G1} = 4$; if Players 1 and 2 take pure strategy α_2, β_1, then the payoff value is $V_{G2} = 8$.

After the event, we can get some facts from the comparison between the potential pure strategy solution and the pure strategy solutions of G_1 and G_2 in the grey matrix game problem.

1. The value of the left endpoint of the grey payoff value of grey matrix game $G(\otimes)$, $LV_G^*(\otimes) = Min\{V_G^*(\otimes)\} = 5$, is more than the payoff value of $G_1 V^*_{G1} = 4$. It means that in a G_1 situation, Player 1 only gets 4 units of market shares, not 5 units. It bring about a risk of overrating 1 unit of game income.
2. The value of the right endpoint of the grey payoff value of grey matrix game $G(\otimes)$, $RV_G^*(\otimes) = Max\{V_G^*(\otimes)\} = 7$, is less than the payoff value of $G_2 V^*_{G2} = 8$. It means that in a G_2 situation, Player 2 gets 8 units of market shares at most, not 7 units. It bring up the risk of underrating 1 unit of game gain.

The risk analysis of Example 7.1 is common in potential pure strategy solutions. Given a grey matrix game, there are only two kinds of risks in its potential pure strategy solution, and the definitions are given as follows.

Definition 7.4 Overrated risk of Player 1: Suppose there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ and the optimal payoff value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$. After the event, the difference between the optimal payoff value $V_{G^*}^k$ in the kth game and the left endpoint value of $V_G^*(\otimes)$, $LV_G^*(\otimes) = a_{ij}^*$, $\sigma_{high.1}^k = -(V_{G^*}^k - LV_G^*(\otimes))$, $(\sigma_{high.1}^k \geq 0)$ is called Player 1's overrated risk in the kth game, which is measured by $\sigma_{high.1}^k$.

Definition 7.5 Maximal overrated risk of Player 1: Suppose there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$. We call the maximal value of all overrated risk values $\sigma_{high.1}^k$ $(k = 1, 2, \ldots, \infty)$ as the maximal overrated risk, denoted as $\sigma_{high.1}^M = \mathrm{Max}_{k=1}^{\infty}\{\sigma_{high.1}^k\}$.

Definition 7.6 Underrated risk of Player 1: Suppose there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ and the optimal pay-off value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$. After the event, we call the difference, $\sigma_{low.1}^k = V_{G^*}^k - RV_G^*(\otimes)$ $(\sigma_{low.1}^k \geq 0)$, between the optimal payoff value $V_{G^*}^k$ of the kth game and the right endpoint value of $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, $RV_G^*(\otimes) = b_{ij}^*$ the underrated risk of Player 1 in the kth game, which is measured by $\sigma_{low.1}^k$.

Definition 7.7 Maximal underrated risk of Player 1: Suppose there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$. We call the maximal value of all underrated risk values $\sigma_{low.1}^k$ $(k=1, 2, \ldots, \infty)$, the maximal underrated risk for Player 1, denoted as $\sigma_{low.1}^M = Max_{k=1}^{\infty}\{\sigma_{low.1}^k\}$.

Theorem 7.1 Risk relations between players: Suppose there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ and the optimal payoff value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$. The overrated risk of Player 1 is equal to the underrated risk of Player 2 for this optimal solution, and vice versa.

Proof: From matrix game theory, we know that the gain of Player 1 is equal to the loss of Player 2. If there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$ and the optimal payoff value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, then it means the payoff value of Player 1 is $[a_{ij}^*, b_{ij}^*]$ and the payoff value of Player 2 is $[-b_{ij}^*, -a_{ij}^*]$.

Player 1's overrated risk value $\sigma_{high.1}^k$ $(k = 1, 2, \ldots, \infty)$ in the kth game means that his real optimal game value in the game is less than $LV_G^*(\otimes) = a_{ij}^*$, the left value of $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, by $\sigma_{high.1}^k$ units, as $y_{k.1}^* = a_{ij}^* - \sigma_{high.1}^k$. Meanwhile, the real optimal game value of Player 2 is more than $RV_{G.2}^*(\otimes) = -a_{ij}^*$, the right value of $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$, by $\sigma_{high.1}^k$ units, as $y_{k.2}^* = \sigma_{high.1}^k - a_{ij}^*$. In other words, Player 2 underrates his payoff value, which brings about underrated risk, and his underrated value is equal to the overrated value of Player 1, $\sigma^k_{low.2} = \sigma^k_{high.1}$.

In a similar way, in the *k*th game, the underrated value of Player 1 is $\sigma^k_{low.1}$ ($k = 1,2,\ldots,\infty$), which means the overrated risk value of Player 2 is $\sigma^k_{high.2} = \sigma^k_{low.1}$.

Given grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$, the overrated risk of Player 1 is equal to the underrated risk of Player 2 in the same grey matrix game, and vice versa.

Inference 7.1 Relations between the maximal overrated and underrated risks: Suppose there exists an optimal potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$ and the optimal payoff value is $V_G^*(\otimes) = [a_{ij}^*, b_{ij}^*]$. The maximal overrated risk of Player 1 is equal to the maximal underrated risk of Player 2, shown as follows:

$$\sigma^M_{high.1} = \operatorname*{Max}_{k=1}^{\infty}\left\{\sigma^k_{high.1}\right\} = \operatorname*{Max}_{k=1}^{\infty}\left\{\sigma^k_{low.2}\right\} = \sigma^M_{low.2}$$

The maximal underrated risk of Player 1 is equal to the maximal overrated risk of Player 2, shown as follows:

$$\sigma^M_{low.1} = \operatorname*{\mathit{Max}}_{k=1}^{\infty}\left\{\sigma^k_{low.1}\right\} = \operatorname*{\mathit{Max}}_{k=1}^{\infty}\left\{\sigma^k_{high.2}\right\} = \sigma^M_{high.2}$$

The proof is simple, like that of the relevant theorems in Ref. 34, so it is omitted here. In fact, the risk of potential optimal pure strategy solution mainly refers to the probability when the situation considered to be impossible to happen occurs, in the process of decision-making according to grey potential.

7.1.2 Measurement of the Optimal Potential Pure Strategy Solution

Suppose that we can solve a potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$. The key problem is whether there is risk in practicing this strategy. If so, how much risk will players undertake? To solve the problem, we give one important lemma.

Lemma 7.1 Most pessimistic and optimistic payoff values of Player 1: Given the grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$, the left and right values of all grey interval elements construct new payoff matrices $\{A_L\}_{m\times n}$ and $\{A_R\}_{m\times n}$ respectively. Then $G_L = \{S_1,S_2, A_L\}$ and $G_R = \{S_1,S_2, A_R\}$ are made by $\{A_L\}_{m\times n}$ and $\{A_R\}_{m\times n}$. Their payoff values V_L^* and V_R^* are respectively equal to the most pessimistic (minimal) value $V_{L(\otimes)}^*$ and the most optimistic (maximal) value $V_{R(\otimes)}^*$ of Player 1.

Proof: We prove $V_L^* = V_{L(\otimes)}^*$ first. In grey matrix game $G(\otimes)$, all the left values of all grey interval numbers construct a new payoff matrix $\{A_L\}_{m\times n}$. Then the game

$G_L = \{S_1, S_2, A_L\}$, decided by $\{A_L\}_{m\times n}$, is a classic accurate math game. Based on matrix game theory, we solve the solutions easily. The optimal strategies of Players 1 and 2 are $T_L\ (x)$ and $T_L\ (y)$, and the payoff value is V_L^*.

Now we prove that V_L^* is the most pessimistic (minimal) payoff value; that is, $V_L^* = V_{L(\otimes)}^*$.

Using reduction to absurdity, if we suppose that V_L^* is not the most pessimistic (minimal) payoff value $V_{L(\otimes)}^*$, then $V_{L(\otimes)}^* < V_L^*$. There is a matrix $\{A_L\}'_{m\times n}$ that is not composed of all values of left endpoints of the grey elements of $A(\otimes)_{m\times n}$ [see Eq. (7.5)]. In the game decided by $\{A_L\}'_{m\times n}$, the payoff value is the most pessimistic (minimal) payoff value $V_{L(\otimes)}^*$ of game $G(\otimes)$, as follows, for $V_{L'}^* = V_{L(\otimes)}^*$.

$$A_L = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} & a_{i-1,k} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,k-1} & a_{i,k} & \cdots & a_{i,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,k-1} & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \tag{7.4}$$

$$A'_L = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} & a_{i-1,k} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,k-1} & a'_{i,k} & \cdots & a_{i,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,k-1} & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \tag{7.5}$$

If we suppose that only one element a'_{ik} in $\{A_L\}'_{m\times n}$ is different from a_{ik} in $\{A_L\}_{m\times n}$ (that is, $a_{ik} < a'_{ik} < b_{ik}$), then according to block matrix game theory, we can divide elements in $\{A_L\}'_{m\times n}$ into some submatrices. We take respectively a_{ik}, a'_{ik} and their following elements in $\{A_L\}_{m\times n}$ and $\{A_L\}'_{m\times n}$ to construct A_{ik} and A'_{ik} [see Eqs. (7.6) and(7.7)] , and divide the other elements to compose matrices A_{LL} and A'_{LL} :

$$A_{ik} = [a_{i,k-1} \quad a_{i,k}] \tag{7.6}$$

$$A'_{ik} = \left[a_{i,k-1} \quad a'_{i,k} \right] \tag{7.7}$$

Here, only payoff values decided by A_{LL} and A'_{LL} may be different, and the other block matrix payoff values are equal to each other. In fact, after matrix blocking, we transfer the measured comparison between V^*_L and $V^*_{L(\otimes)}$ to that between V^*_{LL} and $V^{*'}_{LL}$. When $V^*_{LL} \le V^{*'}_{LL}$, it means that if a_{ik} turns into a'_{ik} ($a_{ik} < a'_{ik}$), then the payoff will not decrease. When $V^*_{LL} > V^{*'}_{LL}$, a_{ik} turns into a'_{ik} ($a_{ik} < a'_{ik}$), the payoff value will decrease.

$$A_{LL} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1s} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ik} & \cdots & A_{is} \\ \vdots & & \vdots & & \vdots \\ A_{t1} & \cdots & A_{tk} & \cdots & A_{ts} \end{bmatrix} \tag{7.8}$$

$$A'_{LL} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1s} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A'_{ik} & \cdots & A_{is} \\ \vdots & & \vdots & & \vdots \\ A_{t1} & \cdots & A_{tk} & \cdots & A_{ts} \end{bmatrix} \tag{7.9}$$

$$V^*_{ik} = \begin{cases} a_{i,k-1}, & a_{i,k-1} \le a_{i,k} \\ a_{i,k}, & a_{i,k} \le a_{i,k-1} \end{cases} \tag{7.10}$$

$$V^{*'}_{ik} = \begin{cases} a_{i,k-1}, & a_{i,k-1} \le a'_{i,k} \\ a'_{i,k}, & a'_{i,k} \le a_{i,k-1} \end{cases} \tag{7.11}$$

According to matrix game theory,[34] the game values of Eqs. (7.6) and (7.7), V^*_{ik} and $V^{*'}_{ik}$, are demonstrated in Eqs. (7.9) and (7.10). From Eqs. (7.10) and (7.11), we know that if $a_{ik} < a'_{ik}$, $V^{*'}_{ik}$ is not less than V^*_{ik}, and only when $a_{i,k} < a'_{i,k} \le a_{i,k-1}$, $V^*_{ik} < V^{*'}_{ik}$. Therefore, if $a_{ik} \le a'_{ik}$ is true in Eqs. (7.6) and (7.7), then $V^*_{ik} \le V^{*'}_{ik}$. In fact, according to block matrix game theory, the game values of payoff matrices $\{A_L\}_{m\times n}$ and $\{A_L\}'_{m\times n}$ can be solved by Eqs. (7.12) and (7.13):

$$A_{LL.V} = \begin{bmatrix} V_{11}^{*} & \cdots & V_{1k}^{*} & \cdots & V_{1s}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{i1}^{*} & \cdots & V_{ik}^{*} & \cdots & V_{is}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{t1}^{*} & \cdots & V_{tk}^{*} & \cdots & V_{ts}^{*} \end{bmatrix} \tag{7.12}$$

$$A_{LL.V}^{/} = \begin{bmatrix} V_{11}^{*} & \cdots & V_{1k}^{*} & \cdots & V_{1s}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{i1}^{*} & \cdots & V_{ik}^{*/} & \cdots & V_{is}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{t1}^{*} & \cdots & V_{tk}^{*} & \cdots & V_{ts}^{*} \end{bmatrix} \tag{7.13}$$

In Eqs. (7.12) and (7.13), according to block matrix game theory, if $V_{ik}^{*} \le V_{ik}^{*/}$, then it is also true that V_{L}^{*} is less than or equal to $V_{L^{/}}^{*}$; that is, $V_{L}^{*} \le V_{L^{/}}^{*}$.

When $a_{ik} \le a_{ik}^{/}$ is true, then $V_{L}^{*} \le V_{L^{/}}^{*}$. Deducing in a similar way, if one or more elements in $\{A_{L}\}_{m\times n}^{/}$ are not less than the corresponding elements in $\{A_{L}\}_{m\times n}$, the payoff values of the games whose payoff matrices are respectively $\{A_{L}\}_{m\times n}$ and $\{A_{L}\}_{m\times n}^{/}$ are in the relationship $V_{L}^{*} \le V_{L^{/}}^{*}$: The payoff value of the game whose payoff matrix is $\{A_{L}\}_{m\times n}$, which is composed of the values of left endpoints of all elements in $A(\otimes)_{m\times n}$, is equal to the most pessimistic (minimal) payoff value $V_{L(\otimes)}^{*}$ of $G(\otimes)$; that is, $V_{L}^{*} = V_{L(\otimes)}^{*}$.

So supposition is wrong and the proposition is right.

In a similar way, we can prove the other parts.

Inference 7.2 Most pessimistic and optimistic payoff values of Player 2: Given the grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$, the most optimistic (maximal) and pessimistic (minimal) payoff values of Player 2 are equal to the minus of the most pessimistic (minimal) and most optimistic (maximal) payoff values of Player 1 respectively; that is, $V_{opt(\otimes)}^{*} = -V_{L(\otimes)}^{*}$ and $V_{pes(\otimes)}^{*} = -V_{R(\otimes)}^{*}$.

Based on the relations between the incomes of the players shown in the payoff matrix game, the proof of the inference above is simple and is therefore omitted here.

Theorem 7.2 Maximal overrated risk of Player 1: Suppose that a problem of potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$ can be solved and the optimal payoff value is $V_{G}^{*}(\otimes) = [a_{ij}^{*}, b_{ij}^{*}]$. The maximal overrated risk value of Player 1, $\sigma_{high.1}^{M}$, is equal to the difference between a_{ij}^{*} and the most pessimistic (minimal) payoff value $V_{L(\otimes)}^{*}$ of $G(\otimes)$; namely, $\sigma_{high.1}^{M} = (a_{ij}^{*} - V_{L(\otimes)}^{*})$.

Theorem 7.3 Maximal underrated risk of Player 1: Suppose that a problem of potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1,S_2, A(\otimes)\}$ can be solved and the optimal payoff value is $V_G^*(\otimes)=[a_{ij}^*,b_{ij}^*]$. The maximal underrated risk value of Player 1, $\sigma_{low.1}^M$, is equal to the difference between b_{ij}^* and the most optimistic (maximal) payoff value $V_{R(\otimes)}^*$ of $G(\otimes)$; namely, $\sigma_{low.1}^M=(V_{R(\otimes)}^*-b_{ij}^*)$.

On the basis of Definitions 3.3 and 3.4 and Lemma 4.1, the proof of the two theorems above is simple and so is omitted here.

Inference 7.3 Maximal overrated and underrated risk of Player 2: Suppose that a problem of a potential pure strategy solution in a grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$ can be solved and the optimal payoff value is $V_G^*(\otimes)=[a_{ij}^*,b_{ij}^*]$. Player 2's maximal overrated risk value $\sigma_{high.2}^M$ is equal to the difference between $(-b_{ij}^*)$ and the most pessimistic (minimal) payoff value $V_{pes(\otimes)}^*=-V_{R(\otimes)}^*$ in the $G(\otimes)$; that is, $\sigma_{high.2}^M=(V_{R(\otimes)}^*-b_{ij}^*)$. Player 2's maximal underrated risk value, $\sigma_{low.2}^M$, is equal to the sum of a_{ij}^* and the most pessimistic (minimal) payoff value $V_{opt(\otimes)}^*=-V_{L(\otimes)}^*$ in the $G(\otimes)$, as $\sigma_{low.2}^M=(a_{ij}^*-V_{L(\otimes)}^*)$.

On the basis of Definitions 3.3, 3.4, Lemma 4.1, and Inference 4.2, the proof is simple and omitted here.

Example 7.2 In Example 7.1, there exists an optimal potential pure strategy solution in a game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$ [as shown in Eq. (7.1)]; solve the maximal overrated and underrated risks of Players 1 and 2.

Solution: In Example 7.1, the payoff value of the optimal potential pure strategy solution is $V_G^*(\otimes)=a_{11}^*(\otimes)=[a_{11}^*,b_{11}^*]=[5,7]$ and the most optimistic (maximal) and pessimistic (minimal) payoff values of Player 1 are respectively $V_{R(\otimes)}^*=V^*G_2=8$ and $V_{L(\otimes)}^*=V^*G_1=4$.

Then, the maximal overrated risk of Player 1 is $\sigma_{high.1}^M=(a_{ij}^*-V_{L(\otimes)}^*)=5-4=1$.
The maximal underrated risk of Player 1 is $\sigma_{low.1}^M=(V_{R(\otimes)}^*-b_{ij}^*)=8-7=1$.
The maximal overrated risk of Player 2 is $\sigma_{high.2}^M=(V_{R(\otimes)}^*-b_{ij}^*)=8-7=1$.
The maximal underrated risk of Player 2 is $\sigma_{low.2}^M=(a_{ij}^*-V_{L(\otimes)}^*)=5-4=1$.

In the grey matrix game $G(\otimes)=\{S_1,S_2,A(\otimes)\}$, it is hard to measure grey interval numbers. Unless this problem can be solved well, it means grey matrix game $G(\otimes)$ hardly goes on. This section builds up the system of determination of the grey potential, which meets the requirement of determining a pure strategy solution in the grey potential situation.

However, the potential measurement between grey numbers isn't the same as the value measurement in grey numbers. They are greatly different from each other. The pure strategy solution in a grey potential meaning is different from that in a classic math meaning. It means there exists risk when making decisions regarding grey potential pure strategy solutions. This section studied the risk, but it did not involve a mixed strategy solution. For example, if there is no potential pure strategy solution, what about the mixed strategy solution? If it has mixed strategy solution, how is it solved? The research into these problems will be elaborated in other sections.

7.1.3 Summary

On the basis of conceptions of equipollent, superior, and inferior potential degrees, this section designed determinant rules of interval grey number potential relations and explored the players' decision-making laws in the conditions of finite knowledge and logos. We also designed grey game decision-making rules when a player chooses the maximum potential degrees of grey game values (the most favorable situation) and in cases where there are likely to be minimum potential degrees of grey game values (the most disadvantageous situation), which is a reliable way that both sides accept. This part of the book also recognized and defined overrated and underrated risks of potential optimal pure strategies in the grey game, and designed arithmetic for determining players' overrated and underrated risks in situations of potential optimal pure strategies.

7.2 Venture and Control Problems of an Optimal Mixed Strategy for a Grey Interval Number Matrix Game

7.2.1 Recognition and Definition of the Overrated and Underrated Risks of the Potential Optimum Mixed Strategy Solution

From Refs. 5 and 13, we know that the optimum mixed strategy solution for the classic matrix game $G = \{S_1, S_2; A\}$ is an accurate white number, but generally it would be an interval grey number for a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$, which means that risks exist in the solution. To recognize and define the risks is the basis of the $G(\otimes)$ research.

Definition 7.8 Overrated and underrated risks of Player 1: Suppose there exists a potential optimum grey mixed solution in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ and the optimal payoff value is $V_G(\otimes) = [LV_G(\otimes), RV_G(\otimes)] = [v_L, v_R]$. After the event, we call the difference between the optimal payoff value $V^k_{G^*}$ in the *k*th game and the left endpoint value of $V_G(\otimes) = [v_L, v_R]$, $LV_G(\otimes) = v_L$, as Player 1's underrated risk in the *k*th game, which is measured by $\sigma^k_{high.1}$; that is, $\sigma^k_{high.1} = RV_G(\otimes) - V^k_G = v_R - V^k_G$, ($\sigma^k_{high.1} \ge 0$). The difference, denoted as $\sigma^k_{high.1} = RV_G(\otimes) - V^k_G = v_R - V^k_G$, ($\sigma^k_{high.1} \ge 0$), which is between the right endpoint value of $V_G(\otimes) = [v_L, v_R]$, $RV_G(\otimes) = v_R$ and V^k_G, is called an overrated risk of Player 1 in the *k*th game, which is measured by $\sigma^k_{high.1}$.

Theorem 7.4 Risk relations between players: Suppose there exists a potential grey mixed strategy solution in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ and the optimal grey payoff value is $V_G(\otimes) = [v_L, v_R]$. The overrated risk of Player 1 is equal to the underrated risk of Player 2 for this optimal solution, and vice versa.

Proof: From matrix game theory, we know that the gain of Player 1 is equal to the loss of Player 2. Suppose there exists a potential pure strategy solution in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ and the potential optimal grey payoff value is $V_G(\otimes) = [v_L, v_R]$. This means that the payoff value of Player 1 is $[v_L, v_R]$ and the payoff value of Player 2 is $[-v_L, -v_R]$.

Player 1's overrated risk value $\sigma^k_{high.1}$ $(k = 1,2,\ldots,\infty)$ in the kth game means that his real optimal game value is less than $RV_G(\otimes) = v_R$, which is the right value of $V_G(\otimes) = [v_L, v_R]$, by $\sigma^k_{high.1} = v_R - V^k_G$ units. Meanwhile, the real optimal game value of Player 2, $V^k_{G.2} = -V^k_G$, is more than $RV_{G.2}(\otimes) = -v_R$, which is the right value of $V_{G.2}(\otimes) = [-v_R, -v_L]$, by $\sigma^k_{low.2} = -V^k_G - RV_{G.2}(\otimes) = v_R - V^k_G$ units. In other words, Player 2 underrates his payoff value, which brings about underrated risk, and his underrated value is equal to the overrated value of Player 1, $\sigma^k_{low.2} = \sigma^k_{high.1}$

In a similar way, the underrated risk value of Player 1 is $\sigma^k_{low.1}$ $(k = 1,2,\ldots,\infty)$ in the kth game, which means the overrated risk value of Player 2 is $\sigma^k_{high.2} = \sigma^k_{low.1}$ at the same time.

The overrated risk of Player 1 is equal to the underrated risk of Player 2 in the same grey matrix game, and vice versa.

Theorem 7.5 Existence of overrated and underrated risks of players: Suppose there exists an optimal grey mixed strategy solution in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ and the optimal grey payoff value is $V_G(\otimes) = [v_L, v_R]$. In the kth game, both the overrated and underrated risks of the players exist in the optimal payoff value V^k_G, which is an accurate white number because the payoff value after the event is unique.

On the basis of Definition 7.8, the proof is simple and so is omitted here. But from Definitions 7.8 and 7.10, we know the difference between V^k_G and $RV_G(\otimes) = v_R$, which is the right value of the optimal grey payoff value in a grey matrix game $G(\otimes)$, is the overrated risk of Player 1; that is, $\sigma^k_{high.1} = RV_G(\otimes) - V^k_G = v_R - V^k_G$. The difference between V^k_G and $LV_G(\otimes) = v_L$, the left value of the optimal grey payoff value in a grey matrix game $G(\otimes)$, is Player 1's underrated risk; that is, $\sigma^k_{low.1} = V^k_G - LV_G(\otimes) = V^k_G - v_L$.

Therefore, the proposition is true.

Example 7.3 There is a digital-TV market almost occupied by two suppliers in an area. Suppose they are preparing for the next round of competition for market share. Both Player 1 and Player 2 have only two strategies: reducing the price or improving the quality. A grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ is used to describe the competition, whose grey payoff value is shown in Eq. (7.14). Solve the overrated and underrated risks of Players 1 and 2 in the kth game while the payoff matrix is A_k [shown as Eq. (7.15)].

As in Ref. 37, we describe the interval grey numbers in Eq. (7.14) in the form of a standard interval grey number [shown in Eq. (7.2)]. According to the method for the grey game strategy and grey payoff value, the grey optimal game strategy solution and payoff value in the grey matrix game $G(\otimes)$ are as follows:

$$A(\otimes) = \begin{bmatrix} 1 & [3,4] \\ [4,6] & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3+\gamma_{12} \\ 4+2\cdot\gamma_{21} & 2 \end{bmatrix} \tag{7.14}$$

$$A_k = \begin{bmatrix} 1 & 4 \\ 5 & 2 \end{bmatrix} \tag{7.15}$$

$$x(\otimes) = \{x_1(\otimes) = [0.40, 0.67] \quad x_2(\otimes) = [0.33, 0.60]\} \tag{7.16}$$

$$y(\otimes) = \{y_1(\otimes) = [0.17, 0.40] \quad y_2(\otimes) = [0.60, 0.83]\} \tag{7.17}$$

$$v^*(\otimes) = \begin{bmatrix} \min\left(\dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le 1}, \\ \max\left(\dfrac{10+4\gamma_{12}+6\gamma_{21}+2\gamma_{12}\gamma_{21}}{4+\gamma_{12}+2\gamma_{21}}\right)_{0\le\gamma_{12},\gamma_{21}\le 1} \end{bmatrix} = [2.50, 3.14]$$

In the *k*th game, the payoff value of A_k is $v_k^* = 3$.In the same game, Player 1's overrated and underrated risk values are $\sigma^k_{high.1} = RV_G(\otimes) - V^k_G = 3.14 - 3 = 0.14$ and $\sigma^k_{low.1} = V^k_G - LV_G(\otimes) = 3 - 2.50 = 0.50$ respectively. Player 2's overrated and underrated risk values are $\sigma^k_{high.1} = RV_G(\otimes) - V^k_G = 3.14 - 3 = 0.14$ and $\sigma^k_{low.2} = V^k_G - LV_{G.2}(\otimes) = -3 - (-3.14) = 0.14$ respectively.

7.2.2 Measurement for Maximal Overrated and Underrated Risks of the Potential Pure Strategy Solution

Given a grey matrix game $G(\otimes) = \{S_1, S_2, A(\otimes)\}$, suppose there exists an optimal grey mixed strategy solution. We are concerned not only with whether there exists risk to practice the strategy but also with how much risk we should undertake.

Definition 7.9 Maximal underrated and overrated risk of Players 1 and 2: Suppose a solution can be found to a potential grey mixed strategy in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$. Among all the underrated and overrated risk values of Players 1 and 2, we call the maximum as their maximal underrated risk values respectively, denoted as $\sigma^M_{high.1} = \mathrm{Max}_{k=1}^{\infty}\{\sigma^k_{high.1}\}$ and $\mathrm{Max}_{k=1}^{\infty}\{\sigma^k_{high.2}\}$.

Theorem 7.6 Most pessimistic and optimistic payoff values of Player 1: Given a grey matrix game, the optimal grey payoff value is $V_G(\otimes) = [v_L, v_R]$. .The left and right values of all grey interval elements construct the new payoff matrices $\{A_L\}_{m\times n}$ and $\{A_R\}_{m\times n}$ respectively. Then $G_L = \{S_1, S_2, A_L\}$ and $G_R = \{S_1, S_2, A_R\}$ are decided by $\{A_L\}_{m\times n}$ and $\{A_R\}_{m\times n}$. Their payoff values $V_{G_L} = v_L$ and $V_{G_R} = v_R$ are respectively equal to the most pessimistic (minimum) value $V^*_{L(\otimes)}$ and the optimistic (maximum) value $V^*_{R(\otimes)}$ of Player 1.

Proof: We prove $V_{GL.1} = V_{G_L} = v_L$ first. In a grey matrix game $G(\otimes)$, all the left endpoint values of all grey interval numbers construct a new payoff matrix $\{A_L\}_{m\times n}$ [see Eq. (7.19)]. Then the game $G_L = \{S_1, S_2, A_L\}$, decided by $\{A_L\}_{m\times n}$, is a classic accurate math game. On the basis of matrix game theory, we can easily find the solutions. The optimal strategies of Players 1 and 2 are $T_L(x)$ and $T_L(y)$, and the payoff value $V_{G_L} = v_L$.

Now we prove that $V_{G_L} = v_L$ is the most pessimistic (minimum) payoff value $V_{GL.1}$; that is, $V_{GL.1} = V_{G_L} = v_L$.

Using reduction to absurdity, if we suppose that $V_{G_L} = v_L$ is not the most pessimistic (minimum) payoff value $V_{GL.1}$, then $V^*_{L(\otimes)} < V^*_L$. There is at least a matrix $\{A_L\}'_{m\times n}$ that is not composed of all values of left endpoints of the grey elements of $A(\otimes)_{m\times n}$. In the game decided by $\{A_L\}'_{m\times n}$, the payoff value, $V'_{G_L} = v'_L$, is the most pessimistic (minimum) payoff value of game $G(\otimes)$; that is, $V_{GL.1} = V_{G_L} = v_L$.

$$A_L = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} & a_{i-1,k} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,k-1} & a_{i,k} & \cdots & a_{i,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,k-1} & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \tag{7.19}$$

$$A'_L = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,k} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} & a_{i-1,k} & \cdots & a_{i-1,n} \\ a_{i,1} & \cdots & a_{i,k-1} & a'_{i,k} & \cdots & a_{i,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,k-1} & a_{m,k} & \cdots & a_{m,n} \end{bmatrix} \tag{7.20}$$

Suppose only one element a'_{ik} in $\{A_L\}'_{m\times n}$ is different from a_{ik} in $\{A_L\}_{m\times n}$; that is, $a_{ik} < a'_{ik} < b_{ik}$. According to block matrix game theory, we can divide elements in $\{A_L\}_{m\times n}$ and $\{A_L\}'_{m\times n}$ into some submatrices. We take respectively a_{ik} and a'_{ik} and their neighboring elements in $\{A_L\}_{m\times n}$ and $\{A_L\}'_{m\times n}$ to construct A_{ik} and A'_{ik} [see Eqs. (7.10) and (7.11)], and divide the other elements to compose matrices to form block matrices A_{LL} and A'_{LL}.

$$A_{ik} = [a_{i,k-1} \quad a_{i,k}] \tag{7.21}$$

$$A'_{ik} = [a_{i,k-1} \quad a'_{i,k}] \tag{7.22}$$

$$A_{LL} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1s} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ik} & \cdots & A_{is} \\ \vdots & & \vdots & & \vdots \\ A_{t1} & \cdots & A_{tk} & \cdots & A_{ts} \end{bmatrix} \tag{7.23}$$

$$A_{LL}^{\prime} = \begin{bmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1s} \\ \vdots & & \vdots & & \vdots \\ A_{i1} & \cdots & A_{ik}^{\prime} & \cdots & A_{is} \\ \vdots & & \vdots & & \vdots \\ A_{t1} & \cdots & A_{tk} & \cdots & A_{ts} \end{bmatrix} \tag{7.24}$$

Here, only payoff values decided by A_{LL} and $A_{LL}^{\prime}$ may possibly be different, and the other block matrix payoff values are equal to each other. If the payoff matrix of the game is decided by Eq. (7.21) or Eq. (7.22), then the optimal payoff values in these problems are shown as Eqs. (7.25) and (7.26) based on matrix game theory.

$$V_{ik}^{*} = \begin{cases} a_{i,k-1}, & a_{i,k-1} \le a_{i,k} \\ a_{i,k}, & a_{i,k} \le a_{i,k-1} \end{cases} \tag{7.25}$$

$$V_{ik}^{*\prime} = \begin{cases} a_{i,k-1}, & a_{i,k-1} \le a_{i,k}^{\prime} \\ a_{i,k}^{\prime}, & a_{i,k}^{\prime} \le a_{i,k-1} \end{cases} \tag{7.26}$$

According to the above two formulas, we know that $V_{ik}^{*\prime} \ge V_{ik}^{*}$ as $a_{ik} < a_{ik}^{\prime}$, and $V_{ik}^{*} < V_{ik}^{*\prime}$ only when $a_{i,k} < a_{i,k}^{\prime} \le a_{i,k-1}$. In summary, if $a_{ik} \le a_{ik}^{\prime}$, then $V_{ik}^{*} \le V_{ik}^{*\prime}$. In fact, according to block matrix game theory, the payoff values of $\{A_L\}_{m\times n}$, $\{A_L\}_{m\times n}^{\prime}$, V_{LL}^{*}, and $V_{LL}^{*\prime}$, can be solved by Eqs. (7.27) and (7.28):

$$A_{LL.V} = \begin{bmatrix} V_{11}^{*} & \cdots & V_{1k}^{*} & \cdots & V_{1s}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{i1}^{*} & \cdots & V_{ik}^{*} & \cdots & V_{is}^{*} \\ \vdots & & \vdots & & \vdots \\ V_{t1}^{*} & \cdots & V_{tk}^{*} & \cdots & V_{ts}^{*} \end{bmatrix} \tag{7.27}$$

$$A'_{LL.V} = \begin{bmatrix} V^*_{11} & \cdots & V^*_{1k} & \cdots & V^*_{1s} \\ \vdots & & \vdots & & \vdots \\ V^*_{i1} & \cdots & V^{*\prime}_{ik} & \cdots & V^*_{is} \\ \vdots & & \vdots & & \vdots \\ V^*_{t1} & \cdots & V^*_{tk} & \cdots & V^*_{ts} \end{bmatrix} \tag{7.28}$$

In a similar way, in Eqs. (7.22) and (7.23), if $V^*_{ik} \le V^{*\prime}_{ik}$, then $V^*_L \le V^*_{L'}$.

If $a_{ik} \le a'_{ik}$ is true, then $V^*_L \le V^*_{L'}$. Deducing in a similar way, if one or more elements in $\{A_L\}'_{m\times n}$ are not less than the corresponding elements in $\{A_L\}_{m\times n}$, the payoff values of the games whose payoff matrices are respectively $\{A_L\}_{m\times n}$ and $\{A_L\}'_{m\times n}$ are in the relationship of $V^*_L \le V^*_{L'}$. In other words, the payoff value of the game whose payoff matrix is $\{A_L\}_{m\times n}$, which is composed of the values of left endpoints of all elements in $A(\otimes)_{m\times n}$, is equal to the most pessimistic (minimum) payoff value $V^*_{L(\otimes)}$ of $G(\otimes)$; that is, $V^*_L = V^*_{L(\otimes)}$.

So the supposition is wrong and the proposition is right.

In the similar way, we can prove the left parts.

Inference 7.4 Most pessimistic and optimistic payoff values of Player 2: Given a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$, the most optimistic (maximum) and pessimistic (minimum) payoff values of Player 2 are equal to the minus of the most pessimistic (minimum) and optimistic (maximum) payoff values of Player 1 respectively; that is, $V_{GR.2} = -V_{GL.1}$ and $V_{GL.2} = -V_{GR.1}$.

Theorem 7.7 Maximal overrated risk equal to maximal underrated risk of players: Suppose there exists a potential mixed strategy solution in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ and the optimal payoff value is $V_G(\otimes) = [v_L, v_R]$. For Players 1 and 2, the maximal overrated risk values, $\sigma^M_{high.1}$ and $\sigma^M_{high.2}$, are equal to maximal underrated risk values, $\sigma^M_{low.1}$ and $\sigma^M_{low.2}$, which is $v_R - v_L$.

Proof: From Theorem 7.6 and Inference 7.4, we know that for Player 1 in a grey matrix game $G(\otimes)$, the minimal profit of each game is the most pessimistic (minimum) payoff value $V_{GL.1} = V_{G_L} = v_L$. That means the risk of the minimal payoff value is his maximum overrated risk, denoted as $\sigma^M_{high.1} = \mathrm{Max}^{\infty}_{k=1}\{\sigma^k_{high.1}\} = v_R - v_L$; the maximal profit of all games is the most optimistic (maximum) payoff value $V_{GR.1} = v_{G_R} = v_R$, which means the risk of the maximal payoff value is his maximal underrated risk, $\sigma^M_{low.1} = \mathrm{Max}^{\infty}_{k=1}\{\sigma^k_{low.1}\} = v_R - v_L$.

In a similar way, for Player 2, his maximal overrated risk value is $\sigma^M_{high.2} = \mathrm{Max}^{\infty}_{k=1}\{\sigma^k_{high.2}\} = v_R - v_L$, and the maximal underrated risk is $\sigma^M_{low.2} = Max^{\infty}_{k=1}\{\sigma^k_{low.2}\} = v_R - v_L$.

In summary, the proposition is right.

According to Theorem 7.7, given a grey matrix game $G(\otimes)$, the maximal overrated and underrated risk values of Players 1 and 2 are all equal to $v_R - v_L$. Here, we set a rule of using $\sigma^M_{high-low} = v_R - v_L$ to represent the players' maximal overrated and underrated risk values in game $G(\otimes)$.

Example 7.4 Solve the maximal overrated and underrated risk values of Players 1 and 2 in Example 7.1. As in Example 7.3, the optimal grey payoff value of $G(\otimes)$ is $V^*(\otimes)=[v_L,v_R]=[2.50,3.14]$. According to Theorem 7.7, the maximal overrated and underrated risk values of Players 1 and 2 are as follows:

$$\sigma^M_{high.1} = \underset{k=1}{\overset{\infty}{\text{Max}}}\left\{\sigma^k_{high.1}\right\} = \sigma^M_{low.1} = \underset{k=1}{\overset{\infty}{\text{Max}}}\left\{\sigma^k_{low.1}\right\} = v_R - v_L = 3.14 - 2.50 = 0.64$$

$$\sigma^M_{high.2} = \underset{k=1}{\overset{\infty}{\text{Max}}}\left\{\sigma^k_{high.2}\right\} = \sigma^M_{low.1} = \underset{k=1}{\overset{\infty}{\text{Max}}}\left\{\sigma^k_{low.1}\right\} = v_R - v_L = 3.14 - 2.50 = 0.64$$

7.2.3 Information Venture Control Ability and Venture Control of Game G(⊗)

In a grey matrix game $G(\otimes) = \{S_1(\otimes),S_2(\otimes); A(\otimes)\}$, which can be solved with an optimal grey mixed strategy solution, there exists risk for players to make decisions on the basis of the solution because people cannot make accurate judgments of the results of the game in advance because they have incomplete information. Therefore, the payoff game matrix is grey.

Theorem 7.8 Conditions of reducing risks in $G(\otimes)$: In a gray matrix game $G(\otimes) = \{S_1(\otimes),S_2(\otimes); A(\otimes)\}$, the relevant condition to reduce risks in the game is that both Player 1 and Player 2 have some effective information to reduce the grey degree of one or more elements in game payoff matrix $A(\otimes)$.

Proof: According to grey system theory,[5] reducing the grey degree of one or more elements in game payoff matrix $A(\otimes)$ means respective internal contractions of the left or right endpoint values.

From Theorem 7.6, when the grey degree of one or more elements in $A(\otimes)$ decreases, which means the contraction of the left or right endpoints, the most pessimistic (minimum) and most optimistic (maximum) payoff values will decrease or stay unchanged.

Thus from Definition 7.8 and Theorem 7.4, we know that the overrated or underrated risk value of Players 1 and 2 may decrease but cannot increase with the grey degree reduction of one or more elements in $A(\otimes)$.

Inference 7.5 Nonrisk conditions in $G(\otimes)$: In a grey matrix game $G(\otimes) = \{S_1(\otimes),S_2(\otimes); A(\otimes)\}$, nonrisk requires the whitenization of all the grey information in $A(\otimes)$, turning $A(\otimes)$ into an accurate whitenization matrix A, and then the overrated and underrated risk values of players will be zero.

The proof is simple, so it is omitted here. According to matrix game theory, given a game whose payoff matrix is an accurate whitenization matrix A, the game value in the kth game is a constant. Thus from Definition 7.4 and Theory 7.5, the overrated and underrated risk values of the players are zero.

Definition 7.10 Information control ability: Suppose there exists some public, efficient information accepted by both Player 1 and Player 2 in a grey matrix game $G(\otimes) = \{S_1(\otimes), S_2(\otimes); A(\otimes)\}$ resulting in the contraction of the left or right endpoint of one or more grey elements to construct a new payoff matrix $A'_I(\otimes)$. We call the difference between the most optimistic and pessimistic game values of $G(\otimes)$ and $G'_I(\otimes)$ as the control ability of the overrated and underrated risk respectively, denoted as P_{IH} and P_{IL}, and the absolute value of the difference between the maximal overrated and underrated risk of $G(\otimes)$ and $G'_I(\otimes)$ as the maximal venture control ability, marked as P_{IM}.

Example 7.5 In Example 7.3, if there exists some public, efficient information that can reduce the grey degree of $a_{21}(\otimes)$ in $A(\otimes)$ by turning $a_{21}(\otimes)$ into $a'_{21}(\otimes) = [4.5, 5.5]$ and $A(\otimes)$ into $A'_I(\otimes)$ [as in Eqs. (7.29) and (7.30)], solve the maximal overrated and underrated venture control abilities of this information, marked as P_{IH} and P_{IL}.

$$A(\otimes) = \begin{bmatrix} a_{11}(\otimes) & a_{12}(\otimes) \\ a_{21}(\otimes) & a_{22}(\otimes) \end{bmatrix} = \begin{bmatrix} 1 & [3,4] \\ [4,6] & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3+\gamma_{12} \\ 4+2\cdot\gamma_{21} & 2 \end{bmatrix} \tag{7.29}$$

$$A'_I(\otimes) = \begin{bmatrix} a_{11}(\otimes) & a_{12}(\otimes) \\ a_{21}(\otimes) & a_{22}(\otimes) \end{bmatrix} = \begin{bmatrix} 1 & [3,4] \\ [4.5,5.5] & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3+\gamma_{12} \\ 4.5+\gamma'_{21} & 2 \end{bmatrix} \tag{7.30}$$

In Example 7.3, the optimal grey payoff value of $G(\otimes)$ is $V^*(\otimes) = [V_L, V_R] = [2.50, 3.14]$. In a similar way, when the payoff matrix is $A'_I(\otimes)$, it can be solved as the optimal grey payoff value $V_I'^{*}(\otimes) = [v'_{IL}, v'_{IR}] = [2.56, 3.08]$.

According to Definition 7.10, this information's overrated and underrated venture control abilities, P_{IH} and P_{IL}, are illustrated in the following formulas:.

$$P_{IH} = v_R - v'_{IR} = 3.14 - 3.08 = 0.06 \tag{7.31}$$

$$P_{IL} = v'_{IL} - v_L = 2.56 - 2.50 = 0.06 \tag{7.32}$$

According to Theorem 7.7, in the situation of $A(\otimes)$ and $A'_I(\otimes)$, the maximal overrated and underrated risks of Players 1 and 2, $\sigma^{M}_{high-low}$ and $\sigma'^{M}_{I-high-low}$, are illustrated in Eqs. (7.33) and (7.34). Then the maximal venture control ability of the information P_{IM} is as shown in Eq. (7.35):

$$\sigma^{M}_{high-low} = v_R - v_L = 3.14 - 2.50 = 0.64 \tag{7.33}$$

$$\sigma'^{M}_{I-high-low} = v'_{IR} - v'_{IL} = 3.08 - 2.56 = 0.52 \tag{7.34}$$

$$P_{IM} = \sigma^{M}_{high-low} - \sigma'^{M}_{I-high-low} = 0.64 - 0.52 = 0.12 \tag{7.35}$$

Almost all the game problems in social and economic life are grey games, and thus are hardly described by classic game theory, which Is based on accurate mathematics. Grey matrix game $G(\otimes) = \{S_1(\otimes),S_2(\otimes); A(\otimes)\}$ illustrated by intervals, is the application of classic game theory in the inaccurate mathematical area. Not only can game models vividly describe the problems in real life, better explaining the social and economic problems, but they also makes game theory more integrated and vivid.

In the process of grey game theory research, the grey matrix game theory is the basis for further research of different grey games. The main difference between grey matrix game $G(\otimes)$ and classic matrix game $G = \{S_1,S_2; A\}$ is that the optimal mixed strategy solution of G is an accurate white number while transforming to an interval grey number in a grey matrix game $G(\otimes)$, which means there is risk in the solution. So study of risk and venture control is important in $G(\otimes)$ research. In this chapter, we discussed the recognition, definition, and measurement of that risk as well as other problems such as venture control ability and venture control of $G(\otimes)$.

7.2.4 Summary

This section defined the overrated and underrated risks of Players 1 and 2 in grey matrix games, as well as other important concepts such as maximal overrated and underrated risks and venture control ability of game information. Many relevant important theories are also proved, such as the risk relations between players, the existence of overrated and underrated risks of players, the most pessimistic and optimistic game values of Players 1 and 2, when maximal overrated and underrated risk value are equal, and conditions for reducing risks and nonrisks. We can better answer the following important questions in the grey matrix game: Does risk exist? How is it recognized? How much risk is there? How do we measure and control the risk?

Chapter 8

Concession and Damping Equilibriums of Duopolistic Strategic Output-Making Based on Limited Rationality and Knowledge

The oligopolistic market is a market mechanism in which only several firms are in competition. It has two main characteristics: there are only several firms (at least two) that monopolize a professional market, and there is not only mutual competition but also compromise between the oligopolies. The Cournot competition model (1838) is the classical model that describes decision-making problems about product competition, and now there are many literatures that discuss it.

One of the key topics for research on oligopolistic competition is a discussion of how the oligopoly can get motion superiority. Gal-Or (1985) did some research on the first-making superiority and the later-making superiority under many different response functions by studying duopoly models.[38] Dowrick (1986) proved that if the oligopolies had different response functions in the competing market, they would strive for the first-making superiority and the later-making superiority.[39] Muceller (1997) studied dependence on the leading oligopoly's first-making superiority in specific industrial cycles.[40] Vander Werf and others (1997) thought different experimental methods had obvious effects on results through comparison research on the experimental methodology of the first-making superiority.[40] Another key research is about the output of oligopolies and social welfare. Haan and Marks (1996) studied the Stackelberg model and the Cournot model from the welfare angle. They pointed out that if there were market-entering barriers, Stackelberg competition cannot certainly enhance welfare.[41] Okuguchi (1999) studied the duopoly model and pointed out that if the response functions are different, the game equilibrium will be different, as well as the profit and ultimate output of the Stackelberg model and Cournot model.[42] Rassenti and others (2000) studied the astringency of the equilibrium solutions of the repeating Cournot competition game.[43] Matsumura (1999) researched the function of the storage quantity in multistage games by analyzing the limited multistages of the Cournot duopoly model.[44] Huck and others (2001) studied the model of the Cournot competition game; they also compared the total quantity and the efficiency of the different models by researching the experimental economic method in duopoly and Cournot models.[45]

The above research mainly paid attention to the duopoly classical competition game model. Now there are many scholars who discuss the multioligopoly problem. Sherali (1983) and others constructed and studied the Stackelberg-Nash-Cournot model of one first-making oligopoly and N later-making oligopolies. They have proved the existence and uniqueness of the equilibrium solution.[46] Afterward, Sherali (1984) carried on an expansion of Stackelberg duopoly model. He also constructed and studied some competition models with many first-making oligopolies and many following oligopolies. He pointed out that the Stackelberg model and the Cournot model are exceptional cases. He concluded that the profit of the first-making oligopoly was higher than the later-making oligopoly in the situation of taking output as a competitive strategy, and the new oligopoly reduces the profits of the original oligopoly.[47] Daughety (1990) analyzed the equilibrium solutions of the Stackelberg game in general circumstances, namely m first-making oligopolies, and $n - m$ later-making oligopolies.[15] Wolf and Smeers (1997) established a stochastic Stackelberg-Nash-Cournot model, discussed equilibrium and the related conclusions of the model, and proved it by applying it to the European fuel gas market.[48]

We can see three main problems based on present research results (see Figure 8.1).

First, rationality and knowledge suppositions of the players don't correspond with reality. Most models assume that players have complete and symmetric decision-making knowledge (information), and the players are also completely rational. However, in reality, almost all players have limited knowledge and rationality.

Second, the assumptions of the players' game purposes do not correspond to reality. Most models in the past believed that the output-making game purpose of the player was to make his maximum profit in that game period. However, in reality, almost all of the output-making game players take their strategy profit as their decision-making starting post.

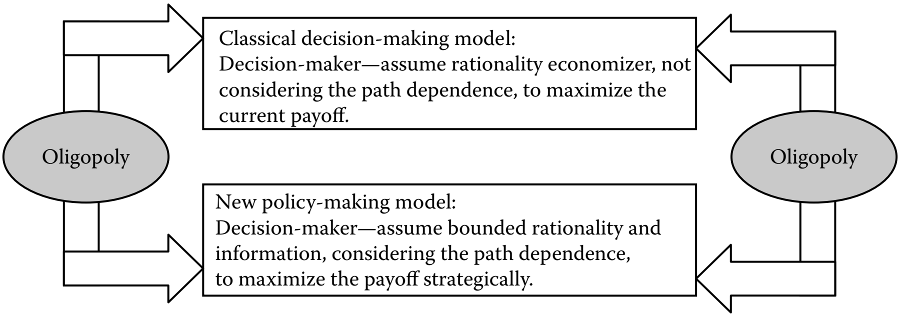


Figure 8.1 Comparison between the classic model and the new oligopoly strategy output-making decision-making model.

Third, assumptions about the path dependence of players do not correspond with reality. Most models have not considered the factor of path dependence. In fact, decision-makers have strong path dependence.

Zhigeng Fang (the author of this chapter) once researched dozens of certain monopolistic enterprises to find out how they made their output decisions. The main points about assumptions of players put forward in this chapter have been confirmed. For example, at the end of twentieth century, the competitive process of strategic output-making and price-making of some large-scale color television enterprises, such as Sichuan Rainbow, Shenzhen Kang Jia, and Nanjing Panda, finally caused the entire industry to suffer losses, which is the best illustration of our viewpoints.

Based on the assumption of decision-makers' bounded rationality and knowledge as well as the assumption of the decision-making path dependence, and considering that the purpose of the players is to maximize the strategy profit, we developed a new duopoly strategic output-making model. We also did some further research on many problems, such as the algorithm of output-making, and the attributes, characteristics, and concession equilibrium of the later decision-maker.

8.1 Duopoly Strategic Output-Making Model Based on the Experienced Ideal Output and the Best Strategy Decision-Making Coefficient

Based on the assumption of Cournot duopoly model, suppose that a product is produced by two firms, Firm 1 and Firm 2. Each firm chooses its quantity at the same time. The quantities are denoted by q_1 and q_2, respectively. The aggregate quantity in the market is therefore $Q = q_1 + q_2$. Let P denote the market clearing price and assume the aggregate output of the market is given by $P = P(Q) = Q_0 - Q$, where

Q_0 is a constant. Assume that there are no fixed costs in the production and the constant marginal costs of production are c_1 and c_2. Based on perfect rationality on both sides, they can choose the quantity that will maximize their profits and will eventually reach Nash equilibrium.

However, duopolies certainly do not have perfect rationality, and one player cannot make sure the other is full of rationality. In addition, considering the competitive strategy, the player may sacrifice the current profit in order to seize more market share, which will kick the rival out of the market when production exceeds the Nash equilibrium. Nash equilibrium based on the current maximum profit, however, may not operate automatically if players have imperfect rationality or other considerations, such as more strategically based output competition in the long run. In this way, duopolies' output-making competitive processes cannot be explained by the Cournot duopoly model. The author surveyed some oligopolistic firms and found that most output-making processes do not follow the Cournot duopoly model but largely depend on the path that reflects their or others' past experiences. They have great dependence on these direct or indirect experiences.

We will still study the duopolistic strategic output-making game as a convenient example. Based on the practical output-making process and considering the sequence when an oligopoly fixes its strategy output, we can get their output-making formulations based on the ideal output and the best strategy extended coefficient, shown in Eq. (8.1).

$$\begin{cases} q_1 = q_{01} + \gamma_1 (Q - q_{01}) & (8.1.1) \\ q_2 = q_{02} + \gamma_2 (Q - q_{02}) & (8.1.2) \\ Q = q_1 + q_2 & (8.1.3) \end{cases} \tag{8.1}$$

where

1. q_1 and q_2 denote the outputs that Oligopoly 1 and Oligopoly 2 will choose; let Q be the market capacity, which is decided by the duopoly's output.
2. q_{01} and q_{02} represent ideal output based on past experiences, respectively, that are formed by their direct experience or others' indirect experience, influenced by the output level of their own, their rival's, and the market's demand conditions. It is believed that under more preferential environmental conditions, productions based on these experiences will lead to more profit. In fact, if the duopoly is a Cournot output-making duopoly (conforming to the assumption of Cournot models and based on the fact that a Cournot oligopoly equilibrium is a Nash equilibrium), we are fully convinced that the duopoly's experienced ideal output is the solution of a Cournot model.

3. γ_1 and γ_2 describe the intended strategy extended when duopolies consider their quantity control; we call them the strategy extended coefficients. It is determined by the decision-maker's practical producing ability, extended production ability, value, decision-making customs, social and economic background, personality, psychological characteristics, moral character, and so on. If Oligopoly *i* first responds to the market, to make strategy extended coefficient first, $\gamma_i, (i = 1,2)$, we call it a first decision-making oligopoly (or initiative); otherwise, we call it a later decision-making oligopoly (passive). A later decision-making oligopoly will find situations of an advanced decision-making oligopoly through other ways (for instance, the oligopoly can be judged according to factors such as its advanced action, prompt signal, the custom, tradition, and individuality of former action, and the situation and background of the present action).
4. The decision-maker will determine the current output in terms of the sum of experienced ideal output $q_{0i}\ (i = 1,2)$ and strategy extended output $\gamma_i (Q - q_{0i}), (i = 1,2)$. $q_{0i}\ (i = 1,2)$ can be decided by comparing the current situation with the historical situation and referring to some historical statistical data. $\gamma_i (Q - q_{0i}), (i = 1,2)$ reflects the strategy extended production in the future market share, and it also can be understood as a strategic contraction (negative expansion), which is decided by the market, plus factors as the strategy expands, and minus factors as the strategy contracts.

Theorem 8.1 When two oligarchic manufacturers determine the output, the value range of strategy extended outputs, γ_1 and γ_1, are $\gamma_1 \in [\frac{-q_{01}}{Q-q_{01}}, 1]$ and $\gamma_2 \in [\frac{-q_{02}}{Q-q_{02}}, 1]$, respectively.

Proof: The output should satisfy the following constraint conditions when two oligarchic manufacturers make decisions.

$$\begin{cases} 0 \le q_1 \le Q & (8.2.1) \\ 0 \le q_2 \le Q & (8.2.2) \\ Q = q_1 + q_2 & (8.2.3) \end{cases} \tag{8.2}$$

From Eqs. (8.1) and (8.2), we can get the value range of strategy extended output, γ_1 and γ_2, which is shown by Eq. (8.3):

$$\begin{cases} \dfrac{-q_{01}}{Q - q_{01}} \le \gamma_1 \le 1 & (8.3.1) \\ \\ \dfrac{-q_{02}}{Q - q_{02}} \le \gamma_2 \le 1 & (8.3.2) \end{cases} \tag{8.3}$$

From Eq. (8.3), the value ranges of γ_1 and γ_2 are $\gamma_1 \in [\frac{-q_{01}}{Q-q_{01}}, 1]$ and $\gamma_2 \in [\frac{-q_{02}}{Q-q_{02}}, 1]$, respectively.

Inference 8.1 When the values of strategy extended output γ_1 and γ_2 are respectively 0 and 1, the two oligarchic manufacturers will take the strategy of the experienced ideal output or the output of market capacity.

The conclusion is obvious based on Eq. (8.1), so the proof is omitted here.

Theorem 8.2 Two oligarchies have the same unit production cost c; namely, $C_1(q_1) = q_1 c$ and $C_2(q_2) = q_2 c$. Their output-making processes satisfy Eq. (8.1), and the experienced ideal output q_{01} and q_{02} equals the Cournot equilibrium output q_1^* and q_2^* respectively. Then the optimal decision of current output is $\gamma_1 = \gamma_2 = 0$ and the optimal output decision is $q_1 = q_1^* = q_{01}, q_2 = q_2^* = q_{02}, q_1 = q_2 = \frac{1}{3}(Q_0 - c)$.

Proof: According to the , as in Eq. (8.1), and the Cournot model, the output decision of a Nash equilibrium is $q_1^* = q_2^* = \frac{1}{3}(Q_0 - c)$. Let $k = Q_0 - c$, then $q_1^* = q_2^* = \frac{k}{3}$, and based on the known conditions in the proposition, we can get $q_{01} = q_{02} = q_0 = \frac{k}{3}$.

From Eq. (8.1), considering the optimal decision of Oligopoly 1 to ensure its maximal profit u_1^*, we can get Eq. (8.4):

$$\begin{aligned}&\gamma_2\left[4q_0^2 - 2q_0(Q_0 - c)\right] + \gamma_2^2\left[-q_0^2 + q_0(Q_0 - c)\right] + \gamma_1\gamma_2\left[-q_0^2 + q_0(Q_0 - c)\right] + \\ &\quad + \gamma_1\left[q_0^2 - q_0(Q_0 - c)\right] - 3q_0^2 + q_0(Q_0 - c) = 0\end{aligned} \tag{8.4}$$

For the same reason, we can get Eq. (8.5):

$$\begin{aligned}&\gamma_1\left[4q_0^2 - 2q_0(Q_0 - c)\right] + \gamma_1^2\left[-q_0^2 + q_0(Q_0 - c)\right] + \gamma_1\gamma_2\left[-q_0^2 + q_0(Q_0 - c)\right] + \\ &\quad + \gamma_2\left[q_0^2 - q_0(Q_0 - c)\right] - 3q_0^2 + q_0(Q_0 - c) = 0\end{aligned} \tag{8.5}$$

Equation (8.4) minus Eq. (8.5) gives Eq. (8.6):

$$\begin{cases}(\gamma_1 - \gamma_2)\left\{3q_0^2 - q_0(Q_0 - c)\right\} + \left(\gamma_1^2 - \gamma_2^2\right)\left[-q_0^2 + q_0(Q_0 - c)\right] = 0 \\ (\gamma_1 - \gamma_2)\left\{\left[3q_0^2 - q_0(Q_0 - c)\right] + (\gamma_1 + \gamma_2)\left[-q_0^2 + q_0(Q_0 - c)\right]\right\} = 0\end{cases} \tag{8.6}$$

The solution of Eq. (8.6) is shown in Eq. (8.7):

$$\begin{cases} y_1 = y_2 & (8.7.1) \\ y_1 = -y_2 & (8.7.2)\end{cases} \tag{8.7}$$

Placing Eq. (8.7.2) into Eq. (8.4) is meaningless. Based on Eqs. (8.4) and (8.7.1), we can get Eq. (8.8):

$$\gamma_1\left[\frac{4k^2}{9}-\frac{2k^2}{3}\right]+\gamma_1^2\left[\frac{-k^2}{9}+\frac{k^2}{3}\right]+\gamma_1^2\left[\frac{-k^2}{9}+\frac{k^2}{3}\right]+\gamma_1\left[\frac{k^2}{9}-\frac{k^2}{3}\right]-\frac{3k^2}{9}-\frac{k^2}{3}=0$$

$$-\frac{2k^2\gamma_1}{9}-\frac{4k^2\gamma_1^2}{9}-\frac{2k^2\gamma_1}{9}=0$$

$$\left(\frac{4k^2\gamma_1}{9}-\frac{4k^2}{9}\right)\gamma_1=0$$

$$\begin{cases}\gamma_1=0\\ \gamma_1^{\prime}=1\end{cases} \tag{8.8}$$

In Eq. (8.5), when $\gamma_1^{\prime}=1$, the formula is meaningless, which demonstrates that Oligopoly 1 has met the market capacity and it is useless to consider an optimal decision of Oligopoly 2 with regard to maximal profit. Then let $\gamma_1=0$ in Eq. (8.1), and we can get $q_1=q_1^*=q_{01}$. It shows the optimal output decision of Oligopoly 1 is $q_1^*=q_{01}=\frac{1}{3}(Q_0-c)$.

From Eqs. (8.7) and (8.8), we may obtain $\gamma_2=0, q_2=q_2^*=q_{02}=\frac{1}{3}(Q_0-c)$.

So for two Cournot oligopolies, the optimal strategy of the current output is $\gamma_1=\gamma_2=0$ and the optimal output strategy is $q_1=q_1^*=q_{01}$, $q_2=q_2^*=q_{02}$, $q_1=q_2=\frac{1}{3}(Q_0-c)$.

Theorem 8.2 shows that the Cournot model is a specific case of the oligarchic model when the relevant parameters of the model are some specific values.

The duopoly strategy of output decision should satisfy Eq. (8.1). If we have to ensure the maximal profit of the current strategy, the total output of the two oligopoly manufactures is shown in Theorem 8.3.

Theorem 8.3 The experienced ideal outputs of the duopolies are assigned respectively $q_{0i}(i=1,2)$. If their output-making process of decision-making satisfies Eq. (8.1), then when faced with different first decision-making oligopolies j $(j=1,2)$, the market product supply Q_{2-j}^* $(j=1,2)$, which benefits from decision-making maximization, is decided by the experienced ideal product q_{0i} $(i=1,2)$ of the first decision-making oligopoly and the unit product cost c_i $(i=1,2)$ of the later decision-making oligopoly, shown by Eq. (8.9.1) and (8.9.2) separately:

$$\begin{cases}Q_{2-j}^*=Q_1^*=\dfrac{q_{02}+(Q_0-c_1)}{2}, j=2 \quad (8.9.1)\\[2ex] Q_{2-j}^*=Q_2^*=\dfrac{q_{01}+(Q_0-c_2)}{2}, j=1 \quad (8.9.2)\end{cases} \tag{8.9}$$

Proof: Suppose Oligopoly 2 is the first decision-making oligopoly—namely, it is assigned the first decision-making coefficient γ_{20}—then we might obtain the best decision-making coefficient $\gamma_1^{(0)}$ for the benefit maximization of Oligopoly 1, and we also can get Eq. (8.10). According to Eq. (8.10), we may obtain Eqs. (8.11) and (8.12):

$$1-\gamma_1^{(0)}=\frac{\left\{\begin{array}{l}-\gamma_{20}^3(q_{02}+c_1-Q_0)(q_{01}-q_{02})+\\+\gamma_{20}\left\{-3q_{01}q_{02}+5q_{02}^2+q_{01}^2-q_{02}(Q_0-c_1)\right\}+\\+\gamma_{20}^2\left\{3q_{01}q_{02}-4q_{02}^2-q_{01}(Q_0-c_1)+2q_{02}(Q_0-c_1)\right\}+\\-2q_{02}^2\end{array}\right\}}{\left\{\begin{array}{l}\gamma_{20}\left\{\left(q_{01}^2-3q_{01}q_{02}+2q_{02}^2\right)+(q_{01}+c_1-Q_0)(q_{01}-2q_{02})\right\}+\\-\gamma_{20}^2\left\{(q_{01}-q_{02})^2+(q_{01}+c_1-Q_0)(q_{01}-q_{02})\right\}+\\+\{q_{02}(q_{01}-q_{02})+q_{02}(q_{01}+c_1-Q_0)\}\end{array}\right\}} \tag{8.10}$$

$$\begin{aligned}&q_{01}\left(1-\gamma_1^{(0)}\right)+q_{02}(1-\gamma_{20})\\&=\frac{\left\{\begin{array}{l}+\gamma_{20}^3\left\{(q_{01}-q_{02})\left[q_{01}q_{02}-q_{02}^2-q_{02}(Q_0-c_1)+q_{01}(Q_0-c_1)\right]\right\}+\\+\gamma_{20}^2\left\{-q_{01}^2q_{02}+4q_{01}q_{02}^2-q_{01}^2(Q_0-c_1)+4q_{01}q_{02}(Q_0-c_1)-3q_{02}^3-3q_{02}^2(Q_0-c_1)\right\}+\\+\gamma_{20}\left\{-q_{01}^2q_{02}-2q_{01}q_{02}^2+3q_{02}^3+q_{01}^3-2q_{01}q_{02}(Q_0-c_1)+3q_{02}^2(Q_0-c_1)\right\}+\\-q_{02}^3-q_{02}^2(Q_0-c_1)\end{array}\right\}}{\left\{\begin{array}{l}\gamma_{20}\left\{\left(q_{01}^2-3q_{01}q_{02}+2q_{02}^2\right)+(q_{01}+c_1-Q_0)(q_{01}-2q_{02})\right\}+\\-\gamma_{20}^2\left\{(q_{01}-q_{02})^2+(q_{01}+c_1-Q_0)(q_{01}-q_{02})\right\}+\\+\{q_{02}(q_{01}-q_{02})+q_{02}(q_{01}+c_1-Q_0)\}\end{array}\right\}}\end{aligned} \tag{8.11}$$

$$1-\gamma_1^{(0)}-\gamma_{20}=\frac{\left\{\begin{array}{l}\gamma_{20}^3\left\{2(q_{01}-q_{02})^2\right\}+\gamma_{20}^2\left\{8q_{01}q_{02}-6q_{02}^2-2q_{01}^2\right\}+\\+\gamma_{20}\left\{-5q_{01}q_{02}+6q_{02}^2+q_{01}^2\right\}-2q_{02}^2\end{array}\right\}}{\left\{\begin{array}{l}\gamma_{20}\left\{\left(q_{01}^2-3q_{01}q_{02}+2q_{02}^2\right)+(q_{01}+c_1-Q_0)(q_{01}-2q_{02})\right\}+\\-\gamma_{20}^2\left\{(q_{01}-q_{02})^2+(q_{01}+c_1-Q_0)(q_{01}-q_{02})\right\}+\\+\{q_{02}(q_{01}-q_{02})+q_{02}(q_{01}+c_1-Q_0)\}\end{array}\right\}} \tag{8.12}$$

According to the above hypothesis of the duopoly output decision-making process, by Eq. (8.1), we can get the supply quantity of the oligopoly that is shown by Eq. (8.13):

$$Q_1^* = \frac{q_{01}(1-\gamma_1)+q_{02}(1-\gamma_2)}{(1-\gamma_1-\gamma_2)} = \frac{q_{01}\left(1-\gamma_1^{(0)}\right)+q_{02}(1-\gamma_{20})}{\left(1-\gamma_1^{(0)}-\gamma_{20}\right)}$$

$$= \frac{\left\{\begin{array}{l} +\gamma_{20}^3\left\{(q_{01}-q_{02})\left[q_{01}q_{02}-q_{02}^2-q_{02}(Q_0-c_1)+q_{01}(Q_0-c_1)\right]\right\}+ \\ +\gamma_{20}^2\left\{-q_{01}^2q_{02}+4q_{01}q_{02}^2-q_{01}^2(Q_0-c_1)+4q_{01}q_{02}(Q_0-c_1)-3q_{02}^3-3q_{02}^2(Q_0-c_1)\right\}+ \\ +\gamma_{20}\left\{-q_{01}^2q_{02}-2q_{01}q_{02}^2+3q_{02}^3+q_{01}^3-2q_{01}q_{02}(Q_0-c_1)+3q_{02}^2(Q_0-c_1)\right\}+ \\ -q_{02}^3-q_{02}^2(Q_0-c_1) \end{array}\right\}}{\left\{\begin{array}{l} \gamma_{20}^3\left\{2(q_{01}-q_{02})^2\right\}+\gamma_{20}^2\left\{8q_{01}q_{02}-6q_{02}^2-2q_{01}^2\right\}+ \\ +\gamma_{20}\left\{-5q_{01}q_{02}+6q_{02}^2+q_{01}^2\right\}-2q_{02}^2 \end{array}\right\}}$$

$$= \frac{\left\{\begin{array}{l} +\gamma_{20}^3\left\{\left[q_{01}^2q_{02}-2q_{01}q_{02}^2+q_{02}^2(Q_0-c_1)+q_{02}^3-2q_{01}q_{02}(Q_0-c_1)+q_{01}^2(Q_0-c_1)\right]\right\}+ \\ +\gamma_{20}^2\left\{-q_{01}^2q_{02}+4q_{01}q_{02}^2-q_{01}^2(Q_0-c_1)+4q_{01}q_{02}(Q_0-c_1)-3q_{02}^3-3q_{02}^2(Q_0-c_1)\right\}+ \\ +\gamma_{20}\left\{-q_{01}^2q_{02}-2q_{01}q_{02}^2+3q_{02}^3+q_{01}^3-2q_{01}q_{02}(Q_0-c_1)+3q_{02}^2(Q_0-c_1)\right\}+ \\ -q_{02}^3-q_{02}^2(Q_0-c_1) \end{array}\right\}}{\left\{\begin{array}{l} \gamma_{20}^3\left\{2(q_{01}-q_{02})^2\right\}+\gamma_{20}^2\left\{8q_{01}q_{02}-6q_{02}^2-2q_{01}^2\right\}+ \\ +\gamma_{20}\left\{-5q_{01}q_{02}+6q_{02}^2+q_{01}^2\right\}-2q_{02}^2 \end{array}\right\}} \tag{8.13}$$

Multiplying the right-hand side of Eq. (8.13) by $\frac{2}{q_{02}+(Q_0-c_1)} \cdot \frac{q_{02}+(Q_0-c_1)}{2}$, we obtain Eq. (8.14):

$$Q_1^* = \frac{2\left\{\begin{array}{l} +\gamma_{20}^3\left\{\left[q_{01}^2q_{02}-2q_{01}q_{02}^2+q_{02}^2(Q_0-c_1)+q_{02}^3-2q_{01}q_{02}(Q_0-c_1)+q_{01}^2(Q_0-c_1)\right]\right\}+ \\ +\gamma_{20}^2\left\{-q_{01}^2q_{02}+4q_{01}q_{02}^2-q_{01}^2(Q_0-c_1)+4q_{01}q_{02}(Q_0-c_1)-3q_{02}^3-3q_{02}^2(Q_0-c_1)\right\}+ \\ +\gamma_{20}\left\{-q_{01}^2q_{02}-2q_{01}q_{02}^2+3q_{02}^3+q_{01}^3-2q_{01}q_{02}(Q_0-c_1)+3q_{02}^2(Q_0-c_1)\right\}+ \\ -q_{02}^3-q_{02}^2(Q_0-c_1) \end{array}\right\}}{[q_{02}+(Q_0-c_1)]\cdot\left\{\begin{array}{l} \gamma_{20}^3\left\{2(q_{01}-q_{02})^2\right\}+\gamma_{20}^2\left\{8q_{01}q_{02}-6q_{02}^2-2q_{01}^2\right\}+ \\ +\gamma_{20}\left\{-5q_{01}q_{02}+6q_{02}^2+q_{01}^2\right\}-2q_{02}^2 \end{array}\right\}}$$

$$\cdot\frac{q_{02}+(Q_0-c_1)}{2} = \frac{q_{02}+(Q_0-c_1)}{2} \tag{8.14}$$

Likewise, we may obtain the market equilibrium output when Oligopoly 2 is the later decision-making oligopoly, as shown in Eq. (8.15):

$$Q_2^* = \frac{q_{01} + (Q_0 - c_2)}{2} \tag{8.15}$$

8.2 Concession Equilibrium of the Later Decision-Maker under Nonstrategic Extended Damping Conditions: Elimination from the Market

In the practical process of duopoly decision-making, considered from the view of maximizing current profit, the advanced decision-making oligopoly will fulfill its strategic aim—occupying more market share—by two paths: the strategic extended coefficient $\gamma_i\ (i = 1,2)$ and the empiric ideal output $q_{0i}(i = 1,2)$. At the same time, the later decision-making oligopoly makes its output decision.

An expanding strategy of production capacity is limited by many factors (for example, practical production capacity, the potential of production expansion, capital expansion ability, management ability, conception, personality, habits, and so on), and we call these the strategic extended damping conditions. We consider this problem in two situations: the first, when there are some damping problems on certain products, where $\gamma_i\ (i = 1,2)$ can be made very close to or equal to 1; and the second when there may exist certain strategic extended damping problems on some products, where $\gamma_i\ (i = 1,2)$ cannot be set very high all at one time. This is mainly because, in reality, producers' production capacity, investing ability, and management ability will be enhanced step by step though construction and study.

We explore the first situation in Theorem 8.4.

Theorem 8.4 There is a duopoly strategy output-making problem. The competitive process of output-making meets Eq. (8.1); moreover, there is no damping limit to the strategic extended process. Provided that an oligopoly's rival makes its decision first, because of its incomplete rationality, greedy psychology, and certain possibility of greedy motion, its strategic extended coefficient is $\gamma_i = \gamma_{i0} > 0\ (i = 1,2)$. Then the oligopoly may possess the best reflection pattern: a concession pattern to deal with its competitor that is based on current maximum profit. A psychological concession, $\gamma_i^{(0)}(i = 1,2)$, reduces output concession, a reduction of its market shares.

Proof: Provided that Oligopoly 2 is the advanced decision-maker, Oligopoly 1 considers the strategic extended coefficient of Oligopoly 2 as a constant, and responds to the advanced decision. Equation (8.9.1) indicates that when Oligopoly 2 responds positively and Oligopoly 1 responses passively, the market supply is Q_1^* based on the current maximum profit from the duopoly's view. We can conclude from Eq. (8.1.2) that $Q = Q_1^*$ is a constant; when Oligopoly 2 becomes greedier, that is to say, $\gamma_2 = \gamma_{20}$ increases and its market share, q_2, increases, at the same time, the market share of Oligopoly 1, $q_1 = Q_1^* - q_2$, decreases. As long as q_{02} and c_1 do not change, Q_1^* is a constant, so when $\gamma_2 = \gamma_{20}$ rises, it will lead to an increase of q_2 and a decrease of q_1. Oligopoly 1 has a decreasing market share and concedes production.

When $Q = Q_1^*$ is invariable, Oligopoly 2 responds positively and become greedier, and $\gamma_2 = \gamma_{20}$ increases. Combined with the above analysis, Oligopoly 1 has to concede output with the target of current maximum profit, and q_1 decreases. We could get Eq. (8.16) from Eq. (8.1). From Eq. (8.16), we could conclude that when q_1 decreases, γ_1 will respond to the decrease: Oligopoly 1 is making a psychological concession.

$$\gamma_1 = \frac{(q_1 - q_{01})}{(Q - q_{01})} \tag{8.16}$$

We can conclude from Theorem 8.4 that under this hypothesis, if the advanced decision-maker seizes the market first, the later decision-maker will concede in psychology and production, which will lead to a concession equilibrium.

Inference 8.2 In Theorem 8.4, there are no strategic extended damping conditions; strategic decisions are based on current maximum profit, so the best strategy for the advanced decision-maker is to eliminate the later decision-maker from the market.

This conclusion of the deduction is obvious, so the process of proof is omitted; only a qualitative explanation is given here.

We can conclude from Theorem 8.4 that if both of the duopolies' strategic decisions are based on current maximum profit, the advanced decision-maker would achieve the advantage of advanced decision-making. In other words, the later decision-maker regards the output of advanced decision-maker as a constant and takes the strategy of production and psychological concession. Combined with the consideration of these two aspects, the later decision-maker will eventually be expelled from the market. One reason is that if the advanced decision-maker chooses a large production (close to or almost equal to the capacity of the market), the later decision-maker will have no available market. Another reason is that when the advanced decision-maker makes early game decisions, the later decision-maker might not be eliminated from the market but, by following the study of decision-making, it will increase the advanced decision-maker's output in order to pursue more strategic decision benefits that would be close to or equal to the capacity of the market, so at last the late decision-maker will be eliminated from the market (see Table 8.1).

Example 8.1 Under nonstrategic extended damping conditions, the advanced strategy extended coefficient γ_{20} increases. The concession-equilibrium simulation of Oligopoly 1 is shown in Table 8.1. Each correlation formula and the parameter values are noted in Table 8.1.

Table 8.1 γ_{20} **Increases,** γ_{20} **the Concession-Equilibrium of Simulation of Oligopoly 1**

γ_{20} (*chosen*)	$\gamma_1^{(0)}$	q_2	q_1	$Q_1^* = q_1 + q_2$	u_1	u_2
–1	1	0	4	4	8	0
–0.8	0.6	0.4	3.6	4	7.2	1.2
–0.6	0.2	0.8	3.2	4	6.4	2.4
–0.4	–0.2	1.2	2.8	4	5.6	3.6
–0.2	–0.6	1.6	2.4	4	4.8	4.8
0	–1	2	2	4	4	6
0.2	–1.4	2.4	1.6	4	3.2	7.2
0.4	–1.8	2.8	1.2	4	2.4	8.4
0.6	–2.4	3.2	0.8	4	1.6	9.6
0.8	–2.6	3.6	0.4	4	0.8	10.8
0.999	–2.998	3.998	0.002	4	0.004	11.994

Note: γ_{10} and γ_{20} express the duopoly's strategic extended coefficients respectively; $\gamma_1^{(0)}$ is the strategic extended response coefficient under the condition of Oligopoly 1's current maximum profit, when the parameter γ_{20} is chosen. Q_1^* represents the market supply of production for the advanced decision-maker. The related variables in the simulation formula can be seen in the corresponding text. $u_1 = q_1 \cdot P(Q) - c_1 \cdot q_1\ u_2 = q_2 \cdot P(Q) - c_2 \cdot q_2$ represents the income values of Oligopolies 1 and 2 separately; the simulation parameter's initial value is $q_{01} = 3, q_{02} = 2, c_1 = 2, c_2 = 1, Q_0 = 8$.

Table 8.1 shows that under the nonstrategic extended damping condition, the concession equilibrium of the later decision-maker is a kind of Nash equilibrium with certain conditions. Based on current strategic maximum profit, the later decision-maker regards the decision of the advanced decision-maker as a constant; in such a situation, the best strategy for the later decision-maker is to exit from the market before he is eliminated from the market.

8.3 Damping Equilibrium of the Advanced Decision-Maker under Strategic Extended Damping Conditions: Giving Up Some Market Share

In this part, we mainly study the duopolies' strategic output-making equilibrium under certain strategic extended damping conditions.

Theorem 8.5 There is a duopoly strategic output-making problem. If the output-making competitive process conforms to Eq. (8.1), and the damping condition restrictions exist in the strategic extended process [the coefficient of strategic extension can only take certain value $\gamma_i = \gamma_{i0}, (i = 1, 2)$], then there is a strategic output-making equilibrium based on the current maximum benefit.

Proof: Suppose Oligopoly 2 makes its decision first and Oligopoly 1 makes its decision later. Because there are some damping constraints in the strategic extended process of Oligopoly 2 (its strategic extended coefficient is a certain constant $\gamma_2 = \gamma_{20}$), the current decision-making benefit of Oligopoly 2 can be described by Eq. (8.17):

$$
\begin{aligned}
u_2 &= q_2 \cdot P(Q) - c_2 \cdot q_2 = q_2 \cdot (Q_0 - Q_1^* - c_2) \\
&= \left(q_{02} + \gamma_{20} \cdot \left(\frac{Q_0 - q_{02} - c_1}{2} \right) \right) \cdot \left(Q_0 - c_2 - \frac{q_{02} + Q_0 - c_1}{2} \right) \\
&= \left(\frac{2q_{02} + (Q_0 - q_{02} - c_1)\gamma_{20}}{2} \right) \cdot \left(\frac{Q_0 - 2c_2 - q_{02} + c_1}{2} \right) \\
&= \frac{1}{4}(2q_{02} + (Q_0 - q_{02} - c_1)\gamma_{20}) \cdot (Q_0 - 2c_2 - q_{02} + c_1)
\end{aligned}
\tag{8.17}
$$

In Eq. (8.17), we get the partial derivatives of its experienced ideal output q_{02}, shown by Eq. (8.18). If we make $\frac{\partial u_2}{\partial q_{02}} = 0$ [shown by Eq. (8.19)], then we can get the empirical ideal output q_{02} of Oligopoly 2 based on current maximum profit, shown by Eq. (8.20).

$$
\begin{aligned}
\frac{\partial u_2}{\partial q_{02}} &= \frac{1}{4}[(2 - \gamma_{20}) \cdot (Q_0 - 2c_2 - q_{02} + c_1) + (-1)(2q_{02} + (Q_0 - q_{02} - c_1)\gamma_{20})] \\
&= \frac{1}{4}[(2 - \gamma_{20}) \cdot (Q_0 - 2c_2 - q_{02} + c_1) - (2q_{02} + (Q_0 - q_{02} - c_1)\gamma_{20})]
\end{aligned}
\tag{8.18}
$$

By making

$$
\frac{\partial u_2}{\partial q_{02}} = \frac{1}{4}[(2 - \gamma_{20}) \cdot (Q_0 - 2c_2 - q_{02} + c_1) - (2q_{02} + (Q_0 - q_{02} - c_1)\gamma_{20})] = 0
$$

then we can get Eq. (8.19) as follows:

$$
\begin{aligned}
&(2 - \gamma_{20}) \cdot (Q_0 - 2c_2 - q_{02} + c_1) - (2q_{02} + (Q_0 - q_{02} - c_1)\gamma_{20}) = 0 \\
&(Q_0 - 2c_2 + c_1)(2 - \gamma_{20}) - (Q_0 - c_1)\gamma_{20} = 2q_{02}(2 - \gamma_{20}) \\
&2q_{02}(2 - \gamma_{20}) = (Q_0 - 2c_2 + c_1)(2 - \gamma_{20}) - (Q_0 - c_1)\gamma_{20}
\end{aligned}
\tag{8.19}
$$

By Eq. (8.19), we can get Eq. (8.20):

$$q_{02}^{*}=\frac{(Q_0-2c_2+c_1)(2-\gamma_{20})-(Q_0-c_1)\gamma_{20}}{2(2-\gamma_{20})} \tag{8.20}$$

By Eqs. (8.1.1) and (8.20), we can get Eq. (8.21); by Eqs. (8.1.2), (8.20), and (8.21), we can get Eq. (8.22); and by Eqs. (8.1.3), (8.21), and (8.22), we can get Eq. (8.23):

$$\begin{aligned}
Q_1^{**}=Q_1^{*}\big|_{q_{02}=q_{02}^{*}}&=\frac{(Q_0-2c_2+c_1)(2-\gamma_{20})-(Q_0-c_1)\gamma_{20}}{4(2-\gamma_{20})}+\frac{(Q_0-c_1)}{2}\\
&=\frac{2(Q_0-2c_2+c_1)-2\gamma_{20}(Q_0-c_2)}{4(2-\gamma_{20})}+\frac{4(Q_0-c_1)-2\gamma_{20}(Q_0-c_1)}{4(2-\gamma_{20})}\\
&=\frac{(3Q_0-2c_2-c_1)-\gamma_{20}(2Q_0-c_2-c_1)}{2(2-\gamma_{20})}
\end{aligned} \tag{8.21}$$

$$q_2^{*}=q_{02}+\gamma_{20}(Q_1^{*}-q_{02})=\frac{2(Q_0-2c_2+c_1)+\gamma_{20}(4c_2-3c_1-Q_0)+\gamma_{20}^{2}(c_1-c_2)}{2(2-\gamma_{20})} \tag{8.22}$$

$$q_1^{*}=Q_1^{*}-q_2^{*}=\frac{(Q_0+2c_2-3c_1)-\gamma_{20}(Q_0-4c_1+3c_2)-\gamma_{20}^{2}(c_1-c_2)}{2(2-\gamma_{20})} \tag{8.23}$$

From Eqs. (8.22) and (8.23), we can get the outputs of Oligopolies 1 and 2 based on the strategic output-making equilibrium of the current maximum profit.

The proof of Theorem 8.5 is a structure process. It proves that strategic extended process has two solution methods based on the oligopoly strategic output-making equilibrium problem of current maximum profit under the damping-limiting condition. The equilibrium of Theorem 8.5 is the result of the damping conditions of the advanced decision-maker, and we call this the equilibrium damping equilibrium of the advanced decision-maker, or damping equilibrium for short.

Although the advanced decision-maker has a decided advantage of first decision-making, the strategic extended damping conditions make it impossible for the advanced decision-maker to occupy the entire market quickly. There is a damping equilibrium to the duopoly that is also a kind of Nash equilibrium with some conditions. Furthermore, there is a relationship between this damping equilibrium and the equilibrium of the classic Cournot model, as shown by Theorem 8.6.

Theorem 8.6 In Theorem 8.5, if both of the duopolies' strategic extended coefficients are 0 (neither oligopoly has a strategic extended tendency)—namely, the strategic extended damping coefficient is so big that both sides are unable to carry on the extended strategy—then the output of the advanced decision-maker, $q_j^*, (j = 1,2)$, is six times the equilibrium output of classic Cournot model $q_j^{*/} (j = 1,2)$, shown by Eq. (8.24):

$$q_j^* = 6q_j^{*/} \tag{8.24}$$

Proof: According to the theory of the classic Cournot model, we can get the duopolies' equilibrium output of a classic Cournot model, shown by Eq. (8.25).

$$\begin{cases} q_1^{*/} = \dfrac{1}{3}(Q_0 - 2c_1 + c_2) & (8.25.1) \\ q_2^{*/} = \dfrac{1}{3}(Q_0 + c_1 - 2c_2) & (8.25.2) \end{cases} \tag{8.25}$$

Suppose Oligopoly 2 makes its decision first, and Oligopoly 1 makes a later decision. According to Eqs. (8.22) and (8.23), we can get the damping equilibrium output of Oligopoly 1 and Oligopoly 2, when $\gamma_{i0} = 0 (i = 1,2)$ and the value of strategic extended damping is infinite, as shown by Eq. (8.26):

$$\begin{cases} q_1^* = Q_0 - 3c_1 + 2c_2 & (8.26.1) \\ q_2^* = 2(Q_0 + c_1 - 2c_2) & (8.26.2) \end{cases} \tag{8.26}$$

Comparing Eq. (8.26.2) with Eq. (8.25.2), we can get Eq. (8.27):

$$q_2^* = 6q_2^{*/} \tag{8.27}$$

According to a similar principle, when Oligopoly 1 makes its decision first, and Oligopoly 2 makes a later decision, the conclusion is also right.

Inference 8.3 In Theorem 8.5, if both the oligopolies' strategic extended coefficients are 0 (neither oligopoly has a strategic extended tendency)—namely, the strategic extended damping coefficient is so big that both sides are unable to carry on the strategy expansion—then there is a relationship between the supply amount of market and the one decided by the classic Cournot model, shown by Eq. (8.28):

$$\begin{cases} Q_1^* - Q^{*/} = \dfrac{7Q_0 - 2c_1 - 5c_2}{3} & (8.28.1) \\ Q_2^* - Q^{*/} = \dfrac{7Q_0 - 5c_1 - 2c_2}{3} & (8.28.2) \end{cases} \tag{8.28}$$

Proof: According to the classic Cournot model, we can get the equilibrium supply amount by the classic Cournot model by adding Eq. (8.25.1) to Eq. (8.25.2).The result is shown by Eq. 29:

$$Q^{*l} = \frac{1}{3}(2Q_0 - c_1 - c_2) \tag{8.29}$$

Suppose that Oligopoly 2 makes its decision first and Oligopoly 1 makes a later decision. By adding Eq. (8.26.1) to Eq. (8.26.2), we can get the equilibrium supply amount of market production based on this model, shown by Eq. (8.30). Equation (8.30) minus Eq. (8.29) gives us Eq. (8.31):

$$Q_1^* = 3Q_0 - 2c_2 - c_1 \tag{8.30}$$

$$Q_1^* - Q^{*l} = \frac{7Q_0 - 2c_1 - 5c_2}{3} \tag{8.31}$$

Then suppose Oligopoly 1 makes its decision first and Oligopoly 2 makes a later decision. In the same way, we can prove that Eq. (8.28.2) is right.

Example 8.2 When strategic extended damping conditions exist, the damping equilibrium simulation result of Oligopoly 2, which makes its decision first, is shown by Table 8.2, and the values of each formula and their parameters are shown in the notes of Table 8.2.

Table 8.2 shows that if there are strategic extended damping conditions, the advanced decision-maker, Oligopoly 2, has the damping equilibrium on the best empirical ideal output and the maximum income, and the values of γ_{20} increase as the curve of u_2 is continually increasing, as shown in Figure 8.2.

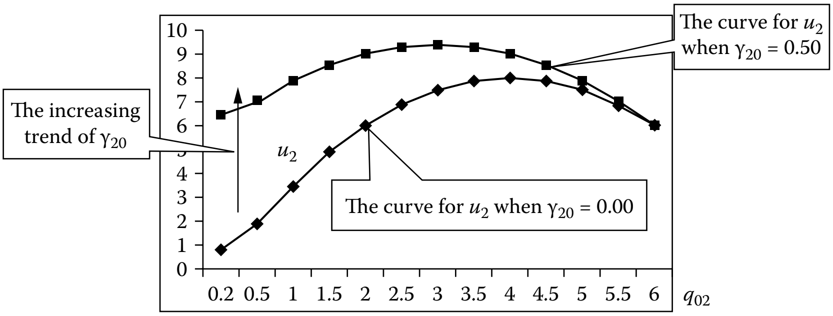


Figure 8.2 Curve of the relationship between q_{02} and u_2 as the values of γ_{20} change.

Table 8.2 Damping Equilibrium Simulation Result of Oligopoly 2 When Strategic Extended Damping Conditions Exist

q_{02} (*chosen*)	γ_{20} (*given*)	$\gamma_1^{(0)}$	q_2	q_1	$Q_1^* = q_1 + q_2$	u_1	u_2
0.200	0.000	0.818	0.200	2.900	3.100	8.410	0.780
0.500		0.600	0.500	2.750	3.250	7.563	1.875
1.000		0.333	1.000	2.500	3.500	6.250	3.500
1.500		0.143	1.500	2.250	3.750	5.063	4.875
2.000		0.000	2.000	2.000	4.000	4.000	6.000
2.500		–0.111	2.500	1.750	4.250	3.063	6.875
3.000		–0.200	3.000	1.500	4.500	2.250	7.500
3.500		–0.273	3.500	1.250	4.750	1.563	7.875
4.000		–0.333	4.000	1.000	5.000	1.000	8.000
4.500		–0.385	4.500	0.750	5.250	0.563	7.875
5.000		–0.429	5.000	0.500	5.500	0.250	7.500
5.500		–0.467	5.500	0.250	5.750	0.063	6.875
6.000		–0.500	6.000	0.000	6.000	0.000	6.000
0.200	0.500	–0.500	1.650	1.450	3.100	4.205	6.435
0.500		–0.500	1.875	1.375	3.250	3.781	7.031
1.000		–0.500	2.250	1.250	3.500	3.125	7.875
1.500		–0.500	2.625	1.125	3.750	2.531	8.531
2.000		–0.500	3.000	1.000	4.000	2.000	9.000
2.500		–0.500	3.375	0.875	4.250	1.531	9.281
3.000		–0.500	3.750	0.7500	4.500	1.125	9.375
3.500		–0.500	4.125	0.625	4.750	0.781	9.281
4.000		–0.500	4.500	0.500	5.000	0.500	9.000
4.500		–0.500	4.875	0.375	5.250	0.281	8.531
5.000		–0.500	5.250	0.250	5.500	0.125	7.875
5.500		–0.500	5.625	0.125	5.750	0.031	7.031
6.000		–0.500	6.000	0.000	6.000	0.000	6.000

Note: γ_{10} and γ_{20} stand for the duopolies' strategic extended coefficients separately; Q_1^* stands for the market production supply amount when Oligopoly 2 makes its decision first; the related variables' simulation formulas can be found in the corresponding text. $u_1 = q_1 \cdot P(Q) - c_1 \cdot q_1$ and $u_2 = q_2 \cdot P(Q) - c_2 \cdot q_2$ stand for the decision-making income of Oligopoly 1 and Oligopoly 2 separately; values of the simulation initial parameters are $q_{01} = 2, c_1 = 2, c_2 = 1, Q_0 = 8$.

8.4 Damping Loss and the Total Damping Cost When the First Decision-Making Oligopoly Has Occupied the Market Completely

In the competitive process of the duopolies' strategic output-making in order to seize the market, if the output of some oligopoly is zero, then we say that this oligopoly has been kicked out of the market. In the duopoly strategy of output-making studied in this section, the first decision-making oligopoly has the absolute initiative authority (the first decision-making advantage). However, if there exists a certain damping in the strategic expansion of the first decision-making oligopoly, then it is impossible for the first decision-making oligopoly to take the method shown by Theorem 8.7 to occupy the market in one step and to obtain the maximal benefit; in order to kick the rival out of the market, the first decision-making oligopoly has to sacrifice present benefit (not act based on maximum benefit decision-making) and incur some expenses, shown by Theorem 8.8. We put forward Theorem 8.7, in order to prove Theorem 8.8.

Theorem 8.7 There is a duopoly strategy output-making problem. Its output-making competitive process satisfies Eq. (8.1), and there is to some extent a damping condition constraint in its strategy expansion process. If oligopoly i $(i=1,2)$ has the absolute advantage of making its decision first in the market, then that oligopoly's output is q_i^{**} $(i=1,2)$ when it occupies the market completely (kicks the rival out of the market), and it experiences ideal output q_{0i} $(i=1,2)$ and the market production supply amount $Q_{2-i}^{*/}$ $(i=1,2)$ based on making the present decision-making maximal benefits equal, and they are also equal to Q_0-c_{2-i} $(i=1,2)$.

Proof: Suppose Oligopoly 2 first makes its decision and has the absolute advantage. Then, according to Theorem 8.3, at this time, the market production supply amount based on making the present decision-making benefit maximal is $Q_1^*=\frac{q_{02}+(Q_0-c_1)}{2}$.

According to the meaning of this proposition, if Oligopoly 2 occupies the whole market, then its output satisfies $q_2^{**}=Q_1^*$. According to Eq. (8.1.2), we can get Eq. (8.32), and then we can get Eq. (8.33) by Eq. (8.32):

$$\begin{cases} q_2^{**}=q_{02}+\gamma_2(Q_1^*-q_{02}) & (8.32.1) \\ q_2^{**}=Q_1^* & (8.32.2) \end{cases} \tag{8.32}$$

$$q_2^{**}=Q_1^*=q_{02} \tag{8.33}$$

According to Eq. (8.14) and Eq. (8.33), we can get Eq. (8.34). By integrating Eqs. (8.33) and (8.34), we can get Eq. (8.35):

$$\begin{aligned} 2q_{02} &= q_{02} + (Q_0 - c_1) \\ q_{02} &= Q_0 - c_1 \end{aligned} \tag{8.34}$$

$$q_2^{**} = Q_1^* = q_{02} = Q_0 - c_1 \tag{8.35}$$

By the same principle, it is known that Eq. (8.36) is right when Oligopoly 1 first makes its decision:

$$q_1^{**} = Q_2^* = q_{01} = Q_0 - c_2 \tag{8.36}$$

Finally, by integrating Eqs. (8.35) and (8.36), it is obvious that the proposition can be proved.

Theorem 8.8 Given a duopoly strategy output-making problem, the competing process of output-making meets Eq. (8.1), and its strategy expansion includes a certain damping constraint. If oligopoly i $(i = 1,2)$ has absolute advantage of first decision-making, the oligopoly that occupies the market completely (kicks every competitor out of the market) will get the value $u_i^{*/} = (Q_0 - c_{2-i}) \cdot (c_{2-i} - c_i)$.

Proof: Provided that Oligopoly 2 is the advanced decision-maker who has the advanced absolute advantage, we could get the value concluded from Eqs. (8.17) and (8.35) when Oligopoly 2 seizes the market completely, shown by Eq. (8.37):

$$\begin{aligned} u_2^{*/} &= q_2 \cdot p(Q) - c_2 \cdot q_2 = q_2 \cdot (Q_0 - Q_1^{*/} - c_2) \\ &= Q_1^{*/} \cdot (Q_0 - Q_1^{*/} - c_2) = (Q_0 - c_1) \cdot (Q_0 - (Q_0 - c_1) - c_2) \\ &= (Q_0 - c_1) \cdot (c_1 - c_2) \end{aligned} \tag{8.37}$$

In the same way, we could get the value when Oligopoly 1 first makes its decision and seizes the market completely, which is shown by Eq. (8.38):

$$u_1^{*/} = (Q_0 - c_2) \cdot (c_2 - c_1) \tag{8.38}$$

Integrating Eq. (8.37) with Eq. (8.38), the original proposition has been proved.

Inference 8.4 In Theorem 8.8, if the first decision-making oligopoly i $(i = 1,2)$ has a marginal cost c_i $(i = 1,2)$ that is higher than the marginal cost c_{2-i} $(i = 1,2)$ of the later decision-making oligopoly, the first decision-making oligopoly could not kick the later decision-making oligopoly out of the market.

The conclusion of this deduction is obvious, so the process of proof is omitted. Here we only give a qualitative explanation.

By Theorem 8.8, if we considered oligopoly i to be the first decision-making oligopoly and its marginal cost of production $c_i\ (i=1,2)$ was higher than its competitor, namely $c_i \geq c_{2-i}\ (i=1,2)$, we could conclude from Theorem 8.8 that if this first decision-making oligopoly produces products that can fulfill the complete market, the value is negative; that is, $u_i^{*/} = (Q_0 - c_{2-i}) \cdot (c_{2-i} - c_i) \leq 0$. The final result is that the first decision-making oligopoly has a loss. We can see that at this time the first decision-making oligopoly could not eliminate the later decision-making oligopoly from the market.

Theorem 8.9 Given a duopoly strategy output-making problem, the competing process of output-making meets the requirement of Eq. (8.1), and there exists a certain damping constraint in the strategy expansion process, if the first decision-making oligopoly $i\ (i=1,2)$ wants to seize the market share completely, there is an associated cost to do so.

Proof: If Oligopoly 2 is the first decision-making oligopoly and has the absolute advantage of making its decision first, then we can conclude the value of Oligopoly 2 when it seizes the market completely, shown in Eq. (8.37). From Theorem 8.7, we could get the damping equilibrium value based on the maximum profit, which is shown in Eq. (8.39):

$$\begin{aligned} u_2^* &= q_2^* \cdot P(Q) - c_2 \cdot q_2^* = q_2^* \cdot (Q_0 - Q_1^* - c_2) \\ &= ((1-\gamma_{20})q_{02}^* + \gamma_{20} Q_1^*) \cdot (Q_0 - Q_1^* - c_2) \end{aligned} \tag{8.39}$$

Comparing Eq. (8.38) with Eq. (8.39), it is obvious that $u_2^{*/} \leq u_2^*$. Because u_2^* is the maximum profit based on all the possible strategy output-making, when there exists damping constraints in the strategy expanding process, $u_2^{*/}$ is only a special situation all around. From Eqs. (8.38) and (8.39), we can get the cost ΔC_2, which is the value when the oligopoly could seize the market completely, shown in Eq. (8.40):

$$\begin{aligned} \Delta C_2 &= u_2^* - u_2^{*/} \\ &= ((1-\gamma_{20})q_{02}^* + \gamma_{20} Q_1^*) \cdot (Q_0 - Q_1^* - c_2) - (Q_0 - c_1) \cdot (c_1 - c_2) \geq 0 \end{aligned} \tag{8.40}$$

In the same way, we could get the cost ΔC_1, when Oligopoly1 makes its decision first and seizes the market completely, shown in Eq. (8.41):

$$\begin{aligned} \Delta C_1 &= u_1^* - u_1^{*/} \\ &= ((1-\gamma_{10})q_{01}^* + \gamma_{10} Q_2^*) \cdot (Q_0 - Q_2^* - c_1) - (Q_0 - c_2) \cdot (c_2 - c_1) \geq 0 \end{aligned} \tag{8.41}$$

Although the first decision-making oligopoly has owned the absolute superiority of seizing the market first, as a result of the existence of the strategic expansion damping constraints, the oligopoly could not occupy the market in one step. The oligopoly has to pay a cost in order to occupy the market. Here the cost ΔC_i $(i = 1,2)$ is called the damping cost of the first decision-making oligopoly that intended to occupy the market completely.

In situations where strategy expansion damping constraints exist, the existence of the damping cost makes the first decision-making oligopoly pay a cost in order to occupy market completely.

Definition 8.1 Total damping cost: The first decision-making oligopoly i $(i = 1,2)$ starts at 0, and he will get the values u_{ij} $(i = 1,2; j = 0,1,2,\ldots,t)$ several times until he completely occupies the market. u_{ik}^* is called the maximum profit when the oligopoly reaches the damping equilibrium at k period. The total sum of u_{ik}^* minus u_{ij} is called the total damping cost, which is shown in Theorem 8.10.

Theorem 8.10 Given a duopoly strategy output-making problem, the competing process of output-making meets Eq. (8.1) and there exists a certain damping constraint in the process of strategic expansion. If the first decision-making oligopoly i $(i = 1,2)$ starts at 0, and terminates at t, this oligopoly must pay a damping cost in order to occupy the market completely.

Proof: Provided the oligopoly is the first decision-making oligopoly i $(i = 1,2)$ and starts at 0, it will experience strategies T and get the value u_{ij} $(i = 1,2; j = 0,1,2,\ldots,t)$ each time until it completely occupies the market at i. Then Oligopoly 2 starts at 0, and experiences strategies T. We define each strategy value as u_{ij} $(i = 1,2; j = 0,1,2,\ldots,t)$. If the oligopoly reaches the damping equilibrium, its equilibrium value is u_{2k}^*, and Δd_{2j} is called the damping loss every time, which is shown in Eq. (8.42):

$$\Delta d_{2j} = u_{2k}^* - u_{2j} (j = 0,1,2,\ldots,t) \tag{8.42}$$

From Eq. (8.42), we could get the total damping cost D_2 in the case where the oligopoly starts at 0, experiences strategies T, and occupies the market at t, which is shown in Eq. (8.43):

$$D_2 = \sum_{j=0}^{t} \Delta d_j = \sum_{j=0}^{t} (u_{2k}^* - u_{2s}) \tag{8.43}$$

In the same way, we could conclude the same solution, provided that Oligopoly 2 is the first decision-making oligopoly.

Table 8.3 Stimulation of Damping Cost in Condition That Decision-Maker Oligopoly 2 Completely Dominates the Market with Different Strategic Expansion Damping

γ_{20} (given)	q_{02}^*	q_2^*	$q_2^{**}=Q_1^*$	u_2^*	u_1^*	$u_2^{*/}$	ΔC_2
0.000	4.000	4.000	6.000	8.000	1.000	6.000	2.000
0.200	3.667	3.900	6.000	8.450	1.089	6.000	2.450
0.300	3.471	3.850	6.000	8.719	1.120	6.000	2.719
0.400	3.250	3.800	6.000	9.025	1.134	6.000	3.025
0.500	3.000	3.750	6.000	9.375	1.125	6.000	3.375
0.600	2.714	3.700	6.000	9.779	1.080	6.000	3.779
0.700	2.385	3.650	6.000	10.248	0.980	6.000	4.248
0.800	2.000	3.600	6.000	10.800	0.800	6.000	4.800
0.900	1.546	3.550	6.000	11.457	0.496	6.000	5.457
1.000	1.000	3.500	6.000	12.250	0.000	6.000	6.250

Note: γ_{20} expresses the strategic expansion coefficient of Oligopoly 2, q_{02}^* and q_2^* respectively express the best experienced ideal output and the production output of Oligopoly 2 under the parameter γ_{20} confirmed; $q_2^{**}=Q_1^*$ expresses the market capability when all the product are produced by Oligopoly 2; $u_2^*, u_2^{*/}$ respectively express the income of Oligopoly 2 when its production outputs are q_2^* and q_2^{**}, $\Delta C_2 = u_2^* - u_2^{*/}$, and u_1^* expresses the income of Oligopoly 1 with its production output q_1^* relative to q_2^*; the initial values of stimulation parameters are $q_{01}=2, c_1=2, c_2=1, Q_0=8$.

Example 8.3 Given a duopoly strategy output-making problem, the competing process of output-making meets Eq. (8.1), and there exists a certain damping constraint in the process of strategic expansion. Provided that Oligopoly 2 makes decisions first, the simulation situation of damping cost in conditions where Oligopoly 2 occupies the market completely with different strategic expansion damping is shown in Table 8.3.

Table 8.3 demonstrates that, since the damping constraint of strategic expansion exists, with an increase of γ_{20}, the income of the first decision maker, Oligopoly 2, when it becomes a complete market occupier, will keep on increasing, as will the damping cost.

Example 8.4 Given a duopoly strategy output-making problem, the competing process of output-making meets Eq. (8.1), and there exists a certain damping constraint in the process of strategic expansion, if Oligopoly 2 makes its decision first, take $\gamma_{20}=0.300$. The experienced ideal output of the first bout is $q_{02}^{(1)}=3.000$, and its experienced ideal output of the sth bout is taken as the actual output of the (s-1)th bout $q_{02}^{(s)}=q_{2,(s-1)}$. Take this iteration until the kth bout, take $u_{1k}=0$, and finish the operation. The simulation situation is shown in Table 8.4.

From Table 8.4, it has been demonstrated that the former decision-maker Oligopoly 2 may kick other oligopolies out of the market by modifying its

Table 8.4 Stimulation of Damping Cost in Conditions That Oligopoly 2, Which Makes Decision in Advance, Taking Possession of Market Completely with $\gamma_{20} = 0.300$

Game Bout s	$q_{02}^{(s)}$ *(given)*	q_{2s}	u_{2s}	u_{1s}	Δd_{2s}
1	1.000	1.750	6.125	4.375	2.594
2	1.750	2.388	7.461	3.161	1.258
3	2.388	2.930	8.221	2.283	0.498
4	2.930	3.391	8.595	1.649	0.124
5	3.391	3.782	8.716	1.191	0.003
6	3.782	4.115	8.678	0.861	0.041
7	4.115	4.398	8.543	0.622	0.176
8	4.398	4.638	8.354	0.449	0.365
9	4.638	4.842	8.140	0.325	0.579
10	4.842	5.016	7.920	0.235	0.799
11	5.016	5.164	7.704	0.169	1.015
12	5.164	5.289	7.500	0.122	1.219
13	5.289	5.396	7.314	0.089	1.405
14	5.396	5.487	7.144	0.064	1.575
15	5.487	5.564	6.991	0.046	1.728
16	5.564	5.629	6.857	0.033	1.862
17	5.629	5.685	6.739	0.024	1.980
18	5.685	5.732	6.635	0.017	2.034
19	5.732	5.772	6.546	0.013	2.173
20	5.772	5.806	6.468	0.009	2.251
21	5.806	5.835	6.401	0.007	2.318
22	5.835	5.860	6.343	0.005	2.376
23	5.860	5.881	6.293	0.003	2.426
24	5.881	5.899	6.250	0.003	2.469
25	5.899	5.914	6.213	0.002	2.506
26	5.914	5.927	6.182	0.001	2.537
27	5.927	5.938	6.155	0.000	2.564
					$D_2 = 40.875$

Notes:

1. All the damping cost when Oligopoly 2 has completely taken possession of the market is $D_2 = \sum_{j=0}^{t} \Delta d_j \sum_{j=0}^{t}(u_{2k}^{*} - u_{2s})$.
2. The income $u_2^{*} = 8.719$ when damping of Oligopoly 2 is uniform in conditions that the strategic expansion coefficient of Oligopoly 2 who makes decision in advance $\gamma_{20} = 0.300$.
3. γ_{20} expresses γ_{20} the strategic expansion coefficient, $q_{02}^{(s)}$ and q_{2s} respectively represent the experienced ideal output and the real output of Oligopoly 2 in the second round, u_{2s} and u_{1s} respectively represent the output of Oligopoly 2 and Oligopoly 1 in the *s*th round; Δd_{2s} represents the damping cost of Oligopoly 2 in the *s*th round.
4. The initial values of parameters in the stimulation are $\gamma_{20} = 0.30, q_{01} = 2, c_1 = 2, c_2 = 1, Q_0 = 8$.

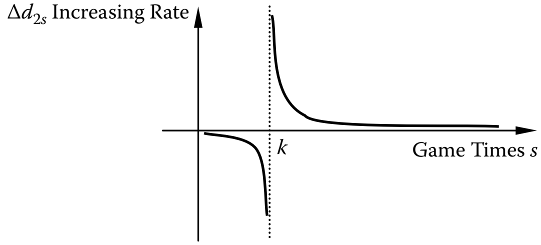


Figure 8.3 $\gamma_{20} = 0.3$, **Δd_{2s} curve of the increasing rate.**

experienced ideal output. However, according to the maximum current profit decision-making under this strategic expansion damping condition, Oligopoly 2 will have to pay for such action.

During each bout of the game, the closer the output from that which can kick the rival out of the market, the lower the rate of output increase will be. Meanwhile, the damping cost has experienced three periods, namely, steady decrease, rapid increase, and slow increase, as shown in Figure 8.3.

There are essential differences between the duopoly output-making model based on bounded knowledge and rationality and the classic Cournot model. The main differences between them are as follows:

1. The difference of the rationality and knowledge of game players. The classic Cournot model argues that decision-makers should boast complete symmetric knowledge (information) and they have complete rationality. The model in this book holds the view that the decision-makers only boast bounded and asymmetric knowledge (information) and they have bounded rationality.
2. The difference of the game players' purposes. The classic Cournot model argues that the output-making purpose of the game players is to maximize their profits, while the model in this book thinks the purpose of the output-making game purpose is to maximize strategic profit.
3. The difference of the game time-order supposition. The game players make decisions simultaneously in the classic Cournot model, while the decision-making of the players in this model is ordered.
4. The difference of the model structures. The classic Cournot model is a model of optimized structure that does not take path dependence into consideration. This model shows that the current decision-making of the decision-maker has strong path dependence to the historical decision-making information, and there is a supposition that the game players will try their best to maximize their own profits. A descriptive game structure model that has strong universality to the realistic decision-making situations is constructed in this chapter. This model well describes the strategic

decision-making problem between the leading company and the subordinating company, the situation where some first-knowing game players have an advantage in acquiring decision-making information while others are at a disadvantage, and the decision-making situation of the classical Cournot model, because when some parameters are fixed, the model degenerates to the classic Cournot model.

In this chapter, we did some further research on the model structure, decision-making algorithms, and the damping equilibrium of the first decision-maker. Some valuable conclusions have been achieved. However, there are also some other problems waiting for us to solve, such as whether the Nash equilibration of the duopoly can be achieved in reality based on the maximum of the strategic profit, under what kinds of circumstances can the Nash equilibration be achieved, what kind of model is the best to make the strategic profit maximum, and so on. We will do some further studies on these issues in other chapters.

8.5 Summary

The chapter reconstructs the assumption of duopoly decision-makers' game purposes, the game time-order supposition, as well as the assumption of bounded rationality and knowledge. Then a descriptive game structure model that has strong universality to the realistic decision-making situations is established. This model describes the strategic decision-making problem between the leading company and the subordinating company and the output decision-making of some first-knowing game players, and proves that the classical Cournot model is just one of its special cases. We did some further research on many problems, such as the algorithm of output-making and its attributes and characteristics, and put forward ten valuable theorems and four important inferences, such as the concession equilibrium of the later decision-maker, elimination from the market, the damping equilibrium of the advanced decision-maker, giving up possession of some market share, and the concepts of damping loss and total damping cost and their relevant algorithm. Furthermore, four simulation examples are used to explore some important propositions and algorithm.

Chapter 9

Nash Equilibrium Analysis Model of Static and Dynamic Games of Grey Pair-Wise Matrices

9.1 Nash Equilibrium of Grey Potential Pure Strategy Analysis of *N*-Person Static Games of Symmetrical Information Loss

Academic circles have broadly accepted two ways to divide game theory research. One, from the angle of the participants' action order, divided game theory into static games and dynamic games. The other, from the angle of one's knowledge about his or her counterpart's characteristics, strategic space, and payment functions, divided it into complete information games and incomplete information games. In essence, the second way mainly concerns the understanding of others' information in the game; complete and incomplete information here means symmetric and asymmetric information respectively.

In addition to "symmetric" and "asymmetric," there possibly exists another situation. Owing to the "grey information" problems such as uncertainty, finite knowledge, and small samples (or little information), all counterparts may not clearly understand the characteristics, strategic space, and payment functions of others and themselves, even based on symmetric information. Here we call it *information loss*. For example, in a comparatively closed color-TV market, two oligopolistic

Table 9.1 Game Value of Two Color-TV Oligopoly Manufacturers Competing for Market Shares Based on Symmetric Information loss

Manufacturer-1	*Manufacturer-2*			
		Strategy-1	*Strategy-2*	…
	Strategy-1	$[a_{11},b_{11}]$, $[c_{11},d_{11}]$	$[a_{12},b_{12}]$, $[c_{12},d_{12}]$	…
	Strategy-2	$[a_{21},b_{21}]$, $[c_{21}, d_{21}]$	$[a_{22},b_{22}]$, $[c_{22},d_{22}]$	…
	…	…, …	…, …	

Notes: The ellipses represent the omitted strategies or profit and loss values.

manufacturers compete for market shares by taking many strategies such as lower prices, good service, more product functions, and higher quality. For various reasons, however, the players cannot accurately estimate their profit and loss in different situations. Even under strict conditions, the profit and loss value of any two games cannot be the same because they are affected by regular and random factors. In a real-life situation, the profit and loss value matrix is grey and lacks information. Table 9.1 demonstrates the profit and loss of two color-TV oligopolistic manufacturers competing for market share based on symmetric information loss. In the table, $[a_{ij},b_{ij}]$ and $[c_{ij},d_{ij}]$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$) mean the income intervals of grey Manufacturer 1 and grey Manufacturer 2 respectively, resulting from a lack of corresponding information about profit and loss.

The division of the game information[1,2] without regard to information loss is a severe mistake by academic circles, and is shown in Table 9.2. Some scholars have studied the game with an information loss premise. In recent years, Professor Sifeng Liu and I put forward and researched problems such as matrix games based on information loss (grey matrix game[1–5]), established the basic concept of grey Nash equilibrium, and constructed the basic framework of grey matrix game theory. However, from the general view of theory system integrity, research in the following theory areas remains blank: static nongrey games based on symmetric information loss, dynamic games based on symmetric information loss, static and dynamic games based on asymmetric information loss, all of which are shown in Table 9.2. This section mainly discusses the concept and analysis method of Nash equilibrium of grey potential pure strategy on N-person static games of symmetrical information loss.

9.1.1 Nash Equilibrium of Grey Potential Pure Strategy

Definition 9.1 Symmetrical information loss of profit and loss: Suppose in a N-person game, for various reasons, there exists lost information of profit and loss values that can only be characterized by interval grey numbers, and this loss of

Table 9.2 Classification of Game Theory and Corresponding Concepts of Equilibrium

Action Order *Information*		*Static*	*Dynamic*
No Loss	Symmetric information	Static game of symmetric information: Nash equilibrium (Nash 1950, 1951)	Dynamic game of symmetric information: subgame perfect Nash equilibrium (Seleten 1965)
	Asymmetric information	Static game of asymmetric information: Bayesian-Nash equilibrium, (Harsany 1967, 1968)	Dynamic game of asymmetric information: perfect Bayesian Nash equilibrium (Seleten 1975; Kreps and Wilson 1982; Furdenberg and Tirole 1991)
Loss	Symmetric information loss	Grey matrix game: grey saddle point (Zhigeng Fang and Sifeng Liu 2003, 2005); Static game of symmetric information loss: equilibrium = ?	Dynamic game of symmetric information loss: equilibrium = ?
	Asymmetric information loss	Static game of asymmetric loss information: equilibrium = ?	Dynamic game of symmetric information loss: equilibrium = ?

Notes: In the table, the question mark ("?") represents a theory blank.

information is a shared knowledge. This situation is called symmetrical information loss of profit and loss values, denoted by $U(\otimes)=\{u_1(\otimes),\ u_2(\otimes),\dots,u_n(\otimes)\}$.

Definition 9.2 Static game of symmetrical information loss of profit and loss: Suppose the profit and loss value is $U(\otimes)=\{u_1(\otimes),u_2(\otimes),\dots,u_n(\otimes)\}$ in an N-person static game, and the corresponding strategic set is $S(\otimes)=(s_1(\otimes),s_2(\otimes),\dots\ s_n(\otimes))$. Then we call it an N-person static game of symmetrical information loss of profit and loss value, demonstrated by $G(\otimes)=\{S(\otimes),\ U(\otimes)\}$, or $G(\otimes)$ for short.

It is necessary to find a convenient measurement for the grey elements in $U(\otimes)$ so that we can analyze the game equilibrium of $G(\otimes)=\{S(\otimes),\ U(\otimes)\}$ without knowing the inner number distribution information of the interval grey numbers. Considering that people without clear inner distribution information of interval numbers compare interval grey numbers is a way of potential comparison, here

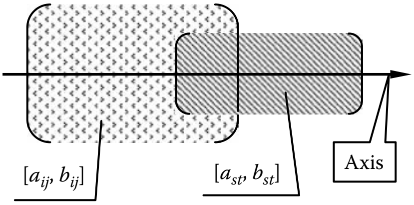


Figure 9.1 Size relation between interval numbers.

we make use of the concept of grey potentials to build and compare the frames of interval grey numbers.

Definition 9.3 Conceptions of superior, inferior, and equipollent potential degrees: For two random interval grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$, $\otimes_{st} \in [a_{st}, b_{st}]$ and $b_{st} \geq b_{ij} \geq a_{st}$ (as shown in Figure 9.1), according to the potential of the endpoints of the two grey numbers in the axis, we can divide the sum set range $\otimes_{ij} \bigcup \otimes_{st}$ into three parts, $[a_{ij}, a_{st}]$, $[a_{st}, b_{ij}]$, and $[b_{ij}, b_{st}]$. $[a_{st}, b_{ij}]$ is the intersection range $\otimes_{ij} \bigcap \otimes_{st}$ between the two interval numbers. Then,

1. We call the intersection range $\otimes_{ij} \bigcap \otimes_{st}$ an equipollent potential zone, $EPD_{ij \to st} = \frac{b_{ij} - a_{st}}{b_{ij} - a_{ij}} \geq 0$, where $ij \to st$ means $\otimes_{ij}$ is compared to $\otimes_{st}$, and is the equipollent potential degree of interval number $\otimes_{ij}$ compared to $\otimes_{st}$; $EPD_{st \to ij} = \frac{b_{ij} - a_{st}}{b_{st} - a_{st}} \geq 0$ is the equipollent potential degree of interval number $\otimes_{ij}$ compared to $\otimes_{st}$.
2. The range $[b_{ij}, b_{st}]$, which is divided by the two right endpoints of two grey numbers and is at the right of the intersection, is called the superior potential zone of grey number $[a_{st}, b_{st}]$ compared to $[a_{ij}, b_{ij}]$. $SPD_{st \to ij} = \frac{b_{st} - b_{ij}}{b_{st} - a_{st}} \geq 0$ is the superior potential degree of the interval number $\otimes_{st}$ compared to $\otimes_{ij}$.
3. The range $[b_{ij}, b_{st}]$, which is decided by the two left endpoints of two grey numbers, and which is on the left of the intersection, is defined as the inferior potential zone of grey number $[a_{ij}, b_{ij}]$ compared to $[a_{st}, b_{st}]$. $IPD_{ij \to st} = -\frac{a_{st} - a_{ij}}{b_{ij} - a_{ij}} \leq 0$ is the inferior potential degree of interval number $\otimes_{ij}$ compared to $\otimes_{st}$.

On the basis of the concepts mentioned above, we can make a determination of the potential degrees among the interval grey numbers without difficulty. Here we give the determination rules.

Definition 9.4 The determination rules of potential relation of interval grey numbers: For two random grey numbers, the sum of the superior and inferior potential degrees of one compared to the other is called the potential difference, or potential in brief. If the potential is positive, it is called a positive potential, and the corresponding grey number is called a superior grey number; if the potential is minus, we call it a minus potential, and the corresponding number is an inferior potential grey number; and if the potential is zero, we call it an equal potential, and call these numbers equipollent potential grey numbers.

From Definition 2.2 we know that, for two grey numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$ and $\otimes_{ij} \in [a_{st}, b_{st}]$,

If $SDP_{ij\to st} + IDP_{ij\to st} > 0$, we say that, compared to $\otimes_{st}$, there exists positive potential in $\otimes_{ij}$; $\otimes_{ij}$ is the superior grey number and $\otimes_{st}$ is the inferior grey number, marked as $\otimes_{ij} > \otimes_{st}$.

If $SDP_{ij\to st} + IDP_{ij\to st} < 0$, we say that, compared to $\otimes_{st}$, there exists minus potential in $\otimes_{ij}$; $\otimes_{ij}$ is the inferior grey number and $\otimes_{st}$ is the superior grey number, denoted as $\otimes_{ij} < \otimes_{st}$.

If $SDP_{ij\to st} + IDP_{ij\to st} = 0$, we say that $\otimes_{ij}$ and $\otimes_{st}$ have equal potential; $\otimes_{ij}$ and $\otimes_{st}$ are equipollent potential grey numbers, marked as $\otimes_{ij} = \otimes_{st}$.

In Definition 9.2, some classical mathematical operators such as >, <, and = are used to describe the potential relationships among grey numbers, but their meanings are not the same as the greater and lesser determinations in classical mathematics.

An important problem is whether the potential set of grey numbers is a total ordered set, so Theorem 9.1 is presented.

Theorem 9.1 Total ordered character of potential comparison: The set of grey potential relations is a total ordered set.

Proof: First of all, the total ordered set has to be a positive set, in accordance with transitivity. We prove the potential transitivity of interval grey numbers first.

Generally, given three interval numbers $\otimes_{ij} \in [a_{ij}, b_{ij}]$, $\otimes_{uv} \in [a_{uv}, b_{uv}]$, and $\otimes_{st} \in [a_{st}, b_{st}]$ (as shown in Figure 9.2), if $\otimes_{ij} < \otimes_{uv}$ and $\otimes_{uv} < \otimes_{st}$, then we need to prove $\otimes_{ij} < \otimes_{st}$.

According to Definition 9.4 and $\otimes_{ij} < \otimes_{uv}$, the difference between the superior and inferior potential degrees of $\otimes_{uv}$ compared to $\otimes_{ij}$ is beyond zero, as shown in

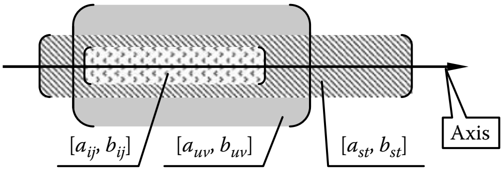


Figure 9.2 Potential transitivity of interval grey numbers.

Eq. (9.1); from $\otimes_{uv} < \otimes_{st}$ (already known), the difference between the superior and inferior potential degree of $\otimes_{st}$ compared to $\otimes_{uv}$ is beyond zero, as shown in Eq. (9.2).

$$SDP_{uv\to ij} - IDP_{uv\to ij} = \frac{b_{uv} - b_{ij}}{b_{uv} - a_{uv}} - \frac{a_{ij} - a_{uv}}{b_{uv} - a_{uv}} = \frac{(b_{uv} + a_{uv}) - (a_{ij} + b_{ij})}{b_{uv} - a_{uv}} > 0 \tag{9.1}$$

$$SDP_{st\to uv} - IDP_{st\to uv} = \frac{b_{st} - b_{uv}}{b_{st} - a_{st}} - \frac{a_{st} - a_{uv}}{b_{st} - a_{st}} = \frac{(b_{st} + a_{st}) - (a_{uv} + b_{uv})}{b_{st} - a_{st}} > 0 \tag{9.2}$$

The difference between the superior and inferior potential degrees of $\otimes_{st}$ compared to $\otimes_{ij}$ can be illustrated in Eq. (9.3):

$$SDP_{st\to ij} - IDP_{st\to ij} = \frac{b_{st} - b_{ij}}{b_{st} - a_{st}} - \frac{a_{st} - a_{ij}}{b_{st} - a_{st}} = \frac{(b_{st} + a_{st}) - (a_{ij} + b_{ij})}{b_{st} - a_{st}} \tag{9.3}$$

According to Eq. (9.1), if $(b_{uv} + a_{uv}) > (a_{ij} + b_{ij})$, then

$$(b_{st} + a_{st}) - (a_{ij} + b_{ij}) > (b_{st} + a_{st}) - (a_{uv} + b_{uv}) \tag{9.4}$$

According to Eq. (9.2),

$$(b_{st} + a_{st}) - (a_{uv} + b_{uv}) > 0 \tag{9.5}$$

Then from Eqs. (9.3), (9.4), and (9.5),

$$SDP_{st\to ij} - IDP_{st\to ij} > 0 \tag{9.6}$$

Therefore, according to Eq. (9.6) and Definition 9.2, we can get $\otimes_{ij} < \otimes_{st}$.

Second, we have to prove any two elements in the total ordered set have to be comparable. In a set of some interval grey numbers $\otimes_{ij}$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$), it is easy to prove that according to Definition 9.4, the two elements must have only one relation, meaning one is more, less than, or equal to the other. So the proof is omitted here.

In summary, the theorem is true.

Definition 9.5 Nash equilibrium of grey potential pure strategy: In an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$, we choose one strategy from every single game player to form a strategy group, marked as $S^*(\otimes) = (s_1^*(\otimes), s_2^*(\otimes), \ldots, s_n^*(\otimes))$, and strategy $s_i^*(\otimes)$ of Player i is the best countermeasure for the other players' strategy group $S_{(i)}^*(\otimes) = (s_1^*(\otimes), s_2^*(\otimes), \ldots, s_{i-1}^*(\otimes), s_{i+1}^*(\otimes), \ldots, s_n^*(\otimes))$ [see Eq. (9.7)]:

$$\begin{aligned} & u_i^*\left(s_1^*(\otimes), s_2^*(\otimes), \ldots, s_{i-1}^*(\otimes), s_i^*(\otimes), s_{i+1}^*(\otimes), \ldots, s_n^*(\otimes)\right) \\ & \geq u_i^*\left(s_1^*(\otimes), s_2^*(\otimes), \ldots, s_{i-1}^*(\otimes), s_i^*(\otimes), s_{ij}(\otimes), \ldots, s_n^*(\otimes)\right) \end{aligned} \tag{9.7}$$

The comparison between strategy profit and loss values, conducted under the comparison rule of grey potential degrees in Definition 9.4, is suitable for any $s_{ij}(\otimes) \in s_i(\otimes)$. Then we call $S^*(\otimes) = (s_1^*(\otimes), s_2^*(\otimes), \ldots, s_n^*(\otimes))$ a Nash equilibrium of grey potential pure strategy for $G(\otimes)$, or grey potential Nash equilibrium in brief.

Table 9.3 Prisoner's Dilemma in the Price War of Two Oligopoly Manufacturers in a Color-TV Market

Manufacturer 1	*Manufacturer 2*		
		Price down (1)	*Price up (2)*
	Price down (1)	[−5,−6], [−4,−5]	[2,3], [−8,−10]
	Price up (2)	[−8,−10], [2,3]	[0,1], [0,1]

Example 9.1 Suppose there is a Prisoner's Dilemma in the price war of two oligopolistic manufacturers in a color-TV market. For various reasons, there exists symmetric information loss of their profit and loss value, which can be illustrated in interval grey numbers (unit: 10 million Yuan), as shown in Table 9.3.

Similar to the analysis of the Prisoner's Dilemma, we can get the price-down strategy of Manufacturers 1 and 2, where (down, down) is the Nash equilibrium point of grey potential pure strategy referring to the determination rules in Definition 9.4. For more details, refer to Section 9.1.3 in this chapter.

9.1.2 Analyzing Methods of Absolute Grey Superior and Inferior Potential Relationships of Game $G(\otimes)$

In the equilibrium analysis of game $G(\otimes)$, if the analysis is based on the absolute grey relationship between the pros and cons, then we call it an absolute grey superior and inferior potential relationship analysis of game $G(\otimes)$, which contains two methods: a dominant pure strategy of grey potential analysis and an iterated elimination of strictly grey potential dominated strategies analysis. We discuss the first method next.

Definition 9.6 Dominant pure strategy of grey potential: In an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$, according to the determination rules of the potential degree relations between interval numbers in Definition 9.4, if the grey potential of the profit brought by one player's strategy is no less than that by other strategies—no matter what strategies the counterparts take—then it is called his dominant pure strategy of grey potential, or grey potential dominant strategy for short.

Definition 9.7 Grey potential-dominant equilibrium of pure strategy: In an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$. If each strategy on one certain strategy set is Grey potential-dominant pure strategy of its game player, the strategy set could be called "Grey potential-dominant equilibrium of pure strategy" of the Game, or grey potential dominant equilibrium.

Theorem 9.2 Grey potential dominant strategy equilibrium is a grey potential Nash equilibrium: If there exists a grey potential dominant strategy equilibrium in an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$, then it is a grey potential Nash equilibrium.

It is obviously true according to Definitions 9.5, 9.6, and 9.7, so the proof is omitted here.

In an N-person game $G(\otimes)$, the grey potential dominant strategy is preferred by all players. From Definitions 9.6 and 9.7, we know that according to the determination rules of the potential degree relationships between interval numbers in Definition 9.4, we could find the grey potential dominant strategy for a player, if there is one.

Example 9.2 Analyze the grey potential dominant strategy and the equilibrium point for all players in Example 9.1.

For Manufacturer 1, the superior potential degree of the profit of strategy (down, down) is $SPD_{11\to21} = 1$, compared to that of strategy (up, down); and the superior potential degree is $SPD_{11\to21} = 1$ when comparing strategy (down, up) and (up, up). So all the interval numbers of his price-down strategy profit are his superior potential grey numbers of the profit brought by his price-up strategies. Thus, price-down is his grey potential dominant strategy.

Similar to the above analysis, for Manufacturer 2, we can get the same conclusion.

In summary, the strategy set (down, down) of the two manufacturers is the equilibrium point of a grey potential dominant strategy. The equilibrium, which reflects the absolute preference of all the players, is very stable and gives a more precise prediction of game results. Being able to find such an equilibrium means a prediction result of the game analysis. However, the optimal strategy changes with other players, which is the basic characteristic and the principle demonstration of the mutual dependency of game relationships, but an equilibrium point doesn't always exist for every player. That is a limitation of the equilibrium analysis of grey potential dominant strategy. Therefore, we put forward a second analysis method of game $G(\otimes)$ based on the absolute grey relationship between the pros and cons of potential choices.

Definition 9.8 Grey potential disadvantaged strategy: In an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$, according to the determination rule of the potential degree relation between interval numbers in Definition 9.4, if we could find a strategy whose grey potential of profit and loss is no higher than others after comparing every pair of strategies, then we call it grey potential disadvantaged strategy.

Definition 9.9 Iterated elimination of strictly grey potential dominated strategies: In an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$, according to the determination rule of the potential degree relation between interval numbers in Definition 9.4, the analysis, which finds and eliminates the grey potential disadvantaged strategy of each player repeatedly based on comparing every two strategies, is called the iterated elimination of strictly grey potential dominated strategies.

Example 9.3 In a game $G(\otimes)$, shown in Table 9.4, find the grey-potential Nash equilibrium point using the method of iterated elimination of strictly grey potential dominated strategies.

According to the analysis method of grey potential dominant strategy equilibrium, we could not find such an equilibrium point in this game. By using the

Table 9.4 Payoff Matrix of a Game Based on Symmetric Information Loss of Profit and Loss

		Player-2		
Player-1		*Strategy-1*	*Strategy-2*	*Strategy-3*
	Strategy-1	[1,1.5], [–1,0.5]	[1,1.5], [3,3.5]	[1,1.5], [–1,0.5]
	Strategy-2	[–1,0.5], [4.5,5]	[–1,0.5], [2.5,3]	[2.5,3], [–1,0.5]

method of iterated elimination of strictly grey potential dominated strategies, however, we could easily get the grey-potential Nash equilibrium: Strategy 1 of Player 1 and Strategy 2 of Player 2; that is, {[1,1.5], [3,3.5]}.

Iterated elimination of strictly grey potential dominated strategies is a more effective and useful analysis method compared with the analysis of grey potential dominant strategy equilibrium, but it still cannot solve all the equilibrium analysis problems because in many grey games there is no relationship between two grey potential disadvantaged strategies.

9.1.3 Analysis of Relative Grey Superior and Inferior Relationships in Game $G(\otimes)$

In real-life situations, although the analysis methods of absolute grey superior and inferior relations between strategies is not useful in many games $G(\otimes)$, we can turn to the relative or conditional grey superior and inferior relationships that may exist between different strategies to find a new method. Reference 15 researched the marking method and the arrow method in classic games. Here, according to Definition 9.4, we use it in the grey potential Nash equilibrium analysis in an N-person game on symmetric information loss of profit and loss cases—namely, the grey marking method and the grey arrow method respectively.

The final goal of each player is to maximize his own profit. In a game $G(\otimes)$ where profit and strategy have mutual influence, the profit of each player depends not only on himself but also on the other's strategy, so when making decisions the players have to take this into consideration. Thus we put forward the corresponding decision-making analysis method—the method of decision analysis.

Theorem 9.3 Grey marking method: Suppose there exists a Nash equilibrium point of grey potential pure strategy in an N-person game $G(\otimes) = \{S(\otimes),U(\otimes)\}$. We can find the point based on the relative superior and inferior potential relations among different strategies as well as the determination rule of potential relationship between interval grey numbers in Definition 9.4.

Proof: Under the conditions in Theorem 9.3, the basic thought is illustrated below.

Find the maximal grey potential number among the strategy values of Player i against every strategy or strategy combinations of other players, if such strategy

Table 9.5 Maximal Grey Potential Profit and Loss of One Player's Strategies Compared to the Other's

	Player 2					
	Strategy	*1*	…	*j*	…	*n*
Player 1	1	$\max\vert_{player-11,1} = a_{11}(\otimes), b_{11}(\otimes)$	…	$a_{1j}(\otimes), b_{1j}(\otimes)$	…	$a_{1n}(\otimes), \max\vert_{player-2n,1} = b_{1ii}(\otimes)$
	…	…	…	…	…	…
	i	$a_{i1}(\otimes), b_{i1}(\otimes)$	…	$\max\vert_{player-1i,j} = a_{ij}(\otimes)$, $\max\vert_{player-2j,i} = b_{ij}(\otimes)$	…	$\max\vert_{player-1i,n} = a_{in}(\otimes), b_{in}(\otimes)$
	…	…	…	…	…	…
	m	$a_{m1}(\otimes)$, $\max\vert_{player-21,m} = b_{m1}(\otimes)$	…	$a_{nij}(\otimes), b_{nij}(\otimes)$	…	$a_{mn}(\otimes), b_{mn}(\otimes)$

combinations exist. We call the corresponding strategy of Player i a relative best strategy, which always exists but may not be unique.

Next, based on the first step, Player i must predict the other players' possible strategies and his own grey potential optimal strategy after considering all choices and decisions.

To simplify the discussion of the problem, we take a two-person game as an example.

Step 1: Find the maximal grey potential number in every strategy result that Player 1 takes against Player 2, and underline it to make a difference, as shown in Table 9.5. In the table, $\max|_{player-1i,j} = a_{ij}(\otimes)$ means Player 1 gets the maximal grey potential game value $a_{ij}(\otimes)$ in Strategy i compared to Strategy j of Player 2; $\max|_{player-1j,i} = b_{ij}(\otimes)$ means Player 2 gets the maximal grey potential game value $b_{ij}(\otimes)$ in Strategy j compared to Strategy i of Player 1.

Step 2: In a similar way, find the maximal grey potential number of in every strategy result Player 2 takes against Player 1, and underline it to make a difference, as shown in Table 9.6.

Step 3: In Table 9.6, if there exists a profit value underlined by both Player 1 and Player 2 at the same time, then it is the Nash equilibrium point of grey potential in this game, and the corresponding strategy is the grey potential optimal strategy.

Step 4: End.

Table 9.6 Nash Equilibrium Point of Grey Potential in the Prisoner's Dilemma in the Price War of Two Color-TV Oligopoly Manufacturers in a Region

		Manufacturer-2	
		Price down (1)	Price up (2)
Manufacturer-1	Price down (1)	[−5, −6], [−4, −5]	⇦ [2, 3], [−8, −10]
	Price up (2)	⇧ [−8, −10], [2, 3]	⇦ ⇧ [0, 1], [0, 1]

Theorem 9.4 Grey potential arrow method: Suppose there exists a Nash equilibrium point of grey potential pure strategy in an N-person game $G(\otimes) = \{S(\otimes), U(\otimes)\}$. On the basis of the determination rule of interval grey numbers potential size relation in Definition 9.4, we could find the equilibrium point and observe whether one player can increase his grey potential profit value by changing only his own strategy in every strategy combination.

Proof: Under the conditions of Theorem 9.4, the basic thought is illustrated below.

Analyze every strategy combination in game $G(\otimes)$ and consider whether one player can increase his grey potential profit value after changing only his own strategy. If he can, add an arrow into the corresponding profit teams leading to the corresponding potential profit value after the strategy changes. Finally, conclude the analysis on every strategy combination and form a predictive result of the game $G(\otimes)$.

To simplify the discussion of the problem, we take a two-person game as an example.

Step 1: Find the improvement of Player 1's potential profit value of Strategy $i, j = 1, 2, \ldots, m$ against every Strategy $j, j = 1, 2, \ldots, n$ of Player 2 and use an arrow to denote the improvement direction.

Step 2: Find the improvement of Player 2's potential profit value of Strategy $j, j = 1, 2, \ldots, n$ against every Strategy $i, i = 1, 2, \ldots, m$ of Player 1 and use an arrow to denote the improvement direction.

Step 3: Find the profit grey number team indicated by only the head of the arrow in all the strategy combinations of Players 1 and 2. According to Definition 9.5, the grey number is the Nash equilibrium point of grey potential and the corresponding strategies are the grey potential optimal strategies.

Step 4: End.

Example 9.4 Apply the grey potential marking method and grey potential arrow method in Example 9.1 to find the Nash equilibrium point of grey potential in the Prisoner's Dilemma case in the price war of two color-TV oligopoly manufacturers in a region.

According to Theorems 9.3 and 9.4, it can be solved by the Nash equilibrium point of grey potential, as in Table 9.6.

From Table 9.6, we know that all the grey game value teams with underlines and arrow heads are the Nash equilibrium points of grey potential, [–5,–6] and [–4,–5] to be exact. The corresponding strategy (price down, price down) is the potential optimal strategy.

9.1.4 Summary

Game theory continuously improves itself by answering real-life questions. Present academic circles divide game theory research on the basis of the players' knowledge of the game information, considering mainly whether it presents as symmetric information, but ignores the problem of information loss. This presents a severe defect. We find that there are still some theory blanks in the present research. This section puts forward the equilibrium analysis of information loss and the existence of its equilibrium point. It also examines the *N*-person static game of symmetrical information loss of profit and loss and establishes the concept of a Nash equilibrium point of grey potential pure strategy. To easily find this equilibrium, many ways are introduced, such as the advantaged and disadvantaged pure strategy analysis of grey potential, the grey potential marking method, and the grey potential arrow method. Figure 9.3 shows the relations between the classic Nash equilibrium and the Nash equilibrium of grey potential pure strategy, which can better explain the problems in real life but with some risks. In other words, under some conditions, unlike the classic Nash equilibrium, the Nash equilibrium of grey potential pure strategy may not happen automatically, which is further discussed together with its risks.

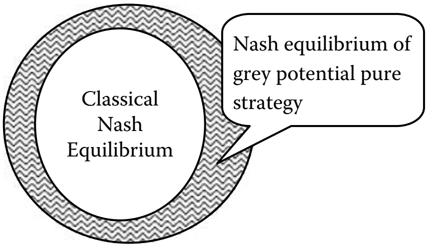


Figure 9.3 Relations between classical Nash equilibrium and Nash equilibrium of grey potential pure strategy.

9.2 Solving the Paradox of the Centipede Game: A New Model of Grey Structured Algorithm of Forward Induction

Backward induction starts analyzing the player's action from the last stage of the dynamic game and moves backward to the corresponding player's action in each former stage until it reaches the initial stage. Complete induction, deductive reasoning, clear thought, and efficient speed have made it a useful method in game theory and game logic research. But backward induction has two basic hypotheses: a rational hypothesis, meaning every decision-maker is rational, and a same expectancy hypothesis, which supposes everyone has a correct prediction regarding each other's action. However, such hypotheses do not conform to people's bounded rationality and are hardly found in real-life situations. Critiques include the famous paradox of the Centipede Game (Rosenthal, 1981).

Since Zermelo first put forward the idea of backward induction in 1913, the method was improved and promoted by Seleten in 1965 and 1975, and has attracted great concern and scholarly interest at home and abroad. Zhang Feng analyzed the origin of its paradox in "Exploration on the Paradox of Backward Induction"; and in "On Paradox from Centipede Game," he discussed limitations and effectiveness. In "Communicative Rationality and Dissolving of the Paradox of Backward Induction," Pan Tianqun introduced the concept of a new cooperative equilibrium in an attempt to eliminate the paradox. Gao Hongzhen and Wang Jiahui studied an improvement to the centipede paradox in "New Developments in Research on Experimental Game Theory." He Wei, Xu Fei, and Chen Jie gave possible results of the Centipede Game by using anticipatory psychology in "A New Viewpoint of Centipede Game: An Application of Anticipate Psychology." Gary Bornstein and others designed the experiment of "Centipede Game with Improving of Seven Points," based on the belief that a group lacks more altruistic spirit and is more selfish than an individual.

Having a general view of all the research mentioned above, almost all the papers relevant to the Centipede Game reached the same conclusion—that the basic reason for optimal choice leading to the worst results is the paradox of individual and collective rationality—but they gave no reasonable explanation for the contradiction in experiment results and people's initiative. More importantly, there is another problem: since neither (1,1) nor (100,100) would possibly be the equilibrium point in the game, does this kind of Nash equilibrium point exist? And if so, how could it be conveniently solved? Until now, academic circles have failed in better realizing and solving this problem; thus, this paradox set the logic of backward induction into doubt.

This section explores the origin of backward induction, and the theory that micro logic is wrong for overall macro logic neglect, or in other words, a player focuses on recent interest rather than long-term interest. The section applies overall system philosophy as well as general thinking and determination rules in designing a modern forward induction of grey structured algorithm from a systematical perspective. It can solve this problem by providing a convenient and highly efficient way to calculate the Nash equilibrium of a multistage dynamic game.

9.2.1 Backward Grey Structured Algorithm of a Dynamic Multistage Game's Profit Value

The first step in designing the forward induction is the grey structured algorithm and the subgame division of a multistage game. To simplify the research, we put forward the following three definitions.

Definition 9.10 Subgame of the original game: In a dynamic game, if the subsequent stages from a stage, except the first one, have a set of initial information and all the required information to form a partial game, then we it a subgame of the original game.

Definition 9.11 Upper-layer, lower-layer, and lowest-layer subgames: In a dynamic game, suppose we can get another subgame B from a subgame A, then we say that game A is the upper-layer subgame compared to B, and B is the lower-layer subgame of A. If no lower-layer subgame can be gotten from a game, then we call that layer the lowest-layer subgame.

One specific characteristic of the lowest-layer subgame is that it has no subgames and there exists a profit vector under any strategy.

Definition 9.12 Guide strategy of an eternal condition for a lower-layer subgame: If there exists a lower-layer subgame in subgame A, the strategy that directs B from A is called the strategy of an external condition of B or the guide strategy for lower-layer subgame, or guide strategy for short. Subgame B is guided by this external condition (guide) strategy.

Example 9.5 In a subgame division of the Rosenthal Centipede Game (Rosenthal, 1981), as shown in Figure 9.4, point out the relations between the upper-layer and lower-layer subgames and the relevant (guide) strategy of external condition.

In the example, if there are N stages in the game, then the lowest-layer subgame is the choice of Player 2 between Strategies A_n and B_n, while its upper-layer subgame is the choice of Player 1 between Strategies A_{n-1} and B_{n-1}, among which

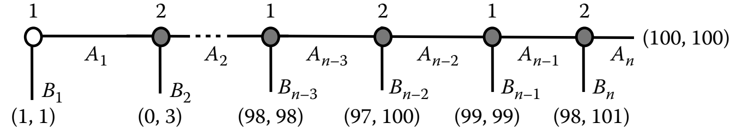


Figure 9.4 Sketch map of Rosenthal Centipede Game (Rosenthal, 1981).

$A_{n\text{-}1}$ is the (guide) strategy of external condition of its lower-layer subgame. The game is constructed by many subgames with upper-layer and lower-layer relations, and the (guide) strategy of external condition of the corresponding lower-layer subgame is denoted as A_i, $i = 2,2,\ldots,n$.

Then we can get a grey structured algorithm of a dynamic multistage game, as shown in Proposition 9.1

Proposition 9.1 In an N-person dynamic game, the grey structured algorithm is demonstrated as follows.

Step 1: Conduct the subgame division of the dynamic game according to Definitions 9.10 and 9.11.

Step 2: Simplify the grey structured algorithm of the lowest-layer game. That is, for Player i under several strategies of subgames, the minimal profit value v_{ij} is $\min\{v_{i1}, v_{i2}, \ldots, v_{iS}\}$, and the maximal profit value is $\max\{v_{i1}, v_{i2}, \ldots, v_{iS}\}$. We take the left and right endpoints of the interval grey numbers respectively to form a profit interval grey number under the (guide) strategy of external condition of the lowest-layer subgame, marked as $[\min\{v_{i1}, v_{i2}, \ldots, v_{iS}\}, \max\{v_{i1}, v_{i2}, \ldots, v_{iS}\}]$. After a similar treatment to all the participants' strategies, the grey structured algorithm of the lowest-layer game will be transformed to the profit vector under its strategy of external condition, $([\min\{v_{11}, \ldots, v_{1S}\}, \max\{v_{11}, \ldots, v_{1S}\}], \ldots, [\min\{v_{K1}, \ldots, v_{KS}\}, \max\{v_{K1}, \ldots, v_{KS}\}])$, as shown in Figure 9.5.

Step 3: After Step 2, the original lowest-layer subgame is simplified, and thus the upper-layer subgame under the strategy of external condition would construct a new lowest-layer subgame. Practice the grey structured algorithm to the new subgame as in Step 2 until the highest-layer game is reached.

Step 4: End.

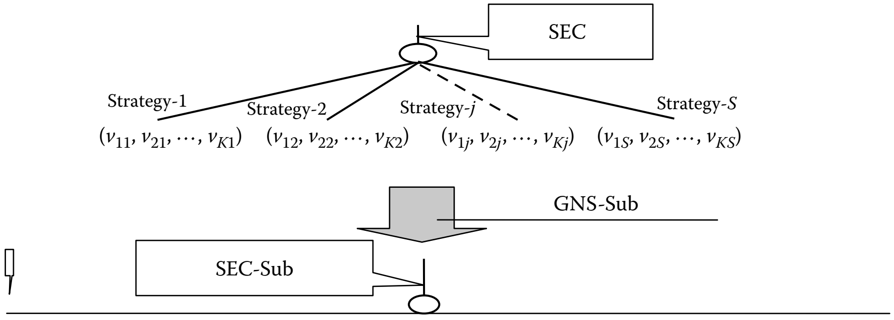


Figure 9.5 Sketch of grey structured algorithm of subgames. SEC: strategy of external condition; SEC-Sub: strategy of external condition in subgames; GNS-Sub: grey number structured in subgames.

Obviously the procedure is true, so the proof is omitted here

$$([\min\{v_{11},\ldots,v_{1S}\}, \max\{v_{11},\ldots,v_{1S}\}],\ldots,[\min\{v_{K1},\ldots,v_{KS}\}, \max\{v_{K1},\ldots,v_{KS}\}])$$

Proposition 9.1 has illustrated the grey structured algorithm of a dynamic multistage game starting from the lowest-layer subgame to the highest-layer subgame. As the linear order is totally in contrast with the time order, it can be considered a backward induction algorithm, to calculate the future possible profit value of the guide strategies.

Example 9.6 Implement the subgame classification and the simplification of the grey structured algorithm in the Rosenthal Centipede Game shown in Example 9.5.

The Rosenthal Centipede Game in Example 9.5 can be divided into $n - 1$ subgames, denoted as G_1 ($i = 2,3,\ldots,x$) respectively, as shown in Figure 9.6.

According to Proposition 9.1, the subgame process can be simplified as shown in Figure 9.6 and the future possible incomes after the grey structured algorithm are illustrated on the guide strategies respectively.

As the grey structured treatment of a dynamic multistage game in Proposition 9.1 formed a new token of dynamic game structure mainly featuring possible future incomes, which are marked on the guide strategy of the lower-layer subgame, we propose Definition 9.13.

Definition 9.13 Guide value lower-layer subgame and the guide value structure of a dynamic game: If all the guide strategies are marked with the possible future profit values in a multistage dynamic game, then the profit value is called as the guide value of lower-layer subgame, guide value for short; the structure of a

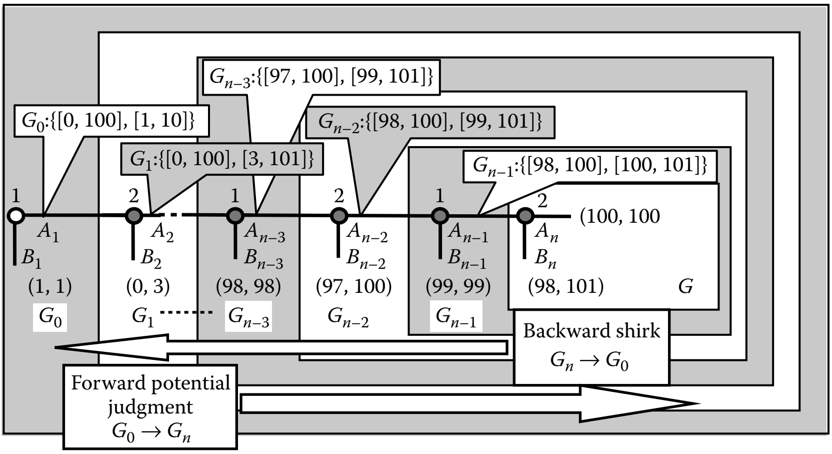


Figure 9.6 Analysis schematic diagram of grey structured algorithm, backward induction.

multistage dynamic game involving guide values is called the guide value structure of a dynamic game in brief.

Definition 9.13 defines a new structure of a multistage dynamic game after the grey structured algorithm. When making decisions at any point in the game, people can compare present and future possible game profit values. They therefore discard rationality in every step in the classic backward induction, which leads to a strange logic with irrational results, as in the Centipede Game. We believe that the classic explanation to the economic phenomenon—the contradiction between individual and population rationality—indicates just one side of the problem. A more essential problem is that of the contradiction and balance between recent and long-term interests, or between present and future interests for players.

9.2.2 The Termination and Guide Nash Equilibrium Analysis of Grey Structured Algorithm of Forward Induction in a Multistage Dynamic Game

According to investigation into this problem over many years, involving hundreds of entrepreneurs, government officers, consulting engineers, technicians, teachers, university and middle-school students, and housewives, we finally drew a conclusion that (in a similar Centipede Game) people will follow the classic Centipede Game if there are few but accurate stages; otherwise, they will make decisions based on a comparison between present and future possible profit values.

In 2005, Zheng Feng, in the article "On Paradox from Centipede Game," concluded that the classic backward induction was useful when the Centipede Game has few stages—three to five, for example. But with an increase in the number of stages, the possibilities of cooperation (or the cooperation stages) on average will increase as well. The inference provided a good example for the view in this section.

After considering decision habits in dynamic games, we propose four hypotheses for a new grey structured algorithm of forward induction, as follows.

Hypothesis 1: In a multistage dynamic game, practice backward induction of grey structured algorithm step by step.

Hypothesis 2: If players do not have clear knowledge about the structure of a multistage dynamic game or the future profit value, they still can estimate the scope of present and future profit values using interval numbers. In other words, they use grey numbers to represent profit values.

Hypothesis 3: Based on the time order of the dynamic game, players evaluate present and future profit values and make a decision according to some determination rules, such as the potential degree of grey numbers[2] and the probability expectation of grey numbers.

Hypothesis 4: Other game hypotheses are required as well.

Based on these four hypotheses, we can analyze the equilibrium solution of a multistage dynamic game in the order of time. Thus we put forward the following definitions.

Definition 9.14 Grey number structured sequential deduction: Given a guide value structure of a dynamic game, if the decision analysis follows the order of time in the game, then we call this balanced analysis a grey number structured sequential deduction.

Definition 9.15 Termination Nash equilibrium and guide Nash equilibrium: Analyze the equilibrium using a grey number structured sequential deduction in a guide value structure of a dynamic game. If the present game realizes the equilibrium at a nonguided value, then the game ends; otherwise, the equilibrium reaches a guide value and the game continues. We call these a termination Nash equilibrium and a guide Nash equilibrium respectively.

Proposition 9.2 Given a two-person dynamic game, a decision in each stage determines whether the game continues. If the current decision-maker realizes the termination Nash equilibrium, then the whole game ends; if he realizes the guide Nash equilibrium, then the current game will induce to its lower-layer subgame.

The conclusion of this proposition is obviously true, so the proof is omitted here.

Proposition 9.3 Given a two-person dynamic game, according to the above four hypotheses, if the current decision-maker evaluates and makes decisions between present and future game profits based on some determination rules, then there exists a termination Nash equilibrium under this rule; in other words, the game algorithm converges.

Proof: In a two-person dynamic game, practice backward induction of the grey structured algorithm step by step according to Proposition 9.1 and mark the future profit values with grey numbers aimed at comparing that value with current income. As shown in Figure 9.7, the first and second profit values demonstrate the profits of Player i and Player j respectively.

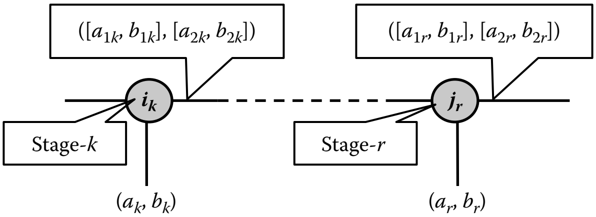


Figure 9.7 Termination Nash equilibrium analysis based on guide value structure of dynamic game.

In Figure 9.7, if Player $i(i = 1, 2)$ is the current decision-maker in stage k, then he will compare the profit $[a_{1k},b_{1k}]$ in stage k with the interval grey number of future possible profit a_k. If players cannot get distribution and value information of the structured interval grey number, then we can get the following judgments based on the rules in Ref. 6. When $a_k < \frac{a_{1k}+b_{1k}}{2}$, then judging from grey potential degree, we can get $a_k > [a_{1k},b_{1k}]$; then (a_k,b_k) is the termination Nash equilibrium of the game.

When $a_k < \frac{a_{1k}+b_{1k}}{2}$, $a_k < [a_{1k},b_{1k}]$ which is based on the determination of grey potential degree, then $([a_{1k},b_{1k}], [a_{2k},b_{2k}])$ is the guide Nash equilibrium of the game.

If the current game realizes the guide Nash equilibrium, it will induce to its lower-layer subgame according to Proposition 9.2.

In a similar way, the termination Nash equilibrium will be realized during a subsequent subgame mainly because the guide value of the current subgame is the grey structured value of all the subsequent subgames. We will prove this by reduction to absurdity.

Proof: Suppose a termination Nash equilibrium in the guide value structure of a dynamic game cannot exist, as in Definition 9.13. According to Proposition 9.2, the game will realize guide Nash equilibrium in its subgames. In every subgame, the current game profit of every player is less than the corresponding guide value, without other size relations, but this is in contradiction with Proposition 9.1 that the guide value of the current subgame is constructed by the grey structured numbers of the subsequent subgames' profit values. So the supposition is wrong. And in an N-stage dynamic game, if the current kth stage realizes the guide Nash equilibrium, then the subsequent $n - k$ stages of the subgame will finally realize the termination Nash equilibrium.

Suppose a player can get distribution and value information of the structured interval grey number. Then every participant can realize the termination Nash equilibrium in that stage based on the corresponding expected value judgment rules.

Proposition 9.3 proves the contraction of the grey number structured sequential deduction and the contraction of the algorithm.

Example 9.7 Apply the grey potential degree and probability expected value judgment rule respectively to solve the termination Nash equilibrium of the Rosenthal Centipede Game in Example 9.6 (see Figure 9.6).

1. Solve the termination Nash equilibrium of a dynamic game based on the grey potential degree judgment rule.

As in Figure 9.6, Player 1 makes a decision first in Stage 1. He will compare the grey potential degrees of 1 unit profit value of guide Strategy A_1 and [0,100] unit profit value of strategy B_1, which is obviously bigger. Then Player 1 takes guide Strategy A_1 and realizes the guide Nash equilibrium. Then Player 2 performs as the decision-maker. Deducing like this until the G_{n-2}th subgame, then Player 2 would be the current decision-maker and the game profit values of guide Strategy A_{n-2} and termination Strategy B_{n-2} are [99,101] (units) and 100 (units) respectively. As the potential degrees of the two interval grey numbers are the same, the two strategies, A_{n-2} and B_{n-2}, are the same for Player 2. He can terminate the process by choosing strategy B_{n-2} and realize the termination Nash equilibrium. In this case, he gets 100 units of profit and Player 1 gets 97 units of profit. He can also choose Strategy A_{n-2} and pass the decision right to Player 1 to realize the guide Nash equilibrium.

For the same reason, we can say that there is no difference between the termination Nash equilibrium Strategy B_{n-1} and the guide termination Nash equilibrium Strategy A_{n-1} in subgame G_{n-1}.Taking Strategy B_{n-1}, both Player 1 and Player 2 get a profit of 99 (units), while if Player 1 takes taking Strategy A_{n-1}, it will induce to the lower-layer subgame and transfer the decision right to Player 2.

Subgame G_n is a fixed game, in which Player 2 chooses the termination strategy B_n and realizes the termination Nash equilibrium. The profit values of Players 1 and 2 are 98 (units) and 100 (units) respectively.

Under this judgment rule, there exist three possible termination Nash equilibriums whose subgames and profit values are G_{n-2} (97, 100), G_{n-1} (99, 99), and G_n (98, 101) respectively.

2. Solve the termination Nash equilibrium of a dynamic game based on the determination rule of expected value.

The interval grey numbers of future possible profits of the guide strategies are actually the discrete grey numbers with a certain appearing probability. For example, in the *k*th stage, the appearing probability of the possible value among the guide interval grey numbers can be estimated through the probability in the future $n - k$ stages.

The judgment of termination and guide Nash equilibrium based on the expected profit values of all the strategies is very simple once the distribution probability of different values in the guide interval grey numbers is known. In the early process of the subgame shown in Figure 9.3, it realizes the guide Nash equilibrium under the determination rule of expected values. We will offer the Nash equilibrium analysis about the last several subgames.

In subgame G_{n-3}: Player 1 is the decision-maker and his interval number of game value is [97, 100], among which the probability of 97, 98, 99, and 100 all equal 0.25, not including this stage. Then the expected guide value is (97+98+99+100)/4 = 98.5 (units) and the profit value of the termination game is 98 (units), so Player 1 will choose the guide Nash equilibrium and pass the decision right to Player 2.

In subgame G_{n-2}: Player 2 is the decision-maker and his interval number of game value is [99, 101], among which the probability of 99, 100, and 101 all equal 0.33, not including this stage. Then the expected guide game value is (99+100+101)/3 = 100 (units) and the profit value of the termination game is 100 (units), so the termination and guide Nash equilibrium are the same to Player 2. If he chooses Strategy B_{n-2} to realize the termination Nash equilibrium, then the profit for each player is 97 (units) and 100 (units) respectively. If he chooses Strategy A_{n-2}, then he will transfer the decision right to Player 1.

In subgame G_{n-1}: Player 1 is the decision-maker and his interval number of game value is [98, 100], among which the probability of 98 and 101 equals 0.50, not including this stage. Then the expected guide value is (98+100)/2 = 99 (units) and the profit value of the termination game is 99 (units), so the two kinds of equilibriums are the same to Player 1. If he chooses Strategy B_{n-1} to realize the termination Nash equilibrium, then the profit for both players is 99 (units). If he chooses Strategy A_{n-1}, then he will transfer the decision right to Player 2.

In subgame G_n: This stage is the lowest-layer subgame so it is easy to know that the game equilibrium is realized when players choose strategy B_n and the profit values of Players 1 and 2 are 98 (units) and 101 (units) respectively.

Under this determination rule, there exist three possible termination Nash equilibriums, whose subgames and profit values are G_{n-2} (97, 100), G_{n-1} (99, 99), and G_n (98, 101) respectively, and the result is the same as the conclusion of the former determination rule.

3. Example of the mechanism of classic backward induction

In subgame G_{n-1}: If Player 1 believes Player 2 will choose Strategy B_n in subgame G_n, then he will choose Strategy B_{n-1} in this stage. Deducing in a similar way, Player 1 will terminate the game by choosing Strategy B_1 in Stage G_1 when both of them win 1 (unit) value.

Therefore, the classic backward induction is a specific conclusion under the special profit probability and a special example of Player 1's choice of the termination Nash equilibrium in the first stage.

9.2.3 Summary

For a long time, the paradox of backward induction disturbed academic circles. Many scholars realized the great difference between the conclusion of the paradox and real life, and they researched this problem from different perspectives. We think the main reasons for this problem lie in the following three aspects:

- In the current game stage of the multistage dynamic game in real life, when people cannot make a precise prediction about a future game, they need to estimate the scope of the possible profit value, using interval numbers to demonstrate, for example; then they will compare such estimated values with the current game profit and make a decision after evaluation.
- The decision of the current game stage in the multistage dynamic game is made after a comparison between current and future game profits, which is neglected by backward induction. The latter just compares current profit with the profit of the former stage.
- The token of the classic multistage dynamic game structure fails to mark the guide strategy with the possible future values, which restricts the players from comparing and analyzing these two values.

For these reasons, the phenomenon of the contradiction between the backward induction paradox and real-life situations shows that backward induction neglects the integrity philosophy and is only concerned with recent interest, and other severe mistakes.

To overcome these fatal shortcomings of backward induction in the multistage dynamic game, this section designed a new model based on grey number structured sequential deduction, which can present a better solution. Meanwhile, it revealed the essential laws in the multistage dynamic game. In dynamic games with few stages or in a short-term dynamic game, classic backward induction can better reveal the essence of the game's strategies, but the mechanism is just a special case of the model in this section that performs under an accurate probability of future profit value.

Chapter 10

Chain Structure Model of Evolutionary Games Based on Limited Rationality and Knowledge

Evolutionary game theory was born because the traditional game theory fails to describe problems of limited (or bounded) rational knowledge. Evolutionary game theory discards full-rationality assumptions and treats the players as living creatures with bounded rationality, who are evolving themselves while competing with each other. It explains the rationally evolutionary processes of some living creatures in nature (J. Maynard Smith, 1982). In the 1960s, biologists used the theory to explain ecosystem phenomena. Especially after the basic concept of an evolutionary stable strategy (ESS) was put forward by Maynard and Price in 1973 and by Maynard in 1974, the theory was broadly applied to ecology, sociology, economy, and other fields. In recent years, papers on applications of evolutionary game theory have occupied an increasingly larger percentage of game theory research.

Traditional analysis methods of evolutional game have some deficiencies. Traditional models lack a kind of suitable structure form that is concretely and vividly like the payoff matrix structure form of a matrix game, the game tree structure form of a dynamic game, and other similar forms to express themselves.

This section designs a chain structure model of evolutionary games based on a symmetric income situation to overcome such limitations. This model could not only be used to analyze stable long-term trends but also to predict possible short-term evolutionary trends. It also satisfactorily describes the process of evolutionary games.

10.1 Chain Structure Model of Evolutionary Games Based on a Symmetric Case

10.1.1 Establishing a Chain Structure Model for Evolutionary Games

In the process of evolutionary games based on a symmetric case, the payoff of each population is independent of its place in the game; the income of one population is no different whether it is Population q or Population r. We will employ a framework with populations of similar members and where at each contest a randomly chosen member of one of the populations plays a game with a randomly chosen member of the other population.

Given an evolutionary game problem based on a symmetric case, we assume that a set of strategies is $s, s = 1, 2, \ldots, m$ During the game, each member of some population chooses a strategy, which is not always the best strategy, after learning from other players or imitating a neighboring strategy. Here, a circle with s is denoted as a strategy of an evolutionary game, where $s, (s = 1, 2, \ldots m)$ Let $p^t_{q,jk}$ $(j, k = 1, 2, \ldots, m)$ show the transfer proportion from original strategy j to a new one, k, in Population q at time t, and $u^t_{q,jk}, (j, k = 1, 2, \ldots, m)$ show the expected income of Population q when Strategy k is taken at time t. An arrow mark whose end denotes the original strategy and whose head indicates the new strategy of the individual in a game, together with $p^t_{q,jk}, (j, k = 1, 2, \ldots, m)$ and $u^t_{q,jk}, (j, k = 1, 2, \ldots, m)$ describe the process of the game. Thus we could get a sketch of an evolutionary game chain model. We employ Player q (or Player r) to denote a randomly chosen member of Population q (or Population r).

We take just a 2 × 2 symmetric case as an example to discuss the model establishing this case. One 2 × 2 payoff matrix based on a common symmetric case is illustrated in Table 10.1. We thus get the sketch of an evolutionary game chain model at time t (see Figure 10.1).

In Figure 10.1, two circles denote Strategy 1 and Strategy 2 respectively. Four arrows show, from time t to time $t + 1$, the probable strategy transfer conditions of populations: some players maintain their original strategies, and some players imitate and study from others and change their strategies. No matter what the game conditions are, the individuals use their payoff strategies to make decisions. If it

Table 10.1 2 × 2 Payoff Matrix Based on a Symmetric Case

Population 2		*Strategy 1*	*Strategy 2*
Population 1	Strategy 1	*a,a*	*b,c*
	Strategy 2	*c,b*	*d,d*

is a self-round arrow, whose head and end direct to the same strategy, it indicates the player keeps his original strategy at time t, and the proportion and expected profit are marked as $p_{ij}^{t}, u_{ij}^{t}, i = j; i, j = 1, 2$ respectively. If not, it means some of the individuals who take this strategy at time t change to other strategies at time $t + 1$, and the notes on the arrows $p_{ij}^{t+1}, u_{ij}^{t+1}, i \neq j; i, j = 1, 2$ indicate the proportion of the population that changes strategies and the corresponding expected profit respectively.

In the discussion below, we assume that at time t the proportion of players who choose Strategy 1 is x, and the remaining $1 - x$ players choose Strategy 2. Therefore, the proportions of individual quantities, the expected income of individuals, and the average expected payments of each player are given in Eqs. (10.1), (10.2), and (10.3) respectively.

Then we consider what these members of each population will do and what their income will be at time $t + 1$.

$$\begin{aligned} p_{11}^{t} &= x \\ u_{11}^{t} &= x \cdot a + (1 - x) \cdot b \end{aligned} \tag{10.1}$$

$$\begin{aligned} p_{22}^{t} &= 1 - x \\ u_{22}^{t} &= x \cdot c + (1 - x) \cdot d \end{aligned} \tag{10.2}$$

$$\bar{u}_t = x \cdot u_{11}^{t} + (1 - x) \cdot u_{22}^{t} \tag{10.3}$$

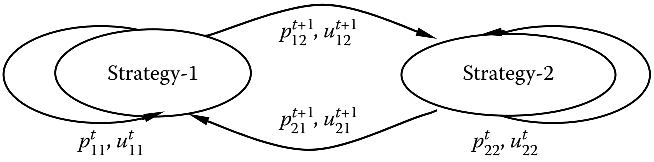


Figure 10.1 Sketch of evolutionary game chain model.

The emphasis of an evolutionary game is a dynamic change of proportion, whose focus is its growth rate. We use positive signs and negative signs to denote the direction of growth. In fact, $p_{i,jk}^{t+1}, u_{i,jk}^{t+1}, j \neq k; j,k = 1,2$ just decides the growth rate. Based on evolutionary theory, we know that the rate of dynamic alteration is decided by the speed at which members in this game learn and imitate. Their speed of learning and imitating mainly lies on two factors:[51] the number of imitators, which influences the scale and the degree of imitation and which could be described by the proportion of players who choose a certain strategy, and how successful the original strategy that was imitated is, which affects how hard the decision is and how much encouragement there is. The more an average payoff for one player is higher than the expected income of imitator, the greater the number of individuals who will imitate him. We could assume that the proportion of strategy transfers in one population from time t to time $t + 1$ is in direct proportion to the quantity of imitators, and also is in direct proportion to the gap between the average expected payment of the population and the expected income of individuals. We presume that both proportionality coefficients are 1.

The states of individuals of each population is described in Eqs. (10.4) and (10.5). Here, $u_{ij}^{t+1} = u_{jj}^{t+1}; i,j = 1,2$ means the payoff of an imitator who transfers his strategy from time t to time $t + 1$, equal to the expected income of the imitated, on the assumption that the cost of learning and imitating is zero. p_{12}^{t+1} and p_{21}^{t+1} respectively mean, after sensing the gap between his profit and the average profit at time t, an individual has taken another strategy at time $t + 1$. The population p_{12}^{t+1} (or p_{21}^{t+1}) who changes their strategy should be in direct proportion to the population who keeps their strategy. $\overline{u}_t - u_{11}^t$ indicates the difference between the expected profit of the player who keeps his strategy and the average group profit at time t. $\frac{\overline{u}_t - u_{11}^t}{\underset{x\in[0,1]}{Max}|\overline{u}_t - u_{11}^t|}$ (or $\frac{\overline{u}_t - u_{22}^t}{\underset{x\in[0,1]}{Max}|\overline{u}_t - u_{22}^t|}$) expresses the proportion of $\overline{u}_t - u_{11}^t$ in its absolute value. Here, we presume that the proportionality coefficients are 1.

$$\begin{aligned} p_{12}^{t+1} &= p_{11}^t \cdot \frac{\overline{u}_t - u_{11}^t}{\underset{x\in[0,1]}{Max}\left|\overline{u}_t - u_{11}^t\right|} \\ u_{12}^{t+1} &= u_{22}^{t+1} \end{aligned} \tag{10.4}$$

$$\begin{aligned} p_{21}^{t+1} &= p_{22}^t \cdot \frac{\overline{u}_t - u_{22}^t}{\underset{x\in[0,1]}{Max}\left|\overline{u}_t - u_{22}^t\right|} \\ u_{21}^{t+1} &= u_{11}^{t+1} \end{aligned} \tag{10.5}$$

We get the parameters of each population who keep their strategy at time *t*+1 [see Eqs. (10.6), (10.7), and (10.8) for more details].

$$
\begin{aligned}
&p_{11}^{t+1} = p_{11}^{t} + p_{21}^{t+1} \\
&u_{11}^{t+1} = p_{11}^{t+1} \cdot a + \left(1 - p_{11}^{t+1}\right) \cdot b
\end{aligned}
\tag{10.6}
$$

$$
\begin{aligned}
&p_{22}^{t+1} = 1 - p_{11}^{t+1} = p_{22}^{t} + p_{12}^{t+1} \\
&u_{22}^{t+1} = p_{11}^{t+1} \cdot c + \left(1 - p_{11}^{t+1}\right) \cdot d
\end{aligned}
\tag{10.7}
$$

$$
\overline{u}_{t+1} = p_{11}^{t+1} \cdot u_{11}^{t+1} + p_{22}^{t+1} \cdot u_{22}^{t+1} \tag{10.8}
$$

The parameters of each population who change strategy are showed in Eqs. (10.9) and (10.10):

$$
\begin{aligned}
&p_{12}^{t+2} = p_{11}^{t+1} \cdot \frac{\overline{u}_{t+1} - u_{11}^{t+1}}{\underset{x \in [0,1]}{Max} \left| \overline{u}_{t+1} - u_{11}^{t+1} \right|} \\
&u_{12}^{t+2} = u_{22}^{t+2}
\end{aligned}
\tag{10.9}
$$

$$
\begin{aligned}
&p_{21}^{t+2} = p_{22}^{t+1} \cdot \frac{\overline{u}_{t+1} - u_{22}^{t+1}}{\underset{x \in [0,1]}{Max} \left| \overline{u}_{t+1} - u_{22}^{t+1} \right|} \\
&u_{21}^{t+2} = u_{11}^{t+2}
\end{aligned}
\tag{10.10}
$$

10.1.2 Imitation of Dynamic Process of Duplication and ESS

Before we consider this imitation experiment, we discuss the important conclusion that if some Strategy *i* (where *i* = 1,2) is the evolutionary dominant strategy of some player, the expected payoff of an individual who chooses this strategy must be higher than the average expected payment of its population $(\overline{u}_t)$; that is, $u_{ii}^{t} \geq \overline{u}_t, i = 1, 2$. According to the definition of evolutionary dominant strategy, this conclusion is absolutely true; otherwise, Strategy *i*, *i* = 1,2, could not be an evolutionary dominant strategy.

In an evolutionary game based on a symmetric case, we find that an individual keeps his strategy only when this strategy is an evolutionary dominant strategy, and others will learn and copy his strategy. It is evident that in the future the proportion of individuals who take this evolutionary dominant strategy will increase and

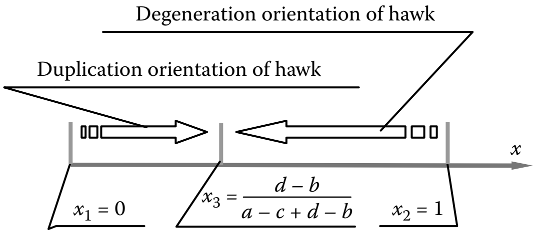


Figure 10.2 Relation of stable points x_1, x_2, x_3 .

at the same time the previously dominant strategy disappears continually. In other words, when no dominant strategy exists, learning and imitating will disappear. At this moment, each individual of this population keeps his strategy and the game is at a comparatively stable state.

For a general 2 × 2 symmetric case, where $p_{21}^{t+1} = 0$, we can calculate the proportions of individuals who take different strategies in a stable state [see Eq. (10.11)]. Equation (10.11) indicates that in this game there are three equilibrium points, including the point decided by ESS (evolutionary stable sets).

$$\begin{aligned} & p_{21}^{t+1} = p_{22}^{t} \cdot \frac{\bar{u}_t - u_{22}^t}{\underset{x\in[0,1]}{Max}\left|\bar{u}_t - u_{22}^t\right|} = 0 \\ & x_1 = 0, \qquad x_2 = 1, \qquad x_3 = \frac{d-b}{a-c+d-b} \end{aligned} \tag{10.11}$$

In Eq. (10.11), stable point x_3, must be in region [0,1], when $d = b$, x_3 is a minimum value 0, and when $a = c$, x_3 is a maximum value 1. Therefore, Figure 10.2 describes the relationships of these three stable points as horizontal coordinates.

Now the problem is, which of these three stable points is the equilibrium decided by ESS? Does a given initial state converge at an equilibrium point? How many generations of replication and evolution does it take, if it is convergent? What happens to the average expected payment of populations during an evolutionary game? In order to answer these questions, we designed a simulation procedure in MATLAB to imitate the famous symmetric Hawk-Dove Game (such as shown in Table 10.2) based on correlated formulas.

Table 10.2 Hawk-Dove Game Payoff Matrix ($v = 2, c = 1,2$)

	Population 2	*Strategy 1*	*Strategy 2*
Population 1	Strategy 1	$\frac{v-c}{2}, \frac{v-c}{2}$	v, 0
	Strategy 2	0, v	$\frac{v}{2}, \frac{v}{2}$

Table 10.3 Evolutionary Process (Initial x = 1.0000 e^{-0004})

Generation ($t + i$)	p_{11}^{t+i}	$\bar{u}_{t+i}$	$\frac{(\bar{u}_{t+1}-\bar{u}_t)}{\lvert\bar{u}_t\rvert}$
0	1.0000 e^{-004}	1.0000	/
1	1.1999 e^{-004}	0.9999	–0.01%
5	2.4862 e^{-004}	0.9998	–0.01%
10	6.1731 e^{-004}	0.9996	–0.02%
15	0.0013	0.9990	–0.06%
20	0.0032	0.9978	–0.12%
25	0.0077	0.9947	–0.31%
30	0.0180	0.9874	–0.73%
35	0.0387	0.9711	–1.36%
40	0.0718	0.9416	–2.95%
45	0.1091	0.9032	–3.84%
50	0.1377	0.8702	–3.30%
55	0.1537	0.8502	–2.00%
60	0.1612	0.8405	–0.97%
65	0.1644	0.8363	–0.42%
70	0.1657	0.8346	–0.17%
75	0.1663	0.8339	–0.07%
80	0.1665	0.8335	–0.04%
85	0.1666	0.8334	–0.01%
86	0.1666	0.8334	0.00%

In Figure 10.2, the value interval is divided into three parts by x_1, x_2, x_3. During imitation, we could choose initial values in regions [x_1, x_3] and [x_2, x_3]. Table 10.3 and Table 10.4 show the imitation conditions.

In Tables 10.3 and 10.4, $\frac{(\bar{u}_{t+1}-\bar{u}_t)}{|\bar{u}_t|}$ means the growth rate of expected average payments of an individual in the game. From the results of the above-mentioned imitation, we discovered there exists a unique equilibrium of ESS, $x_3 = \frac{d-b}{a-c+d-b} = \frac{1-2}{-5-0+1-2} = \frac{1}{6}$. When initial values lie in intervals [x_1, x_3] and [x_2, x_3],

Table 10.4 Evolutionary Process (Initial x = 0.9999)

Generation $(t + i)$	p_{11}^{t+i}	$\bar{u}_{t+i}$	$\frac{(\bar{u}_{t+1}-\bar{u}_t)}{\lvert\bar{u}_t\rvert}$
0	0.9999	–4.9976	/
5	0.9968	–4.9618	0.72%
10	0.9078	–3.9290	20.82%
15	0.3766	0.3720	109.47%
20	0.2182	0.7582	203.82%
25	0.1843	0.8091	6.71%
30	0.1734	0.8243	1.88%
35	0.1693	0.8298	0.67%
40	0.1677	0.8319	0.25%
45	0.1671	0.8328	0.11%
50	0.1668	0.8331	0.04%
55	0.1667	0.8333	0.02%
56	0.1667	0.8333	0.00%

the process of evolution develops to an equilibrium decided by ESS, $x_3 = \frac{1}{6}$, over time (see Figure 10.2).

10.1.3 Initial State and Analysis of Replication Dynamics

In the Hawk-Dove Game, that initial value $p_{11}^{t+i} = p_{11}^{0}$ lies in the interval $[x_1,x_3]$ means the proportion of hawks is less than the stable proportion x_3. In this case, we know from Table 10.3 that the quantity of hawks would grow gradually from its initial value until it reaches stable proportion x_3. We call this process the generating process of hawks or the degenerating process of doves. When initial $p_{11}^{t+i} = p_{11}^{0} = 0.0001$, the generating process of hawks takes 85 generations, during which the expected average payments of individuals of this system, from $\bar{u}_0 = 1.0000$, changes decreasingly to stable value $\bar{u}_{85} = \frac{1}{6}$. With respect to individual hawks, its expected payment increases, while the expected average payment of all individuals of this system decreases. This, from one point of view, shows that an individual's rationality sometimes is in contradiction with the rationality of the colony.

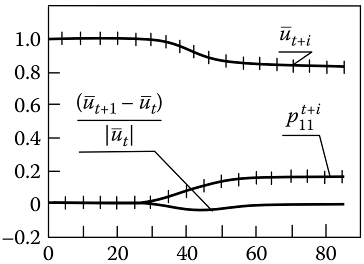


Figure 10.3 Evolutionary process of a hawk.

In the case shown in Table 10.4, the number of hawks goes down gently until it reaches stable proportion x_3. We call this process the degenerating process of hawks or the generating process of doves. When $p_{11}^{t+i} = p_{11}^{0} = 0.9999$, the degenerating process of hawks lasts for 55 generations, during which the expected average payment of individuals, from $\bar{u}_0 = -4.9976$, grows gradually to stable value $\bar{u}_{55} = \frac{1}{6}$. The expected income of individual hawks increases during degeneration, and the expected average payment of all individuals grows. In this condition, the rationality of individuals and that of the colony are in accordance.

Comparing the results in Tables 10.3 and 10.4, we found that it takes 85 generations for hawks to grow from 0.0001 to 0.1667, while it takes only 55 generations to degenerate. Therefore, we could say that the degenerating process of hawks goes more quickly than the generating process of hawks.

On the basis of the above analysis and the imitation result in Tables 10.3 and 10.4, we get the generation, replication, and evolutionary process of hawks and doves respectively, presented in Figure 10.3 and Figure 10.4.

Figures 10.3 and 10.4 respectively illustrate the changes of individual proportion (p_{11}^{t+i}), individual expected average payment ($\bar{u}_{t+i}$), and individual growth rate of expected average payments $\frac{(\bar{u}_{t+1}-\bar{u}_t)}{|\bar{u}_t|}$ of hawks and doves.

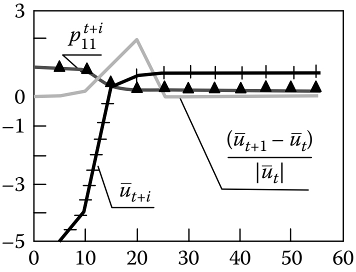


Figure 10.4 Evolutionary process of a dove.

10.1.4 Summary

The chain structure model for an evolutionary game is a good tool to describe the evolutionary process more vividly and is an easy method with which to model and analyze many games of real life. Not only could it be used to study long-term stable trends but it could also be used to predict the probable evolution trajectory in the near future.

10.2 Chain Structure Models of Evolutionary Games Based on an Asymmetric Case

10.2.1 Analysis of the Chain Structure Model of Evolutionary Games

In order to discuss the process of evolutionary games based on an asymmetric case, we will employ a framework with two or more distinct populations with bounded rationality and where, in each contest, a randomly chosen member of one of the populations plays a game with a randomly chosen member of the other population.

Given an evolutionary game based on an asymmetric case, we assume that the choice of strategy of each member of every Population i ($i = 1, 2,\ldots,n$) is j, ($j = 1, 2,\ldots m$); during the game, each member of some population chooses a strategy that is not always the best, after learning from each other or imitating a neighboring strategy. Here, a circle with j is called a strategy of evolutionary game, where j, ($j = 1,2,\ldots m$). Let $p_{i,jk}^t,(j,k = 1,2,\ldots,m)$ be the transfer proportion from original Strategy j to a new one, k, in Population i at time t, and $u_{i,jk}^t,(j,k = 1,2,\ldots,m)$ be the expected income of Population i when Strategy k is taken at time t. An arrow mark whose end denotes the original strategy and whose head indicates the new strategy of an individual in the game, together with $p_{i,jk}^t,(j,k = 1,2,\ldots,m)$ and $u_{i,jk}^t,(j,k = 1,2,\ldots,m)$, describe the process of game. Thus we get sketch of an evolutionary game chain model.

We take a 2 × 2 asymmetric case as an example to discuss a model establishing this case and other problems. One 2 × 2 payoff matrix based on a common asymmetric case is illustrated in Table 10.5. So according to the above-mentioned

Table 10.5 2 × 2 Payoff Matrix Based on an Asymmetric Case

Population 2		*Strategy 1*	*Strategy 2*
Population 1	Strategy 1	*a, b*	*c,d*
	Strategy 2	*e,f*	*g,h*

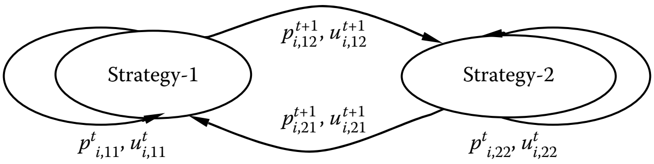


Figure 10.5 Sketch of evolutionary game chain model of a member of Population *i* (*i* = 1,2).

definitions, we get a sketch of an evolutionary game chain model at time t (see Figure 10.5).

In Figure 10.5, the two circles denote Strategy 1 and Strategy 2 respectively. Four arrows show, from time t to time $t + 1$, the probable strategy transfer conditions of populations: some players maintain their original strategies, and some players imitate and study from others and change their strategies. No matter what the condition, individuals make decisions based on their payoff strategy. If it is a self-round arrow, whose head and end direct to the same strategy, it indicates the player keeps his original strategy at time t, and the proportion and expected profit are marked as $p^t_{i,jk}, u^t_{i,jk}, j = k; j,k = 1,2$ respectively. If not, it means some of the individuals who took this strategy at time t change to other strategies at time t + 1, and the notes on the arrows $p^{t+1}_{i,jk}, u^{t+1}_{i,jk}, j \neq k; j,k = 1,2$ indicate the proportion of population that changes strategies and the corresponding expected profit respectively.

Because the order of population exerts a certain influence on the process and on the result of game, an asymmetric game of two populations should be divided into two categories.

Type I: A member of Population i takes action first and Player s acts subsequently. After judging his situation, Player i chooses a strategy that could bring a satisfactory income in the foreseeable future. By analogy, Player s reacts in the condition that Player i had acted. We say that this run of game ends when the two players of game have made choices. In a similar way, they continue the next run of game, and from an initial state finally they reach an equilibrium point that is decided by an evolutionary stable strategy (ESS). Figure 10.6 shows the process.

Type II: Players i and s take actions at the same time. Each of them chooses the strategy that will make him get his expected payoff after thinking about his condition. The process of game is illustrated in Figure 10.7, and this evolutionary game ends at the point of equilibrium decided by evolutionary stable strategy (ESS).

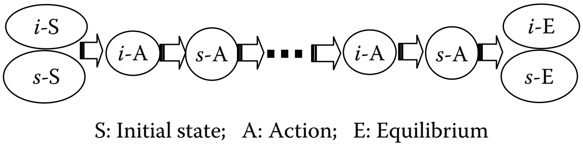


Figure 10.6 Process of game when some player acts (S: initial state; A: action; E: equilibrium).

10.2.2 Establishing Chain Structure Model for Evolutionary Game

Analogous to Section 10.2.1, we could categorize this game into two forms according to the order of action (such as Figure 10.6 and Figure 10.7 show). We consider the situation of one player taking his action first. For convenience, suppose that Player 1 acts first, then Player 2.

Type I: Some player acts first. In the discussion below, we assume that Player 1 acts first and Player 2 acts later. See the payoff matrix in Table 10.5 and the sketch in Figure 10.5, and we presume that at time $t-1$, the proportion of Players 1 who choose Strategy 1 is x, and the remaining $1-x$ of Players 1 choose Strategy 2; the proportion of Players 2 who choose Strategy 1 is y, and the remaining $1-y$ of Players 2 choose Strategy 2. Therefore, the proportions of individual quantity, expected income of individuals, and average expected payments of Players 1 are given in Eqs. (10.12), (10.13), and (10.14) respectively.

$$\begin{aligned} &p_{1,11}^{t-1} = x, p_{1,22}^{t-1} = 1-x \\ &p_{2,11}^{t-1} = y, p_{2,22}^{t-1} = 1-y \end{aligned} \tag{10.12}$$

$$\begin{aligned} &u_{1,11}^{t-1} = y \cdot a + (1-y) \cdot c \\ &u_{1,22}^{t-1} = y \cdot e + (1-y) \cdot g \end{aligned} \tag{10.13}$$

$$\bar{u}_1^{t-1} = x \cdot u_{1,11}^{t-1} + (1-x) \cdot u_{1,22}^{t-1} \tag{10.14}$$

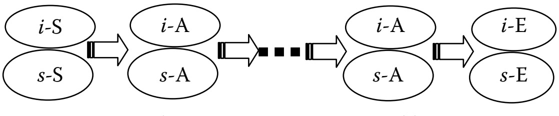


Figure 10.7 Process of game when players act together (S: initial state; A: action; E: equilibrium).

Then we think about what these members with bounded rationality of Population 1 will do to improve their income at time t.

The emphasis of an evolutionary game is a dynamic altered by proportion, whose focus is its growth rate. We use positive signs and negative signs to denote the direction of the growth rate; $p_{i,jk}^{t+1}, u_{i,jk}^{t+1}, j \neq k; j,k = 1,2$ just reflects this rate. Based on evolutionary game theory, we know that the rate of dynamic alteration is decided by the speed at which members in this game learn and imitate. Their speed of learning and imitating mainly lie on two factors: One factor is the quantity of imitators, which influences the scale and the degree of imitation and which could be described by the proportion of choice of certain strategy; the other factor is how successful the imitated is, which affects how hard the judgment is and how much the encouragement there is. The more the average payoff of one player is higher than the expected income of an imitator, the greater the number of individuals who will imitate him. We assume that the proportion of strategy transfer in one population from time $t - 1$ to time t is in direct proportion to the quantity of imitators and also is in direct proportion to the gap between the average expected payment of the population and the expected income of individuals. At the same time, we presume that both proportionality coefficients are 1.

Thus the states of individuals of Population 1 could be described in Eqs. (10.15) and (10.16). Here $u_{1,jk}^{t} = u_{1,jj}^{t}; j,k = 1,2$ means the payoff of an individual who transfers strategy from time $t - 1$ to time t equal to the expected income of the imitated, and that means the cost of learning and imitating is zero. We get the parameters of Population 1 at time t [see Eq.(10.17) for more details]. Similarly, Population 2 acts at time $t + 1$ and its parameters are illustrated in Eqs. (10.18), (10.19), and (10.20). In the same way, Player 1 makes his choice at moment of $t + 2$ based on the situation of time $t + 1$. The repetition of this process continues until both populations reach the point of equilibrium decided by ESS.

$$\begin{aligned} p_{1,12}^{t} &= p_{1,11}^{t-1} \cdot \frac{\overline{u}_1^{t-1} - u_{1,11}^{t-1}}{\underset{x\in[0,1]}{Max}\left|\overline{u}_1^{t-1} - u_{1,11}^{t-1}\right|} \\ p_{1,11}^{t} &= p_{1,11}^{t-1} + p_{1,21}^{t} \end{aligned} \tag{10.15}$$

$$\begin{aligned} p_{1,21}^{t} &= p_{1,22}^{t-1} \cdot \frac{\overline{u}_1^{t-1} - u_{1,22}^{t-1}}{\underset{x\in[0,1]}{Max}\left|\overline{u}_1^{t-1} - u_{1,22}^{t-1}\right|} \\ p_{1,22}^{t} &= 1 - p_{1,11}^{t} = p_{1,22}^{t-1} + p_{1,12}^{t} \end{aligned} \tag{10.16}$$

$$\begin{aligned} u_{1,22}^{t} &= p_{1,22}^{t} \cdot e + \left(1 - p_{1,22}^{t}\right) \cdot g \\ u_{1,11}^{t} &= p_{2,11}^{t-1} \cdot a + \left(1 - p_{2,11}^{t-1}\right) \cdot c \\ \overline{u}_1^{t} &= p_{1,11}^{t} \cdot u_{1,11}^{t} + p_{1,22}^{t} \cdot u_{1,22}^{t} \end{aligned} \tag{10.17}$$

$$\begin{aligned}
&p_{2,11}^{t} = y, p_{2,22}^{t} = 1 - y \\
&u_{2,11}^{t} = p_{1,11}^{t} \cdot b + (1 - p_{1,11}^{t}) \cdot f \\
&u_{2,22}^{t} = p_{2,22}^{t} \cdot d + (1 - p_{2,22}^{t}) \cdot h \\
&\bar{u}_2^t = p_{2,11}^t \cdot u_{2,11}^t + p_{2,22}^t \cdot u_{2,22}^t
\end{aligned} \tag{10.18}$$

$$\begin{aligned}
&p_{2,12}^{t+1} = p_{2,11}^{t} \cdot \frac{\bar{u}_2^t - u_{2,11}^t}{\underset{x \in [0,1]}{Max} \left| \bar{u}_2^t - u_{2,11}^t \right|} \\
&p_{2,21}^{t+1} = p_{2,22}^{t} \cdot \frac{\bar{u}_2^t - u_{2,22}^t}{\underset{x \in [0,1]}{Max} \left| \bar{u}_2^t - u_{2,22}^{t-1} \right|}
\end{aligned} \tag{10.19}$$

$$\begin{aligned}
&p_{2,11}^{t+1} = p_{2,11}^{t} + p_{2,21}^{t+1} \\
&p_{2,22}^{t+1} = p_{2,22}^{t} + p_{2,12}^{t+1} \\
&\bar{u}_2^{t+1} = p_{2,11}^{t+1} \cdot u_{2,11}^{t+1} + p_{2,22}^{t+1} \cdot u_{2,22}^{t+1}
\end{aligned} \tag{10.20}$$

Type II: Both players act at the same time. At time t, members of two populations make their decisions (replicate, imitate, or learn) simultaneously after analyzing the situation of time $t - 1$. The discussion is similar to that of Type I, and we give the parameters of the game in Eqs. (10.21), (10.22), (10.23), (10.24), (10.25), and (10.26):

$$\begin{aligned}
&p_{1,11}^{t-1} = x, \ p_{1,22}^{t-1} = 1 - x \\
&p_{2,11}^{t-1} = y, \ p_{2,22}^{t-1} = 1 - y \\
&u_{1,11}^{t-1} = p_{2,11}^{t-1} \cdot a + p_{2,22}^{t-1} \cdot c \\
&u_{1,22}^{t-1} = p_{2,11}^{t-1} \cdot e + p_{2,22}^{t-1} \cdot g
\end{aligned} \tag{10.21}$$

$$\begin{aligned}
&u_{2,11}^{t-1} = p_{1,11}^{t-1} \cdot b + p_{1,22}^{t-1} \cdot f \\
&u_{2,22}^{t-1} = p_{1,11}^{t-1} \cdot d + p_{1,22}^{t-1} \cdot h \\
&\bar{u}_1^{t-1} = p_{1,11}^{t-1} \cdot u_{1,11}^{t-1} + p_{1,22}^{t-1} \cdot u_{1,22}^{t-1} \\
&\bar{u}_2^{t-1} = p_{2,11}^{t-1} \cdot u_{2,11}^{t-1} + p_{2,22}^{t-1} \cdot u_{2,22}^{t-1}
\end{aligned} \tag{10.22}$$

$$
\begin{aligned}
p_{1,12}^{t} &= p_{1,11}^{t-1} \cdot \frac{\overline{u}_{1}^{t-1} - u_{1,11}^{t-1}}{\underset{x,y\in[0,1]}{Max}\left|\overline{u}_{1}^{t-1} - u_{1,11}^{t-1}\right|} \\
p_{1,21}^{t} &= p_{1,22}^{t-1} \cdot \frac{\overline{u}_{1}^{t-1} - u_{1,22}^{t-1}}{\underset{x,y\in[0,1]}{Max}\left|\overline{u}_{1}^{t-1} - u_{1,22}^{t-1}\right|}
\end{aligned}
\tag{10.23}
$$

$$
\begin{aligned}
p_{2,12}^{t} &= p_{2,11}^{t-1} \cdot \frac{\overline{u}_{2}^{t-1} - u_{2,11}^{t-1}}{\underset{x\in[0,1]}{Max}\left|\overline{u}_{2}^{t-1} - u_{2,11}^{t-1}\right|} \\
p_{2,21}^{t} &= p_{2,22}^{t-1} \cdot \frac{\overline{u}_{2}^{t-1} - u_{2,22}^{t-1}}{\underset{x\in[0,1]}{Max}\left|\overline{u}_{2}^{t-1} - u_{2,22}^{t-1}\right|}
\end{aligned}
\tag{10.24}
$$

$$
\begin{aligned}
& p_{1,11}^{t} = p_{1,11}^{t-1} + p_{1,21}^{t}, \quad p_{1,22}^{t} = 1 - p_{1,11}^{t} = p_{1,22}^{t-1} + p_{1,12}^{t} \\
& u_{1,11}^{t} = p_{2,11}^{t} \cdot a + p_{2,22}^{t} \cdot c, \quad u_{1,22}^{t} = p_{2,11}^{t} \cdot e + p_{2,22}^{t} \cdot g \\
& u_{2,11}^{t} = p_{1,11}^{t} \cdot b + p_{1,22}^{t} \cdot f, \quad u_{2,22}^{t} = p_{1,11}^{t} \cdot d + p_{1,22}^{t} \cdot h
\end{aligned}
\tag{10.25}
$$

$$
\begin{aligned}
p_{2,11}^{t} &= p_{2,11}^{t-1} + p_{2,21}^{t} \\
p_{2,22}^{t} &= 1 - p_{2,11}^{t} = p_{2,22}^{t-1} + p_{2,12}^{t} \\
\overline{u}_{1}^{t} &= p_{1,11}^{t} \cdot u_{1,11}^{t} + p_{1,22}^{t} \cdot u_{1,22}^{t} \\
\overline{u}_{2}^{t} &= p_{2,11}^{t} \cdot u_{2,11}^{t} + p_{2,22}^{t} \cdot u_{2,22}^{t}
\end{aligned}
\tag{10.26}
$$

10.2.3 Imitation of Dynamic Process of Duplication and ESS

Before considering the imitation experiment, let us discuss the important conclusion that if some Strategy j, (j = 1,2), is the evolutionary dominant strategy of some player, the expected payoff of the individual who chooses this strategy must be higher than the average expected payment of its population; that is, $u_{i,jj}^{t} \geq \overline{u}_{i}^{t}, i, j = 1, 2$. According to the definition of evolutionary dominant strategy, this conclusion is absolutely true; otherwise, Strategy j (j = 1,2) couldn't be an evolutionary dominant strategy.

In an evolutionary game based on an asymmetric case, we find that an individual keeps his strategy only when this strategy is an evolutionary dominant strategy, and others will learn and copy his strategy. It is evident that in future the proportion of individuals who take the evolutionary dominant strategy will increase and at the same time, the "dominant" strategy continually disappears. In other words,

Table 10.6 Hawk-Dove Game Payoff Matrix ($v_1 = 10$, $v_2 = 2$, $c = 12$)

Population 2		*Strategy 1*	*Strategy 2*
Population 1	Strategy 1	$\frac{v_1-c}{2},\frac{v_2-c}{2}$	v_1, 0
	Strategy 2	0, v_2	$\frac{v_1}{2},\frac{v_2}{2}$

when no dominant strategy exists, learning and imitating will disappear. At this moment, each individual of this population keeps his strategy and stays in the state of equilibrium decided by ESS.

In our imitation experiment, we take one typical asymmetric Hawk-Dove Game as an example, in which two populations act at the same time. Table 10.6 shows the payoff matrix of that game.

We see the situation of Population 1, in Eq. (10.23), where $p_{1,21}^t = 0$; then we get the equilibrium point, presented in Eq. (10.27). Analogous to Eq. (10.25), where $p_{2,21}^t = 0$, we could work out the equilibrium shown in Eq. (10.28). In fact, $p_{1,21}^t$ and $p_{2,21}^t$ correspond to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively, which are the increase rates in the quantity of individuals who take Strategy 1 in Populations 1 and 2. Therefore, Eqs. (10.27) and (10.28) decide all possible equilibrium points (see Figure 10.6 for more details).

$$x(1-x)[(a-e+g-c)y+c-g]=0,\quad x_1=0,\quad x_2=1,\quad y=\frac{g-c}{a-e+g-c} \tag{10.27}$$

$$y(1-y)[(b-d+h-f)x+f-h]=0,\quad y_1=0,\quad y_2=1,\quad x=\frac{h-f}{b-d+h-f} \tag{10.28}$$

In Figure 10.8, A, B, C, D, and E are all points of equilibrium. Which of them are the points of equilibrium decided by ESS? Does a given initial state converge at an equilibrium point? How many generations of replication and evolution does it take, if it is convergent? What happens to the average expected payment of populations during an evolutionary game? In order to answer these questions, we designed a simulation procedure in MATLAB to imitate the famous asymmetric Hawk-Dove Game based on correlated formulas. The result of experiment is given in Table 10.7.

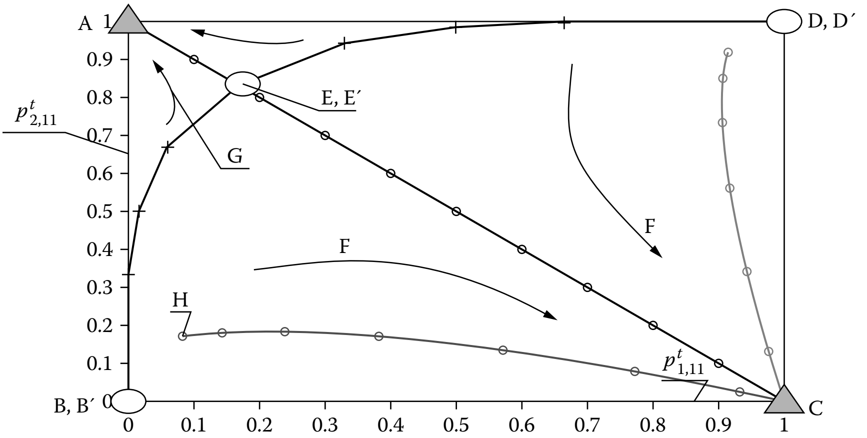


Figure 10.8 Dynamic process, direction of duplication.

10.2.4 Initial State and Analysis of Replication Dynamics

Figure 10.8 shows the equilibrium and simulating situation of this asymmetric Hawk-Dove Game. In Figure 10.8, B, E, and D are unstable equilibrium points, which will not be reached when the initial value has little departure from these equilibriums. Hence we call these three points the single equilibrium points. In reverse, A and C, which we call stable equilibrium points, are equilibrium decided by ESS. All initial states whose values are not equal to that of a single equilibrium point finally will converge on these stable equilibrium points in a period of evolution.

Figure 10.8 illustrates the Hawk Copy Evolutionary Areas (HCEA) of the Populations 1 and 2. The horizontal coordinates correspond to the individual proportion ($p^t_{1,11}$) of hawks of Population 1, and the vertical coordinates correspond to the individual proportion ($p^t_{2,11}$) of hawks of Population 2.

By simulation experiment, we get the critical values presented in Table 10.7. In addition to the unstable equilibrium points (B, E, and D), for the initial states that are in critical values or that are in the zone close to A, the final equilibrium point of ESS is A. Analogously, in addition to the unstable equilibrium points (B, E, and D) and the points in critical values, for the initial states that are in the zone close to C, the eventual equilibrium point of ESS is C. These points of critical value compose a critical value curve of HCEA, which is B′E′D′ in Figure 10.8, and by which area ABCD is divided into two parts, AB′E′D′ and B′CD′E′. The points whose initial values are in area AB′E′D′ must converge to point A in a period of time. The arrow G in this area shows the growth direction of Population 2. The points belonging to area B′CD′E′ (besides points of curve B′CD′E′) must converge to C, and arrow F is the evolutionary direction of Population 1.

Table 10.7 Simulation of Evolutionary Process of Hawk-Dove Game

Generation	*Population*	$p_{i,jk}^{t}$	$\bar{u}_{i}^{t}$
0	1	0.0833	4.5000
	2	0.1667	1.0001
5	1	0.7772	8.1355
	2	0.0777	–0.0616
10	1	1.0000	10.0000
	2	$2.0491e^{-017}$	$-3.2133e^{-016}$
11	1	1.0000	10.000
	2	0.0000	0.0000
0	1	0.9167	–0.0417
	2	0.9167	–4.0417
5	1	0.9748	8.4756
	2	0.1289	–0.5998
10	1	1.0000	10.0000
	2	$2.9131e^{-026}$	$-1.9984e^{-016}$
11	1	1.0000	10.0000
	2	0.0000	0.0000
0	1	0.3333	2.0000
	2	0.6667	$2.4980e^{-016}$
5	1	0.7508	6.3855
	2	0.2492	–0.6242
10	1	1.0000	10.0000
	2	$1.0702e^{-010}$	$-4.2807e^{-010}$
11	1	1.0000	10.000
	2	$2.5196e^{-020}$	$-1.1453e^{-019}$
12	1	1.0000	10.000
	2	0.0000	0.0000

Curve HC corresponds to an example whose initial value are $p_{1,11}^0 = 0.0833$, $p_{2,11}^0 = 0.1667$. HC lies in the growth area of Population 1, and it shows the dominant trend: for Population 1 the quantity of hawks increases through learning, imitation, and replication, while the number of doves reduces; for Population 2, the quantity of doves increases through learning, imitation, and copying, while the number of hawks decreases continuously. In the game described by HC, changes of $p_{1,11}^t$, $p_{2,11}^t$, and the growth rate of average expected payoff of each population $\frac{(\bar{u}_1^{t+1}-\bar{u}_1^t)}{|\bar{u}_1^t|}$ and $\frac{(\bar{u}_2^{t+1}-\bar{u}_2^t)}{|\bar{u}_2^t|}$ are given in Figures 10.7 and 10.8.

In Figure 10.8, the curve H_2C_2 presents the general trend of $p_{2,11}^t$, which reduces gradually to an equilibrium point zero decided by ESS along with evolution. However, the increases of $p_{2,11}^1$ and $p_{2,11}^2$ are 8.34 percent and 2.21 percent respectively in the first and second steps of the dynamic process. This short-time situation is not in accord with the general trend; hence, this inconsistency in the evolution process is called the test-fault of an evolution game. This phenomenon indicates the existence of a test-fault of a bounded-rational population in an evolutionary game. Much evidence could be found to prove it in real life.

10.2.5 Summary

A chain structure model for evolutionary games is a good tool to describe the evolutionary process more vividly and is an easy method to model and analyze many games of real life. Not only can it be used to study long-term stable trends but it also predicts the probable evolution trace in the short term. This section primarily discussed the asymmetric case. Chain structure models for evolutionary games based on asymmetric cases—whose means are flexible, whose forms are lively, and whose theoretical studies and practical applications are abounding—are convenient for further study of game problems. Additional studies on chain structure models of evolutionary games will follow.

10.3 Chain Structure Models of Grey Evolution of Industry Agglomeration and Stability Analysis

10.3.1 Research Background

Industry geographic agglomeration, as a demonstration of an industry's optimal allocation, has become a global economic phenomenon. The study attracts more and more researchers for its influence on modern regional economic growth. Researching the formation mechanism and the process of industry agglomeration is meaningful in a real sense. Game theory has not yet been broadly used in the research of this area, which mainly focuses on issues of cost. Evolutionary game theory was born because development of traditional game theory failed; evolutionary game theory discarded the full-rationality assumptions of classic game theory and treats players as living creatures with bounded rationality who finish self-evolutions while competing with each other.

The existing game chain model[6] based on evolutionary game theory can better explain the generation and development of some social and economic phenomena, not only suitable for predicting one-off games or those of short-term economic equilibrium but also for long-term stable evolutionary trends. But this model failed to consider the incomplete information caused by the actual complexities of an economic environment and realistic decision-making. To answer these questions, this section discusses the evolutionary game of industry agglomeration in the sphere of an unpredictable income based on the grey game theory.[5–10] We consider how companies learn and take actions during the agglomeration process when there exists a grey uncertainty about income levels, and we discuss in depth the realization and mechanism of the equilibrium.

10.3.2 Establishing a Chain Structure Model for the Evolutionry Game of Industry Agglomeration Development

10.3.2.1 Company Learning Mechanisms

Learning is a way to spread knowledge, which embodies the participants' interactions, time, and maintenance to ensure that knowledge is transferred between companies. Initially, the motivation mechanism of an outside environment is the external driving force to learn; when a dominant strategy is taken, the higher the income of one company than the average payoff, the greater the number companies that take this strategy. At that point, the learning ability of company members is the internal driving force, and the larger the population of that group, the higher that learning ability will be.

10.3.2.2 Model Construction

Suppose in an evolutionary game of industry agglomeration, the participants have limited rationality mainly because they use group-decision policy methods for organizational behavior. Then the participants have a low capacity to realize the errors and adjust the strategies, making a change of economic action more like a slow evolution rather than fast learning. During the game, each company of the population chooses a strategy that is not always the best while learning from each other or imitating successful companies' dominant game strategies. Here, a circle with agglomeration demonstrates that a company takes the agglomeration strategy, while a circle with nonagglomeration means it takes the opposite strategy. An arrow mark whose end denotes the original strategy a company takes and whose head indicates the new strategy taken by the same company is used to describe the process of the game together with the transfer proportion between

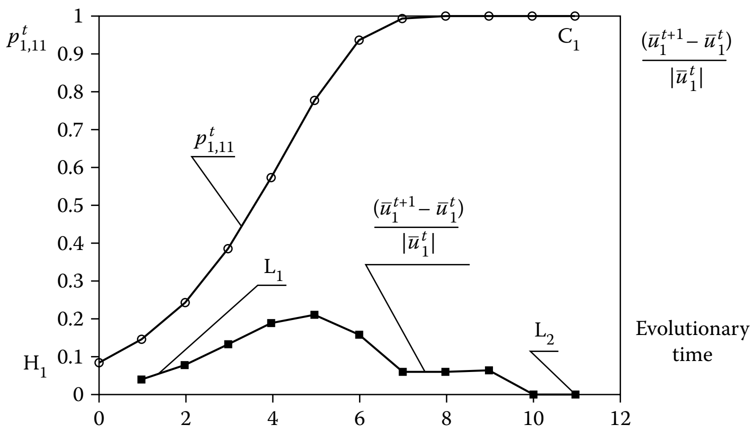


Figure 10.9 Evolutionary game process of Population 1 ($p^0_{1,11} = 0.0833$, $p^0_{2,11} = 0.1667$).

strategies and the respective income. Thus, we get sketch of the game chain model for an evolutionary game of industry agglomeration development, as shown in Figure 10.9. In addition, taking the uncertain income of an industry agglomeration evolution game as an example, the general payoff value matrix is illustrated in Table 10.8.

We assume that at time *t*, the proportion of companies who choose the agglomeration strategy is *x*, and the remaining 1 − *x* companies choose nonagglomeration. According to Figure 10.9, we know the alterations of all the companies in the game from time *t* to time *t* + 2, as they maintain the original strategy, learn from each other, or imitate another.

$$\begin{aligned} p^t_{11} &= x \\ u^t_{11} &= x \cdot \otimes\otimes_{21} + (1-x) \cdot \otimes_2 \\ &= x \cdot (a_1 + (b_1 - a_1)\gamma_1) + (1-x) \cdot (a_2 + (b_2 - a_2)\gamma_2) \end{aligned} \tag{10.29}$$

Table 10.8 Payoff Value Matrix of the Industry Agglomeration Evolutionary Game

Group 2		*Agglomeration*	*Nonagglomeration*
Group 1	Agglomeration	$\otimes_1, \otimes_1$	$\otimes_2, \otimes_3$
	Nonagglomeration	$\otimes_3, \otimes_2$	$\otimes_4, \otimes_4$

$$\begin{aligned} p_{22}^{t} &= 1-x \\ u_{22}^{t} &= x\cdot\otimes_3+(1-x)\cdot\otimes_4 \\ &= x\cdot(a_3+(b_3-a_3)\gamma_3)+(1-x)\cdot(a_4+(b_4-a_4)\gamma_4) \end{aligned} \tag{10.30}$$

$$\bar{u}_t(\otimes) = x\cdot u_{11}^{t}+(1-x)\cdot u_{22}^{t}\ (0\le\gamma_i\le 1) \tag{10.31}$$

$$\begin{aligned} p_{12}^{t+1} &= p_{11}^{t}\cdot\frac{\bar{u}_t(\otimes)-u_{11}^{t}}{|\bar{u}_t(\otimes)|+\left|u_{11}^{t}\right|} \\ u_{12}^{t+1} &= u_{22}^{t+1} \end{aligned} \tag{10.32}$$

$$\begin{aligned} p_{21}^{t+1} &= p_{22}^{t}\cdot\frac{\bar{u}_t(\otimes)-u_{22}^{t}}{|\bar{u}_t(\otimes)|+\left|u_{22}^{t}\right|} \\ u_{21}^{t+1} &= u_{11}^{t+1} \end{aligned} \tag{10.33}$$

$$\begin{aligned} p_{11}^{t+1} &= p_{11}^{t}+p_{21}^{t+1} \\ u_{11}^{t+1} &= p_{11}^{t+1}\cdot\otimes_1+\left(1-p_{11}^{t+1}\right)\cdot\otimes_2 \\ &= p_{11}^{t+1}\cdot(a_1+(b_1-a_1)\gamma_1)+\left(1-p_{11}^{t+1}\right)\cdot(a_2+(b_2-a_2)\gamma_2) \end{aligned} \tag{10.34}$$

$$\begin{aligned} p_{22}^{t+1} &= 1-p_{11}^{t+1}=p_{22}^{t}+p_{12}^{t+1} \\ u_{22}^{t+1} &= p_{11}^{t+1}\cdot\otimes_3+\left(1-p_{11}^{t+1}\right)\cdot\otimes_4 \\ &= p_{11}^{t+1}\cdot(a_3+(b_3-a_3)\gamma_3)+\left(1-p_{11}^{t+1}\right)\cdot(a_4+(b_4-a_4)\gamma_4) \end{aligned} \tag{10.35}$$

$$\bar{u}_{t+1}(\otimes) = p_{11}^{t+1}\cdot u_{11}^{t+1}+p_{22}^{t+1}\cdot u_{22}^{t+1} \tag{10.36}$$

$$\begin{aligned} p_{12}^{t+2} &= p_{11}^{t+1}\cdot\frac{\bar{u}_{t+1}(\otimes)-u_{11}^{t+1}}{|\bar{u}_{t+1}(\otimes)|+\left|u_{11}^{t+1}\right|} \\ u_{12}^{t+2} &= u_{22}^{t+2} \end{aligned} \tag{10.37}$$

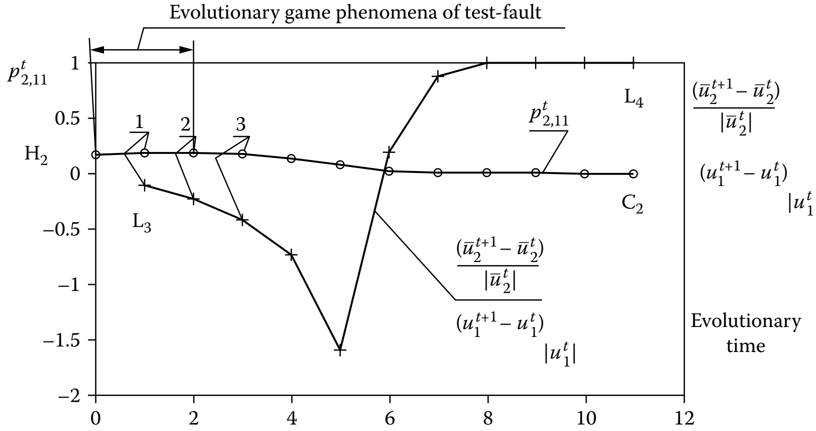


Figure 10.10 Evolutionary game process of Population 2 ($p^0_{1,11}=0.0833$, $p^0_{2,11}=0.1667$).

$$p_{21}^{t+2} = p_{22}^{t+1}\cdot\frac{\bar{u}_{t+1}(\otimes)-u_{22}^{t+1}}{|\bar{u}_{t+1}(\otimes)|+\left|u_{22}^{t+1}\right|}$$

$$u_{21}^{t+2} = u_{11}^{t+2} \tag{10.38}$$

Letting $p_{21}^{t+1}=0$, the equilibrium solution is

$$p_{21}^{t+1} = p_{22}^{t}\cdot\frac{\bar{u}_{t}(\otimes)-u_{22}^{t}}{|\bar{u}_{t}(\otimes)|+\left|u_{22}^{t}\right|}=0,$$

$$x_1 = 0, x_2 = 1,$$

$$x_3 = \frac{a_4+(b_4-a_4)\gamma_4-a_2-(b_2-a_2)\gamma_2}{a_1+(b_1-a_1)\gamma_1-a_3-(b_3-a_3)\gamma_3+a_4+(b_4-a_4)\gamma_4-a_2-(b_2-a_2)\gamma_2} \tag{10.39}$$

The stable interval is as shown in Figure 10.10.

10.3.3 Duplication Dynamic Simulation of the Development of Industry Agglomeration

According to the relevant formulas in a game chain model for an evolutionary game of industry agglomeration development, a simulation program was designed

Table 10.9 Payoff Matrix of Industry Agglomeration Evolutionary Game

Group2		*Agglomeration*	*Nonagglomeration*
Group 1	Agglomeration	[0.5, 1.5], [0.5, 1.5]	0,2
	Nonagglomeration	2,0	[−5,−4], [−5,−4]

using the MATLAB language to simulate the duplication of a dynamic process of industry agglomeration development in a specific problem, shown in Table 10.9, Table 10.10, Table 10.11, Table 10.12, and Table 10.13.

According to the simulation results, there exists a unique grey equilibrium interval of evolutionary stable strategy (ESS in brief) in the evolutionary game of industry agglomeration, $[\min(x_3),\max(x_3)]=[0.7273,0.9091]$, where $\gamma_1=1,\gamma_4=0$, $\min(x_3)=\frac{-5+1-0}{-5+1-0+0.5-2}=0.7273$, and where $\gamma_1=0,\gamma_4=1,$, $\max(x_3)=\frac{-5-0}{-5-0+0.5+1-2}=0.9091$.

10.3.4 Stability Analysis of the Industry Agglomeration Formation and Development

The above experiment shows that the game will finally reach stability in the equilibrium interval independent of the initial value and the length of the evolution generations. Since the agglomeration can promote regional economic development, then why could it not be enlarged infinitely? That discussion follows.

Table 10.10 Evolutionary Process of Industry Agglomeration game ($\min(x_3)$), Initial Value $p_{11}^0=0.0001$

Generation ($t+i$)	p_{11}^{t+i}	$\bar{u}_{t+i}$
Initial State (0)	0.0001	−3.9990
5	0.0032	−3.9681
10	0.1024	−3.0337
15	0.7232	0.3554
20	0.7274	0.3639
22	0.7273	0.3637
23	0.7273	0.3636
25	0.7273	0.3636

Table 10.11 Evolutionary Process of Industry Agglomeration Game (min(x_3)), Initial Value $p_{11}^0 = 0.0001$

Generation ($t + i$)	p_{11}^{t+i}	$\bar{u}_{t+i}$
Initial State(0)	0.0001	–4.9988
5	0.0032	–4.9617
10	0.1024	–3.8289
15	0.8666	1.2688
20	0.8966	1.3379
25	0.9046	1.3544
30	0.9073	1.3601
35	0.9084	1.3622
40	0.9088	1.3631
45	0.9090	1.3634
50	0.9090	1.3635
51	0.9091	1.3636
52	0.9091	1.3636

Table 10.12 Evolutionary Process of Industry Agglomeration Game (min(x_3)), Initial Value $p_{11}^0 = 0.9999$

Generation ($t + i$)	p_{11}^{t+i}	$\bar{u}_{t+i}$
Initial State (0)	0.9999	0.5001
1	0.9998	0.5002
5	0.9904	0.5091
10	0.7255	0.3600
14	0.7272	0.3634
15	0.7273	0.3638
16	0.7272	0.3636
17	0.7273	0.3637
18	0.7273	0.3636
19	0.7273	0.3636

Table 10.13 Evolutionary Process of Industry Agglomeration Game (min(x_3)), Initial Value $p_{11}^0 = 0.9999$

Generation ($t + i$)	p_{11}^{t+i}	$\bar{u}_{t+i}$
Initial State (0)	0.9999	1.4999
1	0.9999	1.4999
5	0.9998	1.4998
10	0.9995	1.4995
15	0.9990	1.4990
20	0.9979	1.4978
25	0.9955	1.4954
30	0.9907	1.4902
35	0.9818	1.4800
40	0.9675	1.4617
45	0.9492	1.4350
50	0.9320	1.4065
55	0.9202	1.3853
60	0.9140	1.3733
65	0.9111	1.3677
70	0.9099	1.3653
75	0.9094	1.3643
80	0.9092	1.3639
83	0.9091	1.3638
84	0.9091	1.3638

The available resources a specific region can provide for the same kinds of companies clustering in a large number are limited, so there exists a maximal density. With the development of the agglomeration, the marginal profit decreases for those new entering companies, and then the increasing speed will slow down. If the population of companies in this region surpasses its maximal density, the population will decrease as a result of the vicious competition for resources. Therefore, the scale of industry agglomeration stays at a dynamic balanced level.

Without considering the influence of external conditions like resources, we use $x(t)$ to represent the number of companies in a region at time t; the relation between the number and its increasing rate is:

$$\frac{dx(t)}{dt} = x' = \alpha x(t)(\alpha \text{ is a constant}) \tag{10.40}$$

The solution is

$$x(t) = x(0)e^{\alpha t} \tag{10.41}$$

Equation (10.41) shows the exponential growth of the number of agglomeration companies in a condition of unlimited resources. There exists a maximal density when growth is restricted by some external conditions, so we update Eq. (10.40). Suppose the maximal density in the region is $k\,(k \in [k_1, k_2])$ and the agglomeration saturation degree at time t is denoted as $\frac{x(t)}{k}$. Then there exists a negative correlation between the increasing rate and the agglomeration saturation degree. The updated formula is as follows:

$$\frac{dx(t)}{dt} = x' = \alpha x(t) - \alpha x(t) \cdot \frac{x(t)}{k} = \alpha x(t)\left(1 - \frac{x(t)}{k}\right) \tag{10.42}$$

The solution is

$$x(t) = \frac{kx(0)e^{\alpha t}}{1 + x(0)e^{\alpha t}} \tag{10.43}$$

1. When $x(0) < k$, $\lim_{t\to\infty} x(t) = k$, the number of agglomeration companies decreases monotonically, corresponding to the dynamic duplication process when the initial value is $x(0) = 0.0001$. During this stage, there is still growth space for agglomeration.
2. When $x(0) = k$, that is the dynamic duplication process when the initial value is in the equilibrium interval. During this stage, the number of agglomeration companies stays at a dynamic balanced level and competition is less.
3. When $x(0) > k$, $\lim_{t\to\infty} x(t) = k$, the number of agglomeration companies increases monotonically, corresponding to the dynamic duplication process when the initial value is $x(0) = 0.9999$. During this stage, because the quantity of companies has surpassed the maximal number the region could endure, so it will decrease as a result of the vicious competition for resources.

The above three situations are illustrated in Figure 10.11, Figure 10.12, and Figure 10.13 respectively.

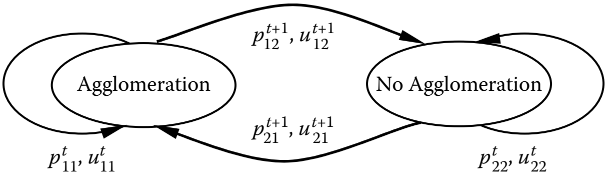


Figure 10.11 Sketch of the game chain model for evolutionary game of the industry agglomeration development.

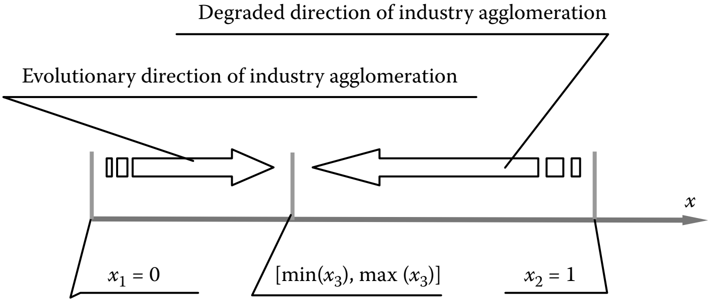


Figure 10.12 Stable interval of industry agglomeration.

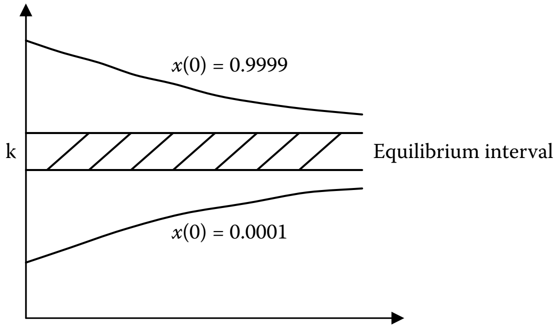


Figure 10.13 Stability analysis of industry agglomeration.

10.3.5 Summary

The game chain model for an evolutionary game of industry agglomeration development can vividly describe the dynamic duplication process and evolution traits of its formation and development. It is not only suitable for stable long-term trends analysis of industry agglomeration but can also be used to predict possible short-term evolutionary trends, providing support for the government of an agglomeration region to make industry policies. The limitation of the model is that it does not consider industry agglomeration as self-affected by its scaled economy effect, meaning the interactive effect between industry agglomeration and regional economic development. This section studied the evolution game of industry agglomeration just on a symmetric case when the payoff value is uncertain; we will do further research on the asymmetric case.

Chapter 11

Bounded Rationality Grey Game Model of First-Price Sealed-Bid Auction

11.1 Optimal Grey Quotation Model Based on an Evaluation of Accurate Value and Experiential Ideal Quotation

Conditions of a classical first-price sealed-bid auction (FPSBA) model are severely restrictive. It is assumed that bidders have an accurate judgment of the value of the auction goods and that they act in a strict rationality that does not affect the value of the goods.

However, bidders do not act in strict rationality in real life. They cannot estimate the value of the goods accurately, and the process of bidding may be affected by factors that have nothing to do with value—for example, aggressive personalities or the capital power of one's rivals.

It is very difficult for bidders to estimate the value of the goods accurately. In addition, if every bid could offer satisfactory profits to the bidder, he could bid at higher prices than the rational quoted prices b_i^* (or b_j^*) in the classical FPSBA model. There are big gaps between the rigorous conditions of the classical model and the processes of the auctions in real life. There is another serious limitation in the classical model: expected utilities are not always the most satisfactory or maximum.

For instance, in the auction, assume a bidder who considers the value of the goods is v_i; the classical model tells us that his optimal (full-rationality) quoted price is $b_i^* = \frac{v_i}{2}$. At the same time, if the probability of clinching a deal is $\text{Prob}(b_j^* < b_i^*)$, then his expected payment is $u_i^* = (v_i - b_i^*)\ \text{Prob}(b_j^* < b_i^*)$. As for the classical model, we can now conceive of a case where Bidder *i*, who is not in strict rationality (or is in bounded rationality), might quote a higher price (b_i^l) than $b_i^* = \frac{v_i}{2}$, that is, $b_i^l > b_i^*$. There may be three reasons for his quoted prices:

1. Just to match the bid price, even though the price (b_i^l) might reduce profits, it is likely to heighten the probability ($\text{Prob}(b_j^* < b_i^l)$) of clinching a deal; that is, if $b_i^* < b_i^l$, then $\text{Prob}(b_j^* < b_i^*) < \text{Prob}(b_j^* < b_i^l)$.
2. There are cases where price b_i^l of the Bidder *i* improves his expected gains; that is, $u_i^l = (v_i - b_i^l)\ \text{Prob}(b_j^* < b_i^l) > u_i^* = (v_i - b_i^*)\ \text{Prob}(b_j^* < b_i^*)$, for reason of the probability $(\text{Prob}(b_j^* < b_i^l))$.
3. In real life, with an eye to the characteristic factors of bidders that have nothing to do with value, we discover that Bidder *i* or *j*, without sufficient reason, believes his rival (Bidder *j* or *i*) is acting in strict rationality, and therefore, their so-called absolute rational quotations are respectively b_i^* and b_j^*.

Considering these problems, and starting with the thoughts and concepts of a grey experience ideal quoted price, the menace of not clinching a deal, and other auction factors, this section constructs and explains FPSBA grey game model based on grey value and the grey experience ideal quoted price of auction goods.

11.1.1 Conditions of Optimal Grey Quotation Model

In a real two-bidder FPSBA, Bidders *i* and *j* may give an experimental ideal quotation b_{i*} and b_j respectively, based on the value of the bid, v_i and v_j (and in a broader sense, take more factors into consideration such as opportunity profit), and any direct or indirect quotation experiences. The values of b_{i*} and b_{j*} are mainly decided by the real-life situation as well as the experiences and individual psychological characters of Bidders *i* and *j*. b_{i*} and b_{j*} do not strictly increase after an increase of v_i and v_j in many cases. After giving values to b_{i*} and b_{j*}, bidders also need to modify them according to the threatened feelings from opponents and some possible information that may be gotten before bidding, such as estimates of an opponent's information, individual psychology characters, and the value type. Bidders *i* and *j* might also quote prices $B_i^{\otimes}$ and $B_j^{\otimes}$, which are thought to bring the maximal expectation utility. For the sake of analyzing the problems more easily, first we set up the conditions of models as follows:

Considering only the actions of two bidders, *i* and *j*:

1. Employ v_i and v_j to denote the evaluation of the bidding goods to Bidders *i* and *j* that are only known to them, but they both know that v_i and v_j all independently abide by an even distribution in interval [0,1].
2. Let b_{i*} and b_{j*} be the respective experience ideal quoted prices of Bidders *i* and *j*.
3. Because of bounded rationality of Bidder *i*, he could quote a higher price ($B_i^\otimes$) than the experience ideal quoted price ($b_{i*}^\otimes$) when he feels enough menace from his rival, which might or might not come from factors relating to the value of the goods. At the same time, also due to the limited knowledge of Bidder *i*, he cannot quote a price ($B_i^\otimes$) that would let his margin be negative number. Therefore, the grey quoted price of Bidder *i* on a foundation of bounded rationality, grey quotation in brief, could be presented in Eq. (11.1):

$$B_i^\otimes = I_i^\otimes \cdot b_{i*} = \left[1, \frac{v_i}{b_{i*}}\right] \cdot b_{i*} = \left\{1 + \left[0, \frac{v_i}{b_{i*}} - 1\right]\right\} \cdot b_{i*} = \left\{1 + \left(\frac{v_i}{b_{i*}} - 1\right) \cdot \gamma_i^\otimes\right\} \cdot b_{i*}$$

$$= \left(1 - \gamma_i^\otimes\right) \cdot b_{i*} + \gamma_i^\otimes \cdot v_i, \quad \left(\gamma_i^\otimes \in [0,1]\right) \tag{11.1}$$

$$B_j^\otimes = J_j^\otimes \cdot b_{j*} = \left[1, \frac{v_j}{b_{j*}}\right] \cdot b_{j*} = \left\{1 + \left[0, \frac{v_j}{b_{j*}} - 1\right]\right\} \cdot b_{j*} = \left\{1 + \left(\frac{v_j}{b_{j*}} - 1\right) \cdot \gamma_j^\otimes\right\} \cdot b_{j*}$$

$$= \left(1 - \gamma_j^\otimes\right) \cdot b_{j*} + \gamma_j^\otimes v_j, \quad \left(\gamma_j^\otimes \in [0,1]\right) \tag{11.2}$$

In Eq. (11.1), the grey quoted price ($B_i^\otimes$) of Bidder *i* is made up of two parts. One part is the grey quoted price coefficient ($I_i^\otimes$), which is designed according to the assumption of bounded rationality of the bidder. The other part is the experience ideal quoted price (b_{i*}). The concise forms of $B_i^\otimes$, by first standard grey transform (or by second standard grey transform), are given in Eq. (11.1). Similarly, we could get the result $B_j^\otimes$ in Eq. (11.2).

With reference to Eq. (11.1), when the value of $\gamma_i^\otimes$, the menace reflection grey coefficient, nears zero, the value of $B_i^\otimes$, the quoted price, will be near b_{i*}. Here, $\gamma_i^\otimes = 0$ means the least menace from a rival. In Eq. (11.2), the condition of the other bidder is similar. The profit of the winner in this FPSBA is $v_i - B_i^\otimes$, if $B_i^\otimes > B_j^\otimes$, or $v_j - B_j^\otimes$, if $B_j^\otimes > B_i^\otimes$, either of which is closer to the experience quoted price, with $\gamma_i^\otimes$ or $\gamma_j^\otimes$ approaching zero. In the extreme condition of $\gamma_i^\otimes$ or $\gamma_j^\otimes$ being equal to zero, the bidder will realize his experience ideal margin, $v_i - b_{i*}$ or $v_j - b_{j*}$.

However, if the value of $\gamma_i^\otimes$ or $\gamma_j^\otimes$ approaches 1, the quoted price ($B_i^\otimes$ or $B_j^\otimes$) will be continuously close to the value (v_i or v_j) placed on it by the bidder. Therefore, the one who could clinch a deal will gain a small profit, which is $v_i - B_i^\otimes$, if $B_i^\otimes > B_j^\otimes$

or $v_j - B_j^{\otimes}$, *if* $B_j^{\otimes} > B_i^{\otimes}$, and which will be less and less, and finally will decrease to zero in extreme situation.

In the bid, Bidder *i* or *j* quotes a high price, $B_i^{\otimes}$ or $B_j^{\otimes}$, that approaches the value, v_i or v_j, depending on the menace from his rival, factors of the bidder, and personal characteristics. The smaller the grey coefficient of the quoted price is, the more satisfactory the bid will be. In the extreme condition of $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ being equal to zero, he will realize his experience ideal margin. On the contrary, the bigger $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ is, the less satisfactory the bid will be. When $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ equals 1, there is no profit margin for the bidder.

In the process of a bid, when Bidder *i* or *j* is unsatisfied with the bidding environment, he will modify b_{i^*} or b_{j^*} by giving $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ a larger number, thus getting less profit. He would realize the experience ideal profit by cutting $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ smaller. Therefore, the grey coefficient of the quoted price, $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$, is actually the modifying coefficient of the experience ideal quotation reflecting the possible menace he feels in that environment. We denote $\gamma_i^{\otimes}$ and $\gamma_j^{\otimes}$ as menace reflection coefficient of Bidders *i* and *j* respectively, and $(1-\gamma_i^{\otimes})$ and $(1-\gamma_j^{\otimes})$ are called the stable coefficient of experience ideal grey quoted price (stable coefficient for short).

Equation (11.1) shows that the grey quoted price $(B_i^{\otimes})$ of Bidder *i* is made up of two parts: $(1-\gamma_i^{\otimes})\cdot b_{i^*}$ is the stable part of experience ideal grey quoted price, and $\gamma_i^{\otimes}\cdot v_i$ is the correcting term of the possible menace. Equation (11.2) illustrates the same state of Bidder *j*.

11.1.2 Design of Grey Quotation and Grey Expected Utility Model

From the results of the above analysis, we can assume the payments of Bidders *i* and *j*, such as Eqs. (11.2) and (11.3) show, respectively. Thinking about both Eq. (11.4) and the probability of clinching a deal $(\text{Prob}\{B_j^{\otimes} < B_i^{\otimes}\})$, we can compute the expected utility of Bidder *i* [see Eq. (11.4)].

$$u_i^{\otimes}\left(B_i^{\otimes}, B_j^{\otimes}; v_i\right) = \begin{cases} v_i - B_i^{\otimes}, & \textit{if} \quad B_i^{\otimes} > B_j^{\otimes} \\ \frac{1}{2}\left(v_i - B_i^{\otimes}\right), & \textit{if} \quad B_i^{\otimes} = B_j^{\otimes} \\ 0, & \textit{if} \quad B_i^{\otimes} < B_j^{\otimes} \end{cases} \tag{11.2}$$

$$u_j^{\otimes}\left(B_j^{\otimes}, B_i^{\otimes}; v_j\right) = \begin{cases} v_j - B_j^{\otimes}, & \textit{if} \quad B_j^{\otimes} > B_i^{\otimes} \\ \frac{1}{2}\left(v_j - B_j^{\otimes}\right), & \textit{if} \quad B_j^{\otimes} = B_i^{\otimes} \\ 0, & \textit{if} \quad B_j^{\otimes} < B_i^{\otimes} \end{cases} \tag{11.3}$$

$$u_i^{\otimes} = \left(v_i - B_i^{\otimes}\right)\text{Prob}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} \tag{11.4}$$

Equation (11.4) presents the probability of clinching the deal,

$$\mathrm{Prob}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} = \mathrm{Prob}\left\{\left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*} + \gamma_j^{\otimes}\cdot v_j < B_i^{\otimes}\right\}$$

$$= \mathrm{Prob}\left\{v_j < \frac{B_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}}{\gamma_j^{\otimes}}\right\} = \frac{B_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}}{\gamma_j^{\otimes}},$$

mainly on the assumption that v_j independently abides by an even distribution in the bounded region [0,1]. Taking the place of probability in Eq. (11.4), we get the result of Eq. (11.5):

$$\begin{aligned}\underset{B_i^{\otimes}}{\mathrm{Max}}\left\{u_i^{\otimes}\right\} &= \left(v_i - B_i^{\otimes}\right)\mathrm{Prob}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} = \left(v_i - B_i^{\otimes}\right)\cdot\frac{B_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}}{\gamma_j^{\otimes}} \\ &= \frac{1}{\gamma_j^{\otimes}}\left(v_i\cdot B_i^{\otimes} - B_i^{\otimes}\cdot B_i^{\otimes}\right) - \frac{1}{\gamma_j^{\otimes}}\left(v_i - B_i^{\otimes}\right)\cdot\left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}\end{aligned} \tag{11.5}$$

Equation (11.6) gives the optimal first-order differential condition of Eq. (11.5). The solution of Eq. (11.5) is illustrated in Eq. (11.7).

$$\frac{\partial\left\{u_i^{\otimes}\right\}}{\partial\left\{B_i^{\otimes}\right\}} = \frac{1}{\gamma_j^{\otimes}}\left(v_i - 2B_i^{\otimes}\right) + \frac{1}{\gamma_j^{\otimes}}\left(1-\gamma_j^{\otimes}\right)\circlearrowright b_{j^*} = 0 \tag{11.6}$$

$$B_{i^*}^{\otimes} = \frac{1}{2}\circlearrowright\left\{v_i + \left(1-\gamma_j^{\otimes}\right)\circlearrowright b_{j^*}\right\} \tag{11.7}$$

$$B_{i^*}^{\otimes} = \begin{cases}\frac{1}{2}\cdot\left\{v_i + \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}\right\}, & if, \frac{1}{2}\{\bullet\} < v_i, and, \frac{1}{2}\{\bullet\} < v_j; \\ v_i, & if, \frac{1}{2}\{\bullet\} \geq v_i, and, \frac{1}{2}\{\bullet\} < v_j; \\ v_j, & if, \frac{1}{2}\{\bullet\} < v_i, and, \frac{1}{2}\{\bullet\} \geq v_j.\end{cases} \tag{11.8}$$

$$B_{j^*}^{\otimes} = \begin{cases}\frac{1}{2}\cdot\left\{v_j + \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*}\right\}, & if, \frac{1}{2}\{\bullet\} < v_j, and, \frac{1}{2}\{\bullet\} < v_i \\ v_j, & if, \frac{1}{2}\{\bullet\} \geq v_j, and, \frac{1}{2}\{\bullet\} < v_i \\ v_i, & if, \frac{1}{2}\{\bullet\} < v_j, and, \frac{1}{2}\{\bullet\} \geq v_i.\end{cases} \tag{11.9}$$

Considering Eq. (11.7) with the factor that the bounded rationality quoted price is not higher than the grey value, we could build the optimal grey quotation model, which is given in Eq. (11.8). Similarly, Eq. (11.9) shows the optimal quotation model of Bidder *j* according to symmetric theory of game.

Assume that Bidder *i* bids obeying the rules presented in Eq. (11.8); how can he calculate the optimal expected utility in this condition? The probability in Eq. (11.4) is turned into the form in Eq. (11.10) by replacing its corresponding variable by Eq. (11.7).

$$\text{Prob}\left\{B_j^\otimes < B_i^\otimes\right\} = \text{Prob}\left\{v_j < \frac{B_i^\otimes - \left(1-\gamma_j^\otimes\right)\cdot b_{j^*}}{\gamma_j^\otimes}\right\}$$

$$= \frac{\frac{1}{2}\circ\left\{v_i + \left(1-\gamma_j^\otimes\right)\cdot b_{j^*}\right\} - \left(1-\gamma_j^\otimes\right)\cdot b_{j^*}}{\gamma_j^\otimes} = \frac{v_i - \left(1-\gamma_j^\otimes\right)\cdot b_{j^*}}{2\cdot\gamma_j^\otimes} \tag{11.10}$$

$$\text{Prob}\left\{B_j^\otimes < B_i^\otimes\right\} = \begin{cases} 0, & if, v_i \le \left(1-\gamma_j^\otimes\right)\cdot b_{j^*}; \\ \dfrac{v_i - \left(1-\gamma_j^\otimes\right)\cdot b_{j^*}}{2\cdot\gamma_j^\otimes}, & if, \left(1-\gamma_j^\otimes\right)\cdot b_{j^*} < v_i < \left(1-\gamma_j^\otimes\right)\cdot b_{j^*} + 2\cdot\gamma_j^\otimes; \\ 1, & if, v_i \ge \left(1-\gamma_j^\otimes\right)\cdot b_{j^*} + 2\cdot\gamma_j^\otimes. \end{cases} \tag{10.11}$$

The probability is between 0 and 1, so $0 \le \text{Prob}\{B_j^\otimes < B_i^\otimes\} \le 1$. We therefore design the quotation probability of Bidder *i* [see Eq. (11.11)]. Now let us discuss their economic meaning. With respect to both the grey probability condition $(B_j^\otimes < B_i^\otimes, (1-\gamma_j^\otimes)\cdot b_{j^*} + \gamma_j^\otimes v_j < B_i^\otimes$ in Eq. (11.4) and the condition that v_j obeys an even distribution interval of [0,1], Figure 11.1 illustrates the win probability of the optimal grey quoted price of Bidder *i*. The shadow in Figure 11.1 expresses the likely quoting range of bidder *j*.

In fact, the range can be divided into three categories, as follows:

1. When the optimal grey quoted price $(B_{i^*}^\otimes)$ of Bidder *i* locates in the left range of the grey interval, $[(1-\gamma_j^\otimes)\cdot b_{j^*}, (1-\gamma_j^\otimes)\cdot b_{j^*} + \gamma_j^\otimes]$ (see Figure 11.1), the win probability of Bidder *i* must be naught. In other words, while $B_i^\otimes \le (1-\gamma_j^\otimes)\cdot b_{j^*}$ is true, then $\text{Prob}\{B_j^\otimes < B_i^\otimes\} = 0$.

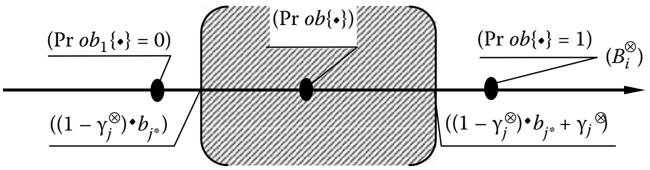


Figure 11.1 Sketch-map of clinching probability of grey quotation ($B_i^{\otimes}$).

2. If the optimal grey quoted price $(B_{i*}^{\otimes})$ of Bidder *i* is included in the range of the grey region, $[(1-\gamma_j^{\otimes})\cdot b_{j*}, (1-\gamma_j^{\otimes})\cdot b_{j*}+\gamma_j^{\otimes}]$ (see Figure 11.1), then the win probability of Bidder *i* should satisfy this equation: $0 \le \text{Prob}\{B_j^{\otimes} < B_i^{\otimes}\} = v_i - (1-\gamma_j^{\otimes})\cdot b_{j*}/2\cdot\gamma_j^{\otimes} \le 1$; that is to say, when $(1-\gamma_j^{\otimes})\cdot b_{j*} < B_i^{\otimes} < (1-\gamma_j^{\otimes})\cdot b_{j*}+\gamma_j^{\otimes}$ is true, then $0 < \text{Prob}\{B_j^{\otimes} < B_i^{\otimes}\} < 1$.
3. In the case that the optimal grey quoted price $(B_{i*}^{\otimes})$ of Bidder *i* lies in the right range of the grey interval, $[(1-\gamma_j^{\otimes})\cdot b_{j*}, (1-\gamma_j^{\otimes})\cdot b_{j*}+\gamma_j^{\otimes}]$ (such as Figure 11.1 shows), the win probability of Bidder *i* will necessarily equal 1; that is, while $B_i^{\otimes} \ge (1-\gamma_j^{\otimes})\cdot b_{j*}+\gamma'$ is true, then $\text{Prob}\{B_j^{\otimes} < B_i^{\otimes}\} = 1$.

Equation (11.11) summarizes the three types mentioned above. The expected utility models of optimal grey quotation of Bidders *i* and *j* ($u_{i*}^{\otimes}$ and $u_{j*}^{\otimes}$) are described in Eqs. (11.12) and (11.13) respectively, which are deduced from Eqs.(11.4), (11.7), and (11.11).

$$u_{i*}^{\otimes} = \begin{cases} 0, & if, v_i \le \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}; \\ \left(v_i - B_{i*}^{\otimes}\right)\text{Prob}\left\{B_{j*}^{\otimes} < B_{i*}^{\otimes}\right\} = \dfrac{\left[v_i - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}\right]^2}{4\cdot\gamma_j^{\otimes}}, & \\ \quad if, \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*} < v_i < \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*} + 2\cdot\gamma_j^{\otimes} & \\ \left(v_i - B_{i*}^{\otimes}\right)\cdot 1 = \left\{v_i - \frac{1}{2}\left[v_i + \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}\right]\right\}\cdot 1 = \frac{1}{2}\left[v_i - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}\right], & \\ \quad if, v_i \ge \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*} + 2\cdot\gamma_j^{\otimes}, and, B_{i*}^{\otimes} > b_{j*} & \end{cases} \tag{11.12}$$

$$u_{j^*}^{\otimes}=\begin{cases}0, & if, v_j \le \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*}; \\ \left(v_j - B_{j^*}^{\otimes}\right)\mathrm{Prob}\left\{B_{i^*}^{\otimes} < B_{j^*}^{\otimes}\right\} = \dfrac{\left[v_j-\left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*}\right]^2}{4\cdot\gamma_i^{\otimes}}, & \\ & if, \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*} < v_j < \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*} + 2\cdot\gamma_i^{\otimes} \\ \left(v_j - B_{j^*}^{\otimes}\right)\cdot 1 = \left\{v_j - \frac{1}{2}\left[v_j + \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*}\right]\right\}\cdot 1 = \frac{1}{2}\left[v_j - \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*}\right], & \\ & if, v_j \ge \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*} + 2\cdot\gamma_i^{\otimes} \end{cases} \tag{11.13}$$

By looking into Eqs. (11.12) and (11.13), we discover that $u_{i^*}^{\otimes}$ and $u_{j^*}^{\otimes}$ do not relate directly with $B_{i^*}^{\otimes}$ and $B_{j^*}^{\otimes}$, and that $u_{i^*}^{\otimes}$ (or $u_{j^*}^{\otimes}$) has relations with v_i (or v_j), with $\gamma_j^{\otimes}$ (or $\gamma_i^{\otimes}$), and with b_{j^*} (or b_{i^*}).

11.1.3 Simulation and Analysis of Optimal Grey Quotation and Grey Expected Utility Model

The states shown in Eqs. (11.8) and (11.9) belong to a Bayesian equilibrium of the grey auction game, whose outcome of equilibrium is related to grey numbers ($\gamma_i^{\otimes}$ and $\gamma_j^{\otimes}$). Here we should pay attention to the values of b_{i^*} and b_{j^*}. From the analysis given above and from a long-term standpoint, b_{i^*} and b_{j^*}, to a high degree, meet the hypothesis that b_{i^*} and b_{j^*} have strictly increasing differentiable relationships to v_i and v_j in a classical model. We draw a conclusion: b_{i^*} and b_{j^*} are consistent with the quotation of a Bayesian equilibrium to a very high degree, $b_{i^*} \approx \frac{v_i}{2}$, $b_{j^*} \approx \frac{v_j}{2}$; the values of b_{i^*} and b_{j^*} fluctuate surrounding $\frac{v_i}{2}$ and $\frac{v_j}{2}$.

Observing that Eqs. (11.8) and (11.9) are symmetric, we take a typical example of Eq. (11.18) to study the optimal grey quotation of Bidder *i* and its corresponding optimal utility $u_{i^*}^{\otimes}$. We use the programming language MATLAB to imitate the values of the optimal grey quotation and the corresponding optimal utility $u_{i^*}^{\otimes}$ of Bidder *i*. The result of this imitation is given in Table 11.1.

In Table 11.1, Prob $\{B_j^{\otimes} < B_i^{\otimes}\}$ shows the win probability of Bidder *i* when he quotes at $B_{i^*}^{\otimes}$; w_i is the percent of the probable margin after Bidder *i* bids at $B_{i^*}^{\otimes}$, and H_{i^*} is the percent of the utility when Bidder *i* bids at $B_{i^*}^{\otimes}$.

Comparing the result of imitation in Table 11.1 with the result of a classical model, the most important conclusions drawn from this section are stated as follows:

Table 11.1 Simulations of $b_{i*}^{\otimes}$ and $u_{i*}^{\otimes}$ of Bidder i

$v_i = 0.4\ (b_{j*} = 0.5v_j)\ Prob\{B_j^{\otimes} < B_i^{\otimes}\}$ $W_i = \frac{v_i - B_{i*}^{\otimes}}{v_i}\ H_{i*} = \frac{u_{i*}}{v_i}$							$v_i = 0.7\ (b_{j*} = 0.5v_j)\ Prob\{B_j^{\otimes} < B_i^{\otimes}\}$ $W_i = \frac{v_i - B_{i*}^{\otimes}}{v_i}\ H_{i*} = \frac{u_{i*}}{v_i}$						
v_j	$\gamma_j^{\otimes}$	$B_{i*}^{\otimes}$	$Prob\{\bullet\}$	$u_{i*}^{\otimes}$	W_j	H_i	v_j	$\gamma_j^{\otimes}$	$B_{i*}^{\otimes}$	$Prob\{\bullet\}$	$u_{i*}^{\otimes}$	W_i	H_i
0.2	0	0.200	1.000	0.200	0.500	0.500	0.6	0	0.500	1.000	0.200	0.286	0.285
	0.3	0.200	1.000	0.200	0.500	0.500		0.3	0.455	0.817	0.200	0.350	0.286
	0.6	0.200	1.000	0.200	0.500	0.500		0.6	0.410	0.483	0.140	0.414	0.200
	0.9	0.200	1.000	0.200	0.500	0.500		0.9	0.365	0.372	0.125	0.479	0.179
0.4	0	0.300	1.000	0.100	0.250	0.25	0.8	0	0.55	1.000	0.15	0.214	0.214
	0.3	0.270	0.433	0.056	0.325	0.140		0.3	0.49	0.700	0.147	0.300	0.210
	0.6	0.24	0.267	0.043	0.400	0.108		0.6	0.430	0.450	0.122	0.386	0.174
	0.9	0.210	0.211	0.040	0.475	0.100		0.9	0.370	0.367	0.121	0.471	0.173

1. For Bidder *i*, $B_{i*}^{\otimes}$ and $u_{i*}^{\otimes}$ will increase and $\gamma_j^{\otimes}$ will decrease, when other factors are fixed. As for Bidder *j*, $B_{j*}^{\otimes}$ and $u_{j*}^{\otimes}$ will enhance if the evaluation of goods of Bidder *i* rises.
2. Under the condition of bounded rationality of bidders, we will see that $B_{i*}^{\otimes}$ or $B_{j*}^{\otimes}$ is always higher than half of v_i or v_j; it is not a fixed value and always changes. According to the classical model, the quotation of bidders is only half of the evaluated value.
3. The ideal quotation way should be like this: Try not to create menace to competitors; in other words, do not let them sense the high value of $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$, then quote a high price thus to bring a comparably high expected utility.
4. Though the quantity of bidders is finite, the auctioneer still receives comparatively high value (v_i or v_j) by taking advantage of the bounded rationality of bidders.
5. In the case that v_i is much greater than v_j, Bidder *i* will achieve satisfactory $u_{i*}^{\otimes}$ if his $B_{i*}^{\otimes}$ is greater than or equal to v_j.
6. Under the condition that v_j is fixed, his $B_{i*}^{\otimes}$ will decrease gradually while $\gamma_j^{\otimes}$ goes up. In spite of an increase of w_i, H_{i*} goes down continuously because Prob$\{B_j^{\otimes} < B_i^{\otimes}\}$ falls off.

11.1.4 Summary

The section, by using the thoughts of grey systems theories, tested the current FPSBA model, which is too restrictive to fit real-life situations. This section designed a grey correction factor of experience ideal quotation and built an optimal grey quotation model based on strict rationality and accurate evaluation. After the first standard grey number transformation of the grey coefficient in the model, we found the bidder's menace reflection grey coefficient and drew some conclusions, as the optimal grey quotation of bidders is not only related to his value but also to the other's and to a menace reflection grey coefficient. The optimal grey quotation varies in different cases, and is generally higher than half of the value. Finally, the model was tested in the MATLAB language, and some valuable conclusions quite different from the classical model were obtained, proposing an optimal bidding method for the bidders.

11.2 Optimal Grey Quotation Model Based on Evaluation of Grey Value and Grey Experiential Ideal Quotation

We have analyzed the defects that currently exist in the FPSBA model and constructed an optimal grey quotation model based on evaluation of accurate value and experiential ideal quotation. However, bidders cannot estimate the exact value of the auction goods. Take a project auction, for example. Even successful bidders do not have a clear knowledge of how much value the project will bring in the future due to uncertain factors and risks. But based on direct or indirect experience, we can still make a believable estimate of the value, using grey numbers, which can be denoted as $v_i^{\otimes}=[v_i^L,v_i^R]$ (or $v_j^{\otimes}=[v_j^L,v_j^R]$). Bidders can get believable grey numbers of the experience ideal quotation in different cases, $b_{i*}^{\otimes}=[b_{i*}^L,b_{i*}^R]$ (or $b_{j*}^{\otimes}=[b_{j*}^L,b_{j*}^R]$), after considering the incomplete information.

We therefore revise the model given in Section 11.1 and construct an optimal grey quotation model based on evaluation of grey value and grey experiential ideal quotation, which is more suitable to real auctions.

11.2.1 Conditions of Optimal Grey Quotation Model

Here, we assume the conditions of the model given in Section 11.1. The evaluations of auction goods by Bidders *i* and *j* are denoted by grey numbers $v_i^{\otimes}=[v_i^L,v_i^R]$, $v_j^{\otimes}=[v_j^L,v_j^R]$ respectively, and we employ two other grey number variables, $b_{i*}^{\otimes}=[b_{i*}^L,b_{i*}^R]$ and $b_{j*}^{\otimes}=[b_{j*}^L,b_{j*}^R]$, to denote the experience ideal quoted prices of Bidders *i* and *j* respectively.

Then we transform these grey value and grey experience quoted prices to their standard forms by first standard grey transform (or by second standard grey transform) [see Eqs. (11.14) and (1.15)]:

$$v_i^{\otimes}=\left[v_i^L,v_i^R\right]=v_i^L+\left[0,v_i^R-v_i^L\right]=v_i^L+\left(v_i^R-v_i^L\right)\cdot[0,1]=v_i^L+\left(v_i^R-v_i^L\right)\cdot\gamma_{vi}^{\otimes} \tag{11.14.1}$$

$$v_j^{\otimes}=\left[v_j^L,v_j^R\right]=v_j^L+\left[0,v_j^R-v_j^L\right]=v_j^L+\left(v_j^R-v_j^L\right)\cdot[0,1]=v_j^L+\left(v_j^R-v_j^L\right)\cdot\gamma_{vj}^{\otimes} \tag{11.14.2}$$

(11.14)

$$b_{i*}^{\otimes} = \left[b_{i*}^{L}, b_{i*}^{R}\right] = b_{i*}^{L} + \left[0, b_{i*}^{R} - b_{i*}^{L}\right] = b_{i*}^{L} + \left(b_{i*}^{R} - b_{i*}^{L}\right)\cdot[0,1] = b_{i*}^{L} + \left(b_{i*}^{R} - b_{i*}^{L}\right)\circ \gamma_{bi}^{\otimes} \tag{11.15.1}$$

$$b_{j*}^{\otimes} = \left[b_{j*}^{L}, b_{j*}^{R}\right] = b_{j*}^{L} + \left[0, b_{j*}^{R} - b_{j*}^{L}\right] = b_{j*}^{L} + \left(b_{j*}^{R} - b_{j*}^{L}\right)\cdot[0,1] = b_{j*}^{L} + \left(b_{j*}^{R} - b_{j*}^{L}\right)\circ \gamma_{bj}^{\otimes} \tag{11.15.2}$$

(11.15)

Note that we still assume that $v_i^{\otimes} = [v_i^L, v_i^R]$ and $v_j^{\otimes} = [v_j^L, v_j^R]$ are known only by Bidders *i* and *j*. $v_i^{\otimes} = [v_i^L, v_i^R]$ and $v_j^{\otimes} = [v_j^L, v_j^R$ figure the types of the bidders. They all know that, when $\gamma_{vi}^{\otimes}$ and $\gamma_{vj}^{\otimes}$ are a certain value, $v_i^{\otimes} = [v_i^L, v_i^R]$ and $v_j^{\otimes} = [v_j^L, v_j^R]$ all independently abide by an even distribution in interval [0,1].

Because of Bidder *i*'s bounded rationality, he could quote a higher price $(B_i^{\otimes})$ than the experience ideal quoted price $(b_{i*}^{\otimes})$ [such as Eq. (11.16.1) shows], when he feels enough menace from his rival, which may come from factors of the value type and the nonvalue type of the bidder. Also due to his bounded rationality, Bidder *i* cannot quote a price $(B_i^{\otimes})$ that would let his margin be a negative number; that is, $B_i^{\otimes} \le v_i^{\otimes}$ no matter what the values of $\gamma_i^{\otimes}$ and $\gamma_{vi}^{\otimes}$ are. Therefore, the grey quoted price $B_i^{\otimes}$ of Bidder *i* on a foundation of bounded rationality (grey quotation for short) could be presented in Eq. (11.16.1).

$$\begin{aligned} B_i^{\otimes} &= I_i^{\otimes}\cdot b_{i*}^{\otimes} = \left[1, \frac{v_i^{\otimes}}{b_{i*}^{\otimes}}\right]\cdot b_{i*}^{\otimes} = \left(1-\gamma_i^{\otimes}\right)\cdot\left(b_{i*}^{L} + \left(b_{i*}^{R} - b_{i*}^{L}\right)\cdot\gamma_{bi}^{\otimes}\right) + \gamma_i^{\otimes}\cdot v_i^{\otimes} \\ &= b_{i*}^{L} + \left(b_{i*}^{R} - b_{i*}^{L}\right)\cdot\gamma_{bi}^{\otimes} - \left[b_{i*}^{L} + \left(b_{i*}^{R} - b_{i*}^{L}\right)\cdot\gamma_{bi}^{\otimes} - v_i^{\otimes}\right]\cdot\gamma_i^{\otimes}, \quad \left(\gamma_i^{\otimes}\in[0,1]\right) \end{aligned} \tag{11.16.1}$$

$$\begin{aligned} B_j^{\otimes} &= I_j^{\otimes}\cdot b_{j*}^{\otimes} = \left[1, \frac{v_j^{\otimes}}{b_{j*}^{\otimes}}\right]\cdot b_{j*}^{\otimes} = \left(1-\gamma_j^{\otimes}\right)\cdot\left(b_{j*}^{L} + \left(b_{j*}^{R} - b_{j*}^{L}\right)\cdot\gamma_{bj}^{\otimes}\right) + \gamma_j^{\otimes}\cdot v_j^{\otimes} \\ &= b_{j*}^{L} + \left(b_{j*}^{R} - b_{j*}^{L}\right)\cdot\gamma_{bj}^{\otimes} - \left[b_{j*}^{L} + \left(b_{j*}^{R} - b_{j*}^{L}\right)\cdot\gamma_{bj}^{\otimes} - v_j^{\otimes}\right]\cdot\gamma_j^{\otimes}, \quad \left(\gamma_j^{\otimes}\in[0,1]\right) \end{aligned} \tag{11.16.2}$$

(11.16)

In Eq. (11.16.1), the grey quoted price $(B_i^{\otimes})$ of bidder *i* is made up of two parts: the grey quoted price coefficient $(I_i^{\otimes})$, which is designed according to the assumption of bidder's bounded rationality, and the experience ideal quoted price $(b_{i*}^{\otimes})$. The concise form of $B_i^{\otimes}$ is given in Eq. (11.16.1). Similarly, we could get the result $B_j^{\otimes}$ in Eq. (11.16.2).

In Eqs. (11.16.1) and (11.16.2), as the value of $\gamma_i^{\otimes}$, an unsuccessful menace reflection grey coefficient, nears zero, and the value of $B_i^{\otimes}$, the quoted price, will be near $b_{i*}^{\otimes}$. The condition of the other bidder is similar. The margin of the winner in this FPSBA is $v_i^{\otimes} - B_i^{\otimes}, \mathit{if}\ B_i^{\otimes} > B_j^{\otimes}$ or $v_j^{\otimes} - B_j^{\otimes}, \mathit{if}\ B_j^{\otimes} > B_i^{\otimes}$, either of which is closer to the experience quoted price with $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ approaching zero. In the extreme condition of $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ being equal to zero, the bidder will realize experience ideal margin, $v_i^{\otimes} - b_{i*}^{\otimes}$ or $v_j^{\otimes} - b_{j*}^{\otimes}$.

However, if the value of $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ approaches 1, the quoted price $(B_i^{\otimes}$ or $B_j^{\otimes})$ will be continuously close to the grey value $(v_i^{\otimes}$ or $v_j^{\otimes})$ in bidder's point. Therefore, the one who could clinch a deal will gain profit, $v_i^{\otimes} - B_i^{\otimes}, \mathit{if}\ B_i^{\otimes} > B_j^{\otimes}$ or $v_j^{\otimes} - B_j^{\otimes}, \mathit{if}\ B_j^{\otimes} > B_i^{\otimes}$, which will be less and less, and will finally decrease to zero in an extreme situation.

We denote $\gamma_i^{\otimes}$ and $\gamma_j^{\otimes}$ as the menace reflection coefficient of Bidders *i* and *j* respectively, and $(1-\gamma_i^{\otimes})$ and $(1-\gamma_j^{\otimes})$ are called the stable coefficients of experience ideal grey quoted price (stable coefficients for short). Equation (11.16.1) shows that the grey quoted price $(B_i^{\otimes})$ of Bidder *i* is made up of two parts: $(1-\gamma_i^{\otimes}) \cdot b_{i*}^{\otimes}$ is the stable part of experience ideal grey quoted price, and $\gamma_i^{\otimes} \cdot v_i^{\otimes}$ is the correction term of an unsuccessful menace. Equation (11.16.2) illustrates the same state of Bidder *j*.

11.2.2 Design of Grey Quotation and Grey Expected Utility Model

From the result of the above analysis, we could assume the payments of Bidders *i* and *j*, such as Eqs. (11.17) and (11.18) show respectively. Thinking about both Eq. (11.17) and the probability of clinching a deal $(\mathrm{Prob}_1^{\otimes}\{B_j^{\otimes} < B_i^{\otimes}\})$, we can calculate the expected utility of Bidder *i* [see Eq. (11.19)]. Equation (11.19) presents the probability of clinching the deal mainly on the assumption that $v_j^{\otimes}$ independently abides by even distribution in bounded region [0,1]. By taking the place of probability in Eq. (11.19) by Eq. (11.20), we get the result of Eq. (11.21):

$$u_i^{\otimes}\left(B_i^{\otimes}, B_j^{\otimes}; v_i\right) = \begin{cases} v_i - B_i^{\otimes}, & \mathit{if}\quad B_i^{\otimes} > B_j^{\otimes} \\ \frac{1}{2}\left(v_i - B_i^{\otimes}\right), & \mathit{if}\quad B_i^{\otimes} = B_j^{\otimes} \\ 0, & \mathit{if}\quad B_i^{\otimes} < B_j^{\otimes} \end{cases} \tag{11.17}$$

$$u_j^{\otimes}\left(B_j^{\otimes}, B_i^{\otimes}; v_j^{\otimes}\right) = \begin{cases} v_j^{\otimes} - B_j^{\otimes}, & \mathit{if}\quad B_j^{\otimes} > B_i^{\otimes} \\ \frac{1}{2}\left(v_j^{\otimes} - B_j^{\otimes}\right), & \mathit{if}\quad B_j^{\otimes} = B_i^{\otimes} \\ 0, & \mathit{if}\quad B_j^{\otimes} < B_i^{\otimes} \end{cases} \tag{11.18}$$

$$u_i^{\otimes} = \left(v_i^{\otimes} - B_i^{\otimes}\right)\text{Prob}_1^{\otimes}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} \tag{11.19}$$

$$\begin{aligned}
\text{Prob}_1^{\otimes}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} &= \text{Prob}_1^{\otimes}\left\{\left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes} + \gamma_j^{\otimes}\cdot v_j^{\otimes} < B_i^{\otimes}\right\} \\
&= \text{Prob}_1^{\otimes}\left\{v_j^{\otimes} < \frac{B_i^{\otimes} - (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes}}{\gamma_j^{\otimes}}\right\} = \frac{B_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}}{\gamma_j^{\otimes}}
\end{aligned} \tag{11.20}$$

$$\begin{aligned}
\underset{B_i^{\otimes}}{\text{Max}}\left\{u_i^{\otimes}\right\} &= \left(v_i^{\otimes} - B_i^{\otimes}\right)\text{Prob}_1^{\otimes}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} = \left(v_i^{\otimes} - B_i^{\otimes}\right)\cdot\frac{B_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}}{\gamma_j^{\otimes}} \\
&= \frac{1}{\gamma_j^{\otimes}}\left(v_i^{\otimes}\cdot B_i^{\otimes} - B_i^{\otimes}\cdot B_i^{\otimes}\right) - \frac{1}{\gamma_j^{\otimes}}\left(v_i^{\otimes} - B_i^{\otimes}\right)\cdot\left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}
\end{aligned} \tag{11.21}$$

Equation (11.22) gives the optimal first-order differential condition of Eq. (11.21). The solution is illustrated in Eq. (11.23):

$$\frac{\partial\left\{u_i^{\otimes}\right\}}{\partial\left\{B_i^{\otimes}\right\}} = \frac{1}{\gamma_j^{\otimes}}\left(v_i^{\otimes} - 2B_i^{\otimes}\right) + \frac{1}{\gamma_j^{\otimes}}\left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes} = 0 \tag{11.22}$$

$$B_{i^*}^{\otimes} = \frac{1}{2}\cdot\left\{v_i^{\otimes} + \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}\right\} \tag{11.23}$$

$$B_{i^*}^{\otimes} = \begin{cases}
\frac{1}{2}\cdot\left\{v_i^{\otimes} + (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes}\right\}, & if, \frac{1}{2}\{\bullet\} < v_i^{\otimes}, and, \frac{1}{2}\{\bullet\} < v_j^{\otimes} \\
v_i^{\otimes}, & if, \frac{1}{2}\{\bullet\} \geq v_i^{\otimes}, and, \frac{1}{2}\{\bullet\} < v_j^{\otimes} \\
v_j^{\otimes}, & if, \frac{1}{2}\{\bullet\} < v_i^{\otimes}, and, \frac{1}{2}\{\bullet\} \geq v_j^{\otimes}
\end{cases} \tag{11.24}$$

$$B_{j^*}^{\otimes} = \begin{cases}
\frac{1}{2}\cdot\left\{v_j^{\otimes} + \left(1-\gamma_i^{\otimes}\right)\cdot b_{i^*}^{\otimes}\right\}, & if, \frac{1}{2}\{\bullet\} < v_j^{\otimes}, and, \frac{1}{2}\{\bullet\} < v_i^{\otimes} \\
v_j^{\otimes}, & if, \frac{1}{2}\{\bullet\} \geq v_j^{\otimes}, and, \frac{1}{2}\{\bullet\} < v_i^{\otimes} \\
v_i^{\otimes}, & if, \frac{1}{2}\{\bullet\} < v_j^{\otimes}, and, \frac{1}{2}\{\bullet\} \geq v_i^{\otimes}
\end{cases} \tag{11.25}$$

Considering Eq. (11.23) with the factor that the bounded rationality quoted price is not higher than the grey value, we could build the optimal grey quotation model given in Eq. (11.24). Here, conditions are judged by means of a standard grey number. Similarly, Eq. (11.25) shows the optimal quotation model of Bidder *j* according to a symmetric theory of games.

Assuming that Bidder *i* bids obeys the rules presented in Eq. (11.24), how does he calculate the optimal grey expected utility in this condition? The probability in Eq. (11.19) is turned into the form of Eq. (11.26) by replacing its corresponding variable by Eq. (11.23):

$$\begin{aligned}\mathrm{Prob}_1^{\otimes}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} &= \mathrm{Prob}_1^{\otimes}\left\{v_j^{\otimes} < \frac{B_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes}}{\gamma_j^{\otimes}}\right\} \\ &= \frac{\frac{1}{2}\cdot\left\{v_i^{\otimes} + \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes}\right\} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes}}{\gamma_j^{\otimes}} = \frac{v_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes}}{2\cdot\gamma_j^{\otimes}}\end{aligned} \tag{11.26}$$

$$\mathrm{Prob}_1^{\otimes}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} = \begin{cases} 0, & if, v_i^{\otimes} \le \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes}; \\ \dfrac{v_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes}}{2\cdot\gamma_j^{\otimes}}, & if, \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes} < v_i^{\otimes} < \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes} + 2\cdot\gamma_j^{\otimes} \\ 1, & if, v_i^{\otimes} \ge \left(1-\gamma_j^{\otimes}\right)\cdot b_{j*}^{\otimes} + 2\cdot\gamma_j^{\otimes} \end{cases} \tag{11.27}$$

The probability is between 0 and 1, $0 \le \mathrm{Prob}\{B_j^{\otimes} < B_i^{\otimes}\} \le 1$. We therefore design the quotation probability of Bidder *i* [see Eq. (11.27)]. Now let us discuss their economic meaning. Both on the grey probability condition $(B_j^{\otimes} < B_i^{\otimes}, (1-\gamma_j^{\otimes}) \circ b_{j*}^{\otimes} + \gamma_j^{\otimes} v_j^{\otimes} < B_i^{\otimes})$ in Eq. (11.20) and on condition that $v_j^{\otimes}$ obeys an even distribution in intervals [0,1], Figure 11.2 illustrates the win probability of the optimal grey quoted price of Bidder *i* in the middle and outside the grey interval. The shaded area in Figure 11.2 expresses the likely quoting range of Bidder *j*.

In fact, the range can be divided into three categories, as follow:

1. When the optimal grey quoted price $(B_{i*}^{\otimes})$ of Bidder *i* is located in the left range of the grey interval, $[(1-\gamma_j^{\otimes})\cdot b_{j*}^{\otimes}, (1-\gamma_j^{\otimes})\cdot b_{j*}^{\otimes} + \gamma_j^{\otimes}]$(see Figure 11.2),

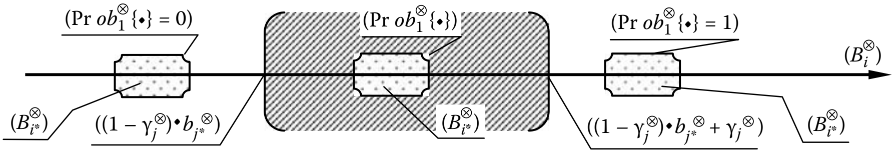


Figure 11.2 Sketch-map of clinching probability of grey quotation $B_i^{\otimes}$.

the win probability of Bidder i must be naught. In other words, while $B_i^{\otimes} \le (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes}$ is true, then $\text{Prob}_1^{\otimes}\{B_j^{\otimes} < B_i^{\otimes}\} = 0$.

2. If the optimal grey quoted price $(B_{i^*}^{\otimes})$ of Bidder i is included in the range of the grey region, $[(1-\gamma_j^{\otimes}) \circ b_{j^*}^{\otimes}, (1-\gamma_j^{\otimes}) \circ b_{j^*}^{\otimes} + \gamma_j^{\otimes}]$ (see Figure 11.2), then the win probability of Bidder i should satisfy this equation: $0 \le \text{Prob}_1^{\otimes}\{B_j^{\otimes} < B_i^{\otimes}\} = v_i^{\otimes} - (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes}/2\cdot\gamma_j^{\otimes} \le 1$; that is to say, when $(1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes} < B_i^{\otimes} < (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes} + \gamma_j^{\otimes}$ is true, then $0 < \text{Prob}_1^{\otimes}\{B_j^{\otimes} < B_i^{\otimes}\} < 1$.
3. In the case that the optimal grey quoted price $(B_{i^*}^{\otimes})$ of Bidder i lies in the right range of the grey interval, $[(1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes}, (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes} + \gamma_j^{\otimes}]$(such as Figure 11.2 shows), the win probability of Bidder i will necessarily equal 1; that is, while $B_i^{\otimes} \ge (1-\gamma_j^{\otimes})\cdot b_{j^*}^{\otimes} + \gamma_j^{\otimes}$ is true, then $\text{Prob}_1^{\otimes}\{B_j^{\otimes} < B_i^{\otimes}\} = 1$.

Equation (11.24) summarizes these three types. The expected utility models of optimal grey quotation of Bidders i and j ($u_{i^*}^{\otimes}$ and $u_{j^*}^{\otimes}$) are described in Eqs. (11.27) and (11.28) respectively, which are deduced from Eqs. (11.20), (11.23), and (11.25).

$$u_{i^*}^{\otimes} = \begin{cases} 0, \quad if, v_i^{\otimes} \le \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}; \\[2ex] \left(v_i^{\otimes} - B_{i^*}^{\otimes}\right)\text{Prob}_1^{\otimes}\left\{B_{j^*}^{\otimes} < B_{i^*}^{\otimes}\right\} = \dfrac{\left[v_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}\right]^2}{4\cdot\gamma_j^{\otimes}}, \\[2ex] \qquad if, \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes} < v_i^{\otimes} < \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes} + 2\cdot\gamma_j^{\otimes} \\[2ex] \left(v_i^{\otimes} - B_{i^*}^{\otimes}\right)\cdot 1 = \left\{v_i^{\otimes} - \frac{1}{2}\left[v_i^{\otimes} + \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}\right]\right\}\cdot 1 = \frac{1}{2}\left[v_i^{\otimes} - \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes}\right], \\[2ex] \qquad if, v_i^{\otimes} \ge \left(1-\gamma_j^{\otimes}\right)\cdot b_{j^*}^{\otimes} + 2\cdot\gamma_j^{\otimes}, and, B_{i^*}^{\otimes} > b_{j^*}^{\otimes} \end{cases} \tag{11.28}$$

$$u_{j*}^{\otimes}=\begin{cases}0, & if, v_j^{\otimes}\le\left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes};\\ \left(v_j^{\otimes}-B_{j*}^{\otimes}\right)\text{Prob}_1^{\otimes}\left\{B_{i*}^{\otimes}<B_{j*}^{\otimes}\right\}=\dfrac{\left[v_j^{\otimes}-\left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes}\right]^2}{4\cdot\gamma_i^{\otimes}}, & \\ & if, \left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes}<v_j^{\otimes}<\left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes}+2\cdot\gamma_i^{\otimes}\\ \left(v_j^{\otimes}-B_{j*}^{\otimes}\right)\cdot 1=\left\{v_j^{\otimes}-\frac{1}{2}\left[v_j^{\otimes}+\left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes}\right]\right\}\cdot 1=\frac{1}{2}\left[v_j^{\otimes}-\left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes}\right] & \\ & if, v_j^{\otimes}\ge\left(1-\gamma_i^{\otimes}\right)\cdot b_{i*}^{\otimes}+2\cdot\gamma_i^{\otimes}\end{cases}\tag{11.29}$$

By looking into Eqs. (11.28) and (11.29), we discover that $u_{i*}^{\otimes}$ and $u_{j*}^{\otimes}$ do not relate directly with $B_{i*}^{\otimes}$ and $B_{j*}^{\otimes}$, and that $u_{i*}^{\otimes}$ (or $u_{j*}^{\otimes}$) has relations with $v_i^{\otimes}$ (or $v_j^{\otimes}$), with $\gamma_j^{\otimes}$(or $\gamma_i^{\otimes}$), and with $b_{j*}^{\otimes}$ (or $b_{i*}^{\otimes}$).

11.2.3 Simulation and Analysis of Optimal Grey Quotation and Grey Expected Utility Model

The states shown in Eqs. (11.24) and (11.25) belong to a Bayesian equilibrium of the grey auction game, whose outcome of equilibrium is related to grey numbers $(b_{i*}^{\otimes},b_{j*}^{\otimes},v_{i*}^{\otimes},v_{j*}^{\otimes},\gamma_i^{\otimes},\gamma_j^{\otimes})$. Here, we should pay attention to the values of $b_{i*}^{\otimes}$ and $b_{j*}^{\otimes}$. From the analysis given above and from a long-term standpoint, $b_{i*}^{\otimes}$ and $b_{j*}^{\otimes}$, to a high degree, meet the hypothesis that $b_{i*}^{\otimes}$ and $b_{j*}^{\otimes}$ have strictly increasing differential relations to $v_i^{\otimes}$ and $v_j^{\otimes}$ in classical model. We draw the conclusion that $b_{i*}^{\otimes}$ and $b_{j*}^{\otimes}$ are consistent with the quotation of a Bayesian equilibrium to a very high degree, $b_{i*}^{\otimes}\approx\Big|_{\gamma_{vi}^{\otimes}=c_{vi}}\frac{v_i^{\otimes}}{2}, b_{j*}^{\otimes}\approx\Big|_{\gamma_{vj}^{\otimes}=c_{vj}}\frac{v_j^{\otimes}}{2}$; in another words, the values of $b_{i*}^{\otimes}$ and $b_{j*}^{\otimes}$ fluctuate surrounding $\frac{v_i^{\otimes}}{2}\Big|_{\gamma_{vi}^{\otimes}=c_{vi}}$ and $\frac{v_j^{\otimes}}{2}\Big|_{\gamma_{vj}^{\otimes}=c_{vj}}$.

Observing that Eqs. (11.24) and (11.25) are symmetric, we can take a typical example of Eq. (11.24) to study the optimal grey quotation of Bidder *i* and its corresponding optimal utility $u_{i*}^{\otimes}$. We use MATLAB to imitate the values of the optimal grey quotation and the corresponding optimal utility $u_{i*}^{\otimes}$ of Bidder *i*. The result of the imitation is given in Table 11.2.

In Table 11.2, $\text{Prob}\{B_j^{\otimes}<B_i^{\otimes}\}$ shows the win probability of Bidder *i* when he quotes at $B_{i*}^{\otimes}$. Here $w_{i*}^{\otimes}$ is the percent of the probable margin after Bidder *i* bids at $B_{i*}^{\otimes}$, and $H_{i*}^{\otimes}$ is the percent of the utility when Bidder *i* bids at $B_{i*}^{\otimes}$. Comparing the results of the imitation in Table 11.2 with the results of a classical model, the most important conclusions drawn from this section are stated as follows:

Table 11.2 Simulations of $b_{i*}^{\otimes}$ and $u_{i*}^{\otimes}$ of Bidder *i*

$v_j^{\otimes}=[0.2,0.4]=0.2+0.2\gamma_{vj}^{\otimes}, v_i^{\otimes}=[0.2,0.3]=0.2+0.1\gamma_{vi}^{\otimes}\quad \left(b_{j*}^{\otimes}=0.5v_j^{\otimes}\right)$ $\mathrm{Prob}_1^{\otimes}\left\{B_j^{\otimes}<B_i^{\otimes}\right\}\quad W_{i*}^{\otimes}=\frac{v_i^{\otimes}-B_{i*}^{\otimes}}{v_i^{\otimes}}\quad H_{i*}^{\otimes}=\frac{u_{i*}^{\otimes}}{v_i^{\otimes}}$						
$v_j^{\otimes}$	$\gamma_j^{\otimes}$	$B_{i*}^{\otimes}$	$Prob_1^{\otimes}\{\bullet\}$	$u_{i*}^{\otimes}$	$W_{i*}^{\otimes}$	$H_{i*}^{\otimes}$
$\gamma_{vj}^{\otimes}=0.3$ $(v_j^{\otimes}=0.206)$	0	[0.1515, 0.2015]	[1.0000, 1.0000]	[0.0485, 0.0985]	[0.2425, 0.3283]	[0.2425, 0.3283]
	0.3	[0.1361, 0.1860]	[0.2132, 0.3798]	[0.0136, 0.0433]	[0.3195, 0.3800]	[0.0680, 0.1443]
	0.6	[0.1206, 0.1706]	[0.1323, 0.2157]	[0.0105, 0.0279]	[0.3970, 0.4313]	[0.0525, 0.0930]
	0.9	[0.1052, 0.1552]	[0.1054, 0.1609]	[0.0100, 0.0233]	[0.4740, 0.4827]	[0.0500, 0.0777]
$\gamma_{vj}^{\otimes}=0.6$ $(v_j^{\otimes}=0.320)$	0	[0.1800, 0.2300]	[1.0000, 1.0000]	[0.0200, 0.0700]	[0.1000, 0.2333]	[0.1000, 0.2333]
	0.3	[0.1560, 0.2060]	[0.1467, 0.3133]	[0.0065, 0.0295]	[0.2200, 0.3133]	[0.0325, 0.0983]
	0.6	[0.1320, 0.1820]	[0.1133, 0.1967]	[0.0077, 0.0232]	[0.3400, 0.3933]	[0.0385, 0.0773]
	0.9	[0.1080, 0.1580]	[0.1022, 0.1578]	[0.0094, 0.0224]	[0.4600, 0.4733]	[0.0470, 0.0747]
$v_j^{\otimes}=[0.6,0.8]=0.6+0.2\gamma_{vj}^{\otimes}, v_i^{\otimes}=[0.7,0.9]=0.7+0.2\gamma_{vi}^{\otimes}\ (b_{j*}^{\otimes}=0.5v_j^{\otimes})$ $\mathrm{Prob}\{B_j^{\otimes}<B_i^{\otimes}\}\ W_{i*}^{\otimes}=\frac{v_i^{\otimes}-B_{i*}^{\otimes}}{v_i^{\otimes}}\ H_{i*}^{\otimes}=\frac{u_{i*}^{\otimes}}{v_i^{\otimes}}$						
$v_j^{\otimes}$	$\gamma_j^{\otimes}$	$B_{i*}^{\otimes}$	$Prob_1^{\otimes}\{\bullet\}$	$u_{i*}^{\otimes}$	$W_{i*}^{\otimes}$	$H_{i*}^{\otimes}$
$\gamma_{vj}^{\otimes}=0.7$ $(v_j^{\otimes}=0.740)$	0	[0.5350, 0.6350]	[1.0000, 1.0000]	[0.1650, 0.2650]	[0.2357, 0.2944]	[0.2357, 0.2944]
	0.3	[0.4795, 0.5795]	[0.7350, 1.0000]	[0.1621, 0.3205]	[0.3150, 0.3561]	[0.2316, 0.3561]
	0.6	[0.4240, 0.5240]	[0.4600, 0.6267]	[0.1270, 0.2356]	[0.3943, 0.4178]	[0.1814, 0.2618]
	0.9	[0.3685, 0.4685]	[0.3683, 0.4794]	[0.1221, 0.2069]	[0.4736, 0.4794]	[0.1744, 0.2299]

(*Continued*)

Table 11.2 Simulations of $b_{i*}^{\otimes}$ and $u_{i*}^{\otimes}$ of Bidder *i* (Continued)

$v_j^{\otimes}$	$\gamma_j^{\otimes}$	$B_{i*}^{\otimes}$	$Prob_1^{\otimes}\{\bullet\}$	$u_{i*}^{\otimes}$	$W_{i*}^{\otimes}$	$H_{i*}^{\otimes}$
$\gamma_{vj}^{\otimes}=0.9$ $(v_j^{\otimes}=0.780)$	0	[0.5450, 0.6450]	[1.0000, 1.0000]	[0.1550, 0.2550]	[0.2214, 0.2833]	[0.2214, 0.2833]
	0.3	[0.4865, 0.5865]	[0.7117, 1.0000]	[0.1519, 0.3135]	[0.3050, 0.3483]	[0.2170, 0.3483]
	0.6	[0.4280, 0.5280]	[0.4533, 0.6200]	[0.1233, 0.2306]	[0.3886, 0.4133]	[0.1761, 0.2562]
	0.9	[0.3695, 0.4695]	[0.3672, 0.4783]	[0.1214, 0.2059]	[0.4721, 0.4783]	[0.1734, 0.2288]

1. For Bidder *i*, $B_{i*}^{\otimes}$ and $u_{i*}^{\otimes}$ will increase and $\gamma_j^{\otimes}$ will decrease, when other factors are fixed. As for Bidder *j*, $B_{j*}^{\otimes}$ and $u_{j*}^{\otimes}$ will enhance if the evaluation of goods of Bidder *i* rises.
2. Under the condition of bounded rationality of bidders, we will see that $B_{i*}^{\otimes}$ or $B_{j*}^{\otimes}$ is always higher than half of v_i or v_j, which is not a fixed value and always changes.
3. The ideal quotation way should be like this: Try not to create menace to competitors; do not let them sense the high value of $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ and then quote a high price thus to bring a comparably higher expected utility.
4. Though the quantity of bidders is finite, the auctioneer still receives comparatively high value (v_i or v_j) by taking advantage of the bounded rationality of bidders.
5. In the case that v_i is much greater than v_j, Bidder *i* will achieve satisfactory $u_{i*}^{\otimes}$ if his $B_{i*}^{\otimes}$ is greater than or equal to v_j.
6. Under the condition that v_i is fixed, a bidder's $B_{i*}^{\otimes}$ will decrease gradually while $\gamma_j^{\otimes}$ goes up. In spite of an increase of $w_{i*}^{\otimes}$, $H_{i*}^{\otimes}$ goes down continuously because $\mathrm{Prob}^{\otimes}\{B_j^{\otimes} < B_i^{\otimes}\}$ falls off.

11.2.4 Summary

There are some of defects in the classical first-price sealed-bid auction (FPSBA) model, which has so many restrictive conditions that it does not fit a real situation. Considering uncertain information, this section designed a grey correction factor of experiential ideal quotation and built an optimal grey quotation model based on strict rationality by using grey systems theories. An optimal grey quotation model based on evaluation of grey value and an experiential ideal quotation was built after concepts of optimal grey quotations and grey expected utilities were put forward. The model was created using MATLAB, and the effectiveness of the simulation is good.

11.3 Choice of Stock Agent: A First-Price Sealed-Bid Auction Game Model Based on Bounded Rationality

After acquiring stock financing, a company will look for a suitable stock agent to sell its stock. Such behaviors between companies and dealers could be described by classical noncooperative game theories, such as the first-price sealed-bid auction model, the second-price sealed-bid auction model, and the two-sided quoted auction model.

However, both rationality and judgment are in high demand in the classical game models; that is, the players not only operate under conditions of complete rationality but also in the same rationality.[54–57] In reality, their rationality and judgment could not satisfy the model's requests because of various factors, such as the actual environment, life experience, knowledge level, emotions, and intellect of the players. There are many differences between the players, which are too many to imagine. Even for players with the same rationality and judgment ability, they act quite differently based on their different backgrounds: different interest groups; facing different political, economic, cultural, and social backgrounds; and different feelings toward various factors. The conclusion may be not suitable for realistic decisions, if we describe the stock dealers by these classical models.

In order to solve this problem, this section discusses the game problems based on the first-price sealed-bid auction between bounded rationality enterprises and stock dealers. Here we assume there are only two stock dealers to participate in a stock auction of a company, namely Dealers *i* and *j*.

In a stock auction with two participants, the value of the auction goods (stocks) is v_i for Dealer *i*. According to the classical first-price sealed-bid auction model,[1] the optimal (rational) quotation of Dealer *i* is $b_i^* = \frac{v_i}{2}$. Assume that the win probability at the time is $\mathrm{Prob}(b_j^* < b_i^*)$, then the expected payment is $u_i^* = (v_i - b_i^*)\mathrm{Prob}(b_j^* < b_i^*)$.

An invited public bidding process on the stock dealership of an enterprise assumes that bounded rationality bidders are in an uncertain decision environment. They not only need to evaluate the products up for bid but also to estimate the threats from their opponents that could make deal fail, shown by Figure 11.3. We can imagine that if Dealer *i* is of bounded rationality, he may quote a higher price $b_i^l > b_i^*$ than $b_i^* = \frac{v_i}{2}$ $b_i^l > b_i^*$ for the following reasons:

1. Although the expected profit u_i^l may be lower than u_i^*, it raises the win possibility if the opponent's quotation is b_j^*; that is, if $b_i^* < b_i^l$, then $\mathrm{prob}(b_j^* < b_i^1) <$ $\mathrm{Prob}(b_j^* < b_i^l)$.
2. If a dealer quotes a higher price to enhance the win possibility, under some situations, the expected profit can be improved to some extent; that is, $u_i^l = (v_i - b_i^l)\mathrm{Prob}(b_j^* < b_i^l) > u_i^* = (v_i - b_i^*)\mathrm{Prob}(b_j^* < b_i^*)$.

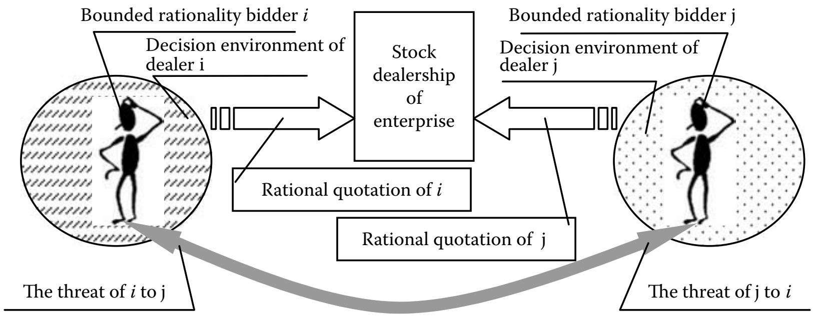


Figure 11.3 Bidding process of bounded rationality dealer and *j* on the stock dealership.

3. In a real situation, considering the nonvalue factors of dealers, they don't have enough reasons to assume the complete rationality of the opponents. So in the bidding process, dealers only quote the so-called absolute price, b_i^* and b_j^*.

Considering the problems in the classical first-price sealed-bid model, this section introduces the concept and thoughts from a more practical perspective, such as experience ideal quotation and menace reflection coefficient of an unsuccessful bid, and studies the first-price sealed-bid auction game model based on bounded rationality.

11.3.1 Construction of Optimal Quoted Price Model of Stock Dealers

In a second-price sealed-bid auction on a listed (or unlisted) company stock dealership, Bidder *i* or *j* is likely to quote his price in consideration of the value, v_i or v_j, he places on the goods, and at the same time make an experience ideal quoted price b_{i*} or b_{j*}, according to his direct or indirect quoted experience. Here, the value v_i or v_j has a broad sense, which includes factors such as the opportunity profit that the bid might bring, the value of b_{i*} and b_{j*} that are mainly decided by the actual situation, and the experiences and individual psychological characters of Bidders *i* and *j*. b_{i*} and b_{j*} do not strictly increase after the increase of v_i and v_j in many cases. After giving values to b_{i*} and b_{j*}, bidders also need to modify them according to the threatened feelings from opponents and some possible information that may be gotten before the bid—for example, estimates of an opponent's information, individual psychology characters, and the value type. Finally, Bidders *i* and *j* quote the prices $B_i^{\otimes}$ and $B_j^{\otimes}$ that they think could bring the maximal expectation utility. For the sake of analyzing the problems more easily, we set up the conditions of models as follows:

1. Presume two bidders, *i* and *j*.
2. Let v_i and v_j respectively denote the estimated values of the bid products of Bidders *i* and *j*. Suppose that only Bidder *i* or *j* himself knows v_i or v_j accurately; that is to say, v_i and v_j represent value for the type of Bidders *i* and *j*, but both bidders know that v_i and v_j obey an even distribution in the zone [0,1] independently.
3. Let b_{i*} and b_{j*} respectively stand for the experience ideal quoted prices of Bidders *i* and *j*.
4. Considering the bidder's limited rationality, he might quote a price $B_i^{\otimes}$ that is higher than his experience ideal quoted price b_{i*}, when Bidder *i* feels that the threat from his opponent is great enough to influence his success in the bid. Here, the threat might come from factors of value type and nonvalue type of the bidder. However, as a result of the bidder's bounded rationality, his quoted price is unlikely to make him get a negative profit; that is to say, $B_i^{\otimes} \le v_i$. Therefore, $B_i^{\otimes}$, the quoted price of Bidder *i* based on bounded rationality (quoted price in short), can be presented in Eq. (11.30.1). The situation of $B_j^{\otimes}$ is similar [expressed by Eq. (11.30.2):

$$B_i^{\otimes} = I_i^{\otimes} . b_{i*} = \left[1, \frac{v_i}{b_{i*}}\right] . b_{i*} = \left\{1 + \left[0, \frac{v_i}{b_{i*}} - 1\right]\right\} . b_{i*} = \left\{1 + \left(\frac{v_i}{b_{i*}} - 1\right) . \gamma_i^{\otimes}\right\} . b_{i*}$$

$$= \left(1 - \gamma_i^{\otimes}\right) . b_{i*} + \gamma_{i*}^{\otimes} . v_i, \left(\gamma_i^{\otimes} \in [0,1]\right) \qquad (11.30.1)$$

$$B_j^{\otimes} = J_j^{\otimes} . b_{j*} = \left[1, \frac{v_j}{b_{j*}}\right] . b_{j*} = \left\{1 + \left[0, \frac{v_j}{b_{j*}} - 1\right]\right\} . b_{j*} = \left\{1 + \left(\frac{v_j}{b_{j*}} - 1\right) . \gamma_j^{\otimes}\right\} . b_{j*}$$

$$= \left(1 - \gamma_j^{\otimes}\right) . b_{j*} + \gamma_{j*}^{\otimes} . v_j, \left(\gamma_j^{\otimes} \in [0,1]\right) \qquad (11.30.2)$$

(11.30)

In the process of a bid, when Bidder *i* or *j* is unsatisfied with the bidding environment, he will modify b_{i*} and b_{j*} by giving $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ a larger number, thus getting less profit, but he would realize his experience ideal profit by cutting $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ smaller. Therefore, the grey coefficient of the quoted price $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$ is actually the modifying coefficient of the experience ideal quotation reflecting the possible menace he feels in that environment. We denote $\gamma_i^{\otimes}$ and $\gamma_j^{\otimes}$ as menace reflection coefficient of Bidders *i* and *j* respectively, and $(1 - \gamma_i^{\otimes})$ and $(1 - \gamma_j^{\otimes})$ are called the stable coefficient of experience ideal grey quoted price (stable coefficient for short). Equation (11.30.1) shows that the grey quoted price $(B_i^{\otimes})$ of Bidder *i* is made up of two parts: $(1 - \gamma_i^{\otimes}) \cdot b_{i*}$ is the stable part of experience ideal grey quoted price, and $\gamma_i^{\otimes} \cdot v_i$ is the correction term of an possible menace. Equation (11.30.1) illustrates the same state for Bidder *j*.

According to the conditions of the model, we can gain the payment of Bidder *i* or *j* [as Eqs. (11.31) and (11.32) show] and the expectation utility of Bidder *i* [as Eq. (11.33) shows]. Based on Eqs. (11.30), (11.31), (11.32), and (11.33) and on theories of probability and economics, we get the probability of closing a deal (Prob$\{B_j^{\otimes} < B_i^{\otimes}\}$), the optimization quoted price ($B_i^{\otimes}$), and its expectation utility ($u_{i^*}^{\otimes}$) model of Bidder *i*, shown in Eqs. (11.34), (11.35), and (11.36) respectively. The same formulas apply to Bidder *j*, which are omitted.

$$u_i^{\otimes}\left(B_i^{\otimes}, B_j^{\otimes}; v_i\right) = \begin{cases} v_i - B_i^{\otimes}, & if\ B_i^{\otimes} > B_j^{\otimes} \\ \frac{1}{2}\left(v_i - B_i^{\otimes}\right), & if\ B_i^{\otimes} = B_j^{\otimes} \\ 0, & if\ B_i^{\otimes} < B_j^{\otimes} \end{cases} \tag{11.31}$$

$$u_j^{\otimes}\left(B_j^{\otimes}, B_i^{\otimes}; v_j\right) = \begin{cases} v_j - B_j^{\otimes}, & if\quad B_j^{\otimes} > B_i^{\otimes} \\ \frac{1}{2}\left(v_j - B_j^{\otimes}\right), & if\quad B_j^{\otimes} = B_i^{\otimes} \\ 0, & if\quad B_j^{\otimes} < B_i^{\otimes} \end{cases} \tag{11.32}$$

$$u_i^{\otimes} = \left(v_i - B_i^{\otimes}\right)\mathrm{Prob}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} \tag{11.33}$$

$$\mathrm{Prob}\left\{B_j^{\otimes} < B_i^{\otimes}\right\} = \begin{cases} 0, & if, v_i \leq \left(1-\gamma_j^{\otimes}\right)b_{j^*}; \\ \dfrac{v_i - \left(1-\gamma_j^{\otimes}\right)b_{j^*}}{2\gamma_j^{\otimes}}, & if, \left(1-\gamma_j^{\otimes}\right)b_{j^*} < v_i < \left(1-\gamma_j^{\otimes}\right)b_{j^*} + 2\gamma_j^{\otimes} \\ 1, & if, v_i \geq \left(1-\gamma_j^{\otimes}\right)b_{j^*} + 2\gamma_j^{\otimes}. \end{cases} \tag{11.34}$$

$$B_{i^*}^{\otimes} = \begin{cases} \frac{1}{2}\left\{v_i + \left(1-\gamma_j^{\otimes}\right)b_{j^*}\right\}, & if, \frac{1}{2}\{\bullet\} < v_i, and, \frac{1}{2}\{\bullet\} < v_j \\ v_i, & if, \frac{1}{2}\{\bullet\} \geq v_i, and, \frac{1}{2}\{\bullet\} < v_j \\ v_j, & if, \frac{1}{2}\{\bullet\} < v_i, and, \frac{1}{2}\{\bullet\} \geq v_j \end{cases} \tag{11.35}$$

$$u_{i^*}^{\otimes}=\begin{cases} 0, \quad if, v_i \le \left(1-\gamma_j^{\otimes}\right)b_{j^*}; \\ \left(v_i - B_{i^*}^{\otimes}\right)\text{Prob}\left\{B_{j^*}^{\otimes} < B_{i^*}^{\otimes}\right\} = \dfrac{\left[v_i - \left(1-\gamma_j^{\otimes}\right)b_{j^*}\right]^2}{4\gamma_j^{\otimes}}, \\ \quad if, \left(1-\gamma_j^{\otimes}\right)b_{j^*} < v_i < \left(1-\gamma_j^{\otimes}\right)b_{j^*} + 2\gamma_j^{\otimes} \\ \left(v_i - B_{i^*}^{\otimes}\right) = \left\{v_i - \dfrac{1}{2}\left[v_i + \left(1-\gamma_j^{\otimes}\right)b_{j^*}\right]\right\} = \dfrac{1}{2}\left[v_i - \left(1-\gamma_j^{\otimes}\right)b_{j^*}\right], \\ \quad if, v_i \ge \left(1-\gamma_j^{\otimes}\right)b_{j^*} + 2\gamma_j^{\otimes}, and, B_{i^*}^{\otimes} > b_{j^*} \end{cases} \tag{11.36}$$

11.3.2 Model Imitation of Optimization Quoted Price and Expectation Utility and Analysis of the Main Conclusions

11.3.2.1 Model Imitation

In the classical FPSBA model with two bidders, the quoted price decided by the Bayesian balance is half of the value that the bid product is thought to have. Therefore, we could deduce that the values of b_{i^*} and b_{j^*} in our model are in accordance with those in classical model to a great extent; that is, $b_{i^*} = \frac{v_i}{2}, b_{j^*} = \frac{v_j}{2}$. In other words, b_{i^*} and b_{j^*} fluctuate around $\frac{v_i}{2}$ and $\frac{v_j}{2}$.

In this section, we only take Bidder i as an example to make an simulation for his optimized quoted price and the maximal utility. For convenience, we still suppose that the values v_i and v_j of a company's stock estimated by Bidders i and j obey uniform distribution in zone [0,1].

Making use of Eqs. (11.30) through (11.36) and considering $b_{i^*} = \frac{v_i}{2}, b_{j^*} = \frac{v_j}{2}$ and the symmetry principle of the game, we get the situation of the game in the background of this particular experience ideal quoted price, shown by Eqs. (11.37)–(11.42). According to Eqs. (11.37)–(11.40), we can get the related formulas between the optimization quoted price and the values v_i or v_j, as well as the threatened reflection coefficient of Bidders i and j, shown by Eqs. (11.43) and (11.44):

$$B_{i^*}^{\otimes}=\begin{cases} \dfrac{1}{2}v_i + \dfrac{1}{4}v_j\left(1-\gamma_j^{\otimes}\right) & if, v_i < 2v_j \\ b_{i^*} & if, v_i \ge 2v_j \end{cases} \tag{11.37}$$

$$B^{\otimes}_{j^*} = \begin{cases} \frac{1}{2}v_j + \frac{1}{4}v_i\left(1-\gamma^{\otimes}_i\right) & if, v_j < 2v_i \\ b_{j^*} & if, v_j \geq 2v_i \end{cases} \tag{11.38}$$

$$\gamma^{\otimes}_j = \begin{cases} 0 & if, v_i < \frac{1}{2}v_j \\ \frac{2v_i}{3v_j} - \frac{1}{3} & if, \frac{1}{2}v_j \leq v_i < 2v_j \\ 1 & if, v_i \geq 2v_j \end{cases} \tag{11.39}$$

$$\gamma^{\otimes}_i = \begin{cases} 0 & if, v_j < \frac{1}{2}v_i \\ \frac{2v_j}{3v_i} - \frac{1}{3} & if, \frac{1}{2}v_i \leq v_j < 2v_i \\ 1 & if, v_j \geq 2v_i \end{cases} \tag{11.40}$$

$$\text{Prob}\left\{B^{\otimes}_j < B^{\otimes}_i\right\} = \begin{cases} 0 & v_j \geq 2v_i \\ v_j & \frac{1}{2}v_i < v_j < 2v_i \\ 1 & v_j \leq \frac{1}{2}v_i \end{cases} \tag{11.41}$$

$$\text{Prob}\left\{B^{\otimes}_i < B^{\otimes}_j\right\} = \begin{cases} 0 & v_i \geq 2v_j \\ v_i & \frac{1}{2}v_j < v_i < 2v_j \\ 1 & v_i \leq \frac{1}{2}v_j \end{cases} \tag{11.42}$$

$$B^{\otimes}_{i^*} = \begin{cases} \frac{1}{2}v_i + \frac{3v_i}{8}\left(\gamma^{\otimes}_i + \frac{1}{3}\right)\left(\frac{4}{3} - \frac{4}{9\gamma^{\otimes}_i + 3}\right) & if, 0 < \gamma^{\otimes}_i \leq 1 \\ b_{i^*} & if, \gamma^{\otimes}_i = 0 \end{cases} \tag{11.43}$$

Table 11.3 Imitation of Optimal Quotation of Bidder i ($B_{i^*}^{\otimes}$) and Variation of Parameters Along with v_j

$v_i = 0.40,\ b_{i^*} = 0.5v_j,$ Prob$\{B_j^{\otimes} < {}_i^{\otimes}\},\ W_i$						
v_j	$B_{i^*}^{\otimes}$	$\gamma_j^{\otimes}$	$\gamma_i^{\otimes}$	Prob{•}	$u_{i^*}^{\otimes}$	W_i
0.100	0.200	1.000	0.000	1.000	0.300	0.750
0.200	0.200	1.000	0.000	1.000	0.300	0.750
0.300	0.233	0.556	0.167	0.300	0.050	0.714
0.400	0.267	0.333	0.333	0.400	0.053	0.500
0.500	0.300	0.200	0.500	0.500	0.050	0.333
0.600	0.333	0.111	0.667	0.600	0.040	0.200
0.700	0.367	0.048	0.833	0.700	0.023	0.091
0.800	0.400	0.000	1.000	0.000	0.000	0.000
0.900	0.400	0.000	1.000	0.000	0.000	0.000

$$B_{j^*}^{\otimes} = \begin{cases} \dfrac{1}{2}v_j + \dfrac{3v_j}{8}\left(\gamma_j^{\otimes} + \dfrac{1}{3}\right)\left(\dfrac{4}{3} - \dfrac{4}{9\gamma_j^{\otimes} + 3}\right) & if, 0 < \gamma_j^{\otimes} \le 1 \\ b_{j^*} & if, \gamma_j^{\otimes} = 0 \end{cases} \tag{11.44}$$

Using MATLAB, we finished the imitation experiment of the parameters of Bidder *i*, including the optimization quoted price $B_{i^*}^{\otimes}$; the threatened reflection coefficient $\gamma_i^{\otimes}$ and $\gamma_j^{\otimes}$; the probability of closing a deal, Prob$\{B_j^{\otimes} < B_i^{\otimes}\}$; the maximal utility $u_{i^*}^{\otimes}$; and $W_i = \frac{v_i - B_{i^*}^{\otimes}}{v_i}$; the results are shown in Table 11.3 and Figure 11.4. W_i is

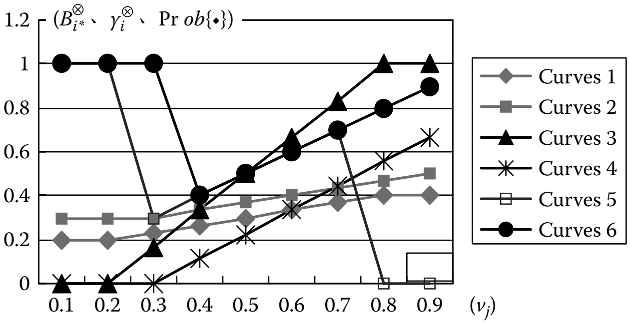


Figure 11.4 Comparison of parameters $B_i^{\otimes}$, $\gamma_i^{\otimes}$, when V_j = 0.40 and V_j = 0.60.

the percentage of the profit that Bidder *i* might acquire from the approved value of the bid product after quoting his optimization quoted price $B_{i^*}^{\otimes}$.

Table 11.3 shows the changes of parameters along with the value of stock dealership Bidder *j* thought might change when $v_i = 0.40$, $b_{i^*} = 0.5v_i$, $b_{j^*} = 0.5v_j$. In Figure 11.4, when $v_i = 0.40$ and $v_i = 0.60$, the fluctuation of the optimization quoted price $B_{i^*}^{\otimes}$ of Bidder *i* is presented by Curves 1 and 2; Curves 3 and 4 reflect the changes of the threatened reflection coefficient $\gamma_i^{\otimes}$ of Bidder *i* ; and Curves 5 and 6 denote the fluctuation of the probability of success Prob$\{B_j^{\otimes} < B_i^{\otimes}\}$ at the optimal quoted price.

11.3.2.2 Analysis of the Imitation Conclusions

The imitation results shown in Table 11.3 and Figure 11.4 reveal the following conclusions compared with the classic model:

1. For Bidder *i*, the more he recognizes the value of the stock dealership, the higher $B_{i^*}^{\otimes}$ and $u_{i^*}^{\otimes}$ will be (see Figure 11.4); if v_i is much higher than v_j (that is, if $v_i \geq 2v_j$), he will quote an experience ideal price b_{i^*}, and then the possibility of closing a deal can reach as high as 1 and win a satisfactory stable utility $u_{i^*}^{\otimes}$. Meanwhile, the menace reflection coefficient of the opponent, $\gamma_j^{\otimes}$, can hardly exert any influence on the process (see Table 11.3).
2. For Bidder *i*, when other factors are fixed, if v_i is not substantially higher than $v_j (\frac{1}{2}v_j \leq v_i \leq 2v_j)$, then his optimal quotation $B_{i^*}^{\otimes}$ and menace reflection coefficient $\gamma_i^{\otimes}$ will increase together with v_j. His $u_{i^*}^{\otimes}$, W_i, win probability Prob$\{B_j^{\otimes} < B_i^{\otimes}\}$ and menace reflection coefficient of the opponent's $\gamma_j^{\otimes}$ will decrease when v_j increases.
3. If the value v_i is much lower than $v_j (v_i \leq \frac{1}{2}v_j)$, then the maximal optimal quotation of Bidder *i* can be reached and the win possibility is nearly zero. It is actually impossible to make a bid, so the best strategy for Bidder *i* is to give up.
4. For Bidder *i*, with the increase of the value v_i, his menace reflection coefficient will decrease, as Curves 3 and 4 shown in Figure 11.4.
5. Because of the bounded rationality of Bidder *i*, the optimal quotation of Bidder *i*, $B_{i^*}^{\otimes}$, is always higher than half of v_i and is always changeable; the case is the same for Bidder *j*. In the classical model, it is considered that the maximal quotation is only half of the value of the auction goods to the bidders.
6. According to the above analysis, the ideal quotation way is to try not to create menace to the opponents; in other words, do not let them sense a high value of $\gamma_i^{\otimes}$ or $\gamma_j^{\otimes}$, and then quote a low price thus to bring a comparably higher expected utility $u_{i^*}^{\otimes}$ (see Table 11.3).

11.3.3 The Strategy in the Bid Process of Stock Dealership

According to theory of the classical FPSBA, in the bid process of stock dealership, the only way for the enterprises to achieve the biggest profit is to invite as many opponents as possible. When quantity of bidders gets to infinite, they could receive the largest profits; in other words, they can almost acquire the whole of the buyer's value. But for all bidders, their highest rational quoted price increases while the number of the bidders goes up. Their quoted prices are equal to the whole value they approve to the bid products, when the number of the bidders gets to infinite.

However, according to the conclusions of this model, in reality—because of the existence of asymmetric information, the limitation of the bidder's knowledge, and bounded rationality—there are great differences between the conclusions of a classical model and a real-life situation. In the process of an auction, enterprises could acquire more buyers' profit not only by attracting more bidders but also by much more active methods to pursue the maximal profit. The following two strategies summarize these measures.

11.3.3.1 Strategy 1: To Enhance Bidder's Approval Degree to Enterprise Stock Dealership

Figure 11.5 shows that the higher approval degree the bidder has to the bid product, the higher bid price he would quote. Generally, the main measures to implement this strategy are as follows:

1. An enterprise should make efforts to improve its management and the business environment, which are the best promotions of its value.
2. Attracting more bidders enhances the possibility of the approval degree to enterprise stock dealership.
3. Appropriate marketing and image propaganda are beneficial to promote bidders' knowledge of the enterprise stock dealership's value.

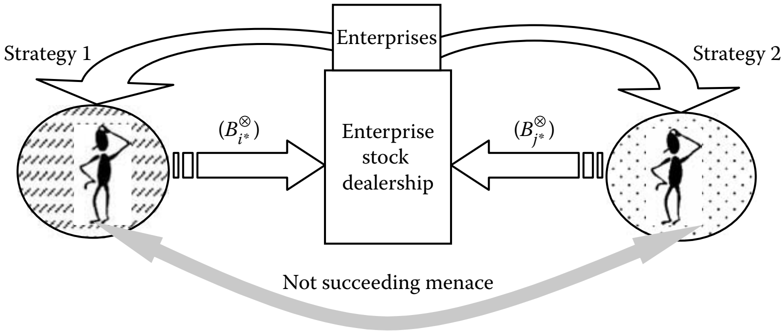


Figure 11.5 Influence of enterprise on its stock dealership bidding process.

11.3.3.2 Strategy 2: To Construct Favorable Environment for Auction of Stock Dealership

A favorable environment should be like that shown in Figure 11.5. Many competitive bidders engage in the bid and the competition is fierce (bidder cooperation cannot be allowed). The main measures to implement this strategy are as follow:

1. Attracting more bidders, especially those with great competitive strength, enhances the competition.
2. Appropriate advertising is beneficial to promote the stock dealership; relevant bidder information, such as knowledge of enterprises' stock dealership's value, competitive strength, and aggressive characteristics, stimulates a competitive atmosphere of the bidders.

11.3.4 Summary

According to economics, choosing the stock dealers by means of auctions can save and promote the reasonable allocation of resources. The first-price sealed-bid auction is an important bidding method, but the classical model involves some severe defects based on the high requirements of a bidder's rationality, judgment ability, and symmetric information. It is impossible to meet these requirements in real situations. The model in this section liberalizes those conditions, believing the enterprise's stock dealership bidding is based on an environment with incomplete information and bounded rationality of the bidders. The optimal quotations are decided by the value of the auction goods to the bidder himself as well as his experience ideal quotation and the threat of an unsuccessful bid if he fails. On the basis of these, the model also reveals that the bidding can be improved by attracting more bidders and taking more strategies and measures in the process.

Chapter 12

Summary and Prospect

12.1 Summary

Because of the incomplete information of the elements (parameters), structure, boundaries, and behavior of the system, together with effects of various random and nonrandom factors, we are unable to precisely judge the potential effects of any strategy in reality. However, our experiences (direct or indirect) allow us to make more precise judgments on the range of effects. Thus, grey game exists universally.

Any game can be considered a grey game. Classic games are a special example of the grey game, while grey game theory is an expansion in the field of incomplete information (or bounded knowledge). Grey games are the broadening of the binding conditions of detailed knowledge (where precise information gains of game strategies can be mastered before implementation) for classic games, so we cannot solve all the problems of this area completely by applying classic game theories to a grey game. The solving processes of a grey game are more difficult and complicated than those for classic games and involve many more factors.

We refer to the structure of classic game theory in research and consider its system of theory structure as the referring system and its theory demand to solve the problem as motivation. Abundant thoughts, theories, models, and methods of systematic science and philosophy, especially grey system theory, are taken into account in solving grey games by applying the models and methods that can be used directly and be brought into the research. Great emphasis is laid on a series of special and complicated problems of grey games raised from the broadening of the binding conditions of the detailed knowledge for classic games. The key problems and different points of this book are shown as the following six aspects:

1. Because of the difficulty of working out the mixed strategy solutions of the grey matrix game led by the demerits of classic grey number operations of arithmetic, this book established a new system of grey number operations of arithmetic to solve the operations of the linear model of grey matrix games, and constructed the grey matrix game model based on pure and mixed strategies. It also proposed and established the concepts and structural system of grey saddle points and grey mixed strategy solutions. The book proved that solutions under the meaning of grey numbers definitely exist for any grey matrix game (namely, the theorem of the largest and smallest grey numbers). A new efficient algorithm of the mixed strategy solutions of grey matrix games—the inverse solution of grey matrix—was established and was proved correct by applying the theorem of matrix algebra algorithm. This book researched the risk of the optimum solution of grey matrix game, the controlling value of information to risk, the strategy of controlling risk, and so forth, according to the special character of grey matrix games and classic matrix games. It brought and solved the problems of risk, measuring and controlling of the optimum mixed solution of grey matrix games caused by a lack of information in the grey matrix game. It established and improved the theoretical system of risk measurement and control in a grey matrix game.
2. This book solved the theoretical mistake of neglecting the whole logic viewpoint of backward induction—the core method for solving classic dynamic games—by applying the whole standpoint of systematic theory and considering human intuition, thinking, and grey systems. It established a new backward induction method for grey structured algorithms based on a systematic whole viewpoint and constructed the concept system of "terminating" and "guiding" Nash equilibrium solutions for multiphase dynamic games as well as the convenient and efficient equilibrium analysis method. As a result, the false theory of the Centipede Game was well analyzed and explained.
3. The book revealed that the most fatal flaw of the classical Cournot oligopoly model as well as the related oligopoly output-making competition model is the supposition of the game players' purpose. In this section, some new suppositions of a duopoly game were established, including the game purpose supposition, the time-order supposition, and bounded rationality and knowledge supposition. A descriptive game structure model that has strong universality to realistic decision-making situations was also constructed. This model could well describe the strategy decision-making problem between a leading company and its subordinating company with the existence of the first-knowing game players. It also could be proved that the classical Cournot game model is a particular kind of this model.

This section brought forward the concepts of damping loss and total damping cost when the first decision-maker completely seizes the whole market; related algorithms were designed.

4. The book proposed one kind of game equilibrium analysis and the existence of an equilibrium point under conditions of incomplete game information. It also established the structure of a still game whose information of profit and loss is correspondingly incomplete, as well as the concepts of equipollence, superior potential degree, and inferior potential degree of grey numbers in the range of profit and loss value; Nash equilibrium of grey potential pure strategy; dominant pure strategy of grey potential; and the position judging rule of grey number relationships among ranges. This part of the book proved that for an *N*-person stable game, if it has incomplete profit and loss information, there exists a Nash equilibrium of grey potential pure strategy, then we can apply the method of grey potential advantaged and disadvantaged strategy, grey potential marking method, and grey potential arrow method in different situations. Then the equilibrium point can be conveniently found and the result of the game can be scientifically predicted.
5. In order to overcome the limitation that traditional evolutionary game models could not compute results of one-shot games or in cases of short-term economic equilibrium, the book designed a chain structure model of evolutionary games based on symmetric income situations. We discovered that individuals in the game are in correlated dependence and in conversion with each other, and established transfer relations between an individual quantity of each population and expected average payments of interrelated parts in each step of the game. The book revealed the symmetric Hawk-Dove Game through simulation experiments.
6. Considering some of defects in the classical first-price sealed-bid auction (FPSBA) model, the book built a bounded rationality optimal grey quotation model based on evaluation of grey value and experiential ideal quotation. A first standard grey transform to grey number of the model helps to find the threat-reacting grey number of bidders. The optimum grey quotation of bidders not only relies on the value itself but also on the value of others and the threat-reacting grey number. The optimum grey quotation of bidders is not just half the recognized value of the object; in general, it is higher than half the recognized value. We get some valuable conclusions that are quite different from those of a classic model through emulation. The best bidding model for bidders is also suggested.

The actual importance of this book's research is in solving a kind of grey game caused by incomplete information, which universally exists and which classic game theory neglects (or is unable to solve), with the help of the abundant thoughts and relating means of the grey system theory; constructing and improving its theoretic system makes it more widely suitable during the process of solving real problems. The conclusions explain and guide us better in real life, and thus the project has important real value.

The theoretical importance of research of the paper is to extend classic game theory to the area of grey systems of bounded knowledge (or incomplete information) and to bring the problem of incomplete information into the area of game theory. As a result, the area of research objects is extended. The theoretical system is enriched and developed, as are the thoughts and theoretical systems. Exploration of the false theory of the Centipede Game, which has puzzled academia for a long time, will largely improve the theoretical system of core method—backward induction, for perfect Nash equilibrium analysis in subgame. This project has had a significant impact on game theory.

12.2 Research Prospects

The thesis mainly involves game theory, grey system theory, and other research domains that apply the grey system idea, theory, and method in game theory to other fields. It is used to resolve the game problem of bounded knowledge (or information loss), which universally exists but is not involved (or cannot be solved) in classical game theory. The subject pulls the missing information problem into game theory domain and makes game theory widespread to the grey system domain of bounded knowledge and poor information.

The publication of *Game Theory and Economy Behavior* by John Von Neumann and Oskar Morgenstern in 1944 marks game theory's beginnings. By the 1950s, cooperation game theory had developed to the peak and noncooperation game theory came into being. Through nearly sixty years of development, noncooperation game theory has become an integrated theory system. It mainly involves full information statistics, full information dynamics, inadequate information statistics, inadequate information dynamic problems, and so on. Within classical game theory always existed a rational confusion problem that could not be solved satisfactorily until now. Evolutionary game theory came into being by giving up the hypothesis of complete rationality of classical game theory, and treats living creature as players with bounded rationality. They complete their evolution while they are competing with each other. Evolutionary game theory also interprets how some players' behaviors take shape (Maynard Smith, 1982). Especially after Maynard and Price (1973), Maynard proposed the evolutionary game's basic concept of evolutionary stable strategy, and it developed rapidly. Nowadays, game theory applies to almost every field of our economic and social lives, and it has had a brilliant result. Through the whole history of the development of game theory, it has perfected itself while answering real-life questions.

Looking back to the development history of the research and application of game theory, today we can say that game theory applies successfully not only to economics but also to many other fields, and it has had a long-lasting influence. For instance, the definition and resolution of the Prisoner's Dilemma have important effects on several subjects and fields such as economics, sociology, politics, crime psychology, and so on. For a long time, scholars have paid special attention to game theory. In recent years, the game theory problem focus on the uncertain game problem has also aroused the attention of academia. This book is an initiative discussion of the (grey) game problem based on the uncertainty of poor information and has gotten some valuable initial results. However, the research of this area is just a start; in the future, it may break through in problems such as the following:

12.2.1 Research on Grey Matrix Game's Optimal Mix Strategy Solution Algorithm

For the problem of grey matrix game $G(\otimes)$, the grey saddle point does not always exist. If it doesn't exist, it will have vital academic and practical meaning regarding how to find the problem's mixed strategy solution.

The difficulties of solving a grey matrix game's mixed strategy solution is that the original grey number operation system cannot meet the needs of the operation; initial research shows that it is easy to amplify the grey mixed strategy solution to a meaningless and abnormal black number with original grey number operation methods. Finding this solution to the operation problem would involve using standard interval grey number rules; transforming interval grey numbers $[a_{ij}, b_{ij}]$ in Eq. (1.1) into the standard form of an interval grey number $a_{ij} + (b_{ij} - a_{ij}) \cdot \gamma_{ij}$, where $i = 1, \ldots, m,\ j = 1, \ldots, n.$ Divide the grey number into white part a_{ij} and grey part $(b_{ij} - a_{ij}) \cdot \gamma_{ij}$, and $\gamma_{ij} \in [0,1]$ represents unit interval grey number. The uncertainty of grey numbers caused by information loss mainly reflects on the obtained number $\gamma_{ij} \in [0,1]$, and establishes the algorithm model of $G(\otimes)$ by linear programming theories of the classical game problem to transform the problem that cannot be solved by original grey number operation system into another optimized problem that depends on $\gamma_{ij} \in [0,1]$. It is a highly effective and optimized algorithm that considers using advanced thoughts of optimizing algorithms such as the inheritance algorithm, the ant flock algorithm, and so on to design the solution of the game's grey mixed strategy.

A new efficient algorithm of mixed strategy solution of a grey matrix game is the algorithm design of "the inverse solution of grey matrix." Although it can solve the $G(\otimes)$ problem by using the linear programmed algorithm model of the optimal mixed strategy solution for a grey matrix game based on the inheritance (or ant flock) algorithm, it is quite complicated and requires a large quantity of work. It is difficult to apply current computer software to this problem. This project is based on the thoughts of matrix theory algorithm, and establishes the inverse solution of a grey matrix, which is a new efficient algorithm of the grey matrix game optimal mixed strategy solution.

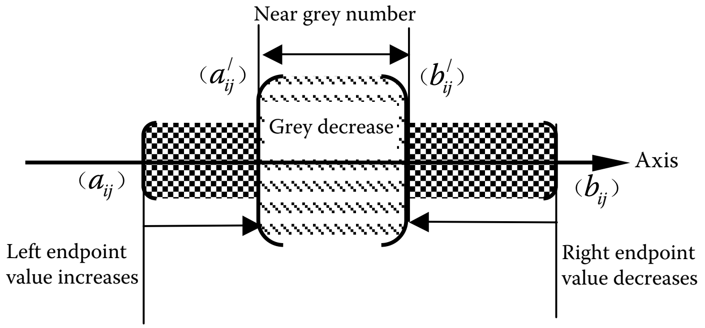
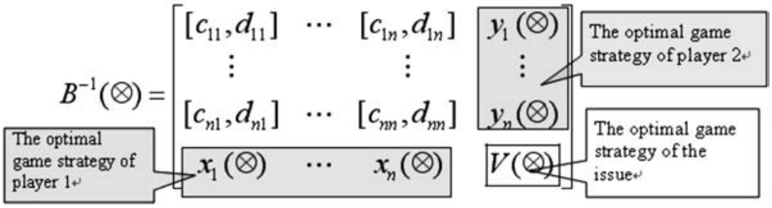


Figure 12.1 Change of the right and left value with grey decrease.

Using the relative theory of "rank" and "inverse matrix" in classical matrix theory and the principle of not adopting an inferior strategy in classical matrix game is another point of research. (To a game player, if his strategy is opposite to his opponent's and the yield from the game is not higher than the other's, then the strategy is called an inferior strategy. During the game process, a game player will not use an inferior strategy.) A buildup of the algorithm of $G(\otimes)$ in which $A(\otimes)$ does a full-rank judgment and the full-rank treatment of a not-full-rank matrix (which increases the virtual game strategy) to obtain the full-rank treatment grey profit and loss matrix $A^{/}(\otimes)$ does not change the nature of original game; all game players' optimal game strategies and the optimal game are not changed. Then construct an inequality group of game problems by using $A^{/}(\otimes)$ to prove that the inverse matrix $B^{-1}(\otimes)$ [see Eq. (12.2)] of $A^{/}(\otimes)$'s expanding matrix $B(\otimes)$ [see Eq. (12.1)] must exist. Its solution (which may include a virtual strategy's solution, and its probability is 0) are all expressed by $B^{-1}(\otimes)$ [(see Eq. 12.2 and Figure 12.1).

$$B(\otimes)=\begin{bmatrix} [a_{11},b_{11}] & \cdots & [a_{1n},b_{1n}] & 1 \\ \vdots & & \vdots & \vdots \\ [a_{n1},b_{n1}] & \cdots & [a_{nn},b_{nn}] & 1 \\ 1 & \cdots & 1 & 0 \end{bmatrix} \tag{12.1}$$



$$B^{-1}(\otimes)=\left[\begin{array}{ccc|c} [c_{11},d_{11}] & \cdots & [c_{1n},d_{1n}] & y_1(\otimes) \\ \vdots & & \vdots & \vdots \\ [c_{n1},d_{n1}] & \cdots & [c_{nn},d_{nn}] & y_n(\otimes) \\ \hline x_1(\otimes) & \cdots & x_n(\otimes) & V(\otimes) \end{array}\right] \tag{12.2}$$

12.2.2 Venture Control Problem of the Optimal Grey Matrix Game Solution

For the game $G(\otimes)$, besides concern about whether it has an optimal solution in terms of grey numbers and how to solve these problems, people may care more about whether there is risk and how big the risk is when making decisions based on these optimal strategy solutions.

Study of risk recognition and definition problems: Consider the maximum (minimum) possible overrated risk of a game player in $G(\otimes)$ as the most optimistic (pessimistic) forecast judgment; however, a player gets the most pessimistic (optimistic) result after he evaluates risk. Define the maximum overrated risk value of Players 1 and 2 as $\sigma^M_{high,1}$ and $\sigma^M_{high,2}$ respectively, and the maximum underrated risk value as $\sigma^M_{low,1}$ and $\sigma^M_{low,2}$, and study the interrelationship between these risk values and the relationship with the optimal grey game value $V_G(\otimes) = [v_L, v_R]$.

Study of risk controlling ability and value of information and venture control problem of $G(\otimes)$: The root of the risk of $G(\otimes)$ is that poor information restricts people from making accurate determinations of the game results of each strategy beforehand and results in the existence of grey elements in game profit and loss value matrix $A(\otimes)$. The contribution of lowering $G(\otimes)$'s risk by game players to decrease the grey degree of some grey elements in $A(\otimes)$ by making use of some information (shown in Figure 12.2) is different from that by the whitenization of $A(\otimes)$'s grey elements (which involves taking advantage of valid information to lower its grey degree). Information is different in relation to $G(\otimes)$'s controlling ability. Further research would involve a study of the value of the information and risk controlling ability; building up a discovery algorithm model of grey elements that is sensitive to the risk in $A(\otimes)$, finding the sensitive grey elements and the key information that influence $G(\otimes)$, and looking for the risk controlling strategy (see Figure 12.2).

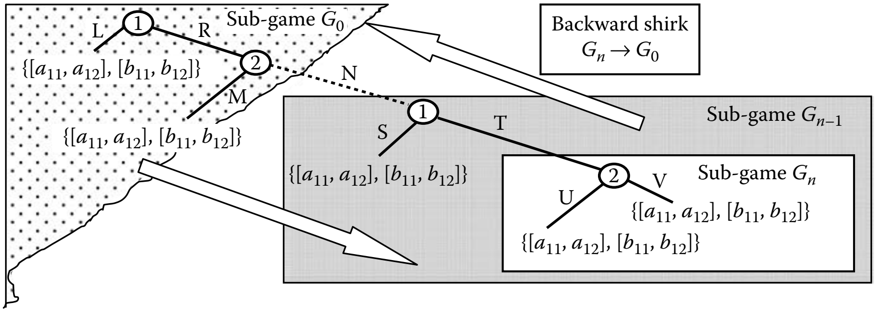


Figure 12.2 Sketch of the solution of the subgame perfect Nash equilibrium by inverted sequential deduction of grey numbers.

12.2.3 Solution of a Subsidiary Game's Grey Potential

This would involve study of the solution of a subsidiary game's grey potential perfect Nash equilibrium and its refined equilibrium, and the risk of a grey potential solution and its control. Based on the thought of classical subsidiary games and the perfect Nash equilibrium, study the perfect Nash equilibrium algorithm of solving subsidiary game by using a grey potential structured deduction and prove its reasonability.

Further investigation would involve making use of classical thought of *thrilling hand equilibrium* and the algorithm rule of interval grey numbers in Ref. 4 to study the calculating method of both static and dynamic games' thrilling hand equilibrium symbolized by a grey interval number.

One could make use of the theory results of the risk of the optimal grey matrix game solution and venture control problem in this book, and study the application of the grey structured algorithm on the recognition and definition of risk when solving subsidiary game perfect Nash equilibrium solutions by backward induction. Study of the cumulative and transmission effect as well as the controlling capacity of the information risk and some relatively essential problems would be necessary to reveal its intrinsic laws.

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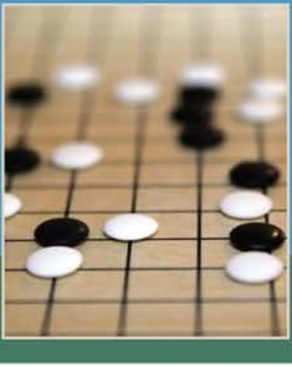
Systems Engineering

GREY GAME THEORY AND ITS APPLICATIONS IN ECONOMIC DECISION-MAKING



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To make the best decisions, you need the best information. However, because most issues in game theory are grey, nearly all recent research has been carried out using a simplified method that considers grey systems as white ones. This often results in a forecasting function that is far from satisfactory when applied to many real situations. ***Grey Game Theory and Its Applications in Economic Decision-Making*** introduces classic game theory into the realm of grey system theory with limited knowledge. The book resolves three theoretical issues:

- A game equilibrium of grey game
- A reasonable explanation for the equilibrium of a grey matrix of static nonmatrix game issues based on incomplete information
- The Centipede Game paradox, which has puzzled theory circles for a long time and greatly enriched and developed the core methods of subgame Nash perfect equilibrium analysis as a result

The book establishes a grey matrix game model based on pure and mixed strategies. The author proposes the concepts of grey saddle points, grey mixed strategy solutions, and their corresponding structures and also puts forward the models and methods of risk measurement and evaluation of optimal grey strategies. He raises and solves the problems of grey matrix games. The book includes definitions of the test rules of information distortion experienced during calculation, the design of tokens based on new interval grey numbers, and new arithmetic laws to manipulate grey numbers. These features combine to provide a practical and efficient tool for forecasting real-life economic problems.

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